Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

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por

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"O esforço supera tudo, inclusive ao talento ". (Romulo Diaz)

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O estudo de problemas elípticos do tipo Kirchhoff-Boussinesq não linear

Resumo

Nesta tese, estudaremos existência e multiplicidade de soluções para a seguinte classe de problemas:

$$(P_i) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V(x)u = f(u) + \beta|u|^{2_{**}-2}u \text{ in } \Omega, \\ u \in H^2 \cap H_0^1(\Omega), \end{cases}$$

onde (P_i) $(i = 1, 2, 3)$ correspondem aos três problemas considerados nos capítulos 1-3, respectivamente, $\Omega \subset \mathbb{R}^N$ é um domínio suave, no caso $\beta = 0$ obtemos $2 < p < 2^* = \frac{2N}{N-2}$, para $N \geq 3$ e o caso $\beta = 1$ consideramos $2_{**} = \frac{2N}{N-4}$ para $N \geq 5$.

O Capítulo 1 é dedicado a provar um resultado de existência de soluções para o problema (P_1) quando $V = 0$ e $\beta = 0$, onde $\Omega \subset \mathbb{R}^4$ é um domínio com fronteira suave, $2 < p < 4$ e f é uma função contínua superlinear com crescimento exponencial subcrítico ou crítico. Aplicamos o método de Nehari para provar o resultado principal.

No Capítulo 2 é dedicado a provar a existência e multiplicidade de soluções para o problema (P_2) quando $V = 0$ e $\beta \in \{0, 1\}$, onde $\Omega \subset \mathbb{R}^N$ é um domínio limitado e suave e f é uma função contínua. Mostramos a existência e multiplicidade de soluções não triviais usando técnicas de minimização na variedade de Nehari, Teorema de Passo da Montanha e Teoria do Gênero.

No Capítulo 3 é dedicado a provar a existência de uma solução de estado fundamental para o problema (P_3) quando $\beta \in \{0, 1\}$. Aqui V e f são funções contínuas com V sendo periódica ou assintótica ao infinito. A função f tem crescimento subcrítico ou critico.

Palavras-chave: Operador biharmônico; p-Laplaciano; problemas do tipo Kirchhoff-Boussinesq; métodos variacionais; crescimento exponencial crítico.

Abstract

In this thesis, we study the existence and multiplicity of solutions for the following class of problems

$$(P_i) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V(x)u = f(u) + \beta |u|^{2_{**}-2}u \text{ in } \Omega, \\ u \in H^2 \cap H_0^1(\Omega), \end{cases}$$

where (P_i) $(i = 1, 2, 3)$ correspond to the three problems we considered in Chapters 1-3, respectively, $\Omega \subset \mathbb{R}^N$ is a smooth domain, in the case $\beta = 0$ we get $2 < p < 2^* = \frac{2N}{N-2}$, for $N \geq 3$ and the case $\beta = 1$ we consider $2_{**} = \frac{2N}{N-4}$ for $N \geq 5$.

The Chapter 1 is devoted to existence result of solutions for the problem (P_1) when $V = 0$ and $\beta = 0$, where $\Omega \subset \mathbb{R}^4$ is a smooth bounded domain, $2 < p < 4$ and f is a superlinear continuous function with exponential subcritical or critical growth. We apply the Nehari manifold method to prove the main results.

In Chapter 2 we establish an existence and multiplicity of solutions for the problem (P_2) when $V = 0$ and $\beta \in \{0, 1\}$, where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain and f is a continuous function. In this chapter, we show the existence and multiplicity of nontrivial solutions by using minimization technique on the Nehari manifold, the Mountain Pass Theorem and Genus theory.

In Chapter 3 is concerned with the existence of a ground state solution for the problem (P_3) when $\beta \in \{0, 1\}$. Here V and f are continuous functions with V being either periodic or asymptotic at infinity to a periodic function. The function f has subcritical or critical growth

Key words: Biharmonic operator; p-Laplacian; Kirchhoff-Boussinesq type problems; variational methods; critical exponential growth.

Contents

Introduction

Partial differential equations are an important branch of mathematics with a wide range of applications and a powerful tool to understand certain areas of applied and purely mathematical aspects. For instance, they arise in physics, engineering, thermodynamics, diffusion, electrodynamics, fluid dynamics, differential geometry, calculus of variations, numerically approximate, etc. Despite the fundamental utility of the partial differential equations, pioneer research interests have a considerable attention to this area and we encounter many types of equations. Especially, since their discovery by Boussinesq in 1872 [23], the Boussinesq equations have represented one of the most powerful tools for modelling some physical phenomena. They are used to describe properties and time evolution of physical systems, the propagation of water waves, the vibration in a string, waves in a plasma, nonlinear lattice wave, Bose-Einstein condensates, geotechnical and road engineering: influence of plasticity, etc (see [21–23, 76, 89] and the references therein for more details).

Later, the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \big| \frac{\partial u}{\partial x} \big|^2 \, dx \Big) \frac{\partial^2 u}{\partial x^2} = 0, \tag{0.0.1}$$

was used to extend the classical D'Alembert's wave equation by Kirchhoff [51] due to the influence of the changing in the length of the string during the vibration. The meaning of the parameters in (0.0.1) are as follows: L denotes the length of the string, h means the area of the cross-section, E stands for the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

The motivation for this thesis comes from evolution partial differential equations, so we begin with a short introduction of such equations. It is worth mentioning that plate equations have received much attention over the recent years. The plate equations originated from engineering mechanics. In the continuous medium mechanics, they are defined as planar structures with very small thickness. In particular, the following plate equation with perturbation of p-Laplacian type has been widely investigated:

$$\begin{cases} u_{tt} + \Delta_x^2 u - \beta \operatorname{div}_x \left(a_x(\nabla u) \right) = F(u, u_t) \quad \text{in } \Omega \times (\tau, \infty), \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \\ u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{in } \partial\Omega \times (\tau, \infty), \end{cases} \tag{0.0.2}$$

where $\Delta^2 = \Delta(\Delta)$ denotes the biharmonic operator, $a(s) \approx |s|^{p-2}s$, $p \geq 2$, and $F(u, u_t)$ represents additional damping and forcing terms.

In $1D$ setting, as noticed in Yang [98], this kind of problem models flows of elastoplastic microstructures. The main physical justifications come from a model of elastoplastic microstructure flow

$$u_{tt} = \varepsilon u_{xxxx} + b(u_x^2)_x, \quad b < 0, \tag{0.0.3}$$

considered by An and Peirce [9].

Also Ma and Pelicer [66] were concerned with a class of weakly damped one-dimensional beam equations with lower order perturbation of p-Laplacian type

$$u_{tt} + \varepsilon u_{xxxx} - (\sigma(u_x))_x + ku_t + f(u) = h \quad \text{in } (0, L) \times \mathbb{R}^+, \tag{0.0.4}$$

where $\sigma(z) \approx |z|^{p-2}z$, $p \geq 2$, $k > 0$ and $f(u)$ and $h(x)$ are forcing terms. The study of the model (0.0.4), including the well-posedness, exponential stability and existence of a finite-dimensional attractor, was carried out.

Qu and Zhou in [73] used the potential well method to obtain global existence and finite time blow-up threshold results for weak solutions of the following equations with a non-local source term:

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{p-1}u\, dx \quad \text{in } \ \Omega \times (0, T), \\ u_x = u_{xxx} = 0 \quad \text{in } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{0.0.5}$$

where $\Omega = (0, a)$, $p > 1$ and $u_0 \in H^2(\Omega)$ with $\displaystyle\int_\Omega u_0 dx = 0$ and $u_0 \neq 0$. The model (0.0.5) is a fourth-order reaction–diffusion equation, which arises in many physical applications, such as thin film theory, lubrication theory, phase transition, etc. They also studied the extinction of the solutions for the problem under some suitable conditions.

The authors Li, Gao and Han [58] investigated the following thin-film equation with the same initial and boundary conditions

$$u_t + u_{xxxx} - (|u_x|^{p-2}u_x)_x = |u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u dx \quad (x, t) \in (0, a) \times (0, T), \tag{0.0.6}$$

where $p > 1$, $q > \max\{1, p-1\}$ and obtained similar results to those in [73] via the potential well method.

Concerning a nonlinear $2D$ plate equation, also known as a Kirchhoff-Boussinesq model, the analysis was discussed by Chueshov and Lasiecka in [29–31]. They focused on the well-posedness and long-time behavior of the following problem by only considering a weak damping

$$\begin{cases} u_{tt} + ku_t + \Delta^2 u = \operatorname{div}\,[f_0(\nabla u)] + \Delta[f_1(u)] - f_2(u), \\ u(x, 0) = u_0(x) \quad u_t(x, 0) = u_1(x), \quad x \in \Omega \end{cases} \tag{0.0.7}$$

defined on a bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$. Here $k \geq 0$ is the damping parameter, the mapping $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and the scalar, sufficiently smooth functions f_1 and f_2 represent (nonlinear) feedback forces acting upon the plate. Their main goal is to study the well-posedness and asymptotic behavior of finite energy solutions associated with (0.0.7).

In order to motivate the structure of nonlinear terms appearing in (0.0.7), we would like to point out that model (0.0.7) arises naturally as the limit in Midlin-Timoshenko (MT) equations which describe the dynamics of a plate that accounts for transverse shear effects (see, e.g., [54], [55, Chap.1] and the references therein). More specifically, the (MT) system is given in the following canonical form:

$$\alpha[v_{tt} + kv_t] - Av + k(v + \Delta u) + h_0(v) + vh_1(u) = 0, \tag{0.0.8}$$

$$u_{tt} + k_0 u_t - k\text{div}(v + \Delta u) + h_2(u), \tag{0.0.9}$$

where $v(x,t) = (v_1(x,t), v_2(x,t))^T$ is a vector function and $u(x,t)$ is a scalar function on $\Omega \times \mathbb{R}^+$. The functions $v_1(x,t)$ and $v_2(x,t)$ represent the angles of deflection of a filament (they are measures of transverse shear effects) and $u(x,t)$ is the bending component (transverse displacement). The parameter $\alpha > 0$ describes rotational inertia of filaments. The terms $\alpha k v_t$ and $k_0 u_t$ represent resistance forces (with the strengths $k > 0$ and $k_0 > 0$). The factor $k > 0$ is the so-called shear modulus. The second order elliptic differential operator A in (0.0.8) is defined by:

$$A = \begin{bmatrix} \partial^2_{x_1} + \dfrac{1-\mu}{2}\partial^2_{x_2} & \dfrac{1+\mu}{2}\partial^2_{x_1 x_2} \\ \dfrac{1-\mu}{2}\partial^2_{x_1 x_2} & \dfrac{1-\mu}{2}\partial^2_{x_1} + \partial^2_{x_2} \end{bmatrix},$$

where $0 < \mu < 1$ is the Poisson ratio. The mapping $h_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and the scalar functions h_1 and h_2 represent feedback forces. Here we note that the presence of the term $vh_1(u)$ destroys the conservative character of these forces and, as we will see later, the energy of the associated dynamical system will be no longer decreasing along trajectories. As we know, the dynamics of Mindlin-Timoshenko plates has been widely investigated in [54, 55] and also in [29] they proved the existence of a compact global attractor and studied its limiting properties when the shear modulus tends to infinity. This limit corresponds to absence of transverse shear which is one of the Kirchhoff hypotheses in the plate theory. Notice that the Mindlin-Timoshenko system (0.0.8)-(0.0.9) subjected to nonlinear damping has been discussed in [31, Chap.7].

In an interesting article, Chueshov and Lasiecka [32] considered the following nonlinear plate equation referred to as Kirchhoff–Boussinesq (K–B) model:

$$w_{tt} + kw_t + \Delta^2 w = \text{div}\left(|\nabla w|^{p-2}\nabla w\right) + \sigma\Delta(w^2) - f(w) \tag{0.0.10}$$

defined on a bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary and a suitable initial data. Here $k \geq 0$ is the damping parameter, the right-hand side of (0.0.10) represents a feedback force acting upon the plate and the parameter σ is nonnegative.

Another high-impact model is the Schrödinger-Kirchhoff-Boussinesq equation with boundary damping

$$M_\alpha w_{tt} + \Delta^2 w + a(x)\Big[g(w_t) - \alpha\text{div}G(\nabla w_t)\Big] = \text{div}(|\nabla w|^2\nabla w) + R(w). \tag{0.0.11}$$

The damping functions $g : \mathbb{R} \to \mathbb{R}_+$ and $G : \mathbb{R}^2 \to \mathbb{R}^2_+$ have the following form

$$g(s) = g_1 s + |s|^{m-1}s \quad \text{and} \quad G(s,\sigma) = G_1(s;\sigma)(s;\sigma) + (|s|^{m-1}s; \sigma^{m-1}\sigma),$$

where g_1 and G_1 are nonnegative constants. The boundary damping is the same as the previous case, i.e., $g_0(s) = g_2 s + |s|^{q-1}s$, $q \geq 1$. The source term R is assumed locally

Lipschitz operator acting from $H^2(\Omega)$ into $L^2(\Omega)$ when $\alpha = 0$ and from $H^2(\Omega)$ into $H^{-1}(\Omega)$ when $\alpha > 0$. The associated energy function has the form

$$E(t) = \frac{1}{2}(\|w_t(t)\|^2 + \alpha\|\nabla w_t(t)\|^2) + \frac{1}{2}\|\Delta w(t)\|^2 + \frac{1}{4}\int_\Omega |\nabla w_t(t)|^4 dx.$$

The well-posedness of (0.0.11) in the case of $\alpha > 0$ is standard. This is due to the fact that $\mathrm{div}(|\nabla w|^2 \nabla w) \in H^{-1}(\Omega)$ for finite energy solutions w. The case $\alpha = 0$ is subtle. Its analysis requires special consideration and depends on linearity of the damping. Let us now pay attention to the qualitative behavior of the system consisting of coupled Boussinesq and Schrödinger equations in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. The resulting system takes the form:

$$w_{tt} + \gamma_1 w_t \Delta^2 w - \Delta(f(w) + |E|^2) = g_1(x) \quad \text{in } \Omega, \tag{0.0.12}$$

$$iE_t + \Delta E + i\gamma_2 E = g_2(x) \quad \text{in } x \in \Omega,\ t > 0, \tag{0.0.13}$$

where $E(x,t)$ and $w(x,t)$ are unknown functions, $E(x,t)$ is complex and $w(x,t)$ is real. Here the above γ_1 and γ_2 are nonnegative parameters and $g_1(x)$ and $g_2(x)$ are given (real and complex) L^2-functions. Equipping equations (0.0.12)-(0.0.13) with the boundary conditions

$$w = \Delta w = 0, \quad E = 0, \quad x \in \partial\Omega \tag{0.0.14}$$

and the initial data

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad E(x,0) = E_0(x) \quad x \in \Omega. \tag{0.0.15}$$

The dynamics of this system was studied in [33] by the previous methods under the following hypotheses when the (nonlinear) function $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and

$$\exists\ C_1, C_2 \geq 0 : \ F(r) = \int_0^r f(\xi)d\xi \geq -C_1 r^2 - C_2, \quad \forall\ |r| \geq r_0, \tag{0.0.16}$$

$$\exists\ M \geq 0,\ p \geq 1 : \ |f'(s)| = M(1 + |s|^{p-1}), \quad \forall\ s \in \mathbb{R}. \tag{0.0.17}$$

The method used in [33] is the same as in the study of the (K-B) system (see (0.0.11) with $\alpha = 0$), and can also be applied to the following Schrödinger-Boussinesq-Kirchfoff model:

$$w_{tt} - \Delta w_t + \Delta^2 w - \mathrm{div}(|\nabla w|^3 \nabla w) - \Delta(w^2 + |E|^2) = g_1(x) \quad \text{in } \Omega, \tag{0.0.18}$$

$$iE_t + \Delta E - wE + i\gamma_2 E = g_2(x) \quad \text{in } x \in \Omega,\ t > 0 \tag{0.0.19}$$

equipped with the boundary conditions (0.0.14) and initial data (0.0.15).

For N-dimensional setting, Yang et al. in [98, 99, 101] considered the strongly damped equation as follows:

$$u_{tt} + \Delta^2 u - \mathrm{div}\left(\sigma(|\nabla u|^2 \nabla u)\right) - \Delta u_t = h(t, u, u_t) \tag{0.0.20}$$

with both clamped and simply supported boundary conditions. Their results were mainly concerned with the global existence and long-time behavior of weak and strong solutions.

In addition, Yang in [99, 100] investigated the regularity and Hausdorff dimensions of global attractors, also the existence of the finite dimensional global and exponential attractors for the dynamical system associated with the Kirchhoff models arising in elastoplastic flow

$$\begin{cases} u_{tt} + \Delta^2 u - \operatorname{div}\left(|\nabla u|^{m-1}\nabla u\right) - \Delta u_t + h(u_t) + g(u) = f(x) \text{ in } \Omega \times (\tau, \infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \\ u_t(x,0) = u_1(x) \quad \text{in } \Omega, \\ \Delta u = u = 0 \quad \text{in } \partial\Omega \times (\tau, \infty), \end{cases} \tag{0.0.21}$$

where $1 \le m < \frac{N+2}{N-2}$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the assumptions on nonlinear terms $h(u_t), g(u)$ and external force term f are suitable.

Yang and Nascimento in [97] focused on the long-time behavior for a class of non-autonomous plate equations with perturbation and strong damping of p-Laplacian type

$$u_{tt} + \Delta^2 u + a_\varepsilon(t)u_t - \Delta_p u - \Delta u_t + f(u) = g(x,t) \tag{0.0.22}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and critical nonlinear terms. The global existence of weak solution which generates a continuous process was proved firstly, then the existence of strong and weak uniform attractors with non-compact external forces was also derived. Moreover, the authors established the upper-semicontinuity of uniform attractors under small perturbations by delicate estimate and contradiction argument.

Andrade D, J Silva and T. F. Ma in [65] studied the energy drop for a class of memory plate equations and lower order perturbation of $p-$Laplacian type

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta_p u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + f(u) = 0 \text{ in } \Omega \times \mathbb{R}^+, \\ \Delta u = u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \end{cases} \tag{0.0.23}$$

They obtained the existence of global solutions and energy decay to the mixed problem

Sun, Liu and Wu in [79] considered the initial boundary value problem for a class of thin-film equations in $\mathbb{R}^N$ with a p-Laplacian term and a nonlocal source term

$$\begin{cases} u_t + \Delta^2 u - \Delta_p u = |u|^{q-2}u + \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{q-2}u dx \text{ in } \Omega \times (\tau, \infty), \\ \dfrac{\partial \Delta u}{\partial \eta} = \dfrac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{0.0.24}$$

They obtained the upper bounds for the blow-up time, proved the existence and blow-up in finite time of solutions for the problem with arbitrarily initial energy.

From now on, we are going to introduce the stationary equations associated with the problems mentioned above of the perturbation of the biharmonic operator by the p-Laplacian operator.

Fourth-order Schrödinger equations have been widely considered due to the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity, see [48,49]. Such equations have been studied from the mathematical view point finding existence, multiplicity, nonexistence results focusing in the nonlinear problems driven by the biharmonic operator. In the same way, biharmonic equations or even their higher version of polyharmonic equations hove received great attention due to their wide application in physic and geometry. In fact, as mathematical modeling, biharmonic equations can be used to describe some physics phenomena, which have many applications such as engineering applications in the deformation of thin plates, the motion of fluids, free boundary problems and nonlinear elasticity. Here we refer interested readers to the historical details in [1,16,17,40]. The difference between these applications lies in the choice of nonlinearities, where types could be considered: polynomial, exponential, logarithmic, singular, etc. On the other hand, there are in some particular aspects of operators and potentials. The latter include positive potential, potential changing sign, singular potential, etc. The fourth order operator has been studied extensively by many authors in recent times.

Let us now pay attention to the following biharmonic nonlinear Schrödinger (NLS) equation

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} - \alpha\Delta^2\Psi - \gamma\Delta\Psi + W(x)\Delta\Psi = \tilde{f}(x,\Psi) + \beta|\Psi|^{2_{**}-2}u \quad \text{in } \mathbb{R}^N, \\ \\ \Psi(0,x) = u \text{ for each } x \in \mathbb{R}^N \text{ where } u \in H^2(\mathbb{R}^N), \end{cases} \tag{0.0.25}$$

where $\alpha > 0$, $\gamma \in \mathbb{R}$, $\beta \in \{0,1\}$ and $W : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential. Assume also that $\tilde{f} : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a continuous nonlinearity. Hence the standing waves to the (NLS) equation given in (0.0.25) are solutions $\Psi : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ in the following form

$$\Psi(x,t) = e^{-ikt}u(x), \quad t \geq 0, \ x \in \mathbb{R}^N, \tag{0.0.26}$$

where $k \in \mathbb{R}$. Then the standing wave Ψ given in (0.0.26) is a solution for the (NLS) equation when $u : \mathbb{R}^N \to \mathbb{R}$ is a solution for problem (0.0.25) assuming that $\tilde{f}(x, \exp(-ikt)u) = e^{-ikt}u(x)f(x,u)$ and $V(x) = W(x) + k$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. Here for the dynamic of the (NLS) equation we refer the interested readers to [12,19,20] for more details.

For the case where Ω is a bounded domain of $\mathbb{R}^N$, the problems involving the biharmonic operator with the Navier boundary condition are modeled in the space $H = H^2(\Omega)\bigcap H_0^1(\Omega)$, in which certain difficulties arise that we will mention below. In general, the principle of the strong maximum is not fulfilled. This fact is one of the difficulties in working with this operator, because given a function $u \in H$, we do not necessarily have the positive and negative parts of this function, respectively $u^+ = \max\{0,u\}$ and $u^- = \min\{0,u\}$, are in H (see [41]). This is a very common method when working with the Laplacian operator, as the space in question is $H_0^1(\Omega)$ and, in this case, for any $u \in H_0^1(\Omega)$, we have that u^+ and u^- also belong to the space $H_0^1(\Omega)$. We also do not have Harnack-type inequalities and the non-positivity, in general, of the Green function of the biharmonic operator. Finding a solution to problem (0.0.25) when Ω is an unbounded domain becomes more difficult due to the lack of compact embedding from $H_0^1(\Omega) \cap H^2(\Omega)$ into $L^p(\Omega)$. In general, the nolinearity of the function f helps overcome this difficulty.

Also elliptic problems involving fourth order operators have been extensively studied. In 1987, Mckenna and Walter [68] modeled nonlinear oscillations occurring in suspension bridges, describing the behavior of each of these oscillations. More precisely, the study involved analyzing an operator of the type $\mathcal{L}u = u_{tt} + u_{xxxx}$. Furthermore, the study of the existence of solutions to the following linear system is still an open problem.

$$\begin{cases} \Delta^2 u + bu = f(x), \quad x \in \Omega, \\ u = \Delta u = 0, x \in \partial\Omega, \end{cases} \tag{0.0.27}$$

where $0 \leq b \leq |\Omega|$.

Another prominent work in the theory is that of Lazer and Mckenna from 1990 [56], in which the authors modeled nonlinear oscillations that occurred specifically on the Golden Gate and Tacoma Narrows bridges. In this paper, the authors described the behavior and type of each of these oscillations, classifying them as vertical, horizontal or even torsional oscillations.

In the case where Ω is a bounded domain on $\mathbb{R}^N$ with smooth $\partial\Omega$ and Navier boundary conditions, Lazer and McKenna in [57] studied the following problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u), \quad x \in \Omega, \\ u = \Delta u = 0, x \in \partial\Omega, \end{cases} \tag{0.0.28}$$

where $f(x,t) = b[(t+1)^+ - 1]$, b is a constant. They used Rabinowitz's global bifurcation method to prove the trivial existence of solutions for $b < \lambda_1(\lambda_1 - c)$ and some showed that there is a positive solution if only if $b = \lambda_1(\lambda_1 - c)$, where λ_1 the principal eigenvalue of $(-\Delta, H_0^1(\Omega)$. In recent times, several authors have studied this type of operator with different boundary conditions and in unbounded domains, for which we can see some works [42, 63, 90, 93, 103] and the references therein. We can also see such an operator with logarithmic or exponential nonlinearity in [7, 62, 75, 85].

The study for $\Omega \subset \mathbb{R}^4$ in this thesis is motivated by Adams (see [7]) who studied the generalized version of Trudinger result to the Sobolev space $W_0^{m,p}(\Omega)$. The hypothesis (f_1) established in Chapter 1 of this thesis, implies that f in has exponential growth which is called critical when $\alpha_0 > 0$. This kind of growth is driven by a Trudinger-Moser type inequalities [26,37]. The concept of criticality in $\mathbb{R}^2$ was introduced by Yadava [95] motivated by the classical Trudinger-Moser inequality (see [69, 88]). By using this notion, perturbed problems involving critical exponential growth were explored in bounded domains (see [71, 72, 85, 87]) and in the whole space (see [26, 75] and references therein). There are many recent works involving a nonlinearity with critical exponential growth, where the existence and concentration of solutions have been considered (e.g., see [6, 36, 38]).

For more detail let us leave the following two theorems as inspiration for Chapter 1 of this thesis.

Theorem 0.0.1. *(Adams, 1988) If m is a positive integer less than N, then there is a constant $\beta_0 = \beta_0(m, N)$ such that for all $u \in C^m(\mathbb{R}^N)$ with support contained in Ω and $p = \frac{N}{m}$*

$$\sup_{u \in W^{m,p}(\Omega): \|\nabla^m u\|_p^p \leq 1} \int_\Omega \exp\left(\beta |u|^{p'}\right) dx \leq \beta_0, \tag{0.0.29}$$

for all $\beta \leq \beta_0 = \beta_0(m, N)$ where

$$\beta_0 = \beta_0(m,N) = \begin{cases} \dfrac{N}{w_{N-1}} \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma\left(\dfrac{m+1}{2}\right)}{\Gamma\left(\dfrac{N-m+1}{2}\right)} \right]^{p'}, & \textit{if } m \textit{ is odd },\\ \dfrac{N}{w_{N-1}} \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma\left(\dfrac{m}{2}\right)}{\Gamma\left(\dfrac{N-m}{2}\right)} \right]^{p'}, & \textit{if } m \textit{ is even }, \end{cases} \tag{0.0.30}$$

and ∇^m stands for the m order gradient of u:

$$\nabla^m u = \begin{cases} \Delta^{m/2} u, & m = 2, 4, 6, \cdots, \\ \nabla \Delta^{(m-1)/2} u, & m = 1, 3, 5, \cdots, \end{cases} \tag{0.0.31}$$

here w_{N-1} denotes the $N-1$ dimensional surface measure of the unit ball in $\mathbb{R}^N$ and $p' = p/(p-1)$ is the conjugate exponent of p. Furthermore, if $\beta > \beta_0$, then there exists a smooth u supported in Ω with $\|\nabla^m u\|_p \leq 1$, for the integral in (0.0.29) can be made as large as desired (see [7, Theorem 1]).

We now consider a space that is of high importance in the proof of Theorems 1.0.1 and 1.0.2 of Chapter 1, which properly contain the space $W_0^{m,\frac{N}{m}}(\Omega)$, $1 < m < N$, which we define by

$$W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega) := \left\{ u \in W^{m,\frac{N}{m}}(\Omega) : \ \Delta^j u = 0 = u \text{ on } \partial\Omega \text{ for } 0 \leq j \leq \frac{m-1}{2} \right\}. \tag{0.0.32}$$

An Adams-type inequality was also proved by C. Tarsi in [86, Theorem 4] on this space, which is as follows:

Theorem 0.0.2. *Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain and $1 < m < N$ be a positive integer. Then*

$$\sup_{u \in W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega) : \|\nabla^m u\|_{\frac{N}{m}} \leq 1} \int_\Omega \exp\left(\beta |u|^{\frac{N}{N-m}}\right) dx \leq \beta_0,$$

moreover, the constant appearing in (0.0.31) is sharp.

The authors Wang and Mao in [91] studied the existence and non-existence of solutions that change sign the following problem

$$\begin{cases} \Delta^2 u - \alpha \nabla(f(\nabla u)) - \gamma \Delta_p u = g(x,u) \text{ in } \ \Omega, \\ \dfrac{\partial(\Delta u)}{\partial \eta} = \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \ \partial\Omega, \end{cases} \tag{0.0.33}$$

where $p > 2$, $\alpha, \gamma \in \mathbb{R}$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$. By using a special function space with the constraint $\displaystyle\int_\Omega u dx = 0$, under suitable assumptions on $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfying $f(0) = 0$ and $g(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, they showed the existence and multiplicity of sign-changing solutions of the above problem via the Mountain Pass Theorem and the Fountain Theorem.

More recently, some authors have investigated stationary Kirchhoff-Boussinesq problems, that is, those in which there is only a single space variable $x \in \Omega$ unlike in evolution problems, where the unknown also depends on the time variable $t \geq 0$. For example, Sun, Liu and Wu [78] were concerned with the following biharmonic equation with p-Laplacian and Neumann boundary condition given by

$$\begin{cases} \Delta^2 u - \gamma \Delta_p u = f(x,u) - \dfrac{\mu}{|\Omega|} \displaystyle\int_\Omega f(y, u(y))dy \;\text{ in }\; \Omega, \\ \dfrac{\partial \Delta u}{\partial \eta} = \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \;\partial\Omega. \end{cases} \tag{0.0.34}$$

Using the fountain theorem, the authors obtained the existence of infinitely many sign-changing high energy solutions. . In that paper was crucial the inequalit

$$F(t) \leq t f(t) + d(x) t^\sigma, \tag{0.0.35}$$

where $d \in L^{\frac{2}{2-\sigma}}(\Omega)$ for $0 < \sigma \leq 2$.

Now we present some results when $\Omega = \mathbb{R}^N$. Recently, biharmonic equations on unbounded domain $\mathbb{R}^N$ have attracted a lot of attention. Especially, the researchers mainly investigated the following problems with the steep potential:

$$\begin{cases} \Delta^2 u - \gamma \Delta u + \lambda V(x) u = f(x,u) \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases} \tag{0.0.36}$$

With the aid of λ, they proved that the energy functional possesses the property of being locally compact, see [59, 63, 90, 102] and their references therein. Especially, Ye and Tang [102] assumed that $f(x,u)$ was superlinear and subcritical at infinity, when λ was large enough, they obtained the existence and multiplicity of nontrivial solutions. Later, Zhang et al. in [104] improved their results and obtained the existence of infinite nontrivial solutions when $\lambda > 0$ was large enough. Badiale, Greco and Rolando [10] obtained two nontrivial solutions for the case $f(x,u) = g(x,u) + \mu\xi(x)|u|^{q-2}u$ when $g(x,u)$, $\xi(x)$ satisfied some assumptions, λ was large enough and μ was small enough. Mao and Zhao [67] considered (0.0.36) with Kirchhoff terms and concave-convex nonlinearities, and proved the existence and multiplicity of solutions by using the variational method.

As for replacing Laplacian with p-Laplacian in (0.0.36), Sun, Chu and Wu in [80] studied the following problem

$$\begin{cases} \Delta^2 u - \gamma \Delta_p u + \lambda V(x) u = f(x,u) \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{0.0.37}$$

where $N \geq 1$, $\gamma \in \mathbb{R}$, $\lambda > 0$ are parameter and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p \geq 2$. Unlike other papers dealing with this problem, the authors allowed γ to be negative. Under suitable assumptions on V and $f(\cdot, u)$ which will be presented later, one can obtain the existence and multiplicity of non-trivial solutions for λ large enough. The proof relies on variational methods and Gagliardo-Nirenberg inequality. Where $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and the potential V satisfy the following conditions:

(V_1) $V \in C(\mathbb{R}^N)$ and $V(x) \geqslant 0$ $\forall x \in \mathbb{R}^N$;

(V_2) There exists $b > 0$ such that the set

$$\{V < b\} := \left\{x \in \mathbb{R}^N, V(x) < b\right\}, \tag{0.0.38}$$

has finite positive Lebesgue measure for $N \geqslant 4$ and

$$|\{V < b\}| < S_\infty^{-2}\left(1 + \frac{A_0^2}{2}\right)^{-1} \quad \text{para} \quad N \leq 3, \tag{0.0.39}$$

where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for the embedding $H^2(\mathbb{R}^N)$ in $L^\infty(\mathbb{R}^N)$ to $N \leq 3$, and A_0 is the Gagliardo-Nirenberg inequality constant;

(V_3) $\Omega = \{x \in \mathbb{R}^N : V(x) = 0\}$ is non-empty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N : V(x) = 0\}$.

(D_1) $f \in C\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ and there is q such that $p < q < 2_*$ and two functions $a, b \in L^\infty\left(\mathbb{R}^N\right)$ satisfying $|a^+|_\infty < \Theta_2^{-1}$ and $b(x) > 0$ in $\overline{\Omega}$ such that

$$\lim_{s \to 0^+} \frac{f(x,s)}{|s|^{q-1}} = a(x) \quad \text{and} \quad \lim_{s \to \infty} \frac{f(x,s)}{|s|^{q-1}} = b(x) \ \text{ uniformly on } \ x \in \mathbb{R}^N.$$

(D_2) There is l such that $1 < l < 2$ and a non-negative function $d \in L^{2/(2-l)}\left(\mathbb{R}^N\right)$ such that

$$pF(x,s) - f(x,s) \leqslant d(x)|s|^l \quad \forall x \in \mathbb{R}^N \text{ and, } \ s \in \mathbb{R},$$

where $F(x,u) = \int_0^u f(x,s)ds$.

(D_1') $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there is q such that $2 < q < p$ and three non-negative functions g_0, $g_1 \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$ with $g_0(x) > 0$ in $\overline{\Omega}$ and $a \in L^\infty(\mathbb{R}^N)$ such that

$$g_0(x)s^{q-1} \leqslant f(x,s) \leqslant a(x)s + g_1(x)s^{q-1},$$

where $p^* := \frac{Np}{N-p}$.

(D_2') There is a non-negative function $g_2 \in L^{p^*/(p^*-q)}(\mathbb{R}^N)$ such that

$$\frac{1}{2}f(x,s)s - F(x,s) \leqslant g_2(x)|s|^q.$$

(D_3) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ with $f(x,s) = 0$ $\forall x \in \mathbb{R}^N$ and $s \leqslant 0$, and there is q such that $1 < q < p$, and two non-negative functions $\tilde{a} \in L^{2/(2-q)}(\mathbb{R}^N)$ for $1 < q < 2$, or $\tilde{a} \in L^\infty(\mathbb{R}^N)$ for $2 \leqslant q < p$ and $\tilde{b} \in L^\infty(\mathbb{R}^N)$ such that

$$-\tilde{a}(x)s^{q-1} \leqslant f(x,s) \leqslant \tilde{a}(x)s^{q-1} + \tilde{b}(x)s^{p-1}.$$

Note that these authors proved the following theorems.

Theorem 0.0.3. *Suppose that $N \geqslant 1$, $2 \leqslant p < 2_*$ and conditions (V_1)-(V_3) are satisfied. Assume that the function f satisfies (D_1) and (D_2). Then there exists $\Lambda_0 > 0$ such that the problem (0.0.37) admits at least one non-trivial solution for all $\lambda \geqslant \Lambda_0$ and $\gamma \geqslant 0$.*

Theorem 0.0.4. *Suppose that $N \geqslant 3$, $2 < p < \min\{N, \frac{2N}{N-2}\}$ and conditions (V_1)-(V_3) are satisfied. We also assume that the function f satisfies (D_1') and (D_2'). Then:*

(i) for $\gamma > 0$, $\Lambda_0 > 0$ such that the equation (0.0.37) *admits at least one non-trivial solution for all $\lambda \geqslant \Lambda_0$;*

(ii) there exists γ_0 , $\Lambda_0 > 0$ such that the equation (0.0.37) *admits at least two non-trivial solutions for all $\lambda \geqslant \Lambda_0$ and $0 < \gamma < \gamma_0$.*

Theorem 0.0.5. *Suppose that $N \geqslant 1, 2 < p < 4$ and the conditions (V_1)-(V_3), (D_2) and (D_3) are satisfied. Then*

(i) if $f(x, s) \geqslant 0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ and $1 < q < 2$, then there is $\Lambda_0, \Pi_0 > 0$ such that for $|\tilde{a}|_{L^{2/(2-q)}} < \Pi_0$, then the equation (0.0.37) *admits at least one non-trivial solution for all $\lambda \geqslant \Lambda_0$ and $\gamma < 0$;*

(ii) if $f(x, s) \geqslant 0$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$ and $1 < q < p$, then there is $\Lambda_0 > 0$ such that the equation (0.0.37) *admits at least one non-trivial solution for all $\lambda \geqslant \Lambda_0$ and $\beta < 0$.*

Theorem 0.0.6. *Suppose that $N \geqslant 1$, $2 < p < 4$ and the conditions (V_1)-(V_2) are satisfied. If $f(x, u) = h(x)|u|^{q-2}u$ with $h \in L^{2/(2-q)}(\mathbb{R}^N)$ and $1 < q < 2$, then there exist Λ^*,$\Pi^* > 0$ such that for $0 < |h^+|_{2/(2-q)} < \Pi^*$, the equation* (0.0.37) *admits at least two non-trivial solutions for all $\lambda \geqslant \Lambda^*$ and $\gamma < 0$.*

The authors Jiang, Sun and their collaborators in [45, 46, 83] studied the existence and multiplicity of solutions when the potential is singular and in the presence of two parameters and λ for the following problem

$$\begin{cases} \Delta^2 u - \gamma \Delta_p u + V_\lambda(x)u = f(x, u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{0.0.40}$$

where $N \geq 1$, $\gamma \in \mathbb{R}$, $\lambda > 0$. This is different from previous works on biharmonic problems and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p \geq 2$ and suppose that $V(x) = \lambda a(x) - b(x)$ with $\lambda > 0$ and $b(x)$ can be singular at the origin, in the particular they allowed γ to be a real number. Under suitable conditions on $V_\lambda(x)$ and $f(x, u)$, the multiplicity of solution was obtained for $\lambda > 0$ sufficiently large. The potentials $a(x)$ and $b(x)$ satisfy the following conditions:

(H_1) $a \in C(\mathbb{R}^N)$ and $a(x) \geqslant 0$ $\forall x \in \mathbb{R}^N$ and there exists $a_0 > 0$ such that the set

$$\{a < a_0\} := \left\{x \in \mathbb{R}^N, a(x) < a_0\right\}, \tag{0.0.41}$$

has finite positive Lebesgue measure for $N \geqslant 4$ and

$$|\{V < b\}| < S_\infty^{-2}\left(1 + \frac{A_0^2}{2}\right)^{-1} \quad \text{para} \quad N \leq 3, \tag{0.0.42}$$

where $|\cdot|$ is the Lebesgue measure, S_∞ is the best Sobolev constant for embedding $H^2(\mathbb{R}^N)$ in $L^\infty(\mathbb{R}^N)$ for $N \leq 3$, and A_0 is the Gagliardo-Nirenberg inequality constant;

(H_2) $\Omega = \{x \in \mathbb{R}^N : a(x) = 0\}$ is non-empty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N : a(x) = 0\}$;

(H_3) $b(x)$ is a measurable function on $\mathbb{R}^N$ and there exists $0 < b(x) < \overline{\theta}$ such that $0 \leq b(x) \leq \frac{b_0}{|x|^4}$ for all $x \in \mathbb{R}^N$, where $\overline{\theta} = \frac{N^2(N-4)^2}{16}$ is a critical Hardy–Sobolev constant.

As for $f(x, u) = 0$, the authors Sun and Wu in [83] studied the existence of nontrivial solutions for a biharmonic equation with p-Laplacian and singular sign-changing potential. And they showed the following theorem:

Theorem 0.0.7. *Assume that $\gamma < 0$, $2 < p \leq \frac{2N}{N-2}$ and the conditions $(H_1), (H_2), f(x,u) = 0$ and (V_3) hold. Then there exists a constant $\Lambda_0 > 0$ such that the problem (0.0.40) admits at least one nontrivial solution for each $\lambda > \Lambda_0$.*

The potential $b(x)$ could be singular at the origin by condition (H_3). Furthermore, the improved Hardy-Sobolev inequality (see [93, Lemma 1.1]) gives

$$\int_{\mathbb{R}^N} b(x)|u|^2 dx \leq b_0 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} dx \leq \frac{b_0}{4} \int_{\mathbb{R}^N} |\Delta u|^2 dx. \tag{0.0.43}$$

Under hypotheses (H_1) and (H_2), $\lambda a(x)$ is called the steep potential well whose depth is controlled by the parameter λ. Such potential was first suggested by Bartsch and Wang [12] in the study of scalar Schrödinger equations. So far, steep potential wells have been introduced to the study of some other types of nonlinear differential equations. In addition, they have also been introduced to the study of some other types of nonlinear differential equations, such as Kirchhoff type equations [81], Schrödinger–Poisson systems [47], biharmonic equations [63, 82], etc.

The authors Sun and Wu in [84] investigated a class of biharmonic equations with p-Laplacian and singular potential as follows:

$$\begin{cases} \Delta^2 u + V_\lambda(x)u - \operatorname{div}(W(x)|\nabla u|^{p-2}\nabla u) = 0 \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases} \tag{0.0.44}$$

where $N \geq 3$, $1 < p < \frac{2N}{N-2}$ except $p = 2$ and $V_\lambda(x) = \lambda a(x) - b(x)$ with $\lambda > 0$. They used (H_1), (H_2) and established the following hypothesis:

(H_4) $W(x)$ is a sign-changing weight function satisfying $W \in L^{\frac{2}{2-p}}(\mathbb{R}^N)$ if $1 < p < 2$ and $W \in L^\infty(\mathbb{R}^N)$ and $\{W(x) > 0\} \cap \Omega$ has finite positive Lebesgue measure if $2 < p < \frac{2N}{N-2}$.

Under suitable hypotheses about the potentials $a(x), b(x)$ and $W(x)$, they used the Nehari method to establish the existence of non-trivial solutions for sufficiently large $\lambda > 0$.

The authors Benhanna and Choutri in [15] studied multiplicity of solutions for fourth-order elliptic equations with p-Laplacian and mixed nonlinearity

$$\begin{cases} \Delta^2 u - \Delta_p u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases} \tag{0.0.45}$$

Using the Mountain Pass Theorem and Ekeland's Variational Principle, they showed the existence of two non-trivial solutions. To overcome the difficulty of convergence of subsequences for the Palais-Smale sequences of the Euler-Lagrange functional, they considered Cerami Sequences. Later, Jiang and Zhai [46] complemented their results, when $\gamma \in \mathbb{R}$ and they replaced $\lambda V(x)$ with the singular potential $V_\lambda(x)$, and obtained the multiplicity of non-trivial solutions.

The authors Silva, Carvalho and Goulart in [77] established the existence of solutions for critical and subcritical nonlinearities by considering a fourth-order elliptic problem defined in the entire space $\mathbb{R}^N$. They also studied a class of potentials and nonlinearities that can be periodic or asymptotically periodic. Note that they considered a general fourth-order elliptic problem where the principal part is given by $\alpha\Delta^2 u + \gamma\Delta u + V(x)u$ where $\alpha > 0$, $\gamma \in \mathbb{R}$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential for the cases γ negative, zero or positive. In this work, they used some precise estimates to demonstrate the compactness of the associated energy functional.

Yang in [96] established a useful embedding inequality in $D^{2,2}(\mathbb{R}^N, \mathbb{R}^2)$. Benefiting from the previous inequality, the author also obtained a nontrivial weak solution to a critical biharmonic system involving p-Laplacian and Hardy potential

$$\begin{cases} \Delta^2 u + \Delta_p u - \theta_1 \dfrac{\eta_1 u}{|x|^4} = \dfrac{\eta_1}{4^*(\alpha)} \dfrac{|u|^{\eta_1 - 2} u |v|^{\eta_2 - 2}}{|x|^4} \text{ in } \mathbb{R}^N, \\ \Delta^2 v + \Delta_p v - \theta_2 \dfrac{\eta_2 v}{|x|^4} = \dfrac{\eta_2}{4^*(\alpha)} \dfrac{|v|^{\eta_1 - 2} v |u|^{\eta_2 - 2}}{|x|^4} \text{ in } \mathbb{R}^N, \\ \quad u, v \in H^2(\mathbb{R}^N), \end{cases} \tag{0.0.46}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $0 < \alpha < 4 < N$, $p = 2^* := \frac{2N}{N-2}$, $\theta_1, \theta_2 < \theta_H = \frac{N^2(N-4)^2}{16}$, $\eta_1, \eta_2 > 1$ and $\eta_1 + \eta_2 = 4^*(\alpha) = \frac{2(N-\alpha)}{N-4}$. Problems involving biharmonic operator and Hardy potential are mathematical models for describing the practical phenomena appeared in physics and engineering (e.g. in static deflection of an elastic plate, clamped plates, quantum cosmology and so on) via variational methods.

Motivated by all of the above, in this thesis, we study the existence, multiplicity of solutions for the following class of problems:

$$(P_i) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V(x)u = f(u) + \beta |u|^{2_{**}-2}u \text{ in } \Omega, \\ \\ u \in H^2 \cap H_0^1(\Omega), \end{cases}$$

where $i = 1, 2, 3$, $\Omega \subset \mathbb{R}^N$ is a smooth domain, f is continuous function, in the case $\beta = 0$ we get $2 < p < 2^* = \frac{2N}{N-2}$ for $N \geq 3$, case $\beta = 1$ we consider $2_{**} = \frac{2N}{N-4}$ for $N \geq 5$ and V is a continuous function.

In Chapter 1, we deal with a problem of type (P_1), where Ω is bounded in $\mathbb{R}^4$, $\beta = 0$ and $V = 0$. More precisely, we study the following problem:

$$(P_1) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u = f(u) \text{ in } \Omega, \\ \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{cases}$$

where the hypotheses about nonlinearity $f : \mathbb{R} \to \mathbb{R}$ are given as follows.

(f_1) There exists $\alpha_0 \geq 0$ such that the function f satisfies

$$\lim_{t \to +\infty} \frac{f(t)}{\exp(\alpha |t|^2)} = \begin{cases} 0 & \text{for } \alpha > \alpha_0, \\ +\infty & \text{for } \alpha < \alpha_0; \end{cases}$$

(f_2) The following limit holds:

$$\lim_{t \to 0^+} \frac{f(t)}{|t|} = 0;$$

(f_3) The function $t \to \dfrac{f(t)}{|t|^{p-2}t}$ is decreasing for $t \in (-\infty, 0)$ and increasing in $t \in (0, +\infty)$;

(f_4) There are $r > p$ and $\tau > 0$ such that

$$f(t) \geq \tau |t|^{r-2} t, \quad \text{for all } t \geq 0$$

The main result of Chapter 1.

First, in this chapter we proof the existence of ground state solution when f has subcritical growth.

Theorem 0.0.8. *Assume that condition* (f_1) *holds with* $\alpha_0 = 0$ *and* (f_2)-(f_3) *hold with* $\tau = 1$. *Then, problem* (P_1) *has a ground state solution.*

Secondly,in this chapter we proof the existence of ground state solution when f has critical growth.

Theorem 0.0.9. *Assume that condition* (f_1) *with* $\alpha_0 > 0$ *and* (f_2)-(f_4) *hold with* τ *sufficiently large. Then, problem* (P_1) *has a ground state solution.*

For the convenience of the readers, the hypotheses in the previous theorems will be stated again in the corresponding chapter.

Notice that all the results obtained in Chapter 1 of this thesis have been published in the following article:
Carlos, Romulo D and Figueiredo, Giovany M, *On an elliptic Kirchhoff–Boussinesq type problems with exponential growth*, Mathematical Methods in the Applied Sciences, (2023). https://https://doi.org/10.1002/mma.9662.

In Chapter 2, we investigate a problem of type (P_2). More presicely, we consider Ω is bounded in $\mathbb{R}^N$, $V = 0$ and $\beta = 0$ or $\beta = 1$ for the following problem

$$(P_2) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u = f(u) + \beta |u|^{2_{**}-2} u \text{ in } \Omega, \\ \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{cases}$$

among other hypotheses, the nonlinear function f satisfies:

$(f_1)'$ There exists $C > 0$ such that

$$|f(t)| \leq C(1 + |t|^{q-1}).$$

The main result of Chapter 2.

First, in this chapter we proof the existence of a ground state solution when f has subcritical growth.

Theorem 0.0.10. *Assume that conditions* $(f_1)'$, (f_2), (f_3), *hold with* $\beta = 0$ *and* $2 < p \leq 2^* < q < 2_{**}$ *or* $2 < p < q \leq 2^*$. *Then, problem* (P_2) *has a ground state solution.*

Secondly, in this chapter we proof the existence of a ground state solution when f has critical growth.

Theorem 0.0.11. *Assume that conditions* $(f_1)'$, (f_2), (f_3), (f_4) *hold with* $\beta = 1$ *and* $2 < p \leq 2^* < q < 2_{**}$ *or* $2 < p < q \leq 2^*$. *Then, problem* (P_2) *has a ground state solution.*

To find multiplicity of solutions, in the next theorems, we will assume that f is equal to a prototype, that is,

$$f(t) = \tau |t|^{q-2} t,$$

and it satisfies the assumptions $(f_1)'$-(f_4).

Thirdly, in this chapter we proof the existence and multiplicity of ground state solution when f has subcritical growth.

Theorem 0.0.12. *Assume that $\beta = 0$, $1 < q < 2 < p \leq 2^*$. Then, there exists $\tau^* > 0$ such that problem (P_2) has infinitely many weak solutions, for all $\tau \in (0, \tau^*)$.*

Fourthly, in this chapter we proof the existence and multiplicity of the ground state solution when f has critical growth.

Theorem 0.0.13. *Assume that $\beta = 1$, $1 < q < 2 < p \leq 2^*$. Then, there exists $\tau^* > 0$ such that problem (P_2) has infinitely many weak solutions, for all $\tau \in (0, \tau^*)$.*

In Chapter 3, we are concern with a problem of type (P_3), where $\Omega = \mathbb{R}^N$, V is periodic, asymptotic periodic and $\beta \in \{0, 1\}$ as follows:

$$(P_3) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V(x)u = f(u) + \beta|u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u(x) \neq 0 \ \ x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

with V being a continuous function and satisfying:

(V_1) There is a $\mathbb{Z}^N$-periodic function $V_{per} : \mathbb{R}^N \to \mathbb{R}$, that is,

$$V_{per}(x + y) = V_{per}(x) \ \text{ for all } \ x \in \mathbb{R}^N \ \text{ and for all } \ y \in \mathbb{Z}^N;$$

(V_2) There is a constant $V_0 > 0$ such that

$$V_{per}(x) \geq V_0 \quad \forall x \in \mathbb{R}^N;$$

(V_3) There is a constant $W_0 > 0$ and a function $W \in L^{N/2}(\mathbb{R}^N)$ with $W(x) \geq 0$ such that

$$V(x) = V_{per}(x) - W(x) \geq W_0 \quad \forall x \in \mathbb{R}^N,$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$.

The main result of Chapter 3.

First, in this chapter we proof the existence of ground state solution when f has subcritical growth and V is not periodic.

Theorem 0.0.14. *Assume that conditions (V_1)-(V_2) and $(f_1), (\tilde{f}_2), (f_3)$ hold with $\beta = 0$. Then, problem (P_3) has a ground state solution.*

Secondly in this chapter we proof the existence of ground state solution when f has critical growth and V is not periodic.

Theorem 0.0.15. *Assume that conditions (V_1)-(V_3) and $(f_1), (\tilde{f}_2), (f_3), (f_4)$ hold with $\beta = 1$. Then, there exist $\tau^* > 0$ such that problem (P_3) has a ground state solution, for all $\tau \geq \tau^*$.*

For the convenience of the readers, the hypotheses in the previous theorems will be stated again in the corresponding chapter.

Notice that all the results obtained in Chapter 3 of this thesis have been published in the following article:
Carlos, Romulo D and Figueiredo, Giovany M, *Nonlinear perturbations of a periodic Kirchhoff–Boussinesq type problems in $\mathbb{R}^N$*, Zeitschrift für angewandte Mathematik und Physik, (2024). https://https://doi.org/10.1007/s00033-023-02161-z.

The present thesis is strongly influenced by the articles [46], [77], [78], [80], [83], [84] and [96]. Below we list the main contributions that we believe this thesis has made.

In Chapter 1, the following novelties have appeared in the study of (P_1):

(1) The argument of Nehari method is sufficient to study the two cases $\Delta^2 \pm \Delta_p$.

(2) We complement the study that can be found in [78], [80] and [83] because, in our results, we show existence and concentration solutions for subcritical and critical exponential growths in a bounded domain Ω in $\mathbb{R}^4$.

As for Chapter 2, we point out the novelties that appeared in the study of (P_2) as follows:

(1) In the articles [78], [83] and [80], the inequality 0.0.35 and the size of the parameter in front of the p-Laplacian operator were crucial in their arguments. In our arguments, the inequality 0.0.35 was not necessary, and there is no parameter in front of the p-Laplacian operator. Furthermore, we are considering two possibilities $\pm\Delta_p$.

(2) Unlike the results found in [78], [83], [80], [96] , here we are considering subcritical and critical growths. Moreover, in order to consider different values for q, we use Nehari method, Mountain Pass Theorem and Genus Theory

(3) Since in the Kirchhoff-Boussinesq type problem appear the term $\pm\Delta_p$, some estimates are more refined. See for example, Proof of Theorem 0.0.10. Furthermore, in the critical case, these two possibilities imply the definition of two Sobolev constants that are used to overcome the difficulty of studying a problem with critical growth. Other refined estimates can be observed in the proofs of Theorems 0.0.12 and Theorem 0.0.13.

As for Chapter 3, we point out the novelties that appeared in the study of (P_3) as follows:

(1) The two cases of $\Delta^2 u \pm \Delta_p u$ that we are studying that are different from those of [46], [78], [80], [83], [84] and [96]. Furthermore we are considering critical and subcritical nonlinear growth without parameters and another kind of potential V.

(2) Furthermore, our results complete the results that can be found in [11], [12], [13], [60] and [77] because we consider a quasilinear and nonhomogeneous operator which is the combination of Biharmonic operator and p-Laplacian operator.

(3) Moreover, we study the critical case which has additional difficulties to overcome lack of compactness produced by the critical exponent.

The major challenge in the study of problems involving this operator lies in estimates with the L-infinity norm. This is because the presence of the biharmonic operator does not allow the use of the Moser iteration method. The presence of the p-Laplacian operator does not permit estimates that can be found in the article [53], for example. The authors of this article already have partial results in this direction, but they are not sufficient for regularity results, for example.

It is worth stressing that ground state solutions play an important role in this thesis, which will be established in most of our theorems. For a better understanding of the readers we will give the following Remark.

Remark 1. *A solution u is a ground state of the equation (P_i) $(i = 1, 2, 3)$ correspond to the three problems that we have consider in Chapters 1-3, respectively, it is the least among all nontrivial critical value of the functional I associated to problems (P_i) $(i = 1, 2, 3)$, namely, u has the least energy among nontrivial solutions. A natural method of searching for the ground state is to minimize the I on the Nehari manifold of equations defined by*

$$\mathcal{N} = \{u \in H\backslash\{0\} : \ I'(u)u = 0\}.$$

The corresponding ground state energy is given by

$$c = \min_{u \in \mathcal{N}} I(u).$$

In the following, I will list two other problems that have developed with other collaborators during the doctoral period.

Beyond the scope of this thesis, on the one hand, in my joint work with Figueiredo and Ruviario in [27], we have studied existence of solutions for the following class of elliptic Kirchhoff-Boussinesq type problems given by

$$\Delta^2 u - \Delta_p u + u = h(u) \ \text{ in } \mathbb{R}^N$$

and

$$\Delta^2 u - \Delta_p u = f(u) \ \text{ in } \mathbb{R}^N,$$

where $2 < p \leq \frac{2N}{N-2}$ if $N \geq 3$, $2_{**} = \infty$ if $N = 3, 4$, $2_{**} = \frac{2N}{N-4}$ if $N \geq 5$, h and f are continuous functions satisfying hypotheses considered by Berestycki and Lions in [18]. More precisely, the problem with the nonlinearity h is related to Positive mass case and the problem with the nonlinearity f is related to Zero mass case. The main argument is to find a Palais-Smale sequence that satisfies a property related to Pohozaev identity, as in [43], which was used for the first time by [44].

On the other hand, in joint work with Figueiredo and Costa, we have studied existence of a ground state solution for the following class of elliptic Kirchhoff-Boussinesq type problems given by

$$\Delta^2 u \pm \Delta_p u + (1 + \lambda V(x))u = f(u) + \beta |u|^{2_{**}-2} u \ \text{ in } \ \mathbb{R}^N,$$

where $2 < p < 2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2_{**} = \infty$ if $N = 3, 4$ and $2_{**} = \frac{2N}{N-4}$ if $N \geq 5$. Here f is a continuous function and the term $1 + \beta V(x)$ is the steep potential well introduced by Bartsch and Wang in [12]. The function f has subcritical growth and behaves like $|u|^{q-2}u$ with $p < q < 2_{**}$. Using variational methods, we have established the existence of a ground state solution in the subcritical case, i.e, $\beta = 0$ and the critical case, i.e, $\beta = 1$.

We remark that the above results give rise to article that was accepted for publication in the Journal Complex Variables and Elliptic Equations:
R. D. Carlos, G. M. Figueiredo and Gustavo S. A. Costa, *Existence and concentration of solutions for a class of biharmonic and p-Laplacian equations with steep potential well,* Complex Variables and Elliptic Equations, (2023).

Notation

In this work we use the following notations:

$\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \cdots \dfrac{\partial u}{\partial x_N}\right)$	gradient of the function u;
$\Delta u = \sum_{i=1}^{N} \dfrac{\partial^2 u}{\partial x_i^2} = \mathrm{div}(\nabla u)$	Laplacian of u;
$\Delta_p u = \mathrm{div}(\|\nabla u\|^{p-2}\nabla u)$	p-Laplacian operator of u;
$\Delta^2 u = \Delta(\Delta u)$	Biharmonic operator of u;
$\rightharpoonup$	weak convergence;
$\rightarrow$	strong convergence;
$a.e.$	almost everywhere;
supp f	support of the function f;
B_R	open ball of radius R centered at 0;
V_{per}	periodic potential ;
X'	dual space of the Banach space X;
$L^s_{\mathrm{loc}}(\mathbb{R}^N)$	space of all classes of functions which are in L^s on every compact subset of $\mathbb{R}^N$;
$\\|\cdot\\|_V$	norm in the normed space X_V;
$\\|\cdot\\|$	norm in the space $H^2(\mathbb{R}^N)$;
$\\|I'(u)\\|_*$	norm of the derivative of I restricted to $\mathcal{V}$ at the point u;
$\mathcal{M}_0$	Nehari manifold of I_0;
$\mathcal{N}$	Nehari manifold of I.

Chapter 1

On elliptic Kirchhoff-Boussinesq type problems with exponential growth

In this chapter, we are concerned with existence of nontrivial solutions for the following class of problems

$$(P_1) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u = f(u) \text{ in } \Omega, \\ \\ u = \Delta u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded smooth domain and $2 < p < 4$ and f is a continuous function satisfying the following:

(f_1) There exists $\alpha_0 \geq 0$ such that the function f satisfies

$$\lim_{t\to+\infty} \frac{f(t)}{\exp(\alpha|t|^2)} = \begin{cases} 0 & \text{for } \alpha > \alpha_0, \\ +\infty & \text{for } \alpha < \alpha_0, \end{cases}$$

(f_2) The following limit holds:

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0.$$

(f_3) The function $t \to \dfrac{f(t)}{|t|^{p-2}t}$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

(f_4) There are $r > p$ and $\tau > 0$ such that

$$f(t) \geq \tau|t|^{r-2}t, \quad \text{for all } t \geq 0.$$

In our first result, we establish the existence of a nontrivial solution for (P_1) in the subcritical case.

Theorem 1.0.1. *Assume that conditions (f_1) with $\alpha_0 = 0$ and (f_2)-(f_3) hold with $\tau = 1$. Then, problem (P_1) has a ground state solution.*

The next result provides the existence of a nontrivial solution when f has critical growth.

Theorem 1.0.2. *Assume that conditions (f_1) with $\alpha_0 > 0$ and (f_2)-(f_4) hold with τ sufficient large. Then, problem (P_1) has a ground state solution.*

We remark that our theorems can be applied for the model non-linearity

$$f(s) = \tau t^{r-1} exp(\alpha_0 t^2).$$

The plan of the chapter is the following: In section 1.1 we describe the variational framework and we prove some technical lemmas. The subcritical case is studied in section 1.2 and the critical case in section 1.3.

1.1 The variational framework and some technical lemmas

Motivated by the works of Adams in [7, Theorem 1], F. Sani in [75, Theorem 4,1] and C. Tarsi in [86, Theorem 3], in this first chapter of the thesis, we will consider the following space $H := H^2(\Omega) \cap H_0^1(\Omega)$. We have a well defined inner product

$$\langle v, w \rangle = \int_\Omega \Delta v \Delta w dx, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega),$$

and associated norm given by

$$\|u\| := \left(\int_\Omega |\Delta u|^2 dx \right)^{1/2}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

From now on we denote by $H = (H^2(\Omega) \cap H_0^1(\Omega), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Let us start with the following important result due to Adams [7, Theorem 1].

Theorem 1.1.1. *For every $u \in H$ and $\alpha > 0$,*

$$\exp(\alpha u^2) \in L^1(\Omega) \tag{1.1.1}$$

and there is a constant $M > 0$ such that

$$\sup_{\|u\| \le 1} \int_\Omega \exp(\alpha u^2) dx \le M, \tag{1.1.2}$$

for every $\alpha \le 32\pi^2$.

Another important result in this chapter is a Gagliardo-Nirenberg interpolation inequality [39], [70]:

Theorem 1.1.2. *Suppose that N, j, m are non-negative integers and that $1 \le \kappa_1, \kappa_2, \kappa_3 \le \infty$ and $\Upsilon \in [0, 1]$ are real numbers such that*

$$\frac{1}{\kappa_1} = \frac{j}{N} + \left(\frac{1}{\kappa_3} - \frac{m}{N} \right) \Upsilon + \frac{(1 - \Upsilon)}{\kappa_2}$$

and

$$\frac{j}{m} \le \Upsilon \le 1.$$

Then, there exist $C_1, C_2 > 0$ independent of u such that

$$|D^j u|_{\kappa_1} \le C_1 |D^m u|_{\kappa_3}^{\Upsilon} |u|_{\kappa_2}^{1-\Upsilon} \quad \forall\, u \in L^{\kappa_2}(\mathbb{R}^N) \cap W^{m,\kappa_3}(\mathbb{R}^N).$$

Under assumptions for Ω be a bounded Lipschitz domain, we have

$$|D^j u|_{\kappa_1} \le C_1 |D^m u|_{\kappa_3}^{\Upsilon} |u|_{\kappa_2}^{1-\Upsilon} + C_2 |u|_s \quad \forall\, u \in L^{\kappa_2}(\Omega) \cap W^{m,\kappa_3}(\Omega) \quad \text{for } s \ge 1.$$

Consider the functional $I : H \to \mathbb{R}$ associated to problem (P_1) is given by

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(u)dx.$$

where $F(t) = \displaystyle\int_0^t f(s)ds$. Since $2 < p < 4$, using Theorem 1.1.2 with $j = 1$, $\kappa_1 = p$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $N = 4$ and $\kappa_3 = 2$, we have that the injection $H \hookrightarrow W_0^{1,p}(\Omega)$ is continuous for $2 \leq p \leq 4$ and, as a consequence, I is well-defined and of C^1 class. Moreover, we get

$$I'(u)\phi = \int_\Omega \Delta u \Delta \phi dx \pm \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx - \int_\Omega f(u)\phi dx,$$

for all $\phi \in H$. Then, the critical points of I are weak solutions of (P_1). The Nehari manifold associated to the functional I is given by

$$\mathcal{N} = \{u \in H\backslash\{0\} : \; I'(u)u = 0\}.$$

Note that, from (f_2), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t)| \leq \varepsilon|t| \tag{1.1.3}$$

and

$$|F(t)| \leq \frac{1}{2}\varepsilon|t|^2, \tag{1.1.4}$$

for all $0 < t \leq \delta$.

Moreover, from (f_1), there exists $K > 0$ such that.

$$|f(t)| \leq \varepsilon \exp(\alpha t^2),$$

for all $t \geq K$ and $\alpha > \alpha_0$. In particular we get

$$|f(t)| \leq \frac{\varepsilon}{K} t \exp(\alpha t^2) \tag{1.1.5}$$

and

$$|F(t)| \leq \frac{\varepsilon}{2\alpha K}\exp(\alpha t^2), \tag{1.1.6}$$

for all $t \geq K$ and $\alpha > \alpha_0$. Consequently, using (1.1.3), (1.1.4), (1.1.5) and (1.1.6), for all $\varepsilon > 0$ and for all $\alpha > \alpha_0$, there exists $C_\varepsilon > 0$ such that

$$\int_\Omega f(u)u dx \leq \varepsilon \int_\Omega |u|^2 dx + C_\varepsilon \int_\Omega |u|^q \exp(\alpha|u|^2)dx \tag{1.1.7}$$

and

$$\int_\Omega F(u)dx \leq \frac{\varepsilon}{2}\int_\Omega |u|^2 dx + \frac{C_\varepsilon}{q}\int_\Omega |u|^q \exp(\alpha|u|^2)dx, \tag{1.1.8}$$

for all $u \in H$ and for all $q \geq 0$. In particular, in this chapter, we will use $q > p$.

Lemma 1.1.3. *If condition (f_3) holds, then the map*

$$s \mapsto sf(s) - pF(s) \text{ is increasing for } s \in (0, \infty)$$

and decreasing for $s \in (-\infty, 0)$. In particular, $sf(s) - pF(s) \geq 0$ for all $s \in \mathbb{R} \setminus \{0\}$.

Proof. Suppose $0 < s < t$. Hence, we obtain

$$\begin{aligned} sf(s) - pF(s) &= \frac{f(s)}{s^{p-1}}s^p - pF(t) + p\int_s^t f(\tau)\mathrm{d}\tau \\ &< \frac{f(t)}{t^{p-1}}s^p - pF(t) + \frac{f(t)}{t^{p-1}}(t^p - s^p) \\ &= tf(t) - pF(t) \end{aligned}$$

The proof in the case $t < s < 0$ is similar and this proves the lemma. □

In the next result we prove that $\mathcal{N}$ is not empty and that I restricted to $\mathcal{N}$ is bounded from below.

Lemma 1.1.4. *For each $u \in H \setminus \{0\}$ there exists a unique $t > 0$ such that $tu \in \mathcal{N}$. Moreover, $I(u) > 0$ for every $u \in \mathcal{N}$.*

Proof. Given $u \in H \setminus \{0\}$, let $\gamma_u(t) = I(tu)$ for $t > 0$. Then $tu \in \mathcal{N}$ if and only if $\gamma'(t) = 0$. Note that, by $\varepsilon > 0$ sufficiently small in (1.1.8) and Sobolev embedding, there exists $C > 0$ such that

$$\begin{aligned} \gamma_u(t) &= \frac{t^2}{2}\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(tu)dx \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)t^2\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \frac{C_\varepsilon S_q^{\frac{-q}{2}}}{q}t^q\int_\Omega |u|^q \exp(\alpha|tu|^2)dx \end{aligned}$$

Using Cauchy–Schwarz's inequality, we get

$$\begin{aligned} \gamma_u(t) \geq{}& \left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)t^2\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx \\ &- \frac{C_\varepsilon S_q^{\frac{-q}{2}}}{q}t^q\left\{\int_\Omega |u|^{2q}dx\right\}^{1/2}\left\{\int_\Omega \exp\left[2\alpha\|tu\|^2\left(\frac{u}{\|u\|}\right)^2\right]dx\right\}^{1/2} \end{aligned}$$

Choosing $\alpha > \alpha_0$, $\varepsilon \in (0,1)$ and $t_1 > 0$ such that $2\alpha\|t_1 u\|^2 \leq 32\pi^2$, using (1.1.2) we get

$$\gamma_u(t) \geq D_1 t^2 \pm D_2 t^p - D_3 t^q,$$

for all $0 < t < t_1$ and for some $D_1, D_2, D_3 > 0$. Thus, since $2 < p < q$, we have $\gamma_u(t) > 0$ for $0 < t_1$ sufficient small.

Now, for all $t > 0$ anf by τ fixed by (f_4), we have

$$\frac{\gamma_u(t)}{t^p} \leq \frac{1}{2t^{p-2}}\|u\|^2 \pm \int_\Omega |\nabla u|^p dx - \tau t^{r-p}\int_\Omega |u|^r dx.$$

Therefore, since $2 < p < r$, we conclude $\lim_{t\to+\infty}\gamma_u(t) = -\infty$. Then, there exists at least one $t(u) > 0$ such that $\gamma_u'(t(u)) = 0$, i.e. $t(u)u \in \mathcal{N}$.

Suppose that there exist $t(u)$ and $s(u)$ such that $t(u)u, s(u)u \in \mathcal{N}$. Then,

$$\pm\int_\Omega |\nabla u|^p dx = \int_\Omega \frac{f(tu)}{t^{p-1}}u dx - \frac{\|u\|^2}{t^{p-2}}$$

and

$$\pm\int_\Omega |\nabla u|^p dx = \int_\Omega \frac{f(su)}{s^{p-1}}u dx - \frac{\|u\|^2}{s^{p-2}}.$$

Since

$$\int_\Omega \left[\frac{f(tu)}{(tu)^{p-1}} - \frac{f(su)}{(su)^{p-1}}\right] u^p dx = \frac{\|u\|^2}{t^{p-2}} - \frac{\|u\|^2}{s^{p-2}}$$

and using (f_3), we conclude that $t = s$. Moreover, note that

$$I(u) = I(u) - \frac{1}{p}I'(u)u = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 + \int_\Omega \left(\frac{1}{p}f(u)u - F(u)\right)dx.$$

From Lemma 1.1.3 we obtain $I(u) > 0$ for all $u \in \mathcal{N}$. □

Lemma 1.1.5. *There exists a constant $C > 0$ such that $\|u\| \geq C > 0$, for every $u \in \mathcal{N}$.*

Proof. Suppose, by contradiction, that there is $(u_n) \subset \mathcal{N}$ such that

$$u_n \to 0 \ \text{ in } \ H. \tag{1.1.9}$$

Since that $(u_n) \subset \mathcal{N}$, we get

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx \ = \ \int_\Omega f(u_n)u_n dx.$$

Then, using (1.1.7), we have

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx \ = \ \int_\Omega f(u_n)u_n dx \leq \varepsilon \int_\Omega |u_n|^2 dx + C\varepsilon \int_\Omega |u_n|^q \exp(\alpha|u_n|^2)dx.$$

Using Sobolev embedding, there exists $C > 0$ such that

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \int_\Omega |\nabla u|^p dx \leq C_\varepsilon \int_\Omega |u_n|^q \exp\left[\alpha\|u_n\|^2\left(\frac{|u_n|}{\|u_n\|}\right)^2\right]dx.$$

Using Cauchy–Schwarz's inequality, we get

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \int_\Omega |\nabla u|^p dx \leq C_\varepsilon \left\{\int_\Omega |u_n|^{2q}dx\right\}^{1/2} \left\{\int_\Omega \exp\left[2\alpha\|u_n\|^2\left(\frac{|u_n|}{\|u_n\|}\right)^2\right]dx\right\}^{1/2}$$

Note that there is $n_0 \in \mathbb{N}$ such that

$$\|u_n\|^2 \leq \frac{16\pi^2}{\alpha_0}, \quad \text{for all } n \geq n_0.$$

Then, from (1.1.2) and Sobolev embedding again, we have

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \int_\Omega |\nabla u|^p dx \ \leq \ MC_\varepsilon \left\{\int_\Omega |u_n|^{2q}dx\right\}^{1/2} \leq MC_\varepsilon\|u_n\|^q.$$

In the case that the second term in I is positive, this inequality implies

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u_n\|^2 \leq \left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u_n\|^2 + \int_\Omega |\nabla u|^p dx \leq MC_\varepsilon\|u_n\|^q.$$

Consequently

$$0 < \left[\frac{(2 - \varepsilon S_2^{-1})}{2MC_\varepsilon}\right]^{1/(q-2)} \leq \|u_n\|. \tag{1.1.10}$$

Since $q > 2$, the above inequality contradicts (1.1.9) and the lemma is proved.

In the case that the second term in I is negative, using the embedding $H \hookrightarrow W_0^{1,p}(\Omega)$, we get

$$\begin{aligned} \left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u_n\|^2 &\leq \frac{S_p^{\frac{-p}{2}}}{p}\|u\|^p + MC_\varepsilon\|u_n\|^q \\ &\leq \max\left[\frac{S_p^{\frac{-p}{2}}}{p}, MC_\varepsilon\right]\|u_n\|^q \end{aligned}$$

Since $2 < p < q$, this implies that

$$0 < \left\{\frac{2 - \varepsilon S_2^{-1}}{2\left[\frac{S_p^{\frac{-p}{2}}}{p}, MC_\varepsilon\right]}\right\}^{1/(q-2)} \leq \|u_n\|$$

and the above inequality contradicts (1.1.9) and the lemma is proved. □

Set $c := \inf_{\mathcal{N}} I$. In the next result we will prove that the minimizing sequences are bounded.

Lemma 1.1.6. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence, then (u_n) is bounded in H.*

Proof. Suppose, by contradiction, that up to a subsequence, we imply $\|u_n\| \to +\infty$ as $n \to +\infty$. Using Lemma 1.1.3, $I(u_n) \to c$ and $I'(u_n)u_n = 0$, we have

$$\begin{aligned} c + o_n(1)\|u_n\| &= I(u_n) - \frac{1}{p}I'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{p}f(u_n)u - F(u_n)\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 \end{aligned}$$

Note that $0 \geq \frac{(p-2)}{2p}$ when $\|u_n\| \to +\infty$. This is a contradiction with $\frac{(p-2)}{2p} > 0$. Therefore, we conclude that (u_n) is bounded on H, the result follows. □

To end this section, let us prove that if the minimum of I on $\mathcal{N}$ is achieved at some $u \in \mathcal{N}$, then u is a critical point of I. This follows from some arguments used in [64].

Lemma 1.1.7. *If $u \in \mathcal{N}$ is such that*

$$I(u) = \min_{\mathcal{N}} I,$$

then $I'(u) = 0$.

Proof. Suppose, by contradiction, that u is not a weak solution of (P_1). Then we find a function $\phi \in C_0^\infty(\Omega)$ such that

$$I'(u)\phi \leq -1.$$

Choose $\varepsilon > 0$ small such that

$$I'(tu + \sigma\phi)\phi \leq -\frac{1}{2}, \quad \text{for } |t-1| + |\sigma| \leq \varepsilon. \tag{1.1.11}$$

Let η be a cut-off function that $\eta(t) = 1$ for $|t-1| \leq \varepsilon/2$ and $\eta(t) = 0$ for $|t-1| \geq \varepsilon$.

Now we estimate $\sup_{t\geq 0} I(tu + \varepsilon\eta(t)\phi)$. Observe that for all $(t,\sigma) \neq (1,0)$ we have $I(tu + \varepsilon\eta\phi) < I(u)$. In fact, for $|t-1| \geq \varepsilon$, we have $I(tu + \varepsilon\eta\phi) = I(tu) < I(u)$ by Lemma 1.1.4. For $0 < |t-1| \leq \varepsilon$, from (1.1.11) we have

$$I(tu + \varepsilon\eta\phi) \;=\; I(tu) + \int_0^1 I'(tu + \sigma\varepsilon\eta(t)\phi)\varepsilon\eta(t)\phi d\sigma \leq I(tu) - \frac{1}{2}\varepsilon\eta(t) \leq I(tu) < I(u).$$

Now for $t = 1$, $I(tu + \varepsilon\eta(t)\phi) = I(u + \varepsilon\eta(1)\phi) \leq I(u) - \frac{1}{2}\varepsilon < I(u)$. Hence, we concluded $\sup_{t\geq 0} I(tu + \varepsilon\eta\phi) \leq c = \inf_{u\in\mathcal{N}} I(u)$. Now it is sufficient to find $\bar{t} > 0$ such that $\bar{t}u + \varepsilon\eta(\bar{t})\phi \in \mathcal{N}$, which is a contradiction by definition of c. For this, consider the function $\Upsilon : [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ given by $\Upsilon(t) = I'(tu + \varepsilon\eta(t)\phi)(tu + \varepsilon\eta(t)\phi)$. Note that $\Upsilon(t) = P(t) - Q(t)$ where P is a polynomial and

$$Q(t) = \int_{\mathbb{R}^N} f(tu + \varepsilon\eta(t)\phi)(tu + \varepsilon\eta(t)\phi)dx.$$

Note that Υ is a continuous function. Moreover, one has $\Upsilon(1-\varepsilon) = I'((1-\varepsilon)u)(1-\varepsilon)u > 0$ and $\Upsilon(1+\varepsilon) = I'((1+\varepsilon)u)(1+\varepsilon)u < 0$, hence we conclude that there exists $\bar{t} \in (1-\varepsilon, 1+\varepsilon)$ such that $\Upsilon(\bar{t}) = 0$.

□

1.2 Subcritical case

The first result in subcritical case is related to the convergence of functions f and F.

Lemma 1.2.1. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for c, then there exists $u \in H$ such that*

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx$$

and

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx.$$

Proof. We prove only the first convergence, because the second follows by the same reasoning. By Lemma 1.1.6, we have

$$\|u_n\| \leq K. \tag{1.2.1}$$

So, up to a subsequence,

$$u_n \to u \;\text{ in }\; L^2(\Omega) \;\text{ and }\; u_n(x) \to u(x) \;\; a.e. \;\; in \;\; \Omega$$

and, by continuity of f,

$$f(u_n(x))u_n(x) \to f(u(x))u(x) \;\; a.e \;\; in \;\; \Omega.$$

It is sufficient to prove that there is $g : \mathbb{R} \to \mathbb{R}$ such that $|f(u_n)u_n| \leq g(u_n)$ with $(g(u_n))$ convergent in $L^1(\Omega)$, because, in this case, using [24, Theorem 4.9 and Theorem 4.2] we get

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx.$$

Note that by the inequalities (1.1.3) and (1.1.5), for any $\varepsilon > 0$, $\alpha > 0$ and $q > 2$, we have

$$f(u_n(x))u_n(x) \leq \varepsilon|u_n(x)|^2 + C_\varepsilon|u_n(x)|^q \exp\Big(\alpha|u_n(x)|^2\Big) := g(u_n(x)).$$

We now prove that $(g(u_n))$ is convergent in $L^1(\Omega)$. First note that

$$\int_\Omega |u_n|^2 dx \to \int_\Omega |u|^2 dx$$

and

$$|u_n|^q \to |u|^q \ \text{ in } \ L^2(\Omega), \tag{1.2.2}$$

since the embedding $H \hookrightarrow L^r(\Omega$ is compact for $r \geq 1$. Moreover, using (1.2.1) and choosing $\alpha = \frac{16\pi^2}{K^2}$, we conclude by Theorem 1.1.1 that

$$\begin{aligned} \int_\Omega \exp\Big(2\alpha|u_n(x)|^2\Big)dx &\leq \int_\Omega \exp\left[2\alpha K^2\left(\frac{|u_n(x)|}{\|u_n\|}\right)^2\right]dx \\ &\leq \int_\Omega \exp\left[32\pi^2\left(\frac{|u_n(x)|}{\|u_n\|}\right)^2\right]dx \\ &\leq M. \end{aligned} \tag{1.2.3}$$

Since

$$\exp\Big(\alpha|u_n(x)|^2\Big) \to \exp\Big(\alpha|u(x)|^2\Big) \ \text{ a.e. in } \ \Omega,$$

we use [50, Lemma 4.8] and conclude that

$$\exp\Big(\alpha|u_n|^2\Big) \rightharpoonup \exp\Big(\alpha|u|^2\Big) \ \text{ in } \ L^2(\Omega). \tag{1.2.4}$$

Now using (1.2.2), (1.2.4) and [50, Lemma 4.8] again, we conclude

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx.$$

□

Lemma 1.2.2. *There exists $u_0 \in \mathcal{N}$ such that $I(u_0) = c$.*

Proof. Consider $(u_n) \subset \mathcal{N}$ a minimizing sequence. Then, by Lemma 1.1.6, (u_n) is bounded in H and, up to a subsequence,

$$u_n \rightharpoonup u_0 \ \text{ in } \ H.$$

We claim that $u_0 \not\equiv 0$. Indeed, if $u_0 \equiv 0$ then, from Lemma 1.2.1, we get

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx = \int_\Omega f(u_n)u_n dx \to 0,$$

which implies

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx \to 0. \tag{1.2.5}$$

From Theorem 1.1.2 for $j = 1$, $\kappa_1 = p$, $m = 2$, $\Upsilon = \frac{1}{2}$, $\kappa_3 = 2$, $N = 4$ and $\kappa_2 = \frac{2p}{4-p}$, there exist $C_1, C_2 > 0$

$$\left(\int_\Omega |\nabla u_n|^p\right)^{1/p} \leq C_1 \left(\int_\Omega |\Delta u_n|^2\right)^{1/4} |u_n|_{\kappa_2}^{1/2} + C_2|u_n|_s$$

for $s \geq 1$. Since

$$u_n \rightharpoonup 0 \ \text{ in } \ H,$$

we have that

$$\pm \int_\Omega |\nabla u_n|^p dx \to 0,$$

which implies by (1.2.5) that $\|u_n\|^2 \to 0$, contradicting Lemma 1.1.5. Then $u_n \rightharpoonup u_0 \neq 0$ in H. Note that $\|u_0\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2$. We are going to prove that

$$\|u_0\|^2 = \lim_{n\to\infty} \|u_n\|^2. \tag{1.2.6}$$

Applying Ekeland's Variational Principle [92, Theorem 8.5], we may suppose that (u_n) is a $(PS)_c$ sequence for I. Suppose, by contradiction, that (1.2.6) does not hold. Using a density argument we have that $I'(u_0)u_0 = 0$, which yields $u_0 \in \mathcal{N}$. From Lemma 1.1.3, Lemma 1.2.1 and Fatou's Lemma, we obtain

$$\begin{aligned} c &= I(u_0) - \frac{1}{p}I'(u_0)u_0 \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \int_\Omega \left(\frac{1}{p}f(u)u - F(u)\right)dx \\ &< \liminf_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \int_\Omega \left(\frac{1}{p}f(u_n)u_n - F(u_n)\right)dx\right] \\ &= \liminf_{n\to+\infty} \left[I(u_n) - \frac{1}{p}I'(u_n)u_n\right] \\ &= c, \end{aligned}$$

which is a contradiction. Hence, $u_n \to u_0$ in H and consequently $I(u_0) = c$. □

1.2.1 Proof of Theorem 1.0.1

Proof. The equality $I'(u_0) = 0$ is a consequence of Lemma 1.1.7. Thus, u_0 is a ground state solution of (P_1). □

1.3 Critical case

The critical case differs from the subcritical one mainly due to the more refined estimates about the minimization level c (Lemma 1.3.1), which permit to state the convergence result of Lemma 1.3.3. Once this convergence is established, the argument for the existence of a weak solution is the same as in the subcritical case. In this section, in order to prove the existence result in the critical case, we consider the auxiliary problem given by

$$(A) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u = |u|^{r-2}u \text{ in } \ \Omega, \\ u = \Delta u = 0, \end{cases}$$

where r is the constant that appears in the hypothesis (f_4). Associated to problem (A), one has the functional

$$I_r(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{r}\int_\Omega |u|^r dx,$$

and the Nehari manifold

$$\mathcal{N}_r = \{u \in H; u \neq 0 : I'_r(u)u = 0\}.$$

Taking $r > 4$, we have $H \hookrightarrow L^r(\Omega)$ with compact embedding. Then, considering problem (A), we can use Theorem 1.0.1 for the case $f(t) = |t|^{r-2}t$ and we conclude that there exists $w_r \in H$ such that

$$I_r(w_r) = c_r, \quad I'_r(w_r) = 0$$

and

$$c_r \geq \left(\frac{r-p}{pr}\right)\int_\Omega |w_r|^r dx, \tag{1.3.1}$$

where $c_r = \inf_{\mathcal{N}_r} I_r$. The next result is an estimate to $c = \inf_{\mathcal{N}} I$.

Lemma 1.3.1. *The value $c = \inf_{\mathcal{N}} I$ satisfies*

$$c \leq \frac{p(r-2)}{2(r-p)}\frac{c_r}{\tau^{2/(r-2)}}.$$

Proof. Note that, considering problem (A) and by the hypothesis (f_4) with $\tau \geq 1$, we have

$$\begin{aligned}\|w_r\|^2 \pm \int_\Omega |\nabla w_r|^p dx &= \int_\Omega |w_r|^r dx \\ &\leq \tau\int_\Omega |w_r|^r dx \\ &\leq \int_\Omega f(w_r)w_r dx.\end{aligned}$$

This inequality implies that $I'(w_r)w_r \leq 0$. Then, there exists $\beta \in (0,1)$ such that $\beta w_r \in \mathcal{N}$. Using (f_4) again, we obtain

$$c \leq I(\beta w_r) \leq \frac{\beta^2}{2}\|w_r\|^2 \pm \frac{\beta^p}{p}\int_\Omega |\nabla w_r|^p dx - \frac{\tau}{r}\beta^r\int_\Omega |w_r|^r dx.$$

Since $\beta \in (0,1)$, we get

$$c \leq \frac{\beta^2}{2}\left[\int_\Omega |\Delta w_r|^2 dx \pm \int_\Omega |\nabla w_r|^p dx\right] - \frac{\tau}{r}\beta^r\int_\Omega |w_r|^r dx.$$

Since $I'_r(w_r) = 0$, we conclude that

$$\begin{aligned}c &\leq \frac{\beta^2}{2}\int_\Omega |w_r|^r dx - \frac{\tau}{r}\beta^r\int_\Omega |w_r|^r dx \\ &= \left[\frac{\beta^2}{2} - \tau\frac{\beta^r}{r}\right]\int_\Omega |w_r|^r dx.\end{aligned}$$

Using (1.3.1), we have

$$c \leq \left[\frac{\beta^2}{2} - \tau\frac{\beta^r}{r}\right]\frac{c_r pr}{(r-p)} \leq \max_{s\geq 0}\left[\frac{s^2}{2} - \tau\frac{s^r}{r}\right]\frac{c_r pr}{(r-p)}.$$

By some lementary algebraic manipulations, we get

$$c \leq \frac{p(r-2)}{2(r-p)}\frac{c_r}{\tau^{2/(r-2)}}.$$

□

Lemma 1.3.2. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence, then*

$$\limsup_{n\to\infty} \|u_n\|^2 \leq \frac{16\pi^2}{\alpha_0}.$$

Proof. Using the estimate from Lemma 1.1.6, we have

$$\|u_n\|^2 \leq \frac{2cp}{(p-2)} + o_n(1).$$

By the estimate on the value c in Lemma 1.3.1, we get

$$\|u_n\|^2 \leq \frac{p^2(r-2)}{(p-2)(r-p)}\frac{c_r}{\tau^{2/(r-2)}} + o_n(1).$$

Taking $\tau > \tau^*$ where τ^* is given by

$$\tau^* = \left(\frac{p^2(r-2)}{(p-2)(r-p)}\frac{c_r}{\left(\frac{16\pi^2}{\alpha_0}\right)^{1/2}}\right)^{(r-2)/2}$$

and the result follows. □

The next result is the counterpart of Lemma 1.2.1 for the critical case.

Lemma 1.3.3. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence, then*

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx$$

and

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx.$$

Proof. We prove only the first convergence, because the second follows by the same reasoning. By the Lemma 1.3.2, we have

$$\limsup_{n\to\infty} \|u_n\|^2 < \frac{16\pi^2}{\alpha_0} \tag{1.3.2}$$

Then, exists $n_0 \in \mathbb{N}$ and $\alpha > \alpha_0$ such that

$$\|u_n\|^2 \leq \frac{16\pi^2}{\alpha_0} \quad \text{for all } n \geq n_0.$$

Up to a subsequence,

$$u_n \to u \ \text{ in } \ L^2(\Omega) \ \text{ and } \ u_n(x) \to u(x) \ \ a.e. \ \ in \ \ \Omega$$

and, from continuity of f,

$$f(u_n(x))u_n(x) \to f(u(x))u(x) \ \ a.e \ \ in \ \ \Omega.$$

Arguing as in Lemma 1.2.1, it is sufficient to prove that there is $g : \mathbb{R} \to \mathbb{R}$ such that $|f(u_n)u_n| \leq g(u_n)$ with $(g(u_n))$ convergent in $L^1(\Omega)$. Note that by the inequalities (1.1.3) and (1.1.5), we have

$$f(u_n(x))u_n(x) \ \ \leq \ \ \varepsilon|u_n(x)|^2 + C_\varepsilon|u_n(x)|^q \exp\Big(\alpha|u_n(x)|^2\Big) =: g(u_n(x)).$$

First note that

$$\int_\Omega |u_n|^2 dx \to \int_\Omega |u|^2 dx$$

and

$$|u_n|^q \to |u|^q \ \text{ in } \ L^2(\Omega). \tag{1.3.3}$$

Moreover, using (2.4.1) and choosing $\alpha > \alpha_0$, we conclude by Theorem 1.1.1 that

$$\int_\Omega \exp\Big(\alpha s|u_n(x)|^2\Big)dx \ \ \leq \ \ \int_\Omega \exp\left[32\pi^2\left(\frac{|u_n(x)|}{\|u_n\|}\right)^2\right]dx \leq M. \tag{1.3.4}$$

Since

$$\exp\Big(\alpha|u_n(x)|^2\Big) \to \exp\Big(\alpha|u(x)|^2\Big) \ \ \text{a.e. in} \ \ \Omega,$$

we use [50, Lemma 4.8] and conclude that

$$\exp\Big(\alpha|u_n|^2\Big) \rightharpoonup \exp\Big(\alpha|u|^2\Big) \ \ \text{in} \ \ L^2(\Omega). \tag{1.3.5}$$

Now using (3.2.5), (3.2.6) and [50, Lemma 4.8] again, we conclude

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx.$$

□

1.3.1 Proof of the Theorem 1.0.2

Proof. Considering Lemma 1.2.2, there exists $u_0 \in \mathcal{N}$ such that $I(u_0) = c$. Now, by Lemma 1.1.7, we conclude $I'(u_0) = 0$. Thus, u_0 has a ground state solution of (P_1). □

Chapter 2

Existence and multiplicity of nontrivial solutions to a class of elliptic Kirchhoff-Boussinesq type problems

In this chapter we are concerned with the existence and multiplicity of nontrivial solutions for the following class of problems

$$(P_2) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u = f(u) + \beta |u|^{2_{**}-2}u \text{ in } \Omega, \\ \\ \Delta u = u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 < p \leq 2^* = \frac{2N}{N-2}$, for $N \geq 3$. In this chapter we also use $2_{**} = \frac{2N}{N-4}$ if $N \geq 5$. We consider the subcritical case $\beta = 0$ and the critical case $\beta = 1$. The hypotheses on the function f are the following:

$(f_1)'$ There exists $C > 0$ such that the function f satisfies

$$|f(t)| \leq C(1 + |t|^{q-1}).$$

(f_2) The following limit holds:

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 0.$$

(f_3) The function $t \to \dfrac{f(t)}{|t|^{p-2}t}$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

(f_4) There are $p < r < 2_{**}$ and $\tau^* > 0$ such that

$$f(t) \geq \tau |t|^{r-2}t,$$

for all $t \geq 0$ and for all $\tau > \tau^*$, where τ^* will be fixed in Lemma 2.3.2.

Our main results are:

Theorem 2.0.1. *Assume that conditions $(f_1)'$, (f_2), (f_3), (f_4) hold with $\beta = 0$ and $2 < p \leq 2^* < q < 2_{**}$ or $2 < p < q \leq 2^*$. Then, problem (P_2) has a ground state solution.*

Theorem 2.0.2. *Assume that conditions* $(f_1)'$, (f_2), (f_3), (f_4) *hold with* $\beta = 1$ *and* $2 < p \leq 2^* < q < 2_{**}$ *or* $2 < p < q \leq 2^*$. *Then, problem* (P_2) *has a ground state solution.*

Since now we intend to find a multiplicity of solutions, in the next theorems, we will assume that f is equal to a prototype that satisfies the assumptions $(f_1)'$, (f_2), (f_3), (f_4), that is,

$$f(t) = \tau |t|^{q-2} t.$$

Theorem 2.0.3. *Assume* $\beta = 0$, $1 < q < 2 < p \leq 2^*$. *Then, there exists* $\tau^* > 0$ *such that problem* (P_2) *has infinitely many weak solutions, for all* $\tau \in (0, \tau^*)$.

Theorem 2.0.4. *Assume* $\beta = 1$, $1 < q < 2 < p \leq 2^*$. *Then, there exists* $\tau^* > 0$ *such that problem* (P_2) *has infinitely many weak solutions, for all* $\tau \in (0, \tau^*)$.

The plan of the chapter is the following: in Section 2.1 we show the variational framework and we prove some technical lemmas. Using Nehari technique in Section 2.2 we study the subcritical with $2 < p \leq 2^* < q < 2_{**}$ or $2 < p < q \leq 2^*$. In section 2.3, using the Mountain Pass Theorem, let us consider the critical case with $2 < p \leq 2^* < q < 2_{**}$ or $2 < p < q \leq 2^*$. In section 2.4 using the Krasnoselskii genus we study of multiplicity with $\beta = 0$ and $1 < q < 2 < p \leq 2^*$. In Section 2.5, we show the existence and multiplicity for the case $\beta = 1$ and $1 < q < 2 < p \leq 2^*$.

2.1 The variational framework and some technical lemmas

Motivated by the work of F. Gazzola, H. C. Grunau and G. Sweers in [41, Theorem 2.30], in this second chapter of the thesis, we will consider the following space $H := H^2(\Omega) \cap H_0^1(\Omega)$. We have a well defined inner product

$$\langle v, w \rangle = \int_\Omega \Delta v \Delta w dx, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega),$$

and associated norm given by

$$\|u\| := \left(\int_\Omega |\Delta u|^2 dx \right)^{1/2}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

From now on we denote by $H = (H^2(\Omega) \cap H_0^1(\Omega), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Consider the functional $I : H \to \mathbb{R}$ associated given by

$$I(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 dx \pm \frac{1}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega F(u) dx - \frac{\beta}{2_{**}} \int_\Omega |u|^{2_{**}} dx.$$

Since $2 < p < 2^*$, using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = 2$, we have that $H \hookrightarrow W_0^{1,p}(\Omega)$ is a continuous embedding and as a consequence we obtain that I is well-defined and of C^1 class. Moreover,

$$I'(u)\phi = \int_\Omega \Delta u \Delta \phi dx \pm \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \int_\Omega f(u) \phi \, dx - \frac{\beta}{2_{**}} \int_\Omega |u|^{2_{**}-2} u \phi dx,$$

for all $\phi \in H$. Then, the critical points of I are weak solution of (P_2). The Nehari manifold associated to the functional I is given by

$$\mathcal{N} = \{u \in H \backslash \{0\} : \ J(u) = 0\},$$

where $J(u) = I'(u)u$, for $u \in H$. Note that, from $(f_1)'$ and (f_2), for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon |t|^{q-1}. \tag{2.1.1}$$

and

$$|F(t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|t|^q. \tag{2.1.2}$$

Since Ω is bounded and f has subcritical growth, if $(u_n) \subset H$ is such that $u_n \rightharpoonup u$ in H, then

$$\int_\Omega f(u_n)u_n dx \to \int_\Omega f(u)u dx \tag{2.1.3}$$

and

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u)dx. \tag{2.1.4}$$

In order to use critical point theory, we firstly derive results related to the Palais-Smale compactness condition. We say that a sequence (u_n) is a Palais-Smale sequence for the functional I at level c_* if

$$I(u_n) \to c_*$$

and

$$I'(u_n) \to 0 \text{ in } (H)',$$

where

$$c_* = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], H) : \eta(0) = 0,\ I(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais-Smale condition ((PS) for short).

2.2 The case $\beta = 0$ and $2 < p \leq 2^* < q < 2_{**}$ or $2 < p < q \leq 2^*$

In the next result we prove that $\mathcal{N}$ is not empty and I restricted to $\mathcal{N}$ is bounded from below.

Lemma 2.2.1. *For each $u \in H$, there exists an unique $t > 0$ such that $tu \in \mathcal{N}$. Moreover, $I(u) > 0$ for every $u \in \mathcal{N}$.*

Proof. Given $u \in H \setminus \{0\}$, let $\gamma_u(t) = I(tu)$ for $t > 0$. Then $tu \in \mathcal{N}$ if and only if ${\gamma'}_u(t) = 0$. Note that, taking $\varepsilon > 0$ sufficiently small in (2.1.2) and using best Sobolev embedding $H \hookrightarrow L^2(\Omega)$ and $H \hookrightarrow L^q(\Omega)$, there exists $S_2^{-1} > 0$ and $S_q^{-\frac{q}{2}}$ such that

$$\begin{aligned}
\gamma_u(t) &= \frac{t^2}{2}\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(tu)dx \\
&\geq t^2\left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \frac{C_\varepsilon}{q}t^q \int_\Omega |u|^q dx \\
&\geq t^2\left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \frac{C_\varepsilon S_q^{\frac{-q}{2}}}{q}t^q\|u\|^q.
\end{aligned}$$

Thus, since $2 < p < q$, we have $\gamma_u(t) > 0$ for all $0 < t$ sufficiently small.

Now, from (f_4) and using $2 < p < r$, for all $\tau > 0$, we have

$$\frac{\gamma_u(t)}{t^p} \leq \frac{1}{2t^{p-2}}\|u\|^2 \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \tau\frac{t^{r-p}}{r}\int_\Omega |u|^r dx.$$

Hence, $\lim_{t\to+\infty} \gamma_u(t) = -\infty$. Then, there exists at least one $t(u) > 0$ such that $\gamma_u'(t(u)) = 0$, i.e. $t(u)u \in \mathcal{N}$. Moreover, in the case $2 < p$, we get

$$\gamma_u'(t) = t^{p-1}\left[\frac{1}{t^{p-2}}\|u\|^2 \pm \int_\Omega |\nabla u|^p dx - \int_\Omega \frac{f(tu)}{t^{p-1}} u dx\right].$$

From (f_3) we conclude that $\frac{\gamma_u'(t)}{t^{p-1}}$ is decreasing. Then, it vanishes exactly once, and consequently there is no other $t > 0$ such that $tu \in \mathcal{N}$. Note, in particular, that $t(u)$ is a global maximum point of γ_u and $\gamma_u(t(u)) > 0$, i.e. $I(t(u)u) > 0$. Since $t(u) = 1$ if $u \in \mathcal{N}$, we deduce that $I(u) > 0$ for every $u \in \mathcal{N}$. □

We set

$$c := \inf_{\mathcal{N}} I.$$

In the next result, we prove that minimizing sequences in $\mathcal{N}$ does not converge to 0.

Lemma 2.2.2. *There exists a constant $C > 0$ such that $0 < C \leq \|u\|$ for every $u \in \mathcal{N}$.*

Proof. Since that $u \in \mathbb{N}$, we get

$$\int_\Omega |\Delta u|^2 dx \pm \int_\Omega |\nabla u|^p dx = \int_\Omega f(u) u dx$$

By the inequality (2.1.1)

$$\int_\Omega |\Delta u|^2 dx \pm \int_\Omega |\nabla u|^p dx \leq \frac{\varepsilon}{2}\int_\Omega |u|^2 dx + \frac{C_\varepsilon}{q}\int_\Omega |u|^q dx$$

Using Sobolev embedding $H \hookrightarrow L^2(\Omega)$ and $H \hookrightarrow L^q(\Omega)$, there exists $S_2^{-1} > 0$ and $S_q^{-\frac{q}{2}}$, we get

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \int_\Omega |\nabla u|^p dx \leq \frac{S_q^{\frac{-q}{2}}}{q} C_\varepsilon \|u\|^q.$$

In the case that the second term in the associated functional I is positive, this inequality implies

$$\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \leq \left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 + \int_\Omega |\nabla u|^p dx \leq \frac{S_q^{\frac{-q}{2}}}{q} C_\varepsilon \|u\|^q.$$

Consequently

$$0 < \left[\frac{(2 - \varepsilon S_2^{-1})q}{2C_\varepsilon S_q^{\frac{-q}{2}}}\right]^{1/(q-2)} \leq \|u\|. \tag{2.2.1}$$

Since $2 < q$ the result follows. In the case that the second term in the associated functional I is negative, using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $k_1 = p$, $\kappa_3 = 2$, $s = q$ and by continuous embedding, we get

$$\begin{aligned}\left(1 - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 &\leq \frac{S_p^{\frac{-p}{2}}}{p}\|u\|^p + \frac{S_q^{\frac{-q}{2}}}{q}C_\varepsilon\|u\|^q \\ &\leq \max\left[\frac{S_p^{\frac{-p}{2}}}{p}, \frac{S_q^{\frac{-q}{2}}}{q}C_\varepsilon\right]\|u\|^q\end{aligned}$$

Since $2 < p < q$, this implies that

$$0 < \left\{\frac{2 - \varepsilon S_2^{-1}}{2\left[\frac{S_p^{\frac{-p}{2}}}{p}, \frac{S_q^{\frac{-q}{2}}}{q}C_\varepsilon\right]}\right\}^{1/(q-2)} \leq \|u\|$$

the lemma is proved. □

In the next lemma, we prove that all minimizing sequences in $\mathcal{N}$ are bounded in H.

Lemma 2.2.3. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence to I, then (u_n) is bounded in H.*

Proof. Note that $I(u_n) \to c$ and $I'(u_n)u_n = 0$. Then, from Lemma 1.1.3, we have

$$\begin{aligned}c + o_n(1)\|u_n\| &= I(u_n) - \frac{1}{p}I'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{p}f(u_n)u - F(u_n)\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2\end{aligned}$$

and the result follows. □

To end up this section, let us prove that if the minimum of I on $\mathcal{N}$ is achieved in some $u \in \mathcal{N}$, then in fact u is a critical point of I. This follows from some arguments used in [64].

Lemma 2.2.4. *If $u_0 \in \mathcal{N}$ is such that*

$$I(u_0) = \min_{\mathcal{N}} I,$$

then $I'(u_0) = 0$.

Proof. The proof follows in the same spirit as in Lemma 1.1.7.

□

2.2.1 Proof of the Theorem 2.0.1

Proof. Consider $(u_n) \subset \mathcal{N}$ a minimizing sequence for c. Then, by the Lemma 2.2.3, (u_n) is bounded in H and, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in} \quad H.$$

We claim that $u_0 \not\equiv 0$. Indeed, if $u_0 \equiv 0$ then, from (2.1.3), we get

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx = \int_\Omega f(u_n)u_n dx \to 0, \tag{2.2.2}$$

which implies

$$\|u_n\|^2 \pm \int_\Omega |\nabla u_n|^p dx \to 0. \tag{2.2.3}$$

From Theorem 1.1.2 for $j = 1$, $\kappa_1 = p$, $m = 2$, $\Upsilon = \frac{1}{2}$, $\kappa_3 = 2$ and $k_2 = \frac{2p}{4-p}$, we have that there exist $C_1, C_2 > 0$

$$\left(\int_\Omega |\nabla u_n|^p\right)^{1/p} \leq C_1 \left(\int_\Omega |\Delta u_n|^2\right)^{1/4} |u_n|_{\kappa_2}^{1/2} + C_2|u_n|_s$$

for $s \geq 1$. Since

$$u_n \rightharpoonup 0 \text{ in } H,$$

we have that

$$\pm \int_\Omega |\nabla u_n|^p dx \to 0,$$

which implies by (2.2.2) that $\|u_n\|^2 \to 0$, contradicting Lemma 2.2.2. Then $u_n \rightharpoonup u_0 \neq 0$ in H.

Note that $\|u_0\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2$. We are going to prove that

$$\|u_0\|^2 = \lim_{n\to\infty} \|u_n\|^2. \tag{2.2.4}$$

Applying Ekeland's Variational Principle [92, Theorem 8.5], we may suppose that (u_n) is a $(PS)_c$ for I. Suppose, by contradiction, that (2.2.4) does not hold. Using a density argument we have that $I'(u_0)u_0 = 0$, where we conclude that $u_0 \in \mathcal{N}$. From Lemma 1.1.3, Fatou's Lemma and (2.1.4) we obtain

$$\begin{aligned}
c &\leq I(u_0) - \frac{1}{p}I'(u_0)u_0 \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_0\|^2 + \int_\Omega \left(\frac{1}{p}f(u_0)u_0 - F(u_0)\right) dx \\
&< \liminf_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{p}f(u_n)u_n - F(u_n)\right) dx\right] \\
&= \liminf_{n\to+\infty} \left[I(u_n) - \frac{1}{p}I'(u_n)u_n\right] = c,
\end{aligned}$$

which is a contradiction. Hence, $u_n \to u_0$ in H and consequently, $I(u_0) = c$.

□

2.3 The case $\beta = 1$ and $2 < p \leq 2^* < q < 2_{**}$ or $2 < p < q \leq 2^*$

In this subsection, we use the Mountain Pass Theorem to show the existence of a solution, taking into consideration the condition (PS) for the functional I is satisfied under certain constants that will be constructed.

Lemma 2.3.1. *The functional I satisfies the following conditions:*
(i) There exist ρ_1, $\rho_2 > 0$ such that:

$$I(u) \geq \rho_2 \ \text{ with } \ \|u\| = \rho_1$$

(ii) There exists $e \in B^c_{\rho_1}(0)$ with $I(e) < 0$.

Proof. i) Note that, taking $\varepsilon > 0$ sufficiently small in (2.1.2) and using Sobolev embedding, $H \hookrightarrow L^2(\Omega)$ and $H \hookrightarrow L^q(\Omega)$, there exists $S_2^{-1} > 0$ and $S_q^{-\frac{q}{2}} > 0$, we get

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(u)dx - \frac{1}{2_{**}}\int_\Omega |u|^{2_{**}}dx \\ &\geq \frac{1}{2}\int_\Omega |\Delta u|^2 dx \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega \left(\frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{q}|u|^q\right)dx - \frac{1}{2_{**}}\int_\Omega |u|^{2_{**}}dx \\ &= \left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{C_\varepsilon}{q} S_q^{\frac{-q}{2}}\|u\|^q - \frac{S^{\frac{-2_{**}}{2}}}{2_{**}}\|u\|^{2_{**}}. \end{aligned}$$

In the case that the second term in the associated functional I is positive

$$I(u) \geq \left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 - \frac{C_\varepsilon}{q} S_q^{\frac{-q}{2}}\|u\|^q - \frac{S^{\frac{-2_{**}}{2}}}{2_{**}}\|u\|^{2_{**}}.$$

Since $2 < q$, we have

$$I(u) \geq \rho_2 > 0, \quad \text{for all } \|u\| = \rho_1,$$

where

$$\rho_2 = \left[\left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right) - \frac{C_\varepsilon}{q} S_q^{\frac{-q}{2}} \rho_1^{q-2} - -\frac{S^{\frac{-2_{**}}{2}}}{2_{**}}\rho_1^{2_{**}-2}\right]\rho_1^2.$$

This establishes (i).

In the case that the second term in the associated functional I is negative, using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = q$ and by continuous embedding, we get

$$I(u) \geq \left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right)\|u\|^2 - \frac{1}{p} S_p^{\frac{-p}{2}}\|u\|^p - \frac{C_\varepsilon}{q} S_q^{\frac{-q}{2}}\|u\|^q - \frac{S^{\frac{-2_{**}}{2}}}{2_{**}}\|u\|^{2_{**}}.$$

where

$$\rho_2 = \left[\left(\frac{1}{2} - \frac{\varepsilon S_2^{-1}}{2}\right) - \frac{1}{p} S_p^{\frac{-p}{2}}\rho_1^{p-2} - \frac{C_\varepsilon}{q} S_q^{\frac{-q}{2}} \rho_1^{q-2} - \frac{S^{\frac{-2_{**}}{2}}}{2_{**}}\rho_1^{2_{**}-2}\right]\rho_1^2.$$

This establishes (i).

(ii) Fixed $\phi \in C_0^\infty(\Omega)$. Now, from (f_4) and for all $\tau > 0$, we have

$$\frac{I(t\phi)}{t^p} \leq \frac{1}{2t^{p-2}}\|\phi\|^2 \pm \frac{1}{p}\int_\Omega |\nabla\phi|^p dx - \tau\frac{t^{r-p}}{r}\int_\Omega |\phi|^r dx - \frac{t^{2_{**}-p}}{2_{**}}\int_\Omega |\psi|^{2_{**}}dx.$$

Since $2 < p < r$, there exists $\bar{t} > 0$ large such that $e = \bar{t}\phi$ satisfies $I(e) < 0$ and $\|e\| > \rho_2$. ∎

We devote the rest of this section to show that c_* is attained by a positive function. As in [25], we are able to compare the minimax level c_* with a suitable number which involves the constants

$$S = \inf_{v\in H, v\neq 0} \left\{ \frac{\displaystyle\int_\Omega |\Delta u|^2 dx}{\left(\displaystyle\int_\Omega |v|^{2_{**}} dx\right)^{2/2_{**}}} \right\}$$

and

$$\overline{S} = \inf_{v\in H, v\neq 0} \left\{ \frac{\displaystyle\int_\Omega |\Delta u|^2 dx + \int_\Omega |\nabla v|^p dx}{\left(\displaystyle\int_\Omega |v|^{2_{**}} dx\right)^{2/2_{**}}} \right\}$$

Lemma 2.3.2. *If the conditions $(f_1)'$, (f_2), (f_3), (f_4) hold, then there exists $\tau_* > 0$ such that*

$$c_\tau < \min\left\{ \left(\frac{1}{p} - \frac{1}{2_{**}}\right) \overline{S}^{N/4}, \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) \min\left\{\frac{1}{C^p}, S\right\}, \left(\frac{1}{2} - \frac{1}{p}\right) \right\}$$

for all $\tau > \tau^*$.

Proof. If we define $\eta_*(t) = te$ for $t \in [0,1]$, where $e = \bar{t}\phi$ is the function given by Lemma 2.3.1. It follows that $\eta_* \in \Gamma$ and thus

$$\begin{aligned} 0 < c_* &\leq \max_{t\in[0,1]} I(\eta_*(t)) \\ &\leq \max_{t\geq 0} I(\eta_*(t)) \\ &\leq \max_{t\geq 0} \left[\frac{t^2}{2}\|e\|^2 \pm \frac{t^p}{p}\int_\Omega |\nabla e|^p dx - \frac{\tau t^r}{r}\int_\Omega |e|^r dx\right]. \end{aligned}$$

In the case that the second term in the associated functional I is negative, we have

$$\begin{aligned} 0 < c_* &\leq \max_{t\geq 0} \left[\frac{t^2}{2}\|e\|^2 - \frac{\tau t^r}{r}\int_\Omega |e|^r dx\right] \\ &= \frac{t_\tau^2}{2}\|e\|^2 - \frac{\tau t_\tau^r}{r}\int_\Omega |e|^r dx, \end{aligned}$$

where

$$t_\tau = \left[\frac{\|e\|^2}{\tau\displaystyle\int_\Omega |e|^r dx}\right]^{1/(r-2)}.$$

Then,

$$\begin{aligned} 0 < c_* &\leq \frac{1}{2}\left[\frac{\|e\|^2}{\tau\displaystyle\int_\Omega |e|^r dx}\right]^{2/(r-2)} \|e\|^2 - \tau\frac{1}{r}\left[\frac{\|e\|^2}{\tau\displaystyle\int_\Omega |e|^r dx}\right]^{r/(r-2)} \int_\Omega |e|^r dx \\ &= \left(\frac{1}{2} - \frac{1}{r}\right) \frac{\left[\|e\|^2\right]^{r/(r-2)}}{\left[\displaystyle\int_\Omega |e|^r dx\right]^{2/(r-2)}} \frac{1}{\tau^{2/(r-2)}}. \end{aligned}$$

For

$$\tau^* = \left[\left(\frac{1}{2} - \frac{1}{r} \right) \frac{\left[\|e\|^2 \right]^{r/(r-2)}}{\left[\int_\Omega |e|^r dx \right]^{2/(r-2)}} \frac{4p}{(p-2) \min \left\{ \frac{1}{C^p}, S \right\}} \right]^{(r-2)/2},$$

we have $\tau > \tau^*$ for

$$c_\tau < \frac{1}{2} \left(\frac{1}{2} - \frac{1}{p} \right) \min \left\{ \frac{1}{C^p}, S \right\}.$$

In the case that the second term in the associated functional I is positive, we have

$$\begin{aligned} 0 < c_* &\leq \max_{t \in [0,1]} I(\eta_*(t)) \\ &\leq \max_{t \in [0,1]} \left[\left(\frac{p+2}{p} \right) \frac{t^2}{2} \left(\|e\|^2 + \int_\Omega |\nabla e|^p dx \right) - \frac{\tau t^r}{r} \int_\Omega |e|^r dx \right] \\ &= \left(\frac{p+2}{p} \right) \frac{t_\tau^2}{2} \left(\|e\|^2 + \int_\Omega |\nabla e|^p dx \right) - \frac{\tau t_\tau^r}{r} \int_\Omega |e|^r dx, \end{aligned}$$

where

$$t_\tau = \left[\frac{\|e\|^2 + \int_\Omega |\nabla e|^p dx}{\tau \int_\Omega |e|^r dx} \right]^{1/(r-2)}.$$

Then,

$$\begin{aligned} 0 < c_* &\leq \left(\frac{p+2}{p} \right) \left[\frac{\|e\|^2 + \int_\Omega |\nabla e|^p dx}{\tau \int_\Omega |e|^r dx} \right]^{2/(r-2)} \left(\|e\|^2 + \int_\Omega |\nabla e|^p dx \right) \\ &\quad - \tau \frac{1}{r} \left[\frac{\left(\|e\|^2 + \int_\Omega |\nabla e|^p dx \right)}{\tau \int_\Omega |e|^r dx} \right]^{r/(r-2)} \int_\Omega |e|^r dx \\ &= \left(\frac{p+2}{p} - \frac{1}{r} \right) \frac{\left[\|e\|^2 + \int_\Omega |\nabla e|^p dx \right]^{r/(r-2)}}{\left[\int_\Omega |e|^r dx \right]^{2/(r-2)}} \frac{1}{\tau^{2/(r-2)}}. \end{aligned}$$

For

$$\tau^* = \left[\left(\frac{p+2}{p} - \frac{1}{r} \right) \frac{\left[\|e\|^2 + \int_\Omega |\nabla e|^p dx \right]^{r/(r-2)}}{\left[\int_\Omega |e|^r dx \right]^{2/(r-2)}} \frac{2_{**} p}{(2_{**} - p) \overline{S}^{N/4}} \right]^{\frac{(r-2)}{2}}$$

we have $\tau > \tau^*$ for

$$c_* < \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \overline{S}^{N/4}.$$

□

Lemma 2.3.3. *Let (u_n) be a sequence in H such that $I(u_n) \to c_*$ and $I'(u_n) \to 0$ as $n \to \infty$. Then*

(i) $u_n \rightharpoonup u$ *in* H*;*

(ii) *The weak limit $u \in H$ is a critical point of I, that is, $I'(u) = 0$;*

(iii) $u_n \to u$ *in* H.

Proof. Now we prove (i). Note that from Lemma 1.1.3, we have

$$\begin{aligned} c + o_n(1)\|u_n\| &= I(u_n) - \frac{1}{p}I'(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \int_\Omega \left(\frac{1}{p}f(u_n)u - F(u_n)\right)dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_\Omega |u_n|^{2_{**}}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2, \end{aligned} \tag{2.3.1}$$

which implies that (u_n) is bounded in H. Then, up to a subsequence, $u_n \rightharpoonup u$ in H.

Now we prove (ii). As a consequence of weak convergence, for all $\phi \in H$, we have that

$$\int_\Omega \Delta u_n \Delta\phi dx = \int_\Omega \Delta u \Delta \phi dx + o_n(1).$$

From [50, Lemme 4.8], for all $\phi \in H$, we get

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi dx = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx + o_n(1)$$

and

$$\int_\Omega |\, u_n|^{2_{**}-2}u_n\phi dx = \int_\Omega |u|^{2_{**}-2}u\phi dx + o_n(1).$$

Since Ω is bounded and f has subcritical growth, for all $\phi \in H$, we obtain,

$$\int_\Omega f(u_n)\phi dx = \int_\Omega f(u)\phi dx + o_n(1).$$

Since $I'(u_n)\phi = o_n(1)$, for all $\phi \in H$, using these convergence above, $I'(u)\phi = 0$, for all $\phi \in H$.

Now we prove $iii)$. Consider $v_n = u_n - u$. Then, from [50, Lemma 4.6] and arguing as [5, Lemma 3.1]), we have

$$I(v_n) = I(u_n) - I(u) + o_n(1) = c_* - I(u) = \widetilde{c}.$$

We also know that the following expression is true

$$\int_\Omega [F(u_n) - F(u) - F(u_n - u)]dx = o_n(1) \tag{2.3.2}$$

and

$$\sup_{\varphi \in H, \|\varphi\|=1} \int_\Omega [f(u_n) - f(u) - f(u_n - u)]\varphi dx = o_n(1). \tag{2.3.3}$$

Here, we only show (2.3.2) since the verification of (2.3.3) is similar. Let $v_n := u_n - u$. Then, $v_n \rightharpoonup 0$ in H and $u(x) \to u(x)$ $a.e.$ in Ω. It follows from (2.1.1) that

$$\begin{aligned}
|F(v_n+u)-F(u)| &\leq \int_0^1 |f(v_n+tu)u|dt \\
&\leq \int_0^1 \Big(\varepsilon|v_n+tu||\overline{u}| + C(\varepsilon)|v_n+tu|^{q-1}|u|\Big)dx \\
&\leq C\Big(\varepsilon|v_n||u| + \varepsilon|u|^2 + C(\varepsilon)|v_n|^{q-1}|u| + C(\varepsilon)|u|^q\Big).
\end{aligned}$$

By Young inequality, we get that

$$|F(v_n+u)-F(u)| \leq C\Big(\varepsilon|v_n|^2 + \varepsilon|u|^2 + C(\varepsilon)|v_n|^q + C(\varepsilon)|u|^q\Big).$$

With combining with (2.1.2) yields that

$$|F(v_n+u)-F(v_n)-F(u)| \leq C\Big(\varepsilon|v_n|^2 + \varepsilon|u|^2 + C(\varepsilon)|v_n|^q + C(\varepsilon)|u|^q\Big), \quad \forall n \in \mathbb{N}.$$

Let

$$R_n(x) := \max\left\{|F(v_n+u)-F(v_n)-F(u)| - C(\varepsilon)(|v_n|^2 + |v_n|^q), 0\right\}.$$

Then $0 \leq R_n(x) \leq C(\varepsilon)(|v_n|^2 + |u|^q) \in L^1(\Omega)$. Thus, the Lebesgue dominated convergence theorem implies that

$$\int_\Omega R_n(x)dx \to 0, \quad \text{as } n \to \infty. \tag{2.3.4}$$

Furthermore, by the definition of $R_n(x)$, we have

$$|F(v_n+u)-F(v_n)-F(u)| \leq C(\varepsilon)(|v_n|^2 + |v_n|^q) + R_n(x), \quad \forall n \in \mathbb{N}$$

which together with (2.1.1) and (2.3.4) shows that

$$\begin{aligned}
\int_\Omega |F(v_n+u)-F(v_n)-F(u)|dx &\leq C(\varepsilon)(\|v_n\|_2^2 + \|v_n\|_q^q) + \varepsilon \\
&\leq C(\varepsilon)(\|v_n\|^2 + \|v_n\|^q) + \varepsilon \leq C\varepsilon
\end{aligned}$$

for n sufficiently large. Hence, (2.3.2) holds.

By the weak convergence, (2.3.2) and [3, Theorem 1] , it follows that

$$\begin{aligned}
I(v_n) - I(u_n) + I(u) &= \frac{1}{2}\int_\Omega (|\Delta u_n - \Delta u|^2 - |\Delta u_n|^2 + |\Delta u|^2)dx \\
&= \frac{1}{p}\int_\Omega (|\nabla u_n - \nabla u|^p - |\nabla u_n|^p + |\nabla u|^p)dx \\
&\quad - \int_\Omega (F(u_n-u) - F(u_n) + F(u))\,dx \\
&= \langle u, u\rangle - \langle u_n, u\rangle + o_n(1)
\end{aligned} \tag{2.3.5}$$

taking limit,

$$I(v_n) \to c_* - I(u) = \widetilde{c}., \quad \text{as } n \to +\infty.$$

In order to prove that

$$I'(v_n) \to 0, \ n \to \infty, \ \text{in } H'.$$

Note that

$$\int_\Omega [f(u_n) - f(u) - f(v_n)]\phi dx = o_n(1) \tag{2.3.6}$$

and

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi dx = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi dx + o_n(1) \tag{2.3.7}$$

and

$$\int_\Omega |\ u_n|^{2_{**}-2} u_n \phi dx = \int_\Omega |u|^{2_{**}-2} u\phi dx + o_n(1). \tag{2.3.8}$$

From the weak convergence, (2.3.6), (2.3.7) and [3, Theorem 1] , it follows that

$$\begin{aligned}
I'(v_n)\varphi - I'(u_n)\varphi &= \int_{\mathbb{R}^N} ((\Delta u_n - \Delta u)\Delta\varphi - \Delta u_n \Delta\varphi)dx \\
&\quad \pm \int_\Omega \Big(|\nabla(u_n - u)|^{p-2}\nabla(u_n - u) - |\nabla u_n|^{p-2}\nabla u_n)\nabla\phi dx \\
&\quad - \int_{\mathbb{R}^N} (f(u_n - u)\varphi - f(u_n)\varphi)dx \\
&\quad + \int_\Omega (|(u_n - u)|^{2_{**}-2}(u_n - u) - |u_n|^{2_{**}-2}u_n)\phi dx \\
&= -\langle u, \varphi\rangle + \int_\Omega f(u)\varphi dx \\
&= -I'(u)\varphi
\end{aligned}$$

taking limit,

Note that for (2.3.1), $c_* \geq \widetilde{c}$. Moreover, for all $\phi \in H$, we get

$$I'(v_n)\phi = I'(u_n)\phi - I'(u)\phi = o_n(1),$$

which implies that

$$\|v_n\|^2 \pm \int_\Omega |\nabla v_n|^p dx = \int_\Omega |v_n|^{2_{**}} dx + o_n(1). \tag{2.3.9}$$

In the case that the second term in (2.3.9) is positive, we have

$$\|v_n\|^2 + \int_\Omega |\nabla v_n|^p dx \to \overline{L} \ \text{ and } \ \int_\Omega |v_n|^{2_{**}} dx \to \overline{L}.$$

Note that

$$\begin{aligned}
\widetilde{c} + o_n(1) &= \frac{1}{2}\|v_n\|^2 + \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}} dx \\
&\geq \frac{1}{p}\|v_n\|^2 + \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}} dx \\
&= \frac{1}{p}\left[\|v_n\|^2 + \int_\Omega |\nabla v_n|^p dx\right] - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}} dx.
\end{aligned}$$

In this case, considering $\overline{L} > 0$ we obtain

$$\widetilde{c} \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{L}.$$

Now using the definition of $\overline{S}$, we have

$$\overline{S} \leq \frac{\|v_n\|^2 + \displaystyle\int_\Omega |\nabla v_n|^p dx}{\left(\displaystyle\int_\Omega |v_n|^{2_{**}}\right)^{2/2_{**}}} = \overline{L}^{4/N} + o_n(1).$$

Then, $\overline{S}^{N/4} \leq \overline{L}$. We conclude

$$c_* \geq \widetilde{c} \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4},$$

which is a contradiction with Lemma 2.3.2. Then, $\overline{L} = 0$ and $\|v_n\|^2 \to 0$, which implies that $u_n \to u$ in H.

In the case that the second term in (2.3.9) is negative, we have

$$\|u_n\|^2 \to \widetilde{L} \quad \text{and} \quad \int_\Omega |u_n|^{2_{**}} dx + \int_\Omega |\nabla u_n|^p dx \to \widetilde{L}. \tag{2.3.10}$$

with $\widetilde{L} = \widetilde{L}_1 + \widetilde{L}_2$, where

$$\int_\Omega |\nabla u_n|^p dx \to \widetilde{L}_1 \quad \text{and} \quad \int_\Omega |u_n|^{2_{**}} dx \to \widetilde{L}_2$$

Since $2 < p < 2_{**}$ and $\displaystyle\int_{\mathbb{R}^N} F(u_n)dx \to 0$, we have

$$\begin{aligned}
\widetilde{c} + o_n(1) &= \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}} dx \\
&\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^p dx \\
&= \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\left[\int_\Omega |v_n|^{2_{**}} dx + \int_\Omega |\nabla v_n|^p dx\right].
\end{aligned}$$

Letting $n \to \infty$ and using (2.3.10) we get

$$\widetilde{c} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\widetilde{L}. \tag{2.3.11}$$

If $\widetilde{L} \geq 1$, then,

$$\widetilde{c} \geq \left(\frac{1}{2} - \frac{1}{p}\right), \tag{2.3.12}$$

which is a contradiction by the hypotheses.
On the other hand, we can use the definition of S to get

$$S\left(\int_\Omega |u_n|^{2_{**}} dx\right)^{2/2_{**}} \leq \int_\Omega |\Delta u_n|^2 dx \tag{2.3.13}$$

Now from Theorem 1.1.2, there exits $C > 0$ such that

$$\left(\int_\Omega |\nabla u_n|^p dx\right)^{1/p} \leq C\left(\int_\Omega |\Delta u_n|^2 dx\right)^{1/2}.$$

Then,

$$\frac{1}{C^p}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^{2/p} \leq \int_{\mathbb{R}^N} |\Delta u_n|^2 dx \tag{2.3.14}$$

Using (2.3.13) and (2.3.14) we obtain

$$\|u_n\|^2 \quad \geq \quad \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\left(\int_\Omega |\nabla u_n|^p dx\right)^{2/p} + \left(\int_\Omega |u_n|^{2_{**}} dx\right)^{2/2_{**}}\right\}.$$

Suppose that $0 < \widetilde{L} < 1$. In this case $0 < \widetilde{L}_1, \widetilde{L}_2 < 1$. Then, since $2 < p < 2_{**}$, we have

$$\|u_n\|^2 \quad \geq \quad \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\int_\Omega |\nabla u_n|^p dx + \int_\Omega |u_n|^{2_{**}} dx\right\}^{2/p}.$$

Taking the limit we conclude that

$$\widetilde{L} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\widetilde{L}^{2/p}$$

Since $\widetilde{L} > 0$, we obtain

$$\widetilde{L}^{(p-2)/p} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}$$

and from (2.3.11) that

$$c_* \geq \widetilde{c} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}^{p/(p-2)},$$

which is a contradiction with Lemma 2.3.2. Then, $\widetilde{L} = 0$ and $\|v_n\|^2 \to 0$, which implies that $u_n \to u$ in H. □

2.3.1 Proof of Theorem 2.0.2

Proof. The proof of Theorem 2.0.2 is a consequence of Lemma 2.3.1 and Lemma 2.3.3. □

2.4 The case $\beta = 0$ and $1 < q < 2 < p \leq 2^*$

We will start by considering some basic notions on the Krasnoselskii genus which we will use in the proofs of our main results.

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If k does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.

In the sequel we will establish only the properties of the genus that will be used through this work. More information on this subject may be found in the references [2], [28], [35] and [52].

Proposition 2.4.1. *Let A and B be sets in $\mathfrak{A}$.*

(i) If there exists an odd application $\varphi \in C(A, B)$ then $\gamma(A) \leq \gamma(B)$.

(ii) If there exists an odd homeomorphism $\varphi : A \to B$ then $\gamma(A) = \gamma(B)$. (iii) If A is a compact set, then there exists a neighborhood $K \in \mathfrak{A}$ of A such that $\gamma(A) = \gamma(K)$.

(iv) If $\gamma(B) < \infty$, then $\gamma(\overline{A \backslash B}) \geq \gamma(A) - \gamma(B)$.

Proposition 2.4.2. *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 2.4.3. *$\gamma(\mathcal{S}^{N-1}) = N$ where $\mathcal{S}^{N-1}$ is a unit sphere of $\mathbb{R}^N$.*

Proposition 2.4.4. *If $K \in \mathfrak{A}$, $0 \notin K$ and $\gamma(K) \geq 2$, then K has infinitely many points.*

The proofs of these results can be found, for example, in Proposition 7.5, Remark 7.6 and Proposition 7.7 from [74]. We now present a result due to Clark [34].

Theorem 2.4.5. *Let $J \in C^1(X, \mathbb{R}l)$ be a functional satisfying the Palais-Smale condition. Furthermore, let us suppose that*
(A_1) J is bounded from below and even;
(A_2) there is a compact set $K \in \mathfrak{A}$ such that $\gamma(K) = k$ and $\sup\limits_{x \in K} J(x) < J(0)$. Then J possesses at least k pairs of distinct critical points and their corresponding critical values c_j are less than $J(0)$.

Since we intend to find a multiplicity of solutions, in the next sections, we will assume that f is equal to a prototype that satisfies the assumptions $(f_1)'$, (f_2), (f_3), (f_4), that is,

$$f(t) = \tau |t|^{q-2} t.$$

In this section we study some properties related to the C^1functional I_τ given by

$$I_\tau(u) = \frac{1}{2}\|u\|^2 \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\tau}{q}\int_\Omega |u|^q dx.$$

2.4.1 The case that the second term in the associated functional I_τ is positive

Lemma 2.4.6. *I_τ is bounded from below.*

Proof. From Sobolev embedding, there is a positive contant $C > 0$ such that

$$\begin{aligned} I_\tau(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\tau}{q}\int_\Omega |u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - C\frac{\tau}{q}\|u\|^q, \end{aligned}$$

showing that I_τ is coercive and, therefore, I is bounded from below. □

Lemma 2.4.7. *I_τ satisfies the (PS) condition.*

Proof. Let (u_n) be a sequence in H such that

$$I_\tau(u_n) \to c \ \text{ and } \ {I'}_\tau(u_n) \to 0.$$

Since I_τ is coercive, we conclude that (u_n) is bounded in H. Thus, passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup u \ \text{ in } \ H$$

$$u_n \to u \;\text{ in }\; L^q(\Omega)$$

and

$$u_n(x) \to u(x) \;\text{ a.e in }\; \Omega.$$

Thus, from convergence in $L^q(\Omega)$ we get

$$\int_\Omega |u_n|^q\, dx - \int_\Omega |u_n|^{q-2} u_n u\, dx = o_n(1), \tag{2.4.1}$$

and from the weak convergence

$$\int_\Omega \Delta u_n \Delta u dx - \|u\|^2 = o_n(1). \tag{2.4.2}$$

Hence, from (2.4.1) and (2.4.2) we obtain

$$\begin{aligned} 0 \le \|u_n - u\|^2 &\le \|u_n - u\|^2 + C_p \int_\Omega |\nabla(u_n - u)|^p dx \\ &\le \|u_n\|^2 - \int_\Omega \Delta u_n \Delta u dx + \int_\Omega |\nabla u_n|^p dx - \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla u dx \\ &\le {I_\tau}'(u_n) u_n - {I_\tau}'(u_n) u + o_n(1) \\ &= o_n(1), \end{aligned}$$

where C_p is a constant which appears the standard inequality in given by

$$(|x|^{p-2}x - |y|^{p-2}y)(x-y) \ge C_p |x-y|^p,$$

if $p \ge 2$. Thus, we conclude that $u_n \to u$ in H and the proof is complete. □

2.4.2 Proof of Theorem 2.0.3 in the case that the second term in the associated functional I is positive

Proof. Let $X_k = span\{e_1, e_2, ..., e_k\}$ be a subspace of H with $\dim X_k = k$. Thus, since that X_k is continuously embedded in $L^q(\Omega)$ and using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \le \Upsilon \le 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = q$, we get that the norms of H and $L^q(\Omega)$ are equivalent on X_k and there exists a positive constant $C(k)$ which depends on k, such that

$$C(k)\|u\|^q \le \int_\Omega |u|^q\, dx, \quad \text{for all } \; u \in X_k.$$

Thus, we conclude that

$$I_\tau(u) \le \frac{1}{2}\|u\|^2 + \widetilde{C}\frac{1}{p}\|u\|^p - \tau C(k)\frac{1}{q}\|u\|^q,$$

for some $\widetilde{C} > 0$. Let $0 < R < 1$ and $u \in H$ be such that $\|u\| \le R$. Thus

$$\begin{aligned} I_\tau(u) &\le \frac{1}{2}\|u\|^2 + \widetilde{C}\frac{1}{p}\|u\|^p - \tau C(k)\frac{1}{q}\|u\|^q \\ &\le \|u\|^q \left[\left(\frac{1}{2} + \widetilde{C}\frac{1}{p} \right) \|u\|^{2-q} - \tau C(k) \frac{1}{q} \right]. \end{aligned}$$

Since $1 < q < 2$, choosing $0 < R < \min\left\{1, \left(\frac{\tau C(k)2p}{q(p+2\widetilde{C})}\right)^{\frac{1}{2-q}}\right\}$ we have

$$\begin{aligned} I_\tau(u) &\leq R^r\left[\left(\frac{1}{2}+\widetilde{C}\frac{1}{p}\right)R^{2-q} - \tau C(k)\frac{1}{q}\right] \\ &< 0 = I_\tau(0), \end{aligned}$$

for all $u \in K = \{u \in X_k : \|u\| = R\}$ and for all $\tau > 0$. This inequality implies

$$\sup_{u\in K} I_\tau(u) < 0 = I_\tau(0).$$

Since X_k and $\mathbb{R}^k$ are isomorphic and K and S^{k-1} are homeomorphic, we conclude that $\gamma(K) = k$. Moreover, I_τ is even. By Clark's Theorem (Theorem 2.4.5), I_τ has at least k pairs of different critical points. Since k is arbitrary, we found infinitely many critical points of I_τ, that is, problem (P_2) has infinitely many solutions. □

We point out that in order to apply the Clark's Theorem in the previous proof, we use Lemmas 2.4.6 and 2.4.7.

2.4.3 The case that the second term in the associated functional I_τ is negative

Since I_τ is not bounded from below, in this case, to apply genus theory, we will need to make a truncation in the functional I_τ. In fact, the idea is to get a truncated functional J_τ such that critical points u of J_τ with $J_\tau(u) < 0$ are also critical points of I_τ.

Using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = q$ and by continuous embedding, there are positive constants $C, \widetilde{C} > 0$ such that

$$I_\tau(u) \geq \frac{1}{2}\|u\|^2 - \frac{\widetilde{C}}{p}\|u\|^p - C\frac{\tau}{q}\|u\|^q = g(\|u\|), \tag{2.4.3}$$

where $g(t) = \frac{1}{2}t^2 - \frac{\widetilde{C}}{p}t^p - C\frac{\tau}{q}t^q$. So, there exists $\tau^* > 0$ such that, if $\tau \in (0, \tau^*)$, then g attains its positive maximum. We denote by $0 < R_0(\tau) < R_1(\tau)$ the unique two zeros of g. The next lemma is essential to construct the truncated functional.

Lemma 2.4.8. *$R_0(\tau) \to 0$ as $\tau \to 0$.*

Proof. Indeed, from $g(R_0(\tau)) = 0$ and $g'(R_0(\tau)) > 0$, we have

$$\frac{1}{2}R_0(\tau)^2 = C\frac{\tau}{q}R_0(\tau)^q + \frac{\widetilde{C}}{p}R_0(\tau)^p \tag{2.4.4}$$

and

$$R_0(\tau) > \tau C R_0(\tau)^{q-1} + \widetilde{C}R_0(\tau)^{p-1}, \tag{2.4.5}$$

for all $\tau \in (0, \tau^*)$. From (2.4.4), we conclude that $R_0(\tau)$ is bounded. Suppose that $R_0(\tau) \to R_0 > 0$ as $\tau \to 0$. Then,

$$\frac{1}{2}R_0^2 = \frac{\widetilde{C}}{p}R_0^p \text{ and } R_0 > \widetilde{C}R_0(\tau)^{p-1},$$

a contradiction, because $2 < p$. Therefore $R_0 = 0$. ■

Now we consider the following truncation in the functional I_τ. From Lemma 2.4.8, we have $R_0(\tau) < 1$ for small τ. So $R_0(\tau) < R_1(\tau)$ and we can take $\phi \in C_0^\infty([0,+\infty))$, $0 \le \phi(t) \le 1$, for all $t \in [0,+\infty)$, such that

$$\phi(t) = \begin{cases} 1\,, t \in [0, R_0(\tau)], \\ 0\,, t \in [R_1(\tau), +\infty). \end{cases}$$

We define the functional

$$J_\tau(u) = \frac{1}{2}\|u\|^2 - \tau\frac{1}{q}\int_\Omega |u|^q\,dx - \phi(\|u\|)\frac{1}{p}\int_\Omega |\nabla u|^p\,dx.$$

Note that $J \in C^1(H,\mathbb{R})$ and, as in (2.4.3), $J_\tau(u) \ge \overline{g}(\|u\|)$, for all $u \in H$, where

$$\overline{g}(t) = \frac{1}{2}t^2 - \frac{\tau}{q}t^q - \frac{\widetilde{C}}{p}\phi(t)t^p \ge 0,\ \forall\, t \in (0, R_1(\tau)]. \tag{2.4.6}$$

By definition, if $\|u\| \le R_0(\tau)$ then $J_\tau(u) = I_\tau(u)$ and if $\|u\|^2 \ge R_1(\tau)$, then

$$J_\tau(u) = \frac{1}{2}\|u\|^2 - \frac{\tau}{q}\int_\Omega |u|^q dx.$$

Thus, we conclude that the functional J_τ is coercive and, hence, J_τ is bounded below.

Now, we will show that J_τ satisfy the local Palais-Smale condition.

Lemma 2.4.9. *If $J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(v) = I_\tau(v)$, for all v in a small enough neighborhood of u. Moreover, J_τ verifies a local Palais-Smale condition for $c < 0$.*

Proof. Since $\overline{g}(\|u\|) \le J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(u) = I_\tau(u)$. Moreover, since J_τ is a functional continuous, we conclude that $J_\tau(v) = I_\tau(v)$, for all $v \in B_{R_0/2}(0)$. Moreover, if (u_n) is a sequence such that $J_\tau(u_n) \to c < 0$ and $J'(u_n) \to 0$, for n sufficiently large, $I_\tau(u_n) = J_\tau(u_n) \to c < 0$ and $I_\tau{}'(u_n) = J_\tau{}'(u_n) \to 0$. Since that J_τ is coercive, we get that (u_n) is bounded in H. The result follows using the same argument used in Lemma 2.4.7. □

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional J. Thus, for each real number ϵ, we consider the set

$$J_\tau^{-\varepsilon} = \{u \in H : J_\tau(u) \le -\varepsilon\} \in \mathfrak{A}.$$

Lemma 2.4.10. *Given $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$ such that*

$$\gamma(J_\tau^{-\varepsilon}) \ge k.$$

Proof. Given $k \in \mathbb{N}$, we can consider a k-dimensional subspace

$$X_k = span\{e_1, \dots, e_k\}$$

of H, such that

$$\int_\Omega |u|^q dx \le C(k)\|u\|^q,\ \forall\, u \in X_k.$$

Thus,

$$J_\tau(u) \le \frac{1}{2}\|u\|^2 - \tau\frac{1}{q}C(k)\|u\|^q.$$

We can argue exactly as proof of Theorem 2.0.3 to conclude there exists $R \in (0,1)$ small enough, such that defining $K = \{u \in X_k : \|u\| = R\}$, we get

$$J_\tau(u) \leq \sup_{u\in K} J_\tau(u) = -\varepsilon < J_\tau(0) = 0, \ \forall\, u \in K,$$

for some $\varepsilon > 0$. Since $\gamma(K) = k$ and $K \subset J_\tau^{-\varepsilon}$, it follows from (i) in Proposition 2.4.1, that $\gamma(J_\tau^{-\varepsilon}) \geq k$. □

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset H : C \in \mathfrak{A} \ \text{ and } \ \gamma(C) \geq k\},$$

$$K_c = \{u \in H : {J_\tau}'(u) = 0 \ \text{ and } \ J_\tau(u) = c\}$$

and the number

$$c_k = \inf_{C\in\Gamma_k} \sup_{u\in C} J_\tau(u).$$

Lemma 2.4.11. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 2.5.7, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J_\tau^{-\epsilon}) \geq k$. Moreover, $0 \notin J_\tau^{-\varepsilon}$ and $J_\tau^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u\in J_\tau^{-\varepsilon}} J_\tau(u) \leq -\varepsilon.$$

Hence,

$$-\infty < c_k = \inf_{C\in\Gamma_k} \sup_{u\in C} J_\tau(u) \leq \sup_{u\in J_\tau^{-\varepsilon}} J_\tau(u) \leq -\varepsilon < 0.$$

□

The next lemma allows us to prove the existence of critical points of J_τ.

Lemma 2.4.12. *If $c = c_k = c_{k+1} = \ldots = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\tau^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\tau \in (0, \tau^*)$.

Proof. Since $c = c_k = c_{k+1} = \ldots = c_{k+r} < 0$, from Lemma 2.5.6 and Lemma 2.5.4, we get that K_c is a compact set. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\tau^0$ because $c < 0$. By the deformation lemma [14] we have an odd homeomorphism $\eta : X \to X$ such that $\eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\tau^{c+\delta} \subset J_\tau^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u\in A} J_\tau(u) < c + \delta$, that is, $A \subset J_\tau^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}. \tag{2.4.7}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (2.5.10). Hence, this lemma is proved. □

2.4.4 Proof of Theorem 2.0.3 in the case that the second term in the associated functional I is negative

Proof. If $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ with $c_i \neq c_j$, since each c_k is critical value of J_τ, the we obtain infinitely many critical points of J_τ.

On the other hand, if there are two constants $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = ... = c_{k+r}$ and from Lemma 2.5.9, we have

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all for all $\tau^* > 0$. From Proposition 2.4.4, K_c has infinitely many points, that is, problem (P_2) has infinitely many solutions. □

2.5 The case $\beta = 1$ and $1 < q < 2 < p \leq 2^*$

In this section we study some properties related to the C^1 functional I_τ given by

$$I_\tau(u) = \frac{1}{2}\|u\|^2 \pm \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\tau}{q}\int_\Omega |u|^q dx - \frac{1}{2_{**}}\int_\Omega |u|^{2_{**}} dx.$$

For this, we need the following technical result. We denote by S the best constant of the Sobolev embedding $H \hookrightarrow L^{2_{**}}(\Omega)$.

Lemma 2.5.1. *Let (u_n) be a sequence in H such that $I_\tau(u_n) \to c_*$ and $I_\tau{}'(u_n) \to 0$ as $n \to \infty$. Then*

- *(i) $u_n \rightharpoonup u$ in H;*
- *(ii) The weak limit $u \in H$ is a critical point of u, that is, $I_\tau{}'(u) = 0$;*
- *(iii) There exists a positive constant $M > 0$ such that $I_\tau(u) \geq -\tau^{2_{**}/(2_{**}-q)}M$, where M depends p, q, N and Ω;*
- *(iv) If*

$$c_* < \min\{T_1, T_2, T_3\}, \tag{2.5.1}$$

where

$$T_1 = \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4} - \tau^{2_{**}/(2_{**}-q)}M,$$

$$T_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{C^p}, S\right\} - \tau^{2_{**}/(2_{**}-q)}M$$

and

$$T_3 = \left(\frac{1}{2} - \frac{1}{p}\right) - \tau^{2_{**}/(2_{**}-q)}M.$$

Then $u_n \to u$ in H.

Proof. Now we prove (i). Note that from Lemma 1.1.3, we have

$$\begin{aligned} c_* + o_n(1)\|u_n\| &= I_\tau(u_n) - \frac{1}{p}I'_\tau(u_n)u_n \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 + \left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_\Omega |u_n|^{2_{**}}dx \\ & \quad -\tau\left(\frac{1}{q}-\frac{1}{p}\right)\int_\Omega |u_n|^q dx. \end{aligned} \tag{2.5.2}$$

From Sobelev embedding, there exists $C>0$ such that

$$\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 \leq \tau C\left(\frac{1}{q}-\frac{1}{p}\right)\|u_n\|^q + c + o_n(1)\|u_n\|.$$

Since $1<q<2$, we have that (u_n) is bounded in H. Then, up to a subsequence, $u_n \rightharpoonup u$ in H.

Now we prove (ii). As a consequence of weak convergence, for all $\phi \in H$, we have that

$$\int_\Omega \Delta u_n \Delta\phi dx = \int_\Omega \Delta u\Delta\phi dx + o_n(1).$$

From [50, Lemme 4.8], for all $\phi\in H$, we get

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\phi dx = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx + o_n(1)$$

and

$$\int_\Omega |\ u_n|^{2_{**}-2}u_n\phi dx = \int_\Omega |u|^{2_{**}-2}u\phi dx + o_n(1).$$

Since Ω is bounded and f has subcritical growth, for all $\phi\in H$, we obtain,

$$\int_\Omega |u_n|^{q-2}u_n\phi dx = \int_\Omega |u|^{q-2}u\phi dx + o_n(1).$$

Since $I_\tau(u_n)\phi = o_n(1)$, for all $\phi\in H$, using these convergence above, $I_\tau'(u)\phi = 0$, for all $\phi\in H$.

In order to prove (iii), note that

$$\begin{aligned} I_\tau(u) &= I_\tau(u) - \frac{1}{p}I_\tau'(u)u \\ &\geq \left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_\Omega |u|^{2_{**}}dx - \tau\left(\frac{1}{q}-\frac{1}{p}\right)\int_\Omega |u|^q dx. \end{aligned}$$

Using Holder's inequality we get

$$I_\tau(u) \geq \left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_\Omega |u|^{2_{**}}dx - \tau\left(\frac{1}{q}-\frac{1}{p}\right)|\Omega|^{(2_{**}-q)/2_{**}}\left(\int_\Omega |u|^{2_{**}}dx\right)^{q/2_{**}}.$$

Let

$$\Sigma(t) = \left(\frac{1}{p}-\frac{1}{2_{**}}\right)t^{2_{**}} - \tau\left(\frac{1}{q}-\frac{1}{p}\right)|\Omega|^{(2_{**}-q)/2_{**}}t^q.$$

This function attains its absolute minimum, for $t>0$, at the point

$$t_0 = \left[\tau|\Omega|^{\frac{(2_{**}-q)}{2_{**}}}\frac{(p-q)}{(2_{**}-p)}\right]^{2_{**}/(2_{**}-q)}$$

Thus, we conclude that

$$I_\tau(u) \geq -\tau^{2_{**}/(2_{**}-q)}M,$$

wehere

$$M = \left(\frac{1}{q} - \frac{1}{p}\right)\left[|\Omega|^{\frac{(2_{**}-q)}{2_{**}}}\frac{(p-q)}{(2_{**}-p)}\right]^{q/(2_{**}-q)}|\Omega|^{\frac{2^*-q}{2^*}}.$$

Now we prove (iv). Consider $v_n = u_n - u$. Then, from [50, Lemme 4.6] and arguing as [5, Lemma 3.1]), we have

$$I_\tau(v_n) = I_\tau(u_n) - I_\tau(u) + o_n(1) = c_* - I(u).$$

Moreover, for all $\phi \in H$, we get

$$I'_\tau(v_n)\phi = I_\tau{}'(u_n)\phi - I'_\tau(u)\phi = o_n(1),$$

which implies that

$$\|v_n\|^2 \pm \int_\Omega |\nabla v_n|^p dx = \int_\Omega |v_n|^{2_{**}}dx + o_n(1). \tag{2.5.3}$$

In the case that the second term in (2.5.3) is positive, we have

$$\|v_n\|^2 + \int_\Omega |\nabla v_n|^p dx \to \overline{L} \ \text{ and } \ \int_\Omega |v_n|^{2_{**}}dx \to \overline{L}.$$

Note that

$$\begin{aligned} c_* - I_\tau(u_n) + o_n(1) &= \frac{1}{2}\|v_n\|^2 + \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}}dx \\ &\geq \frac{1}{p}\|v_n\|^2 + \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}}dx \\ &\geq \frac{1}{p}\left[\|v_n\|^2 + \int_\Omega |\nabla v_n|^p dx\right] - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}}dx. \end{aligned}$$

In this case, considering $\overline{L} > 0$ we obtain

$$c_* - I_\tau(u) \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{L}.$$

Now using the definition of $\overline{S}$, we have

$$\overline{S} \leq \frac{\|v_n\|^2 + \displaystyle\int_\Omega |\nabla v_n|^p dx}{\left(\displaystyle\int_\Omega |v_n|^{2_{**}}\right)^{2/2_{**}}} = \overline{L}^{4/N} + o_n(1).$$

Then, $\overline{S}^{N/4} \leq \overline{L}$. We conclude

$$c_* \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4} + I_\tau(u).$$

Using the item (iii), we get

$$c_* \geq \overline{S}^{N/4} - \tau^{2_{**}/(2_{**}-q)}M,$$

which is a contradiction with (2.5.12). Then, $\overline{L} = 0$ and $\|v_n\|^2 \to 0$, which implies that $u_n \to u$ in H.

In the case that the second term in (2.5.3) is negative, we have

$$\|v_n\|^2 \to \widetilde{L} \quad \text{and} \quad \int_\Omega |v_n|^{2_{**}} dx + \int_\Omega |\nabla v_n|^p dx \to \widetilde{L}.$$

with $\widetilde{L} = \widetilde{L}_1 + \widetilde{L}_2$, where

$$\int_\Omega |\nabla u_n|^p dx \to \widetilde{L}_1 \quad \text{and} \quad \int_\Omega |u_n|^{2_{**}} dx \to \widetilde{L}_2$$

Since $2 < p < 2_{**}$ and $\int_{\mathbb{R}^N} F(u_n)dx \to 0$, we have

$$\begin{aligned} c_* - I_\tau(u) &= \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^{2_{**}} dx \\ &\geq \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{2_{**}}\int_\Omega |v_n|^p dx \\ &= \frac{1}{2}\|v_n\|^2 - \frac{1}{p}\left[\int_\Omega |v_n|^{2_{**}} dx + \int_\Omega |\nabla v_n|^p dx\right]. \end{aligned}$$

Letting $n \to \infty$ and using (2.3.10) we get

$$c_* - I_\tau(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\widetilde{L}. \tag{2.5.4}$$

If $\widetilde{L} \geq 1$, then,

$$c_* - I_\tau(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right), \tag{2.5.5}$$

which is a contradiction by the hypotheses.
On the other hand, we can use the definition of S to get

$$S\left(\int_\Omega |u_n|^{2_{**}} dx\right)^{2/2_{**}} \leq \int_\Omega |\Delta u_n|^2 dx \tag{2.5.6}$$

Now from Theorem 1.1.2, there exits $C > 0$ such that

$$\left(\int_\Omega |\nabla u_n|^p dx\right)^{1/p} \leq C\left(\int_\Omega |\Delta u_n|^2 dx\right)^{1/2}.$$

Then,

$$\frac{1}{C^p}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^{2/p} \leq \int_{\mathbb{R}^N} |\Delta u_n|^2 dx \tag{2.5.7}$$

Using (2.5.6) and (2.5.7) we obtain

$$\|u_n\|^2 \quad \geq \quad \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\left(\int_\Omega |\nabla u_n|^p dx\right)^{2/p} + \left(\int_\Omega |u_n|^{2_{**}} dx\right)^{2/2_{**}}\right\}.$$

Suppose that $0 < \widetilde{L} < 1$. In this case $0 < \widetilde{L}_1, \widetilde{L}_2 < 1$. Then, since $2 < p < 2_{**}$, we have

$$\|u_n\|^2 \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\int_\Omega |\nabla u_n|^p dx + \int_\Omega |u_n|^{2_{**}} dx\right\}^{2/p}.$$

Taking the limit we conclude that

$$\widetilde{L} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\widetilde{L}^{2/p}$$

Since $\widetilde{L} > 0$, we obtain

$$\widetilde{L}^{(p-2)/p} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}$$

and from (2.5.4) we get

$$c_* - I_\tau(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}^{p/(p-2)},$$

Using the item (iii), we get

$$c_* \geq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}^{p/(p-2)} - \tau^{2_{**}/(2_{**}-q)}M,$$

which is a contradiction with (2.5.1). Then, $\widetilde{L} = 0$ and $\|v_n\|^2 \to 0$, which implies that $u_n \to u$ in H. □

Remark 2. *Note that there exists $\tau^* > 0$ such that, for all $\tau \in (0, \tau^*)$, then*

$$\overline{S}^{N/4} - \tau^{2_{**}/(2_{**}-q)}M > 0$$

and

$$\min\left\{\frac{1}{C^p}, S\right\}^{p/(p-2)} - \tau^{2_{**}/(2_{**}-q)}M - \tau^{2_{**}/(2_{**}-q)}M > 0.$$

2.5.1 The case that the second term in the associated functional I_τ is positive

Since I_τ is not bounded below, arguing as subsection 5.3, we will make a truncation in the functional I_τ as follows:

From Sobolev's embedding, there exists $C > 0$ such that

$$I_\tau(u) \geq \frac{1}{2}\|u\|^2 - C\frac{\tau}{q}\|u\|^q - \frac{1}{2_{**}S^{2_{**}/2}}\|u\|^{2_{**}} = g(\|u\|), \tag{2.5.8}$$

where S is the best constant of the Sobolev embedding $H \hookrightarrow L^{2_{**}}(\Omega)$ and

$$g(t) = \frac{1}{2}t^2 - C\frac{\tau}{q}t^q - \frac{1}{2_{**}S^{2_{**}/2}}t^{2_{**}}.$$

Hence, there exists $\tau^* > 0$ such that, if $\tau \in (0, \tau^*)$, then g attains its positive maximum.

Let us assume $\tau \in (0, \tau^*)$, denoting by $R_0(\tau) < R_1(\tau)$ the only roots of g, we make the following of the truncation I. Take $\phi \in C_0^\infty([0, +\infty))$, $0 \leq \phi(t) \leq 1$, for all $t \in [0, +\infty)$, such that $\phi(t) = 1$ if $t \in [0, R_0(\tau)]$ and $\phi(t) = 0$ if $t \in [R_1(\tau), +\infty)$. Now, we consider the truncated functional

$$J_\tau(u) = \frac{1}{2}\|u\|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\tau}{q}\int_\Omega |u|^q\, dx - \phi(\|u\|^2)\frac{1}{2_{**}}\int_\Omega |u|^{2_{**}}\, dx.$$

Note that $J_\tau \in C^1(H, \mathbb{R})$ and, as in (2.5.8), $J_\tau(u) \geq \overline{g}(\|u\|)$, where

$$\overline{g}(t) = \frac{1}{2}t^2 - C\frac{\tau}{q}t^q - \phi(t)\frac{1}{2_{**}S^{2_{**}/2}}t^{2_{**}}.$$

Note that, if $\|u\|^2 \leq R_0$, then $J_\tau(u) = I_\tau(u)$ and if $\|u\|^2 \geq R_1$, then

$$J_\tau(u) = \frac{1}{2}\|u\|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\tau}{q}\int_\Omega |u|^q\, dx.$$

Thus, we conclude that the functional J_τ is coercive and, hence, J_τ is bounded below. Now, we will show that J_τ satisfy the Palais-Smale condition.

Lemma 2.5.2. *If $J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(v) = I_\tau(v)$, for all v in a small enough neighborhood of u. Moreover, J_τ verifies a Palais-Smale condition for $c < 0$.*

Proof. Since $\overline{g}(\|u\|) \leq J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(u) = I_\tau(u)$. Moreover, since J_τ is a functional continuous, we conclude that $J_\tau(v) = I_\tau(v)$, for all $v \in B_{R_0/2}(0)$. Moreover, if (u_n) is a sequence such that $J_\tau(u_n) \to c < 0$ and $J_\tau{}'(u_n) \to 0$, for n sufficiently large, $I_\tau(u_n) = J_\tau(u_n) \to c < 0$ and $I_\tau{}'(u_n) = J_\tau{}'(u_n) \to 0$. Since that J_τ is coercive, we get that (u_n) is bounded in H. From Lemma 2.5.1 and Remark 2, for τ sufficiently small,

$$c < 0 < \min\{T_1, T_2, T_3\}, \tag{2.5.9}$$

where

$$T_1 = \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4} - \tau^{2_{**}/(2_{**}-q)}M,$$

$$T_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{C^p}, S\right\} - \tau^{2_{**}/(2_{**}-q)}M$$

and

$$T_3 = \left(\frac{1}{2} - \frac{1}{p}\right) - \tau^{2_{**}/(2_{**}-q)}M.$$

and, hence, up to a subsequence, (u_n) is strongly convergent in H. □

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional J_τ.

Lemma 2.5.3. *Given $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$ such that*

$$\gamma(J_\tau^{-\varepsilon}) \geq k,$$

where $J_\tau^{-\varepsilon} = \{u \in X : J_\tau(u) \leq -\varepsilon\}$.

Proof. Fix $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of H. Thus, there exists $C(k) > 0$ such that

$$C(k)\|u\|^q \leq \int_\Omega |u|^q \, dx,$$

for all $u \in X_k$.

Considering $\overline{\rho} > 0$ such that $\|u\| = \overline{\rho} < 1$ and using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon < 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = q$ and by continuous embedding, there are positive constants $C, \widetilde{C} > 0$ such that , we derive that

$$\begin{aligned} J_\tau(u) &\leq \left(\frac{1}{2} + \frac{\widetilde{C}}{p}\right)\overline{\rho}^2 - \frac{\tau}{q}C(k)\overline{\rho}^q \\ &= \overline{\rho}^q \left[\left(\frac{1}{2} + \frac{\widetilde{C}}{p}\right)\overline{\rho}^{2-q} - \frac{\tau}{q}C(k)\right]. \end{aligned}$$

Choosing

$$\overline{\rho} < \min\left\{1, \left[\frac{2p\tau C(k)}{q(p+\widetilde{C})}\right]^{1/(2-q)}\right\},$$

there exists $\varepsilon = \varepsilon(k)$ such that

$$J_\tau(u) < -\varepsilon,$$

for all $u \in X_k$ and with $u \in \mathcal{S}$, where $\mathcal{S} = \{u \in X_k : \|u\| = \overline{\rho}\}$. Hence, we conclude that $\mathcal{S} \subset J_\tau^{-\varepsilon}$. Since $J_\tau^{-\varepsilon}$ is symmetric and closed, from Corollary 2.4.3,

$$\gamma(J_\tau^{-\varepsilon}) \geq \gamma(\mathcal{S}) = k.$$

□

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset H : C \text{ is closed}, C = -C \text{ and } \gamma(C) \geq k\},$$

$$K_c = \{u \in H : {J_\tau}'(u) = 0 \text{ and } J_\tau(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\tau(u).$$

Lemma 2.5.4. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 2.5.7, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J_\tau^{-\varepsilon}) \geq k$. Moreover, $0 \notin J_\tau^{-\varepsilon}$ and $J_\tau^{-\varepsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J_\tau^{-\varepsilon}} J_\tau(u) \leq -\varepsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\tau(u) \leq \sup_{u \in J_\tau^{-\epsilon}} J_\tau(u) \leq -\varepsilon < 0.$$

□

The next Lemma allows us to prove the existence of critical points of J.

Lemma 2.5.5. *If $c = c_k = c_{k+1} = ... = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\tau^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\tau \in (0, \tau^*)$.

Proof. Since $c = c_k = c_{k+1} = ... = c_{k+r} < 0$, from Lemma 2.5.1 and Lemma 2.5.4, we get that K_c is a compactness set. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\tau^0$ because $c < 0$. By the deformation lemma [14] we have an odd homeomorphism $\eta : H \to H$ such that $\eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\tau^{c+\delta} \subset J_\tau^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u\in A} J_\tau(u) < c + \delta$, that is, $A \subset J^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}. \tag{2.5.10}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (2.5.10). Hence, this lemma is proved. □

2.5.2 Proof of Theorem 2.0.4 in the case that the second term in the associated functional I_τ is positive

Proof. If $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ with $c_i \neq c_j$, since each c_k is critical value of J_τ, the we obtain infinitely many critical points of J and, hance problem (P_2) has infinitely many solutions.

On the other hand, if there are two constants $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = ... = c_{k+r}$ and from Lemma 2.5.9, there exists $\tau^* > 0$ such that

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all $\tau \in (0, \tau^*)$. From Proposition 2.4.4, K_c has infinitely many points, that is, problem (P_2) has infinitely many solutions. □

2.5.3 The case that the second term in the associated functional I_τ is negative

Since I_τ is not bounded below, arguing as subsection 5.3, we will make a truncation in the functional I_τ as follows:

Using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $\kappa_1 = p$, $\kappa_3 = 2$, $s = q$ and by continuous embedding, there are positive constants $C, \widetilde{C} > 0$ such that

$$I_\tau(u) \geq \frac{1}{2}\|u\|^2 - \frac{\widetilde{C}}{p}\|u\|^p - C\frac{\tau}{q}\|u\|^q - \frac{1}{2_{**}S^{2_{**}/2}}\|u\|^{2_{**}} = g(\|u\|), \tag{2.5.11}$$

where

$$g(t) = \frac{1}{2}t^2 - \frac{\widetilde{C}}{p}t^p - C\frac{\tau}{q}t^q - \frac{1}{2_{**}S^{2_{**}/2}}t^{2_{**}}.$$

Hence, there exists $\tau^* > 0$ such that, if $\tau \in (0, \tau^*)$, then g attains its positive maximum.

Let us assume $\tau \in (0, \tau^*)$, denoting by $R_0(\tau) < R_1(\tau)$ the only roots of g, we make the following of the truncation I_τ. Take $\phi \in C_0^\infty([0, +\infty))$, $0 \leq \phi(t) \leq 1$, for all $t \in [0, +\infty)$, such that $\phi(t) = 1$ if $t \in [0, R_0(\tau)]$ and $\phi(t) = 0$ if $t \in [R_1(\tau), +\infty)$. Now, we consider the truncated functional

$$J_\tau(u) = \frac{1}{2}\|u\|^2 - \frac{\tau}{q}\int_\Omega |u|^q dx - \phi(\|u\|^2)\left[\frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{2_{**}}\int_\Omega |u|^{2_{**}} dx\right].$$

Note that $J_\tau \in C^1(H,\mathbb{R})$ and, as in (2.5.11), $J_\tau(u) \geq \overline{g}(\|u\|)$, where

$$\overline{g}(t) = \frac{1}{2}t^2 - C\frac{\tau}{q}t^q - \phi(t^2)\left[\frac{\widetilde{C}}{p}t^p + \frac{1}{2_{**}S^{2_{**}/2}}t^{2_{**}}\right].$$

Note that, if $\|u\|^2 \leq R_0$, then $J_\tau(u) = I_\tau(u)$ and if $\|u\|^2 \geq R_1$, then

$$J_\tau(u) = \frac{1}{2}\|u\|^2 - \frac{\tau}{q}\int_\Omega |u|^q \, dx.$$

Thus, we conclude that the functional J_τ is coercive and, hence, J_τ is bounded below. Now, we will show that J_τ satisfy the Palais-Smale condition.

Lemma 2.5.6. *If $J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(v) = I_\tau(v)$, for all v in a small enough neighborhood of u. Moreover, J_τ verifies a Palais-Smale condition for $c < 0$.*

Proof. Since $\overline{g}(\|u\|) \leq J_\tau(u) < 0$, then $\|u\|^2 < R_0(\tau)$ and $J_\tau(u) = I_\tau(u)$. Moreover, since J_τ is a functional continuous, we conclude that $J_\tau(v) = I_\tau(v)$, for all $v \in B_{R_0/2}(0)$. Moreover, if (u_n) is a sequence such that $J_\tau(u_n) \to c < 0$ and $J_\tau{}'(u_n) \to 0$, for n sufficiently large, $I_\tau(u_n) = J_\tau(u_n) \to c < 0$ and $I_\tau{}'(u_n) = J_\tau{}'(u_n) \to 0$. Since that J_τ is coercive, we get that (u_n) is bounded in H. From Lemma 2.5.1 and Remark 2, for τ sufficiently small,

$$c < 0 < \min\{T_1, T_2, T_3\}, \tag{2.5.12}$$

where

$$T_1 = \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4} - \tau^{2_{**}/(2_{**}-q)}M,$$

$$T_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)\min\left\{\frac{1}{C^p}, S\right\} - \tau^{2_{**}/(2_{**}-q)}M$$

and

$$T_3 = \left(\frac{1}{2} - \frac{1}{p}\right) - \tau^{2_{**}/(2_{**}-q)}M.$$

and, hence, up to a subsequence, (u_n) is strongly convergent in H. □

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional J_τ.

Lemma 2.5.7. *Given $k \in \mathbb{N}$, there exists $\varepsilon = \varepsilon(k) > 0$ such that*

$$\gamma(J_\tau^{-\varepsilon}) \geq k,$$

where $J_\tau^{-\varepsilon} = \{u \in X : J_\tau(u) \leq -\varepsilon\}$.

Proof. Fix $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of H. Thus, there exists $C(k) > 0$ such that

$$C(k)\|u\|^q \leq \int_\Omega |u|^q \, dx,$$

for all $u \in X_k$.

Considering $\overline{\rho} > 0$ such that $\|u\| = \overline{\rho} < 1$, we derive that

$$\begin{aligned} J_\tau(u) &\leq \frac{1}{2}\overline{\rho}^2 - \frac{\tau}{q}C(k)\overline{\rho}^q \\ &= \overline{\rho}^q \left[\frac{1}{2}\overline{\rho}^{2-q} - \frac{\tau}{q}C(k)\right]. \end{aligned}$$

Choosing

$$\overline{\rho} < \min\left\{1, \left[\frac{2\tau C(k)}{q}\right]^{1/(2-q)}\right\},$$

there exists $\varepsilon = \varepsilon(k)$ such that

$$J_\tau(u) < -\varepsilon,$$

for all $u \in X_k$ and with $u \in \mathcal{S}$, where $\mathcal{S} = \{u \in X_k : \|u\| = \overline{\rho}\}$. Hence, we conclude that $\mathcal{S} \subset J_\tau^{-\epsilon}$. Since $J_\tau^{-\epsilon}$ is symmetric and closed, from Corollary 2.4.3,

$$\gamma(J_\tau^{-\varepsilon}) \geq \gamma(\mathcal{S}) = k.$$

□

We define now, for each $k \in \mathbb{N}$, the sets

$$\Gamma_k = \{C \subset H : C \text{ is closed}, C = -C \text{ and } \gamma(C) \geq k\},$$

$$K_c = \{u \in H : J_\tau{}'(u) = 0 \text{ and } J_\tau(u) = c\}$$

and the number

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\tau(u).$$

Lemma 2.5.8. *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof. From Lemma 2.5.7, for each $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(J_\tau^{-\epsilon}) \geq k$. Moreover, $0 \notin J_\tau^{-\varepsilon}$ and $J_\tau^{-\varepsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J_\tau^{-\varepsilon}} J_\tau(u) \leq -\varepsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\tau(u) \leq \sup_{u \in J_\tau^{-\varepsilon}} J_\tau(u) \leq -\varepsilon < 0.$$

□

The next Lemma allows us to prove the existence of critical points of J.

Lemma 2.5.9. *If $c = c_k = c_{k+1} = ... = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\tau^* > 0$ such that*

$$\gamma(K_c) \geq r + 1,$$

for $\tau \in (0, \tau^*)$.

Proof. Since $c = c_k = c_{k+1} = ... = c_{k+r} < 0$, from Lemma 2.5.1 and Lemma 2.5.8, we get that K_c is a compactness set. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set U with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\tau^0$ because $c < 0$. By the deformation lemma [14] we have an odd homeomorphism $\eta : H \to H$ such that $\eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\tau^{c+\delta} \subset J_\tau^0$

and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} J_\tau(u) < c + \delta$, that is, $A \subset J^{c+\delta}$ and

$$\eta(A - U) \subset \eta(J_\tau^{c+\delta} - U) \subset J_\tau^{c-\delta}. \tag{2.5.13}$$

But $\gamma(\overline{A - U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A - U})) \geq \gamma(\overline{A - U}) \geq k$. Then $\eta(\overline{A - U}) \in \Gamma_k$ and this contradicts (2.5.13). Hence, this lemma is proved. □

2.5.4 Proof of Theorem 2.0.4 in the case that the second term in the associated functional I_τ is negative

Proof. If $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ with $c_i \neq c_j$, since each c_k is critical value of J_τ, the we obtain infinitely many critical points of J_τ and, hance problem (P_2) has infinitely many solutions.

On the other hand, if there are two constants $c_k = c_{k+r}$, then $c = c_k = c_{k+1} = ... = c_{k+r}$ and from Lemma 2.5.9, there exists $\tau^* > 0$ such that

$$\gamma(K_c) \geq r + 1 \geq 2$$

for all $\tau \in (0, \tau^{**})$. From Proposition 2.4.4, K_c has infinitely many points, that is, problem (P_2) has infinitely many solutions. □

Chapter 3

Nonlinear perturbations of a periodic Kirchhoff-Boussinesq type problems in $\mathbb{R}^N$

The purpose of this chapter is to investigate the existence of nontrivial solutions for the following class of problems given by

$$(P_3) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V(x)u = f(u) + \beta |u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u(x) \neq 0 \ \ x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{cases}$$

in the case $\beta = 0$ we get $2 < p < 2^* = \frac{2N}{N-2}$, for $N \geq 3$ and the case $\beta = 1$ we consider $2_{**} = \frac{2N}{N-4}$ for $N \geq 5$. V is a continuous function satisfying:

(V_1) There is a $\mathbb{Z}^N$- periodic function $V_{per} : \mathbb{R}^N \to \mathbb{R}$, that is,

$$V_{per}(x+y) = V_{per}(x) \ \text{ for all } \ x \in \mathbb{R}^N \ \text{ and for all } \ y \in \mathbb{Z}^N.$$

(V_2) There is a constant $V_0 > 0$ such that

$$V_{per}(x) \geq V_0 \quad \forall x \in \mathbb{R}^N.$$

(V_3) There are constant $W_0 > 0$ and a function $W \in L^{N/2}(\mathbb{R}^N)$ with $W(x) \geq 0$ such that

$$V(x) = V_{per}(x) - W(x) \geq W_0 \quad \forall x \in \mathbb{R}^N,$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$.

On the continuous function f, we assume that:

(f_1) Moreover, we also suppose that

$$\lim_{|t| \to 0} \frac{|f(t)|}{|t|} = 0.$$

$(\tilde{f_2})$ There exists $q \in (p, 2_{**})$ such that

$$\lim_{|t| \to \infty} \frac{|f(t)|}{|t|^{q-1}} = 0.$$

(f_3) The function $t \to \dfrac{f(t)}{|t|^{p-2}t}$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$.

(f_4) There are $r > p$ and $\tau^* \geq 0$ such that

$$f(t) \geq \tau |t|^{r-2}t,$$

for all $t \geq 0$ and for all $\tau > \tau^*$, where τ^* will be fixed in Lemma 3.2.1.

A typical examples

Example 3.0.1. *A function satisfying the conditions* $(f_1), (\tilde{f}_2), (f_3), (f_4)$ *is*

$$f(t) = \sum_{i=1}^{N} C_i |t|^{q_i - 2} t$$

with $p < q_i < 2_{**}$, C_i *are positive constants for each* $2 \leq i \leq N$.

Example 3.0.2. *A function satisfying the conditions* (V_1)-(V_3) *is*

$$V_{per}(x) = 2 + \sin[2\pi(x_1, x_2, \cdots, x_N)]. \tag{3.0.1}$$

It is easy to verify that V_{per} *is a 1-periodic continuous function. Moreover, the function*

$$V(x) = \left(1 - \frac{1}{2(1 + |x|)}\right) V_{per}(x), \quad \textit{for } x \in \mathbb{R}^N \tag{3.0.2}$$

satisfies our hypotheses $(V_1) - (V_3)$.

Our main results are the following :

Theorem 3.0.3. *(Subcritical case)*
Assume that conditions (V_1)-(V_3) *and* $(f_1), (\tilde{f}_2), (f_3), (f_4)$ *hold with* $\beta = 0$. *Then, problem* (P_3) *has a ground state solution.*

Theorem 3.0.4. *(Critical case)*
Assume that conditions (V_1)-(V_3) *and* $(f_1), (\tilde{f}_2), (f_3), (f_4)$ *hold with* $\beta = 1$. *Then, problem* (P_3) *has a ground state solution, for all* $\tau \geq \tau^*$.

This chapter is organized as follows. In section 3.1 we show the existence of a ground state solution to the problem when $V = V_{per}$, that is, V is a periodic potential with sub-critical nonlinear growth. In section 3.2 we consider the periodic potential with critical nonlinear growth. In section 3.3 we study the ground state solution for the non-periodic V potential.

3.1 The periodic problem

In this section, we study the existence of solution for the following periodic problem

$$(\widehat{P}) \qquad \begin{cases} \Delta^2 u \pm \Delta_p u + V_{per}(x)u = f(u) + \beta |u|^{2_{**}-2}u \text{ in } \mathbb{R}^N, \\ u(x) \neq 0 \text{ in } \mathbb{R}^N \\ u \in H^2(\mathbb{R}^N). \end{cases}$$

where $\beta \in \{0, 1\}$. Moreover, we assume that the conditions $(V_1)-(V_2)$ and $(f_1), (\tilde{f}_2), (f_3), (f_4)$ hold.

In this subsection we consider the space $H^2(\mathbb{R}^N)$ with with the inner product

$$\langle u, v\rangle_{per} := \int_{\mathbb{R}^N} (\Delta u \Delta v + V_{per}(x) uv)\, dx, \quad \text{for all } u, v \in H^2(\mathbb{R}^N)$$

and associated norm given by

$$\|u\|_{per} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} V_{per}(x)|u|^2 dx \right)^{1/2}$$

Note that $H^2(\mathbb{R}^N)$ is a Hilbert space.

From now on, we say that $u \in H^2(\mathbb{R}^N)$ is a weak solution of the problem $(\widehat{P})$ if

$$\begin{aligned} &\int_{\mathbb{R}^N} \Delta u \Delta \phi dx \pm \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi dx + \int_{\mathbb{R}^N} V_{per}(x) u \phi dx \\ &- \int_{\mathbb{R}^N} f(u)\phi \, dx - \beta \int_{\mathbb{R}^N} |u|^{2_{**}-2} u\phi \, dx = 0 \end{aligned}$$

for all $\phi \in H^2(\mathbb{R}^N)$.

Note that from $(f_1) - (\tilde{f}_2)$, given $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that

$$|f(t)| \le \varepsilon |t| + C(\varepsilon)|t|^{q-1}. \tag{3.1.1}$$

and

$$|F(t)| \le \frac{\varepsilon}{2}|t|^2 + \frac{C(\varepsilon)}{q}|t|^q. \tag{3.1.2}$$

Consider the functional $I_{per} : H^2(\mathbb{R}^N) \to \mathbb{R}$ associated given by

$$\begin{aligned} I_{per}(u) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{2}\int_{\mathbb{R}^N} V_{per}(x)|u|^2 dx \\ & - \int_{\mathbb{R}^N} F(u) dx - \frac{\beta}{2_{**}} \int_{\mathbb{R}^N} |u|^{2_{**}} dx. \end{aligned}$$

Since $2 < p \le 2^*$, using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \le \Upsilon \le 1$, $\kappa_1 = p$, $\kappa_3 = 2$, we have that $H^2(\mathbb{R}^N) \hookrightarrow W^{1,p}(\mathbb{R}^N)$. Moreover, from (3.1.2), I_{per} is well-defined and of C^1 class and

$$\begin{aligned} I'_{per}(u)\phi \quad = \quad & \int_{\mathbb{R}^N} \Delta u \Delta \phi dx \pm \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx + \int_{\mathbb{R}^N} V_{per}(x)|u|^2 dx \\ & - \int_{\mathbb{R}^N} f(u)\phi \, dx - \beta \int_{\mathbb{R}^N} |u|^{2_{**}-2} u \phi dx, \end{aligned}$$

for all $\phi \in H^2(\mathbb{R}^N)$. Then, the critical points of I_{per} are weak solution of $(\widehat{P})$.

Remark 3. *A direct computation shows that $\|\cdot\|$ is a norm in $H^2(\mathbb{R}^N)$, with it is equivalent to the usual norm of $H^2(\mathbb{R}^N)$, because V_{per} is bounded from below and above in whole $\mathbb{R}^N$.*

The next two lemmas show that functional I_{per} verifies the mountain pass geometry.

Lemma 3.1.1. *Assume that (f_1) and (f_2) hold. Then, there exist positive numbers ρ and α such that,*

$$I_{per}(u) \ge \alpha > 0, \;\; \forall u \in H^2(\mathbb{R}^N) : \|u\|_{per} = r.$$

Proof. Note that, taking $\varepsilon > 0$ sufficiently small in (3.1.1) and using Sobolev embedding, there exists $C_1, C_2 > 0$ such that

$$\begin{aligned} I_{per}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + V_{per}(x)|u|^2)dx \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx \\ &\quad -\int_{\mathbb{R}^N}F(u)dx - \frac{\beta}{2_{**}}\int_{\mathbb{R}^N}|u|^{2_{**}}dx \\ &\geq \frac{1}{2}\|u\|_{per}^2 \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|u|^2 dx \\ &\quad -\frac{C(\varepsilon)}{q}\int_{\mathbb{R}^N}|u|^q dx - \frac{\beta}{2_{**}}\int_{\mathbb{R}^N}|u|^{2_{**}}dx \\ &= \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\|u\|_{per}^2 \pm \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx - \frac{C_1C(\varepsilon)}{q}\|u\|_{per}^q - \frac{C_2\beta}{2_{**}}\|u\|_{per}^{2_{**}} \end{aligned}$$

In the case that the second term in the associated functional I_{per} is positive, since $2 < p < q < 2_{**}$, we have

$$I_{per}(u) \geq \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\|u\|_{per}^2 - \frac{C_1C(\varepsilon)}{q}\|u\|_{per}^q - \frac{C_2\beta}{2_{**}}\|u\|_{per}^{2_{**}}$$

Hence, by choosing $\varepsilon \in (0,1)$, there exists a small $r > 0$ such that

$$I_{per}(u) \geq \rho > 0, \quad \text{for all } \|u\|_{per} = r,$$

where

$$\rho = \left[\left(\frac{1}{2}-\frac{\varepsilon}{2}\right) - \frac{C_1C(\varepsilon)}{q}r^{q-2} - \frac{C_2\beta}{2_{**}}r^{2_{**}-2}\right]r^2$$

This establishes (i).

In the case that the second term in the associated functional I_{per} is negative, using Theorem 1.1.2 for $j = 1$, $m = 2$, $\frac{1}{2} \leq \Upsilon \leq 1$, $\kappa_1 = p$, $\kappa_3 = 2$, and by continuous embedding, we get

$$I_{per}(u) \geq \left(\frac{1}{2}-\frac{\varepsilon}{2}\right)\|u\|_{per}^2 - \frac{\widetilde{C}}{p}\|u\|^p - \frac{C_1C(\varepsilon)}{q}\|u\|_{per}^q - \frac{C_2\beta}{2_{**}}\|u\|_{per}^{2_{**}}$$

for some $\widetilde{C} > 0$. Hence, by choosing $\varepsilon \in (0,1)$ and using $2 < p < q < 2_{**}$, there exists a small $r > 0$ such that

$$I_{per}(u) \geq \rho > 0, \quad \text{for all } \|u\|_{per} = r,$$

where

$$\rho = \left[\left(\frac{1}{2}-\frac{\varepsilon}{2}\right) - \frac{\widetilde{C}}{p}r^{p-2} - \frac{C_1C(\varepsilon)}{q}r^{q-2} - \frac{C_2\gamma}{2_{**}}r^{2_{**}-2}\right]r^2.$$

This establishes (i) □

Lemma 3.1.2. *Assume that (f_4) hold. Then, there exists $e \in H^2(\mathbb{R}^N)$ such that $I_{per}(e) < 0$ and $\|e\| > \rho$.*

Proof. Fixed $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \neq 0$ and $t \in \mathbb{R} \setminus 0$.

$$I_{per}(t\phi) \quad = \quad \frac{t^2}{2}\|\phi\|_{per}^2 \pm \frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla\phi|^p dx - \int_{\mathbb{R}^N}F(t\phi)dx - \frac{\beta t^{2_{**}}}{2_{**}}\int_{\mathbb{R}^N}|\phi|^{2_{**}}dx$$

In the case that the second term in the associated functional I_{per} is positive, from (f_4) and for all $\tau > 0$, we have

$$\begin{aligned}\frac{I_{per}(t\phi)}{t^p} \quad &\leq \quad \frac{1}{2t^{p-2}}\|\phi\|_{per}^2 + \frac{1}{p}\int_{\text{supp}\psi}|\nabla\phi|^p dx - \tau\frac{t^{r-p}}{r}\int_{\text{supp}\psi}|\phi|^r dx \\ &\quad -C_4|\text{supp}\psi| - \frac{\beta t^{2_{**}-p}}{r}\int_{\text{supp}\psi}|\phi|^{2_{**}}dx\end{aligned}$$

Since $2 < p < r < 2_{**}$, there exists $\bar{t} > 0$ large such that $e = \bar{t}\phi$ satisfies $I_{per}(e) < 0$ and $\|e\|_{per} > \rho$.

In the case that the second term in the associated functional I_{per} is negative, from (f_4) and for all $\tau > 0$, we have

$$\frac{I_{per}(t\phi)}{t^p} \quad \leq \quad \frac{1}{2t^{p-2}}\|\phi\|_{per}^2 - \tau\frac{t^{r-p}}{r}\int_{\text{supp}\psi}|\phi|^r dx - C_4|\text{supp}\psi| - \frac{\beta t^{2_{**}-p}}{r}\int_{\text{supp}\psi}|\phi|^{2_{**}}dx$$

Since $2 < p < r < 2_{**}$, there exists $\bar{t} > 0$ large such that $\bar{e} = \bar{t}\phi$ satisfies $I_{per}(\bar{e}) < 0$ and $\|\bar{e}\|_{per} > \rho$. □

Using a version of the Mountain Pass Theorem without (PS) condition found in [92], there exists a sequence $(u_n) \subset H^2(\mathbb{R}^N)$ satisfying

$$I_{per}(u_n) \to c_{per} \quad \text{and} \quad I'_{per}(u_n) \to 0,$$

where

$$c_{per} = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I_{per}(\eta(t)) > 0$$

and

$$\Gamma_{per} := \{\eta \in C([0,1], H^2(\mathbb{R}^N)) : \eta(0) = 0,\ I_{per}(\eta(1)) < 0\}.$$

The above sequence is called a $(PS)_{c_{per}}$ sequence for I_{per}.

The next lemma is a key point in our arguments, because it establishes an important characterization involving the mountain pass level for elliptic problem. Hereafter, $\mathcal{M}_{per}$ denotes the Nehari Manifolds, associated with I_{per}, that is,

$$\mathcal{M}_{per} = \{u \in H^2(\mathbb{R}^N) \setminus \{0\} \ / \ I'_{per}(u)u = 0\}. \tag{3.1.3}$$

Lemma 3.1.3. *Assume that (V_1) and (f_1)- (f_4) hold. Then, for each $u \in H^2(\mathbb{R}^N)$ with $u \neq 0$, there exists a unique $t_0 = t_0(u) > 0$ such that $t_0u \in \mathcal{M}_{per}$ and $I_{per}(t_0u) = \max_{t\geq 0} I_{per}(tu)$. Moreover $c_{per} = \overline{c}_{per} = \widehat{c}_{per} > 0$, where*

$$\overline{c}_{per} = \inf_{u\in H^2(\mathbb{R}^N)\setminus\{0\}}\max_{t\geq 0} I_{per}(tu)$$

and

$$\widehat{c}_{per} = \inf_{\mathcal{M}_{per}} I_{per}.$$

Proof. Given $u \in H^2(\mathbb{R}^N)$ with $u \neq 0$, let $h_u : (0,\infty) \to \mathbb{R}$ as $h_u(t) = I_{per}(tu)$ for $t > 0$. Then $tu \in \mathcal{M}_{per}$ if and only if $h'{}_u(t) = 0$. Note that, taking $\varepsilon > 0$ sufficiently small in (3.1.1) and using Sobolev embedding, there exists $C_1, C_2 > 0$ such that

$$\begin{aligned}
\gamma_u(t) &= \frac{t^2}{2}\|u\|_{per}^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(tu)dx - \frac{t^{2_{**}}}{2_{**}}\int_\Omega |u|^{2_{**}}dx \\
&\geq t^2\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\|u\|_{per}^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \frac{C(\varepsilon)}{q}t^q\int_\Omega |u|^q dx - \frac{t^{2_{**}}}{2_{**}}\int_\Omega |u|^{2_{**}}dx \\
&\geq t^2\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\|u\|_{per}^2 \pm \frac{t^p}{p}\int_\Omega |\nabla u|^p dx - \frac{C_1C(\varepsilon)}{q}t^q\|u\|_{per}^q - \frac{\beta C_2}{2_{**}}\|u\|_{per}^{2_{**}}
\end{aligned}$$

Thus, since $2 < p < q < 2_{**}$, we have $h_u(t) > 0$ for all $0 < t$ sufficiently small.

Now, from (f_4) and using $2 < p < r$, for all $\tau > 0$, we have

$$\frac{h_u(t)}{t^p} \leq \frac{1}{2t^{p-2}}\|u\|_{per}^2 \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx - \tau\frac{t^{r-p}}{r}\int_{\mathbb{R}^N} |u|^r dx.$$

Hence, $\lim_{t\to+\infty} \gamma_u(t) = -\infty$. Then, there exists at least one $t_0(u) > 0$ such that $h'_u(t_0(u)) = 0$, i.e. $t_0(u)u \in \mathcal{M}$. Moreover, in the case $2 < p$, we get

$$h'_u(t) = t^{p-1}\left[\frac{1}{t^{p-2}}\|u\|^2 \pm \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} \frac{f(tu)}{t^{p-1}}udx - t^{2_{**}-p}\int_{\mathbb{R}^N} |u|^{2_{**}}dx\right].$$

From (f_4) we conclude that $\frac{h'_u(t)}{t^{p-1}}$ is decreasing. Then, it vanishes exactly once, and consequently there is no other $t > 0$ such that $tu \in \mathcal{M}$. Note, in particular, that $t_0(u)$ is a global maximum point of h_u and $h_u(t(u)) > 0$, i.e. $I_{per}(t_0(u)u) > 0$. Since $t_0(u) = 1$ if $u \in \mathcal{M}_{per}$, we deduce that $I_{per}(u) > 0$ for every $u \in \mathcal{M}_{per}$. Now, the proof follows by using similar arguments found in Willem [92]. □

The following result presents an interesting property involving the $(PS)_{c_{per}}$ sequences of I_{per}, for the subcritical case, that is, $\beta = 0$.

Lemma 3.1.4. *Let $(u_n) \subset H^2(\mathbb{R}^N)$ be a $(\mathrm{PS})_{c_{per}}$ sequence for I_{per} with $u_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$. If $\beta = 0$, there is a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R,\ \eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\, dx \geq \eta > 0.$$

Proof. Suppose that the lemma does not hold. Then, a result due to Lions [61, Lemma I.1] gives $u_n \to 0$ in $L^q(\mathbb{R}^N)$. This limit combined with (3.1.1) yields $\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0$. Thus, since $I'_{per}(u_n)u_n = o_n(1)$, we obtain

$$\|u_n\|_{per}^2 \pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx = o_n(1).$$

From Theorem 1.1.2 for $j = 1$, $\kappa_1 = p$, $m = 2$, $\Upsilon = \frac{1}{2}$, $\kappa_3 = 2$ and $k_2 = \frac{2p}{4-p}$, we have that there exists $C_1 > 0$ such that

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^p\right)^{1/p} \leq C_1\left(\int_{\mathbb{R}^N} |\Delta u_n|^2\right)^{1/4} |u_n|_{\kappa_2}^{1/2}.$$

Since
$$u_n \rightharpoonup 0 \ \text{ in } \ H^2(\mathbb{R}^N),$$
we have that
$$\pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx \to 0,$$
which implies that $\|u_n\|^2_{per} \to 0$, leading to $c_{per} = 0$, which is an absurd in view of Lemma 3.1.3. □

The next result establishes the existence of solution for problem $(\widehat{P})$ for the subcritical case, that is, $\beta = 0$.

Theorem 3.1.5. *Assume that conditions* (V_1)-(V_2) *and* $(f_1), (\tilde{f}_2), (f_3), (f_4)$ *hold. Then, problem* $(\widehat{P})$ *with* $\beta = 0$ *has a ground state solution, for all* $\tau > 0$.

Proof. Using a version of the Mountain Pass Theorem without (PS) condition found in [92], there exists a sequence $(u_n) \subset H^2(\mathbb{R}^N)$ satisfying
$$I_{per}(u_n) \to c_{per} \ \text{ and } \ I'_{per}(u_n) \to 0.$$
Then, from Lemma 1.1.3, we have
$$\begin{aligned} c_{per} + o_n(1)\|u_n\|_{per} &= I_{per}(u_n) - \frac{1}{p} I'{}_{per}(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{per} + \int_\Omega \left(\frac{1}{p} f(u_n)u - F(u_n)\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{per}, \end{aligned} \tag{3.1.4}$$
Hence (u_n) is bounded in $H^2(\mathbb{R}^N)$. From boundedness of (u_n), there are a subsequence of (u_n), still denoted by itself, $u \in H^2(\mathbb{R}^N)$ verifying
$$u_n \rightharpoonup u \text{ in } H^2(\mathbb{R}^N),$$
$$u_n(x) \to u(x), \ \text{a.e } \mathbb{R}^N,$$
$$u_n \to u \text{ in } L^t_{loc}(\mathbb{R}^N) \text{ for all } t \in (2, 2_{**}).$$
Without loss of generality, we can assume that $u \neq 0$, because by Lemma 3.1.4, there exist $\eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that
$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \eta > 0. \tag{3.1.5}$$
A direct computation shows that we can assume $(y_n) \subset \mathbb{Z}^N$. Considering $v_n(x) = u_n(x+y_n)$, once that V_{per} is $\mathbb{Z}^N$-periodic function, we have that (v_n) is also bounded in $H^2(\mathbb{R}^N)$ and its weak limit denoted by v is nontrivial, because the last inequality together Sobolev embedding implies that
$$\int_{B_R(0)} |v|^2 \, dx \geq \eta > 0.$$
Furthermore, a routine calculus leads to
$$I_{per}(v_n) \to c_{per} \text{ and } I'{}_{per}(v_n) = o_n(1).$$

As a consequence of weak convergence, for all $\phi \in H^2(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} \Delta v_n \Delta \phi dx + \int_{\mathbb{R}^N} V_{per}(x) v_n \phi dx = \int_{\mathbb{R}^N} \Delta v \Delta \phi dx + \int_{\mathbb{R}^N} V_{per}(x) v \phi dx + o_n(1).$$

From [50, Lemme 4.8], for all $\phi \in H^2(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2} \nabla v_n \nabla \phi dx = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \phi dx + o_n(1)$$

and by a density argument, for all $\phi \in H^2(\mathbb{R}^N)$, we obtain,

$$\int_{\mathbb{R}^N} f(v_n) \phi dx = \int_{\mathbb{R}^N} f(v) \phi dx + o_n(1).$$

Since $I'_{per}(v_n)\phi = o_n(1)$, for all $\phi \in H^2(\mathbb{R}^N)$, using these convergence above, $I'_{per}(v)\phi = 0$, for all $\phi \in H^2(\mathbb{R}^N)$, and the theorem proved. □

3.2 Critical and periodic case

In this section motivated by [25], we study the case where the nonlinearity has a critical growth, that is, $\beta = 1$. To this end, we begin studying the behavior of mountain pass level c_τ related to the parameter τ. We denote that S in the best constant to the Sobolev embedding $D^{2,2}(\mathbb{R}^N) \hookrightarrow L^{2_{**}}(\mathbb{R}^N)$, namely

$$S = \inf_{0 \neq u \in D^{2,2}(\mathbb{R}^N)} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2_{**}} dx \right)^{\frac{2}{2_{**}}}} \right\}$$

The infimum $S > 0$ is archived by the functions

$$U_{\varepsilon, x_0}(x) := a_N \varepsilon^{\frac{4-N}{2}} \left(\frac{1}{\varepsilon^2 + |x - x_0|^2} \right)^{\frac{N-4}{2}},$$

where $\varepsilon > 0$ and $a_N = ((N+2)N(N-2)(N-4))^{\frac{N-4}{8}}$ for $N \geq 5$. Now, we defined the following constant

$$\overline{S} = \inf_{v \in H^2(\mathbb{R}^N), v \neq 0} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^p dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2_{**}} dx \right)^{2/2_{**}}} \right\} \tag{3.2.1}$$

Notice that $\overline{S} > 0$. In fact, since that

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2_{**}} dx \right)^{2/2_{**}}} < \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^p dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2_{**}} dx \right)^{2/2_{**}}} \tag{3.2.2}$$

This implies that

$$
\begin{aligned}
0 < S &= \inf_{0 \neq u \in D^{2,2}(\mathbb{R}^N)} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2_{**}} dx\right)^{\frac{2}{2_{**}}}} \right\} \\
&\leq \inf_{v \in H^2(\mathbb{R}^N), v \neq 0} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^p dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2_{**}} dx\right)^{2/2_{**}}} \right\} \\
&= \overline{S}
\end{aligned}
$$

Consequently, we get $0 < S \leq \overline{S}$.

Lemma 3.2.1. *If the conditions $(f_1), (\tilde{f}_2), (f_3), (f_4)$ hold with $\beta = 1$, then there exists $\tau_* > 0$ such that*

$$
c_{\tau,per} < \min\left\{ \left(\frac{1}{p} - \frac{1}{2_{**}}\right) \overline{S}^{N/4}, \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) \min\left\{\frac{1}{C^p}, S\right\}, \left(\frac{1}{2} - \frac{1}{p}\right) \right\}
$$

for all $\tau > \tau^*$.

Proof. If we define $\eta_*(t) = te$ for $t \in [0,1]$, where $e = \bar{t}\phi$ is the function given by Lemma 3.1.2. It follows that $\eta_* \in \Gamma$ and thus

$$
\begin{aligned}
0 < c_{\tau,per} &\leq \max_{t \geq 0} I_{per}(\eta_*(t)) \\
&\leq \max_{t \geq 0} \left[\frac{t^2}{2} \|e\|_{per}^2 \pm \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla e|^p dx - \frac{\tau t^r}{r} \int_{\mathbb{R}^N} |e|^r dx \right].
\end{aligned}
$$

In the case that the second term in the associated functional I_{per} is negative, we have

$$
0 < c_{\tau,per} \leq \max_{t \geq 0} \left[\frac{t^2}{2} \|e\|_{per}^2 - \frac{\tau t^r}{r} \int_{\mathbb{R}^N} |e|^r dx \right] = \frac{t_\tau^2}{2} \|e\|_{per}^2 - \frac{\tau t_\tau^r}{r} \int_{\mathbb{R}^N} |e|^r dx,
$$

where

$$
t_\tau = \left[\frac{\|e\|_{per}^2}{\tau \displaystyle\int_{\mathbb{R}^N} |e|^r dx} \right]^{1/(r-2)}
$$

Then,

$$
\begin{aligned}
0 < c_{\tau,per} &\leq \frac{1}{2} \left[\frac{\|e\|_{per}^2}{\tau \displaystyle\int_{\mathbb{R}^N} |e|^r dx} \right]^{2/(r-2)} \|e\|_{per}^2 - \tau \frac{1}{r} \left[\frac{\|e\|_{per}^2}{\tau \displaystyle\int_{\mathbb{R}^N} |e|^r dx} \right]^{r/(r-2)} \int_{\mathbb{R}^N} |e|^r dx \\
&= \left(\frac{1}{2} - \frac{1}{r}\right) \frac{\left[\|e\|_{per}^2\right]^{r/(r-2)}}{\left[\displaystyle\int_{\mathbb{R}^N} |e|^r dx\right]^{2/(r-2)}} \frac{1}{\tau^{2/(r-2)}}.
\end{aligned}
$$

For

$$\tau^* = \left[\left(\frac{1}{2} - \frac{1}{r}\right) \frac{\left[\|e\|_{per}^2\right]^{r/(r-2)}}{\left[\int_{\mathbb{R}^N} |e|^r dx\right]^{2/(r-2)}} \frac{4p}{(p-2)\min\left\{\frac{1}{C^p}, S\right\}} \right]^{(r-2)/2},$$

for all $\tau > \tau^*$, we have that,

$$c_{\tau,per} < \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) \min\left\{\frac{1}{C^p}, S\right\}.$$

In the case that the second term in the associated functional I_{per} is positive, we have

$$\begin{aligned} 0 \;<\; & c_{\tau,per} \leq \max_{t\in[0,1]} I_{per}(\eta_*(t)) \\ \leq\; & \max_{t\in[0,1]} \left[\left(\frac{p+2}{p}\right) \frac{t^2}{2} \left(\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx \right) - \frac{\tau t^r}{r} \int_{\mathbb{R}^N} |e|^r dx \right] \\ =\; & \left(\frac{p+2}{p}\right) \frac{t_\tau^2}{2} \left(\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx \right) - \frac{\tau t_\tau^r}{r} \int_{\mathbb{R}^N} |e|^r dx, \end{aligned}$$

where

$$t_\tau = \left[\frac{\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx}{\tau \int_{\mathbb{R}^N} |e|^r dx} \right]^{1/(r-2)}$$

Then,

$$\begin{aligned} 0 < c_{\tau,per} \;\leq\; & \left(\frac{p+2}{p}\right) \left[\frac{\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx}{\tau \int_{\mathbb{R}^N} |e|^r dx} \right]^{2/(r-2)} \left(\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx \right) \\ & -\tau \frac{1}{r} \left[\frac{\left(\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx \right)}{\tau \int_{\mathbb{R}^N} |e|^r dx} \right]^{r/(r-2)} \int_{\mathbb{R}^N} |e|^r dx \\ =\; & \left(\frac{p+2}{p} - \frac{1}{r}\right) \frac{\left[\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx\right]^{r/(r-2)}}{\left[\int_{\mathbb{R}^N} |e|^r dx\right]^{2/(r-2)}} \frac{1}{\tau^{2/(r-2)}}. \end{aligned}$$

For

$$\tau^* = \left[\left(\frac{p+2}{p} - \frac{1}{r}\right) \frac{\left[\|e\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla e|^p dx\right]^{r/(r-2)}}{\left[\int_{\mathbb{R}^N} |e|^r dx\right]^{2/(r-2)}} \frac{2_{**}p}{(2_{**} - p)\overline{S}^{N/4}} \right]^{\frac{(r-2)}{r}}$$

for all $\tau > \tau^*$, we have that,

$$c_{\tau,per} < \left(\frac{1}{p} - \frac{1}{2_{**}}\right) \overline{S}^{N/4}.$$

□

Lemma 3.2.2. *Let (u_n) be a sequence in $H^2(\mathbb{R}^N)$ such that $I_{per}(u_n) \to c_{\tau,per}$ and ${I'}_{per}(u_n) \to 0$ as $n \to \infty$. Then*

(i) $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$;

(ii) The weak limit $u \in H^2(\mathbb{R}^N)$ is a critical point of u, that is, ${I'}_{per}(u) = 0$.

Proof. Now we prove (i). Note that from Lemma 1.1.3, we have

$$\begin{aligned} c_{per} + o_n(1)\|u_n\|_{per} &= I_{per}(u_n) - \frac{1}{p}{I'}_{per}(u_n)u_n \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{per} + \int_\Omega \left(\frac{1}{p}f(u_n)u - F(u_n)\right)dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\int_{\mathbb{R}^N} |u_n|^{2_{**}}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2_{per} \end{aligned} \tag{3.2.3}$$

which implies that (u_n) is bounded in $H^2(\mathbb{R}^N)$.

Now we prove (ii). Hence (u_n) is bounded in $H^2(\mathbb{R}^N)$. From boundedness of (u_n), there are a subsequence of (u_n), still denoted by itself, $u \in H^2(\mathbb{R}^N)$ and using [8, 94] verifying

$$\begin{cases} u_n \rightharpoonup u \text{ in } H^2(\mathbb{R}^N); \\ \nabla u_n(x) \to \nabla u(x) \ a.e \text{ in } \mathbb{R}^N; \\ \Delta u_n(x) \to \Delta u(x) \ a.e \text{ in } \mathbb{R}^N; \\ u_n \to u, \quad \text{strongly in } L^t_{loc}(\mathbb{R}^N), \text{ for } 2 \leq t < 2_{**} \\ u_n(x) \to u(x) \ a.e \text{ in } \mathbb{R}^N. \end{cases} \tag{3.2.4}$$

Afirmation 1. The following convergent are valid

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \varphi dx = \int_{\mathbb{R}^N} |\nabla|^{p-2}\nabla u \varphi dx + o_n(1)$$

First let's show the convergence in $C_0^\infty(\mathbb{R}^N)$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

Indeed, consider $h_n(x) = |\nabla u_n(x)|^{p-2}\nabla u_n(x)$ and $h(x) = |\nabla u(x)|^{p-2}\nabla u(x)$,

$$|\nabla u_n(x)|^{p-2}\nabla u_n(x)\varphi(x) \to |\nabla u(x)|^{p-2}\nabla u(x)\varphi(x), \quad a.e. \quad x \in supp\varphi$$

up to a subsequence

$$u_n(x) \to u(x), \quad a.e. \quad x \in supp\varphi$$

hence

$$h(u_n(x))\varphi(x) \to h(u(x))\varphi(x), \quad a.e. \quad x \in supp\varphi.$$

It is sufficiently to prove that there is $g : \mathbb{R} \to \mathbb{R}$ such that $|h(u_n)\varphi| \leq g(u_n)$ with $(g(u_n))$ convergent in $L^1(supp\varphi)$, because, in this case, using [24, Theorem 4.9 and Theorem 4.2] we get

$$\int_{supp\varphi} h(u_n)\varphi \ dx \to \int_{supp\varphi} h(u)\varphi \ dx.$$

Note that by the inequality (3.1.1) we have

$$|h(u_n(x))\varphi(x)| \leq |\nabla u_n(x)|^{p-1}|\varphi(x)| := g(u_n(x)).$$

Considering $s, s' > 1$ such that $\dfrac{1}{s} + \dfrac{1}{s'} = 1$, we get

$$\varphi \to \varphi \quad \text{in} \quad L^{s'}(supp\varphi), \tag{3.2.5}$$

we use [50, Lemma 4.8] and conclude that

$$|\nabla u_n(x)|^{p-1} \rightharpoonup |\nabla u(x)|^{p-1} \quad \text{in} \quad L^{s}(supp\varphi). \tag{3.2.6}$$

Now using (3.2.5), (3.2.6) and [50, Lemma 4.8] again, we conclude

$$\int_{supp\varphi} h(u_n)\varphi dx \to \int_{supp\varphi} h(\overline{u})\varphi dx.$$

Therefore

$$\int_{\mathbb{R}^N} |\nabla u_n(x)|^{p-2}\nabla u_n(x)\varphi(x) \to \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\varphi(x), \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

By a density argument, for all $\phi \in H^2(\mathbb{R}^N)$, we obtain,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\varphi = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\varphi + o_n(1).$$

Afirmation 2. The following convergent are valid

$$\int_{\mathbb{R}^N} f(u_n)\phi dx = \int_{\mathbb{R}^N} f(u)\phi dx + o_n(1).$$

Let's also show the convergence in $C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} f(u_n)\phi dx \to \int_{\mathbb{R}^N} f(u)\phi dx \quad \forall \phi C_0^\infty(\mathbb{R}^N)$$

Indeed. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^2(\mathbb{R}^N)$, fixed $\varphi \in C_0^\infty(\mathbb{R}^N)$, then

$$f(u_n(x))\varphi(x) \to f(u(x))\varphi(x), \quad a.e. \quad x \in supp\varphi$$

and

$$|f(u_n(x))\varphi(x)| \leq (|u_n(x)| + C(\varepsilon)|u_n(x)|^{q-1})\varphi(x), \quad a.e. \quad x \in supp\varphi.$$

By Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n(x))\varphi(x)dx \to \int_{\mathbb{R}^N} f(u(x))\varphi(x)dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N)dx.$$

By a density argument, for all $\phi \in H^2(\mathbb{R}^N)$, we obtain,

$$\int_{\mathbb{R}^N} f(u_n)\phi dx = \int_{\mathbb{R}^N} f(u)\phi dx + o_n(1).$$

On the other hand gives weak convergence of $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \Delta u_n \Delta\phi dx + \int_{\mathbb{R}^N} V_{per}(x)u_n\phi dx \to \int_{\mathbb{R}^N} \Delta u\Delta\phi dx + \int_{\mathbb{R}^N} V_{per}(x)u\phi dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N).$$

Also, we get

$$\int_{\mathbb{R}^N} |\, u_n|^{2_{**}-2}u_n\phi dx = \int_{\mathbb{R}^N} |u|^{2_{**}-2}u\phi dx + o_n(1).$$

Since $I'_{per}(u_n)\phi = o_n(1)$, for all $\phi \in H^2(\mathbb{R}^N)$, using these convergence above, $I'_{per}(u)\phi = 0$, for all $\phi \in H^2(\mathbb{R}^N)$. □

The next lemma is a version of Lemma 3.1.4 for the critical case, that is, $\beta = 1$.

Lemma 3.2.3. *Let $(u_n) \subset H^2(\mathbb{R}^N)$ be a $(\mathrm{PS})_{c_{\tau,per}}$ sequence for I_{per} with $u_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$ and $\tau \geq \tau_*$. If $\beta = 1$, there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants R, $\eta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \eta > 0.$$

Proof. Suppose that the lemma does not hold. Then, it follows from [61, Lemma I.1] that $u_n \to 0$ in $L^q(\mathbb{R}^N)$, and thus, $\displaystyle\int_{\mathbb{R}^N} f(u_n)u_n = o_n(1)$. Recalling that the limit $I'_{per}(u_n)u_n = o_n(1)$ implies that which implies that

$$\|u_n\|^2_{per} \pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + o_n(1). \tag{3.2.7}$$

In the case that the second term in (3.2.7) is positive, we have

$$\|u_n\|^2_{per} + \int_{\mathbb{R}^N} |\nabla u_n|^p dx \to \overline{L} \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \to \overline{L}.$$

Note that

$$\begin{aligned} c_{\tau,per} + o_n(1) &\geq \frac{1}{2}\|u_n\|^2_{per} + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{1}{2_{**}}\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \\ &\geq \frac{1}{p}\left[\|u_n\|^2_{per} + \int_{\mathbb{R}^N} |\nabla u_n|^p dx\right] - \frac{1}{2_{**}}\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx. \end{aligned}$$

Since $c_{\tau,per} > 0$, we have that $\overline{L} > 0$. Then,

$$c_{\tau,per} \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{L}.$$

Now using the definition of $\overline{S}$, we have

$$\overline{S} \leq \frac{\|u_n\|^2_{per} + \displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^p dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_n|^{2_{**}}\right)^{2/2_{**}}} = \overline{L}^{4/N} + o_n(1).$$

Then, $\overline{S}^{N/4} \leq \overline{L}$. We conclude

$$c_{\tau,per} \geq \left(\frac{1}{p} - \frac{1}{2_{**}}\right)\overline{S}^{N/4},$$

which is a contradiction with Lemma 3.2.1.

In the case that the second term in (3.2.7) is negative, we have

$$\|u_n\|^2_{per} \to \widetilde{L} \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + \int_{\mathbb{R}^N} |\nabla u_n|^p dx \to \widetilde{L}. \tag{3.2.8}$$

With $\widetilde{L} = \widetilde{L}_1 + \widetilde{L}_2$, where

$$\int_{\mathbb{R}^N} |\nabla u_n|^p dx \to \widetilde{L}_1 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \to \widetilde{L}_2$$

Since $2 < p < 2_{**}$ and $\int_{\mathbb{R}^N} F(u_n)dx \to 0$, we have

$$\begin{aligned} c_\tau + o_n(1) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{1}{2_{**}}\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \\ &\geq \frac{1}{2}\|u_n\|_{per}^2 - \left[\frac{1}{p}\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right] \\ &= \frac{1}{2}\|u_n\|_{per}^2 - \frac{1}{p}\left[\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + \int_{\mathbb{R}^N} |\nabla u_n|^p dx\right]. \end{aligned}$$

Letting $n \to \infty$ and using (3.2.8) we get

$$c_{\tau,per} \geq \left(\frac{1}{2} - \frac{1}{p}\right)\widetilde{L}. \tag{3.2.9}$$

If $\widetilde{L} \geq 1$, then,

$$c_{\tau,per} \geq \left(\frac{1}{2} - \frac{1}{p}\right), \tag{3.2.10}$$

which is a contradition by the hypotheses.
On the other hand, we can use the definition of S to get

$$S\left(\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right)^{2/2_{**}} \leq \int_{\mathbb{R}^N} |\Delta u_n|^2 dx \leq \|u_n\|_{per}^2 \tag{3.2.11}$$

Now from Theorem 1.1.2, there exits $C > 0$ such that

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^N} |\Delta u_n|^2 dx\right)^{1/2}.$$

Then,

$$\frac{1}{C^p}\left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^{2/p} \leq \int_{\mathbb{R}^N} |\Delta u_n|^2 dx \leq \|u_n\|_{per}^2 \tag{3.2.12}$$

Using (3.2.11) and (3.2.12) we obtain

$$\|u_n\|_{per}^2 \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx\right)^{2/p} + \left(\int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right)^{2/2_{**}}\right\}.$$

Suppose that $0 < \widetilde{L} < 1$. In this case $0 < \widetilde{L}_1, \widetilde{L}_2 < 1$. Then, since $2 < p < 2_{**}$, we have

$$\|u_n\|_{per}^2 \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\left\{\int_{\mathbb{R}^N} |\nabla u_n|^p dx + \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx\right\}^{2/p}.$$

Taking the limit we conclude that

$$\widetilde{L} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}\widetilde{L}^{2/p}$$

Since $\widetilde{L} > 0$, we obtain

$$\widetilde{L}^{(p-2)/p} \geq \frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}$$

and from (3.2.9) that

$$c_\tau \geq \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{2}\min\left\{\frac{1}{C^p}, S\right\}^{p/(p-2)},$$

which is a contradiction. Hence $\widetilde{L} = 0$ and therefore (i) holds. □

Theorem 3.2.4. *(Critical case)*
Assume that conditions (V_1)*-*(V_3) *and* $(f_1), (\tilde{f_2}), (f_3), (f_4)$ *hold. Then, there is* $\tau_* > 0$ *such that* $(\widehat{P})$ *with* $\beta = 1$*, has a ground state solution for all* $\tau \geq \tau_*$*.*

Proof. This is a consequence of the Lemma 3.2.3 and minor adaptations of the Theorem 3.1.5. □

3.3 Variational framework on nonperiodic problem

The main tool used to prove Theorems 3.0.3 and 3.0.4 is the variational method, in which solutions to (P_3) are critical points of the functional given by

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx \pm \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx \\ &\quad - \int_{\mathbb{R}^N} F(u)dx - \frac{\beta}{2_{**}}\int_{\mathbb{R}^N} |u|^{2_{**}} dx. \end{aligned}$$

It is well known that I is well defined on the Hilbert space E given by

$$E = \left\{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < \infty \right\}$$

with the inner product

$$\langle u, v\rangle_* := \int_{\mathbb{R}^N} (\Delta u \Delta v + V(x)uv) \, dx, \quad \text{for all } u, v \in E$$

and associated norm given by

$$\|u\|_*^2 = \int_{\mathbb{R}^N} (|\Delta u|^2 + V(x)|u|^2) dx \quad \text{for all } u \in E$$

and I belongs to $C^1(H^2(\mathbb{R}^N), \mathbb{R})$ with

$$\begin{aligned} I'(u)\phi &= \int_{\mathbb{R}^N} \Delta u \Delta \phi + \int_{\mathbb{R}^N} V(x)u\phi \, dx \pm \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \phi dx \\ &\quad - \int_{\mathbb{R}^N} f(u)\phi \, dx - \gamma \int_{\mathbb{R}^N} |u|^{2_{**}-2} u\phi \, dx, \end{aligned}$$

for all $\phi \in E$. In this section, $\mathcal{N}$ denotes the Nehari manifold related to I, that is,

$$\mathcal{N} = \left\{ u \in H^2(\mathbb{R}^N) \setminus \{0\} : I'(u)u = 0 \right\}.$$

Arguing as Lemma 3.1.1 and 3.1.2, it is easy to prove that the functional I has the mountain pass geometry. Thus, using again a version of the Mountain Pass Theorem without (PS) condition that can be found in found in [92], there exists a $(PS)_d$ sequence $(u_n) \subset E$, that is, a sequence satisfying

$$I(u_n) \to d \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$d = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I(\eta(t)) > 0.$$

Arguing as Lemma 3.1.3, it follows that

$$d = \inf_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} I(tu) = \inf_{\mathcal{N}} I.$$

The next lemma shows an important inequality involving the levels d and c_{per}.

Lemma 3.3.1. *The levels d and c_{per} verify the inequality*

$$d < c_{per},$$

for all $\tau > \gamma\,\tau_*$.

Proof. From Theorems 3.1.5 and 3.2.4, there is $u \in \mathcal{M}_{per}$ such that $I_{per}(u) = c_{per}$ and $I'_{per}(u) = 0$ for all $\tau > \gamma\,\tau_*$. Hence, there is $\widehat{t} > 0$ such that $\widehat{t}u \in \mathcal{N}$, and so,

$$0 < d \le \sup_{t \ge 0} I(tu) = I(\widehat{t}u).$$

Using (V_3), we conclude that

$$0 < d \le I(\widehat{t}u) < I_{per}(\widehat{t}u) \le \sup_{t\ge 0} I_{per}(tu) = I_{per}(u) = c_{per}.$$

□

Remark 4. *If $u_n \rightharpoonup 0$ in E and $W \in L^{N/2}(\mathbb{R}^N)$. Then,*

$$\int_{\mathbb{R}^N} W(x)|u_n|^2 dx \to 0, \quad n \to +\infty.$$

Proof. Since that $u_n \rightharpoonup 0$ in E, then $u_n(x) \to 0$ a.e. in $\mathbb{R}^N$, this implies

$$|u_n(x)|^2 \to 0 \quad \text{a.e. in } \mathbb{R}^N \tag{3.3.1}$$

For other hand, since that $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \le r \le 2_{**}$, in particular is valid for $L^{2^*}(\mathbb{R}^N)$, consequently $\|u_n\|_{2^*} \le M$ for all n in $\mathbb{N}$, this implies

$$(|u_n|^2)_n, \quad \text{is bounded in } L^{N/(N-2)}(\mathbb{R}^N) \tag{3.3.2}$$

By (3.3.1) and (3.3.2), we have

$$|u_n|^2 \rightharpoonup 0, \quad \text{in } L^{N/(N-2)}(\mathbb{R}^N) \tag{3.3.3}$$

By definition of weak convergence, it follows that

$$\int_{\mathbb{R}^N} h|u_n|^2 dx \to 0, \quad \forall h \in L^s(\mathbb{R}^N) \tag{3.3.4}$$

where $s = (\frac{N}{N-2})' = \frac{N}{2}$, in particular considering $h = W$ we have

$$\int_{\mathbb{R}^N} W(x)|u_n|^2 dx \to 0 \tag{3.3.5}$$

and so,

$$\begin{aligned} |I_{per}(u_n) - I(u_n)| &= \frac{1}{2}\left|\int_{\mathbb{R}^N} (V_{per}(x) - V(x))|u_n|^2 dx\right| \\ &= \frac{1}{2}\left|\int_{\mathbb{R}^N} W(x)|u_n|^2 dx\right| \\ &= o_n(1), \end{aligned} \tag{3.3.6}$$

□

Lemma 3.3.2. *Let $(u_n) \subset E$ be a $(PS)_d$ sequence for I in E such that $u_n \rightharpoonup 0$ weakly in E. If $u_n \nrightarrow 0$ strongly in E, then*

$$c_{per} \leq d.$$

Proof. Affirmation, (t_n) is bounded. In fact, since $t_n u_n \in \mathcal{M}_{per}$, we get $I'_{per}(t_n u_n)(t_n u_n) = 0$, this is

$$0 = t_n^2 \|u_n\|_{per}^2 \pm t_n^p \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \int_{\mathbb{R}^N} f(t_n u_n)(t_n u_n) dx - \beta t_n^{2_{**}} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx,$$

this implies

$$t_n^2 \|u_n\|_{per}^2 \pm t_n^p \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} f(t_n u_n)(t_n u_n) dx + \beta t_n^{2_{**}} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx.$$

In the case that the first term in the associated equality is positive, we have

$$\begin{aligned} t_n^{2-p} \|u_n\|_{per}^2 + \int_{\mathbb{R}^N} |\nabla u_n|^p dx &= \int_{\mathbb{R}^N} \frac{f(t_n u_n)}{(t_n u_n)^{p-1}} u_n^p dx + \beta t_n^{2_{**}-p} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \\ &\geq \beta t_n^{2_{**}-p} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \end{aligned}$$

Assuming by contradictions that $t_n \to +\infty$, we get $C > +\infty$, this is contradictions. Therefore (t_n) is bounded.
In the case that the first term in the associated equality is negative, we have

$$\begin{aligned} \|u_n\|_{per}^2 &= \int_{\mathbb{R}^N} \frac{f(t_n u_n)}{t_n u_n} u_n dx + \beta t_n^{2_{**}-2} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + t_n^{p-2} \int_{\mathbb{R}^N} |\nabla u_n|^p dx \\ &\geq \beta t_n^{2_{**}-p} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx \end{aligned}$$

Assuming by contradictions that $t_n \to +\infty$, we get $C_1 > +\infty$, this is contradictions. Therefore (t_n) is bounded. Since that (t_n) is bounded, there are exist $\bar{t} \geq 0$, such that $t_n \to \bar{t}$.

We start by proving that

$$\limsup_{n\to\infty} t_n \leq 1. \tag{3.3.7}$$

In fact, suppose by contradiction that there exist a subsequence (u_n) and $\delta > 0$ such that

$$t_n \geq 1 + \delta, \quad \forall n \in \mathbb{N}. \tag{3.3.8}$$

Using the facts that $I'(u_n)u_n = o_n(1)$ for all $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}^N} (|\Delta u_n|^2 + V(x)u_n^2) dx \pm \int_{\mathbb{R}^N} |\nabla u_n|^p dx = \int_{\mathbb{R}^N} f(u_n) u_n dx + \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + o_n(1), \tag{3.3.9}$$

and $I'_{per}(t_n u_n) t_n u_n = o_n(1)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & t_n^2 \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V_{per} u_n^2) dx \pm t_n^p \int_{\mathbb{R}^N} |\nabla u_n|^p dx \\ &= \int_{\mathbb{R}^N} f(t_n u_n)(t_n u_n) dx + t_n^{2_{**}} \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx + o_n(1), \end{aligned}$$

then

$$t_n^{2-p}\int_{\mathbb{R}^N}(|\Delta u_n|^2+V_{per}u_n^2)dx \pm \int_{\mathbb{R}^N}|\nabla u_n|^p dx$$
$$= \int_{\mathbb{R}^N}\frac{f(t_nu_n)}{(t_nu_n)^{p-1}}u_n^p dx + t_n^{2_{**}-p}\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx + o_n(1), \tag{3.3.10}$$

Hence (3.3.9) and (3.3.10)

$$\begin{aligned}
&(t_n^{2-p}-1)\int_{\mathbb{R}^N}|\Delta u_n|^2dx + \int_{\mathbb{R}^N}(t_n^{2-p}V_{per}(x)-V(x))dx\\
&= \int_{\mathbb{R}^N}\frac{f(t_nu_n)}{(t_nu_n)^{p-1}}u_n^pdx - \int_{\mathbb{R}^N}f(u_n)u_ndx + (t_n^{2_{**}-p}-1)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx + o_n(1)\\
&= \int_{\mathbb{R}^N}\left(\frac{f(t_nu_n)}{(t_nu_n)^{p-1}} - \frac{f(u_n)}{u_n^{p-1}}\right)u_n^pdx + (t_n^{2_{**}-p}-1)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx + o_n(1)
\end{aligned}$$

using (3.3.8), we get $t_n^{2-p}\le 1$, we get

$$\int_{\mathbb{R}^N}(t_n^{2-p}V_{per}(x)-V(x))|u_n|^2dx \le \int_{\mathbb{R}^N}(V_{per}(x)-V(x))|u_n|^2dx \tag{3.3.11}$$

and

$$(t_n^{2-p}-1)\int_{\mathbb{R}^N}|\Delta u_n|^2dx \le (t_n^{2-p}-1)\int_{\mathbb{R}^N}|\Delta u_n|^2dx \le 0 \tag{3.3.12}$$

for other hand, using $t_n^{2_{**}-p}\ge(1+\delta)^{2_{**}-p}\ge 1+\delta$, we get $t_n^{2_{**}-p}-1\ge\delta$

$$(t_n^{2_{**}-p}-1)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx \ge \delta\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx \tag{3.3.13}$$

consequently, using (3.3.11), (3.3.12) and (3.3.13), we get

$$\begin{aligned}
\int_{\mathbb{R}^N}(V_{per}(x)-V(x))|u_n|^2dx &\ge \int_{\mathbb{R}^N}\left(\frac{f(t_nu_n)}{(t_nu_n)^{p-1}} - \frac{f(u_n)}{u_n^{p-1}}\right)u_n^pdx + \delta\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx\\
&\ge \int_{\mathbb{R}^N}\left(\frac{f(t_nu_n)}{(t_nu_n)^{p-1}} - \frac{f(u_n)}{u_n^{p-1}}\right)u_n^pdx
\end{aligned} \tag{3.3.14}$$

By (3.3.5), (3.3.6) and (V_3)

$$\begin{aligned}
\int_{\mathbb{R}^N}\left(\frac{f(t_nu_n)}{(t_nu_n)^{p-1}} - \frac{f(u_n)}{u_n^{p-1}}\right)u_n^pdx &\le \int_{\mathbb{R}^N}(V_{per}(x)-V(x))|u_n|^2dx\\
&= \int_{\mathbb{R}^N}W(x)|u_n|^2dx\\
&= o_n(1)
\end{aligned} \tag{3.3.15}$$

On the other hand, we can use the fact $u_n \nrightarrow 0$ to obtain $R_1,\beta>0$ and a sequence $(y_n)\subset\mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int_{B_{R_1}(y_n)}|u_n|^2dx \ge \beta > 0. \tag{3.3.16}$$

Let $v_n(x)=v_n(x+y_n)$ and note that $v_n)$ is a bounded sequence in E. Hence $v_n \rightharpoonup v$ in E along a subsequence. By (3.3.16), $v\neq 0$ in a positive measure subset $\Lambda\subset B_{R_1}(0)$.

$$\int_\Lambda\left(\frac{f((1+\delta)v_n)}{((1+\delta)v_n)^{p-1}} - \frac{f(v_n)}{v_n^{p-1}}\right)v_n^p \le \int_{\mathbb{R}^N}\left(\frac{f(t_nu_n)}{(t_nu_n)^{p-1}} - \frac{f(u_n)}{u_n^{p-1}}\right)u_n^pdx \le o_n(1)$$

Taking the limit as $n \to +\infty$ and using Fatou Lemma, (f_3), (3.3.8) and (3.3.15) it follows that

$$
\begin{aligned}
0 &< \int_\Lambda \left(\frac{f((1+\delta)v)}{((1+\delta)v)^{p-1}} - \frac{f(v)}{v^{p-1}} \right) v^p dx \\
&= \int_\Lambda \liminf_{n\to+\infty} \left(\frac{f((1+\delta)v_n)}{((1+\delta)v_n)^{p-1}} - \frac{f(v_n)}{v_n^{p-1}} \right) v_n^p dx \\
&\leq \liminf_{n\to+\infty} \int_\Lambda \left(\frac{f((1+\delta)v_n)}{((1+\delta)v_n)^{p-1}} - \frac{f(v_n)}{v_n^{p-1}} \right) v_n^p dx \\
&= 0,
\end{aligned}
$$

we obtain a contradictions. Therefore $\lim_{n\to\infty} t_n \leq 1$.

Therefore, we have two cases to consider:

(i) $\lim_{n\to\infty} t_n = t < 1$;

(ii) $\lim_{n\to\infty} t_n = t = 1$.

If (i) occurs, then there exists a subsequence (u_n) such that $t_n \to t < 1$. We can also consider that $t_n < 1$ for all $n \in \mathbb{N}$, we get

$$
\begin{aligned}
d + o_n(1) &= I(u_n) - \frac{1}{p} I^{'}(u_n)(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} (|\Delta u_n|^2 + V(x)|u_n|^2) dx + \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u_n)u_n - F(u) \right) dx \\
&\quad + \beta \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx.
\end{aligned}
$$

Recalling that $t_n u_n \in \mathcal{M}_{per}$, and using (f_3) we obtain

$$
\begin{aligned}
c_{per} &\leq I_{per}(t_n u_n) - \frac{1}{p} I'_{per}(t_n u_n) t_n u_n \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \|t_n u_n\|_{per}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{p} f(t_n u_n) t_n u_n - F(s_n v_n) \right) dx \\
&\quad + \beta \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \int_{\mathbb{R}^N} |u_n t_n|^{2_{**}} dx \\
&\leq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_{per}^2 + \int_{\mathbb{R}^N} \left(\frac{1}{p} f(u_n) u_n - F(u_n) \right) dx \\
&\quad + \beta \left(\frac{1}{p} - \frac{1}{2_{**}} \right) \int_{\mathbb{R}^N} |u_n|^{2_{**}} dx.
\end{aligned}
$$

On the other hand, we can use (3.3.5) in the above inequality to get

$$\begin{aligned}
c_{per} &\leq \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}|\Delta u_n|^2dx+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}V_{per}(x)|u_n|^2dx \\
&\quad +\int_{\mathbb{R}^N}\left(\frac{1}{p}f(u_n)u_n-F(u_n)\right)dx+\beta\left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx \\
&\leq \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}|\Delta u_n|^2dx+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}(V(x)+W(x))|u_n|^2dx \\
&\quad +\int_{\mathbb{R}^N}\left(\frac{1}{p}f(u_n)u_n-F(u_n)\right)dx+\beta\left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx \\
&\leq \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}(|\Delta u_n|^2+V(x)|u_n|^2)dx+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}W(x)|u_n|^2dx \\
&\quad +\int_{\mathbb{R}^N}\left(\frac{1}{p}f(u_n)u_n-F(u_n)\right)dx+\beta\left(\frac{1}{p}-\frac{1}{2_{**}}\right)\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx \\
&= I(u_n)-\frac{1}{p}I'(u_n)u_n+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}W(x)|u_n|^2dx \\
&\leq d+\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}W(x)|u_n|^2dx \\
&= d+\left(\frac{1}{2}-\frac{1}{p}\right)o_n(1) \\
&= d+o_n(1).
\end{aligned}$$

Taking $n\to\infty$, we obtain thad $c_{per}\leq d$ as required, Which is the desired conclusion.

Suppose that (ii) holds. Up to a subsequence, we may suppose tha $t_n\to 1$. Taking into account that $I(u_n)\to d$, we get

$$d+o_n(1)=I(u_n)=I_{per}(t_nu_n)+I(u_n)-I_{per}(t_nu_n)$$

which implies that

$$d+o_n(1)=I(u_n)\geq c_{per}+I(u_n)-I_{per}(t_nu_n) \tag{3.3.17}$$

Therefore, it remains to prove that $I(u_n)-I_{per}(t_nu_n)=o_n(1)$. Note that

$$\begin{aligned}
I(u_n)-I_{per}(t_nu_n) &= \frac{1}{2}(1-t_n^2)\int_{\mathbb{R}^N}|\Delta u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}(V(x)-t_n^2V_{per}(x))|u_n|^2dx \\
&\quad \pm\frac{1}{p}(1-t_n^p)\int_{\mathbb{R}^N}|\nabla u_n|^pdx+\int_{\mathbb{R}^N}(F(t_nu_n)-F(u_n))dx \\
&\quad +\beta\frac{1}{2_{**}}(1-t_n^{2_{**}})\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx
\end{aligned}$$

Using (V_3), we get

$$\begin{aligned}
I(u_n)-I_{per}(t_nu_n) &= \frac{1}{2}(1-t_n^2)\int_{\mathbb{R}^N}|\Delta u_n|^2dx+\frac{1}{2}(1-t_n^2)\int_{\mathbb{R}^N}V_{per}(x)|u_n|^2dx \\
&\quad \pm\frac{1}{p}(1-t_n^p)\int_{\mathbb{R}^N}|\nabla u_n|^pdx+\int_{\mathbb{R}^N}(F(t_nu_n)-F(u_n))dx \\
&\quad +\frac{\beta}{2_{**}}(1-t_n^{2_{**}})\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx-\frac{t_n^2}{2}\int_{\mathbb{R}^N}W(x)|u_n|^2dx
\end{aligned} \tag{3.3.18}$$

Since (u_n) is bounded E and $t_n \to 1$, we get

$$\frac{1}{2}(1-t_n^2)\int_{\mathbb{R}^N}|\Delta u_n|^2dx = o_n(1),$$

$$\frac{1}{p}(1-t_n^p)\int_{\mathbb{R}^N}|\nabla u_n|^pdx = o_n(1),$$

$$\frac{1}{2}(1-t_n^2)\int_{\mathbb{R}^N}V_{per}(x)|u_n|^2dx = o_n(1)$$

and

$$\frac{1}{2_{**}}(1-t_n^{2_{**}})\int_{\mathbb{R}^N}|u_n|^{2_{**}}dx = o_n(1).$$

Hence, using that (v_n) is bounded and Sobolev embedding, yields

$$I(u_n) - I_{per}(t_nu_n) \quad \geq \quad o_n(1) + \int_{\mathbb{R}^N}(F(t_nu_n) - F(u_n))dx, \tag{3.3.19}$$

By the mean value theorem

$$\int_{\mathbb{R}^N}|F(t_nu_n) - F(u_n)|dx \leq C_1|1-t_n|\int_{\mathbb{R}^N}|u_n|^2dx + C_2|1-t_n^q|\int_{\mathbb{R}^N}|u_n|^qdx$$

Thus, we can use the bounded of sequence (u_n) and $t_n \to 1$

$$\int_{\mathbb{R}^N}|F(t_nu_n) - F(u_n)|dx = o_n(1) \tag{3.3.20}$$

By (3.3.17), (3.3.19) and (3.3.20), we have

$$c_{per} \leq d,$$

and the result follows after passing to the limit $n \to \infty$. □

3.3.1 Proof of Theorems 3.0.3 and 3.0.4

Proof. Let (u_n) be a $(PS)_d$ sequence for I. Arguing as in the proof of Theorem 3.1.5, we can prove that (u_n) is bounded in E. Thus, there exists $u \in E$ such that

$$u_n \rightharpoonup u \;\text{ in }\; E.$$

Hence, if $u \neq 0$, arguing again as in the proof of the Theorem 3.1.5, u is a ground state solution of the problem (P_3) for the cases $\beta = 0$ and $\beta = 1$.

Now, we will prove that $u = 0$ can not occur. Indeed, if $u = 0$, then $u_n \rightharpoonup 0$ in E. On one hand, since $W \in L^{N/2}(\mathbb{R}^N)$, the Holder's inequality leads to

$$\int_{\mathbb{R}^N}W(x)|u_n|^2\,dx \to 0,$$

and so,

$$|I_{per}(u_n) - I(u_n)| = \left|\int_{\mathbb{R}^N}W(x)|u_n|^2\,dx\right| = o_n(1),$$

showing that

$$I_{per}(u_n) \to d. \tag{3.3.21}$$

On the other hand, taking $\phi \in E$ with $\|\phi\| \leq 1$, we obtain

$$|(I'_{per}(u_n) - I'(u_n))\phi| \leq C \left(\int_{\mathbb{R}^N} |W||u_n|^2 \, dx \right)^{1/2} = o_n(1).$$

Thus,

$$I'_{per}(u_n) = o_n(1). \tag{3.3.22}$$

Let $t_n > 0$ such that $t_n u_n \in \mathcal{M}$. Using Lemma 3.3.2, it follows that $t_n \to 1$. Therefore,

$$c_{per} \leq I_{per}(t_n u_n) = I_{per}(u_n) + o_n(1) = d + o_n(1).$$

Letting $n \to +\infty$, we get

$$c_{per} \leq d$$

obtaining a contradiction with Lemma 3.3.1. This completes the proof of the Theorems 3.0.3 and 3.0.4. □

Appendix A

Some Classical Results

This section is devoted to recall some classical results that were used throughout this work. As this section is just for viewing the results, we will not give any demonstrations.

Theorem A.0.1. *(Dominated Convergence Theorem, Lebesgue). Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*

(a) $f_n(x) \to f(x)$ *a.e. in* Ω,

(b) there is a function $g \in L^1(\Omega)$ *such that for all* n, $|f_n(x)| \leq g(x)$ *a.e. on* Ω.

Then, $f \in L^1(\Omega)$ *and* $\|f_n - f\| \to 0$.

Proof. See [50] □

Theorem A.0.2. *(Fatou's lemma). Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*

(a) for all n, $f_n(x) \geq 0$ *a.e. in* Ω,

(b) $\sup_n \int f_n < +\infty$

For almost all $x \in \Omega$ *we set* $f(x) \liminf_{n\to+\infty} f_n(x) \leq +\infty$ *Then,* $f \in L^1(\Omega)$ *and*

$$\int f(x)dx \leq \liminf_{n\to+\infty} \int f_n(x)dx$$

Proof. See [3] □

Theorem A.0.3. *Let* $1 < p < +\infty$ *and* (f_n) *is bounded in* $L^p(\Omega)$ *and* $f_n(x) \to f(x)$ *a.e. in* Ω *Then,* $f_n \rightharpoonup f$ *in* $L^p(\Omega)$.

Proof. See [50] □

Theorem A.0.4. *Let* (f_n) *be a sequence in* $L^p(\Omega)$ *and let* $f \in L^p(\Omega)$ *be such that* $\|f_p - f\|_p \to 0$. *Then, there exist a subsequence* (f_{n_k}) *and a function* $h \in L^p(\Omega)$ *such that*

(a) $f_{n_k}(x) \to f(x)$ *a.e. in* Ω,

(b) $|f_{n_k}(x)| \leq h(x)\ \forall\ n \in \mathbb{N}$ *a.e. on* Ω.

Proof. See [24] □

Theorem A.0.5. *Assume that* $f, g \in L^p(\Omega)$ *with* $1 \leq p \leq +\infty$. *Then,* $f + g \in L^p(\Omega)$ *and*

$$\left(\int |f+g|^p dx\right)^{1/p} \leq \left(\int |f|^p dx\right)^{1/p} + \left(\int |g|^p dx\right)^{1/p}$$

Proof. See [24] □

Theorem A.0.6. *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p \le +\infty$. Then $f.g \in L^1(\Omega)$ and*

$$\int |fg|dx \le \Big(\int |f|^p dx\Big)^{1/p} \Big(\int |g|^{p'} dx\Big)^{1/p'}$$

Proof. See [24] □

Theorem A.0.7. *Let $\Omega \subset \mathbb{R}^N$ with $n \in \mathbb{N}^+$ be a bounded domain and $\partial\Omega \in C^{2m,\gamma}$. Then there exist the following embeddings for $p \in [1, \infty)$ and $m \in \mathbb{N}^+$:*

(i) *for $(m-1)p < n < mp$: $W^{2m,p}(\Omega) \hookrightarrow C^{m,\mu}(\overline{\Omega})$ with $0 < \mu \le m - \frac{n}{p}$,*

(ii) *for $n < (m-1)p$: $W^{2m,p}(\Omega) \hookrightarrow C^{m,\mu}(\overline{\Omega})$ with $0 < \mu < 1$,*

(iii) *for $n < 2mp$: $W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $p \le q \le p_n^* := \infty$,*

(iii) *for $n = 2mp$: $W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $p \le q < p_n^* := \infty$,*

(iv) *for $n > 2mp$: $W^{2m,p}(\Omega) \hookrightarrow L^q(\Omega)$ with $p \le q < p_n^* = \frac{np}{n-2mp}$.*

Proof. See [24]. □

Lemma A.0.8. *Let $x, y \in \mathbb{R}^N$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product in $\mathbb{R}^N$*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge C_p|x - y|^p \quad \text{if } p \ge 2$$

or

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge \frac{C_p|x - y|^p}{(|x| + |y|)^{2-p}} \quad \text{if } 2 > p > 1.$$

Definition A.0.1. *Let V and H be Hilbert spaces such that $V \subset H$ with injective, dense and continuous embedding. Let V' denote the dual space of V; a scheme of this type (namely $V \subset H \subset V'$) is called a Hilbert triple.*

Definition A.0.2. *We say that a bounded domain $\Omega \subset \mathbb{R}^N$ satisfies an outer ball condition if for each $y \in \partial\Omega$ there exists a ball $B \subset \mathbb{R}^N \setminus \{0\}$ such that $y \in \partial B$. We say that it satisfies a uniform outer ball condition if the radius of the ball B can be taken independently of $y \in \partial\Omega$.*

Theorem A.0.9. *Assume that $\Omega \subset \mathbb{R}^N$ is a Lipschitz bounded domain which satisfies a uniform outer ball condition. Then the space $H^2(\Omega) \cap H_0^1(\Omega)$ becomes a Hilbert space when endowed with the scalar product*

$$\langle v, w\rangle = \int_\Omega \Delta v \Delta w dx, \quad \forall u, v \in H^2(\Omega) \cap H_0^1(\Omega),$$

This scalar product induces a norm equivalent to $\|\cdot\|_{H^2}$.

Proof. See [41, Theorem 2.30]. □

Lemma A.0.10. *The best constant $D^{2,2}(\mathbb{R}^N)$ to $L^{2_{**}}(\mathbb{R}^N)$*

$$S = \inf_{0\neq u\in D^{2,2}(\mathbb{R}^N)} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{2_{**}} dx \right)^{\frac{2}{2_{**}}}} \right\}$$

is a minimum and (up to nontrivial real multiples) it is attained only on the functions

$$U_{x_0}(x) := \frac{a_N}{(1+|x-x_0|^2)^{\frac{N-4}{2}}},$$

and it is also reached by dilations and translations of the function,

$$U_{\varepsilon,x_0}(x) := a_N \varepsilon^{\frac{4-N}{2}} \left(\frac{1}{\varepsilon^2 + |x-x_0|^2} \right)^{\frac{N-4}{2}},$$

where $\varepsilon > 0$ and $a_N = ((N+2)N(N-2)(N-4))^{\frac{N-4}{8}}$ for $N \geq 5$. Also U_{ε,x_0} are the only positive solutions of the equation satisfies the following problem

$$\begin{cases} \Delta^2 u = u^{\frac{N+4}{N-4}} \quad in\ \mathbb{R}^N \\ u \in D^{2,2}(\mathbb{R}^N). \end{cases} \tag{A.0.1}$$

Also note

$$S^{\frac{N}{4}} = \int_{\mathbb{R}^N} |\Delta u_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |u_\varepsilon|^{2_{**}} dx \tag{A.0.2}$$

Consider a cutoff $\rho(x) \geq 0$ function, smooth, as that, $\rho(x) = 1$ if $x \in B_r(x_0)$, $\rho(x) = 0$ if $x \in B_{2r}(x_0)$; where we take $r > 0$. More precisely, define

$$u_\varepsilon(x) = \rho(x) U_\varepsilon(x)$$

We denote by $O(\varepsilon^\alpha)$ a function (which depends on ε) such that there is a constant $C > 0$ (independent of ε) satisfying $|\varepsilon^{-\alpha} O(\varepsilon^\alpha)| \leq C$ for ε small,

$$\int_\Omega |\Delta u_\varepsilon|^2 dx = S^{\frac{N}{4}} + O(\varepsilon^N) \tag{A.0.3}$$

$$\left(\int_\Omega |u_\varepsilon|^{2^{**}} dx \right)^{\frac{2}{2_{**}}} = S^{\frac{N}{4}} + O(\varepsilon^{N-4}) \tag{A.0.4}$$

and for some $C > 0$

$$\int_\Omega |u_\varepsilon|^r dx = \begin{cases} C\varepsilon^{\frac{(N-4)r}{2}} + O(\varepsilon^{\frac{(N-4)r}{2}}) & \text{if } r < \frac{N}{N-4}, \\ C\varepsilon^{N-\frac{(N-4)r}{2}} |\ln \varepsilon| + O(\varepsilon^{N-\frac{(N-4)r}{2}} |\ln \varepsilon|) & \text{if } r = \frac{N}{N-4}, \\ C\varepsilon^{N-\frac{(N-4)r}{2}} + O(\varepsilon^{N-\frac{(N-4)r}{2}}) & \text{if } r > \frac{N}{N-4}. \end{cases} \tag{A.0.5}$$

Then, we have $\|v_\varepsilon\|^2 = \frac{\|u_\varepsilon\|^2}{\|u_\varepsilon\|^2_{2_{**}}}$, this implies that $\|v_\varepsilon\|^2 = S + O(\varepsilon^{N-4})$.

Proof. See ([4] and [41]). □

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