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Regularizing Effect for a Class of Maxwell-Schrödinger Systems

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Aos vinte e dois dias do mês de março do ano de dois mil e vinte e quatro, instalou-se a banca examinadora de Tese de Doutorado do(a) aluno(a) Ayana Pinheiro de Castro Santana, matrícula 20/0065645. A banca examinadora foi composta pelos professores Dr(a) Liliane de Almeida Maia (membro interno - MAT/UnB), Dr(a) João Vitor da Silva (membro externo à instituição - UNICAMP), Dr(a) Edcarlos Domingos da Silva (membro externo à instituição - UFG), Dr(a) Mayra Soares Costa Rodrigues (Suplente - membro interno -MAT/UnB) e Dr(a). Luís Henrique de Miranda (orientador/presidente - MAT/UnB). O(A) discente apresentou o trabalho intitulado Regularizing Effect for a Class of Maxwell – Schrödinger Systems.

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“Sonner or later you’re going to realize just as I did that there’s a difference between knowing the path and walking the path."

The Matrix

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Resumo

Efeito Regularizante para uma Classe de Sistemas de Maxwell-Schrödinger

Neste trabalho provamos a existência e regularidade de soluções fracas para os seguintes sistemas:

Maxwell-Schrödinger

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + g(x,u,v) = f \quad \text{em} \quad \Omega; \\ -\mathrm{div}(M(x)\nabla v) = h(x,u,v) \quad \text{em} \quad \Omega; \\ \quad u = v = 0 \quad \text{sobre} \quad \partial\Omega. \end{cases}$$

Kirchhoff–Maxwell-Schrödinger

$$\begin{cases} -\mathrm{div}\big((M(x) + ||\nabla u||^{\sigma}_{L^{\sigma}})\nabla u\big) + g(x,u,v) = f \quad \text{em } \Omega; \\ -\mathrm{div}(M(x)\nabla v) = h(x,u,v) \quad \text{em } \Omega; \\ u = v = 0 \text{ sobre } \partial\Omega, \end{cases}$$

onde Ω é um subconjunto aberto limitado do $\mathbb{R}^N$, com $N > 2$, $f \in L^m(\Omega)$ onde $m > 1$ e g, h são duas funções Carathéodory. Mostraremos que sob condições apropriadas em g e h, que existem soluções cuja somabilidade escapam à regularidade prevista pela teoria clássica de Stampacchia, dando origem ao chamado efeito regularizante.

Palavras-Chave: Regularidade; EDP Elíptica; Efeito Regularizante; Existência de Solução.

Abstract

In this work we prove the existence and regularity of weak solutions for the following systems:

Maxwell-Schrödinger

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + g(x,u,v) = f \ \text{ in } \ \Omega \\ -\mathrm{div}(M(x)\nabla v) = h(x,u,v) \ \text{ in } \ \Omega; \\ \quad u = v = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Kirchhoff–Maxwell-Schrödinger

$$\begin{cases} -\mathrm{div}\big((M(x) + ||\nabla u||^{\sigma}_{L^{\sigma}})\nabla u\big) + g(x,u,v) = f \quad \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla v) = h(x,u,v) \quad \text{in } \Omega; \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, for $N > 2$, $f \in L^m(\Omega)$, where $m > 1$ and g, h are two Carathéodory functions. We prove that under appropriate conditions on g and h, there exist solutions which escape the predicted regularity by the classical Stampacchia theory giving rise to the so-called regularizing effect.

Keywords: Regularity; Elliptic PDE; Regularizing Effect; Existence of Solution.

Contents

Notations

- Ω .... open bounded set of $\mathbb{R}^N$
- $\partial\Omega$ .... boundary of Ω
- meas .... Lebesgue measure in $\mathbb{R}^N$
- a.e. .... almost everywhere with respect to the Lebesgue measure
- $C_0^\infty(\Omega)$ .... space of infinitely differentiable functions $u : \Omega \to \mathbb{R}$ with compact support in Ω
- m' .... $\frac{m}{m-1}$
- m^* .... $\frac{Nm}{N-m}$
- $(m^*)'$ .... $\frac{Nm}{N(m-1)+m}$
- m^{**} .... $\frac{Nm}{N-2m}$
- $L^m(\Omega)$ .... space of functions f such that $|f|^m$ is Lebesgue integrable over $\Omega \subset \mathbb{R}^N$
- $W^{1,m}(\Omega)$ .... Sobolev space of functions $u : \Omega \to \mathbb{R}$ in $L^m(\Omega)$ such that ∇u exists weak sense with $|\nabla u|$ in $L^m(\Omega)$
- $W_0^{1,m}(\Omega)$ .... the closure of $C_0^\infty(\Omega)$ in $W^{1,m}(\Omega)$
- $W^{-1,m}(\Omega)$ .... $\big(W_0^{1,m'}(\Omega)\big)'$
- $T_k(s)$ .... $\max(-k, \min(k, s))$
- $G_k(s)$ .... $s - T_k(s)$
- $\mathrm{sgn}(u)$ .... $\begin{cases} u/|u|, & \text{if } \ u \neq 0 \\ 0, & \text{if } \ u = 0 \end{cases}$
- $\mathcal{X}_A(x)$ .... $\begin{cases} 1, & \text{if } \ x \in A, \ \text{where } A \subset \mathbb{R}^N \\ 0, & \text{otherwise} \end{cases}$

Introduction

In the present work, we investigate the existence and regularity of positive solutions a classe of systems Maxwell-Schrodinger systems, considering at first a local version and later on the nonlocal equivalent. More precisely, we have considered a local Maxwell-Schrödinger system

$$\begin{cases} -\text{div}(M(x)\nabla u) + g(x,u,v) = f \ \text{ in } \ \Omega; \\ -\text{div}(M(x)\nabla v) = h(x,u,v) \ \text{ in } \ \Omega; \\ \quad u = v = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{P}$$

and a nonlocal Kirchhoff–Maxwell-Schrödinger system

$$\begin{cases} -\text{div}\big((M(x) + ||\nabla u||^{\sigma}_{L^{\sigma}})\nabla u\big) + g(x,u,v) = f \quad \text{in } \Omega; \\ -\text{div}(M(x)\nabla v) = h(x,u,v) \quad \text{in } \Omega; \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{K}$$

For the sake of convenience, we will discuss these cases separately, below.

0.1 Local System

The general idea regarding these systems is that due to the strong coupling between both equations, solutions have zones where they are more regular than expected from the classic regularity theory. This phenomenon has been studied since the seminal work [4] followed by several interesting contributions [2, 5, 7–9, 13] produced by Boccardo, Orsina, Arcoya, Durastanti, among others. The basic idea is that the solutions to a certain class of problems are more regular than what would be guaranteed by the standard regularity results for the decoupled equations of its system. For instance, it happens that even when the data f is very irregular, e.g., if $f \in L^m(\Omega)$ with $m < (2^*)'$, it is possible to guarantee the existence of energy solutions in $W_0^{1,2}(\Omega)$, see [4, 7].

In other to clarify the ideas and to present some of the background concerning Regularizing Effects for Maxwell–Schrödinger equations, let us briefly discuss the papers which have most inspired the present work, namely [4, 6, 7] and [13]. For instance, consider the following Dirichlet problem

$$\begin{cases} -\text{div}(M(x)\nabla u) + Av|u|^{r-2} = f, \quad u \ \text{ in } W_0^{1,2}(\Omega); \\ -\text{div}(M(x)\nabla v) = |u|^r, \quad v \ \text{ in } W_0^{1,2}(\Omega). \end{cases} \tag{1.1}$$

for $f \in L^m(\Omega)$, with $f \geqslant 0$, $m \geqslant 1$, $A > 0$, $M(x)$ is a uniformly elliptic bounded measurable symmetric matrix, and $\Omega \subset \mathbb{R}^N$, $N > 2$, is an open bounded domain. In [4], for a more general class of equations, the author proved that if $f \in L^m(\Omega)$, with $m \geqslant (2^*)'$, then there exists an energy solution $(u,v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, despite that the right-hand side of the second equation does not belong to the dual space $W^{-1,2}(\Omega)$. For this, the strategy is to obtain keen a priori estimates by a clever choice of test functions to an approximated variational PDE, then the proof is finished by means of standard compactness arguments. Moreover, for the specific case of (1.1), the author found out additional regularizing zones for the parameters. As a matter of fact, it was shown that for $2 \leqslant m < \frac{(r-1)N}{2r}$, with $r > \frac{2^*}{2}$, then $u \in L^{rm}(\Omega)$. Remark that, in the light of the classic Stampacchia's theory, see [17,18], by regarding u solely as the solution of the first equation of (1.1), its expected regularity would be $L^{m^{**}}(\Omega)$. However, under the latter conditions $rm > m^{**}$, so that the coupling in (1.1) gives u some extra regularity. Later on, in [7], the authors refined the latter result. Indeed, they proved that:

(i) if $r > 2^*$ and $f \in L^m(\Omega)$, with $r' \leqslant m < (2^*)'$, then (1.1) has a solution $(u,v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ where $u \in L^{\tau_1}(\Omega)$ and $\tau_1 = \max(m(r-1), m^{**})$;

(ii) if $1 < r < 2^*$ and $\max\left(\frac{Nr}{N+2r}, 1\right) < m < (2^*)'$, then (1.1) has a solution $(u,v) \in W_0^{1,m^*}(\Omega) \times W_0^{1,\tau_2}(\Omega)$ where $\tau_2 = \min\left(\frac{Nm}{Nr-2mr-m}, 2\right)$;

Succeeding, [4,7], in [13], the author proposes the study of the following quasilinear elliptic system

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + Av^{\theta+1}|u|^{r-2}u = f, & u \text{ in } W_0^{1,p}(\Omega); \\ -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = |u|^r v^\theta, & v \text{ in } W_0^{1,p}(\Omega), \end{cases} \tag{1.2}$$

where $1 < p < N$ and $0 \leqslant \theta < p-1$. Remark that although it is a p-Laplacian system, for $\theta = 0$ its zeroth order nonlinear term reduce to (1.1).

We stress that, for the case $\theta = 0$, the author shows existence and Regularizing Effects even if the source f does not belong $W_0^{-1,p}(\Omega)$. Actually, the author proves that if $f \in L^m(\Omega)$ with $(r+1)' \leqslant m < (2^*)'$ there exists a weak solution $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ of system (1.2).

In [13] the author also shows existence in the dual case, that is, if $f \in L^m(\Omega)$ with $m \geqslant (p^*)'$ there exists $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ solution, with $A > 0$, $r > 1$ and $0 \leqslant \theta < p-1$.

Nevertheless, the method used to prove the results established in [13] were not sufficient to guarantee a regularizing effect in the case where $\theta > 0$.

In addition to the latter results, there have been other contributions to the investigation of regularizing effects phenomena in general. For instance, without the ambition of being complete, we refer the reader to [8], which is divided in two parts, where the first one consists in a survey for the theory and in the second one some contributions for different classes of Dirichlet systems are presented. Further, we mention [9], for regularizing effects of positive solutions of a symmetric version of (1.1), or [2] where under an interesting growth assumption for the source term f, the authors obtain extra regularity for the solutions.

Regarding the present paper, we have decided to address (P) and to revisit part of the questions raised in the past theory adapted for our problem. If on one hand, sometimes our results are valid for a different kind of differential operators, e.g. if we contrast to [13], on the other hand, our contributions are valid for a different class of zero order nonlinearities, e.g., if we compare to [9]. Yet, despite that we deal with a slightly different type of system, we tried to investigate certain aspects of the theory which were not fully disclosed, at least for (P), to the best of our knowledge. More precisely, in order to better explain our main results, late us state our basic hypotheses.

0.1.1 Assumptions

Below, we describe the basic assumptions for our manuscript. Indeed, throughout the entire text we will always assume that $\Omega \subset \mathbb{R}^N$ is an open bounded subset, where $N > 2$. Remark that ask no smootheness on $\partial\Omega$. We also consider the real paramethers $r > 1$ and $\theta \in (0, \frac{4}{N-2})$, and we take $f \in L^m(\Omega)$, for $m > 1$.

Regarding the semilinear part of System (P), we consider that $g, h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are both Carathéodory satisfying the properties below:

(a) there exist c_1, $c_2 > 0$ such that

$$c_1|s|^{r-1}|t|^{\theta+1} \leqslant |g(x,s,t)| \leqslant c_2|s|^{r-1}|t|^{\theta+1}; \tag{P$_1$}$$

(b) $g(x,s,t)$ is monotone is s, i.e.,

$$(g(x,s_1,t) - g(x,s_2,t))(s_1 - s_2) \geqslant 0 \quad \forall\ s_1, s_2, t \in \mathbb{R} \ \text{ a.e. } x \text{ in } \Omega; \tag{P$_2$}$$

(c) there exist d_1, $d_2 > 0$ such that

$$d_1|s|^r|t|^\theta \leqslant |h(x,s,t)| \leqslant d_2|s|^r|t|^\theta; \tag{P$_3$}$$

(d) $h(.,.,.)$ is non-negative

$$h(x,s,t) \geqslant 0 \quad \forall\ s,t \in \mathbb{R}, \quad \text{a.e. } x \text{ in } \Omega. \tag{P$_4$}$$

Remark that by (P$_1$), $g(x,0,t) = 0$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, so that (P$_2$), there holds that

$$g(x,s,t)s \geqslant 0 \quad \forall\ s,t \in \mathbb{R} \ \text{ a.e. } x \text{ in } \Omega. \tag{P$_2'$}$$

Regarding the differential operators in (P), we assume that $M(x)$ is a symmetric measurable matrix such that $M \in W^{1,\infty}$ and there exist $\alpha, \beta \in \mathbb{R}^+$ satisfying

$$\alpha|\xi|^2 \leqslant M(x)\xi \cdot \xi\ ,\ |M(x)| \leqslant \beta \ \text{ for every } \ \xi \in \mathbb{R}^N. \tag{P$_5$}$$

Now we are in position to introduce our main contributions.

0.1.2 Main results for the local Maxwell- Schrödinger system

Our main contributions are twofold. First we considered nonlinearities which are more general and second we addressed the regularizing effect in the case where there is the presence of a second parameter of coupling on the nonlinearities, as it was conjectured in [13]. However, as it turned out, we discovered the presence of a ramification on the gain of regularity depending on the interplay between the data f and the coupling parameters, below the known results for systems related to (P). We point out that, in order to do that, we considered the Laplacian-like version of the Maxwell-Schrödinger system. We also introduce a definition which, in our view, slightly simplifies the explanation of the concept of gain of regularity.

In the literature, we say that there exist regularizing effects in a solution of a problem or a system, whenever its regularity escapes the predicted one according to the standard Stampacchia or Calderón-Zygmund theories. In order to summarize this justification, we introduce the following definition on "regularized solutions".

Definition 0.1. *Let $F \in L^m(\Omega)$ where $1 \leqslant m < \frac{N}{2}$. Consider w a distributional solution of*

$$-div(M(x)\nabla w) = F(x). \tag{1}$$

a) If $w \in L^s(\Omega)$ where $s > m^{**}$ we say w is Lebesgue regularized.

b) If $w \in W_0^{1,t}(\Omega)$ where $t > m^*$ we say that w is Sobolev regularized.

Despite that the justification of the latter definition is tacit, for the convenience of the reader we explain it. Indeed, for instance, we know by Stamppachia's classical regularity theory that, if $F \in L^m(\Omega)$ with $1 \leqslant m < \frac{N}{2}$, then the distributional solution of problem (1) belongs to $L^{m^{**}}(\Omega)$, for instance see [6] Chapters 6 and 11. Thus, if $w \in L^s(\Omega)$ with $s > m^{**}$, then $L^s(\Omega) \hookrightarrow L^{m^{**}}(\Omega)$, properly. In this case, we have regularizing effect for the Lebesgue summability of the solution w, or in short, w is Lebesgue regularized. Further, regarding the regularity of the gradients, there are two basic scenarios. If $1 \leqslant m \leqslant (2^*)'$ then once again from Stampacchia's theory if w solves (1) then $w \in W_0^{1,m^*}(\Omega)$. Further, if $(2^*)' < m < \frac{N}{2}$ and if $\partial\Omega$ and $M(x)$ are sufficiently smooth, then by the Calderón-Zygmund theory, see [12] Chapters 5 and 10, we have $w \in W_0^{1,m^*}(\Omega)$. Finally, remark that the restriction $1 \leqslant m < \frac{N}{2}$ is considered in order to stay away from known issues concerning the regularity of the gradients when $m > \frac{N}{2}$, for instance see [3].

In our first theorem, we address the existence and higher regularity for positive solutions of (P) in the case that the summability of the source is above the threshold $(r + \theta + 1)'$.

Theorem 0.1. *Let $f \in L^m(\Omega)$, where $f \geqslant 0$ a.e. in Ω, $m \geqslant (r + \theta + 1)'$, $r > 1$, and $0 < \theta < \frac{4}{N-2}$. Then there exists a weak solution (u, v) for (P), with $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$, $u \geqslant 0$ a.e. in Ω and $v \in W_0^{1,2}(\Omega)$, $v \geqslant 0$ a.e in Ω.*

Now, in the spirit of Definition 0.1, we will detail the gain of regularity in Lebesgue or Sobolev spaces for the solutions of (P) given by Theorem 0.1.

Corollary 0.1. *Let (u, v) be the weak solution of (P), given by Theorem 0.1.*

(A) If $r + \theta + 1 > 2^*$ and $(r + \theta + 1)' \leqslant m < (2^*)'$, then u is Lebesgue and Sobolev regularized.

(B) If $r + \theta + 1 > 2^*$ and $(2^*)' \leqslant m < \frac{N(r+\theta+1)}{N+2(r+\theta+1)}$ then u is Lebesgue regularized.

(C) If $2^* < r + \theta + 1 \leqslant \frac{2^*(\theta+1)}{\theta}$ then v is Sobolev regularized.

Regarding Theorem 0.1, we partially address a conjecture left in [13] by R. Durastanti, where the case $m \geqslant (r + \theta + 1)'$ was proposed. Indeed, we have proved that the Lebesgue regularity indicated in [13] is achieved for a class of zeroth-order nonlinear terms dominated by two variable polynomials depending on both unknowns, i.e., $\theta > 0$. Remark that, if on one hand we have considered linear differential operators with nonsmooth coefficients instead of the p-Laplacian, on the other hand, our nonlinear coupling satisfies properties (P_1) - (P_4) .

In addition, our approach allowed us to investigate another regime of regularity for the source term. Our inquiry also considers existence and regularity of solutions for the case where the summability of the source is below $(r + \theta + 1)'$, i.e., we address problem (P), if the source f is positive and belongs to $L^m(\Omega)$ with $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < (r + \theta + 1)'$.

Theorem 0.2. *Let $f \in L^m(\Omega)$, where $f \geqslant 0$ a.e. in Ω, $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < (r+\theta+1)'$, $r > 1$ and $0 < \theta < \delta$, for $\delta = min\{\frac{N+2}{3N-2}, \frac{4}{N-2}, \frac{1}{2}\}$. Then there exists a solution (u, v) for (P), with $u \in W_0^{1,p}(\Omega) \cap L^{r-\theta+1}(\Omega)$, $u \geqslant 0$ a.e. in Ω and $v \in W_0^{1,q}(\Omega)$, $v \geqslant 0$ a.e. in Ω, where $p = \frac{2(r-\theta+1)}{(r+\theta+1)}$ and $q = \frac{2N(1-\theta)}{N-2\theta}$. Furthermore, if $r \geqslant \frac{N+2}{N-2}$, then $(u, v) \in W_0^{1,q}(\Omega) \times W_0^{1,q}(\Omega)$.*

Once again, in the guidelines of Definition 0.1, we now depict the regularizing effect zones guaranteed by Theorem 0.2.

Corollary 0.2. *Let (u, v) be the weak solution of (P), given by Theorem 0.2*

(A) If $r - \theta + 1 > 2^*$ suppose that

$$\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)} \leqslant m < (2^*)'.$$

Then u is Lebesgue regularized.

(B) If $r - \theta + 1 > 2^*$ suppose that

$$\Big(\frac{r-\theta+1}{1-2\theta}\Big)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

Then u is Sobolev and Lebesgue regularized.

(C) If $2^* \geqslant r - \theta + 1 > 2^*(1-\theta)$ suppose that

$$\Big(\frac{r-\theta+1}{1-2\theta}\Big)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

Then u is Sobolev and Lebesgue regularized.

(D) If $r - \theta + 1 > 2^*(1-\theta)$ then v is Sobolev regularized.

We will present proofs for Corollaries 0.1 and 0.2 in Section 5. For now, some remarks are in order.

Remark 0.1. *(i) The intervals established for m in Corollary 0.1 and 0.2 are not empty. Indeed, for m in the first two items of the Corollary 0.2 there follows*

(a) Since $2^*(1-2\theta) < 2^* < r - \theta + 1$ there follows

$$\Big(\frac{r-\theta+1}{1-2\theta}\Big)' < (2^*)' \iff 2^*(1-2\theta) < r-\theta+1.$$

(b) Analogously, $2^*(1-\theta) < 2^* < r - \theta + 1$ there follows

$$\Big(\frac{r-\theta+1}{1-2\theta}\Big)' < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)} \iff 2^*(1-\theta) < r-\theta+1.$$

In addition, the interval established for m in the third item of Corollary 0.2, is also non-empty. Actually, if $r+\theta+1 \geqslant 2^* \geqslant r-\theta+1$, then there holds that $(r+\theta+1)' < (2^*)'$ and

$$r-\theta+1 < 2^* \iff \frac{N(r-\theta+1)}{N+2(r-\theta+1)} < (2^*)'.$$

Further, if $2^* \geqslant r+\theta+1 > r-\theta+1$, then there holds that $(r+\theta+1)' \geqslant (2^*)'$ and

$$r-\theta+1 < 2^* \iff \frac{N(r-\theta+1)}{N+2(r-\theta+1)} < (2^*)'.$$

(ii) We observe that for $r-\theta+1 > 2^*$, as $r+\theta+1 > r-\theta+1$ then $r+\theta+1 > 2^* \iff (r+\theta+1)' < (2^*)'$, in this case, the hypothesis established in m by Theorem 0.2, allows us to conclude that $m < (r+\theta+1)' < (2^*)'$. This means that, if we take $r-\theta+1 > 2^*$, the intervals determined for m in the first two items of the Corollary 0.2 are non-empty.

Finally, let us stress that, being inspired by the classical approach of the school of G. Stampacchia, L. Boccardo, among others, see [4, 5, 7, 17, 18] and the references therein, the main ingredient for our results was based on a carefully choice of tailored test functions. Indeed, by means of subtle modifications on the test functions we were able to address the regimes of regularity described in Theorems 0.1 and 0.2, see Lemmas 2.4 and 2.5. After that, we follow the standard approach of determining a priori estimates for solutions of an approximate problem and then passing to the limit.

0.2 The nonlocal System

In addition, we decided to investigate the influence of nonlocal terms on the regularizing zones to Maxwell-Schrödinger equations. Indeed, let us consider the following nonlocal system

$$\begin{cases} -\mathrm{div}\big((M(x)+||\nabla u||^{\sigma}_{L^{\sigma}})\nabla u\big)+g(x,u,v)=f \quad \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla v)=h(x,u,v) \quad \text{in } \Omega; \\ u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{K}$$

which from now on will be called Kirchhoff-Maxwell-Schrödinger system.

We investigate existence and regularity of positive solutions, assuming that Ω is an open bounded subset of $\mathbb{R}^N$, for $N>2$, $f\in L^m(\Omega)$ with $m\geqslant 1$, $r>1$. Moreover, g and $h:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are Carathéodory and satisfy hypotheses (P_1) - (P_2') and $M:\Omega\to\mathbb{R}^N\times\mathbb{R}^N$ is a bounded measurable matrix satisfying (P_5), see page 12.
From now on, we denote $p=\frac{2(r-\theta+1)}{r+\theta+1}$ and suppose

$$\sigma=\begin{cases} 2 & \text{if } m\geqslant (r+\theta+1)', \\ p & \text{if } m<(r+\theta+1)'. \end{cases} \tag{2}$$

Our motivation to address (K) comes from [10]. In this work, the authors studied existence and certain properties of solutions for the following Kirchhoff–Maxwell–Schrödinger system

$$\begin{cases} -\mathrm{div}\Big(\big(a(x)+||\nabla u||^{2}_{L^{2}}\big)\nabla u\Big)+v|u|^{r-2}u=f \quad \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla v)=|u|^{r} \quad \text{in } \Omega; \\ u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{K_1}$$

where once again $\Omega\subset\mathbb{R}^N$, is bounded open, $N>2$, $f\in L^m(\Omega)$ with $m\geqslant 1$, M is a bounded measurable matrix satisfying (P_5) and, $r>1$, $a:\Omega\to\mathbb{R}$ is a measurable function such that there exist $0<\alpha<\beta$ for

$$0<\alpha\leqslant a(x)\leqslant\beta \quad \text{a.e. in } \Omega.$$

Following the standard strategy, in order to analyze (K_1), the authors obtain approximate solutions, and under certain conditions for r and m, ensure interesting a priori estimates that combined with some compactness arguments, allow the passage to the limit on this approximate version of (K_1). As a consequence, they showed that solutions of the first equation satisfy $u\in L^s(\Omega)$ where $s=\max\{m^{**},\frac{m(2r+1)}{m+1}\}$. In particular, if $(r+1)'\leqslant m<\frac{N}{2}$, u is Lebesgue regularized, see Definition 0.1 page 12. Moreover, since u is bounded in $L^s(\Omega)$, it is easy to see $|u|^r$ is bounded in $L^{\frac{s}{r}}(\Omega)$, which implies that v is Sobolev regularized since $\frac{s}{r}<(2^*)'$. For small r, outside the regularizing zone of m, the existence of positive solution was shown. More specifically they show that if $1 < r < 2^* - 1$ and $m \geqslant (2^*)'$ there exists $(u, v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ solutions of (K_1).

Let us also mention that in [10], in the case of poor data, it is proved that an atypical phenomenon occurs for the nonlocal Maxwell-Schrödinger system. Indeed, due to the presence of the nonlocal Kirchhoff-type term

$$-\mathrm{div}\Big(\|\nabla u\|_{L^2}^2 \nabla u\Big),$$

if $f \in L^1(\Omega) \setminus W^{-1,2}(\Omega)$ and $1 < r < 2^* - 1$, given (u_k, v_k) solutions for an approximate version of (K_1), then $\{u_k\}$ is unbounded in $W_0^{1,2}(\Omega)$. Accordingly, for the case $N = 6$ the authors prove that $\{u_k\}$ is bounded in $W_0^{1,q}(\Omega)$, for $1 < q < \frac{18}{11}$, whereas $u_k \to 0$ strongly in $W_0^{1,q}(\Omega)$. In addition, they proved that $||\nabla u_k||_{L^2}^2 u_k \rightharpoonup w$ weakly in $W_0^{1,\rho}(\Omega)$, where $1 < \rho < \frac{6}{5}$ and w is the entropy solution of

$$\begin{cases} -\Delta w = f & \text{in } \Omega \\ \quad w = 0 & \text{on } \partial\Omega. \end{cases}$$

With regard to this chapter, we decided to investigate an associated version of problem (K_1) by introducing certain modifications. As a matter of fact, we replaced the semilinear parts of the original system by g and h, both satisfying hypotheses (P_1) - (P_4). Moreover, we have decided to consider the nonlocal effects in terms of the regularity of f, instead of fixing an energy Kirchhoff-type term. For this, we employed some of the the results obtained for the local version of our (K_1), i.e., problem (P) which was addressed in the previous chapter. This motivates the choice of a nonlocal Kirchhoff-type term given by $||\nabla u||_{L^\sigma}^\sigma$ with σ satisfying (3.1). By doing so, we avoid the degeneracy of the regularity below $(2^*)'$ and obtain the regularizing effect results for the nonlocal case which are compatible with the local ones.

0.2.1 Main results for Kirchhoff–Maxwell–Schrödinger systems

In this case, our main contributions were to ensure the existence regularizing effects under appropriate conditions r, θ and m. That is, if $r + \theta > 2^* - 1$ and $(r + \theta + 1)' < m < (2^*)'$, then u is Lebesgue and Sobolev regularized. Recall that specifically for $\theta = 0$ the coupling term in (K) encompasses (K_1), and in this case we obtain results similar to [10], proving the existence of a weak solution $(u, v) \in W_0^{1,2}(\Omega)$. Regarding v, we have observed that since $u \in L^{r+\theta+1}(\Omega)$ and $v \in L^{2^*}(\Omega)$, we obtain $h(x, v, u) \in L^s(\Omega)$ where $s = \frac{2^*(r+\theta+1)}{2^*r+\theta(r+\theta+1)}$ which implies that, if $s < (2^*)'$ then v is Sobolev regularized since $r+\theta+1 < \left(\frac{2^*}{\theta+1}\right)'$, see Theorem 0.3, below.

Theorem 0.3. *Let $f \in L^m(\Omega)$, where $f \geqslant 0$ a.e in Ω, $m \geqslant (r + \theta + 1)'$, $r > 1$ and $0 < \theta < \frac{4}{N-2}$. Then there exists a weak solution (u, v) for (K), with $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$, $u \geqslant 0$ a.e in Ω and $v \in W_0^{1,2}(\Omega)$, $v \geqslant 0$ in Ω.*

Furthermore, in the present work, as a byproduct of our modification in the nonlocal term, we were able to address the case when the source summability's is below $(r + \theta + 1)'$. Indeed, as it turned out, see Theorem 0.4, there exist regularizing effect zones for solutions (u, v), under certain conditions for r, θ and m. Actually, we prove that if $r - \theta + 1 > 2^*$, then u is Lebesgue and Sobolev regularized since $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}$ and Lebesgue regularized since $\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)} < m < (2^*)'$. We also deduce that if $2^* >$

$r-\theta+1 > 2^*(1-\theta)$ then u is Lebesgue and Sobolev regularized since $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}$. Regarding v, we have observed that since $u \in L^{r-\theta+1}(\Omega)$ and $v \in L^{q^*}(\Omega)$ we obtain $h(x,u,v) \in L^t(\Omega)$ with $t = \frac{2^*(r-\theta+1)(1-\theta)}{2^*r(1-\theta)+\theta(r-\theta+1)}$ which implies that, if $q > t^*$ then v is Sobolev regularized since $r-\theta+1 > 2^*(1-\theta)$.

Theorem 0.4. *Let $f \in L^m(\Omega)$, where $f \geqslant 0$ a.e. in Ω, $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < (r+\theta+1)'$, $r > 1$ and $0 < \theta < \delta$, for $\delta = min\{\frac{N+2}{3N-2}, \frac{4}{N-2}, \frac{1}{2}\}$. Then there exists a solution (u,v) for (K), with $u \in W_0^{1,p}(\Omega) \cap L^{r-\theta+1}(\Omega)$, $u \geqslant 0$ a.e. in Ω and $v \in W_0^{1,q}(\Omega)$, $v \geqslant 0$ a.e. in Ω, where $p = \frac{2(r-\theta+1)}{(r+\theta+1)}$ and $q = \frac{2N(1-\theta)}{N-2\theta}$. Furthermore, if $r \geqslant \frac{N+2}{N-2}$, then $(u,v) \in W_0^{1,q}(\Omega) \times W_0^{1,q}(\Omega)$.*

We summarize the latter results in the following figures.

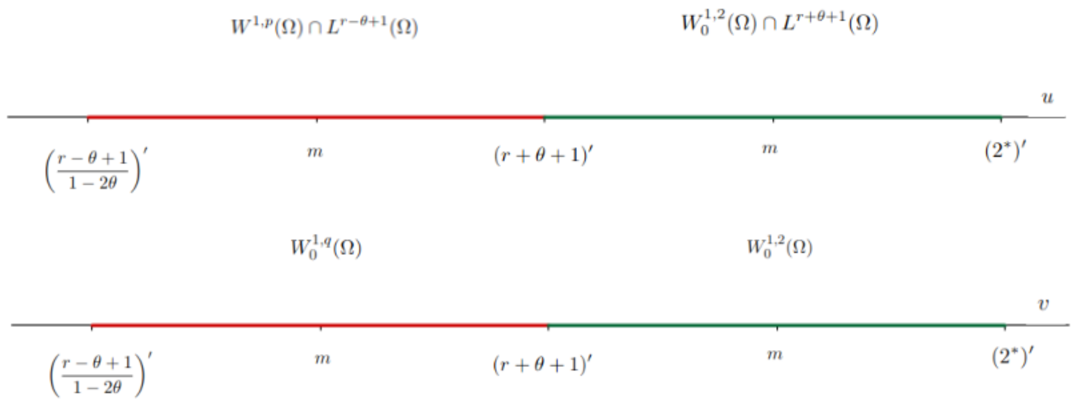


Figure 1: The summability results for (u,v) given by Theorems 0.3 and 0.4.

Remark 0.2. *An important fact to note is that if $v = 0$ then (P) system reduces the equation $-div(M(x)\nabla u) = f$ with Dirichlet condition, which, according to Stampacchia's classical theory, has a solution $u \in W_0^{1,m^*}(\Omega) \cap L^{m^{**}}(\Omega)$ to $f \in L^m(\Omega)$ with $m < (2^*)'$, that being the case, we couldn't have to $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$, whereas $2 > m^*$ and $r+\theta+1 > m^{**}$. Therefore v cannot be null under the tracks where u is Lebesgue and Sobolev regularized.*

0.3 Organization of the thesis

In the first chapter, we introduce the fundamental tools for the development and understanding of the present thesis. The Stampacchia theory is discussed in detail, what includes certain truncations used to establish some classical regularity results for the solutions of linear problems

$$\begin{cases} -\mathrm{div}(M(x)\nabla w) = z & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

For the sake of convenience, we decided to include certain regularity results established for weak and distributional solution of the linear problem. Although being classic, they play a fundamental role in understanding the main results of the work.

In the second chapter, we will start with the preliminary results of convergence that will simplify the proofs of the main results of this chapter, see Theorems 0.2 and 0.1. Thereafter we show existence of approximate solutions to an approximate problem. One of the key points of our work is the careful choice of the test functions in order to, combined with the coupling term of the system, generate "better a priori" estimates for the approximate solutions, see Lemmas 2.4 and 2.5. Through the estimates we can conclude the main theorems and their respective corollaries, whose purpose is to highlight the areas where the solutions are more regular then expected.

In the third chapter, we dedicate our efforts to the study of the Kirchhoff – Maxwell Schrödinger systems, where we circumvent the lack of regularizing effect caused by the nonlocal term $||\nabla u||_{L^2}^2$ ver [10], adding the nonlocal term $||\nabla u||_{L^\sigma}^\sigma$ in the first equation of (P), where the condition established under σ was motivated by the study done for the local system. As the problem thus posed, we show the existence and regularity for solutions, see Theorem 0.3 and 0.4.

We understand that for an initial reading of this work, it is necessary to have a knowledge of prior functional analysis theories, measure and partial differential equations. However, we address some key results used in the work, see Appendices A and B. We highlight the chain rule version for Lipschitz and local Lipschitz functions, where we present our demonstration.

Chapter 1

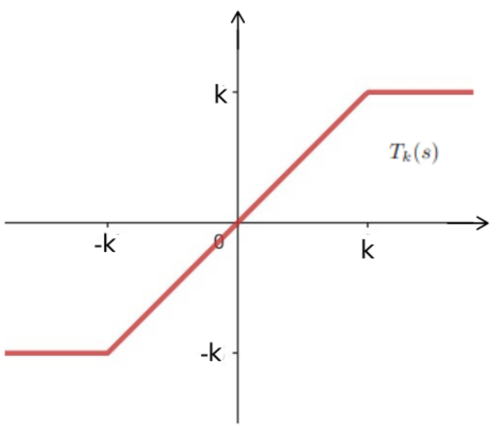
Stampacchia classical theory

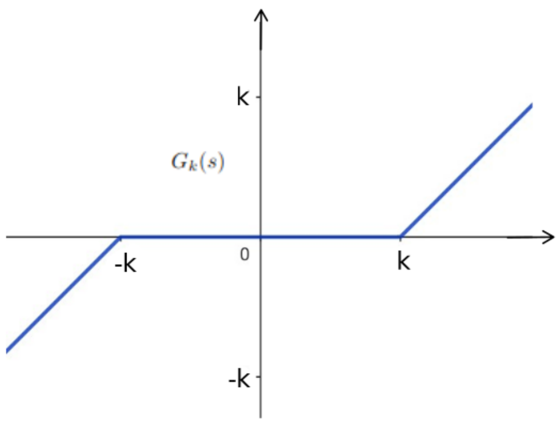
The Stampacchia classical theory plays an important role in the development of this work. We dedicate this chapter to present some of these regularity results for the solution of the linear problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla w) = z & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

We will present our version of the proof of the existence of a distributional solution in the case where source z does not belong to the dual of $W_0^{1,2}(\Omega)$. Specifically, if $z \in L^m(\Omega)$ with $1 < m < (2^*)'$, then $w \in W_0^{1,m^*}(\Omega)$ and if $z \in L^1(\Omega)$ then $w \in W_0^{1,q}(\Omega)$ where $q < \frac{N}{N-1}$. In contrast, these results highlight the importance of the regularizing effect obtained in this work, since we will show through the Theorems 0.3 and 0.4 the existence of an energy solution for the systems (P) and (K), even when the source $z \in L^m(\Omega)$ with $m < (2^*)'$.

We consider some important notations for the development of this work. As well as the use of the well-known Stampacchia truncation functions, whose definitions as well as its graphs are given as there follows. For $k > 0$ we have $T_k(s) = \max(-k, \min(k, s))$ and $G_k(s) = s - T_k(s)$.





Furthermore, very often we will make use of the Sobolev critical exponent, which for $1 \leqslant m < N$, we will denote by $m^* = \frac{Nm}{N-m}$ and $m^{**} = \frac{Nm}{N-2m}$.

Lemma 1.1. *Let $u \in W_0^{1,p}(\Omega)$ where $1 \leqslant p < \infty$ and $k > 0$. Then* *(i)* $T_k(u)$, $G_k(u)$ *are Lipschitz and* $T_k(0) = G_k(0) = 0$*;*

(ii) $T_k(u), G_k(u) \in W_0^{1,p}(\Omega)$ *and*

$$\nabla T_k(u) = \begin{cases} \nabla u, & |u| \leq k \\ 0, & |u| > k. \end{cases}$$

$$\nabla G_k(u) = \begin{cases} 0, & |u| \leq k \\ \nabla u, & |u| > k. \end{cases}$$

Proof. **(i)** By definition of T_k and G_k it is clear that $T_k(0) = G_k(0) = 0$

(ii) Combining the earlier item with by Theorem(4.2) we have $\nabla T_k(u) = T_k'(u)\nabla(u)$ and $\nabla G_k(u) = G_k'(u)\nabla(u)$ where

$$T_k'(u) = \begin{cases} 1, & |u| \leq k \\ 0, & |u| > k. \end{cases} \quad \text{and} \quad G_k'(u) = \begin{cases} 0, & |u| \leq k \\ 1, & |u| > k. \end{cases}$$

However $\nabla u = 0$ a.e. in $\{x \in \Omega;\ |u(x)| = k\}$. Indeed, if $v = u - k$, then $v = 0 \iff u = k$. Since $v^+ = \max\{0, v\}$, $v^- = -\min\{0, v\}$, $\nabla v = \nabla v^+ - \nabla v^-$,

$$\nabla v^+ = \begin{cases} \nabla v, & v > 0 \\ 0, & v \leq 0. \end{cases}$$

$$\nabla v^- = \begin{cases} -\nabla v, & v < 0 \\ 0, & v \geq 0. \end{cases}$$

Hence $\nabla v = 0$ a.e. in $\{x \in \Omega; v(x) = 0\} \iff \nabla u = 0$ a.e. in $\{x \in \Omega; u(x) = k\}$. Analogously for $v = u + k$ we get $\nabla u = 0$ a.e in $\{x \in \Omega; u(x) = k\}$. Thus

$$\nabla T_k(u) = T_k'(u)\nabla(u) = \begin{cases} \nabla u, & |u| \leq k \\ 0, & |u| > k. \end{cases}$$

Repeating the above argument for G_k we have completed the result. □

The following Lemmas can be found in [6], as well as their respective proofs. Even so, we chose to present our version for such demonstrations.

Lemma 1.2. *Let* $f \in L^1(\Omega)$ *and* $l(k) = \int_\Omega |G_k(f)|$*. Then* $l(k)$ *is differentiable a.e and* $l'(k) = -\text{meas}(A_k)$ *where* $A_k = \{x \in \Omega, |f(x)| > k\}$*.*

Proof. Consider $A_k^+ = \{f - k > 0\}$, $A_k^- = \{-(f + k) > 0\}$ and

$$l_+(k) = \int_{A_k^+} (f - k) \text{ and } l_-(k) = \int_{A_k^-} -(f + k).$$

To get the result, just show that l_+ is differentiable with respect to k. Consequently the general case followed, since l can be written as $l(k) = l_+(k) + l_-(k)$, the result follows.

Note that $l_+(k)$ is differentiable a.e, since it is monotone. Thus to calculate its derivative, take $h \in \mathbb{R}^+$, and so the difference quotient of l_+ is

$$\begin{aligned} \frac{l_+(k+h) - l_+(k)}{h} &= \frac{1}{h}\left(\int_{A_{k+h}^+} (f - k - h) - \int_{A_k^+} (f - k)\right) \\ &= \frac{1}{h}\left(\int_{A_{k+h}^+} (f - k) - \int_{A_{k+h}^+} h - \int_{A_k^+} (f - k)\right). \end{aligned} \tag{1.1}$$

As $A_k^+ = A_{k+h}^+ \cup \{k < f \leqslant k+h\}$ we have

$$\int_{A_{k+h}^+} (f-k) - \int_{A_k^+} (f-k) = \int_{A_{k+h}^+} (f-k) - \left[\int_{A_{k+h}^+} (f-k) + \int_{\{k<w\leqslant k+h\}} (f-k) \right]$$
$$= -\int_{\{k<f\leqslant k+h\}} (f-k).$$

Thus, substituting the result obtained above in (1.1) there follows that

$$\frac{l_+(k+h) - l_+(k)}{h} = \frac{1}{h}\left(-\int_{A_{k+h}^+} h - \int_{\{k<f\leqslant k+h\}} (f-k) \right)$$
$$= -\int_\Omega \mathcal{X}_{\{f>k+h\}} - \frac{1}{h}\int_{\{k<f\leqslant k+h\}} (f-k)$$

Consequently, taking $h \to 0$ we get

$$l'_+(k) = \lim_{h\to 0} \frac{l_+(k+h) - l_+(k)}{h} = -\lim_{h\to 0}\int_\Omega \mathcal{X}_{\{f>k+h\}} = -\text{meas}(\{f > k\}) = -\text{meas}(A_k^+),$$

since

$$0 \leqslant \int_{\{k<f\leqslant k+h\}} (f-k) \leqslant \int_{\{k<f\leqslant k+h\}} h \iff 0 \leqslant \frac{1}{h}\int_{\{k<f\leqslant k+h\}} (f-k) \leqslant \int_\Omega \mathcal{X}_{\{k<f\leqslant k+h\}}$$
$$\iff 0 \geqslant -\frac{1}{h}\int_{\{k<f<k+h\}} (f-k) \geqslant -\int_\Omega \mathcal{X}_{\{k<f\leqslant k+h\}},$$

which implies that

$$\lim_{h\to 0} -\frac{1}{h}\int_{\{k<f<k+h\}} (f-k) = 0.$$

Analogously $l'_-(k) = -\text{meas}(A_k^+)$, therefore the result follows. □

The next lemma is a crucial tool used to reach regularity of the type $L^\infty(\Omega)$.

Lemma 1.3. *Let $f \in L^1(\Omega)$ such that $l(k) \leqslant C\text{meas}(A_k)^\alpha$ for every k, where $\alpha > 1$ and $C > 0$. Then $f \in L^\infty(\Omega)$ and there exists a constant $\gamma = \gamma(\alpha, \Omega, ||f||_{L^1})$ such that*

$$||f||_{L^\infty} \leqslant C\gamma.$$

Proof. Combining the result of the previous lemma with the hypothesis of this lemma, we have

$$l(k) \leqslant C\text{meas}(A_k)^\alpha = C(-l'(k))^\alpha$$

where it goes from

$$1 \leqslant -C^{\frac{1}{\alpha}}\frac{l'(k)}{l(k)^{\frac{1}{\alpha}}} \iff -\frac{1}{C^{\frac{1}{\alpha}}} \geqslant \frac{l'(k)}{l(k)^{\frac{1}{\alpha}}}.$$

Integrating the last inequality on $(0,k)$ and by fundamental theorem of calculus we get

$$-\frac{k}{C^{\frac{1}{\alpha}}} = -\int_0^k \frac{1}{C^{\frac{1}{\alpha}}}dt \geqslant -\int_0^k l'(t)l(t)^{-\frac{1}{\alpha}}dt = \frac{1}{(1-\frac{1}{\alpha})}\int_0^k \frac{d}{dt}(l(t))^{1-\frac{1}{\alpha}}dt$$
$$= \frac{l(k)^{1-\frac{1}{\alpha}} - l(0)^{1-\frac{1}{\alpha}}}{(1-\frac{1}{\alpha})}.$$

Now, note that

$$l(0) = \int_\Omega |G_0(f)| = \int_\Omega |f - T_0(f)| = \int_\Omega |f|,$$

and combining the above equality with the previous inequality we obtain

$$-\Big(1 - \frac{1}{\alpha}\Big)\frac{k}{C^{\frac{1}{\alpha}}} \geqslant l(k)^{1-\frac{1}{\alpha}} - ||f||_{L^1}^{1-\frac{1}{\alpha}},$$

hence

$$-\Big(1 - \frac{1}{\alpha}\Big)\frac{k}{C^{\frac{1}{\alpha}}} + ||f||_{L^1}^{1-\frac{1}{\alpha}} \geqslant l(k)^{1-\frac{1}{\alpha}} \quad \forall\, k > 0.$$

In particular taking $\widetilde{k}$ such that $l(\widetilde{k}) = 0$, this is, $\widetilde{k} = \frac{C^{\frac{1}{\alpha}}||f||_{L^1}^{1-\frac{1}{\alpha}}}{\left(1-\frac{1}{\alpha}\right)}$ by defining the truncation function we get

$$|f| \leqslant \widetilde{k} = \frac{C^{\frac{1}{\alpha}}||f||_{L^1}^{1-\frac{1}{\alpha}}}{\left(1 - \frac{1}{\alpha}\right)}$$

and by Hölder's inequality we may conclude that

$$|f| \leqslant \widetilde{k} = \frac{C^{\frac{1}{\alpha}}||f||_{L^1}^{1-\frac{1}{\alpha}}\text{meas}(\Omega)(1-\frac{1}{\alpha})}{\left(1 - \frac{1}{\alpha}\right)}.$$

Therefore $||f||_{L^\infty} \leqslant C\gamma$ where $\gamma = ||f||_{L^1}^{1-\frac{1}{\alpha}}\text{meas}(\Omega)^{1-\frac{1}{\alpha}}\left(1 - \frac{1}{\alpha}\right)$.

□

We will begin our studies by showing the existence and unity of solution to the linear problem. Thereafter, in the company of the results established earlier, we can present some regularity results by Stampacchia, together with the existence of a distributional solution in the case in which $f \in L^m(\Omega)$ with $1 \leqslant m < (2^*)'$. These results, as well as others, can be seen in [6].

The regularity results presented in this section clarify the regularizing effect we obtained in the solutions of (P).

1.1 The linear problem

Consider the linear problem

$$\begin{cases} -\text{div}(M(x)\nabla w) = z \quad \text{in } \Omega \\ \quad w = 0 \quad \text{on } \partial\Omega \end{cases} \tag{P_L}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, with $N > 2$, $z \in L^m(\Omega)$ with $m \geqslant 1$ and $M(x)$ is a symmetric measurable matrix satisfying (P_5).

The existence results that we will present are based on Functional Analysis results by Lax-Milgram and Stampacchia, as we shall see in the next section, see [6].

Theorem 1.1. *Let $z \in L^m(\Omega)$ with $m \geqslant (2^*)'$. Then there exists a unique weak solution $w \in W_0^{1,2}(\Omega)$ to problem (P_L). In other words, there exists a unique $w \in W_0^{1,2}(\Omega)$ such that*

$$\int_\Omega M(x)\nabla w \cdot \nabla\varphi = \int_\Omega z\varphi, \qquad \forall \varphi \in w \in W_0^{1,2}(\Omega).$$

Proof. Define $B : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by $B(w,\varphi) = \int_\Omega M(x)\nabla w \cdot \nabla\varphi$. Note that:

(i) It is easily seen that B is bilinear.

(ii) B is continuous. Indeed, since M is bounded by hypothesis, the Cauchy - Schwarz inequality gives

$$|B(w,\varphi)| = \Big|\int_\Omega M(x)\nabla w \cdot \nabla\varphi\Big| \leqslant \beta||\nabla w||_{L^2}||\nabla\varphi||_{L^2} \quad \forall w,\varphi \in W_0^{1,2}(\Omega).$$

(iii) B is coercive. Indeed, since M have the property of ellipticity, we get

$$|B(w,w)| = \Big|\int_\Omega M(x)\nabla w \cdot \nabla w\Big| \geqslant \alpha||\nabla w||_{L^2}||\nabla w||_{L^2}$$

Thus by the Theorem Lax - Milgram, give $z \in W^{-1,2}(\Omega)$ there exists a unique $w \in W_0^{1,2}(\Omega)$ such that

$$B(w,\varphi) =< z,\varphi >_{W^{-1,2},W_0^{1,2}} \qquad \forall\varphi \in W_0^{1,2}(\Omega).$$

□

Remark 1.1. *We note that, if z belongs $L^m(\Omega)$ with $m \geqslant (2^*)'$, the linear problem(P_L) can be weakly formulated in the sense that for each $\varphi \in W_0^{1,2}(\Omega)$ the application $z \longmapsto \int_\Omega z\varphi$ defines a functional in $W_0^{1,2}(\Omega)$. As long as $m > (2^*)' \iff m' < (2^*)$, by Hölder and Sobolev inequality, we get*

$$\Big|\int_\Omega z\varphi\Big| \leqslant ||z||_{L^m}||\varphi||_{L^{m'}} \leqslant ||z||_{L^m}||\varphi||_{L^{2^*}} \leqslant C||z||_{L^m}||\varphi||_{W_0^{1,2}} \qquad \forall\varphi \in W_0^{1,2}(\Omega).$$

1.2 Stampacchia's classic regularity results of solutions to problem (P_L)

In this section we will present some regularity results by Stampacchia for the problem (P_L). We will see that the regularity of the solution depends on the regularity of the source. Let us assume that the source is a function z belonging to a Lebesgue space to show the following results:

- If $z \in L^m(\Omega)$ with $(2^*)' \leqslant m < \frac{N}{2}$ then $w \in L^{m^{**}}(\Omega)$. Where this result is obtained by taking the test function $\frac{|T_k(w)|^{2\lambda}T_k(w)}{2\lambda+1}$ combined with the hypothesis ellipticity of M, Sobolev's embedding, Hölder's inequality and Fatou's Lemma.

- If $z \in L^m(\Omega)$ with $m > \frac{N}{2}$, then $w \in L^\infty(\Omega)$. In this case, the result is obtained by taking the test function $G_k(w)$ in the formulation weakly of the problem (P_L) combined with the hypothesis ellipticity of M, Hölder's inequality and Lemma 1.3.

Remark 1.2. *We also emphasize that the legitimacy of the test functions taken follows from Theorem 4.2, because the truncation functions are Lipschitz and $w \in W_0^{1,2}(\Omega)$. Moreover, if we consider*

$$\varphi(t) = \frac{|t|^2\lambda t}{2\lambda+1} = \begin{cases} \frac{t^{2\lambda+1}}{2\lambda+1}, & t \geqslant 0; \\ \frac{-|t|^{2\lambda+1}}{2\lambda+1}, & t < 0. \end{cases}$$

Then $\frac{|T_k(w)|^{2\lambda}T_k(w)}{2\lambda+1}$ *is a valid test function. Indeed, note that*

$$\varphi'(t) = \begin{cases} t^{2\lambda}, & t \geqslant 0; \\ |t|^{2\lambda}, & t < 0. \end{cases}$$

Where, for $t < 0$ *we have*

$$\varphi'(t) = \frac{-(2\lambda+1)|t|^{2\lambda}}{2\lambda+1} \cdot (-1)$$

and, since $|t| = -t$*, when* $t < 0$ *which implies that* $\frac{d}{dt}|t| = -1$*. Moreover, we get*

$$\varphi'(0) = \lim_{t\to 0} \frac{\varphi(t) - \varphi(0)}{t - 0} = \lim_{t\to 0} \frac{\varphi(t)}{t} = 0$$

because

$$\varphi'(0) = \lim_{t\to 0^+} \frac{\varphi(t)}{t} = \lim_{t\to 0^+} \frac{t^{2\lambda+1}}{2\lambda+1} \cdot \frac{1}{t} = \frac{1}{2\lambda+1} \cdot \lim_{t\to 0^+} t^{2\lambda} = 0$$

and

$$\varphi'(0) = \lim_{t\to 0^-} \frac{\varphi(t)}{t} = \lim_{t\to 0^-} \frac{-|t|^{2\lambda+1}}{2\lambda+1} \cdot \frac{1}{t} = \frac{1}{2\lambda+1} \cdot \lim_{t\to 0^-} |t|^{2\lambda} = 0,$$

thus φ *is class* C^1*. Since* $T_k(u) \in W_0^{1,2}(\Omega)$ *we conclude that* $\varphi = \frac{|T_k(w)|^{2\lambda}T_k(w)}{2\lambda+1}$ *is a legitimate test function.*

The starting we will assume that the source is a function $z \in L^m(\Omega)$ with $(2^*)' \leqslant m < \frac{N}{2}$.

Theorem 1.2. *Let* $z \in L^m(\Omega)$ *with* $(2^*)' \leqslant m < \frac{N}{2}$*. Then every solution* $w \in W_0^{1,2}(\Omega)$ *to problem* (P_L) *belongs to* $L^{m^{**}}(\Omega)$*. In addition, we have the following estimate*

$$||w||_{L^{m^{**}}} \leqslant C||z||_{L^m}$$

where $C = C(N, m, \alpha)$*.*

Proof. By considering $\varphi = \frac{|T_k(w)|^{2\lambda}T_k(w)}{2\lambda+1}$ as a test function in the weak formulation of the problem (P_L) with $\lambda > 0$, there follows that

$$\int_\Omega M(x)\nabla w \cdot \nabla T_k(w)|T_k(w)|^{2\lambda} = \frac{1}{2\lambda+1}\int_\Omega z|T_k(w)|^{2\lambda}T_k(w). \tag{1.2}$$

Using the ellipticity hypothesis of M for the left - hand side, so that

$$\begin{aligned} \int_\Omega M(x)\nabla w \cdot \nabla T_k(w)|T_k(w)|^{2\lambda} \geqslant \int_{|w|>k} M(x)\nabla w \cdot \nabla w|T_k(w)|^{2\lambda} &\geqslant \alpha\int_{|w|>k} |\nabla w|^2|T_k(w)|^{2\lambda} \\ &= \alpha\int_\Omega |\nabla T_k(w)|^2|T_k(w)|^{2\lambda}. \end{aligned} \tag{1.3}$$

Now, note that

$$\frac{|\nabla T_k(w)|^{(\lambda+1)2}}{(\lambda+1)^2} = |T_k(w)|^{2\lambda}|\nabla T_k(w)|^2. \tag{1.4}$$

Thus, replacing equality (1.9) in (1.3) and using the Sobolev embedding we obtain

$$\int_\Omega M(x)\nabla w \cdot \nabla T_k(w)|T_k(w)|^{2\lambda} \geqslant \frac{\alpha S^2}{(\lambda+1)^2}\Big(\int_\Omega |T_k(w)|^{(\lambda+1)2^*}\Big)^{\frac{2}{2^*}}. \tag{1.5}$$

By using Hölder inequality with exponent m in the right - hand side of (1.2), we get

$$\frac{1}{2\lambda+1}\int_\Omega z|T_k(w)|^{2\lambda}T_k(w) \leqslant \frac{1}{2\lambda+1}||z||_{L^m}\Big(\int_\Omega |T_k(w)|^{(2\lambda+1)m'}\Big)^{\frac{1}{m'}} \tag{1.6}$$

Putting together estimates (1.5) and (1.6) we have

$$\frac{\alpha S^2}{(\lambda+1)^2}\Big(\int_\Omega |T_k(w)|^{(\lambda+1)2^*}\Big)^{\frac{2}{2^*}} \leqslant \frac{1}{2\lambda+1}||z||_{L^m}\Big(\int_\Omega |T_k(w)|^{(2\lambda+1)m'}\Big)^{\frac{1}{m'}}.$$

Choose λ such that $(\lambda+1)2^* = (2\lambda+1)m'$, that is, $\lambda = \frac{-mN+2N-2m}{4m-2N}$ and using that $\frac{2}{2^*} - \frac{1}{m'} = \frac{1}{m^{**}}$ and $(\lambda+1)2^* = \frac{mN}{N-2m}$ we have

$$\Big(\int_\Omega |T_k(w)|^{m^{**}}\Big)^{\frac{1}{m^{**}}} \leqslant C||z||_{L^m}$$

How the truncation function converges to identity when $k \to \infty$, so by Fatou's Lemma conclude

$$||w||_{L^{m^{**}}} \leqslant C||z||_{L^m},$$

where $C = C(\alpha, S, m, N)$.

□

We can now pass to the regularity of the solutions in the case where $z \in L^m(\Omega)$ with $m > \frac{N}{2}$.

Theorem 1.3. *Let $z \in L^m(\Omega)$ with $m > \frac{N}{2}$. Then every solution $w \in W_0^{1,2}(\Omega)$ to problem (P_L) is bounded. Moreover the estimate*

$$||w||_{L^\infty} \leqslant C||z||_{L^m}$$

holds, where $C = C(N, \alpha, m)$.

Proof. By considering $\varphi = G_k(w)$ as a test function in the weak formulation of problem (P_L). From the ellipticity hypothesis of M and Sobolev inequality we have

$$\alpha S^2\Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{2}{2^*}} \leqslant \alpha\int_\Omega |\nabla G_k(w)|^2 \leqslant \int_\Omega zG_k(w). \tag{1.7}$$

Applying the Hölder inequality in $\int_\Omega zG_k(w)$ with exponent 2^* and $(2^*)'$, we get

$$\begin{aligned}\int_\Omega zG_k(w) &\leqslant \Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{2}{2^*}}\Big(\int_{A_k} |z|^{(2^*)'}\Big)^{\frac{1}{(2^*)'}}\\ &\leqslant \Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{2}{2^*}}||z||_{L^m}\mathrm{meas}(A_k)^{[1-\frac{(2^*)'}{m}]\frac{1}{(2^*)'}}.\end{aligned} \tag{1.8}$$

Now combining the estimates (1.7) and (1.8) thus give

$$\alpha S^2\Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{2}{2^*}} \leqslant ||z||_{L^m}\Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{1}{2^*}}\mathrm{meas}(A_k)^{[1-\frac{(2^*)'}{m}]\frac{1}{(2^*)'}}$$

that is,

$$\Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{1}{2^*}} \leqslant ||z||_{L^m}\mathrm{meas}(A_k)^{[1-\frac{(2^*)'}{m}]\frac{1}{(2^*)'}}. \tag{1.9}$$

Again by Holder's inequality with exponent 2^* and $(2^*)'$ one has

$$\int_\Omega |G_k(w)| \leqslant \Big(\int_\Omega |G_k(w)|^{2^*}\Big)^{\frac{1}{2^*}} \mathrm{meas}(A_k)^{\frac{1}{(2^*)'}}$$

and so by (1.9) implies that

$$\int_\Omega |G_k(w)| \leqslant \frac{1}{\alpha S^2}||z||_{L^m}\mathrm{meas}(A_k)^{1+\frac{2}{N}+\frac{1}{m}}.$$

Therefore, by Lemma 1.3 with $\alpha = 1 + \frac{2}{N} - \frac{1}{m}$ and $C = \frac{1}{\alpha S^2}||z||_{L^m}$ gives the result.

□

1.3 Distributional solutions

In this section, we study the existence of distributional solutions to problem (P_L). The initial step is to consider the approximate problem, which the solution is guaranteed by the Theorem 1.1. Then we will show some a priori estimates, which made it possible to pass the limit in the approximate problem.

For the case where the z source of problem (P_L) does not belong to the dual of $W_0^{1,2}(\Omega)$, consider the following definition.

Definition 1.1. *Let $z \in L^m(\Omega)$ with $m < (2^*)'$, we say that a function $u \in W_0^{1,1}(\Omega)$ is distributional solution to the problem (P_L) if*

$$\int_\Omega M(x)\nabla w \cdot \nabla\varphi = \int_\Omega z\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

For the case where $z \in L^m(\Omega)$ with $1 \leqslant m < (2^*)'$, we will show the following results:

- If $z \in L^m(\Omega)$ with $1 < m < (2^*)'$ then $w \in W_0^{1,m^*}(\Omega)$. In this case, the result is obtained by taking the test function

$$[(1+|w_k|)^{2\gamma-1} - 1] \cdot \frac{T_\epsilon(w_k)}{\epsilon} \quad \text{with} \quad \gamma > \frac{1}{2}.$$

- If $z \in L^1(\Omega)$ then there exists a distributional solution $w \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$. Where this result is obtained by taking the test function

$$[(1+|w_k|)^{2\gamma-1} - 1] \cdot \frac{T_\epsilon(w_k)}{\epsilon} \quad \text{with} \quad \gamma < \frac{1}{2}.$$

In the following, we will make an observation, which will establish some important facts about these test functions taken.

Remark 1.3. *Let $b(t) = [(1+|t|)^{2\gamma-1} - 1]$.*

(1) If $\gamma < \frac{1}{2}$, then $b(t) < 0$ and $|b(t)| \leqslant 1$. In fact, on one hand since $1+|t| > 1 \implies (1+|t|)^{2\gamma-1} < 1 \iff (1+|t|)^{2\gamma-1} - 1 < 0$, that is $b(t) < 0$. On the other hand, since $1+|t| > 1 \implies 0 < \frac{1}{1+|t|} < 1 \implies 0 < \frac{1}{(1+|t|)^{1-2\gamma}} < 1$, hence

$$-1 < (1+|t|)^{2\gamma-1} - 1 < 0 \implies |b(t)| \leqslant 1.$$

(2) *If* $\gamma > \frac{1}{2}$*, then* $b(t) > 0$*. Indeed, as* $1 + |t| > 1 \implies (1 + |t|)^{2\gamma-1} > 1 \iff (1 + |t|)^{2\gamma-1} - 1 > 0$*, that is,* $b(t) > 0$*.*

(3) *b is locally Lipschitz continuous. In fact, note that*

$$b'(t) = (2\gamma - 1)sgn(t) = (2\gamma - 1)(1 + |t|)^{2\gamma-2} \quad \forall t \neq 0.$$

On one hand

$$b'(0) = \lim_{t\to 0} b'(t) = (2\gamma - 1)\lim_{t\to 0}(1 + |t|)^{2(\gamma-1)},$$

on the other hand

$$b'(0) = \lim_{t\to 0^+}(1 + |t|)^{2(\gamma-1)} = 1 \quad \text{and} \quad b'(0) = \lim_{t\to 0^-} -(1 + |t|)^{2(\gamma-1)} = -1.$$

Thus b is not differentiable in $t = 0$*.*

However, b' *is continuous in intervals* $]-\infty, 0]$ *and* $[0, +\infty[$*. So that* $t > 0$ *we have:*

For Similarly, taking $b|_{[0,t]} : [0, t] \to \mathbb{R}$*, by the Mean Value Theorem, there exists* $c_t \in (0, t)$ *such that*

$$|b(t) - b(0)| = |b'(c_t)||t| \leqslant M|t|$$

where $M = \sup_{c_t\in(0,t)} |b'(c_t)|$ *which exists, therefore* $b|_{[0,t]} : [0, t] \to \mathbb{R}$ *is continuous. Similarly, for* $t < 0$ *we have* $b|_{[t,0]} : [t, 0] \to \mathbb{R}$ *is continuous. Therefore b is locally Lipschitz.*

(4) *Let us remember that, since the truncation function is Lipschitz, if* $u \in W_0^{1,2}(\Omega)$*, then by Theorem 4.2 we have* $T_k(u) \in W_0^{1,2}(\Omega)$*. Moreover, since b is locally lipschitz, by Corollary 4.1 there follows that*

$$\nabla\big(b(u)T_k(u)\big) = b'(u)\frac{T_k(u)}{\varepsilon}\nabla u + \frac{T_k'(u)}{\varepsilon}b(u)\nabla u,$$

where

$$b'(u_k) = \begin{cases} (2\gamma - 1)(1 + |u_k|)^{2\gamma-2}, & u > 0 \\ -(2\gamma - 1)(1 + |u_k|)^{2\gamma-2}, & u < 0. \end{cases}$$

In other words, $b'(u_k)sgn(u_k) = (2\gamma - 1)(1 + |u_k|)^{2\gamma-2}$*.*

1.3.1 A priori estimates

Let $z \in L^m(\Omega)$ with $1 \leqslant m < (2^*)'$. We consider the following approximate problem

$$\begin{cases} -\text{div}(M(x)\nabla w_k) = z_k & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_L'}$$

where $z_k \in L^\infty(\Omega)$ such that $z_k \to z$ in $L^m(\Omega)$ and $||z_k||_{L^m} \leqslant ||z||_{L^m}$. The existence of solution w_k for every k, follows from Theorem 1.1. Moreover w_k belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Lemma 1.4. *Let* $z \in L^m(\Omega)$ *with* $1 < m < (2^*)'$*. Then the sequence of the solutions* w_k *to problems* (P_L') *satisfies*

$$||w_k||_{L^{m^{**}}} + ||w_k||_{W_0^{1,m^*}} \leqslant C$$

where $C = C(S, \alpha, m, ||z||_{L^1}, ||z||_{L^m})$*.*

Proof. Initially we will prove that the sequence w_k is bounded in $L^{m^{**}}(\Omega)$.

By consider $\varphi = b(w_k)\frac{T_\epsilon(w_k)}{\epsilon}$ as a test function in (P'_L), where $b(w_k) = [(1+|w_k|)^{2\gamma-1}-1]$, $\gamma > \frac{1}{2}$ and $\epsilon > 0$.

$$\int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\frac{T_\epsilon(w_k)}{\epsilon} + \int_\Omega M(x)\nabla w_k \cdot \nabla w_k \frac{T'_\epsilon(w_k)}{\epsilon} b(w_k) = \int_\Omega z_k b(w_k)\frac{T_\epsilon(w_k)}{\epsilon}.$$

Note that, since $\gamma > \frac{1}{2}$ there follows that $b(w_k) > 0$. Thus, using the ellipticity hypothesis of M we get

$$0 \leqslant \alpha \int_\Omega |\nabla w_k|^2 b(w_k)\frac{T'_\epsilon(w_k)}{\epsilon} \leqslant \int_\Omega M(x)\nabla w_k \cdot \nabla w_k b(w_k)\frac{T'_\epsilon(w_k)}{\epsilon}.$$

Hence, discarding the positive term and talking $\epsilon \to \infty$ there follows that

$$\int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\mathrm{sgn}(w_k) \leqslant \int_\Omega z_k b(w_k)\mathrm{sgn}(w_k). \tag{1.10}$$

Developing the term on the right - hand side, so that

$$\int_\Omega z_k b(w_k)\mathrm{sgn}(w_k) \leqslant \int_\Omega |z|[(1+|w_k|)^{2\gamma-1} - 1] \leqslant \int_\Omega |z|[(1+|w_k|)^{2\gamma-1} + 1]$$

by Hölder's inequality with exponent m we get

$$\int_\Omega z_k b(w_k)\mathrm{sgn}(w_k) \leqslant ||f||_{L^1} + ||f||_{L^m}\Big(\int_\Omega (1+|w_k|)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}. \tag{1.11}$$

Now, using the ellipticity hypothesis of M and Sobolev Embedding in the left-hand side of (1.10), we have

$$\alpha \int_\Omega |\nabla w_k|^2 (2\lambda-1)(1+|w_k|)^{2\lambda-2} \leqslant \int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\mathrm{sgn}(w_k). \tag{1.12}$$

Note that

$$\Big|\frac{\nabla(1+|w_k|)^\lambda}{\lambda}\Big|^2 = (1+|w_k|)^{2\lambda-2}\cdot|\nabla|w_k||^2,$$

hence

$$\frac{\alpha(2\lambda-1)}{\lambda^2}\int_\Omega |\nabla(1+|w_k|)^\lambda|^2 \leqslant \int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\mathrm{sgn}(w_k)$$

and by the Sobolev Embedding we get

$$\frac{\alpha S(2\lambda-1)}{\lambda^2}\Big(\int_\Omega [1+|w_k|)^\lambda]^{2^*}\Big)^{\frac{2}{2^*}} \leqslant \int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\mathrm{sgn}(w_k).$$

By (1.10) combining the above inequality with (1.11) we have

$$\frac{\alpha S(2\lambda-1)}{\lambda^2}\Big(\int_\Omega (1+|w_k|)^\lambda)^{2^*}\Big)^{\frac{2}{2^*}} \leqslant ||f||_{L^1} + ||f||_{L^m}\Big(\int_\Omega (1+|w_k|)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}. \tag{1.13}$$

It is natural to choose an adequate λ in order to guarantee that certain crucial exponents coincide. Indeed, fixing λ such that $\lambda 2^* = (2\lambda-1)m'$, that is, $\lambda = \frac{m^{**}}{2^*}$ there follows that

$$\frac{\alpha S(2\lambda-1)}{\lambda^2}\Big(\int_\Omega (1+|w_k|)^{m^{**}}\Big)^{\frac{2}{2^*}} \leqslant ||f||_{L^1} + ||f||_{L^m}\Big(\int_\Omega (1+|w_k|)^{m^{**}}\Big)^{\frac{1}{m'}}.$$

Since $\frac{2}{2^*} > \frac{1}{m'}$ we obtain

$$\int_\Omega |w_k|^{m^{**}} \leqslant \int_\Omega |1 + w_k|^{m^{**}} \leqslant C, \tag{1.14}$$

where $C = C(S, \alpha, m, ||z||_{L^1}, ||z||_{L^m})$.

Now we will prove that the sequence w_k is bounded in $W_0^{1,m^*}(\Omega)$. By combining (1.11) and (1.12) we have

$$\begin{aligned} ||f||_{L^1} + ||f||_{L^m}\Big(\int_\Omega (1 + |w_k|)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}} &\geqslant \alpha \int_\Omega |\nabla w_k|^2 (2\lambda - 1)(1 + |w_k|)^{2\lambda-2} \\ &= \alpha(2\lambda - 1)\int_\Omega \frac{|\nabla w_k|^2}{(1 + |w_k|)^{2(1-\lambda)}} \end{aligned}$$

by (1.14) there follows that

$$\int_\Omega \frac{|\nabla w_k|^2}{(1 + |w_k|)^{2(1-\lambda)}} \quad \text{is bounded.}$$

Suppose that $\lambda < 1$ and $1 < m < (2^*)'$. Let $1 \leqslant q < 2$, writing

$$\int_\Omega |\nabla w_k|^q = \int_\Omega \frac{|\nabla w_k|^q}{(1 + |w_k|)^{2(1-\lambda)\frac{q}{2}}}(1 + |w_k|)^{2(1-\lambda)\frac{q}{2}}$$

by Hölder's inequality with exponent $\frac{2}{q}$ we get

$$\int_\Omega |\nabla w_k|^q \leqslant \Big(\int_\Omega \frac{|\nabla w_k|^q}{(1 + |w_k|)^{2(1-\lambda)\frac{q}{2}}}\Big)^{\frac{q}{2}}\Big(\int_\Omega (1 + |w_k|)^{\frac{(1-\lambda)2q}{2-q}}\Big)^{1-\frac{q}{2}}$$

taking $m^* = q$ and $\frac{(1-\lambda)2q}{2-q} = m^{**}$. Thus by estimates (1.14) we have

$$||w_k||_{W_0^{1,m^*}} \leqslant C.$$

$\square$

Lemma 1.5. *Let $z \in L^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$||w_k||_{W_0^{1,q}(\Omega)} < C \quad \textit{where} \quad q < \frac{N}{N-1}.$$

Proof. By taking $\varphi = b(w_k)\frac{T_\epsilon(w_k)}{\epsilon}$ as a test function in (P_L'), where $b(w_k) = [(1+|w_k|)^{2\gamma-1} - 1]$, $\gamma < \frac{1}{2}$ and $\epsilon > 0$. There follows that

$$\int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\frac{T_\epsilon(w_k)}{\epsilon} + \int_\Omega M(x)\nabla w_k \cdot \nabla w_k \frac{T_\epsilon'(w_k)}{\epsilon} b(w_k) = \int_\Omega z_k b(w_k)\frac{T_\epsilon(w_k)}{\epsilon},$$

hence

$$\int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\frac{T_\epsilon(w_k)}{\epsilon} = \int_\Omega z_k b(w_k)\frac{T_\epsilon(w_k)}{\epsilon} - \int_\Omega M(x)\nabla w_k \cdot \nabla w_k \frac{T_\epsilon'(w_k)}{\epsilon} b(w_k),$$

discarding the positive term and then taking $\epsilon \to \infty$, we get

$$\int_\Omega M(x)\nabla w_k \cdot \nabla w_k b'(w_k)\ sgn(w_k) \geqslant \int_\Omega z_k b(w_k) sgn(w_k)$$

that is

$$(2\gamma-1)\int_\Omega M(x)\nabla w_k\cdot\nabla w_k(1+|w_k|)^{2\gamma-2}\geqslant\int_\Omega z_k b(w_k)sgn(w_k).$$

Since $(2\gamma-1)<0$ and $b(w_k)<0$, implies that

$$\frac{1}{2\gamma-1}\int_\Omega z_k b(w_k)sgn(w_k)>0.$$

Thus, by the ellipticity hypothesis of M and $|b(w_k)|\leqslant 1$, we have

$$\int_\Omega\frac{|\nabla w_k|^2}{(1+|w_k|)^{2(\gamma-1)}}\leqslant C||z||_{L^m}. \tag{1.15}$$

Now note that

$$\int_\Omega|\nabla w_k|^q=\int_\Omega\frac{|\nabla w_k|^q}{(1+|w_k|)^{2(1-\gamma)q/2}}(1+|w_k|)^{2(1-\gamma)q/2}.$$

By Hölder inequality with exponent $\frac{2}{q}$ on the right side of the equality above and Sobolev embedding on the left - hand we obtain

$$S^q\Big(\int_\Omega|w_k|^{q^*}\Big)^{\frac{q}{q^*}}\leqslant\int_\Omega|\nabla w_k|^q\leqslant\Big(\int_\Omega\frac{|\nabla w_k|^2}{(1+|w_k|)^{2(1-\gamma)}}\Big)^{\frac{q}{2}}\Big(\int_\Omega(1+|w_k|)^{\frac{(1-\gamma)2q}{2-q}}\Big)^{1-\frac{q}{2}}$$

by inequality (1.15) implies that

$$S^q\Big(\int_\Omega|w_k|^{q^*}\Big)^{\frac{q}{q^*}}\leqslant\int_\Omega|\nabla w_k|^q\leqslant C\Big(\int_\Omega(1+|w_k|)^{\frac{(1-\gamma)2q}{2-q}}\Big)^{1-\frac{q}{2}}$$

thence

$$S^q\Big(\int_\Omega|w_k|^{q^*}\Big)^{\frac{q}{q^*}}\leqslant\int_\Omega|\nabla w_k|^q\leqslant C+C\Big(\int_\Omega|w_k|^{\frac{(1-\gamma)2q}{2-q}}\Big)^{1-\frac{q}{2}}. \tag{1.16}$$

Specify γ such that $\frac{(1-\gamma)2q}{2-q}=q^*$, that is, $\gamma=\frac{q(N-2)}{2(N-q)}$. Since $\gamma<\frac{1}{2}$, that implies $q<\frac{N}{N-1}$. In this way

$$S^q\Big(\int_\Omega|w_k|^{q^*}\Big)^{\frac{q}{q^*}}\leqslant C+C\Big(\int_\Omega|w_k|^{q^*}\Big)^{1-\frac{q}{2}}$$

and so the sequence $\int_\Omega|w_k|^{q^*}$ is bounded. Hence the right - hand side of (1.16) is uniformly bounded and $\int_\Omega|\nabla w_k|^q$ too. □

1.3.2 Existence of distributional solutions

From the estimates obtained above, we can show the existence of distributional solutions for cases in which $z\in L^m(\Omega)$ with $1<m<(2^*)'$, and $z\in L^1(\Omega)$.

Theorem 1.4. *Let $z\in L^m(\Omega)$ where $1<m<(2^*)'$. Then there exists a distributional solution $w\in W_0^{1,m^*}(\Omega)$ for (P_L).*

Proof. According to Lemma (1.4), there exists a subsequence that we will denote $\{w_k\}$ and w in $W_0^{1,m^*}(\Omega)$ such that

$$w_k\rightharpoonup w \text{ weakly in } W_0^{1,m^*}(\Omega) \text{ and } \nabla w_k\rightharpoonup\nabla w \text{ weakly in } \big(L^{m^*}(\Omega)\big)^N.$$

Thus, since $M(x)$ is symmetric matrix and bounded, by taking the limit on the weak formulation of problem (P'_L), we obtain

$$\int_\Omega M(x)\nabla w \cdot \nabla\varphi = \int_\Omega z\varphi \quad \forall\varphi \in C_0^\infty(\Omega).$$

Therefore, w is a distributional solution for (P_L). □

Theorem 1.5. *Let $z \in L^1(\Omega)$. Then there exists a distributional solution $w \in W_0^{1,q}(\Omega)$ for (P_L) where $q < \frac{N}{N-1}$.*

Proof. By Lemma 1.5, there exists a subsequence that we will denote $\{w_k\}$ and $w \in W_0^{1,q}(\Omega)$ such that

$$w_k \rightharpoonup w \text{ weakly in } W_0^{1,q}(\Omega) \text{ and } \nabla w_k \rightharpoonup \nabla w \text{ weakly in } \left(L^q(\Omega)\right)^N.$$

Thus, since $M(x)\nabla\varphi \in \left(L^{q'}(\Omega)\right)^N$, by taking the limit on the weak formulation of problem (P'_L), we obtain

$$\int_\Omega M(x)\nabla w \cdot \nabla\varphi = \int_\Omega z\varphi \quad \forall\varphi \in C_0^\infty(\Omega).$$

Therefore, w is a distributional solution for (P_L). □

Chapter 2

Regularizing Effect for a Class of Maxwell-Schrödinger Systems

2.1 Preliminaries

In this section we provide certain technical results that are used in the present chapter. Despite that not all of them are new, for the convenience of the reader we decided to keep some proofs. We begin by defining for $\tau > 0$ the following truncations

$$g_\tau(x,t,s) = \begin{cases} g(x,t,s) & \text{if } \ |g(x,t,s)| \leqslant \tau; \\ \tau \ \mathrm{sgn}(g(x,t,s)) & \text{if } \ |g(x,t,s)| > \tau \end{cases}$$

$$h_\tau(x,t,s) = \begin{cases} h(x,t,s) & \text{if } \ |h(x,t,s)| \leqslant \tau; \\ \tau \ \mathrm{sgn}(h(x,t,s)) & \text{if } \ |h(x,t,s)| > \tau. \end{cases}$$

From the definitions, it is clear that

$$\begin{cases} |g_\tau(x,t,s)| = \min\{\tau, |g(x,t,s)|\}; \\ |h_\tau(x,t,s)| = \min\{\tau, |h(x,t,s)|\}. \end{cases} \tag{2.1}$$

Moreover, it is clear that h_τ and g_τ satisfy the hypotheses (P_2) and (P_4) respectively.

Concerning problem (P), in our manuscript, we will consider the following definition of solution:

Definition 2.1. *We say that (u,v) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, for $p > 1$, is a distributional solution for problem* (P) *if and only if*

$$\begin{cases} \int_\Omega M(x)\nabla u \cdot \nabla\varphi + \int_\Omega g(x,u,v)\varphi = \int_\Omega f\varphi \qquad \forall\ \varphi \in C_c^\infty(\Omega) \\ \int_\Omega M(x)\nabla v \cdot \nabla\psi = \int_\Omega h(x,u,v)\psi \qquad \forall\ \psi \in C_c^\infty(\Omega). \end{cases} \tag{P_F}$$

where $g(.,u,v) \in L^1_{loc}(\Omega)$ and $h(.,u,v) \in L^1_{loc}(\Omega)$.

From now on, $C > 0$ will denote a general constant, which may vary from line to line, and may depend only on the data, i.e., $C = C\big(\alpha, \beta, \theta, \Omega, c_1, c_2, d_1, d_2, r, N\big) > 0$. Sometimes, in order to simplify the notation, we will denote $C = C(f) > 0$ in order to stress that C depends on $\|f\|_{L^m}$.

As a fist step, we provide certain convergence results which will be employed in the proofs of our main results. As it will be clear, by cropping certain technical details from Theorem 0.2 and 0.1 we simplify their proofs. We just mention that the difference between cases (ii) and (iii), comes from the fact that, naturally, when f is less regular, the estimates on the mixed terms become also less regular, see Lemmas 2.4 and 2.5.

Lemma 2.1. *Let $f \in L^m(\Omega)$ with $m \geqslant 1$, $\{u_k\}$ bounded in $W_0^{1,s}(\Omega)$ and $\{v_k\}$ bounded in $W_0^{1,t}(\Omega)$, where $s \geqslant t \geqslant \frac{N(\theta+1)}{N+\theta+1}$ and h, g be two Carathéodory functions satisfying (P$_1$) and (P$_3$). Then, there exist u and v in $W_0^{1,t}(\Omega)$ such that, up to subsequences relabeled the same:*

(i) *If* $\int_{\{|u_k|>n\}} |u_k|^{r-1}|v_k|^{\theta+1} \leqslant C \int_{\{|u_k|>n\}} |f|$,
then $g(x,u_k,v_k) \to g(x,u,v)$ in $L^1(\Omega)$.

(ii) *If* $\int_{\{|v_k|>n\}} |u_k|^{r}|v_k|^{\theta+1} \leq C$ *and* $\{|u_k|^r\}$ *is uniformly integrable*
then $h(x,u_k,v_k) \to h(x,u,v)$ in $L^1(\Omega)$.

(iii) *If* $\int_{\{|v_k|>n\}} |u_k|^{r}|v_k|^{1-\theta} \leq C$ *and* $\{|u_k|^r\}$ *is uniformly integrable*
then $h(x,u_k,v_k) \to h(x,u,v)$ in $L^1(\Omega)$.

Proof. (i) Since $\frac{N(\theta+1)}{N+\theta+1} \leqslant t < s$, then $\theta+1 \leqslant t^* < s^*$, and also $u, v \in W^{1,t}(\Omega)$ such that, up to subsequences

$$\begin{cases} u_k \rightharpoonup u \ \text{ weakly in } \ W_0^{1,s}(\Omega) \\ v_k \rightharpoonup v \ \text{ weakly in } \ W_0^{1,t}(\Omega), \\ u_k \to u \quad \text{in } L^{t^*}(\Omega), \ \text{and a.e in } \Omega \\ v_k \to v \quad \text{in } L^{t^*}(\Omega), \ \text{and a.e in } \Omega. \end{cases} \tag{2.2}$$

Of course, we also have $g(x,u_k,v_k) \to g(x,u,v)$ a.e in Ω. Further, from (P$_1$) and *(i)*, given $E \subset \Omega$, a measurable set, there follows that

$$\begin{aligned} \int_E |g(x,u_k,v_k)| &\leqslant c_2 \int_E |u_k|^{r-1}|v_k|^{\theta+1} \\ &= c_2 \int_{E\cap\{|u_k|\leqslant n\}} |u_k|^{r-1}|v_k|^{\theta+1} + c_2 \int_{E\cap\{|u_k|>n\}} |u_k|^{r-1}|v_k|^{\theta+1} \\ &\leqslant c_2 n^{r-1} \int_E |v_k|^{\theta+1} + c_2 \int_{\{|u_k|>n\}} |u_k|^{r-1}|v_k|^{\theta+1} \\ &\leqslant c_2 n^{r-1} \int_E |v_k|^{\theta+1} + C \int_{\{|u_k|>n\}} |f|. \end{aligned}$$

Claim 1. Given $\sigma > 0$, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ with $\text{meas}(E) < \delta$, such that

$$C \int_{\{|u_k|>n\}} |f| \leqslant \frac{\sigma}{2} \quad \text{and} \quad n^{r-1} \int_E |v_k|^{\theta+1} \leqslant \frac{\sigma}{2}.$$

Note that, by Hölder's inequality

$$C \int_{\{|u_k|>n\}} |f| \leqslant C\|f\|_{L^m} \cdot \text{meas}(\{|u_k| > n\})^{\frac{1}{m'}}.$$

Moreover, by considering C_0 such that

$$C_0 > \int_\Omega |u_k|^{t^*} \geqslant \int_{\{|u_k|>n\}} |u_k|^{t^*} > \int_{\{|u_k|>n\}} n^{t^*} = n^{t^*}\text{meas}(\{|u_k| > n\})$$

hence

$$\text{meas}(\{|u_k| > n\}) < \frac{C_0}{n^{t^*}} \to 0, \text{ when } n \to \infty.$$

That is, for all $\sigma_1 > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$, we have

$$\text{meas}(\{|u_k| > n\}) \leqslant \sigma_1, \ \forall \ k \in \mathbb{N}.$$

Thus, by taking $\sigma_1 = \frac{\sigma}{2C(\|f\|_{L^m}+1)}$, for $n > n_0$ fixed we have that

$$C \int_{\{|u_k|>n_0\}} |f| \leqslant \frac{\sigma}{2}.$$

On the other hand, as $\theta + 1 < t^*$, there follows that

$$\|v_k - v\|_{L^{\theta+1}}^{\theta+1} \leqslant C\|v_k - v\|_{L^{t^*}}^{\theta+1} \to 0, \text{ when } k \to \infty.$$

In this way, by the Vitali Theorem, see 4.4, there exists $\delta > 0$ such that $\text{meas}(E) < \delta$ implies that

$$\int_E |v_k|^{\theta+1} < n_0^{r-1}\sigma_2, \quad \forall \ \sigma_2.$$

In this fashion, by taking $\sigma_2 = \frac{\sigma}{2n_0^{r-1}}$, we get

$$\int_E |v_k|^{\theta+1} < \frac{\sigma}{2}, \text{ proving our claim.}$$

At this point, let us stress that by **Claim 1**, we have

$$\exists \ \delta > 0; \quad \text{meas}(E) < \delta \ \text{ implies } \ \int_E |g(x, u_k, v_k)| < \sigma, \ \ \forall \ \sigma > 0,$$

so that, by the Vitali Theorem, we get $g(x, u_k, v_k) \to g(x, u, v)$ in $L^1(\Omega)$.

(ii) Now, remark that by (2.2) we have $h(x, u_k, v_k) \to h(x, u, v)$ a.e in Ω, it up to subsequences relabeled the same. Moreover, for $\sigma > 0$, given $E \subset \Omega$, by hypothesis (P_3) combined with *(i)* we have that

$$\begin{aligned}
\int_E |h(x, u_k, v_k)| &\leqslant d_2 \int_E |u_k|^r |v_k|^\theta \\
&= d_2 \int_{E\cap\{|v_k|\leqslant n\}} |u_k|^r |v_k|^\theta + d_2 \int_{E\cap\{|v_k|>n\}} |u_k|^r |v_k|^{\theta+1-1} \\
&\leqslant d_2 n^\theta \int_E |u_k|^r + \frac{d_2}{n} \int_{\{|v_k|>n\}} |u_k|^r |v_k|^{\theta+1} \\
&\leqslant d_2 n^\theta \int_E |u_k|^r + \frac{d_2 C}{n},
\end{aligned}$$

for all $n \in \mathbb{N}$. In particular, by taking n_0 such that $\frac{C}{n_0} < \frac{\sigma}{2d_2}$, we arrive at

$$\int_E |h(x, u_k, v_k)| \leqslant d_2 n_0^\theta \int_E |u_k|^r + \frac{\sigma}{2}.$$

However, since $\{|u_k|^r\}$ is by hypothesis uniformly integrable, there exist $\delta > 0$ for which, if $\text{meas}(E) < \delta$, one has

$$\int_E |u_k|^r < \frac{\sigma}{2d_2 n_0^\theta}, \quad \forall \ k, n \in \mathbb{N},$$

and then

$$\int_E |h(x,u_k,v_k)| < \sigma,$$

so that $h(x,u_k,v_k) \to h(x,u,v)$ in $L^1(\Omega)$, by the Vitali Theorem.

(iii) This proof is very similar to the last one, nevertheless, in the current case, the argument for the second convergence, is more delicate since now we lost some regularity of the estimates of the mixed terms. In any case, observe that for $\gamma = 1-2\theta$, since $\int_{\{|v_k|>n\}} |u_k|^r |v_k|^{1-\theta} \le C$ one has that

$$\begin{aligned}
\int_E |h(x,u_k,v_k)| &\leqslant d_2 \int_E |u_k|^r |v_k|^\theta \\
&= d_2 \int_{E\cap\{|v_k|\leqslant n\}} |u_k|^r |v_k|^\theta + d_2 \int_{E\cap\{|v_k|>n\}} |u_k|^r |v_k|^{\theta+\gamma-\gamma} \\
&\leqslant d_2 n^\theta \int_E |u_k|^r + \frac{d_2}{n^\gamma} \int_{\{|v_k|>n\}} |u_k|^r |v_k|^{\theta+\gamma} \\
&\leqslant d_2 n^\theta \int_E |u_k|^r + \frac{d_2 C}{n^\gamma}, \quad \text{for all } n \in \mathbb{N},
\end{aligned}$$

so that by repeating the same argument as in *(ii)* we prove that $\{h(x,u_k,v_k)\}$ is uniformly integrable and the result follows once more by the Vitali Theorem.

$\square$

Lemma 2.2. *Let $u,v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $u \geqslant 0$ a.e. in Ω, and let $\varphi_\varepsilon(t) = (t+\varepsilon)^\gamma - \varepsilon^\gamma$ be a Lipschitz function for all $t>0$, where $\varepsilon>0$, $0<\gamma<1$ and $H : \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying*

$$|H(x,t,s)| \leqslant C|t|^{\sigma_1}|s|^{\sigma_2}, \ \textit{for } \sigma_i > 0, i = 1,2.$$

Then

(a) $\gamma \int_\Omega |\nabla u|^2 u^{\gamma-1} \leqslant \lim_{\varepsilon\to0^+} \int_\Omega \nabla u \cdot \nabla\varphi_\varepsilon(u)$,

(b) $\int_\Omega H(x,u,v)u^\gamma = \lim_{\varepsilon\to0^+} \int_\Omega H(x,u,v)\varphi_\varepsilon(u)$,

where $\varphi_\varepsilon = \varphi_\varepsilon(u)$

Proof. *(a)* Remark that $u(x) > 0$ a.e. in Ω_+, so that by setting

$$\omega_u = \begin{cases} u^{\gamma-1} & \text{a.e. in } \Omega_+; \\ +\infty & \text{a.e. in } \Omega\backslash\Omega_+, \end{cases}$$

ω_u is measurable and well–defined. Moreover, since $|\nabla u| = 0$ a.e. in $\Omega\backslash\Omega_+$, by using the real extended line arithmetic rules, we have $|\nabla u|^2\omega_u = 0$ a.e. in $\Omega\backslash\Omega_+$. Thus,

$$\int_\Omega |\nabla u|^2 \omega_u = \int_{\Omega_+} |\nabla u|^2 u^\gamma.$$

Now, observe that $0 \leqslant (u+\varepsilon)^{\gamma-1} < u^{\gamma-1}$ and $(u+\varepsilon)^{\gamma-1} \to \omega_u$ a.e. in Ω when $\varepsilon\to0^+$, there follows from the Fatou Lemma that

$$\begin{aligned}
\gamma \int_{\Omega_+} |\nabla u|^2 u^{\gamma-1} &\leqslant \liminf_{\varepsilon\to0^+} \gamma \int_\Omega |\nabla u|^2 (u+\varepsilon)^{\gamma-1} \\
&= \liminf_{\varepsilon\to0^+} \int_\Omega \nabla u \cdot \nabla\varphi_\varepsilon(u).
\end{aligned}$$

(b) Note that

$$
\begin{aligned}
|H(x,u,v)[(u+\varepsilon)^{\gamma}-\varepsilon^{\gamma}]| &\leqslant |H(x,u,v)|[(u+\varepsilon)^{\gamma}+\varepsilon^{\gamma}] \\
&\leqslant c_1|u|^{\sigma_1}|v|^{\sigma_2}[(u+\varepsilon)^{\gamma}+\varepsilon^{\gamma}].
\end{aligned}
$$

Suppose $0<\varepsilon\leqslant 1$, we have

$$
|H(x,u,v)[(u+\varepsilon)^{\gamma}-\varepsilon^{\gamma}]| \leqslant C\|u\|_{L^\infty}^{\sigma_1}\|v\|_{L^\infty}^{\sigma_2}[(\|u\|_{L^\infty}+1)+1].
$$

Moreover,

$$
H(x,u,v)[(u+\varepsilon)^{\gamma}-\varepsilon^{\gamma}] \to H(x,u,v)u^{\gamma}, \quad \text{a.e. in} \quad \Omega, \text{ when } \quad \varepsilon\to 0^+.
$$

Thus, by the Lebesgue Dominated Convergence Theorem

$$
\int_\Omega H(x,u,v)u^{\gamma} = \lim_{\varepsilon\to 0^+}\int_\Omega H(x,u,v)\varphi_\varepsilon(u).
$$

□

Remark 2.1. *It is clear that by arguing in an analogous manner to the proof of Lemma, since $v>0$ a.e. in Ω we also have the validity of*

(a) $\gamma\int_\Omega |\nabla v|^2 v^{\gamma-1} \leqslant \lim_{\varepsilon\to 0^+}\int_\Omega \nabla v\cdot\nabla\varphi_\varepsilon(v)$,

(b) $\int_\Omega H(x,u,v)v^{\gamma} = \lim_{\varepsilon\to 0^+}\int_\Omega H(x,u,v)\varphi_\varepsilon(v)$.

Now, by using hipotheses (P_1)-(P_5), and a standard argument based on the Schauder Fixed Point Theorem, we obtain existence of solutions to preliminary version of (P). We emphasize that to construct a well-defined operator whose fixed points are weak solutions, we use (P_2).

Proposition 2.1. *Let $\Phi\in L^\infty(\Omega)$. Then there exists a weak solution $(u,v)\in W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ of the system*

$$
\begin{cases}
-div(M(x)\nabla u)+g_\tau(x,u,v)=\Phi & in\ \Omega\\
-div(M(x)\nabla v)=h_\tau(x,u,v)+\frac{1}{\tau} & in\ \Omega\\
\ \ u=v=0 \quad on\ \partial\Omega.
\end{cases} \tag{P_A}
$$

Proof. Fix $\zeta\in W_0^{1,2}(\Omega)$. We will show that there exists $u=S(\zeta)\in W_0^{1,2}(\Omega)$ such that

$$
\int_\Omega M(x)\nabla u\cdot\nabla\varphi+\int_\Omega g_\tau(x,u,\zeta)\varphi=\int_\Omega \Phi\varphi \qquad \forall\ \varphi\in W_0^{1,2}(\Omega). \tag{I}
$$

From the classical PDE theory the linear problem (P_L) has a unique weak solution. That is, there exists an unique $w\in W_0^{1,2}(\Omega)$ such that

$$
\int_\Omega M(x)\nabla w\cdot\nabla\varphi=\int_\Omega z\varphi=\langle z,\varphi\rangle_{W^{-1,2},W_0^{1,2}} \qquad \forall\ \varphi\in W_0^{1,2}(\Omega).
$$

Thus the solution operator $G:W^{-1,2}(\Omega)\to W_0^{1,2}(\Omega)$, given by $G(z)=w$, is **linear** and **continuous**.

Now consider the operator $F_\tau : L^2(\Omega) \to L^2(\Omega)$ defined by $F_\tau(z) = \Phi - g_\tau(\cdot, z, \zeta)\ \ \forall\, z \in L^2(\Omega)$ where $\zeta \in W_0^{1,2}(\Omega)$ is fixed. Note that, F_τ **is well defined.** Indeed,

$$|F_\tau(z)| = |\Phi - g_\tau(\cdot, z, \zeta)| \leqslant |\Phi| + |g_\tau(\cdot, z, \zeta)| \leqslant |\Phi| + \tau, \quad \forall\, z \in L^2(\Omega),$$

there follows that

$$\begin{aligned}\|F_\tau(z)\|_{L^2}^2 &\leqslant \int_\Omega [|\Phi| + \tau]^2 \\ &\leqslant \|\Phi\|_{L^\infty}^2 \mathrm{meas}(\Omega) + 2\tau\|\Phi\|_{L^\infty}\mathrm{meas}(\Omega) + \tau^2\mathrm{meas}(\Omega) < +\infty.\end{aligned}$$

Moreover, F_τ **is continuous**. In fact, let $z_j, z \in L^2(\Omega)$ such that $\|z_j - z\|_{L^2} \to 0$ when $j \to \infty$, we must that $\|F_\tau(z_j) - F_\tau(z)\|_{L^2} \to 0$ when $j \to \infty$.

On one hand, since $z_j \to z$ in $L^2(\Omega)$, then up to a subsequence $z_j \to z$ a.e. in Ω, accordingly as g_τ is continuous in z, we get $g_\tau(x, z_j, \zeta) \to g_\tau(x, z, \zeta)$ a.e. in Ω. On the other hand

$$|g_\tau(x, z_j, \zeta)| \leqslant \tau, \quad \text{a.e. } x \in \Omega \ \ \forall\, j \in \mathbb{N}.$$

Hence, by the Lebesgue Dominated Convergence Theorem

$$\|g_\tau(x, z_j, \zeta) - g_\tau(x, z, \zeta)\|_{L^2} \to 0, \ \text{when } j \to \infty.$$

Thus

$$\begin{aligned}\|F_\tau(z_j) - F_\tau(z)\|_{L^2}^2 &= \int_\Omega |\Phi - g_\tau(x, z_j, \zeta) - \Phi + g_\tau(x, z, \zeta)|^2 \\ &= \|g_\tau(x, z_j, \zeta) - g_\tau(x, z, \zeta)\|_{L^2}^2 \to 0, \quad \text{when } j \to \infty.\end{aligned}$$

The idea now is to show that the operator

$$G \circ F_\tau : L^2(\Omega) \to L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

is in the hypotheses of Schauder's Fixed Point Theorem, in order to conclude that there is $u \in L^2(\Omega)$ such that $u = G \circ F_\tau(u)$.

$$u = (i_k \circ G \circ i_c \circ F_\tau)(u) = G(F_\tau(u))$$

where

$$\begin{aligned}&i_c : L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega) \ \ \text{continuous embedding and} \\ &i_k : W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \ \ \text{compact embedding (Rellich-Kondrachov).}\end{aligned}$$

Since $u \in Im(G)$, then $u \in W_0^{1,2}(\Omega)$. Thus, given $\widetilde{u} \in L^2(\Omega)$ there exists $w \in W_0^{1,2}(\Omega)$ such that $w = G(F_\tau(\widetilde{u}))$.

$$\int_\Omega M(x)\nabla w \cdot \nabla\varphi = \int_\Omega \Phi\varphi - \int_\Omega g_\tau(x, \widetilde{u}, \zeta)\varphi \ \ \forall\, \varphi \in W_0^{1,2}(\Omega).$$

Taking $\varphi = w$ in the weak formulation of the fisrt equation of (2.1) and using the hypothesis (P_5) and Hölder's inequality, there follows that

$$\begin{aligned}\alpha\int_\Omega |\nabla w|^2 &\leqslant \int_\Omega |\Phi||w| + \int_\Omega |g_\tau(x, \widetilde{u}, \zeta)||w| \leqslant \|\Phi\|_{L^\infty}\int_\Omega |w| + \tau\int_\Omega |w| \\ &\leqslant \|\Phi\|_{L^\infty}\|w\|_{L^2}\mathrm{meas}(\Omega)^{1/2} + \tau\|w\|_{L^2}\mathrm{meas}(\Omega)^{1/2} = [\|\Phi\|_{L^\infty} + \tau]\,\mathrm{meas}(\Omega)^{1/2}\|w\|_{L^2},\end{aligned}$$

hence

$$\|\nabla w\|_{L^2}^2 \leqslant [\|\Phi\|_{L^\infty} + \tau]\,\mathrm{meas}(\Omega)^{1/2}\|w\|_{L^2}. \tag{2.3}$$

In particular by Poincaré inequality we get

$$\|w\|_{W_0^{1,2}} \leqslant R_1 \quad \text{and} \quad \|w\|_{L^2} \leqslant R_1, \tag{2.4}$$

where $R_1 = C\,[\|\Phi\|_{L^\infty} + \tau]\,\mathrm{meas}(\Omega)^{1/2}$ and C is Poincaré constant.

Let $\widetilde{B} = \widetilde{B}(0, R_1) = \{w \in L^2(\Omega);\ \|w\|_{L^2} \leqslant R_1\}$. Then of estimate in $w \in L^2(\Omega)$, namely (2.4), there follows that $G(F_\tau(\widetilde{B})) \subset \widetilde{B}$.

It is clean that $G \circ F_\tau$ is **continuous**, because G and F_τ are continuous. It remains to be shown that $G \circ F_\tau(\widetilde{B})$ is **relatively compact** in $L^2(\Omega)$, i.e., $\overline{G(F_\tau(\widetilde{B}))}$ is compact in $L^2(\Omega)$. In fact, since F_τ is continuous and bounded, for each $k \in \mathbb{N}$, we get

$$F_\tau(\widetilde{B}) \text{ is closed in } L^2(\Omega) \text{ and } F_\tau(\widetilde{B}) \text{ is bounded in } L^2(\Omega),$$

by continuous embedding we have

$$i_c(F_\tau(\widetilde{B})) = F_\tau(\widetilde{B}) \text{ is closed in } L^2(\Omega) \text{ and } i_c(F_\tau(\widetilde{B})) = F_\tau(\widetilde{B}) \text{ is bounded in } L^2(\Omega).$$

As G is linear and continuous, there follows that

$$G(F_\tau(\widetilde{B})) \text{ is closed in } W_0^{1,2}(\Omega) \text{ and } G(F_\tau(\widetilde{B})) \text{ is bounded in } W_0^{1,2}(\Omega).$$

Thus by Rellich-Kondrachov we obtain $\overline{G(F_\tau(\widetilde{B}))}$ is compact in $L^2(\Omega)$.
Therefore, by Schauder's Fixed Point Theorem, there exists $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega M(x)\nabla u \cdot \nabla\varphi + \int_\Omega g_\tau(x, u, \zeta)\varphi = \int_\Omega \Phi\varphi \quad \forall\ \varphi \in W_0^{1,2}(\Omega).$$

Furthermore, by an analogous argument, given $u \in W_0^{1,2}(\Omega)$ fixed, there exists $\eta = T(u) \in W_0^{1,2}(\Omega)$ satisfying

$$\int_\Omega M(x)\nabla\eta \cdot \nabla\psi = \int_\Omega \Big(h_\tau(x, u, \eta) + \frac{1}{\tau}\Big)\psi \quad \forall\ \psi \in W_0^{1,2}(\Omega).$$

Taking $\psi = \eta$ in the weak formulation of the second equation of (2.1) given above and using the hypothesis (P_5), Hölder's and inequality we get

$$\begin{aligned}\alpha\int_\Omega |\nabla\eta|^2 = \int_\Omega \Big(h_\tau(x, u, \eta) + \frac{1}{\tau}\Big)\eta &\leqslant \tau\|\eta\|_{L^2}\mathrm{meas}(\Omega)^{1/2} + \frac{1}{\tau}\|\eta\|_{L^2}\mathrm{meas}(\Omega)^{1/2} \\ &= \Big(\frac{\tau^2 + 1}{\tau}\Big)\|\eta\|_{L^2}\mathrm{meas}(\Omega)^{1/2}\end{aligned}$$

thus

$$\|\eta\|_{W_0^{1,2}} \leqslant R_2 \quad \text{and} \quad \|\eta\|_{L^2} \leqslant R_2, \tag{2.5}$$

where $R_2 = C\Big(\frac{\tau^2+1}{\alpha\tau}\Big)\mathrm{med}(\Omega)^{1/2}$ and C is Poincaré constant.

We are now going to prove that $T \circ S$ satisfies the assumptions of Schauder's fixed point Theorem. By estimates (2.4) and (2.5) we consider

$$B = B(0, R) = \{u \in L^2(\Omega);\ \|u\|_{L^2} \leqslant\ R\}$$

where $R = \max\{R_1, R_2\}$. Note that, B is invariant by $T \circ B$ i.e. $T(S(B)) \subset B$. In fact, given $w \in B$, we have that $S(w) = u$, solution of first equation of (P_A), so by (2.4) we get $S(w) = u \in B$, thus for this u we have $\eta = T(u) = T(S(w)) \in B$, because holds (2.5).

Besides, $T \circ S$ is **continuous**. In fact, let $\zeta_k \to \zeta$ in $L^2(\Omega)$. Since u_k is bounded in $W_0^{1,2}(\Omega)$ we have

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega) \quad \text{up to a subsequence,}$$
$$u_k \to u \quad \text{in } L^2(\Omega) \quad \text{up to a subsequence.}$$

As $u_k = S(\zeta_k)$, there follows that

$$\int_\Omega M(x)\nabla u_k \cdot \nabla\varphi + \int_\Omega g_\tau(x, u_k, \zeta_k)\varphi = \int_\Omega \Phi\varphi \quad \forall\, \varphi \in W_0^{1,2}(\Omega). \tag{2.6}$$

Note that by weak convergence of $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ we get

$$\int_\Omega M(x)\nabla u_k \cdot \nabla\varphi \to \int_\Omega M(x)\nabla u \cdot \nabla\varphi.$$

Since $u_k \to u$ and $\zeta_k \to \zeta$ in $L^2(\Omega)$, up to a subsequence we have

$$u_k(x) \to u(x) \quad \text{and } \zeta_k(x) \to \zeta(x) \text{ a.e} \quad \text{in } \Omega.$$

As g_τ is continuous, there follow that $g_\tau(x, u_k, \zeta_k) \to g_\tau(x, u, \zeta)$ a.e in Ω. Moreover

$$|g_\tau(x, u_k, \zeta_k)| \leqslant \tau \quad \text{a.e} \quad \text{in } \Omega \quad \forall k \in \mathbb{N},$$

and by Dominated Convergence Theorem $\|g_\tau(x, u_k, \zeta_k) - g_\tau(x, u, \zeta)\|_{L^2} \to 0$, when $k \to \infty$. Thus, passing the limit in (2.6) we get

$$\int_\Omega M(x)\nabla u \cdot \nabla\varphi + \int_\Omega g_\tau(x, u, \zeta)\varphi = \int_\Omega \Phi\varphi \quad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

That is, $u = S(\zeta)$ which implies that S is continuous.

Since $\eta_k = T(u_k)$, by estimates (2.5) we have

$$\eta_k \rightharpoonup \eta \quad \text{in } W_0^{1,2}(\Omega) \quad \text{up to a subsequence,}$$
$$\eta_k \to \eta \quad \text{in } L^2(\Omega) \quad \text{up to a subsequence.}$$

Thus passing the limit in

$$\int_\Omega M(x)\nabla \eta_k \cdot \nabla\psi = \int_\Omega \Big(h_\tau(x, u, \eta) + \frac{1}{\tau}\Big)\psi \quad \forall\, \psi \in W_0^{1,2}(\Omega),$$

we obtain

$$\int_\Omega M(x)\nabla \eta \cdot \nabla\psi = \int_\Omega \Big(h_\tau(x, u, \eta) + \frac{1}{\tau}\Big)\psi \quad \forall\, \psi \in W_0^{1,2}(\Omega).$$

That is, $\eta = T(u) = T(S(\zeta))$, thus $T \circ S$ is continuous.

And to finish we have $T \circ S(B)$ is **relatively compact** in $L^2(\Omega)$, i.e., $\overline{T(S(B))}$ is compact in $L^2(\Omega)$. Indeed, as we saw earlier B is invariant by $T \circ S$ which implies that given $\widetilde{w}_k \in B$ we get $\widetilde{\eta}_k = T(\widetilde{u}_k) = T(S(\widetilde{w}_k)) \in B$. Since $\widetilde{\eta}_k$ and $\widetilde{u}_k$ are limited in $W_0^{1,2}(\Omega)$ there exist u and $\eta \in W_0^{1,2}(\Omega)$ such that

$$\widetilde{u}_k, \widetilde{\eta}_k \rightharpoonup \widetilde{u}, \widetilde{\eta} \quad \text{in } W_0^{1,2}(\Omega) \quad \text{up to a subsequence,}$$
$$\widetilde{u}_k, \widetilde{\eta}_k \to \widetilde{u}, \widetilde{\eta} \quad \text{in } L^2(\Omega) \quad \text{up to a subsequence.}$$

Thus as $T \circ S$ is continuous there follows that $||\widetilde{\eta}_k - \widetilde{\eta}||_{L^2} \to 0$ when $k \to \infty$. Therefore, by Schauder's Fixed Point Theorem, there exists $v \in B \subset W_0^{1,2}(\Omega)$ such that $v = T(S(v)) = T(u)$. $\square$

2.2 Approximate Problem

This section comprehends the heart of the contributions on the present chapter. Indeed, we explore the choice of tailored test functions for a favorable approximate version of (P) in order to obtain certain key estimates, see Lemmas 2.4 and 2.5.

Nevertheless, we start by obtaining estimates for a first version of approximate problem, i.e., that the solutions given by Proposition 2.1 are bounded in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. We stress that, surprisingly, in order to obtain L^∞ estimates for v, we have to impose that $\theta < \frac{4}{N-2}$.

Lemma 2.3. *Let $\Phi \in L^m(\Omega)$ with $m \geqslant 1$ and let (u, v) be the solution of system (P_A) given by Proposition 2.1. Then there exists a constant $C > 0$ such that*

(i) if $\Phi \in L^m(\Omega)$ for $m \geqslant (2^*)'$ then

$$\|u\|_{W_0^{1,2}} + \|v\|_{W_0^{1,2}} \leqslant C\|\Phi\|_{L^m};$$

(ii) if $\Phi \in L^m(\Omega)$ for $m > \frac{N}{2}$ and $0 < \theta < \frac{4}{N-2}$ then

$$\|u\|_{L^\infty} + \|v\|_{L^\infty} \leqslant C\|\Phi\|_{L^m}.$$

Proof. By taking $\varphi = u$ in the weak formulation of the first equation of (P_A), and by combining the ellipticity of M with Hölder's and Sobolev's inequalities we end up with

$$\begin{aligned} \|u\|_{L^{2^*}} &\leqslant C\|\Phi\|_{L^m}, \\ \|\nabla u\|_{L^2}^2 &\leqslant C\|\Phi\|_{L^m} \text{ and} \\ \int_\Omega g_\tau(x,u,v)u &\leqslant C\|\Phi\|_{L^m}. \end{aligned} \tag{2.7}$$

Further, we claim that

$$\int_\Omega |u|^r|v|^{\theta+1} \leqslant C\|\Phi\|_{L^m}. \tag{2.8}$$

In fact, by (P_2') and the definition of g_τ, see (2.1), it is clear that

$$\int_\Omega g_\tau(x,u,v)u \geqslant \int_{\{|g|\leqslant\tau\}} g(x,u,v)u,$$

and thus, from (2.7), we get

$$\int_{\{|g|\leqslant\tau\}} g(x,u,v)u \leqslant C\|\Phi\|_{L^m}.$$

In this way, if we recall (P_1), by taking $\tau \to +\infty$ in the latter inequality, as a direct application of Fatou's lemma we arrive at

$$\int_\Omega |u|^r|v|^{\theta+1} \leqslant \int_\Omega g(x,u,v)u \leqslant C\|\Phi\|_{L^m},$$

proving our claim.

Moreover, by taking $\psi = v$ in the weak formulation of the second equation of (P_A), by combining (P_3) and the ellipticity of M, we have

$$\begin{aligned} \alpha\int_\Omega |\nabla v|^2 \leqslant \int_\Omega \Big(h_\tau(x,u,v) + \frac{1}{\tau}\Big)v &\leqslant \int_\Omega |h_\tau(x,u,v)||v| + \frac{1}{\tau}\int_\Omega |v| \\ &\leqslant d_1\int_\Omega |u|^r|v|^{\theta+1} + \frac{C}{\tau}||v||_{W_0^{1,2}}. \end{aligned}$$

Thus, taking $\tau \to 0$ and by (2.8) there follows that $\int_\Omega |\nabla v|^2 \leqslant C\|\Phi\|_{L^m}$ and hence, by combining (2.7) and the last inequality, we get

$$\int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2 \leqslant C\|\Phi\|_{L^m},$$

which proves (i).

Now we proceed to the $L^\infty(\Omega)$ estimates. For this, let us recall the definition of one the standard Stampacchia's truncation, $G_k(s) = (|s|+k)^+\text{sign}(s)$. Then, we take $\varphi = G_k(u)$ in the weak formulation of the first equation of (P_A), obtaining

$$\int_\Omega M(x)\nabla u \cdot \nabla u G_k'(u) + \int_\Omega g_\tau(x,u,v)G_k(u) = \int_\Omega \Phi G_k(u). \tag{2.9}$$

In addition, notice that clearly, there holds

$$\begin{aligned}\int_\Omega g_\tau(x,u,v)G_k(u) &= \int_{\{|u|>k\}} g_\tau(x,u,v)G_k(u)\\ &= \int_{\{u>k\}} g_\tau(x,u,v)(u-k) + \int_{\{u<-k\}} g_\tau(x,u,v)(u+k).\end{aligned}$$

Moreover, as a straightforward consequence of (2.1) and the definition of $G_k(.)$, we have

$$\int_\Omega g_\tau(x,u,v)G_k(u) \geqslant 0.$$

Thus, from the latter inequality, by using the ellipticity of M and Hölder's inequality on the right-hand side of (2.9), we get

$$\alpha\int_\Omega |\nabla G_k(u)|^2 \leqslant \int_\Omega \Phi G_k(u) \leqslant \Big(\int_{A_K^u} |\Phi|^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}}\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}},$$

where $A_k^u = \{|u|>k\}$.

Additionally, recall that by Sobolev's and Hölder's inequalities there follows

$$\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{2}{2^*}} \leqslant \Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}}\|\Phi\|_{L^m}\text{meas}(A_k^u)^{[1-\frac{2N}{(N+2)m}]\frac{N+2}{2N}},$$

so that by the latter inequalities we arrive at

$$\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{2}{2^*}} \leqslant C|\Phi\|_{L^m}\text{meas}(A_k^u)^{[1-\frac{2N}{(N+2)m}]\frac{N+2}{2N}}. \tag{2.10}$$

Moreover, by Hölder's inequality and (2.10), we have

$$\int_\Omega |G_k(u)| = \int_{A_k^u} |G_k(u)| \leqslant \text{meas}(A_k^u)^{\frac{N+2}{2N}}\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}} \leqslant C\|\Phi\|_{L^m}\text{meas}(A_k^u)^\alpha.$$

Where $\alpha = 1 + \frac{2}{N} - \frac{1}{m} > 1$, since $m > \frac{N}{2}$. Hence, by Lemma 6.2 in [6] p. 49,

$$u \in L^\infty(\Omega) \text{ and } \|u\|_{L^\infty} \leqslant C\|\Phi\|_{L^m},$$

where we stress that the restriction of θ was not used.

Finally, we handle the $L^\infty(\Omega)$ estimates on v. Indeed, if we take $\psi = G_k(v)$ in the second equation of (P_A), by combining (P_3), the ellipticity of M and the $L^\infty(\Omega)$ estimate for u, we obtain

$$\alpha \int_\Omega |\nabla G_k(v)|^2 \leqslant \int_\Omega \Big(h_\tau(x,u,v) + \frac{1}{\tau}\Big) G_k(v) \leqslant \int_{A_k^v} |u|^r |v|^\theta |G_k(v)| + \frac{1}{\tau}\int_\Omega |G_k(v)|$$
$$\leqslant d_1 \|u\|_{L^\infty}^r \int_{A_k^v} |v|^\theta |G_k(v)| + \frac{2C}{\tau} ||v||_{W_0^{1,2}} \mathrm{meas}(\Omega), \ \text{ where } A_k^v = \{|v| > k\}.$$

Taking $\tau \to +\infty$ we have

$$\alpha \int_\Omega |\nabla G_k(v)|^2 \leqslant d_1 \|u\|_{L^\infty}^r \int_{A_k^v} |v|^\theta |G_k(v)|.$$

Further, by a straightforward combination of Sobolev's and Hölder's inequalities we get

$$\alpha S^2 \left(\int_\Omega |G_k(v)|^{2^*}\right)^{\frac{2}{2^*}} \leqslant d_1 \|u\|_{L^\infty}^r \Big(\int_{A_k^v} |v|^{(2^*)'\theta}\Big)^{\frac{1}{(2^*)'}} \left(\int_\Omega |G_k(v)|^{2^*}\right)^{\frac{1}{2^*}},$$

so that

$$S^2 \left(\int_\Omega |G_k(v)|^{2^*}\right)^{\frac{1}{2^*}} \leqslant \Big(\int_{A_k^v} |v|^{(2^*)'\theta}\Big)^{\frac{1}{(2^*)'}},$$

and then, by applying once again the Hölder inequality on the right-hand side, with the exponents $\frac{2^*}{(2^*)'\theta}$ and $\frac{2^*}{2^*-\theta(2^*)'}$, we arrive at

$$\left(\int_\Omega |G_k(v)|^{2^*}\right)^{\frac{1}{2^*}} \leqslant C \mathrm{meas}(A_k^v)^{\left[\frac{2^*-\theta(2^*)'}{2^*}\right]\cdot\frac{1}{(2^*)'}} \Big(\int_\Omega |v|^{2^*}\Big)^{\frac{\theta}{2^*}}$$
$$\leqslant C \|v\|_{W_0^{1,2}}^\theta \mathrm{meas}(A_k^v)^{\left[\frac{2^*-\theta(2^*)'}{2^*}\right]\cdot\frac{1}{(2^*)'}}.$$

Nonetheless, recall that

$$\int_\Omega |G_k(v)| = \int_{A_k^v} |G_k(v)| \leqslant \mathrm{meas}(A_k^v)^{\frac{1}{(2^*)'}} \left(\int_\Omega |G_k(v)|^{2^*}\right)^{\frac{1}{2^*}}.$$

Thence, the combination between the latter inequalities implies

$$\int_\Omega |G_k(v)| = \int_{A_k^v} |G_k(v)| \leqslant \mathrm{meas}(A_k^v)^\alpha,$$

where $\alpha = \frac{1}{(2^*)'}\left[1 + \frac{2^*-\theta(2^*)'}{2^*}\right] > 1$, since $\frac{4}{N-2} > \theta$. Therefore, once again by invoking Lemma 1.3, we have $v \in L^\infty(\Omega)$ and

$$\|v\|_{L^\infty} \leqslant C\|\Phi\|_{L^m}.$$

□

With these tools at hand, we are able to prove existence of suitable solutions for a more favorable approximate version of our problem which will be explored in our investigation of (P).

Proposition 2.2. *Let $\{f_k\}$ be a sequence of $L^\infty(\Omega)$ functions strongly convergent to f in $L^m(\Omega)$, $m \geqslant 1$, for which $|f_k| \leqslant |f|$ a.e. in Ω. Then, there exists $(u_k, v_k) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, solution to*

$$\begin{cases} -div(M(x)\nabla u_k) + g(x, u_k, v_k) = f_k \\ -div(M(x)\nabla v_k) = h(x, u_k, v_k) + \frac{1}{\tau_k}, \\ \quad u_k = v_k = 0 \quad on \quad \partial\Omega, \end{cases} \tag{AP}$$

where $\tau_k > 0$ and $\tau_k \to +\infty$ if $k \to +\infty$. Moreover, if $f \geq 0$ a.e. in Ω then $u_k \geqslant 0$ a.e. in Ω and $v_k > 0$ a.e. in Ω.

Proof. Given $k > 0$ consider $\tau > (c_1 + d_1)C^{r+\theta}k^{r+\theta}$, where C is given in Lemma 2.3 and c_1, d_1 in (P_1), (P_3). Let us recall the standard truncation $T_k(s) = \max(-k, \min(s, k))$ and then take $f_k = T_k(f)$. Thus, for $\Phi = f_k$, by combining Proposition 2.1 and Lemma 2.3, we obtain a couple $(u_k, v_k) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ solution for (P_A).

Observe that $g_\tau(., u_k, v_k) = g(., u_k, v_k)$ and $h_\tau(., u_k, v_k) = h(., u_k, v_k)$ a.e. in Ω. As a matter of fact, from P_1, Lemma 2.3 item (ii) and the choice of f_k, we have

$$\begin{aligned} |g(x, u_k, v_k)| &\leqslant c_1|u|^{r-1}|v|^{\theta+1} \\ &\leqslant c_1 C^{r+\theta}\|f_k\|_{L^\infty}^{r+\theta} \\ &\leqslant c_1 C^{r+\theta}k^{r+\theta}. \end{aligned}$$

Thence, since by the choice of $\tau > c_1 C^{r+\theta}k^{r+\theta}$, from (2.1) we have that g_τ coincides with g. Analogously using the hypothesis P_3 we conclude that $h_\tau = h$.

Finally, remark that if $f \geqslant 0$ a.e. in Ω then $f_k \geqslant 0$ a.e. in Ω. Thus taking $\varphi = u_k^- = -\max(-u_k, 0)$ in the first equation of (AP), we have that

$$\int_\Omega M(x)\nabla[u_k^+ - u_k^-] \cdot \nabla u_k^- + \int_\Omega g(x, u_k, v_k)u_k^- = \int_\Omega f u_k^-,$$

by using the hypothesis (P_5) we get

$$-\alpha \int_\Omega |\nabla u_k^-|^2 + \int_\Omega g(x, u_k, v_k)u_k^- \geqslant \int_\Omega f u_k^-,$$

Now, note that

$$\int_\Omega g(x, u_k, v_k)u_k^- = \int_{\{u_k<0\}} g(x, u_k, v_k)u_k^- = -\int_{\{u_k<0\}} g(x, u_k, v_k)u_k \leqslant 0$$

where the last integral is negative, by hypothesis (P_2'). Which implies that

$$0 \geqslant -\alpha \int_\Omega |\nabla u_k^-|^2 \geqslant \int_\Omega f u_k^- \geqslant 0.$$

Thus

$$0 \leqslant \int_\Omega |\nabla u_k^-|^2 \leqslant 0,$$

thence

$$\|u_k^-\|_{W_0^{1,2}}^2 = \|\nabla u_k^-\|_{L^2}^2 = 0 \quad \Longrightarrow \quad u_k^- = 0.$$

Consequently we get $u_k \geq 0$ a.e. in Ω.

Now we prove that $v_k > 0$ a.e. in Ω. In fact, consider $w_k \in C^{1,\alpha}(\Omega)$, with $0 \leqslant \alpha < 1$, the solution of

$$\begin{cases} -\operatorname{div}(M(x)\nabla w_k) = \frac{1}{\tau_k} & \text{in } \Omega, \\ \quad w_k = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.11}$$

Remark that, since $M \in W^{1,\infty}(\Omega)$ the existence of this w_k is standard, for instance see Corollary 8.36 in [15]. Then, by a straightforward application of the Strong Maximum Principle of Vasquez in (2.11), see Theorem 4 in [20], we obtain that $w_k > 0$ in Ω. After that, let us stress that

$$-\operatorname{div}(M(x)\nabla v_k) = h(x, u_k, v_k) + \frac{1}{\tau_k} \geqslant -\operatorname{div}(M(x)\nabla w_k)$$

then by the Comparison Principle $v_k \geqslant w_k$ a.e. in Ω so that $v_k > 0$ a.e. in Ω. □

2.3 Estimates

At this point, we are finally ready to obtain a set of delicate uniform a priori estimates which play a key role in our results. First, we adress the "energectic case"

$$m \geqslant (r + \theta + 1)'.$$

Let us stress that we strongly use the fact that one of our approximate solutions of the decoupled system is strictly positive, i.e., v_k satisfies $v_k > 0$ a.e. in Ω.

Lemma 2.4. *Let $f \in L^m(\Omega)$ where $m \geqslant (r + \theta + 1)'$, and $f \geqslant 0$ a.e in Ω, $r > 1$ and $0 < \theta < 1$. Then*

$$\|u_k\|^2_{W_0^{1,2}} + \|v_k\|^2_{W_0^{1,2}} + \int_\Omega u_k^{r+\theta+1} + \int_\Omega u_k^r v_k^{\theta+1} \leqslant C\Big(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\Big), \tag{2.12}$$

where $C > 0$ and $\tau_k \to \infty$ if $k \to \infty$.

Proof. Let us take $\varepsilon > 0$ and by considering $\psi = u_k^{\theta+1}(v_k + \varepsilon)^{-\theta}$ as a test function in the second equation of (AP), after dropping the positive term, we get

$$\int_\Omega h(x, u_k, v_k) u_k^{\theta+1}(v_k + \varepsilon)^{-\theta} \leqslant (\theta + 1)\int_\Omega M(x)\nabla v_k \cdot \nabla u_k u_k^\theta (v_k + \varepsilon)^{-\theta}$$
$$- \theta \int_\Omega M(x)\nabla v_k \cdot \nabla v_k u_k^{\theta+1}(v_k + \varepsilon)^{-(\theta+1)}.$$

Then by (P_3), (P_4) and (P_5), it is clear that

$$\int_\Omega u_k^{r+\theta+1} v_k^\theta (v_k + \varepsilon)^{-\theta} + \alpha\theta \int_\Omega |\nabla v_k|^2 u_k^{\theta+1}(v_k + \varepsilon)^{-(\theta+1)}$$
$$\leqslant (\theta + 1)\beta \int_\Omega |\nabla v_k||\nabla u_k| u_k^\theta (v_k + \varepsilon)^{-\theta}. \tag{2.13}$$

Before letting $\epsilon \to 0$, let us consider $\Omega_1 = \{x \in \Omega;\ \frac{u_k}{v_k+\varepsilon} \leqslant 1\}$ and $\Omega_2 = \{x \in \Omega;\ \frac{u_k}{v_k+\varepsilon} > 1\}$, so that $\Omega = \Omega_1 \cup \Omega_2$. In this way, by Young's inequality

$$(\theta + 1)\beta \int_\Omega |\nabla v_k||\nabla u_k| u_k^\theta (v_k + \varepsilon)^{-\theta} \leqslant C_\eta (\theta + 1)\beta \int_\Omega |\nabla u_k|^2 + \eta(\theta + 1)\beta \int_\Omega |\nabla v_k|^2 \frac{u_k^{2\theta}}{(v_k + \varepsilon)^{2\theta}}$$

Now, remark that since $\frac{u_k}{v_k+\varepsilon} > 1$ in Ω_2 and $2\theta < \theta + 1$, there follows that

$$(\theta+1)\beta \int_\Omega |\nabla v_k||\nabla u_k| u_k^\theta (v_k+\varepsilon)^{-\theta} \leqslant C_\eta(\theta+1)\beta \int_\Omega |\nabla u_k|^2 + \eta(\theta+1)\beta \int_{\Omega_1} |\nabla v_k|^2 + \eta(\theta+1)\beta \int_{\Omega_2} |\nabla v_k|^2 \frac{u_k^{\theta+1}}{(v_k+\varepsilon)^{\theta+1}}$$

and hence, by combining the above estimate with (2.13), we have

$$\int_\Omega u_k^{r+\theta+1} v_k^\theta (v_k+\varepsilon)^{-\theta} + \alpha\theta \int_\Omega |\nabla v_k|^2 u_k^{\theta+1}(v_k+\varepsilon)^{-(\theta+1)} \leqslant C_\eta(\theta+1)\beta \int_\Omega |\nabla u_k|^2 + \eta(\theta+1)\beta \int_\Omega |\nabla v_k|^2 + \eta(\theta+1)\beta \int_\Omega |\nabla v_k|^2 \frac{u_k^{\theta+1}}{(v_k+\varepsilon)^{\theta+1}}.$$

Thus, by taking $\eta = \frac{\theta\alpha}{(\theta+1)\beta}$, and $C = \max\{C_\eta(\theta+1)\beta, \theta\alpha\}$ and by the Fatou Lemma, we arrive at

$$\int_\Omega u_k^{r+\theta+1} \leqslant \liminf_{\varepsilon\to 0} \int_\Omega u_k^{r+\theta+1} v_k^\theta (v_k+\varepsilon)^{-\theta} \leqslant C\left[\int_\Omega |\nabla v_k|^2 + \int_\Omega |\nabla u_k|^2\right]. \tag{2.14}$$

Now, let us proceed to the other estimates. Indeed, by choosing $\varphi = u_k$ in the first equation of (AP), it is clear that

$$\alpha \int_\Omega |\nabla u_k|^2 + \int_\Omega g(x,u_k,v_k)u_k \leqslant \int_\Omega f_k u_k. \tag{2.15}$$

In particular, by combining Hölder's inequality with

$$c_1 \int_\Omega u_k^r v_k^{\theta+1} \leqslant \int_\Omega g(x,u_k,v_k)u_k \leqslant \int_\Omega f u_k.$$

then

$$\int_\Omega u_k^r v_k^{\theta+1} \leqslant C\|f\|_{L^m}\|u_k\|_{L^{r+\theta+1}}, \tag{2.16}$$

where we strongly used that $f \in L^m(\Omega)$ for $m \geq (r+\theta+1)'$.

Further, by taking $\psi = v_k$ in the second equation of (AP), from(P_3) and (P_4), it is clear that

$$\alpha \int_\Omega |\nabla v_k|^2 \leqslant \int_\Omega h(x,u_k,v_k)v_k \leqslant d_2 \int_\Omega u_k^r v_k^{\theta+1} + \frac{1}{\tau_k}\int_\Omega v_k \leqslant C\|f\|_{L^m}\|u_k\|_{L^{r+\theta+1}} + \frac{C}{\tau_k}\|\nabla v_k\|_{L^2}.$$

However, it is clear that

$$\frac{C}{\tau_k}\|\nabla v_k\|_{L^2} \leq \frac{C_\alpha}{\tau_k^2} + \frac{\alpha}{2}\|\nabla v_k\|_{L^2}^2$$

whereas, by combining with (2.15), give us

$$\int_\Omega |\nabla u_k|^2 + \int_\Omega |\nabla v_k|^2 \leqslant C\big(\|f\|_{L^m}\|u_k\|_{L^{r+\theta+1}} + \frac{1}{\tau_k^2}\big). \tag{2.17}$$

In particular, by (2.14) and Young's inequality, we obtain

$$\|u_k\|_{L^{r+\theta+1}}^{r+\theta+1} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\bigg). \tag{2.18}$$

Therefore, gathering (2.16), (2.18) with (2.17), we finally have

$$\|u_k\|_{W_0^{1,2}}^2 + \|v_k\|_{W_0^{1,2}}^2 + \int_\Omega u_k^r v_k^{\theta+1} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\bigg),$$

where we stress that $\tau_k \to \infty$ if $k \to \infty$. □

Lemma 2.5. *Let $f \in L^m(\Omega)$ with $f \geqslant 0$, $\left(\frac{r-\theta+1}{1-2\theta}\right)' \leqslant m < (r+\theta+1)'$ and $0 < \theta < \frac{1}{2}$. Then*

$$\|u_k\|^p_{W_0^{1,p}} + \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^r v_k^{1-\theta} \leqslant C\left(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\right) \text{ and}$$
$$\|v_k\|^q_{W_0^{1,q}} \leqslant C\left(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\right)^{\frac{N}{N-2\theta}} \tag{2.19}$$

where $\tau_k \to \infty$ if $k \to \infty$, $p = \frac{2(r-\theta+1)}{r+\theta+1}$, $q = \frac{2N(1-\theta)}{N-2\theta}$, and $C > 0$. Moreover, if $r \geqslant \frac{N+2}{N-2}$, then $\{u_k\}$ and $\{v_k\}$ are bounded in $W_0^{1,q}(\Omega)$.

Proof. Let us consider $\varphi_\varepsilon = (u_k+\varepsilon)^\gamma - \varepsilon^\gamma$, $0 < \epsilon \leq 1$, $0 < \gamma < 1$, as a test function in the first equation of (AP), so that

$$\alpha\gamma\int_\Omega |\nabla u_k|^2(u_k+\varepsilon)^{\gamma-1} + \int_\Omega g(x,u_k,v_k)\left[(u_k+\varepsilon)^\gamma - \varepsilon^\gamma\right] \leqslant \int_\Omega f_k\left[(u_k+\varepsilon)^\gamma - \varepsilon^\gamma\right]. \tag{2.20}$$

Then, by using that $u_k \in L^\infty(\Omega)$ and the Dominated Convergence Theorem, it is clear that

$$\lim_{\varepsilon\to 0}\int_\Omega f_k[(u_k+\varepsilon)^\gamma - \varepsilon^\gamma] = \int_\Omega f_k u_k^\gamma.$$

Thus, by taking $\varepsilon \to 0$ in (2.20), by recalling Lemma 2.2 and (P_1), we end up with

$$\alpha\gamma\int_{\Omega_+^k} |\nabla u_k|^2 u_k^{\gamma-1} + c_1\int_\Omega u_k^{r-1+\gamma}v_k^{\theta+1} \leqslant \int_\Omega f u_k^\gamma < \infty, \tag{2.21}$$

which is finite for every fixed $k \in \mathbb{N}$, where $\Omega_+^k = \{u_k > 0\}$. Now, we address the set of estimates arising as a byproduct of **the coupling** between both equations of our system. We must stress that for us, the fact that $v_k > 0$ a.e. in Ω will be crucial. As a matter of fact, by considering $\psi = (v_k+\varepsilon)^\gamma - \varepsilon^\gamma$ in the second equation of (AP) it is clear that

$$\alpha\gamma\int_\Omega |\nabla v_k|^2(v_k+\varepsilon)^{\gamma-1} \leqslant \int_\Omega h(x,u_k,v_k)[(v_k+\varepsilon)^\gamma - \varepsilon^\gamma] + \frac{1}{\tau_k}\int_\Omega (v_k+\varepsilon)^\gamma - \varepsilon^\gamma. \tag{2.22}$$

Then, once more, by taking $\varepsilon \to 0$ and by recalling (P_3), from Lemma 2.2, we obtain

$$\alpha\gamma\int_\Omega |\nabla v_k|^2 v_k^{\gamma-1} \leqslant \int_\Omega h(x,u_k,v_k)v_k^\gamma \leqslant d_2\int_\Omega u_k^r v_k^{\gamma+\theta} + \frac{1}{\tau_k}\int_\Omega v_k^\gamma \tag{2.23}$$

Further, take $\psi = (u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta}$ in the second equation of (AP). By dropping the positive term, one has that

$$\begin{aligned}\int_\Omega h(x,u_k,v_k)(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta} &\leqslant (\gamma+\theta)\int_\Omega M(x)\nabla v_k\cdot\nabla u_k\ (u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta}\\ &\quad - \theta\int_\Omega M(x)\nabla v_k\cdot\nabla v_k(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}\end{aligned}$$

Then, by (P_3) and (P_4) it is clear that

$$\begin{aligned}c_1\int_\Omega u_k^r v_k^\theta(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta} &+ \alpha\theta\int_\Omega |\nabla v_k|^2(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}\\ &\leqslant \beta(\gamma+\theta)\int_\Omega |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta}\\ &= \beta(\gamma+\theta)\int_{\Omega_+^k} |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta},\end{aligned} \tag{2.24}$$

since $\nabla u_k = 0$ a.e. in the set $u_k = 0$, where $\Omega_+^k = \{u_k > 0\}$.

Given $\eta > 0$, by Young's inequality, we have that

$$\begin{aligned}\int_{\Omega_+^k} |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta} &\leqslant \int_{\Omega_+^k} |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}|v_k|^{-\theta}\\ &\leqslant \eta\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + C_\eta \int_{\Omega_+^k} |\nabla u_k|^2 (u_k+\varepsilon)^{2(\gamma+\theta-1)}.\end{aligned}$$

By combining the above inequality and (2.24) we get

$$\begin{aligned}c_1\int_\Omega u_k^r v_k^\theta (u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta} + \alpha\theta\int_\Omega |\nabla v_k|^2 (u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}\\ \leqslant \beta(\gamma+\theta)\eta\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + \beta(\gamma+\theta)C_\eta\int_{\Omega_+^k}|\nabla u_k|^2(u_k+\varepsilon)^{2(\gamma+\theta-1)}.\end{aligned}$$

At this point, it is natural to choose an adequate γ in order to guarantee that certain crucial exponents coincide, what allows us to explore the coupling between the equations. Indeed, by fixing $\gamma = 1-2\theta$ so that $2(\gamma+\theta-1) = \gamma-1 = -2\theta$, after dropping the positive term there follows that

$$\begin{aligned}c_1\int_\Omega u_k^r v_k^\theta (u_k+\varepsilon)^{1-\theta}(v_k+\varepsilon)^{-\theta} \leqslant \beta(1-\theta)\eta\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\\ + \beta(1-\theta)C_\eta\int_{\Omega_+^k}|\nabla u_k|^2(u_k+\varepsilon)^{-2\theta}.\end{aligned} \tag{2.25}$$

However, remark that by (2.21), for every k fixed, we have $|\nabla u_k|^2 u_k^{-2\theta} \in L^1(\Omega_+^k)$. Thus by taking $\varepsilon \to 0$ in (2.25), employing the Fatou Lemma combined with the Dominated Convergence Theorem, we arrive at

$$c_1\int_\Omega u_k^{r-\theta+1} \leqslant \beta(1-\theta)\eta\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + \beta(1-\theta)C_\eta\int_{\Omega_+^k}|\nabla u_k|^2 u_k^{-2\theta}. \tag{2.26}$$

Now, it is clear that

$$\int_{\{u_k\leqslant v_k\}} u_k^r v_k^{1-\theta} \leqslant \int_{\{u_k\leqslant v_k\}} u_k^r u_k^{-2\theta} v_k^{\theta+1} \leqslant \int_\Omega u_k^{r-2\theta}v_k^{\theta+1} \text{ and } \int_{\{u_k\geqslant v_k\}} u_k^r v_k^{1-\theta} \leqslant \int_\Omega u_k^{r-\theta+1}.$$

The latter estimates clearly guarantee that

$$\int_\Omega u_k^r v_k^{1-\theta} \leqslant \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^{r-2\theta}v_k^{\theta+1}. \tag{2.27}$$

Thus, by gathering (2.23), (2.26) and (2.27), and by using (2.21) twice, since $\gamma+\theta = 1-\theta$, there follows

$$\begin{aligned}\frac{\alpha(1-2\theta)}{d_2}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} &\leqslant \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^{r-1+\gamma}v_k^{\theta+1} + \frac{1}{\tau_k}\int_\Omega v_k^{1-2\theta}\\ &\leqslant \frac{\beta(1-\theta)\eta}{c_1}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + C\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\int_\Omega v_k^{1-2\theta},\end{aligned}$$

where we combined (2.21) and (2.26) in the right-hand side. If we consider $\eta = \frac{\alpha(1-2\theta)c_1}{2\beta(1-\theta)d_2}$, it is clear that

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\int_\Omega v_k^{1-2\theta} < \infty, \tag{2.28}$$

which is finite for every k fixed.

At this point, for the sake of simplicity, observe that, given $\varepsilon > 0$, by the locally Lipschitz Chain Rule, $|\nabla v_k|^2(v_k+\varepsilon)^{-2\theta} = \frac{1}{(1-\theta)^2}|\nabla(v_k+\varepsilon)^{1-\theta}|^2$. Thence, by combining the Sobolev Embedding, the Fatou Lemma and the Dominated Convergence Theorem, we have

$$\begin{aligned}\left(\int_\Omega v_k^{(1-\theta)2^*}\right)^{\frac{2}{2^*}} &\leqslant \liminf_{\varepsilon\to 0^+}\left(\int_\Omega (v_k+\varepsilon)^{(1-\theta)2^*}\right)^{\frac{2}{2^*}} \\ &\leqslant \liminf_{\varepsilon\to 0^+} C\left(\int_\Omega |\nabla(v_k+\varepsilon)^{1-\theta}|^2\right)^{\frac{2}{2^*}} \\ &= \int_\Omega |\nabla v_k|^2 v_k^{-2\theta},\end{aligned} \tag{2.29}$$

where in the latter estimates we used that $|\nabla v_k|^2 v_k^{-2\theta} \in L^1(\Omega)$ for every $k \in \mathbb{N}$, fixed. Moreover, observe that $\frac{(1-\theta)2^*}{1-2\theta} > 1$ and thus, by combining Hölder's inequality for $\frac{(1-\theta)2^*}{1-2\theta}$ and $\frac{(1-\theta)2N}{N+2-4\theta}$, with the last estimate, we arrive at

$$\begin{aligned}\int_\Omega v_k^{1-2\theta} &\leq C\bigg(\int_\Omega v_k^{(1-\theta)2^*}\bigg)^{\frac{1-2\theta}{(1-\theta)2^*}} \\ &\leq C\bigg(\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\bigg)^{\frac{1-2\theta}{2-2\theta}}.\end{aligned} \tag{2.30}$$

Further, from (2.28) and (2.30) with Young's inequality for $\frac{2-2\theta}{1-2\theta}$ and $2-2\theta$, we obtain

$$\begin{aligned}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} &\leqslant C\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k} C\bigg(\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\bigg)^{\frac{1-2\theta}{2-2\theta}} \\ &\leqslant C\bigg(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\bigg) + \frac{1}{2\tau_k}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\end{aligned}$$

which clearly guarantees that

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C\bigg(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\bigg). \tag{2.31}$$

Now, by (2.26), (2.23) and (2.31), we get

$$\begin{aligned}\int_\Omega u_k^{r-\theta+1} &\leqslant C\Big(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\Big) \\ &\leqslant C\Big(\|f\|_{L^m} ||u_k||^{1-2\theta}_{L^{(1-2\theta)m'}} + \frac{1}{\tau_k}\Big) \\ &\leqslant C\Big(\|f\|_{L^m} ||u_k||^{1-2\theta}_{L^{r-\theta+1}} + \frac{1}{\tau_k}\Big),\end{aligned}$$

where we used that $(1-2\theta)m' = \gamma m' \leqslant r-\theta+1$, for $m' \leqslant \frac{r-\theta+1}{1-2\theta}$. Further, by means of another application of the Young inequality, for $\frac{r-\theta+1}{1-2\theta}$ and $\frac{r-\theta+1}{r+\theta}$, after straightforward compensations, we end up with

$$\int_\Omega u_k^{r-\theta+1} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\bigg). \tag{2.32}$$

In particular, by (2.31), we have

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\bigg). \tag{2.33}$$

Further, by gathering (2.21), (2.27), with an analogous argument used to prove (2.33) we obtain

$$\int_\Omega u_k^r v_k^{1-\theta} \leq C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big).$$

As a final step, we will handle the coupling estimates for the gradients. Indeed, on one hand, observe that for $1 \leqslant p < 2$, by Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we get

$$\begin{aligned}\int_\Omega |\nabla u_k|^p = \int_\Omega \frac{|\nabla u_k|^p}{(u_k+\varepsilon)^{\theta p}}(u_k+\varepsilon)^{\theta p} &\leqslant \Big(\int_\Omega |\nabla u_k|^2 (u_k+\varepsilon)^{-2\theta}\Big)^{\frac{p}{2}} \cdot \Big(\int_\Omega (u_k+\varepsilon)^{\frac{2\theta p}{2-p}}\Big)^{\frac{2-p}{2}} \\ &= \Big(\int_{\Omega_+^k} |\nabla u_k|^2 (u_k+\varepsilon)^{-2\theta}\Big)^{\frac{p}{2}} \cdot \Big(\int_\Omega (u_k+\varepsilon)^{\frac{2\theta p}{2-p}}\Big)^{\frac{2-p}{2}}.\end{aligned}$$

Then, since by (2.21), $|\nabla u_k|^2 u_k^{-2\theta} \in L^1(\Omega_+^k)$, from the Lebesgue Convergence Theorem, we can take $\varepsilon \to 0$ so that

$$\int_\Omega |\nabla u_k|^p \leqslant \Big(\int_{\Omega_+^k} |\nabla u_k|^2 u_k^{-2\theta}\Big)^{\frac{p}{2}} \cdot \Big(\int_\Omega u_k^{\frac{2\theta p}{2-p}}\Big)^{\frac{2-p}{2}}.$$

By choosing $\frac{2\theta p}{2-p} = r-\theta+1$, i.e., $p = \frac{2(r-\theta+1)}{r+\theta+1}$, from (2.21) and (2.32), there follows that

$$\int_\Omega |\nabla u_k|^p \leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big).$$

Finally, by combining (3.26) and (2.33), we already know that

$$\Big(\int_\Omega v_k^{(1-\theta)2^*}\Big)^{\frac{2}{2^*}} \leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big). \tag{2.34}$$

Moreover, recalling that $v_k > 0$ a.e. in Ω, and then for $1 \leqslant q < 2$ by Hölder's inequality with exponent $\frac{2}{q}$, we get

$$\int_\Omega |\nabla v_k|^q \leqslant \Big(\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\Big)^{\frac{q}{2}} \cdot \Big(\int_\Omega v_k^{\frac{2\theta q}{2-q}}\Big)^{\frac{2-q}{2}}.$$

Hence, so that it is enough to choose $(1-\theta)2^* = \frac{2\theta q}{2-q}$, i.e., $q = \frac{2N(1-\theta)}{N-2\theta}$, and by (2.33) and (2.34), after straightforward computations, one arrives at

$$\int_\Omega |\nabla v_k|^q \leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big)^{\frac{N}{N-2\theta}}, \text{ where } C > 0.$$

In addition, remark that by the choice of q, if $r \geqslant \frac{N+2}{N-2}$, then $q < p$. Thus, by the gradient estimates obtained above, its clear that both $\{u_k\}$ and $\{v_k\}$ are bounded in $W_0^{1,q}(\Omega)$.

$\square$

2.4 Proof of Theorems 0.1 and 0.2

Theorem 0.1. *Let $f \in L^m(\Omega)$, where $f \geqslant 0$ a.e. in Ω, $m \geqslant (r+\theta+1)'$, $r > 1$, and $0 < \theta < \frac{4}{N-2}$. Then there exists a weak solution (u, v) for (P), with $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$, $u \geqslant 0$ a.e. in Ω and $v \in W_0^{1,2}(\Omega)$, $v \geqslant 0$ a.e in Ω.*

Proof. By Lemma 2.4, there exist $\{u_k\}$, $\{v_k\} \subset W_0^{1,2}(\Omega)$, and u, v in $W_0^{1,2}(\Omega)$ such that, up to subsequences relabeled the same,

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega),\ u_k \to u \text{ in } L^{s_1}(\Omega), \text{ and a.e. in } \Omega;$$
$$v_k \rightharpoonup v \text{ weakly in } W_0^{1,2}(\Omega),\ v_k \to v \text{ in } L^{s_2}(\Omega), \text{ and a.e. in } \Omega,$$

where $u \geqslant 0$, $v \geqslant 0$ a.e. in Ω, and

$$\|u\|^2_{W_0^{1,2}} + \|v\|^2_{W_0^{1,2}} + \int_\Omega u^{r+\theta+1} + \int_\Omega u^r v^{\theta+1} \leqslant C\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}},$$

and $s_1 < \max\{2^*, r+\theta+1\}$ and $s_2 < 2^*$.

In order to prove that the couple satisfies (P_F) it is enough to show that

$$g(x,u_k,v_k) \to g(x,u,v) \text{ in } L^1(\Omega) \text{ and } h(x,u_k,v_k) \to h(x,u,v) \text{ in } L^1(\Omega).$$

For this, take $\lambda > 0$, and consider $\varphi = \frac{T_\lambda(G_n(u_k))}{\lambda}$ as a test function in the first equation of (AP) so that

$$\int_\Omega M(x)\nabla u_k \cdot \nabla u_k G_n'(u_k)\frac{T_\lambda'(G_n(u_k))}{\lambda} + \int_\Omega g(x,u_k,v_k)\frac{T_\lambda(G_n(u_k))}{\lambda} \leq \int_{u_k>n} f_k. \tag{2.35}$$

However, since T_k and G_n are both monotone, by using (P_5) it is clear that

$$\int_\Omega M(x)\nabla u_k \cdot \nabla u_k G_n'(u_k)\frac{T_\lambda'(G_n(u_k))}{\lambda} \geqslant \alpha \int_\Omega |\nabla u_k|^2 G_n'(u_k)\frac{T_\lambda'(G_n(u_k))}{\lambda} \geqslant 0.$$

Further, by (P_2') and by the very definition of T_λ and G_n

$$\int_\Omega g(x,u_k,v_k)\frac{T_\lambda(G_n(u_k))}{\lambda} \geqslant \int_{\{u_k>n+\lambda\}} g(x,u_k,v_k),$$

Hence, by combining the latter estimates with (2.35), we get

$$\int_{\{u_k>n+\lambda\}} g(x,u_k,v_k) \leqslant \int_{\{u_k>n\}} f,$$

and then, by letting $\lambda \to 0$, from (P_1), we have

$$\int_{\{u_k>n\}} u_k^{r-1}v_k^{\theta+1} \leqslant C\int_{\{u_k>n\}} f.$$

In this way, as a direct consequence of Lemma 2.1, for $s = t = 2$ there follows that $g(x,u_k,v_k) \to g(x,u,v)$ in $L^1(\Omega)$, where clearly $2 > \frac{N(\theta+1)}{N+\theta+1}$, for all $0<\theta<1$.

It remains to prove that $h(x,u_k,v_k) \to h(x,u,v)$ in $L^1(\Omega)$. First observe that we can consider up to subsequences relabeled the same, that $\frac{1}{\tau_k} < 1$. Indeed, remark that by estimate (3.11) and Hölder's inequality,

$$\begin{aligned}\int_E |u_k|^r &\leqslant \left(\int_E |u_k|^{r+\theta+1}\right)^{\frac{r}{r+\theta+1}} \cdot \operatorname{meas}(E)^{\frac{\theta+1}{r+\theta+1}} \\ &\leqslant C\left(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + 1\right)^{\frac{r}{r+\theta+1}} (\operatorname{meas}(E))^{\frac{\theta+1}{r+\theta+1}},\end{aligned}$$

and then u_k^r is clearly uniformly integrable. Moreover, by (3.11) it is clear that

$$\int_{\{v_k>n\}} u_k^r v_k^{\theta+1} \leq C\Big(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}}+1\Big),$$

clearly uniform with respect to k. Thence, once again by Lemma 2.1 for $s=t=2$ we have

$$h(x,u_k,v_k) \to h(x,u,v) \text{ in } L^1(\Omega).$$

And also, it is trivial that

$$\int_\Omega \frac{1}{\tau_k}\psi \to 0.$$

Therefore, by letting $k\to+\infty$ in (AP), we arrive at

$$\int_\Omega M(x)\nabla u\cdot\nabla\varphi + \int_\Omega g(x,u,v)\varphi = \int_\Omega f\varphi \quad \forall\, \varphi\in C_c^\infty(\Omega)$$
$$\int_\Omega M(x)\nabla v\cdot\nabla\psi = \int_\Omega h(x,u,v)\psi \qquad \forall\, \psi\in C_c^\infty(\Omega),$$

where $g(x,u,v), h(x,u,v)\in L^1(\Omega)$ and the result follows. □

Theorem 0.2. *Let $f\in L^m(\Omega)$, where $f\geqslant 0$ a.e. in Ω, $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < (r+\theta+1)'$, $r>1$ and $0<\theta<\delta$, for $\delta = min\{\frac{N+2}{3N-2},\frac{4}{N-2},\frac{1}{2}\}$. Then there exists a solution (u,v) for (P), with $u\in W_0^{1,p}(\Omega)\cap L^{r-\theta+1}(\Omega)$, $u\geqslant 0$ a.e. in Ω and $v\in W_0^{1,q}(\Omega)$, $v\geqslant 0$ a.e. in Ω, where $p=\frac{2(r-\theta+1)}{(r+\theta+1)}$ and $q=\frac{2N(1-\theta)}{N-2\theta}$. Furthermore, if $r\geqslant\frac{N+2}{N-2}$, then $(u,v)\in W_0^{1,q}(\Omega)\times W_0^{1,q}(\Omega)$.*

Proof. Since this proof is analogous to the last one, we will omit certain details. In this fashion, from Lemma 2.5, there exist $\{u_k\}\subset W_0^{1,p}(\Omega)$, $\{v_k\}\subset W_0^{1,q}(\Omega)$, and $u\in W_0^{1,p}(\Omega)$, $v\in W_0^{1,q}(\Omega)$ such that, up to subsequences relabeled the same,

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\ u_k\to u \text{ in } L^{s_1}(\Omega), \text{ and a.e. in } \Omega;$$
$$v_k \rightharpoonup v \text{ weakly in } W_0^{1,q}(\Omega),\ v_k\to v \text{ in } L^{s_2}(\Omega), \text{ and a.e. in } \Omega,$$

where $p=\frac{2(r-\theta-1)}{r+\theta+1}$, $q=\frac{2N(1-\theta)}{N-2\theta}$, $u\geqslant 0$, $v\geqslant 0$ a.e. in Ω, and

$$\|u\|_{W_0^{1,p}}^p + \int_\Omega u^{r-\theta+1} + \int_\Omega u^r v^{1-\theta} \leqslant C\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} \text{ and } \|v\|_{W_0^{1,q}}^q \leq C\|f\|_{L^m}^{\frac{N(r-\theta+1)}{(N-2\theta)(r+\theta)}}, \tag{2.36}$$

and $s_1<\max\{p^*, r-\theta+1\}$ and $s_2<q^*$.

Once more, we still have to prove that $g(x,u_k,v_k)\to g(x,u,v)$ and $h(x,u_k,v_k)\to h(x,u,v)$ strongly in $L^1(\Omega)$, whereas the proof of the first convergence is the same, i.e., we take $\varphi=\frac{T_\lambda(G_n(u_k)}{\lambda}$ in the first equation of (AP), and after the same computations analogous to the last case, we get

$$\int_{\{|u_k|>n\}} |u_k|^{r-1}|v_k|^{\theta+1} \leqslant C\int_{\{|u_k|>n\}}|f|.$$

and then, by (3.37) and Lemma 2.1 we obtain that $g(x,u_k,v_k)\to g(x,u,v)$ in $L^1(\Omega)$.

Finally, once again by (3.37) we know that $\{|u_k|^r\}$ is uniformly integrable. Indeed, note that by Lemma 2.5, since $\theta<\min\{\frac{N+2}{3N-2},\frac{4}{N-2},\frac{1}{2}\}$,

$$\int_E u_k^r \leqslant \left(\int_E u_k^{r-\theta+1}\right)^{\frac{r}{r-\theta+1}}\cdot \text{meas}(E)^{\frac{1-\theta}{r-\theta+1}}$$
$$\leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}}+1\Big)^{\frac{r}{r-\theta+1}}\Big(\text{meas}(E)\Big)^{\frac{1-\theta}{r-\theta+1}},$$

and then, $\{|u_k|^r\}$ is uniformly integrable. Further, by (3.17), we also have that

$$\int_{\{v_k>n\}} u_k^r v_k^{1-\theta} \leqslant C\left(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}}+1\right)$$

so that, once again by Lemma 2.1 there follows that $h(u,u_k,v_k) \to h(x,u,v)$ a.e in $L^1(\Omega)$.

Therefore, by passing to the limit as $k \to +\infty$ in the first and second equations of (AP), we obtain that (u,v) is a solution of (P). □

2.5 Regularizing Effects

In this section there are presented the proofs of Corollaries 0.1 and 0.2, where we inspect the conditions on the data determining the presence or absence of regularizing effects of the solutions of (P), i.e., in the light of Definition 0.1, we show whether the solutions are Lebesgue or Sobolev regularized. Despite being direct, for the convenience of the reader we give the details .

Corollary 0.1. *Let (u,v) be the weak solution of (P), given by Theorem 0.1.*

(A) If $r+\theta+1>2^*$ and $(r+\theta+1)' \leqslant m < (2^*)'$, then u is Lebesgue and Sobolev regularized.

(B) If $r+\theta+1>2^*$ and $(2^*)' \leqslant m < \frac{N(r+\theta+1)}{N+2(r+\theta+1)}$ then u is Lebesgue regularized.

(C) If $2^* < r+\theta+1 \leqslant \frac{2^*(\theta+1)}{\theta}$ then v is Sobolev regularized.

Proof. (A) It is easy to see that $r+\theta+1>2^* \iff (r+\theta+1)' < (2^*)'$.

Thus, if $(r+\theta+1)' < m < (2^*)'$, by Theorem 0.1 we have to $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$. Moreover, since $r+\theta+1 > 2^*$ we obtain the following continuous immersion $L^{r+\theta+1}(\Omega) \subset L^{2^*}(\Omega)$. As $m<(2^*)' \iff 2^*>m^{**}$, implies that $L^{2^*}(\Omega) \subset L^{m^{**}}(\Omega)$ is the continuous immersion. Therefore, we have also a regularizing effect for the Lebesgue summability of the solution u.

(B) Knowing that $r+\theta+1>2^* \iff (r+\theta+1)' < (2^*)'$ and $m \geqslant (2^*)'$, there follows $m > (r+\theta+1)'$, then by Theorem 0.1 we get $u \in W_0^{1,2}(\Omega) \cap L^{r+\theta+1}(\Omega)$. Moreover, it is easy to see that

$$r+\theta+1 > m^{**} \iff m < \frac{N(r+\theta+1)}{N+2(r+\theta+1)}.$$

Therefore, we have a regularizing effect for the Lebesgue summability of the solution u.

(C) By the Theorem 0.1 we know that $u \in L^{r+\theta+1}(\Omega)$ and $v \in L^{2^*}(\Omega)$, so it is easy to see that $u^r \in L^{\frac{r+\theta+1}{r}}(\Omega)$ and $|v|^\theta \in L^{\frac{2^*}{\theta}}(\Omega)$. Thus, by interpolation inequality, we get

$$\begin{aligned}||h(x,u,v)||_{L^s} = \left(\int_\Omega |h(x,u,v)|^s\right)^{\frac{1}{s}} &\leqslant \left(d_1^s \int_\Omega ||u|^r|v|^\theta|^s\right)^{\frac{1}{s}} \\ &\leqslant d_2||u^r||_{L^{\frac{r+\theta+1}{r}}}||v^\theta||_{L^{\frac{2^*}{\theta}}} < \infty,\end{aligned}$$

where for $s \geqslant 1$

$$\frac{1}{s} = \frac{r}{r+\theta+1} + \frac{\theta}{2^*} \implies s = \frac{2^*(r+\theta+1)}{2^*r+\theta(r+\theta+1)}.$$

It is enough to prove that $1 \leqslant s < (2^*)'$, or equivalently,

$$\frac{2^*}{2^*-\theta} \leqslant \frac{r+\theta+1}{r} < \frac{2^*}{2^*-(\theta+1)}$$

and thus v is going to be Sobolev regularized. For this, remark that, since $\left(\frac{r+\theta+1}{r}\right)' = \frac{r+\theta+1}{\theta+1}$, $\left(\frac{2^*}{2^*-\theta}\right)' = \frac{2^*}{\theta}$ and $\left(\frac{2^*}{2^*-(\theta+1)}\right)' = \frac{2^*}{\theta+1}$, and by hypotheses, we have

$$2^* < r+\theta+1 \leqslant \frac{2^*(\theta+1)}{\theta},$$

we prove that v is Sobolev regularized. □

Corollary 0.2. *Let (u,v) be the weak solution of (P), given by Theorem 0.2*

(A) If $r-\theta+1 > 2^*$ suppose that

$$\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)} \leqslant m < (2^*)'.$$

Then u is Lebesgue regularized.

(B) If $r-\theta+1 > 2^*$ suppose that

$$\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

Then u is Sobolev and Lebesgue regularized.

(C) If $2^* \geqslant r-\theta+1 > 2^*(1-\theta)$ suppose that

$$\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

Then u is Sobolev and Lebesgue regularized.

(D) If $r-\theta+1 > 2^*(1-\theta)$ then v is Sobolev regularized.

Proof. (A) First, remark that $u \in W_0^{1,p}(\Omega)$, by the Sobolev embedding we get $u \in L^{p^*}(\Omega)$. Thus, since $r-\theta+1 > 2^* > 2^*(1-\theta)$, there follows

$$p^* < r-\theta+1 \iff 2^*(1-\theta) < r-\theta+1.$$

Thus, in order to prove that u is Lebesgue regularized, it is enough to contrast $r-\theta+1$ with m^{**}. Indeed, observe that $m < (2^*)' \iff m^{**} < 2^*$. However, by hypothesis we know that $r-\theta+1 > 2^*$ which implies that $r-\theta+1 > m^{**}$, that is, $L^{r-\theta+1}(\Omega) \subset L^{m^{**}}(\Omega)$, strictly, and thence, u is Lebesgue regularized.

(B) In order to show that u is Sobolev regularized, remark that it is enough prove that $p > m^*$. Indeed, it is clear that

$$p > m^* \iff m < \frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

Thus, since $u \in W^{1,p}(\Omega)$ we obtain that u is Sobolev regularized. Moreover, once again to prove that u is Lebesgue regularized we shall concentrate in $L^{r-\theta+1}(\Omega)$, since $p^* < r-\theta+1$. In this fashion, remark that by straightforward computations,

$$\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)} < (2^*)' \text{ and } 2^* > m^{**},$$

that last one being a direct consequence of $m < (2^*)'$. Hence, as $r-\theta+1 > 2^*$, there follows that $r-\theta+1 > m^{**}$, and therefore, u is Lebesgue regularized.

(C) It is easy to see that $r-\theta+1>2^*(1-\theta)$ is equivalent to

$$\frac{N(r-\theta+1)}{N+2(r-\theta+1)}>\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}.$$

However, by hypothesis

$$\frac{2N(r-\theta+1)}{N(r+\theta+1)+2(r-\theta+1)}>m,$$

what in particular guarantees that $p>m^*$, i.e., since $u\in W_0^{1,p}(\Omega)$, it is Sobolev regularized. Finally, as a consequence of the latter inequalities, we have

$$\frac{N(r-\theta+1)}{N+2(r-\theta+1)}>m, \text{ so that } r-\theta+1>m^{**},$$

and consequently, u is also Lebesgue regularized.

(D) Notice that $q^*=2^*(1-\theta)$, where $q=\frac{2N(1-\theta)}{N-2\theta}$. Moreover, for $p=\frac{2(r-\theta+1)}{r+\theta+1}$, by Theorem 0.2 we know that $u\in L^{r-\theta+1}(\Omega)\cap W_0^{1,p}(\Omega)$ and $v\in L^{q^*}(\Omega)$, or equivalently, $u^r\in L^{\frac{r-\theta+1}{r}}(\Omega)$ and $v^\theta\in L^{\frac{q^*}{\theta}}(\Omega)$. Now, by combining ($\mathrm{P}_3$) with the standard interpolation inequality for Lebesgue Spaces, it is obvious that

$$h(x,u,v)\in L^s(\Omega) \text{ where } s=\frac{2^*(r-\theta+1)(1-\theta)}{2^*r(1-\theta)+\theta(r-\theta+1)}.$$

In order to show that v is Sobolev regularized, remark that it is enough prove that $q>s^*$ or equivalently $q_*>s$ where $q_*=\frac{2N(1-\theta)}{N+2(1-2\theta)}$. In fact, it is clear that $r-\theta+1>2^*(1-\theta)\iff q_*>s$ and the result follows.

□

To conclude this chapter, a natural example of satisfying non-linearities (P_1)-(P_4) we may consider $g(x,s,t)=|s|^{r-2}st^+$ and $h(x,s,t)=|s|^r$, the classical Maxwell-Schrödinger case, or $g(x,s,t)=|s|^{r-2}|t|^\theta st^+$ and $h(x,s,t)=|s|^r|t|^\theta$. For another interesting example, consider $\eta>0$, $V_i(x)$, bounded and measurable, such that $V_i(x)\geqslant e_i>0$ a.e. in Ω, $i=1,2$, and then set

$$g(x,s,t)=\frac{V_1(x)|s|^{r-2}|t|^\theta(s^2(\eta+1)+\eta t^2)st^+}{s^2+t^2} \text{ and } h(x,s,t)=\frac{V_2(x)|s|^r|t|^\theta(s^2+t^2(\eta+1))}{s^2+t^2}.$$

Chapter 3

Regularizing Effect for a Class of Kirchhoff-Maxwell-Schrödinger Systems

In this chapter we analyze existence and regularity of solutions to the nonlocal counterpart of the system from Chapter 2. Indeed, we consider

$$\begin{cases} -\mathrm{div}\big((M(x) + ||\nabla u||^{\sigma}_{L^{\sigma}})\nabla u\big) + g(x,u,v) = f \quad \text{in } \Omega; \\ -\mathrm{div}(M(x)\nabla v) = h(x,u,v) \quad \text{in } \Omega; \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{K}$$

where $p = \frac{2(r-\theta+1)}{r+\theta+1}$ and

$$\sigma = \begin{cases} 2 & \text{if } \ m \geqslant (r+\theta+1)', \\ p & \text{if } \ m < (r+\theta+1)'. \end{cases} \tag{3.1}$$

For the sake of completeness, let us recall that Ω is an open bounded subset of $\mathbb{R}^N$, for $N > 2$, $f \in L^m(\Omega)$ with $m \geqslant 1$, $r > 1$, and once again $g, h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory and satisfy hypotheses (P_1) - (P_2') and $M : \Omega \to \mathbb{R}^N \times \mathbb{R}^N$ is a bounded measurable matrix satisfying (P_5), see page 12.

3.1 Approximate Problem

Once again, our approach is based on showing the existence of a solution to problem (K) in the case $f \in L^\infty(\Omega)$. In order to do that, we will employ the Minty-Browder Theorem, see Appendix B, Theorem 4.6.

Our starting point is the following continuity result.

Lemma 3.1. *Let $f \in L^\infty(\Omega)$ and $u_k \in W_0^{1,2}(\Omega)$ such that*

a) $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$.

b) u_k satisfies

$$\int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2(\Omega)}\big)\nabla u_k \cdot \nabla\varphi + \int_\Omega g_\tau(x,u_k,\zeta_k)\varphi = \int_\Omega f\varphi \quad \forall\, \varphi \in W_0^{1,2}(\Omega).$$

wehere ζ_k is fixed for each k. Then $u_k \to u$ in $W_0^{1,2}(\Omega)$.

Proof. Talking $\varphi = (u_k - u)$ we get

$$\int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2(\Omega)}\big)\nabla u_k \cdot \nabla(u_k - u) + \int_\Omega g_\tau(x, u_k, \zeta_k)(u_k - u) = \int_\Omega f(u_k - u).$$

Adding and subtracting the term

$$\int_\Omega \big(M(x) + ||\nabla u_k||^2\big)\nabla u \cdot \nabla(u_k - u),$$

by (P_5) and discarding the positive term we have

$$\alpha \int_\Omega |\nabla(u_k - u)|^2 + \int_\Omega g_\tau(x, u_k, \zeta_k)(u_k - u) \leqslant \int_\Omega |f||u_k - u| + \int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2}\big)\nabla u \cdot \nabla(u_k - u). \tag{3.2}$$

As $f \in L^\infty(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$ up to subsequence, there follows that

$$\lim_{n\to\infty} \int_\Omega f(u_k - u) = 0.$$

On one hand, since $M(x)\nabla u \in \big(L^2(\Omega)\big)^N$, $(\nabla u_k - u) \rightharpoonup 0$ weakly in $\big(L^2(\Omega)\big)^N$ and $\{||\nabla u_k||^2_{L^2}\}_{k=1}^\infty$ is bounded, we get

$$\lim_{k\to\infty} \int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2}\big)\nabla u \cdot \nabla(u_k - u) = \lim_{k\to\infty} \int_\Omega M(x)\nabla u \cdot \nabla(u_k - u) + \lim_{k\to\infty} ||\nabla u_k||^2_{L^2} \int_\Omega \nabla u \cdot \nabla(u_k - u) = 0.$$

On the one hand, as $|g_\tau(x, u_k, \zeta_k)| < \tau$ and $u_k \to u \in L^1(\Omega)$ up to subsequences, we have

$$\lim_{k\to\infty} \int_\Omega g_\tau(x, u_k, \zeta_k)(u_k - u) = 0.$$

Thus, taking limit on (3.2), we obtain

$$\alpha \lim_{k\to\infty} \int_\Omega |\nabla(u_k - u)|^2 = 0.$$

Which proves the result. □

Now we will focus on obtaining the existence of solutions for the first equation in (K). Indeed, we will proceed as follows. For fixed $\zeta \in L^2(\Omega)$ we will use the Minty - Browder Theorem, see 4.6, to ensure that there exists weak solution $u = S(\zeta) \in W_0^{1,2}(\Omega)$. Further, by considering $u \in W_0^{1,2}(\Omega)$ fixed, we will show that there exists $\eta = T(u)$ solution of the second equation of (K), and then, finally, in order to conclude that there exists a weak solution $(u, v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, we will use Schauder´s fixed point Theorem, see 4.5. By doing so, we will ensure that there exists $v \in W_0^{1,2}(\Omega)$ such that $v = T(S(v)) = T(u)$. We also emphasize that the use of hypothesis (P_4) is crucial for proof that the operator $T \circ S$ it is well-defined.

Proposition 3.1. *Let $\Psi \in L^\infty(\Omega)$. There exists a weak solution $(u,v) \in W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ of the system*

$$\begin{cases} -div\big((M(x)+||\nabla u||_{L^2}^2)\nabla u\big) + g_\tau(x,u,v) = \Psi \quad in\ \Omega; \\ -div(M(x)\nabla v) = h_\tau(x,u,v) + \frac{1}{\tau} \qquad in\ \Omega; \\ u = v = 0 \ on\ \partial\Omega. \end{cases} \tag{K'}$$

Moreover, there exists a constant $C>0$ such that

(i) if $\Psi \in L^m(\Omega)$ for $m \geqslant (2^*)'$ then

$$\|u\|_{W_0^{1,2}} + \|v\|_{W_0^{1,2}} \leqslant C\|\Psi\|_{L^m};$$

(ii) if $\Psi \in L^m(\Omega)$ for $m > \frac{N}{2}$ and $0<\theta<\frac{4}{N-2}$ then

$$\|u\|_{L^\infty} + \|v\|_{L^\infty} \leqslant C\|\Psi\|_{L^m}.$$

Proof. Fixing ζ in $L^2(\Omega)$. We will show that there exists $u = S(\zeta) \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega (M(x)+||\nabla u||_{L^2(\Omega)}^2)\nabla u\cdot\nabla\varphi + \int_\Omega g_\tau(x,u,\zeta)\varphi = \int_\Omega \Psi\varphi \quad \forall\ \varphi \in W_0^{1,2}(\Omega). \tag{3.3}$$

Consider the operator $A : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ defined by

$$(A(u),\varphi) = \int_\Omega \big(M(x)+||\nabla u||_{L^2(\Omega)}^2\big)\nabla u\cdot\nabla\varphi + \int_\Omega g_\tau(x,u,\zeta)\varphi - \int_\Omega \Psi\varphi.$$

We will prove that A satisfies the hypotheses of Minty-Browder theorem, i.e. A be a pseudomonotone coercive operator.

Note that, A is coercive. In fact, by (P_5), (P_2'), Hölder's and Poincaré's inequality we have

$$\begin{aligned}(A(u),u) &= \int_\Omega \big(M(x)+||\nabla u||_{L^2(\Omega)}^2\big)\nabla u\cdot\nabla u + \int_\Omega g_\tau(x,u,\zeta)u - \int_\Omega \Psi u \\ &\geqslant \alpha||u||_{W_0^{1,2}}^2 + ||u||_{W_0^{1,2}}^4 - C||\Psi||_{L^\infty}||u||_{W_0^{1,2}}.\end{aligned}$$

Thus

$$\frac{(A(u),u)}{||u||_{W_0^{1,2}}} \to +\infty \quad \text{as} \quad ||u||_{W_0^{1,2}} \to +\infty.$$

Moreover, by (P_5), Hölder's, Poincaré's and Young's inequality we get

$$\begin{aligned}|(A(u),w)| &= \Big|\int_\Omega \big(M(x)+||\nabla u||_{L^2(\Omega)}^2\big)\nabla u\cdot\nabla w + \int_\Omega g_\tau(x,u,\zeta)w - \int_\Omega \Psi w\Big| \\ &\leqslant \Big(\frac{\beta+||\nabla u||_{L^2}^2}{2}\Big)\Big[||w||_{W_0^{1,2}}^2 + ||w||_{W_0^{1,2}}^2\Big] + \tau C||w||_{W_0^{1,2}}^2 + C||\Psi||_{L^\infty}||w||_{W_0^{1,2}}\end{aligned}$$

and so $||A(u)||_{W^{-1,2}}$ is bounded if $||u||_{W_0^{1,2}}$ is bounded.

Now, assume that $u_k \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$ and $0 \geqslant \limsup_{k\to\infty}\big(A(u_k),u_k-u\big)$. We will prove that for every $w \in W_0^{1,2}(\Omega)$ we have

$$\liminf_{k\to\infty}\big(A(u_k),u_k-w\big) \geqslant \big(A(u),u-w\big).$$

We remark that

$$(A(u_k), u_k - u) = \int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2}\big)\nabla u_k \cdot \nabla(u_k - u) + \int_\Omega g_\tau(x, u_k, \zeta)(u_k - u) - \int_\Omega \Psi(u_k - u).$$

Adding and Subtracting the term

$$\int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2(\Omega)}\big)\nabla u \cdot \nabla(u_k - u),$$

so by (P$_5$) and discarding the positive term, we have

$$(A(u_k), u_k - u) \geqslant \alpha \int_\Omega |\nabla(u_k - u)|^2 + \int_\Omega g_\tau(x, u_k, \zeta)(u_k - u) - \int_\Omega \Psi(u_k - u) + \int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2(\Omega)}\big)\nabla u \cdot \nabla(u_k - u).$$

As $\Psi \in L^\infty(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$ there follows that

$$\lim_{k\to\infty} \int_\Omega \Psi(u_k - u) = 0.$$

Moreover, on one hand since $M(x)\nabla u \in \big(L^2(\Omega)\big)^N$, $\nabla(u_k - u) \rightharpoonup 0$ weakly in $\big(L^2(\Omega)\big)^N$ and $\big\{||\nabla u_k||^2_{L^2}\big\}_{k=1}^\infty$ is bounded, we get

$$\lim_{k\to\infty} \int_\Omega \big(M(x) + ||\nabla u_k||^2_{L^2(\Omega)}\big)\nabla u \cdot \nabla(u_k - u) = \lim_{k\to\infty} \int_\Omega M(x)\nabla u \cdot \nabla(u_k - u) + \lim_{k\to\infty} ||\nabla u_k||^2_{L^2} \int_\Omega \nabla u \cdot \nabla(u_k - u) = 0.$$

On the one hand, as $|g_\tau(x, u_k, \zeta_k)| < \tau$ and $u_k \to u \in L^1(\Omega)$ up to subsequences, we have

$$\lim_{k\to\infty} \int_\Omega g_\tau(x, u_k, \zeta_k)(u_k - u) = 0.$$

Thus

$$0 \geqslant \limsup_{k\to\infty} \big(A(u_k), u_k - u\big) \geqslant \limsup_{k\to\infty} \alpha \int_\Omega |\nabla(u_k - u)|^2 \geqslant \liminf_{k\to\infty} \alpha \int_\Omega |\nabla(u_k - u)|^2 \geqslant 0,$$

hence $||u_k - u||_{W_0^{1,2}} \to 0$ in $W_0^{1,2}(\Omega)$, which implies that

$$\liminf_{k\to\infty} \big(A(u_k), u_k - w\big) = (A(u), u - w).$$

Therefore A is pseudomonotone, by Minty-Browder Theorem, see 4.6, there exists $u \in W_0^{1,2}(\Omega)$ satisfying (3.3).

From an analogous argument used in the Proposition 2.1, given $u \in W^{1,2}(\Omega)$ fixed there exist $\eta = T(u) \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega M(x)\nabla\eta \cdot \nabla\psi = \int_\Omega h_\tau(x, u, \eta)\psi \quad \forall\ \psi \in W_0^{1,2}(\Omega). \tag{3.4}$$

Now, taking $\varphi = u$ in (3.3), and discarding the positive term, using the hypothesis (P_5), (P_2') and by Hölder's and Poincaré inequality, we have

$$\alpha \int_\Omega |\nabla u|^2 \leqslant C||\Psi||_{L^\infty}||\nabla u||_{L^2}. \tag{3.5}$$

In addition, taking $\psi = \eta$ in (3.4), by Hölder's and Poincaré inequality

$$\alpha \int_\Omega |\nabla \eta|^2 \leqslant C\tau \ \text{meas}(\Omega)^{\frac{1}{2}}||\nabla \eta||_{L^2}.$$

Choosing $R = \max\{C||\Psi||_{L^\infty}, C\tau \text{meas}(\Omega)^{\frac{1}{2}}\}$ by (3.5) and the above inequality, we get

$$||\nabla u||_{L^2} \leqslant R \text{ and } ||\nabla \eta||_{L^2} \leqslant R. \tag{3.6}$$

In particular, by Poincaré inequality $||u||_{L^2} \leqslant R$ and $||\eta||_{L^2} \leqslant R$.

Let $B = B(0,R) = \{u \in L^2(\Omega); ||u||_{L^2} \leqslant R\}$. We will prove that the operator $T \circ S$ defined in B satisfies the hypothesis of Schauder´s fixed point Theorem. By estimates (3.6), it is easy to see that B is inariant from $T \circ S$. However continuity and compactness are more delicate to check, so we will give a little more detail in your verification.

Let $\zeta_k \to \zeta$ in $L^2(\Omega)$ such that $u_k = S(\zeta_k)$ i.e. u_k satisfies

$$\int_\Omega (M(x) + ||\nabla u_k||^2_{L^2})\nabla u_k \cdot \nabla \varphi + \int_\Omega g_\tau(x, u_k, \zeta_k)\varphi = \int_\Omega \Psi \varphi. \tag{3.7}$$

Since u_k is bounded in $W_0^{1,2}(\Omega)$, up to a subsequence, we have

$$\begin{cases} u_k \rightharpoonup u \ \text{ weakly in } \ W_0^{1,2}(\Omega) \\ u_k \to u \ \text{ in } L^2(\Omega), \text{ and a.e in } \Omega. \end{cases}$$

Thus, by making $k \to \infty$ in (3.7), by Lemma 3.1, and fact that h_τ is a Carathéodory function, we have $u = S(\zeta)$.

By estimates (3.6) up to a subsequence we get

$$\begin{cases} \eta_k \rightharpoonup \eta \ \text{ weakly in } \ W_0^{1,2}(\Omega) \\ \eta_k \to \eta \ \text{ in } L^2(\Omega), \text{ and a.e in } \Omega. \end{cases} \tag{3.8}$$

Since $\eta_k = T(u_k)$, that is,

$$\int_\Omega M(x)\nabla \eta_k \cdot \nabla \psi = \int_\Omega \left(h_\tau(x, u_k, \eta_k) + \frac{1}{\tau}\right)\psi \ \ \forall \ \psi \in W_0^{1,2}(\Omega).$$

Thus by (3.8) and fact that g_τ is a Carathéodory function, making $k \to \infty$ in equality above we obtain $\eta = T(u)$.

Moreover, it is clear that $T \circ S(B)$ is **relatively compact** in $L^2(\Omega)$, i.e., $\overline{T(S(B))}$ is compact in $L^2(\Omega)$. Once, as we saw earlier B is invariant by $T \circ S$ and $T \circ S$ is continuous. Therefore, by Schauder's Fixed Point Theorem, there exists $v \in B \subset W_0^{1,2}(\Omega)$ such that $v = T(S(v)) = T(u)$.

Now, we will prove that if $\Psi \in L^m(\Omega)$ with $m \geqslant (2^*)'$ there exists a constant $C > 0$ such that

$$||u||_{W_0^{1,2}} + ||v||_{W_0^{1,2}} \leqslant C||\Psi||_{L^m}.$$

In fact, taking $\varphi = u$ in the weak formulation of the first equation of (K'), and discarding the positive term, using the hypothesis (P$_5$),(P$'_2$), (P$_2$) and since $m \geqslant (2^*)'$ by Hölder's and Sobolev's inequality , we have

$$\alpha \int_\Omega |\nabla u|^2 + c_2 \int |u|^r |v|^{\theta+1} \leqslant C||\Psi||_{L^m}||u||_{L^{2^*}} \leqslant C||\Psi||_{L^m}||\nabla u||_{L^2}$$

hence

$$||u||_{W_0^{1,2}} \leqslant C||\Psi||_{L^m} \text{ and } \int |u|^r |v|^{\theta+1} \leqslant C||\Psi||^2_{L^m}$$

In addition, taking $\psi = v$ in the weak formulation of the second equation of (K'), by hypothesis (P$_5$) ,(P$_3$) and the above estimate we get

$$||v||_{W_0^{1,2}} \leqslant C||\Psi||_{L^m}.$$

Finally, let is show that if $\Psi \in L^m(\Omega)$ for $m > \frac{N}{2}$ and $0 < \theta < \frac{4}{N-2}$.

$$||u||_{L^\infty} + ||v||_{L^\infty} < C.$$

For this, we take $\varphi = G_k(u)$ in the weak formulation of the first equation of (K') obtaining

$$\int_\Omega \big(M(x) + ||\nabla u||_\sigma^\sigma\big)\nabla u \cdot \nabla u G'_k(u) + \int_\Omega g_\tau(x,u,v)G_k(u) = \int_\Omega \Psi G_k(u)$$

by definition of G_k, we have

$$\int_\Omega g_\tau(x,u,v)G_k(u) \geqslant 0 \quad \text{ and } \quad ||\nabla u||^\sigma_{L^\sigma} \int_\Omega |\nabla u|^2 G'_k(u) \geqslant 0.$$

Thus, discarding the positive term using the ellipticity of M and Hölder's inequelity with exponent $\frac{2N}{N+2}$, we get

$$\alpha \int_\Omega |\nabla G_k(u)|^2 \leqslant \int_\Omega \Psi G_k(u) \leqslant \Big(\int_{A_k^u} |\Psi|^{\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}} \Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}}$$

where $A_k^u = \{|u| > k\}$. Firthermore, recall that by Sobolev's and Hölder's with exponent $\frac{(N+2)m}{2N}$ there follows

$$\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{2}{2^*}} \leqslant \Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}} ||\Psi||_{L^m}\text{meas}(A_k^u)^{(1-\frac{2N}{(N+2)m})\frac{N+2}{2N}}$$

so that,

$$\Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}} \leqslant C||\Psi||_{L^m}\text{meas}(A_k^u)^{(1-\frac{2N}{(N+2)m})\frac{N+2}{2N}} \tag{3.9}$$

hence by Hölder's inequality and (3.9) we have

$$\int_\Omega |G_k(u)| = \int_{A_k^u} |G_k(u)| \leqslant \text{meas}(A_k^u)^{\frac{N+2}{2N}} \Big(\int_\Omega |G_k(u)|^{2^*}\Big)^{\frac{1}{2^*}} \leqslant C||\Psi||_{L^m}\text{meas}(A_k^u)^\alpha$$

where $\alpha = 1 + \frac{1}{2} - \frac{1}{m} > 1$, since $m > \frac{N}{2}$. Therefore, by Lemma 1.3 we obtain $||u||_{L^\infty} < C||\Psi||_{L^m}$. Analogous to Lemma 2.3 item (ii), we obtain the estimate $L^\infty(\Omega)$ for v.

□

Since $g_\tau(., u_k, v_k) = g(., u_k, v_k)$ and $h_\tau(., u_k, v_k) = h(., u_k, v_k)$ a.e. in Ω, for simplicity, we consider the following approximate problem.

Proposition 3.2. *Let $\{f_k\}$ be a sequence of $L^\infty(\Omega)$ functions strongly convergent to f in $L^m(\Omega)$, $m \geqslant 1$, for which $|f_k| \leqslant |f|$ a.e. in Ω. Then, there exists $(u_k, v_k) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, solution to*

$$\begin{cases} -div\big((M(x) + ||\nabla u_k||_{L^2}^2)\nabla u_k\big) + g(x, u_k, v_k) = f_k \\ -div(M(x)\nabla v_k) = h(x, u_k, v_k) + \frac{1}{\tau_k}. \end{cases} \tag{K_A}$$

where $\tau_k > 0$ and $\tau_k \to +\infty$ if $k \to +\infty$. Moreover, if $f \geq 0$ a.e. in Ω then $u_k \geqslant 0$ a.e. in Ω and $v_k > 0$ a.e. in Ω.

Proof. Given $k > 0$ consider $\tau > (c_1 + d_1)C^{r+\theta}k^{r+\theta}$, where C is given in Lemma 2.3 and c_1, d_1 in (P_1), (P_3). Let us recall the standard truncation $T_k(s) = \max(-k, \min(s, k))$ and then take $f_k = T_k(f)$. Thus, for $\Psi = f_k$, by combining Proposition 3.1, we obtain a couple $(u_k, v_k) \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ solution of (K'). Observe that $g_\tau(., u_k, v_k) = g(., u_k, v_k)$ and $h_\tau(., u_k, v_k) = h(., u_k, v_k)$ a.e. in Ω. As a matter of fact, from P_1, Proposition 3.1 item (ii) and the choice of f_k, we have

$$\begin{aligned} |g(x, u_k, v_k)| &\leqslant c_1|u|^{r-1}|v|^{\theta+1} \\ &\leqslant c_1 C^{r+\theta}\|f_k\|_{L^\infty}^{r+\theta} \\ &\leqslant c_1 C^{r+\theta}k^{r+\theta}. \end{aligned}$$

Thence, since by the choice of $\tau > c_1 C^{r+\theta}k^{r+\theta}$, from (2.1) we have that g_τ coincides with g. Analogously using the hypothesis P_3 we conclude that $h_\tau = h$.

Finally, remark that if $f \geqslant 0$ a.e. in Ω then $f_k \geqslant 0$ a.e. in Ω. Thus taking $\varphi = u_k^- = -\max(-u_k, 0)$ in the first equation of (AP), we have that

$$\int_\Omega (M(x) + ||\nabla u_k||_{L^2}^2)\nabla[u_k^+ - u_k^-] \cdot \nabla u_k^- + \int_\Omega g(x, u_k, v_k)u_k^- = \int_\Omega f u_k^-,$$

by using the hypothesis (P_5) we get

$$-(\alpha + ||\nabla u_k||_{L^2}^2)\int_\Omega |\nabla u_k^-|^2 + \int_\Omega g(x, u_k, v_k)u_k^- \geqslant \int_\Omega f u_k^-,$$

Now, note that

$$\int_\Omega g(x, u_k, v_k)u_k^- = \int_{\{u_k<0\}} g(x, u_k, v_k)u_k^- = -\int_{\{u_k<0\}} g(x, u_k, v_k)u_k \leqslant 0$$

where the last integral is negative, by hypothesis (P_2'). Which implies that

$$0 \geqslant -(\alpha + ||\nabla u_k||_{L^2}^2)\int_\Omega |\nabla u_k^-|^2 \geqslant \int_\Omega f u_k^- \geqslant 0.$$

Thus

$$0 \leqslant \int_\Omega |\nabla u_k^-|^2 \leqslant 0,$$

thence $\|u_k^-\|_{W_0^{1,2}}^2 = \|\nabla u_k^-\|_{L^2}^2 = 0$ which implies that $u_k^- = 0$. Consequently we get $u_k \geq 0$ a.e. in Ω.

Now we prove that $v_k > 0$ a.e. in Ω. In fact, consider $w_k \in C^{1,\alpha}(\Omega)$, with $0 \leqslant \alpha < 1$, the solution of

$$\begin{cases} -\mathrm{div}(M(x)\nabla w_k) = \frac{1}{\tau_k} \quad \text{in } \Omega, \\ \quad w_k = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.10}$$

Remark that, since $M \in W^{1,\infty}(\Omega)$ the existence of this w_k is standard, for instance see Corollary 8.36 in [15]. Then, by a straightforward application of the Strong Maximum Principle of Vasquez in (3.10), see Theorem 4 in [20], we obtain that $w_k > 0$ in Ω.After that, let us stress that

$$-\mathrm{div}(M(x)\nabla v_k) = h(x, u_k, v_k) + \frac{1}{\tau_k} \geqslant -\mathrm{div}(M(x)\nabla w_k)$$

then by the Comparison Principle $v_k \geqslant w_k$ a.e. in Ω so that $v_k > 0$ a.e. in Ω. □

3.2 Estimates

Initially we will get uniform a priori estimates that play a key role in our results. First, we adress the "energy case"

$$m \geqslant (r + \theta + 1)'.$$

In this case, the non-local term in the second is $||\nabla u||^2_{L^2}$.

Lemma 3.2. *Let $f \in L^m(\Omega)$ where $m \geqslant (r + \theta + 1)'$, and $f \geqslant 0$ a.e in Ω, $r > 1$ and $0 < \theta < 1$. Then*

$$\|u_k\|^2_{W_0^{1,2}} + \|v_k\|^2_{W_0^{1,2}} + \int_\Omega u_k^{r+\theta+1} + \int_\Omega u_k^r v_k^{\theta+1} \leqslant C\Big(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\Big) \tag{3.11}$$

where $C > 0$ and $\tau_k \to \infty$ if $k \to \infty$.

Proof. By taking $\varphi = u_k^{\theta+1}(v_k + \epsilon)^{-\theta}$ in the second equation of (K_A), after dropping the positive term, from analogous manner to Lemma 2.4 we obtain

$$\int_\Omega u_k^{r+\theta+1} \leqslant C\Big(\int_\Omega |\nabla u_k|^2 + \int_\Omega |\nabla v_k|^2\Big) \tag{3.12}$$

Now let us proceed to the other estimates. Indeed by choosing $\varphi = u_k$ in the first equation of (K_A) we get

$$\int_\Omega M(x)\nabla u_k \cdot \nabla u_k + ||\nabla u_k||^2_{L^2} \int_\Omega \nabla u_k \cdot \nabla u_k + \int_\Omega g(x, u_k, v_k)u_k = \int_\Omega f_k u_k$$

discarding the term positive and using the hypothesis (P_5) we have

$$\alpha \int_\Omega |\nabla u_k|^2 + \int_\Omega g(x, u_k, v_k)u_k \leqslant \int_\Omega f_k u_k. \tag{3.13}$$

Hence, since $f \in L^m(\Omega)$ with $m \geqslant (r + \theta + 1)'$ by Hölder's inequality on the right - hand side of the above inequality and by (P_1) we get

$$\int_\Omega u_k^r v_k^{\theta+1} \leqslant C||f||_{L^m}||u_k||_{L^{r+\theta+1}}. \tag{3.14}$$

By talking $\psi = v_k$ in the second equation of (K_A) and using the hypothesis (P_3),(P_4), (P_5) and (3.14) we have

$$\alpha \int_\Omega |\nabla v_k|^2 \leqslant \int_\Omega \Big(h(x,u_k,v_k) + \frac{1}{\tau_k}\Big)v_k \leqslant d_2 \int_\Omega u_k^r v_k^{\theta+1} + \frac{1}{\tau_k}\int_\Omega v_k$$
$$\leqslant C\|f\|_{L^m}\|u_k\|_{L^{r+\theta+1}} + \frac{C}{\tau_k}\|\nabla v_k\|_{L^2}.$$

However, it is clear that

$$\frac{C}{\tau_k}\|\nabla v_k\|_{L^2} \leq \frac{C_\alpha}{\tau_k^2} + \frac{\alpha}{2}\|\nabla v_k\|_{L^2}^2$$

then by (3.13) we get

$$\int_\Omega |\nabla u_k|^2 + \int_\Omega |\nabla v_k|^2 \leqslant C\big(\|f\|_{L^m}\|u_k\|_{L^{r+\theta+1}} + \frac{1}{\tau_k^2}\big). \tag{3.15}$$

In particular, by combining (3.12) with (3.15) and Young's inequality, we obtain

$$\|u_k\|_{L^{r+\theta+1}}^{r+\theta+1} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\bigg). \tag{3.16}$$

Therefore, by combining (3.14), (3.16) with (3.15), we finally have get

$$\|u_k\|_{W_0^{1,2}}^2 + \|v_k\|_{W_0^{1,2}}^2 + \int_\Omega u_k^r v_k^{\theta+1} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}} + \frac{1}{\tau_k^2}\bigg),$$

where we stress that $\tau_k \to \infty$ if $k \to \infty$. $\square$

The next lemma will establish estimates under a weaker regime for source, specifically for $f \in L^m(\Omega)$ where

$$\Big(\frac{r-\theta+1}{1-2\theta}\Big)' < m < (r+\theta+1)'.$$

In contrast to Lemma 3.2, we consider another set of test functions to compensate for the additional uniqueness in our system. In this case, the non-local term in the second is $||\nabla u||_{L^p}^p$.

Lemma 3.3. *Let f be a positive function in $L^m(\Omega)$ with $\left(\frac{r-\theta+1}{1-2\theta}\right)' < m < (r+\theta+1)'$ and $0<\theta<\frac{1}{2}$. Then*

$$\|u_k\|_{W_0^{1,p}}^p + \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^r v_k^{1-\theta} \leqslant C\bigg(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\bigg) \quad \text{and}$$
$$\|v_k\|_{W_0^{1,q}}^q \leqslant C\bigg(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\bigg)^{\frac{N}{N-2\theta}} \tag{3.17}$$

where $p = \frac{2(r-\theta+1)}{r+\theta+1}$, $q = \frac{2N(1-\theta)}{N-2\theta}$ and $C>0$, where $\tau_k \to \infty$ if $k \to \infty$. Moreover, if $r \geqslant \frac{N+2}{N-2}$, then $\{u_k\}$ and $\{v_k\}$ are bounded in $W_0^{1,q}(\Omega)$.

Proof. Consider $\varphi_\varepsilon = (u+\varepsilon)^\gamma - \varepsilon^\gamma$ where $0<\epsilon\leqslant 1$, as a test function in the first equation of (K_A), we get

$$\int_\Omega (M(x) + ||\nabla u_k||_{L^p}^p)\nabla u_k \cdot \nabla u_k (u_k+\varepsilon)^\gamma + \int_\Omega g(x,u_k,v_k)[(u_k+\epsilon)^\gamma - \varepsilon^\gamma]$$
$$= \int_\Omega f_k[(u_k+\varepsilon)^\gamma - \epsilon^\gamma]. \tag{3.18}$$

Note that

$$\gamma\int_\Omega (M(x)+||\nabla u_k||_{L^p}^p)\nabla u_k\cdot\nabla u_k(u_k+\varepsilon)^{\gamma-1}=\gamma\int_\Omega M(x)\nabla u_k\cdot\nabla u_k(u_k+\varepsilon)^{\gamma-1}\\ +\gamma||\nabla u_k||_{L^p}^p\int_\Omega |\nabla u_k|^2(u_k+\varepsilon)^{\gamma-1}$$

discarding the positive term and using the ellipcity hypothesis of M we have

$$\gamma\int_\Omega (M(x)+||\nabla u_k||_{L^p}^p)\nabla u_k\cdot\nabla u_k(u_k+\varepsilon)^{\gamma-1}\geqslant\alpha\gamma\int_\Omega |\nabla u_k|^2(u_k+\varepsilon)^{\gamma-1}.$$

Combining the above inequality with (3.18) we have

$$\alpha\gamma\int_\Omega |\nabla u_k|^2(u_k+\varepsilon)^{\gamma-1}+\int_\Omega g(x,u_k,v_k)[(u_k+\varepsilon)^\gamma-\epsilon^\gamma]\leqslant\int_\Omega f_k[(u_k+\varepsilon)^\gamma-\varepsilon^\gamma].$$

Since $u_k\in L^\infty(\Omega)$ and $0<\varepsilon\leqslant 1$ for the right hand side we have $|f_k|[(u_k+\varepsilon)^\gamma-\varepsilon^\gamma]|\leqslant |f|\cdot[(\|u_k\|_{L^\infty}+\varepsilon)^\gamma+\varepsilon^\gamma]$ and $f_k[(u_k+\varepsilon)^\gamma-\varepsilon^\gamma]\to f_k u_k^\gamma$ a.e. in Ω when $\varepsilon\to 0$. Thus, by the Dominated Convergence Theorem

$$\int_\Omega f_k u_k^\gamma=\lim_{\varepsilon\to 0^+}\int_\Omega f_k[(u_k+\varepsilon)^\gamma-\varepsilon^\gamma].$$

Thus, by taking $\varepsilon\to 0$ in (2.20), by recalling Lemma 2.2 and (P_1), we end up with

$$\alpha\gamma\int_{\Omega_+^k}|\nabla u_k|^2u_k^{\gamma-1}+c_2\int_\Omega u_k^{r-1+\gamma}v_k^{\theta+1}\leqslant\int_\Omega |f|u_k^\gamma<\infty,\tag{3.19}$$

which is finite for every fixed $k\in\mathbb{N}$, where $\Omega_+^k=\{u_k>0\}$.

Using the coupling terms of the two equations together with the appropriate choice of test function, we will obtain fundamental estimates to conclude our result. Indeed, by considering $\psi_\varepsilon=(v_k+\varepsilon)^\gamma-\varepsilon^\gamma$ in the second equation of (K_A) it is clear that

$$\alpha\gamma\int_\Omega |\nabla v_k|^2(v_k+\varepsilon)^{\gamma-1}\leqslant\int_\Omega\Big(h(x,u_k,v_k)+\frac{1}{\tau_k}\Big)[(v_k+\varepsilon)^\gamma-\varepsilon^\gamma].$$

Then, once more, since $v_k>0$ a.e. in Ω, by taking $\varepsilon\to 0$ and by recalling (P_3), from Lemma 2.2, we obtain

$$\alpha\gamma\int_\Omega |\nabla v_k|^2v_k^{\gamma-1}\leqslant\int_\Omega\Big(h(x,u_k,v_k)+\frac{1}{\tau_k}\Big)v_k^\gamma\leqslant d_2\int_\Omega u_k^rv_k^{\gamma+\theta}+\frac{1}{\tau_k}\int_\Omega v_k^\gamma.\tag{3.20}$$

Further, take $\psi=(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta}$ in the second equation of (K_A) and discarding the positive term, one has that

$$\int_\Omega h(x,u_k,v_k)(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta}\leqslant(\gamma+\theta)\int_\Omega M(x)\nabla v_k\cdot\nabla u_k\ (u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta}\\ -\theta\int_\Omega M(x)\nabla v_k\cdot\nabla v_k(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}$$

Then, by (P_3), (P_4) and (P_5) it is clear that

$$\begin{aligned}c_1\int_\Omega u_k^rv_k^\theta(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta}&+\alpha\theta\int_\Omega|\nabla v_k|^2(u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}\\ &\leqslant\beta(\gamma+\theta)\int_\Omega|\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta}\\ &=\beta(\gamma+\theta)\int_{\Omega_+^k}|\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta},\end{aligned}\tag{3.21}$$

since $\nabla u_k = 0$ a.e. in the set $u_k = 0$, where $\Omega_+^k = \{u_k > 0\}$.

Given $\eta > 0$, from Young's inequality, we have that

$$\int_{\Omega_+^k} |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}(v_k+\varepsilon)^{-\theta} \leqslant \int_{\Omega_+^k} |\nabla v_k||\nabla u_k|(u_k+\varepsilon)^{\gamma+\theta-1}v_k^{-\theta}$$
$$\leqslant \eta \int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + C_\eta \int_{\Omega_+^k} |\nabla u_k|^2 (u_k+\varepsilon)^{2(\gamma+\theta-1)}.$$

By combining the above inequality and (3.21) we get

$$c_1 \int_\Omega u_k^r v_k^\theta (u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-\theta} + \alpha\theta \int_\Omega |\nabla v_k|^2 (u_k+\varepsilon)^{\gamma+\theta}(v_k+\varepsilon)^{-(\theta+1)}$$
$$\leqslant \beta(\gamma+\theta)\eta \int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + \beta(\gamma+\theta)C_\eta \int_{\Omega_+^k} |\nabla u_k|^2 (u_k+\varepsilon)^{2(\gamma+\theta-1)}.$$

At this point, it is natural to choose an adequate γ in order to guarantee that certain crucial exponents coincide, what allows us to explore the coupling between the equations. Indeed, by fixing $\gamma = 1-2\theta$ so that $2(\gamma+\theta-1) = \gamma - 1 = -2\theta$, after dropping the positive term there follows that

$$c_1 \int_\Omega u_k^r v_k^\theta (u_k+\varepsilon)^{1-\theta}(v_k+\varepsilon)^{-\theta} \leqslant \beta(1-\theta)\eta \int_\Omega |\nabla v_k|^2 v_k^{-2\theta}$$
$$+ \beta(1-\theta)C_\eta \int_{\Omega_+^k} |\nabla u_k|^2 (u_k+\varepsilon)^{-2\theta}. \tag{3.22}$$

However, remark that by (3.19) for every k fixed, we have $|\nabla u_k|^2 u_k^{-2\theta} \in L^1(\Omega_+^k)$. Thus by taking $\varepsilon \to 0$ in (3.22), employing the Fatou Lemma combined with the Dominated Convergence Theorem, we arrive at

$$c_1 \int_\Omega u_k^{r-\theta+1} \leqslant \beta(1-\theta)\eta \int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + \beta(1-\theta)C_\eta \int_{\Omega_+^k} |\nabla u_k|^2 u_k^{-2\theta}. \tag{3.23}$$

Now observe that

$$\int_\Omega u_k^{r-1+\gamma} v_k^{\theta+1} = \int_\Omega u_k^{r-2\theta} v_k^{\theta+1} \geqslant \int_{\{u_k \leqslant v_k\}} u_k^r u_k^{-2\theta} v_k^{\theta+1} \geqslant \int_{\{u_k \leqslant v_k\}} u_k^r v_k^{1-\theta}$$

and

$$\int_\Omega u_k^{r+1-\theta} = \int_\Omega u_k^r u_k^{1-\theta} \geqslant \int_{\{u_k \geqslant v_k\}} u_k^r v_k^{1-\theta} = \int_{\{u_k \geqslant v_k\}} u_k^r v_k^{1-\theta},$$

The latter estimates clearly guarantee that

$$\int_\Omega u_k^r v_k^{1-\theta} \leqslant \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^{r-2\theta} v_k^{\theta+1}. \tag{3.24}$$

Thus, by gathering (3.20), (3.23) and (3.24), and by using (3.19) twice, since $\gamma+\theta = 1-\theta$, there follows

$$\frac{\alpha(1-2\theta)}{d_2} \int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant \int_\Omega u_k^{r-\theta+1} + \int_\Omega u_k^{r-1+\gamma} v_k^{\theta+1} + \frac{1}{\tau_k} \int_\Omega v_k^{1-2\theta}$$
$$\leqslant \frac{\beta(1-\theta)\eta}{c_1} \int_\Omega |\nabla v_k|^2 v_k^{-2\theta} + C \int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k} \int_\Omega v_k^{1-2\theta},$$

where we combined (3.19) and (3.23) in the right-hand side. If we consider $\eta = \frac{\alpha(1-2\theta)c_1}{2\beta(1-\theta)d_2}$, it is easy to see that

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C \int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\int_\Omega v_k^{1-2\theta} < \infty, \tag{3.25}$$

which is finite for every k fixed.

At this point, for the sake of simplicity, observe that, given $\varepsilon > 0$, by the locally Lipschitz Chain Rule, $|\nabla v_k|^2 (v_k+\varepsilon)^{-2\theta} = \frac{1}{(1-\theta)^2}|\nabla(v_k+\varepsilon)^{1-\theta}|^2$. Thence, by combining the Sobolev Embedding, the Fatou Lemma and the Dominated Convergence Theorem, we have

$$\begin{aligned}\left(\int_\Omega v_k^{(1-\theta)2^*}\right)^{\frac{2}{2^*}} &\leqslant \liminf_{\varepsilon\to 0^+}\left(\int_\Omega (v_k+\varepsilon)^{(1-\theta)2^*}\right)^{\frac{2}{2^*}}\\ &\leqslant \liminf_{\varepsilon\to 0^+} C\left(\int_\Omega |\nabla(v_k+\varepsilon)^{1-\theta}|^2\right)^{\frac{2}{2^*}}\\ &= \int_\Omega |\nabla v_k|^2 v_k^{-2\theta},\end{aligned} \tag{3.26}$$

where in the latter estimates we used that $|\nabla v_k|^2 v_k^{-2\theta} \in L^1(\Omega)$ for every $k \in \mathbb{N}$, fixed. Moreover, observe that $\frac{(1-\theta)2^*}{1-2\theta} > 1$ and thus, by combining Hölder's inequality for $\frac{(1-\theta)2^*}{1-2\theta}$ and $\frac{(1-\theta)2N}{N+2-4\theta}$, with the last estimate, we arrive at

$$\begin{aligned}\int_\Omega v_k^{1-2\theta} &\leq C\Big(\int_\Omega v_k^{(1-\theta)2^*}\Big)^{\frac{1-2\theta}{(1-\theta)2^*}}\\ &\leq C\Big(\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\Big)^{\frac{1-2\theta}{2-2\theta}}.\end{aligned} \tag{3.27}$$

Further, from (3.25) and (3.27) with Young's inequality for $\frac{2-2\theta}{1-2\theta}$ and $2-2\theta$, we obtain

$$\begin{aligned}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} &\leqslant C\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k} C\Big(\int_\Omega |\nabla v_k|^2 v_k^{-2\theta}\Big)^{\frac{1-2\theta}{2-2\theta}}\\ &\leqslant C\Big(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\Big) + \frac{1}{2\tau_k}\int_\Omega |\nabla v_k|^2 v_k^{-2\theta},\end{aligned}$$

which clearly guarantees that

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C\Big(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\Big). \tag{3.28}$$

Now, by (3.23), (3.19) and (3.28), we get

$$\begin{aligned}\int_\Omega u_k^{r-\theta+1} &\leqslant C\Big(\int_\Omega f u_k^{1-2\theta} + \frac{1}{\tau_k}\Big)\\ &\leqslant C\Big(\|f\|_{L^m} ||u_k||^{1-2\theta}_{L^{(1-2\theta)m'}} + \frac{1}{\tau_k}\Big)\\ &\leqslant C\Big(\|f\|_{L^m} ||u_k||^{1-2\theta}_{L^{r-\theta+1}} + \frac{1}{\tau_k}\Big),\end{aligned}$$

where we used that $(1-2\theta)m' = \gamma m' \leqslant r-\theta+1$, for $m' \leqslant \frac{r-\theta+1}{1-2\theta}$. Further, by means of another application of the Young inequality, for $\frac{r-\theta+1}{1-2\theta}$ and $\frac{r-\theta+1}{r+\theta}$, after straightforward compensations, we end up with

$$\int_\Omega u_k^{r-\theta+1} \leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big). \tag{3.29}$$

In particular, by combining the above estimate with (3.28) we get

$$\int_\Omega |\nabla v_k|^2 v_k^{-2\theta} \leqslant C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big). \tag{3.30}$$

Further, by gathering (3.24), (3.19), with an analogous argument used to prove (3.30) we obtain

$$\int_\Omega u_k^r v_k^{1-\theta} \leq C\Big(\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} + \frac{1}{\tau_k}\Big).$$

Accordingly, remark that by the choice of q, if $r \geqslant \frac{N+2}{N-2}$, then $q < p$. Therefore, by the gradient estimates obtained above, its clear that both $\{u_k\}$ and $\{v_k\}$ are bounded in $W_0^{1,q}(\Omega)$. □

Incorporating estimates obtained by the Lemmas 3.2 and 3.3, we are able to prove Theorems 0.3 and 0.4.

3.3 Proof of Theorem 0.3

Proof. By Lemma 3.2, there exist subsequences still indexed by u_k and v_k and functions u and v in $W_0^{1,2}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega),\ u_k \to u \text{ in } L^{\sigma_1}(\Omega), \text{ and a.e. in } \Omega; \\ v_k \rightharpoonup v \text{ weakly in } W_0^{1,2}(\Omega),\ v_k \to v \text{ in } L^{\sigma_2}(\Omega), \text{ and a.e. in } \Omega, \end{cases} \tag{3.31}$$

where $u \geqslant 0$, $v \geqslant 0$ a.e. in Ω, and

$$\|u\|_{W_0^{1,2}}^2 + \|v\|_{W_0^{1,2}}^2 + \int_\Omega u^{r+\theta+1} + \int_\Omega u^r v^{\theta+1} \leqslant C\|f\|_{L^m}^{\frac{r+\theta+1}{r+\theta}},$$

and $\sigma_1 < \max\{2^*, r+\theta+1\}$ and $\sigma_2 < 2^*$. Analogous to the proof of the Theorem 0.1 we get

$$g(x,u_k,v_k) \to g(x,u,v) \text{ in } L^1(\Omega) \ \text{ and } \ h(x,u_k,v_k) \to h(x,u,v) \text{ in } L^1(\Omega). \tag{3.32}$$

By setting $\Gamma_k = ||\nabla u_k||_{L^2}^2$, since $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$, we have that $\{\Gamma_k\}_k^\infty$ is a bounded sequence of real numbers, which we may suppose converging to a certain real number Γ. It is clear that

$$\int_\Omega M(x)\nabla u_k \cdot \nabla\varphi \to \int_\Omega M(x)\nabla u \cdot \nabla\varphi.$$

By the latter arguments, in particular, we alredy have that

$$\begin{cases} -\text{div}\big((M(x)+\Gamma)\nabla u\big) + g(x,u,v) = f \ \text{ a.e. in } \Omega. \\ -\text{div}\big(M(x)\nabla v\big) = h(x,u,v) \ \text{ a.e. in } \Omega. \end{cases} \tag{3.33}$$

So to prove the theorem, we have to show that

$$\int_\Omega (M(x)+\Gamma_k)\nabla u_k \cdot \nabla\varphi \to \int_\Omega (M(x)+||\nabla u||_{L^2}^2)\nabla u \cdot \nabla\varphi.$$

We claim that $\Gamma \neq 0$. Indeed, otherwise we would have $u = 0$, which by (3.33) implies that $f \in L^\infty(\Omega)$, which creates a contradiction, because $f \in L^m(\Omega)$ with $m < (2^*)'$. Thus, since Γ_k is a nonnegative sequence and $\Gamma \neq 0$, there follows that $\Gamma > 0$.

Now we prove that $\Gamma = \|\nabla u\|_{L^2}^2$. First of all, let us remark that by (P$_5$)

$$\begin{aligned}\int_\Omega \big(M(x)+\Gamma_k\big)\nabla u_k\cdot\nabla\big(u_k-T_n(u)\big) &= \int_\Omega \big(M(x)+\Gamma_k\big)\nabla\big(u_k-T_n(u)\big)\cdot\nabla\big(u_k-T_n(u)\big)\\ &\quad+\int_\Omega \big(M(x)+\Gamma_k\big)\nabla T_n(u)\cdot\nabla\big(u_k-T_n(u)\big)\\ &\geqslant (\alpha+\Gamma_k)\int_\Omega |\nabla\big(u_k-T_n(u)\big)|^2\\ &\quad+\int_\Omega \big(M(x)+\Gamma_k\big)\nabla T_n(u)\cdot\nabla\big(u_k-T_n(u)\big).\end{aligned}$$

Further, by taking $\varphi = u_k - T_n(u)$ as test function in the first equation of (K_A) we get

$$\int_\Omega \big(M(x)+\Gamma_k\big)\nabla u_k\cdot\nabla(u_k-T_n(u)) + \int_\Omega g(x,u_k,v_k)(u_k-T_n(u)) = \int_\Omega f_k(u_k-T_n(u)).$$

In this fashion, by combining the last identity and the latter inequality we end up with

$$\begin{aligned}\alpha\int_\Omega |\nabla(u_k-T_n(u))|^2 + \int_\Omega g(x,u_k,v_k)(u_k-T_n(u)) &\leqslant \int_\Omega f_k(u_k-T_n(u))\\ &\quad-\int_\Omega \big(M(x)+\Gamma_k\big)\nabla T_n(u)\cdot\nabla(u_k-T_n(u))\end{aligned} \tag{3.34}$$

Now, note that by the Fatou Lemma, (3.32) and (P$_2'$) we have

$$0\leqslant \int_{\{|u|>n\}} g(x,u,v)u \leqslant \int_\Omega g(x,u,v)G_n(u) \leqslant \liminf_{k\to\infty}\int_\Omega g(x,u_k,v_k)(u_k-T_n(u)).$$

Moreover, remark that, up to subsequences, $f_k(u_k-T_n(u)) \to fG_n(u)$ in $L^1(\Omega)$. As a matter of fact, by using that $m \geqslant (r+\theta+1)'$ and that $\{u_k - T_n(u)\}$ is bounded in $L^{r+\theta+1}(\Omega)$, given $E\subset\Omega$, measurable, Hölder's inequality guarantees that

$$\int_E |f_k||u_k-T_n(u)| \leqslant C\Big(\int_E |f_k|^m\Big)^{\frac{1}{m}}.$$

However, by the very choice of f_k we have that $f_k\to f$ in $L^m(\Omega)$, so that, $\{|f_k|^m\}$ in uniformly integrable, and, by the latter inequality we have that $\{f_k\big(u_k-T_n(u)\big)\}$, in uniformly integrable. Since it clear that $f_k(u_k-T_n(u))$ a.e. in Ω, hence, by the Vitali Convergence theorem, our claim holds true. In particular, there follows that

$$\lim_{k\to\infty}\int_\Omega f_k(u_k-T_n(u)) = \int_\Omega fG_n(u).$$

Now, recall that $\nabla T_n(u)\cdot\nabla G_n(u) = |\nabla u|^2 T_n'(u)G_n'(u) = 0$ a.e. in Ω, since $T_n'(s)G_n'(s)=0$ for a.e. $s\in\mathbb{R}$. Then, by (3.31),

$$\lim_{k\to\infty}\int_\Omega \big(M(x)+\Gamma_k\big)\nabla T_n(u)\nabla(u_k-T_n(u)) = \int_\Omega \big(M(x)+\Gamma\big)\nabla T_n(u)\nabla G_n(u) = 0.$$

Thus, by plugging the latter convergences in (3.40) we obtain

$$\lim_{k\to\infty}\alpha\int_\Omega |\nabla(u_k-T_n(u))|^2 \leqslant \int_\Omega fG_n(u). \tag{3.35}$$

However, we realise that we can write $u = T_n(u) + G_n(u)$, so that

$$\int_\Omega |\nabla(u_k - u)|^2 = \int_\Omega |\nabla(u_k - T_n(u) + G_n(u))|^2 \leqslant 2\int_\Omega |\nabla(u_k - T_n(u))|^2 + 2\int_\Omega |\nabla G_n(u)|^2$$

taking the limit when $k \to \infty$ on the above inequality and by (3.35) we get

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega |\nabla(u_k - u)|^2 &\leqslant \lim_{k\to\infty} 2\int_\Omega |\nabla(u_k - T_n(u))|^2 + 2\int_\Omega |\nabla G_n(u)|^2 \\ &\leqslant \frac{2}{\alpha}\int_\Omega fG_n(u) + 2\int_\Omega |\nabla G_n(u)|^2 \\ &\leqslant \frac{4}{\alpha}\int_{|u|>n} fu + 2\int_{|u|>n} |\nabla u|^2 \end{aligned} \tag{3.36}$$

Nevertheless, since $\operatorname{meas}(\{|u| > n\}) \to 0$ as $n \to \infty$, then given $\epsilon > 0$, there exists $n_\epsilon > 0$, such that for $n > n_\epsilon$ we have

$$\frac{2}{\alpha}\int_\Omega fG_n(u) + 2\int_\Omega |\nabla G_n(u)|^2 < \epsilon.$$

Consequently, since ϵ is arbitrary by (3.36) there follows that $\lim_{k\to\infty} \int_\Omega |\nabla(u_k - u)|^2 = 0$, which means $u_k \to u$ in $W_0^{1,2}(\Omega)$.

Therefore by (3.33) we can pass the limit on the approximate problem (K_A) and get that (u, v) are solutions of the problem (K). $\square$

3.4 Proof of Theorem 0.4

Proof. By Lemma 3.3, there exist $\{u_k\} \subset W_0^{1,p}(\Omega)$, $\{v_k\} \subset W_0^{1,q}(\Omega)$, and $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,q}(\Omega)$ such that, up to subsequences relabeled the same,

$$\begin{aligned} &u_k \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\ u_k \to u \text{ in } L^{s_1}(\Omega),\ \text{and a.e. in } \Omega; \\ &v_k \rightharpoonup v \text{ weakly in } W_0^{1,q}(\Omega),\ v_k \to v \text{ in } L^{s_2}(\Omega),\ \text{and a.e. in } \Omega, \end{aligned}$$

where $p = \frac{2(r-\theta-1)}{r+\theta+1}$, $q = \frac{2N(1-\theta)}{N-2\theta}$, $u \geqslant 0$, $v \geqslant 0$ a.e. in Ω, and

$$\|u\|^p_{W_0^{1,p}} + \int_\Omega u^{r-\theta+1} + \int_\Omega u^r v^{1-\theta} \leqslant C\|f\|_{L^m}^{\frac{r-\theta+1}{r+\theta}} \text{ and } \|v\|^q_{W_0^{1,q}} \leq C\|f\|_{L^m}^{\frac{N(r-\theta+1)}{(N-2\theta)(r+\theta)}}, \tag{3.37}$$

and $s_1 < \max\{p^*, r-\theta+1\}$ and $s_2 < q^*$. Similar to the proof of the Theorem 0.1 we get

$$g(x, u_k, v_k) \to g(x, u, v) \text{ in } L^1(\Omega) \ \text{ and } \ h(x, u_k, v_k) \to h(x, u, v) \text{ in } L^1(\Omega). \tag{3.38}$$

Although the argument for obtaining the convergence of the term local $||\nabla u_k||^p_{L^p}$ in the case $m < (r+\theta+1)'$, be equal to the one used in the Theorem 0.3, some subtle difficulties arise because we are in a regime of weaker regularity for f. Thus, we chose to present the details.

Setting $\Lambda_k = ||\nabla u_k||^p_{L^p}$, since $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$, we have $\{\Lambda_k\}_{k=1}^\infty$ is a sequence of real numbers, which we way suppose converges to some real number Λ. In this case, we also observed that $\Lambda \neq 0$. Actually, otherwise we would have $u = 0$, which by (3.33) implies that $f \in L^\infty(\Omega)$, which creates a contradiction, because $f \in L^m(\Omega)$ with $m < (r+\theta+1)'$. Thus, since Λ_k is a nonnegative sequence and $\Lambda \neq 0$, there follows that $\Lambda > 0$.

To finalise the proof we show $\Lambda = \|\nabla u\|^p_{L^p}$. First of all, let us remark that by (P_5)

$$\int_\Omega \big(M(x)+\Lambda_k\big)\nabla u_k\cdot\nabla\big(u_k-T_n(u)\big) \geqslant (\alpha+\Lambda_k)\int_\Omega \big|\nabla\big(u_k-T_n(u)\big)\big|^2 + \int_\Omega \big(M(x)+\Lambda_k\big)\nabla T_n(u)\cdot\nabla\big(u_k-T_n(u)\big). \tag{3.39}$$

Now for $n>0$ taking $\varphi = u_k - T_k(u)$ as test function in the first equation (K_A) we get

$$\int_\Omega (M(x)+\Lambda_k)\nabla u_k\cdot\nabla(u_k-T_n(u)) + \int_\Omega g(x,u_k,v_k)(u_k-T_n(u)) = \int_\Omega f_k(u_k-T_n(u)).$$

In this way, by combining the last identity with (3.39) we get

$$\alpha\int_\Omega |\nabla(u_k-T_n(u))|^2 + \int_\Omega g(x,u_k,v_k)(u_k-T_n(u)) \leqslant \int_\Omega f_k(u_k-T_n(u)) - \int_\Omega \big(M(x)+\Lambda_k\big)\nabla T_n(u)\cdot\nabla(u_k-T_n(u)). \tag{3.40}$$

On one hand by the Fatou Lemma, (3.38) and (P'_2) we have

$$0\leqslant \int_{\{|u|>n\}} g(x,u,v)u \leqslant \int_\Omega g(x,u,v)G_n(u) \leqslant \liminf_{k\to\infty}\int_\Omega g(x,u_k,v_k)(u_k-T_n(u)),$$

on the other hand, since $m' < r-\theta+1$, as $\{u_k\}$ is uniformly bounded in $L^{r-\theta+1}(\Omega)$ there follows that $\{u_k - T_n(u)\}$ is bounded in $L^{m'}(\Omega)$. Consequently up to subsequence $u_k - T_n(u) \rightharpoonup G_n(u)$ weakly in $L^{m'}(\Omega)$. Thus, as $f_k \to f$ in $L^m(\Omega)$ we have

$$\lim_{k\to\infty}\int_\Omega f_k(u_k-T_n(u)) = \int_\Omega fG_n(u).$$

Moreover, since $\{u_k\}$ is bounded in $W_0^{1,p}(\Omega)$ there follows that $\nabla(u_k-T_n(u)) \rightharpoonup \nabla(G_n(u))$ in $(L^p(\Omega))^N$, so

$$\lim_{k\to\infty}\int_\Omega (M(x)+\Lambda_k)\nabla T_n(u)\cdot\nabla(u_k-T_n(u)) = \int_\Omega (M(x)+\Lambda)\nabla T_n(u)\cdot\nabla(u-T_n(u)) = 0$$

for the reason that $\nabla T_n(u)\cdot\nabla G_n(u) = |\nabla u|^2 T'_n(u)G'_n(u) = 0$ a.e. in Ω, since $T'_n(s)G'_n(s)=0$ for a.e. $s\in\mathbb{R}$. Thus, using that $p<2$ by (3.40) we obtain

$$C\lim_{k\to\infty}\alpha\int_\Omega |\nabla(u_k-T_n(u))|^p \leqslant \lim_{k\to\infty}\alpha\int_\Omega |\nabla(u_k-T_n(u))|^2 \leqslant \int_\Omega fG_n(u). \tag{3.41}$$

However, we realise that we can write $u = T_n(u)+G_n(u)$, so that

$$\int_\Omega |\nabla(u_k-u)|^p = \int_\Omega |\nabla(u_k-(T_n(u)+G_n(u))|^p \leqslant 2^p\int_\Omega |\nabla(u_k-T_n(u))|^p + 2^p\int_\Omega |\nabla G_n(u)|^p.$$

Hence, by taking the limit and then from (3.41) there follows that

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |\nabla(u_k-u)|^p &\leqslant 2^p\lim_{k\to\infty}\int_\Omega |\nabla(u_k-T_n(u))|^p + 2^p\int_\Omega |\nabla G_n(u)|^p\\ &\leqslant \frac{2^p}{\alpha}\int_\Omega fG_n(u) + 2^p\int_\Omega |\nabla G_n(u)|^p\\ &= \frac{2^{p+1}}{\alpha}\int_{|u|>n} fu + 2^p\int_{|u|>n}|\nabla u|^p.\end{aligned} \tag{3.42}$$

Nevertheless, since meas$(\{|u| > n\}) \to 0$ as $n \to \infty$, then given $\epsilon > 0$, there exists $n_\epsilon > 0$, such that for $n > n_\epsilon$ we have

$$\frac{2^{p+1}}{\alpha} \int_{|u|>n} fu + 2^p \int_{|u|>n} |\nabla u|^p < \epsilon.$$

Consequently, since ϵ is arbitrary by (3.42) we conclude that $\lim_{k\to\infty} \int_\Omega |\nabla(u_k - u)|^p = 0$. Therefore, we can take the limit on the approximate problem (K_A) and get that (u, v) are solutions of the problem (K). $\square$

Chapter 4

Appendix A

The following lemma is a fundamental instrument to obtain the chain rule of function in Sobolev space.

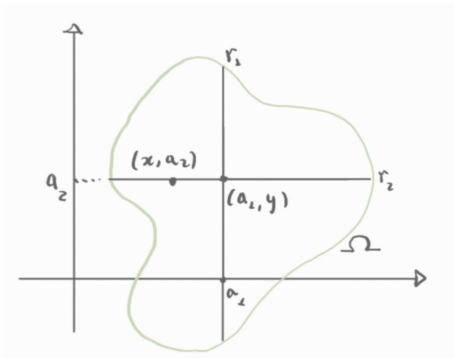
Definition 4.1. *A function $u : [a,b] \to \mathbb{R}$ is said to be absolutely continuous if for every $\epsilon > 0$ there exist $\delta > 0$ such that if $a = x_1 < y_1 \leqslant x_2 < y_2 \leqslant \ldots \leqslant x_m < y_m = b$ is a partition of $[a,b]$ with*

$$\sum_{k=1}^{l} |b_k - a_k| \leqslant \delta.$$

Then

$$\sum_{k=1}^{l} |u(b_k) - u(a_k)| \leqslant \epsilon.$$

Moreover, if $u : \Omega \to \mathbb{R}$ where Ω be an open set in $\mathbb{R}^{\mathbb{N}}$ we say that is s absolutely continuous on almost all line segments in Ω parallel to the coordinate axes if $u(a_1, \ldots, a_{i-1}, \cdot, a_{i+1}, \ldots, a_N) : I \to \mathbb{R}$ is absolutely continuous for each $i = 1, \ldots, N$, e.g. in the case where $N = 2$, u restricted to the remains r_1 and r_2 is absolutely continuous.



Lemma 4.1. *Let $u \in L^p(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ where $p \geqslant 1$. If, and only if, u has a representative $\overline{u}$ that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes and whose (classical) partial derivatives belong to $L^p(\Omega)$.*

Proof. The reader is referred to [11, p. 293] or [19, p. 44]. □

Theorem 4.1. ***(Rademacher)*** *Let Φ be locally Lipschitz continuous in Ω. Then Φ is differentiable almost everywhere in Ω. See [14, p. 296].*

The proof of the next result where $\Phi \in C^1(\mathbb{R})$ can be easily found at [11, 14] and [19]. However the proof is subtly omitted when Φ is a Lipschitz continuous function of $\mathbb{R}$ into itself. That being the case, we have chosen to present our demonstration for such a result.

Theorem 4.2. *(Chain Rule Sobolev). Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $\Phi(0) = 0$ and $u \in W^{1,p}(\Omega)$ where $1 < p < \infty$. Then $\Phi \circ u \in W_0^{1,p}(\Omega)$ and*

$$\frac{\partial(\Phi \circ u)(x)}{\partial x_i} = \Phi'(u(x))\frac{\partial u(x)}{\partial x_i}.$$

Proof. As the space $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$, given $u \in W_0^{1,p}(\Omega)$ there exists $u_k \in C_c^\infty(\Omega)$ such that $u_k \to u$ in $W^{1,p}(\Omega)$. Consider $v_k = \Phi \circ u_k$. Since u_k has compact support and $\Phi(0) = 0$, there follows that v_k has compact support. In addition, as Φ is Lipschitz continuous and u_k is a smooth function with compact support, we get

$$|v_k(x) - v_k(y)| = |\Phi(u_k(x)) - \Phi(u_k(y))| \leqslant L|u_k(x) - u_k(y)| \leqslant L_k|x - y|$$

which implies that v_k is Lipschitz continuous and consequently absolutely continuous on almost all line segments in Ω parallel to the coordinate axes. Hence taking $x + te_i$ where e_i is the i^{th} coordinate vector for $i = 1, \ldots, N$, we have

$$\left|\frac{\partial v_k}{\partial x_i}(x)\right| = \lim_{t\to\infty} \frac{|v_k(x+te_i) - v_k(x)|}{|t|} \leqslant \lim_{t\to\infty} \frac{L|u_k(x+te_i) - u_k(x)|}{|t|} = L\left|\frac{\partial u_k}{\partial x_i}(x)\right|,$$

consequently $\dfrac{\partial v_k}{\partial x_i}(x) \in L^p(\Omega)$. Thus, by Lemma 4.1 we obtain $v_k \in W_0^{1,p}(\Omega)$.

Now, note that

$$|v_k(x) - \Phi(u(x))| = |\Phi(u_k(x)) - \Phi(u(x))| \leqslant L|u_k(x) - u(x)|$$

it follows that $v_k \to \Phi \circ u$ in $L^p(\Omega)$ when $k \to \infty$. Since $\left\{\dfrac{\partial v_k}{\partial x_i}\right\}$ is bounded for each $1 \leqslant k \leqslant n$, hence $\dfrac{\partial v_k}{\partial x_i}(x) \rightharpoonup \omega_i$ weakly in $L^p(\Omega)$ up to subsequence. Thus give $\psi \in C_c^\infty(\Omega)$, there follows that

$$\int_\Omega \psi \cdot \omega_i = \lim_{k\to\infty} \int_\Omega \psi \frac{\partial v_k}{\partial x_i} = -\lim_{k\to\infty} \int_\Omega v_k \frac{\partial \psi}{\partial x_i} = -\int_\Omega v \frac{\partial \psi}{\partial x_i}$$

by definition of a weak derivative we have $\omega_i = \frac{\partial v}{\partial x_i}$, and so $\Phi \circ u \in W_0^{1,p}(\Omega)$.

Finally, as Φ is Lipschitz by Theorem 4.1 we may conclude that

$$\begin{aligned}
\int_\Omega (\Phi \circ u)\frac{\partial \psi}{\partial x_i} &= \lim_{j\to\infty} \int_\Omega (\Phi \circ u_j)\frac{\partial \psi}{\partial x_i} \\
&= -\lim_{j\to\infty} \int_\Omega \frac{\partial(\Phi \circ u_j)}{\partial x_i}\psi \\
&= -\lim_{j\to\infty} \int_\Omega (\Phi' \circ u_j)\frac{\partial u_j}{\partial x_i}\psi \\
&= -\int_\Omega (\Phi' \circ u)\frac{\partial u}{\partial x_i}\psi \quad \forall\, \psi \in C_0^\infty.
\end{aligned}$$

Therefore by definition of a weak derivative we get

$$\frac{\partial(\Phi \circ u)(x)}{\partial x_i} = \Phi'(u(x))\frac{\partial u(x)}{\partial x_i}.$$

$\square$

In order to ensure the legitimacy of the test functions taken to obtain the priori estimates, we established a stronger version of the Chain Rule.

Corollary 4.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function such that $\Phi(0) = 0$ and $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ such that $||u||_{L^\infty} < C$ where $1 < p < \infty$. Then $\Phi \circ u \in W_0^{1,p}(\Omega)$ and*

$$\frac{\partial(\Phi \circ u)(x)}{\partial x_i} = \Phi'(u(x))\frac{\partial u(x)}{\partial x_i} \text{ a.e. in } \Omega$$

Proof. Consider $\Psi = \Phi \circ T_k$ where T_k is Stampacchia truncation function defined at the beginning of Chapter 1. Note that Ψ is a locally continuous function, Indeed, give $x \in \mathbb{R}$, let V be a neighborhood, such that $T_k(x) \in V$. As Φ by hypothesis is locally Lipschitz and T_k is Lipschitz, there follows that

$$|\Psi(x) - \Psi(y)| = |\Phi(T_k(x)) - \Phi(T_k(y))| \leqslant L|T_k(x) - T_k(y)| \leqslant L|x - y| \quad \forall y \in \mathbb{R}.$$

Moreover, since $||u||_{L^\infty}$ and $\text{meas}(\Omega) < \infty$ we get

$$|\Psi(u)| = |\Psi(u) - \Psi(0)| = |\Phi(T_k(u)) - \Phi(T_k(0))| \leqslant L|T_k(x) - 0| = L|u| \leqslant C$$

where we conclude $\Psi \circ u \in L^\infty(\Omega)$. Thus, by Theorem 4.2 we have $\Psi(u) \in W^{1,p}(\Omega)$ and

$$\frac{\partial \Psi}{\partial x_i}(u(x)) = \Psi'(u(x))\frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega.$$

Taking $||u||_{L^\infty} < k$, since $T'(u) = 1$ in $\{x \in \Omega; |u(x)| < k\}$ we obtain

$$\Psi'(u(x)) = \Phi'(T_k(u(x)))T_k'(u(x)) = \Phi'(u(x)).$$

In this way, as $\Psi' \circ u = \Phi' \circ u$ a.e. in Ω, there follows that

$$\frac{\partial(\Phi \circ u)(x)}{\partial x_i} = \Phi'(u(x))\frac{\partial u(x)}{\partial x_i} \text{ a.e. in } \Omega.$$

$\square$

Let us look at some results of the Measure Theory that were important for the development of the work.

Definition 4.2. *Let (Ω, Σ, μ) be a measure spaces and u_k, u functions in Ω from $\mathbb{R}$, Σ - measurable.*

(i) *$u_k \to u$ a.e. in Ω if there exists a set Ω_0 in Ω with $\mu(\Omega_0) = 0$ such that for each $\varepsilon > 0$ and $x \in \Omega \backslash \Omega_0$ there exists a natural number $N(\varepsilon, x)$ such that if $k \geqslant N(\varepsilon, x)$, then $|u_k(x) - u(x)| < \varepsilon$.*

(ii) *Let $L^p(\Omega) = L^p(\Omega, \Sigma, \mu)$ with $1 \leqslant p < \infty$. A sequence $u_k \to u$ in $L^p(\Omega)$ if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $k \geqslant N(\varepsilon)$, then*

$$||u_k - u||_{L^p} = \Big(\int_\Omega |u_k - u|^p\Big)^{\frac{1}{p}} < \varepsilon.$$

By Egoroff Theorem establishes a condition for achieving near-uniform convergences, based on the hypothesis that the sequence converges almost everywhere. Even though it is a classic result of the measure theory, we chose to state and present its demonstration.

Theorem 4.3 (Egoroff). *Let $\mu(\Omega) < \infty$ and $u_k \to u$ a.e. in Ω. Then u_k converges almost uniformly to u and in measure.*

Proof. If $u_k \to u$ a.e. $x \in \Omega$ then there exists $M \subset \Omega$ with $\mu(M) = 0$ such that $\forall \varepsilon > 0$ and $\forall x \in \Omega \backslash M$ there exists $N(\varepsilon, x) \in \mathbb{N}$ such that $k > N(\varepsilon, x)$ there is

$$|u_k(x) - u(x)| < \varepsilon.$$

Given $m, n \in \mathbb{N}$, define

$$E_k(m) = \bigcup_{j=k}^{\infty} \{x \in \Omega;\ |u_j(x) - u(x)| > m^{-1}\}.$$

Note that, for fixed m

$$E_k(m) = \{x \in \Omega;\ |u_k(x) - u(x)| > m^{-1}\} \bigcup \bigcup_{k=n+1}^{\infty} \{x \in \Omega;\ |u_k(x) - u(x)| > m^{-1}.\}$$

So $E_{k+1}(m) \subset E_k(m)$, moreover clearly $E_k(m) \in \Sigma$. As by hypothesis $u_k \to u$ a.e. we have that

$$\bigcap_{k=1}^{\infty} E_k(m) = \emptyset.$$

Thus since $\mu(\Omega) < +\infty$ and $E_{k+1}(m) \subset E_k(m)$ we obtain

$$0 = \lim_{n\to\infty} \mu\left(\bigcap_{k=1}^{\infty} E_k(m)\right) = \lim_{k\to\infty} \mu(E_k(m)).$$

Now given $\delta > 0$, let k_n such that $\mu(E_{j_k}(m)) < \frac{\delta}{2^k}$ and $E_\delta = \bigcup_{k=1}^{\infty} E_{j_k(m)}$. Note that $E_\delta \in \sigma$ and

$$\mu(E_\delta) = \mu\left(\bigcap_{k=1}^{\infty} E_{j_k}(m)\right) = \sum_{k=1}^{\infty} \mu(E_{j_k}(m)) < \delta.$$

Then if $x \notin E_\delta$ follows that $x \in E_{j_k}(m)$, so for $j > j_k$ we get

$$|u_k(x) - u(x)| < \frac{1}{m}\ \forall x \in \Omega \backslash E_\delta.$$

$\square$

The next result is a consequence of the Egoroff Theorem and quite useful for proving convergence of nonlinear terms.

Theorem 4.4 (Vitali). *Let $\{u_k\}, u \in L^p(\Omega)$ where $1 \leqslant p < \infty$. Suppose that*

(i) $u_k \to u$ *a.e. in* Ω

(ii) $\lim_{\mu(E)\to 0} \int_\Omega |u_k|^p = 0$ *uniformly in k, where $E \subset \Omega$ is a mensurable subset.*

Then $u_k \to u$ in $L^p(\Omega)$.

Proof. Consider $E \subset \Omega$ a mensurable subset, so that

$$\begin{aligned}\int_\Omega |u_k - u|^p &= \int_{\Omega\setminus E} |u_k - u|^p + \int_E |u_k - u|^p \\ &\leqslant \int_{\Omega\setminus E} |u_k - u|^p + 2^{p-1}\int_E (|u_k|^p - |u|^p).\end{aligned}$$

Given $\varepsilon > 0$, by hypothesis (ii) there exist $\delta_1(\varepsilon) > 0$ such that $\mu(E) < \delta_1(\varepsilon)$ implies that

$$\int_E |u_k|^p < \frac{\varepsilon}{2^{p+1}} \quad \forall\ k.$$

Moreover, as $f \in L^p(\Omega)$ there exists $\delta_2(\varepsilon) > 0$ such that $\mu(E) < \delta_2(\varepsilon)$ implies that

$$\int_E |u|^p < \frac{\varepsilon}{2^{p+1}}.$$

Taking $\delta = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ by hypothesis (i) and Theorem 4.3 exists $N(\varepsilon) \in \mathbb{N}$ such that $k > N(\varepsilon)$ we have

$$\int_{\Omega\setminus E} |u_k - u|^p \frac{\varepsilon}{2} \quad \text{where} \quad \text{meas}(\Omega) < \delta.$$

Therefore since $k > N(\varepsilon)$ we obtain

$$\int_\Omega |u_k - u|^p < \varepsilon$$

that is $u_k \to u$ in $L^p(\Omega)$. □

Now we will introduce a fixed-point result for operators over Banach spaces. Schauder's well-known Theorem. See [16].

Before starting Schauder's theorem, we need the following definition:

Definition 4.3. *Let E a Banach space. A map $T : E \to E$ is completely continuous if it is continuous and if, for every bounded subset $B \subset E$, $\overline{T(B)}$ is compact.*

Theorem 4.5. *Let $T : K \subset E \to E$ be a completely continuous map, where K is a convex, bounded, closed and invariant subset of E. Then T has a fixed point in K.*

The main tool used in the proof Proposition 3.1 is an important result, a kind of Minty-Browder Theorem. Below, we recall the definition of pseudomonotone operators

Definition 4.4. *Let E be a Banach space, E' dual space and $A : E \to E'$ an operator. We say that A is pseudomonotone if $u_k \to u$ in E and*

$$\limsup_{k\to+\infty} < Au_k, u_k - u > \leqslant 0,$$

then, $\liminf_{k\to+\infty} < Au_k, u_k - v > \geqslant < Au, u - v > \quad \forall v \in E$.

Theorem 4.6. *(Minty - Browder). Let E be a reflexive and separable Banach space and $A : E \to E'$ an operator satisfying*

(ii) A is coercive, i.e.,

$$\frac{< Au, u >}{||u||_E} \to \infty \quad \textit{as} \quad ||u||_E \to \infty;$$

(ii) A is bounded and continuous;

(iii) A is pseudomotone.

Then, A is surjective, that is , $A(E) = E'$.

Proof. For a detailed proof of this result, we recommend that the reader see [6, p. 38]. □

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