Ends of complete gradient $\rho-$Einstein solitons

Hector Andrés Rosero García

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Advisor:
Prof. Dr. João Paulo dos Santos
Co-advisor:
Prof. Dr. Valter Borges Sampaio Junior

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Hector Andres Rosero Garcia*

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Prof. Dr. João Paulo dos Santos – MAT/UnB (Orientador)

Prof. Dr. Tarcísio Castro Silva – MAT/UnB (Membro)

Prof. Dr. Ernani de Sousa Ribeiro Junior – UFC (Membro)

Prof. Dr. Rondinelle Marcolino Batista – UFPI (Membro)

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A mis abuelos...

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Abstract

In this thesis we consider ends of complete gradient ρ-Einstein solitons by adapting and extending the techniques used to describe ends of Ricci solitons. For shrinking Schouten solitons we show that there is at most one f-non-parabolic end, where f stands for the potential function. Also, under an appropriate bound on the scalar curvature, we show that all ends of a shrinking Schouten soliton are non-parabolic. With no additional assumptions, we show that an expanding Schouten soliton must be connected at infinity, that is, it has only one end, unless it is a rigid Ricci soliton. Regarding $\rho-$Einstein solitons with $\rho \in \left[0, \frac{1}{2(n-1)}\right)$, we provide bounds on the scalar curvature for a shrinking soliton to be connected at infinity.

Keywords: gradient ρ-Einstein solitons, ends, parabolicity, connectedness at infinity.

Resumo

Título: Fins de solitons $\rho-$Einstein gradiente completos

Nesta tese, consideramos fins de solitons $\rho-$Einstein gradiente completos, adaptando e estendendo técnicas usadas para descrever fins de solitons de Ricci. Para Schouten solitons shrinking, mostramos que existe no máximo um fim $f-$não-parabólico, em que f é a função potencial do soliton. Também, sob limitantes apropriados da curvatura escalar, mostramos que todos os fins de um soliton Schouten shrinking devem ser não-parabólicos. Sem hipóteses adicionais, mostramos que um soliton Schouten expanding é conexo no infinito, isto é, possui apenas um fim, a menos que seja um soliton de Ricci rígido. Quanto aos $\rho-$ Einstein solitons com $\rho \in \left[0, \frac{1}{2(n-1)}\right)$, fornecemos limitantes na curvatura escalar para que um soliton shrinking seja conexo no infinito.

Palavras-chave: $\rho-$Einstein solitons gradiente, fins, parabolicidade, conexidade no infinito.

Contents

Introduction

In this thesis, we approach an interesting problem in geometry which has, as most of them, an inviting, simple statement, and can be easily understood when working with simple geometrical objects such as curves or surfaces. However, its solution can become quite technical and counter-intuitive when applied to more complex geometrical objects such as high dimensional manifolds. Namely, our main objective is to count ends of certain manifolds or, more precisely, to establish conditions for them to have only one end. In simple terms, an end of a manifold is the "remaining part" we get when we subtract a big enough compact from it. The simpler examples can be seen if we consider a plane or a cylinder. In the first case, by subtracting any compact we still get a connected component with a "hole" in it. It is natural then to say that a plane has only one end (see Figure 1 (left)). In the case of a cylinder C, if we subtract a "small" compact Ω_0 from it, it is possible we can also get a unique connected component. Nevertheless, if the compact taken out is big enough (for example, a ball whose diameter exceeds the length of a geodesic circle on C), we will end up with two different connected components and, no matter how bigger we take out a second compact $\Omega_1 \supset \Omega_0$, we will still have two connected non-compact components remaining. In this case, we conclude C is a 2-dimensional manifold with two ends. A final example of a manifold with three ends can be seen in Figure 1 (right), reproduced from an example given in [1].

In general, manifolds can have as many ends as we would like, independent of their dimension, even infinitely many of them. Interesting examples of this kind are given in the work of Mazet *et al.* in [37], where the authors present minimal $2-$dimensional surfaces with infinitely many ends. Therefore, it becomes of interest to classify certain $n-$dimensional manifolds depending on whether they are connected at infinity (*i.e.* it has only one end) or not.

Regarding the manifolds we would like to classify, we will focus throughout this work on solitons of certain flows, that have been widely studied in recent years. Such solitons

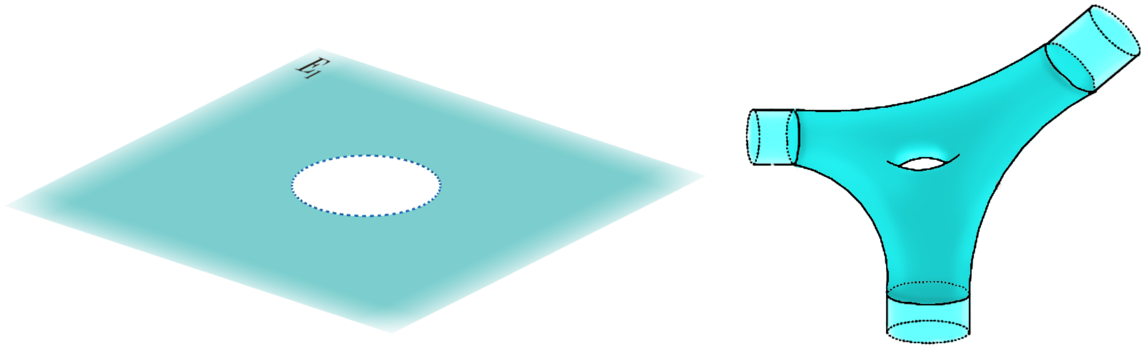

Figure 1: A One end (left) and a Three ends manifold (right).

are generalizations of the well-known gradient Ricci solitons, manifolds where the equation

$$\mathrm{Ric} + \nabla^2 f = \lambda g,$$

is satisfied for a certain function $f \in C^2(M)$ and $\lambda \in \mathbb{R}$. Ricci solitons have been widely studied since the 1980's decade when introduced by Richard Hamilton in [26] (among other papers by the same author). They act like a natural generalization of Einstein metrics, in which the equation $R_{ij} = \lambda g_{ij}$ is satisfied by the components of the Riemannian curvature and the Riemannian metric. Ricci solitons appear as self-similar solutions of the Ricci flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}$$

introduced by Hamilton himself in [25] (see also [8]), and gained even more interest from the mathematicians after used by Hamilton and (more remarkably) Perelman to solve the famous *Millennium problem* of the Poincaré's conjecture ([10, 49, 50]). Since their appearance, they have been classified in many senses and we refer the interested reader to [8,27,52] for an extensive list of results and classifications. Subsequently, several generalizations to the Ricci flow have appeared in literature with the aim of generalizing Ricci soliton classifications. Most of such generalizations are inspired by the Ricci-Bourguignon flow, first introduced by Jean-Pierre Bourguignon in his early 80's work [5] (published one year before Hamilton's work) and given by the evolution equation

$$\frac{\partial}{\partial t} g = -2\left(\mathrm{Ric} - \rho R g\right),$$

from which Ricci solitons themselves are a particular case. Ricci-Bourguignon flow has been studied by Catino *et al.* in [11] where the authors give results on the existence of solutions and curvature estimates. In particular, Catino and co-authors proved that self-similar solutions of the Ricci-Bourguignon flow satisfy the soliton equation

$$\mathrm{Ric} + \nabla^2 f = (\rho R + \lambda)\, g$$

and are known as $\rho-$Einstein solitons. Solitons defined above, as well as other solitons derived from the Ricci-Bourguignon equation, are regarded as *shrinking*, *steady*, or *expanding* depending on whether λ is positive, zero, or negative, respectively. We observe that due to the rescaling properties of the tensors, it is usual to find in the literature the use of *normalized solitons*, this is, the assumption of $\lambda = -1/2, 0$ or $1/2$ when M is expanding, steady or shrinking respectively, instead of the more general assumptions $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. We will not assume such normalization in this work.

The function f satisfying the soliton equations above (and other gradient soliton equations related to the Ricci-Bourguignon flow) is known as the *potential function* of the soliton M, and it plays an important role on the study of the soliton itself. In that sense, an important amount of the geometric analysis developed for solitons is based on their view as smooth metric measure spaces (M, g, e^{-f}) more than just the Riemannian manifolds (M, g), and, as will be seen in this work, classification of ends can be analyzed from either point of view in order to count the ends of M. While the ends' counting brings topological information about a Ricci soliton that is of interest itself, it also brings information on the construction of new examples of Ricci solitons. One example of this comes from the results of Ovidiu Munteanu and Jiaping Wang in [40], where they proved a steady Ricci soliton must be connected at infinity. As they observe, this also leads to conclude that a new example of steady Ricci soliton cannot be constructed as the connected sum of two steady Ricci solitons (see also [43]). This relation with the construction of new examples gives additional motivation to obtain topological information from more general solitons, in our case, of Schouten and $\rho-$Einstein type.

As for the ends' classification and subsequent counting, methods can become quite technical when working with Ricci solitons and other solitons in general. Several authors such as Li, Munteanu and Wang ([31, 33, 39–44, 56], among others) have developed an important amount of techniques based on geometric analysis in order to characterize and count ends of manifolds and, particularly, Ricci solitons. It is worth mentioning that the number of ends of manifolds has been studied widely as-well in non-soliton structures. In early 70's, Cheeger and Gromoll [16] proved their famous *splitting* theorem, which states that a manifold M^n with non-negative Ricci curvature must be isometric to a Riemannian product $\mathbb{R}^{n-k} \times N^k$, where N does not contain a line, this, in turn, implies such manifold must have at most two ends. Still in that sense, in [6], Cai proved that any Riemannian manifold with non-negative Ricci curvature outside a compact must have finitely many ends. More recently, in [2], Batista and collaborators have proved that a steady $m-$quasi-Einstein manifold must be connected at infinity and have shown the existence of non-parabolic ends for the shrinking case.

Here, we aim to apply techniques from geometric analysis for *Schouten* solitons, defined as self-similar solutions of the soliton equation

$$\mathrm{Ric} + \nabla^2 f = \left(\frac{R}{2(n-1)} + \lambda \right) g,$$

and the aforementioned shrinking $\rho-$*Einstein* solitons, satisfying $\mathrm{Ric} + \nabla^2 f = (\rho R + \lambda)\, g$ with $\lambda > 0$.

As it can be seen, Ricci and Schouten solitons arise as particular cases of $\rho-$Einstein solitons for $\rho = 0$ and $\rho = \frac{1}{2(n-1)}$ respectively, and, as well as the Ricci soliton case, the quantity $\rho = \frac{1}{2(n-1)}$ itself is far from being an arbitrary choice and indeed is often found in the study of solitons as an interesting limit-case. An example of this is given by Hamilton in [27]: If the initial metric (soliton at $t = 0$) is an Einstein metric with $\lambda > 0$ and positive scalar curvature, it will shrink on time under the flow by a time-dependent parameter, and the time limit for a unitary sphere $S^n(1)$, from $t = 0$ until its collapse into a point, is exactly $T = \frac{1}{2(n-1)}$. Another example is in fact matter of this work, as our main result is proven in Chapter 3 for $\rho-$Einstein solitons with $\rho \in \left[0, \frac{1}{2(n-1)}\right)$ but the technique therein cannot be replicated for the limit case $\rho = \frac{1}{2(n-1)}$. We refer the interested reader to the work of Borges in [4] for further information and rigidity results on Schouten solitons.

In the context described above, we now describe the general structure and the main results found in our work.

This thesis is divided into three chapters. In the first one, the notation that will be used is established, as well as some definitions and basic results on Riemannian geometry, solitons and smooth metric spaces, paying special attention to those that will be used more often on the remaining chapters.

Chapter 2 will present the results on shrinking and expanding Schouten solitons. In [40], Munteanu and Wang proved that a smooth metric measure space with positive Ricci curvature has, at most, one $f-$non-parabolic end (see Definition 1.4.3). Later, in [39], Munteanu and Sesum showed that if (M, g) is a gradient shrinking Ricci soliton with scalar curvature $R \leq \alpha < \frac{n}{2} - 1$ for some constant α, then all ends of M are non-parabolic and if M is Kälher, then it is connected at infinity. As pointed out by Li in [29], without the assumption of M being Kähler we can not expect to infer any further information about the geometry of M at infinity. Indeed, an explicit counterexample to the Riemannian case is given in [28], where the authors also prove connectedness at infinity of Riemannian shrinking Ricci solitons with additional hypotheses over the Ricci tensor, R and the potential function f. With this in mind, as results for shrinking Schouten solitons, we prove the next theorems.

Theorem 1. *Let (M, g, f, λ) with $\lambda \geq 0$ be a shrinking Schouten soliton with f non-constant and such that for some constant α we have $\delta \leq R \leq \alpha < \frac{2}{n}(n-1)(n-2)\lambda$, then all ends of M are non-parabolic.*

The upper bound on R is optimal, and the result no longer holds if we consider $R = \frac{2}{n}(n-1)(n-2)\lambda$ as parabolic examples of Schouten shrinkers with such constant scalar curvature are known to exist (see Remark 2.1). We also prove the next bound for the number of $f-$non-parabolic ends:

Theorem 2. *Let (M, g, f, λ), $\lambda > 0$ be a shrinking Schouten soliton with f non-constant. Then M has, at most, one f-non-parabolic end.*

It is natural to wonder whether the additional hypothesis of M being Kähler would lead to a generalization of the result on [39]. Unfortunately the answer is negative. Although the hypothesis of M Kähler does lead to conclude the connectedness at infinity, the result is trivial and does not provide any generalization. This is because any gradient Kähler *almost* Ricci soliton, which includes the Schouten case, is in fact a gradient Kähler Ricci soliton (see [36, Proposition 3.1]), so this is already covered by Munteanu and Sesum's aforementioned result.

In regards to the expanding case, Munteanu and Wang proved in [41] that a complete gradient expanding Ricci soliton with scalar curvature $R \geq -\frac{n-1}{2}$ must be either connected at infinity or isometric to the product of a one-dimensional Gaussian soliton (see Example 1.1) with a compact Einstein manifold. In section 2.2, we extend this result to the Schouten case and prove the next theorem.

Theorem 3. *Let (M, g, f, λ) be a complete non-trivial expanding Schouten soliton. Then either M is connected at infinity or it is isometric to $\mathbb{R}^1 \times N^{n-1}$ where $\mathbb{R}^1$ is the Gaussian Ricci soliton of dimension 1 and N is a compact Einstein manifold.*

In particular, one can conclude from the theorem above that new examples of non-trivial non-rigid expanding Schouten solitons can not be constructed as connected sums of two or more of such solitons.

It is worth noticing that Munteanu and Wang's hypothesis over R is not necessary in our case as the lower bound is already guaranteed for Schouten solitons (see [3] or Theorem 1.D here).

In the final chapter of this thesis, we focus our attention on shrinking $\rho-$Einstein solitons (M, g, λ, ρ) for $\rho \in \left[0, \frac{1}{2(n-1)}\right)$. Our main goal is to prove that, under an upper bound assumption of the scalar curvature of M, we can assure the soliton is connected at infinity. In [44], the authors prove that a complete gradient shrinking Ricci soliton of dimension $n \geq 4$ with scalar curvature $R \leq n/3$ has exactly one end[1]. Here we generalize this fact to shrinking gradient $\rho-$Einstein solitons with a suitable bound condition over their scalar curvature R. The chapter is divided into two sections, containing the two main results, that lead to this generalization. First, in section 3.1, we show that under a general boundedness condition, $\rho-$Einstein solitons can only have $\varphi-$non-parabolic ends, where $\varphi = -af$ for a an arbitrary positive constant. Next, in section 3.2 we show that, under a particular choice of the upper bound of R, the $\rho-$Einstein soliton must be connected at infinity. Namely, we have the next two theorems:

Theorem 4. *Let (M, g, f, λ) be a shrinking gradient $\rho-$Einstein soliton with scalar curvature $0 \leq R \leq K$ for some positive constant K. Then all ends of M are $\varphi-$non-parabolic.*

Theorem 5. *Let (M^n, g, f, λ) be a shrinking gradient $\rho-$Einstein soliton with $n \geq 4$, non-negative scalar curvature satisfying*

$$R \leq \frac{2n(n-3)\lambda}{3(n-3) - (3n^2 - 12n + 5)\rho}, \tag{1}$$

and such that $\rho \in \left[0, \frac{1}{2(n-1)}\right)$. Then M has only one end.

Notice that the result of Munteanu and Wang in [44] is effectively covered by Theorem 5 by making $\rho = 0$.

[1]As in [44] is used a normalized constant $\lambda = 1/2$, the condition over R on Munteanu's result should be read as $R \leq 2n\lambda/3$ in the context of our work.

Chapter 1

Preliminaries

We introduce now some basic concepts and notation we will use on this thesis, details on definitions with no explicit reference can be found in classical literature such as [15, 19, 51], among others.

1.1 Notation and basic definitions.

Throughout this work, we will consider (M^n, g) to be an n-dimensional Riemannian manifold with differentiable boundary ∂M, that may or not be empty, and ν to be the outer unitary vector field normal to ∂M. We will also denote here the Ricci tensor and scalar curvature of M by Ric and $R = \text{Trace}(\text{Ric})$ respectively. From the Cauchy-Schwarz inequality, it is clear

$$|\text{Ric}|^2 = \sum_{i,j} R_{ij}^2 \geq \frac{1}{n} \sum_i R_{ii}^2 = \frac{R^2}{n},$$

recalling

$$R_{ij} = \sum_k R_{ikj}^k = \sum_{k,l} \left(\Gamma_{ij}^l \Gamma_{kl}^k - \Gamma_{kj}^l \Gamma_{il}^k \right) + \sum_k \left(\frac{\partial}{\partial x_k} \Gamma_{ij}^k - \frac{\partial}{\partial x_i} \Gamma_{kj}^k \right),$$

with Γ_{ij}^k standing for the Christoffel symbols of the Riemannian connection, ∇, on (M, g).

We denote by $C(M)$ the set of continuous functions over M and by $C^d(M)$ the set of functions over M that are continuously differentiable up to, at least, order d. Setting as $\chi(M)$ the set of C^∞ vector fields over M we have the following definitions:

Definition 1.1.1. *Let $f \in C^1(M)$. The gradient of f, ∇f, is defined at a point $p \in M$*

as the unique vector field such that

$$\langle \nabla f(p), v\rangle = df_p(v),\ \ p \in M,\ \ v \in T_pM,$$

if $\{e_i\}_{i=1}^n$ *is a local geodesic frame defined on a neighborhood of* p*, then* ∇f *can be expressed as*

$$\nabla f(p) = \sum_{i=1}^{n}(e_1(f))e_i(p).$$

Proposition 1.1.1. ***The co-area formula*** *[47,48, Corolary 2.3]: Let* (M, g) *be a smooth manifold with volume element* dV_M*, consider* $p \in M$*, and let* $f \in C^1(M)$ *be a function with no critical points. Then for any measurable function* $\phi : M \to \mathbb{R}$ *we have*

$$\int\limits_M \phi(p)dV_M(p) = \int\limits_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{\phi(p)}{|\nabla f(p)|} dV_{f^{-1}(t)}(p)\right) dt. \tag{1.1}$$

In particular, if $\phi = 1$*, then the volume of* M *is given by*

$$Vol(M) = \int\limits_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{1}{|\nabla f(p)|} dV_{f^{-1}(t)}(p)\right) dt.$$

Definition 1.1.2. *Let* $X, Y \in \chi(M)$ *and let* $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ *be the Riemannian connection of* M*. The divergent of* X*, noted by* $\mathrm{div} f$*, is defined as the trace of the linear application* $Y(p) \to \nabla_Y X(p)$*. In terms of a local geodesic frame* $\{e_i\}$*,* $\mathrm{div}X$ *can be expressed locally as*

$$\mathrm{div}X(p) = \sum_{i=1}^{n}(e_i(f_i))(p),$$

where f_i *are the coordinate functions of the local expression of the field* X*:*

$$X = \sum_{i=1}^{n} f_i e_i.$$

Definition 1.1.3. *The Laplace (also known as Laplace-Beltrami) operator* $\Delta : C^2(M) \to C^2(M)$ *is defined on* M *as*

$$\Delta f = \mathrm{div}\nabla f,\ \ f \in C^2(M).$$

In terms of a local geodesic frame $\{e_i\}$*,*

$$\Delta f(p) = \sum_{i=1}^{n} e_i(e_i(f))(p).$$

A function $f \in C^2(M)$ is said to be *harmonic* if $\Delta f = 0$, and is said to be *super-harmonic* (res. *sub-harmonic*) whenever $\Delta f \leq 0$ (resp. $\Delta f \geq 0$)

Definition 1.1.4. *The Hessian of $f \in C(M)$ on the point $p \in M$ is defined as the linear operator*

$$\begin{aligned} \text{Hess} f : T_pM &\to T_pM \\ Y &\mapsto (\text{Hess} f)(Y) = \nabla_Y(\nabla f). \end{aligned}$$

As a tensor, the Hessian of $f \in C(M)$ is given by:

$$\begin{aligned} \text{Hess}(f) := \nabla^2 f : \chi(M) \times \chi(M) &\to \mathbb{R} \\ (X, Y) &\mapsto \langle \nabla_X \nabla f, Y \rangle, \end{aligned}$$

it can be shown that the definition above is equivalent to

$$\nabla^2 f(X, Y) = X(Yf) - (\nabla_X Y)f, \;\; X, Y \in T_pM.$$

An alternative definition of the Laplacian can be given in terms of the trace of Hessian tensor, this is, if $\{e_i\}_{i=1}^n$ forms an orthonormal base for T_pM, then

$$\Delta f = \text{Tr}(\nabla^2 f) = \sum_i \nabla^2 f(e_i, e_i).$$

From the Cauchy-Schwarz inequality, it is true that

$$(\Delta f)^2 \geq \frac{|\nabla^2 f|^2}{n}.$$

In order to ease the reading, we will write $|\nabla^2 f|^2$ in Einstein notation as $|f_{ij}|^2$ as long as there is no room for ambiguities.

Kato's inequality: If f is a smooth function over a Riemannian inequality (M, g), then

$$|\nabla|\nabla f||^2 \leq |f_{ij}|^2.$$

Let M, ∂M, ν be as above. Let dx be the n-dimensional volume element on M and $d\overline{x}$ be the $(n-1)$-dimensional volume element on ∂M. The following classical results hold.

Theorem 1.1. Divergence theorem. [14] *Let $X \in \chi(M)$, then*

$$\int\limits_M \text{div} X \; dx = \int\limits_{\partial M} \langle X, \nu \rangle \; d\overline{x}.$$

Theorem 1.2. Green's identity. *Let $f \in C^1(M)$ and $g \in C^2(M)$, and let $\partial_\nu g = \langle \nabla g, \nu \rangle$ be the partial derivative of g with respect to outer normal vector field ν on ∂M. Then*

$$\int_M [f\Delta g + \langle \nabla f, \nabla g \rangle] \; dx = \int_{\partial M} f \partial_\nu g \; d\overline{x}.$$

The Identity above is a direct consequence of the divergence theorem. An important relation between the elements presented above is the following formula:

Theorem 1.3. Bochner's formula. *Let M be a Riemannian manifold, let* Ric *stand for the Ricci curvature tensor of M and let $f \in C(M)$, then*

$$\frac{1}{2}\Delta|\nabla f|^2 = \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla f, \nabla(\Delta f) \rangle + |\nabla^2 f|^2. \tag{1.2}$$

The proof of Bochner's formula (also known as Bochner-Lichnerowitz) comes directly from the definitions of the Laplacian, gradient, Hessian and Ricci curvature tensor given above.

Consider now a smooth domain Ω with boundary $\partial\Omega$, given a partial differential problem (P) over $\overline{\Omega}$, we call $\mathcal{H}_{(P)}(\Omega) \subset C(\Omega)$ the space of admissible solutions (or *trial functions*) of (P), composed by functions that satisfy the "shape" of problem (P), in particular the boundary conditions given. Two well-known PD problems involving Laplace operator are the Dirichlet and Neumann problems, where we look for solutions $u \in \mathcal{H}(\Omega)$ that satisfy:

$$\begin{cases} \Delta u = f(u,x), & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u = f(u,x), & \text{in } \Omega, \\ \partial_i u = g_i, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

Dirichlet boundary conditions **Neumann boundary conditions**

where f and g are given arbitrary functions in $C(\Omega)$. In this sense, Dirichlet and Neumann eigenvalue problems are classically stated on manifolds with boundary as

$$\begin{cases} \Delta u = \lambda u, & \text{in } M \\ u = 0, & \text{on } \partial M \end{cases} \qquad \begin{cases} \Delta u = \mu u, & \text{in } M \\ \partial_\nu u = 0, & \text{on } \partial M. \end{cases} \tag{1.4}$$

Dirichlet eigenvalue problem **Neumann eigenvalue problem**

It is well known that the problems above have infinitely many countable solutions of the form $(\lambda_i, u_i) \in \mathbb{R} \times C(M)$, referred to as the i-th eigenvalue of the problem and its *associated eigenfunction.*

Classical results on eigenvalue problems are the Rayleigh inequality, and the Min-max principle; we show here their statements for the case of the Dirichlet and Neumann eigenvalue problems. Analogous statements apply to other eigenvalue problems we will work with in upcoming chapters.

Consider the Lebesgue space of functions $L^2(M) = \{f \in C(M) : \int_M f^2 \leq +\infty\}$ endowed with the inner product

$$\langle f, g \rangle_{L^2(M)} = \int_M fg \, dx, \tag{1.5}$$

and let $H^2(M)$ be the (Sobolev) space of functions whose derivatives belong to $L^2(M)$ up to, at least, order 2, this is

$$H^2(M) = \left\{ f \in C(M) : \int\limits_M \left(f^2 + \sum_i (f_{,i})^2 + \sum_{i,j} (f_{,ij})^2 \right) dx \leq +\infty \right\}.$$

In the next theorem, $\mathcal{H}(M)$ will represent either $\mathcal{H}_{Dir.}(M) = \{u \in H^2(M) : u = 0 \text{ on } \partial M\}$ or $\mathcal{H}_{Neu.}(M) = \{u \in H^2(M) : \partial_\nu u = 0 \text{ on } \partial M\}$, where *Dir.* and *Neu.* stand for the Dirichlet and Neumann problems respectively, and the η_i's will stand for the Dirichlet or Neumann eigenvalues according to the case.

Theorem 1.4. Rayleigh theorem *[14] Let M be a Riemannian manifold, and consider an eigenvalue problem with function space $\mathcal{H}(M)$ and eigenvalues*

$$\eta_1 \leq \eta_2 \leq \cdots, \tag{1.6}$$

where each eigenvalue is repeated according to its multiplicity. Then, for any $f \in \mathcal{H}(M) - \{0\}$, we have

$$\eta_1 \leq \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

with equality holding if and only if f is an eigenfunction associated to η_1. Furthermore, if $\{u_1, u_2, \cdots\}$ is a complete orthonormal basis of $L^2(M)$ such that u_i is an eigenfunction associated to η_i for each $i = 1, 2, ...$, then for $f \in \mathcal{H}(M) - \{0\}$ satisfying $f \perp u_1$, ..., $f \perp u_{k-1}$ with respect to norm (1.5),

$$\eta_k \leq \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

with equality holding if and only if f is an eigenfunction associated to η_k.

Theorem 1.5. Min-max principle. *[14] Given $v_1, ..., v_{k-1} \in L^2(M)$, let*

$$\rho = \inf_{f \neq 0} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

where f varies over the orthogonal subspace of $\mathrm{Span}\{v_1, ..., v_{k-1}\}$ *in $\mathcal{H}(M)$. Then, the eigenvalues in (1.6) satisfy*

$$\rho \leq \eta_k.$$

Of course, if $v_1, ..., v_{k-1}$ are orthonormal eigenfunctions associated to $\eta_1, ..., \eta_{k-1}$, then $\rho = \eta_k$.

This way, the first nonzero Dirichlet eigenvalue is given by

$$\lambda_1 = \min_{\substack{u \in H_0^2(M), \\ u \neq cte}} \frac{\int_M |\nabla u|^2}{\int_M u^2}, \tag{1.7}$$

which is known as the Rayleigh quotient of the Dirichlet problem or the *variational characterization* of λ_1. On the other hand, the first nonzero Neumann eigenvalue variational characterization is given by

$$\mu_1 = \min_{\substack{u \in H^2(M), \\ u \neq cte, \partial_\nu u|_{\partial M} = 0}} \frac{\int_M |\nabla u|^2}{\int_M u^2}, \tag{1.8}$$

as it can be easily seen, when working over the same manifold, $\mathcal{H}_{Dir}(M) \subset \mathcal{H}_{Neu}(M)$, thus $\mu_1(M) \leq \lambda_1(M)$. It is worth mentioning that the theorems above imply that both Dirichlet and Neumann eigenvalue problems have discrete non-decreasing divergent spectra,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty,$$

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \nearrow \infty.$$

In general, given $p \geq 1$, we define the metric space $L^p(M)$ as the set of $p-$integrable functions

$$\left\{ f \in C(M) : \int_M f^p \leq \infty \right\},$$

endowed with the norm

$$||f||_p = \left(\int_M f^p \right)^{1/p}.$$

The classical Poincaré inequality states that, for $1 \leq p < \infty$ and a bounded domain Ω

there is a constant C depending only on p and Ω such that

$$||u||_{L^p(\Omega)} \leq C||\nabla u||_{L^p(\Omega)},$$

for any u in the Sobolev space $W_0^p(\Omega)$. From the Rayleigh quotient given above, it is clear that the first Dirichlet eigenvalue of the Laplacian optimizes the Poincaré inequality, that is

$$||u||_{L^p(\Omega)} \leq \lambda^{-1}||\nabla u||_{L^p(\Omega)},$$

for any $u \in W_0^p(\Omega)$.

1.2 Smooth metric measure spaces

A smooth metric measure space is a Riemannian manifold (M, g) with (possibly empty) boundary ∂M and endowed with a weighted volume form $\mathrm{e}^{-f}dv$, where $f \in C(M)$ and dv is the Riemannian volume form induced by the metric g, and it is often noted as the triple $(M, g, e^{-f}dv)$. In such space, we can extend the geometric elements defined on the previous section as follows. The *weighted Laplacian operator* Δ_f, also known as $f-$Laplacian, is defined by

$$\Delta_f \cdot = \Delta \cdot - \langle \nabla f, \nabla \cdot \rangle,$$

where Δ represents the Laplace operator acting on functions of $C^2(M)$. If $\Delta_f h = 0$ for $h \in C^2(M)$ we say h is an $f-$harmonic function.

By noticing

$$\mathrm{div}(e^{-\phi}v\nabla u) = v\Delta u e^{-\phi} + \langle \nabla u, \nabla v\rangle e^{-\phi} - v\langle \nabla\phi, \nabla u\rangle e^{-\phi},$$

we can conclude Δ_f satisfies a Green's identity in the form

$$\int\limits_M \left(v\Delta_f u + \langle \nabla v, \nabla u\rangle\right) \mathrm{e}^{-f}dv = \int\limits_{\partial M} v\partial_\nu u \mathrm{e}^{-f}d\overline{v}, \tag{1.9}$$

where ν is the outer normal vector field over ∂M and $d\overline{v}$ is the volume element on the boundary of M. It was first shown by Liu and Ma in [35] that the following Bochner formula also applies to the weighted Laplacian

$$\Delta_f|\nabla h|^2 = 2|\nabla^2 h|^2 + 2\langle \nabla h, \nabla\Delta_f h\rangle + 2\mathrm{Ric}_f(\nabla h, \nabla h), \tag{1.10}$$

for all functions $h \in C(M)$, where

$$\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$$

is the so-called Bakry-Emery Ricci tensor over M. Most of the classical results and definitions, including those given in the previous section, are extended to the weighted Laplacian and, in consequence, for smooth metric measures spaces in general without much effort.

In addition to this, we include here a couple of technical results involving smooth metric measure spaces with positive Bakry-Emery Ricci tensor. Let x_0 be a fixed point in M. First, we have the next Sobolev type inequality

Theorem 1.A. *[40] Let $(M, g, e^{-f}dv)$ be a smooth metric measure space with $Ric_f \geq 0$. Then there are constants $\nu > 2$, c_1 and c_2, all depending only on n, such that*

$$\left(\int_{B_{x_0}(R)} |\varphi - \varphi_{B(x_0,R)}|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq c_1 e^{c_2 A} \frac{R^2}{V(B_{x_0}(R))^{2/\nu}} \int_{B_{x_0}(R)} |\nabla \varphi|^2$$

for any $\varphi \in C^\infty(B_{x_0}(R))$, where $\varphi_{B(x_0,R)} := V^{-1}(B_{x_0}(R)) \int_{B_{x_0}(R)} \varphi$, and

$$A := A(f) = \left| \sup_{x \in B_{x_0}(3R)} |f(x)| - \inf_{x \in B_{x_0}(3R)} |f(x)| \right|$$

is the oscillation of $|f|$ on the ball $B_{x_0}(3R)$.

Remark 1.1. *Despite we are not going to use it explicitly in our work, inequality from Theorem 1.A is an important reference on the structure of the proof of Theorem 4 (Theorem 3.1 on Chapter 3) and is included here for the sake of completeness.*

Finally, we also recall the next estimate over the volume of the unitary ball centered on an arbitrary $x \in M$:

Theorem 1.6. *[41, 42]. Let (M, g, e^{-f}) be a complete smooth metric measure space of dimension n. Assume $\mathrm{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$. Then there exists a constant $C_0 > 0$ such that*

$$V(B_x(1)) \geq e^{-c_0 \sqrt{R \ln R}} V(B_{x_0}(1))$$

for all $R = d(x, x_0) > 2$. The constant C_0 depend only on n and $f(x_0)$.

1.3 Solitons

We call a *soliton* to a manifold that is a somehow self-similar solution of a particular evolution equation. Given the rising interest in Ricci flow in the last decades, most studied evolution equations involve Ricci curvature prescribed as a function of the Riemannian metric of the manifold. Simple examples of solitons are the Einstein manifolds, which satisfy the equation $\mathrm{Ric} = \lambda g$, for a constant $\lambda \in \mathbb{R}$. More notable examples are the well-known Ricci solitons, which are manifolds (M, g) such that there is a smooth vector field V satisfying

$$\mathrm{Ric} = \lambda g - \frac{1}{2}\mathcal{L}_V g \tag{1.11}$$

for some constant $\lambda \in \mathbb{R}$, where $\mathcal{L}_V$ stands for the Lie derivative with respect to the field V. If there is a function $f : M \to \mathbb{R}$ such that $\nabla f = V$, then we can write

$$\mathrm{Ric} + \nabla^2 f = \lambda g$$

and M is called a *gradient Ricci soliton.* The simplest example occurs whenever $V = 0$ or $\nabla f = 0$ as we get $\mathrm{Ric} = \lambda g$ and recover the aforementioned *Einstein solitons.* Another well-known example of a Ricci soliton is the *Gaussian Ricci solitons* constructed as follows:

Example 1.1. *Let g be the canonical metric in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$f(x) = \frac{1}{2}\lambda||x||^2.$$

Since $\mathrm{Ric} = 0$ *and* $\nabla^2 f = \lambda g$, *we get* (M^n, g, f, λ) *is a gradient Ricci soliton known as the Gaussian Ricci soliton of dimension n.*

Definition 1.3.1. *Let (M^n, g) be a Riemannian manifold with $n \geq 3$, scalar curvature R and let $\rho \in \mathbb{R}$. We say that M is a $\rho-$Einstein soliton if there is a smooth vector field V such that*

$$Ric + \frac{1}{2}\mathcal{L}_V g - \rho R g = \lambda g.$$

If there is a function $f : M \to \mathbb{R}$ such that $V = \nabla f$ on the definition above, then M is called a gradient $\rho-$Einstein soliton and the soliton equation is written as

$$\mathrm{Ric} + \nabla^2 f = (\rho R + \lambda)g.$$

Solitons are regarded as *shrinking*, *steady* or *expanding* depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. A gradient soliton is said to be *trivial* whenever $\nabla^2 f$ is null (either for f being constant or for ∇f being a parallel field). By changing the nature of λ in (1.11) (and subsequent definitions) to be a smooth function we get what is known as an *almost Ricci soliton* (respectively *almost gradient Ricci* soliton, *almost Einstein* soliton, etc.). In [13], the authors shown some useful identities for gradient $\rho-$Einstein solitons we will use in the final chapter of this work:

Theorem 1.B. *[13] Let (M^n, g, f, λ), $n \geq 3$, be a gradient $\rho-$Einstein soliton. Then, the following identities hold*

$$\begin{aligned} \Delta f =& (n\rho - 1)R + n\lambda, \\ (1 - 2(n-1)\rho)\nabla R =& 2\mathrm{Ric}(\nabla f), \\ (1 - 2(n-1)\rho)\Delta R =& \langle \nabla R, \nabla f\rangle + 2(\rho R^2 - |\mathrm{Ric}|^2 + \lambda R). \end{aligned}$$

Second equation on theorem above can be simply seen as

$$(1 - 2(n-1)\rho)\langle \nabla R, X\rangle = 2Ric(\nabla f, X),$$

for any $X \in \chi(M)$.

Remark 1.2. *Theorem 1.B brings up one of the main differences when dealing with more general solitons instead of Ricci type ones. While on Ricci solitons we have the well-knwon Hamilton identities that are base for plenty of geometric and analytic results about the soliton, on the $\rho-$Einstein soliton case (with $\rho \neq 0$) such identity does not hold, and is necessary to adapt the identities and techniques used for Ricci solitons (or even give additional hypothesis over M) in order to amend its absence.*

Additionally, in [13], the authors show upper and lower bounds for the square norm of the gradient of the potential function f in terms of f itself:

Theorem 1.C. *[13] Let (M^n, g, f, λ) be a gradient shrinking $\rho-$Einstein soliton with $\rho > 0$, scalar curvature $R \geq 0$ and such that $|R| < K$ for some positive constant K. Then, either f is constant or there exist positive real constants $\alpha, \beta, \epsilon, \delta$ such that*

$$\alpha f(r) - \beta \leq |\nabla f|^2(r) \leq \epsilon f(r) + \delta,$$

where r is the distance to a connected component $\Sigma_0 \subset M$ of some regular level set of f.

Regarding the growth of f, Munteanu *et al.* gave in [42] the next estimate for more general smooth metric measure spaces not necessarily shrinkers:

Proposition 1.3.1. *[42] Let (M, g, e^{-f}) be a complete smooth metric measure space of dimension n. Assume $\mathrm{Ric}_f \geq \frac{1}{2}$ and $|\nabla f|^2 \leq f$. Then there exists a constant $a > 0$ such that*

$$\frac{1}{4}(d(x, x_0) - a)^2 \leq f(x) \leq \frac{1}{4}(d(x, x_0) + a)^2,$$

for any $x \in M$ and $d(x, x_0) \geq r_0$. The constants a and r_0 depend only on n and $f(x_0)$.

In general, if $|\nabla f|^2 \leq \epsilon f + \delta$ in the proposition above, then $f(x) \leq \epsilon/4(d(x, x_0) + a)^2$ for any $x \in M$ with $d(x, x_0) \geq r_0$, for some positive constants a and r_0 depending on n and $f(x_0)$.

Volume of geodesic balls $B_x(\rho)$ in smooth metric measure spaces given as in Proposition 1.3.1 (see e.g [17, 41, 42, 44]), also satisfy

$$c_0\rho \leq Vol(B_x(\rho)) \leq c(n)\rho^n$$

for $\rho > 1$, where the constant $c(n)$ depends only in n and c_0 depends on n and the *Perelman's entropy*

$$\mu(g) = \int_M e^{-f} < \infty.$$

It is worth mentioning that, while this *weighted volume* is finite, that is not necessarily true for the Riemannian volume, which, for example, has been proven to be infinite on the Shrinking Ricci soliton case (see [9, Theorem 3.1] or [41, Lemma 6.2]).

A particular case of $\rho-$Einstein soliton we will work with is the Schouten soliton, given by making $\rho = \frac{1}{2(n-1)}$:

Definition 1.3.2. *A manifold (M, g, f, λ) is said to be a gradient Schouten soliton if its Ricci curvature* Ric*, potential function f and scalar curvature R satisfy*

$$\mathrm{Ric} + \nabla^2 f = \left(\frac{R}{2(n-1)} + \lambda\right) g, \tag{1.12}$$

where $\nabla^2 f$ represents the Hessian of f. Furthermore, M is said to be a shrinking (resp. steady or expanding) Schouten soliton if $\lambda > 0$ (resp. $\lambda = 0$ or $\lambda < 0$).

Examples of Schouten solitons, among other interesting results we will use here, were given by Borges on [3] and Catino in ([11, 12]). In particular, the next results give useful additional properties for f and R on a complete non-steady Schouten soliton

Theorem 1.D. *[3] Let $(M^n, g, f, \lambda \neq 0)$ be a complete non-compact Schouten soliton with f non-constant. If $\lambda > 0$ (respectively $\lambda < 0$), then the potential function f attains* *a global minimum (resp. maximum) and is unbounded above (resp. below). Furthermore*

$$0 \leq \lambda R \leq 2(n-1)\lambda^2, \tag{1.13}$$

$$2\lambda(f - f_0) \leq |\nabla f|^2 \leq 4\lambda(f - f_0), \tag{1.14}$$

where $f_0 = \min_M f$ *(resp.* $f_0 = \max_M f$*).*

Theorem 1.E. *[12] If* (M^n, g, f, λ) *is a gradient Schouten soliton, then*

$$\Delta f = n\lambda - \frac{n-2}{2(n-1)}R,$$
$$\mathrm{Ric}(\nabla f, X) = 0, \forall X \in \mathcal{X}(M),$$
$$\langle \nabla f, \nabla R\rangle + \left(\frac{R}{n-1} + 2\lambda\right) R = 2|\mathrm{Ric}|^2.$$

Besides Einstein manifolds, which are the simplest examples of Schouten solitons, we can consider the following:

Example 1.2. *[3] Let* $n \geq 3$, $\lambda > 0$ *and consider an Einstein manifold* (N^k, g) *of dimension* $k \leq n$ *with scalar curvature*

$$R = \frac{2(n-1)k\lambda}{2(n-1)-k}.$$

Then $(\mathbb{R}^{n-k} \times_\Gamma N^k, g', f, \lambda)$ *is an* n *dimensional Schouten soliton with product metric* $g' = \langle , \rangle_{\mathbb{R}} + g$*. Here* Γ *acts freely on* N *and by orthogonal transformations on* $\mathbb{R}^{n-k}$ *and*

$$f(x,p) = \frac{1}{2}\left(\frac{R}{2(n-1)} + \lambda\right) ||x||^2_{\mathbb{R}}.$$

In the context of the example above we have the next

Definition 1.3.3. *Solitons isometric to the product* $\mathbb{R}^k \times_\Gamma N$ *where* N *is an Einstein manifold and* Γ *acts freely on* N *and by orthogonal transformations on* $\mathbb{R}^k$ *are known as* Rigid solitons*.*

It was proven by Catino *et al.* in [12] that every complete gradient steady Schouten soliton is actually trivial, and, for dimension $n = 3$, any shrinking Schouten soliton is isometric to a finite quotient of either $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{S}^2$. Regarding non-rigid examples, in [12] the authors construct examples of rotationally symmetric gradient steady $\rho-$Einstein solitons with $\rho < 1/2(n-1)$ or $\rho \geq 1/(n-1)$ that are warped products with positive sectional curvature. They also prove that for $n = 3$ it is actually the unique example of a soliton of its type up to homotheties and that such examples does not exist if $1/2(n-1) \leq \rho < 1/(n-1)$, which in particular includes the Schouten soliton case (we refer the reader to Theorems 1.3 and 4.3 therein for further details).

1.4 Ends of a manifold

As it was mentioned before, the main objective of this work is to count the ends of some particular solitons, properly speaking, given a Riemannian manifold (M, g) we have the following

Definition 1.4.1. *An end E with respect to a compact subset $\Omega \subset M$ is an unbounded connected component of $M \setminus \Omega$. The number of ends with respect to Ω, denoted by $N_\Omega(M)$, is the number of unbounded connected components of $M \setminus \Omega$.*

Clearly, if $\Omega' \subset \Omega$ is compact then $N_{\Omega'} \leq N_\Omega$. Hence, given $\{\Omega_i\}$ a compact exhaustion of M, the sequence $N_{\Omega_i}(M)$ is monotonically non-decreasing. Furthermore, if this sequence is bounded, we say M has *finitely many ends* and we denote the number of ends of M by

$$N(M) = \max_{i \to \infty} N_{\Omega_i}(M).$$

Given a manifold M^n it is possible to "add new ends" to it as a connected sum. Roughly speaking, given a second manifold M_2, we can remove the interior of sets $U_1 \cong S^n \subset M$ and $U_2 \cong S^n \subset M_2$ and then "glue" together their $(n-1)-$dimensional boundaries by an orientation reversing metamorphism, resulting on the *connected sum* manifold $M \# M_2$ with $N(M \# M_2) = N(M) + N(M_2)$. It is also possible to sum ends of different $n-$manifolds as a non-compact connected sum known as *end sum*. The sum end is made by choosing to remove subsets $H_1 \subset M_1, H_2 \subset M_2$ homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty)$ such that we can glue the truncated ends together, the result is a manifold with $N(M) + N(M_2) - 2$ ends. We refer the interested reader to [7, 22, 38] and the introduction in [45] for more details and examples on connected sum and end sum of manifolds.

In [31], the author gives a detailed study about functions over the ends of a manifold M which provides a powerful method for counting the number of ends a manifold has. The main idea of the technique is to classify the ends according to their *parabolicity* (which we will define shortly) and subsequently find geometric conditions for M to estimate the number of parabolic and non-parabolic ends. Such parabolicity classification is based on the idea of parabolicity of manifolds and can be given from several mathematical points of view (geometric, analytic, group theory, ...) which lead to several equivalent definitions of what a parabolic end is, here we enumerate those we will use more often (for an algebraic point of view on parabolic manifolds we refer the reader to the work of Troyanov in [54]).

Definition 1.4.2. *[31] A Green's function $G(x,y)$ is a function $G : M \times M \setminus D \to \mathbb{R}$, where $D = \{(x,x) | x \in M\}$, such that*

$$\int_M G(x,y)\Delta f(y)dy = -f(y) \tag{1.15}$$

for all functions f satisfying the Dirichlet boundary condition $f|_{\partial M} = 0$.

In general, existence of Green functions is guaranteed for compact manifolds and the following properties apply:

$$\begin{aligned} G(x,z) = G(z,x), \ \forall (x,z) \in M \times M \backslash D, \\ \Delta_x \int_M G(x,y)f(y)dy = \int_M G(x,y)\Delta f(y)dy = -f(y). \end{aligned} \tag{1.16}$$

Existence and construction of such G is detailed in Chapter 17 of [31]. In the study of Riemann surfaces, non-compact manifolds are classified as *parabolic* or *hyperbolic* depending on the non-existence (resp. existence) of positive super-harmonic functions (see e.g. [20]), inspired in such classification, we have the next definition:

Definition 1.4.3. *[31] A complete manifold is said to be parabolic if it does not admit a positive Green's function. Otherwise it is said to be non-parabolic.*

A simple example of parabolic and non-parabolic manifolds can be found on the works of Grigor'yan *et al.* [23,24], where the authors give an interesting approach to parabolicity of manifolds both from harmonic functions and group theory: $\mathbb{R}^n$ with $n \geq 2$ is parabolic if and only if $n = 2$. This is because for $\mathbb{R}^n$ with $n \geq 3$ we can construct bounded super-harmonic functions u with $\int \Delta u < 0$ (see [24, Theorem 1.1] for further details).

For complete manifolds, the next result gives us the existence of Green's functions and a simplified way to define a parabolic manifold:

Theorem 1.F. *[31] Let M^n be a complete manifold without boundary. There exists a Green's function $G(x,y)$ which is smooth in $(M \times M)\backslash D$ satisfying properties (1.15) and (1.16). Moreover, $G(x,y)$ can be taken to be positive if and only if there exists a positive super-harmonic function f on $M \backslash B_p(\rho)$ with the property that*

$$\liminf_{x \to \infty} f(x) < \inf_{x \in \partial B_p(\rho)} f(x),$$

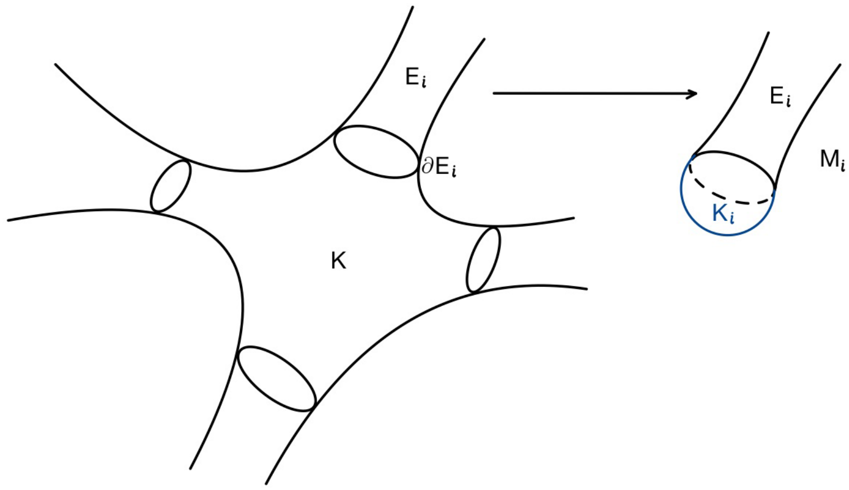

Figure 1.1: A manifold with multiple ends.

where $B_p(\rho)$ is a ball centered in $p \in M$ of radius $\rho > 0$.

This motivates the next alternative definition.

Definition 1.4.4. *[31] An end E is said to be parabolic if it does not admit a positive harmonic function f satisfying*

$$\liminf_{y \to E(\infty)} f(y) = 0,$$

where $E(\infty)$ denotes the infinity of E. Otherwise, E is said to be non-parabolic and the function f is said to be a barrier function of E.

Recently, in [23], the authors gave an alternative definition that may help the reader to understand ends and their classification from a more geometric point of view: Let $M_1, M_2, ..., M_k$, $k \geq 2$ be complete non-compact manifolds of dimension n.

Definition 1.4.5. *[23] We say M^n is a manifold with k ends, and we write*

$$M = M_1 \# \cdots \# M_k,$$

if there is a compact $K \subset M$ such that $M \backslash K$ consists of k components $E_1, ..., E_k$ where each E_i is isometric to $M_i \backslash K_i$ for some compact $K_i \subset M_i$. We call each E_i an end of M (see Figure 1.1[1])*. For a manifold with ends, we say E_i is parabolic (resp. non-parabolic) if M_i is a parabolic (resp. non-parabolic) manifold.*

We can then divide our ends-counting objective into counting the number of parabolic and non-parabolic ends on a manifold M.

[1]Reproduction from [23].

Definition 1.4.6. *[31] Let $\{\Omega_i\}$ be a compact exhaustion of M. We denote by $N^0_{\Omega_i}(M)$ the number of non-parabolic ends of M with respect to the compact set Ω_i. If the sequence $\{N^0_{\Omega_i}(M)\}$ is bounded, we say M has finitely many non-parabolic ends, and we denote by $N^0(M) = \lim_{i\to\infty} N^0_{\Omega_i}(M)$ the number of non-parabolic ends of M.*

Definition 1.4.7. *[31] Let $\{\Omega_i\}$ be a compact exhaustion of M. We denote by $N'_{\Omega_i}(M)$ the number of parabolic ends of M with respect to the compact set Ω_i. If the sequence $\{N'_{\Omega_i}(M)\}$ is bounded, we say M has finitely many parabolic ends, and we denote by $N'(M) = \lim_{i\to\infty} N'_{\Omega_i}(M)$ the number of parabolic ends of M.*

If E is an end of a manifold M, we will denote by $E(\rho)$ the set $E \cap B_p(\rho)$for a fixed point $p \in M$.

In [32], the authors state another notable relation between the number of (parabolic and non-parabolic) ends of a manifold (M, g) with the dimension of spaces of harmonic functions over M. Prior to this result, we need the following definitions:

Definition 1.4.8. *[31] The space $\mathcal{H}^\infty(M)$ is the linear space of all bounded harmonic functions defined on M.*

Definition 1.4.9. *[31] The space $\mathcal{H}^+$ is the linear space spanned by the set of all positive harmonic functions on M.*

For a complete manifold (M, g), we have the following additional definitions

Definition 1.4.10. *[31] The space $\mathcal{H}^\infty_D(M)$ is the space of all bounded harmonic functions on M with finite Dirichlet integral, this is*

$$u \in \mathcal{H}^\infty_D(M) \Longleftrightarrow \int_M |\nabla u|^2 < \infty.$$

Definition 1.4.11. *[31] The space $\mathcal{H}^0(M)$ is the linear space spanned by the set of harmonic functions on M which are bounded on one side at each end. This is, the space $\mathcal{H}^0$ is spanned by harmonic functions u such that there is a compact subset $\Omega \subset M$ for which u is bounded from above or below on each of the ends corresponding to Ω.*

From definitions above, it is clear that $c(M) \subset \mathcal{H}^\infty_D(M) \subset \mathcal{H}^\infty(M) \subset \mathcal{H}^+(M) \subset \mathcal{H}^0(M)$, where $c(M)$ is the set of all constant functions over M. Li and Tam's result in [32] estimate the number of parabolic and non-parabolic ends in terms of the dimension of spaces defined above:

Theorem 1.G. *[32] Let M be a complete manifold. If $dim\mathcal{H}^0 < \infty$, then M must have finitely many ends. In particular,*

$$N'(M) + N^0(M) \leq dim\mathcal{H}^0(M).$$

Moreover, if $N^0(M) \geq 1$, then

$$N'(M) + N^0(M) \leq dim\mathcal{H}^+(M)$$

and

$$N^0(M) \leq dim\mathcal{H}^\infty_D(M).$$

Last inequality on theorem above implies that, if M has more than one non-parabolic end, then there exist non-constant harmonic functions on M with finite Dirichlet integral over M. This fact will be used strongly throughout this work. Furthermore, theory above works fine when working with conformal measures. In this sense, as it has been widely explored on the literature (e.g. [40–42,44,56]), existence of several ends can be generalized from the Theorem above, namely, if a smooth metric measure space (M, g, e^φ) has more than one $\varphi-$non-parabolic end, then there exists a non-constant $\varphi-$harmonic function u

$$\Delta_\varphi u = \Delta u - \langle \nabla\varphi, \nabla u\rangle = 0,$$

with finite Dirichlet integral

$$\int_M |\nabla u| e^\varphi < \infty.$$

Regarding parabolic ends, in [31] the author prove the next

Theorem 1.H. *[31, Theorem 20.7] Let $x_0 \in M$ and let E be a parabolic end with respect to $B_{x_0}(\rho_0)$. Let ρ_i be an increasing sequence such that $\rho_0 < \rho_1 < \cdots < \rho_i \to \infty$. Then there exists a sequence of constants $C_i \to \infty$ such that the sequence of positive harmonic functions g_i defined in $E_{x_0}(\rho_i)$, satisfying*

$$\begin{cases} g_i = 0 & on\ \partial E, \\ g_i = C_i & on\ \partial B_{x_0}(\rho_i) \cap E, \end{cases}$$

has a convergent sub-sequence that converges uniformly on compact subsets of $E \cup \partial E$ to a positive harmonic function g such that

$$g = 0 \ on\ \partial E$$

and

$$\sup_{y \in E} g = \infty.$$

According to Li and Munteanu ([31, 43]), theorem above is due to Nakai's work [46]. Notice that up to translations, if E is a parabolic end, we can assure the existence of a function g satisfying $g = C$ on ∂E for an arbitrary constant $C \geq 0$ and such that g diverges to infinity at the *infinity of* E, $E(\infty)$. This characterization will be particularly useful to prove non-parabolicity of ends of $\rho-$Einstein solitons in the final chapter of this thesis.

Chapter

2

Ends of Schouten solitons

As mentioned in the Introduction chapter, Schouten solitons are the limit case of $\rho-$Einstein solitons for $\rho^* = \frac{1}{2(n-1)}$. An interesting motivation to consider Schouten solitons is linked with the existence of flow solutions. While for $\rho < \rho^*$ and M compact the existence of short-time solutions for the Ricci-Bourguignon flow is guaranteed for any initial metric, solutions do not exist in general when $\rho > \rho^*$, and the existence of short time solutions for $\rho = \rho^*$ is still unknown (see [11]). Schouten solitons can also be found when working with other flows: recently, in [53], the authors considered the Riemann flow

$$\frac{\partial}{\partial t}G = -2Rm_g$$

where $G = \frac{1}{2}g \odot g$ is the Kulkarni-Nomizu product, also known as the *bialternate product Riemannian metric* over M (see [53,55] for further details), and proved that a Riemannian manifold (M^n, g) is a gradient Riemann soliton with potential vector field X if and only if (M^n, g) is a Schouten soliton with potential vector field $V = (n-2)X$ and null Weyl tensor. These classic and modern results on Schouten solitons, among a plenty of others found in literature, make them a case of great interest in soliton analysis and manifold classification theory, and provide a motivation for us to dedicate this chapter to the study of shrinking and expanding Schouten solitons connectedness at infinity and other relevant information about their ends.

2.1 Ends of shrinking Schouten solitons

In this section we present two main results. First, in Theorem 2.1, we show all ends of a shrinking Schouten soliton with a particular bound on the scalar curvature are non-parabolic. Next, in Theorem 2.2, we show a shrinking Schouten soliton has at most one $f-$non-parabolic end. It is worth noticing that this does not imply shrinking Schouten solitons satisfying such condition on R are connected at infinity, since the theorems are referring to different measures.

Let (M, g, f, λ) be a gradient Schouten soliton. Consider $p \in M$ a regular point of f and let $a_p : (\omega_1(p), \omega_2(p)) \to M$ be a maximal integral curve of the field $\frac{\nabla f}{|\nabla f|^2}$ through p. When the choice of p is irrelevant, we write $a : (\omega_1, \omega_2) \to M$. It is known (see [3]) that $(f \circ a)'(s) = 1$ for all $s \in (\omega_1, \omega_2)$, and for any $[s_1, s_2] \subset (\omega_1, \omega_2)$ we that

$$(f \circ a)(s_2) - (f \circ a)(s_1) = s_2 - s_1, \tag{2.1}$$

i.e. $f \circ a$ is an affine function of s.

The next proposition gives upper and lower bounds for $|\nabla f|^2$, given a bound for the scalar curvature of a gradient Schouten (either shrinking or expanding) soliton M.

Proposition 2.1.1. *Let (M, g, f, λ), be a gradient Schouten soliton with f non-constant and such that for some constants α, δ we have $\delta\lambda \leq R\lambda \leq \alpha\lambda$. Let p be a regular point of f (i.e. a point such that $\nabla f(p) \neq 0$). Then*

$$\left(\frac{\delta}{n-1} + 2\lambda\right) \lambda f(p) \leq \lambda |\nabla f|^2(p) \leq \left(\frac{\alpha}{n-1} + 2\lambda\right) \lambda f(p). \tag{2.2}$$

Proof: Let $a(s)$, $s \in (\omega_1, \omega_2)$, be a maximal integral curve of $\frac{\nabla f}{|\nabla f|^2}$, and let

$$b(s) = |\nabla f(a(s))|^2.$$

Notice that, since $d(|\nabla f|^2)(X) = 2\nabla^2 f(X, \nabla f)$, we have from the Schouten equation (1.12) that

$$\begin{aligned}
\operatorname{Ric}(a'(s), \nabla f) + \frac{1}{2} d(|\nabla f|^2)(a'(s)) &= \left(\frac{R}{2(n-1)} + \lambda\right) \langle \nabla f, a'(s) \rangle \\
&= \left(\frac{R}{2(n-1)} + \lambda\right) df(a'(s)).
\end{aligned}$$

Given $\mathrm{Ric}(a'(s), \nabla f) = 0$ (Theorem 1.E) and $(f \circ a)'(s) = 1$, we conclude

$$b'(s) = d(|\nabla f|^2)a'(s) = \left(\frac{R}{n-1} + 2\lambda\right). \tag{2.3}$$

By integrating (2.3) over $[s_1, s] \subset (\omega_1, \omega_2)$ we get,

$$\lambda \int_{s_1}^{s} \left(\frac{R}{n-1} + 2\lambda\right) = \lambda \int_{s_1}^{s} b'(s) = \lambda \left(b(s) - b(s_1)\right),$$

recalling that $\lambda\delta \leq \lambda R \leq \lambda\alpha$, one sees that

$$\left(\frac{\lambda\delta}{n-1} + 2\lambda^2\right)(s - s_1) \leq \lambda\left(b(s) - b(s_1)\right) \leq \left(\frac{\lambda\alpha}{n-1} + 2\lambda^2\right)(s - s_1).$$

It follows from (2.1) that the inequality above is equivalent to

$$\left(\frac{\delta}{n-1} + 2\lambda\right)\lambda(f(a(s)) - f(a(s_1))) \leq \lambda\left(b(s) - b(s_1)\right) \leq \left(\frac{\alpha}{n-1} + 2\lambda\right)\lambda(f(a(s)) - f(a(s_1))). \tag{2.4}$$

Let $s_0 \in (\omega_1, \omega_2)$ be such that $\lim_{s \to s_0} f(a(s)) = f_0$, where $f_0 = \min_{p \in M} f(p)$, and, consequently, $\lim_{s \to s_0} b(s) = 0$. Then, by doing $a(s) = p$ and $s_1 \to s_0$ in (2.4), we have

$$\left(\frac{\delta}{n-1} + 2\lambda\right)\lambda(f(p) - f_0) \leq \lambda|\nabla f|^2(p) \leq \left(\frac{\alpha}{n-1} + 2\lambda\right)\lambda(f(p) - f_0),$$

with $f_0 = f(a(s_0))$. Noticing that, if f satisfies the Schouten soliton, then also does $f + c$ for any constant c, we can assume without loss of generality that $f_0 = 0$ and the proposition is therefore proved.

□

The next result, proved in [3], shows f is bounded by the distance function over M

Theorem 2.I. *Let (M^n, g, f, λ) be a complete non-compact shrinking Schouten soliton with f non-constant, and let $f_0 = \min_{p \in M} f(p)$ and $p_0 \in M$. Then*

$$\frac{\lambda}{4}(d(p) - C_1)^2 + f_0 \leq f(p) \leq \lambda(d(p) + C_2)^2 + f_0,$$

where C_1 and C_2 are positive constants depending on λ and the geometry of the soliton on the unit ball $B_{p_0}(1)$ and $d(p) = d(p, p_0) > 2$.

We are now ready to state the first main result of this chapter.

Theorem 2.1. *Let (M, g, f, λ) be a shrinking Schouten soliton with f non-constant and such that for some constant α we have $\delta \le R \le \alpha < \frac{2}{n}(n-1)(n-2)\lambda$, then all ends of M are non-parabolic.*

Proof: Taking the trace over (1.12) we know that

$$\Delta f = \left(\frac{n}{2(n-1)} - 1\right) R + n\lambda.$$

Since $\delta \le R \le \alpha$, this implies

$$n\lambda - \frac{n-2}{2(n-1)}\alpha \le \Delta f \le n\lambda - \frac{n-2}{2(n-1)}\delta.$$

Thus, by taking

$$a = \frac{2(n-1)(n-2)\lambda - n\alpha}{2(\alpha + 2\lambda)} > 0,$$

we get from (2.2) that

$$\begin{aligned} \Delta f^{-a} &= -af^{-a-1}\Delta f + a(a+1)|\nabla f|^2 f^{-a-2} \\ &\le -af^{-a-1}\left(n\lambda - \frac{n-2}{2(n-1)}\alpha\right) + a(a+1)\left(\frac{\alpha}{n-1} + 2\lambda\right) f^{-a-1} \\ &= af^{-a-1}\left[(a+1)\left(\frac{\alpha}{n-1} + 2\lambda\right) - \left(n\lambda - \frac{n-2}{2(n-1)}\alpha\right)\right] = 0. \end{aligned} \tag{2.5}$$

On the other hand, from Theorem 2.I there is a constant $C > 0$ such that $C(d(p))^2 < f(p)$, where $d(p)$ is the distance function from a fixed point $p_0 \in M$, thus

$$f^{-a} \to 0, \quad \text{as} \quad p \to +\infty,$$

which proves there is a positive super-harmonic function which converges to zero at infinity. By definition 1.4.4, we conclude that all ends in M must be non-parabolic.

□

Remark 2.1. *It is worth mentioning that the upper bound for R on the theorem above is optimal in the sense that, if we take $R = \frac{2}{n}(n-1)(n-2)\lambda$ the result no longer holds. An immediate counter-example is the Schouten soliton of Example 1.2 with $k = n-2$, which is parabolic.*

Theorem 2.2. *Let (M, g, f, λ) be a shrinking Schouten soliton with f non-constant. Then M has, at most, one f-non-parabolic end.*

Proof: Suppose M has two $f-$non-parabolic ends. Then, from [32] (Theorem 1.G in Preliminaries) M admits a positive non-constant bounded f-harmonic function v such that

$$\int_M |\nabla v|^2 e^{-f} < \infty,$$

Recalling $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$ we get from the Schouten soliton equation that

$$\mathrm{Ric}_f = \left(\frac{R}{2(n-1)} + \lambda\right) g.$$

Thus, by the Bochner formula for the weighted Laplacian we conclude

$$\begin{aligned} \frac{1}{2}\Delta_f |\nabla v|^2 =& |\nabla^2 v|^2 + \langle \nabla \Delta_f v, \nabla v\rangle + \mathrm{Ric}_f(\nabla v, \nabla v) \\ =& |\nabla^2 v|^2 + \left(\frac{R}{2(n-1)} + \lambda\right) |\nabla v|^2. \end{aligned} \tag{2.6}$$

Let ϕ be a cut-off function over M such that $\phi = 1$ on $B_p(r)$ and $\phi = 0$ outside $B_p(2r)$, for $p \in M$ and $r > 0$, then, integrating over M we get from the divergence theorem and the Schwarz and Young inequalities that

$$\begin{aligned} 2\int_M |\nabla^2 v|^2 \phi^2 \mathrm{e}^{-f} + \int_M \left(\frac{R}{(n-1)} + 2\lambda\right) |\nabla v|^2 \phi^2 \mathrm{e}^{-f} &= \int_M \left(\Delta_f |\nabla v|^2\right) \phi^2 \mathrm{e}^{-f} \\ &= -\int_M \langle \nabla |\nabla v|^2, \nabla \phi^2\rangle \mathrm{e}^{-f} = -\int_M \langle 2\nabla v \nabla^2 v, 2\phi \nabla \phi\rangle \mathrm{e}^{-f} \\ &= -4\int_M \phi \nabla^2 v \langle \nabla v, \nabla \phi\rangle \mathrm{e}^{-f} \leq 4\int_M |\phi||\nabla^2 v||\nabla v||\nabla \phi| \mathrm{e}^{-f} \\ &\leq 2\int_M |\nabla^2 v|^2 \phi^2 \mathrm{e}^{-f} + 2\int_M |\nabla v|^2 |\nabla \phi|^2 \mathrm{e}^{-f}. \end{aligned} \tag{2.7}$$

Recalling $R > 0$, expression above implies

$$\lambda \int_M |\nabla v|^2 \phi^2 \mathrm{e}^{-f} \leq \int_M |\nabla v|^2 |\nabla \phi|^2 \mathrm{e}^{-f}.$$

Since $\int_M |\nabla v|^2 e^{-f} < \infty$, the right side must tend to zero as $r \to \infty$ (because $|\nabla \phi| \to 0$), this forces $|\nabla v|^2 = 0$ as λ is not null. Therefore v must be constant, which is a contradiction. Thus M must have at most one $f-$non-parabolic end, as we wanted to prove.

□

Remark 2.2. *Techniques above, nonetheless, do not allow us to affirm all ends of shrinking Schouten solitons are in fact f-non-parabolic, and we can only conclude so far that if M is a non-compact shrinking Schouten soliton with scalar curvature satisfying $\delta \leq R \leq \alpha < \frac{2}{n}(n-1)(n-2)\lambda$ and no $f-$parabolic ends, then M is connected at infinity.*

Recently, Munteanu and Wang [44] have found a bound on the scalar curvature which allowed them to conclude connectedness at infinity of a Shrinking Ricci soliton satisfying such bound. Nevertheless, this technique does not apply directly to shrinking Schouten solitons. In the next chapter we will see that under reasonable assumptions such techniques can be adapted to show a similar result for certain shrinking ρ-Einstein solitons, however, the computations will also show where the argument fails to handle the Schouten case.

2.2 Ends of expanding Schouten solitons

In this section, we aim to determine necessary conditions for a non-trivial expanding Schouten soliton to be connected at infinity. The main result (Theorem 2.7) states that the soliton must be either isometric to $N \times_\Gamma \mathbb{R}^k$, with N being an Einstein manifold, or have only one end. To prove this fact, we are going to need first some auxiliary results. To begin with, Li and Wang gave in [34] the next Poincaré inequality:

Proposition 2.2.1. *[34, Prop. 1.1] Let M be a complete Riemannian manifold. If there exists a non-negative function h defined on M that is not identically 0 and satisfies $\Delta h(x) \leq -\sigma(x)h(x)$, then the weighted Poincaré inequality*

$$\int_M \sigma(x)\phi^2(x) \leq \int_M |\nabla\phi|^2(x)$$

must be valid for all compactly supported smooth functions ϕ over M.

As we will require to work with M from the point of view of a smooth metric measure space, we can follow the ideas of the proof in [34] and extend Proposition above to weighted manifolds as follows:

Proposition 2.2.2. *Let (M, g, e^{-f}) be a complete smooth metric measure space. If there exists a non-negative function $h \neq 0$ defined on M satisfying $\Delta_f h(x) \leq -\sigma(x)h(x)$, then*

$$\int_M \sigma(x)\phi^2(x)e^{-f} \leq \int_M |\nabla\phi|^2(x)e^{-f}$$

must be valid for all compactly supported smooth functions ϕ over M.

Proof: Let $D \subset M$ be a smooth compact subdomain of M. Let $\lambda_\sigma(D)$ be the first Dirichlet eigenvalue of the operator $\Delta_f + \sigma(x)$.

Variational characterization of $\lambda_\sigma(D)$ is given by

$$\lambda_\sigma(D) = \min_{\substack{w \in C_0^\infty(D) \\ w \neq cte}} \left\{ \frac{\int_D |\nabla w|^2 e^{-f} - \int_D \sigma w^2 e^{-f}}{\int_D w^2 e^{-f}} \right\}. \tag{2.8}$$

Let u be the first eigenfunction associated to $\lambda_\sigma(D)$, that is

$$\Delta_f u(x) + \sigma(x)u(x) = -\lambda_\sigma(D)u(x), \text{ on } D,$$

$$u(x) = 0, \text{ on } \partial D.$$

We may assume $u \geq 0$ and $u > 0$ in the interior of D. By Hopf's maximum principle (see e.g. [21]) we know $\partial_\nu u \leq 0$ and thus, Green's identity for the $f-$Laplacian yields

$$\int_D u\Delta_f h e^{-f} - \int_D h\Delta_f u e^{-f} = \int_{\partial D} u\partial_\nu h e^{-f} - \int_{\partial D} h\partial_\nu u e^{-f} \geq 0.$$

On the other hand, since $\Delta_f h(x) \leq -\sigma(x)h(x)$ we have

$$\begin{aligned} u(x)\Delta_f h(x) - h(x)\Delta_f u(x) \leq & - u(x)\sigma(x)h(x) + h(x)(\lambda_\sigma(D)u(x) + \sigma(x)u(x)) \\ = & \lambda_\sigma(D)u(x)h(x) \end{aligned}$$

Since $u > 0$ and $h \neq 0$ this implies $\lambda_\sigma(D) \geq 0$, we have from (2.8) that

$$0 \leq \lambda_\sigma(D) \int_D \phi^2(x)e^{-f} \leq \int_D |\nabla \phi|^2(x)e^{-f} - \int_D \sigma(x)\phi^2(x)e^{-f}$$

for any $\phi \in C_0^\infty(D)$, since D is any arbitrary subdomain in M, the result follows.

□

With the aid of the result above, we may prove the following lemma.

Lemma 2.3. *Let (M, g, f, λ) be a complete non-trivial expanding Schouten soliton. Let*

$$\sigma = \frac{n-2}{2(n-1)}R - n\lambda,$$

then $\sigma \geq 0$ *and*

$$\int_M \sigma\phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f}$$

for any $\phi \in C_0^\infty(M)$.

Proof: From Theorem 1.D, we know that $2(n-1)\lambda \leq R \leq 0$, then,

$$\sigma \geq \frac{2(n-2)(n-1)\lambda}{2(n-1)} - n\lambda = -2\lambda \geq 0.$$

By taking the trace over (1.12) we have

$$\Delta_f(\mathrm{e}^f) = \left(\Delta_f(f) + |\nabla f|^2\right)\mathrm{e}^f = (\Delta f)\mathrm{e}^f = -\left(\frac{n-2}{2(n-1)}R - n\lambda\right)\mathrm{e}^f = -\sigma\mathrm{e}^f$$

and the lemma follows from Proposition (2.2.1). □

The proof of Theorem 2.7 is, as usual, divided into two main arguments.First, we show that under certain hypothesis M can only have $f-$non-parabolic ends. Next on, we prove that there can only be one $f-$non-parabolic end in M, concluding M must be therefore connected at infinity or rigid. For the first part of the proof, we will proceed by contradiction and assume there are $f-$parabolic ends in M. To reach such contradiction, we will need three technical results we present next.

From here on, let E be an end of M, r be the distance function over M from a fixed point $p \in M$, $r(x) := d(x,p)$, and set $E(\rho) := E \cap B_p(\rho)$. We would like to mention that Lemmas 2.4 and 2.5 are an extension to smooth metric measure spaces of the results of Li and Wang in [31] and have already been used implicitly by Munteanu and Wang in [43]. Given no explicit proof was found during our bibliographic search, we have decided to include one here for the sake of completeness following the ideas in [31].

Lemma 2.4. *Let* (M,g,f) *be a complete smooth metric measure space. Suppose* E *is an end of* M *respect to* $B_p(\rho_0)$. *Let* $\mu_1(E) > 0$ *be the infimum of the Dirichlet spectrum of* Δ_f *over* E *and let* u *be a non-negative function defined on* E *satisfying*

$$\Delta_f u^k \geq c_0 u^k,$$

for some constant $c_0 > 0$ *and each* $k \geq 1$. *If* u *satisfies the growth condition*

$$\int_{E(\rho)} u^{2k} e^{-2\left(\sqrt{c_0+\mu_1(E)}\right)r} e^{-f} = o(\rho)$$

then for each $0 < \delta < 1$ *there is a constant* C *depending on* δ*,* k *and* μ_1 *such that*

$$\int_{E\setminus E(\rho)} u^{2k}e^{-f} \leq Ce^{-2\delta\sqrt{c_0+\mu_1(E)}(\rho-\rho_0)} \int_{E(\rho_0)\setminus E(\rho_0-1)} u^{2k}e^{-f}.$$

Proof: Let $\phi(r(x))$ be a non-negative cutoff function defined on E with $r(x)$ the geodesic distance function to the fixed point p. Then for any function $h(r(x))$,

$$\begin{aligned}
\int_E |\nabla(\phi e^h u^k)|^2 e^{-f} &= \int_E \langle \nabla(\phi e^h u^k), \nabla(\phi e^h u^k)\rangle e^{-f} \\
&= \int_E \left(|\nabla(\phi e^h)|^2 u^{2k} + \phi^2 e^{2h}|\nabla u^k|^2 + 2\phi e^h u^k \langle \nabla u^k, \nabla(\phi e^h)\rangle\right) e^{-f} \\
&= \int_E \left(|\nabla(\phi e^h)|^2 u^{2k} + \phi^2 e^{2h}|\nabla u^k|^2 + \frac{1}{2}\langle \nabla u^{2k}, \nabla(\phi^2 e^{2h})\rangle\right) e^{-f}
\end{aligned}$$

From Green's identity and the hypothesis over the $f-$Laplacian, we have

$$\begin{aligned}
\int_E |\nabla(\phi e^h u^k)|^2 e^{-f} &= \int_E \left(|\nabla(\phi e^h)|^2 u^{2k} + \phi^2 e^{2h}|\nabla u^k|^2 - \frac{1}{2}\phi^2 e^{2h}\Delta_f(u^{2k})\right) e^{-f} \\
&= \int_E \left[|\nabla(\phi e^h)|^2 u^{2k} + \phi^2 e^{2h}\left(|\nabla u^k|^2 - (u^k\Delta_f u^k + |\nabla u^k|^2)\right)\right] e^{-f} \\
&\leq \int_E \left[|\nabla(\phi e^h)|^2 u^{2k} - c_0\phi^2 e^{2h} u^{2k}\right] e^{-f}.
\end{aligned}$$

The variational definition of $\mu_1(E)$ implies

$$\mu_1(E)\int_E \phi^2 e^{2h} u^{2k} e^{-f} \leq \int_E |\nabla(\phi e^h u^k)|^2 e^{-f},$$

thus, we get from inequalities above that

$$\begin{aligned}
(\mu_1(E) + c_0)\int_E \phi^2 e^{2h} u^{2k} e^{-f} &\leq \int_E |\nabla(\phi e^h)|^2 u^{2k} e^{-f} \\
&= \int_E \langle e^h\nabla\phi + \phi e^h \nabla h, e^h\nabla\phi + \phi e^h\nabla h\rangle u^{2k} e^{-f} \\
&= \int_E e^{2h}\left(|\nabla\phi|^2 + 2\phi\langle\nabla\phi, \nabla h\rangle + \phi^2|\nabla h|^2\right) u^{2k} e^{-f}
\end{aligned} \tag{2.9}$$

Now, we pick suitable functions ϕ and h as follows. Let

$$\phi = \begin{cases} r(x) - \rho_0, & \text{in } E(\rho_0+1)\backslash E(\rho_0), \\ 1, & \text{in } E(\rho)\backslash E(\rho_0+1), \\ \rho^{-1}(2\rho - r(x)), & \text{in } E(2\rho)\backslash E(\rho), \\ 0, & \text{in } E\backslash E(2\rho), \end{cases}$$

which yields

$$\nabla\phi = \begin{cases} \nabla r(x), & \text{in } E(\rho_0+1)\backslash E(\rho_0), \\ 0, & \text{in } E(\rho)\backslash E(\rho_0+1), \\ -\rho^{-1}\nabla r(x), & \text{in } E(2\rho)\backslash E(\rho), \\ 0, & \text{in } E\backslash E(2\rho), \end{cases}$$

and

$$|\nabla\phi|^2 = \begin{cases} 1, & \text{in } E(\rho_0+1)\backslash E(\rho_0), \\ 0, & \text{in } E(\rho)\backslash E(\rho_0+1), \\ \rho^{-2}, & \text{in } E(2\rho)\backslash E(\rho), \\ 0, & \text{in } E\backslash E(2\rho). \end{cases}$$

Setting $a =: \sqrt{c_0 + \mu_1(E)}$, let A be a constant such that $A > (\rho_0+1)(1+\delta)a$, and let h be defined by

$$h(r) = \begin{cases} \delta a r, & \text{if } r < \frac{A}{(1+\delta)a}, \\ A - ar, & \text{if } r \geq \frac{A}{(1+\delta)a}. \end{cases}$$

This way

$$\nabla h(r) = \begin{cases} \delta a \nabla r, & \text{if } r < \frac{A}{(1+\delta)a}, \\ -a\nabla r, & \text{if } r \geq \frac{A}{(1+\delta)a}, \end{cases}$$

and, for $\rho > \frac{A}{(1+\delta)a}$,

$$\langle \nabla h(r), \nabla\phi\rangle = \begin{cases} \delta a, & \text{in } E(\rho_0+1)\backslash E(\rho_0), \\ \rho^{-1}a, & \text{in } E(2\rho)\backslash E(\rho), \\ 0, & \text{otherwise.} \end{cases}$$

In order to simplify the notation, set $B = A((1+\delta)a)^{-1}$. Plugging the definitions above into (2.9) and integrating over the respective domains, we have

$$
\begin{aligned}
a^2 \int_E \phi^2 e^{2h} u^{2k} e^{-f} \leq & \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f} + \rho^{-2} \int_{E(2\rho)\setminus E(\rho)} e^{2h} u^{2k} e^{-f} \\
& + 2\delta a \int_{E(\rho_0+1)\setminus E(\rho_0)} (r(x)-\rho_0) e^{2h} u^{2k} e^{-f} + 2\rho^{-2} a \int_{E(2\rho)\setminus E(\rho)} (2\rho - r(x)) e^{2h} u^{2k} e^{-f} \\
& + \delta^2 a^2 \int_{E(B)\setminus E(\rho_0)} \phi^2 e^{2h} u^{2k} e^{-f} + a^2 \int_{E(2\rho)\setminus E(B)} \phi^2 e^{2h} u^{2k} e^{-f} \\
\leq & \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f} + \rho^{-2} \int_{E(2\rho)\setminus E(\rho)} e^{2h} u^{2k} e^{-f} + 2\delta a \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f} \\
& + 2\rho^{-1} a \int_{E(2\rho)\setminus E(\rho)} e^{2h} u^{2k} e^{-f} + \delta^2 a^2 \int_{E(B)\setminus E(\rho_0)} \phi^2 e^{2h} u^{2k} e^{-f} + a^2 \int_{E(2\rho)\setminus E(B)} \phi^2 e^{2h} u^{2k} e^{-f}.
\end{aligned}
\tag{2.10}
$$

By noticing that

$$
\int_{E\setminus[E(2\rho)\setminus E(B)]} \phi^2 e^{2h} u^{2k} e^{-f} = \int_{E(B)} \phi^2 e^{2h} u^{2k} e^{-f},
$$

inequality (2.10) implies

$$
\begin{aligned}
a^2 \int_{E(B)} \phi^2 e^{2h} u^{2k} e^{-f} \leq & \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f} + \rho^{-2} \int_{E(2\rho)\setminus E(\rho)} e^{2h} u^{2k} e^{-f} \\
& + 2\delta a \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f} + 2\rho^{-1} a \int_{E(2\rho)\setminus E(\rho)} e^{2h} u^{2k} e^{-f} \\
& + \delta^2 a^2 \int_{E(B)\setminus E(\rho_0)} \phi^2 e^{2h} u^{2k} e^{-f}.
\end{aligned}
\tag{2.11}
$$

Recalling $\rho > B > \rho_0 + 1$ and the definition of ϕ, we know

$$
\int_{E(B)\setminus E(\rho_0+1)} \phi^2 e^{2h} u^{2k} e^{-f} = \int_{E(B)\setminus E(\rho_0+1)} e^{2h} u^{2k} e^{-f} \leq \int_{E(B)} \phi^2 e^{2h} u^{2k} e^{-f}
$$

and

$$
\int_{E(\rho_0+1)\setminus E(\rho_0)} \phi^2 e^{2h} u^{2k} e^{-f} \leq \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h} u^{2k} e^{-f}.
$$

Thus, we can estimate the integral of last term on the right side of (2.11) as

$$\begin{aligned}\int_{E(B)\setminus E(\rho_0)} \phi^2 e^{2h}u^{2k}e^{-f} &= \int_{E(B)\setminus E(\rho_0+1)} \phi^2 e^{2h}u^{2k}e^{-f} + \int_{E(\rho_0+1)\setminus E(\rho_0)} \phi^2 e^{2h}u^{2k}e^{-f}\\ &\leq \int_{E(B)} \phi^2 e^{2h}u^{2k}e^{-f} + \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h}u^{2k}e^{-f}.\end{aligned}$$

Plugging this into (2.11), we get

$$\begin{aligned}a^2 \int_{E(B)\setminus E(\rho_0+1)} e^{2h}u^{2k}e^{-f} \leq &\int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h}u^{2k}e^{-f} + \rho^{-2}\int_{E(2\rho)\setminus E(\rho)} e^{2h}u^{2k}e^{-f}\\ &+ 2\delta a \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h}u^{2k}e^{-f} + 2\rho^{-1}a \int_{E(2\rho)\setminus E(\rho)} e^{2h}u^{2k}e^{-f}\\ &+ \delta^2 a^2 \int_{E(B)\setminus E(\rho_0+1)} e^{2h}u^{2k}e^{-f} + \delta^2 a^2 \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h}u^{2k}e^{-f}.\end{aligned}$$

Now we can group the integrals over the same regions and apply the definition of h to conclude

$$\begin{aligned}(1-\delta^2)a^2 \int_{E(B)\setminus E(\rho_0+1)} e^{2\delta ar}u^{2k}e^{-f} \leq &(\delta a+1)^2 \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2\delta ar}u^{2k}e^{-f}\\ &+ (\rho^{-2}+2\rho^{-1}a)\int_{E(2\rho)\setminus E(\rho)} e^{2(A-ar)}u^{2k}e^{-f}.\end{aligned}$$

From the hypothesis on the growth of u, the second term on the right side above tends to 0 as ρ tends to infinity, thus for ρ large enough, we have

$$(1-\delta^2)a^2 \int_{E(B))\setminus E(\rho_0+1)} e^{2\delta ar}u^{2k}e^{-f} \leq (\delta a+1)^2 \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2h}u^{2k}e^{-f}.$$

Heeding the right-hand side does not depend on B, we can make $B\to\infty$, to get

$$\int_{E\setminus E(\rho_0+1)} e^{2\delta ar}u^{2k}e^{-f} \leq \frac{(\delta a+1)^2}{(1-\delta)a^2}\int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2\delta ar}u^{2k}e^{-f}. \tag{2.12}$$

Since $\rho > \rho_0 + 1$,

$$e^{2\delta\rho} \int\limits_{E\backslash E(\rho)} u^{2k}e^{-f} \leq \int\limits_{E\backslash E(\rho)} e^{2\delta ar}u^{2k}e^{-f} \leq C(a,\delta)e^{2\delta a(\rho_0+1)} \int\limits_{E(\rho_0+1)\backslash E(\rho_0)} u^{2k}e^{-f},$$

which, by renaming ρ_0, is equivalent to

$$\int\limits_{E\backslash E(\rho)} u^{2k}e^{-f} \leq C(a,\delta)e^{-2\delta a(\rho-\rho_0)} \int\limits_{E(\rho_0)\backslash E(\rho_0-1)} u^{2k}e^{-f},$$

and the proof is complete.

□

Lemma 2.5. *Under the hypothesis of Lemma 2.4, setting $c_0 = k$, u also satisfies the decay estimate*

$$\int\limits_{E(\rho+1)\backslash E(\rho)} u^2e^{-f} \leq C(a)(a + (\rho-\rho_0)^{-1})e^{-2a\rho} \int\limits_{E(\rho_0+1)\backslash E(\rho_0)} e^{2ar}u^2e^{-f}, \qquad (2.13)$$

for some constant $C(a)$ depending on $a = \sqrt{k + \mu_1(E)}$, for each integer $k \geq 1$, and for all $\rho \geq 2(\rho_0 + 1)$.

Proof: Recalling equation (2.9)

$$a^2 \int\limits_E \phi^2e^{2h}u^{2k}e^{-f} \leq \int\limits_E \left(e^{2h}|\nabla\phi|^2 + 2\phi e^{2h}\langle\nabla\phi,\nabla h\rangle + \phi^2e^{2h}|\nabla h|^2\right)u^{2k}e^{-f},$$

and taking $h = ar$, we have

$$a^2 \int\limits_E \phi^2e^{2ar}u^{2k}e^{-f} \leq \int\limits_E \left(e^{2ar}|\nabla\phi|^2 + 2a\phi e^{2ar}\langle\nabla\phi,\nabla r\rangle + a^2\phi^2e^{2ar}\right)u^{2k}e^{-f},$$

which is equivalent to

$$-2a\int\limits_E \phi e^{2ar}\langle\nabla\phi,\nabla r\rangle u^{2k}e^{-f} \leq \int\limits_E e^{2ar}|\nabla\phi|^2u^{2k}e^{-f}. \qquad (2.14)$$

For $\rho_0 < \rho_1 < \rho$, let $\phi(x) = 0$ in $E\backslash E(\rho)$ and let

$$\phi(x) = \begin{cases} \frac{r(x)-\rho_0}{\rho_1-\rho_0}, & \text{in } E(\rho_1)\backslash E(\rho_0), \\ \frac{\rho-r(x)}{\rho-\rho_1}, & \text{in } E(\rho)\backslash E(\rho_1), \end{cases}$$

then

$$\nabla\phi = \begin{cases} \frac{\nabla r}{\rho_1-\rho_0}, & \text{in } E(\rho_1)\backslash E(\rho_0), \\ \frac{-\nabla r}{\rho-\rho_1}, & \text{in } E(\rho)\backslash E(\rho_1), \end{cases} \qquad |\nabla\phi|^2 = \begin{cases} \frac{1}{(\rho_1-\rho_0)^2}, & \text{in } E(\rho_1)\backslash E(\rho_0), \\ \frac{1}{(\rho-\rho_1)^2}, & \text{in } E(\rho)\backslash E(\rho_1); \end{cases}$$

and

$$\langle\nabla\phi, \nabla r\rangle = \begin{cases} \frac{1}{\rho_1-\rho_0}, & \text{in } E(\rho_1)\backslash E(\rho_0), \\ \frac{-1}{\rho-\rho_1}, & \text{in } E(\rho)\backslash E(\rho_1), \end{cases}$$

with all of them null outside $E(\rho)$. By applying this into (2.14) we have

$$\begin{aligned} \frac{2a}{(\rho-\rho_1)^2}\int\limits_{E(\rho)\backslash E(\rho_1)} (\rho-r)e^{2ar}u^{2k}e^{-f} - \frac{2a}{(\rho_1-\rho_0)^2}\int\limits_{E(\rho_1)\backslash E(\rho_0)} (r-\rho_0)e^{2ar}u^{2k}e^{-f} \\ \leq \frac{1}{(\rho-\rho_1)^2}\int\limits_{E(\rho)\backslash E(\rho_1)} e^{2ar}u^{2k}e^{-f} + \frac{1}{(\rho_1-\rho_0)^2}\int\limits_{E(\rho_1)\backslash E(\rho_0)} e^{2ar}u^{2k}e^{-f}. \end{aligned}$$

On the other hand, for any $0 < t < \rho-\rho_1$, we have $E(\rho-t)\backslash E(\rho_1) \subset E(\rho)\backslash E(\rho_1)$ and $t < \rho - r$ on $E(\rho-t)\backslash E(\rho_1)$, then

$$\frac{2at}{(\rho-\rho_1)^2}\int\limits_{E(\rho-t)\backslash E(\rho_1)} e^{2ar}u^{2k}e^{-f} \leq \frac{2a}{(\rho-\rho_1)^2}\int\limits_{E(\rho)\backslash E(\rho_1)} (\rho-r)e^{2ar}u^{2k}e^{-f},$$

which implies

$$\begin{aligned} \frac{2at}{(\rho-\rho_1)^2}\int\limits_{E(\rho-t)\backslash E(\rho_1)} e^{2ar}u^{2k}e^{-f} &\leq \frac{1}{(\rho-\rho_1)^2}\int\limits_{E(\rho)\backslash E(\rho_1)} e^{2ar}u^{2k}e^{-f} \\ &+ \frac{1}{(\rho_1-\rho_0)^2}\int\limits_{E(\rho_1)\backslash E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{2a}{(\rho_1-\rho_0)^2}\int\limits_{E(\rho_1)\backslash E(\rho_0)} (r-\rho_0)e^{2ar}u^{2k}e^{-f} \\ &\leq \left(\frac{1}{(\rho_1-\rho_0)^2} + \frac{2a}{\rho_1-\rho_0}\right)\int\limits_{E(\rho_1)\backslash E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{1}{(\rho-\rho_1)^2}\int\limits_{E(\rho)\backslash E(\rho_1)} e^{2ar}u^{2k}e^{-f}. \end{aligned} \tag{2.15}$$

By taking $\rho_1 = \rho_0 + 1$ and $t = a^{-1}$, inequality above is equivalent to

$$\begin{aligned} \frac{2}{(\rho-\rho_0-1)^2}\int\limits_{E(\rho-a^{-1})\backslash E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \leq \\ (1+2a)\int\limits_{E(\rho_0+1)\backslash E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{1}{(\rho-\rho_0-1)^2}\int\limits_{E(\rho)\backslash E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \end{aligned}$$

which, by renaming $\rho := \rho - a^{-1}$, yields

$$\begin{aligned}\int_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \leq& \frac{1+2a}{2}(\rho + a^{-1} - \rho_0 - 1)^2 \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} \\ &+ \frac{1}{2}\int_{E(\rho+a^{-1})\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \\ \leq& \frac{1+2a}{2}(\rho + a^{-1})^2 \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{1}{2}\int_{E(\rho+a^{-1})\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f}.\end{aligned} \tag{2.16}$$

Setting

$$g(\rho) = \int_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f},$$

we can rewrite (2.16) as

$$g(\rho) \leq I_1(\rho + a^{-1})^2 + \frac{1}{2}g(\rho + a^{-1}), \tag{2.17}$$

where

$$I_1 = \frac{1+2a}{2}\int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}$$

is independent of ρ. Notice we can apply the same inequality (2.17) on the term $g(\rho+a^{-1})$ obtaining the iteration

$$g(\rho) \leq I_1(\rho + a^{-1})^2 + \frac{1}{2}I_1(\rho + 2a^{-1})^2 + \frac{1}{4}g(\rho + 2a^{-1})$$

and, by iterating m times, we have

$$\begin{aligned} g(\rho) \leq& I_1 \sum_{i=1}^{m} \frac{(\rho + ia^{-1})^2}{2^{i-1}} + \frac{1}{2^m}g(\rho + ma^{-1}) \\ \leq& I_1\rho^2 \sum_{i=1}^{\infty} \frac{(1 + ia^{-1})^2}{2^{i-1}} + \frac{1}{2^m}g(\rho + ma^{-1}) \\ \leq& I_2\rho^2 + \frac{1}{2^m}g(\rho + ma^{-1})\end{aligned} \tag{2.18}$$

where

$$I_2 = \sum_{i=1}^{\infty} \frac{(1+ia^{-1})^2}{2^{i-1}} I_1 = \sum_{i=1}^{\infty} \frac{(1+2a)(1+ia^{-1})^2}{2^i} \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar} u^{2k} e^{-f};$$

however, from (2.12) and for any $\delta < 1$, second term on the right-hand side of (2.18) is bounded by

$$\begin{aligned} \frac{1}{2^m} g(\rho + {}^{m}/_{a}) =& \frac{1}{2^m} \int_{E(\rho+{}^{m}/_{a})\setminus E(\rho_0+1)} e^{2ar} u^{2k} e^{-f} \le \frac{e^{2a(\rho+{}^{m}/_{a})(1-\delta)}}{2^m} \int_{E(\rho+{}^{m}/_{a})\setminus E(\rho_0+1)} e^{2\delta ar} u^{2k} e^{-f} \\ \le& \frac{e^{2a(\rho+{}^{m}/_{a})(1-\delta)}}{2^m} \int_{E(\rho)\setminus E(\rho_0+1)} e^{2\delta ar} u^{2k} e^{-f} \le C_1 2^{-m} e^{2a(\rho+{}^{m}/_{a})(1-\delta)}, \end{aligned}$$

where

$$C_1 = C(a,\delta) e^{2\delta a(\rho_0+1)} \int_{E(\rho_0+1)\setminus E(\rho_0)} u^{2k} e^{-f}.$$

Then, by choosing $2(1-\delta) < \ln 2$ we get $2^{-1} e^{2(1-\delta)} < 1$ and

$$g(\rho + {}^{m}/_{a}) \le C_1 2^{-m} \left(e^{2(1-\delta)}\right)^{a\rho+m} = C_2 (2^{-1} e^{2(1-\delta)})^m \to 0, \text{ as } m \to \infty.$$

Thus, we conclude from (2.18) that for ρ large enough

$$\int_{E(\rho)\setminus E(\rho_0+1)} e^{2ar} u^{2k} e^{-f} \le I_2 \rho^2. \tag{2.19}$$

Using once again inequality (2.15) with $\rho_1 = \rho_0 + 1$ and $t = \rho/2$ we get

$$\begin{aligned} \frac{a\rho}{(\rho-\rho_0-1)^2} \int_{E(\rho/2)\setminus E(\rho_0+1)} e^{2ar} u^{2k} e^{-f} \le& (1+2a) \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar} u^{2k} e^{-f} \\ &+ \frac{1}{(\rho-\rho_0-1)^2} \int_{E(\rho)\setminus E(\rho_0+1)} e^{2ar} u^{2k} e^{-f} \end{aligned}$$

or, equivalently,

$$a\rho \int\limits_{E(\rho/2)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \leq (1+2a)\,(\rho-\rho_0-1)^2 \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \int\limits_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f}.$$

Applying inequality (2.19) to the second term of the right side we have

$$a\rho \int\limits_{E(\rho/2)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \leq (1+2a)\,(\rho-\rho_0-1)^2 \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + I_2\rho^2,$$

or, equivalently,

$$\begin{aligned}\int\limits_{E(\frac{\rho}{2})\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} \leq& \frac{1+2a}{a}\left(\frac{(\rho-\rho_0-1)^2}{\rho^2}+\sum_{i=1}^{\infty}\frac{(1+ia^{-1})^2}{2^i}\right)\rho \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}\\ \leq& \frac{1+2a}{a}\left(1+\sum_{i=1}^{\infty}\frac{(1+ia^{-1})^2}{2^i}\right)\rho \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}.\end{aligned} \tag{2.20}$$

Recalling u is bounded, we know

$$\begin{aligned}\int\limits_{E(\rho)} e^{2ar}u^{2k}e^{-f} =& \int\limits_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} + \int\limits_{E(\rho_0+1)} e^{2ar}u^{2k}e^{-f}\\ \leq& \int\limits_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} + e^{2a(\rho_0+1)} \int\limits_{E(\rho_0+1)} u^{2k}e^{-f}\\ \leq& \int\limits_{E(\rho)\setminus E(\rho_0+1)} e^{2ar}u^{2k}e^{-f} + C_3(a)\end{aligned}$$

for some constant $C_3(a)$ depending only on a. Thus, for ρ large enough, (2.20) implies

$$\int\limits_{E(\rho)} e^{2ar}u^{2k}e^{-f} \leq C(a)\rho \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}, \tag{2.21}$$

for some constant $C(a)$ depending only of a.

In order to prove (2.13), we apply once again equation (2.15) for $t = 2a^{-1}$ and $\rho_1 =$

$\rho - 4a^{-1}$, obtaining

$$\frac{a^2}{4}\int_{E(\rho-2/a)\setminus E(\rho-4/a)} e^{2ar}u^{2k}e^{-f} \leq \left(\frac{1}{(\rho - 4/a - \rho_0)^2} + \frac{2a}{\rho - 4/a - \rho_0}\right)\int_{E(\rho-4/a)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{a^2}{16}\int_{E(\rho)\setminus E(\rho-4/a)} e^{2ar}u^{2k}e^{-f},$$

this is,

$$\int_{E(\rho-2/a)\setminus E(\rho-4/a)} e^{2ar}u^{2k}e^{-f} \leq \left(\frac{4}{a^2(\rho - 4/a - \rho_0)^2} + \frac{8}{a(\rho - 4/a - \rho_0)}\right)\int_{E(\rho-4/a)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{1}{4}\int_{E(\rho)\setminus E(\rho-4/a)} e^{2ar}u^{2k}e^{-f}. \tag{2.22}$$

From (2.21), we can bound the first term on in the right side of (2.22) by

$$C(a)(a + (\rho - \rho_0 - 4a^{-1})^{-1})\int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f},$$

and by renaming $\rho := \rho - 4a^{-1}$, we have

$$\int_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq C(a)(a + (\rho - \rho_0)^{-1})\int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} + \frac{1}{4}\int_{E(\rho+4a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f}. \tag{2.23}$$

Noticing

$$\int_{E(\rho+4a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} = \int_{E(\rho+4a^{-1})\setminus E(\rho+2a^{-1})} e^{2ar}u^{2k}e^{-f} + \int_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f},$$

we have (2.23) is equivalent to

$$\begin{aligned}\frac{3}{4}\int\limits_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq & C(a)(a+(\rho-\rho_0)^{-1})\int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} \\ & + \frac{1}{4}\int\limits_{E(\rho+4a^{-1})\setminus E(\rho+2a^{-1})} e^{2ar}u^{2k}e^{-f},\end{aligned}$$

which implies

$$\begin{aligned}\int\limits_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq & C(a)(a+(\rho-\rho_0)^{-1})\int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f} \\ & + \frac{1}{3}\int\limits_{E(\rho+4a^{-1})\setminus E(\rho+2a^{-1})} e^{2ar}u^{2k}e^{-f}.\end{aligned}$$

As it was done before to obtain (2.18), we can iterate this inequality on itself to obtain

$$\begin{aligned}\int\limits_{E(\rho+2/a)\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq & C(a)(a+(\rho-\rho_0)^{-1})I_3 \\ & + \frac{1}{3}\left(C(a)(a+(\rho-\rho_0)^{-1})I_3 + \frac{1}{3}\int\limits_{E(\rho+6/a)\setminus E(\rho+4/a)} e^{2ar}u^{2k}e^{-f}\right)\end{aligned}$$

where

$$I_3 = \int\limits_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f},$$

then, by iterating m times, one sees that

$$\begin{aligned}\int\limits_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq & C(a)(a+(\rho-\rho_0)^{-1})\sum_{i=1}^{m}\frac{1}{3^{i-1}}I_3 \\ & + \frac{1}{3^m}\int\limits_{E(\rho+2(m+1)a^{-1})\setminus E(\rho+2ma^{-1})} e^{2ar}u^{2k}e^{-f}.\end{aligned}$$

However, by (2.21), second term on the right is bounded by

$$\frac{1}{3^m}\int\limits_{E(\rho+2(m+1)a^{-1})\setminus E(\rho+2ma^{-1})} e^{2ar}u^{2k}e^{-f} \leq \frac{C(a)}{3^m}(\rho+2(m+1)a^{-1})I_3$$

which goes to 0 as $m \to \infty$. Hence, for a constant C depending on a, we conclude that

$$\int_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f} \leq C(a)(a+(\rho-\rho_0)^{-1}) \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}. \tag{2.24}$$

Since

$$e^{2a\rho} \int_{E(\rho+2a^{-1})\setminus E(\rho)} u^{2k}e^{-f} \leq \int_{E(\rho+2a^{-1})\setminus E(\rho)} e^{2ar}u^{2k}e^{-f},$$

equation (2.24) implies

$$\int_{E(\rho+2a^{-1})\setminus E(\rho)} u^{2k}e^{-f} \leq C(a+(\rho-\rho_0)^{-1})e^{-2a\rho} \int_{E(\rho_0+1)\setminus E(\rho_0)} e^{2ar}u^{2k}e^{-f}, \tag{2.25}$$

for $\rho + 4a^{-1} \geq 2(\rho_0+1)$ which proves the lemma for any $a \leq 2$. If $a > 2$, right side of (2.13) is still valid for a new suitable C: take the ceiling $\lceil a/2 \rceil$ and notice we can divide the interval $[\rho, \rho+1]$ into $\lceil a/2 \rceil$ components $[\rho, \rho+\lceil a/2\rceil^{-1}], [\rho+\lceil a/2\rceil^{-1}, \rho+2\lceil a/2\rceil^{-1}]\ldots$, each one of size $2a^{-1}$, since

$$(a+(\rho+k\lceil a/2\rceil^{-1}-\rho_0)^{-1})e^{-2a(\rho+k\lceil a/2\rceil^{-1})} \leq (a+(\rho-\rho_0)^{-1})e^{-2a\rho},$$

we can apply (2.25) in each one of these intervals and sum the estimate $\lceil a/2 \rceil$ times. □

Let $V(E)$ denote the volume of an end E and $V_E(\rho)$ denote the volume of the set $E(\rho)$. The following decay estimate for the weighted volume of a parabolic end is an extension to smooth metric measure spaces of the results of Li and Wang given in [33], and [31, Theorem 22.1], and is the last result we are going to need for the first part of the proof of Theorem 2.7:

Lemma 2.6. *Let E be a parabolic end of (M, g, λ, f), a complete smooth metric space with first Dirichlet eigenvalue of the weighted $f-$Laplacian $\mu_1 > 0$. Then, E must have exponential volume decay given by*

$$V(E) - V_E(\rho) \leq C(V_E(\rho_0+1) - V_E(\rho_0))e^{-2(\rho-\rho_0)\sqrt{\mu_1(E)}}, \tag{2.26}$$

for $\rho \geq \rho_0 + 1$ and some constant $C > 0$ depending on μ_1.

Proof: First, notice that equation (2.13) of Lemma 2.5 directly implies

$$\int_{E(\rho+1)\backslash E(\rho)} u^2 e^{-f} \leq C(a)(a+(\rho-\rho_0)^{-1})e^{-2a(\rho-\rho_0)} \int_{E(\rho_0+1)\backslash E(\rho_0)} u^2 e^{-f}.$$

Let u_ρ be the harmonic function on $E(\rho)$ such that $u_\rho = 1$ on ∂E and $u_\rho = 0$ on $\partial E(\rho)$. The assumption of E being parabolic implies that u_ρ converges to $u = 1$ as $\rho \to \infty$ (existence of such function and its convergence is discussed in detail in [31, Chapter 20] and it is a consequence of Theorems 17.1, 20.6 and 20.7 therein). Hence, we have that for ρ large enough,

$$\int_{E(\rho+1)\backslash E(\rho)} u^2 e^{-f} \approx \int_{E(\rho+1)\backslash E(\rho)} e^{-f} = V_E(\rho+1) - V_E(\rho),$$

and, from the estimate of Lemma (2.5), we get the volume decay estimate

$$V_E(\rho+1) - V_E(\rho) \leq C(a+(\rho-\rho_0)^{-1})(V_E(\rho_0+1) - V_E(\rho_0))e^{-2(\rho-\rho_0)\sqrt{\mu_1(E)}}.$$

Let now $\rho = \rho + i$ for $i = 0, 1, ...$ on the estimate above, then, by summing over i and taking suitable constants C, we have

$$\begin{aligned} V(E) - V_E(\rho) =& \sum_{i=0}^{\infty} (V_E(\rho+i+1) - V_E(\rho+i)) \\ \leq& C(V_E(\rho_0+1) - V_E(\rho_0)) \sum_{i=0}^{\infty} (a+(\rho+i-\rho_0)^{-1})e^{-2(\rho+i-\rho_0)\sqrt{\mu_1(E)}} \\ \leq& C(V_E(\rho_0+1) - V_E(\rho_0))e^{-2(\rho-\rho_0)\sqrt{\mu_1(E)}} \sum_{i=0}^{\infty} (a+(\rho-\rho_0)^{-1})e^{-2i\sqrt{\mu_1(E)}} \\ \leq& C(V_E(\rho_0+1) - V_E(\rho_0))e^{-2(\rho-\rho_0)\sqrt{\mu_1(E)}} \end{aligned} \tag{2.27}$$

as we wanted to prove. □

For the final part of the proof of Theorem 2.7, we will use the next result found in [41]:

Theorem 2.J. *Let (M, g, f) be a complete gradient expanding Ricci soliton. Assume that the scalar curvature satisfies $R \geq \frac{1}{2}(1-n)$, then either M is connected at infinity or $M = \mathbb{R} \times N^{n-1}$, where N is a compact Einstein manifold and $\mathbb{R}$ is the Gaussian expanding soliton.*

Recalling Theorem 2.J is stated on the context of normalized solitons, the condition over R shall be read as $R \geq (n-1)\lambda_0$ in the context of our work, where, λ_0 stands for the expanding argument of the soliton. We are now ready to state and prove the main result of this section. It asserts that a non-trivial and non-rigid expanding Schouten soliton has, at most, one end:

Theorem 2.7. *Let (M, g, f, λ) be a complete non-trivial expanding Schouten soliton. Then either M is connected at infinity or it is isometric to $\mathbb{R}^1 \times N^{n-1}$, where N is a compact Einstein manifold and $\mathbb{R}^1$ is the one-dimensional Gaussian expanding soliton.*

Proof: Considering weight function σ as in Lemma 2.3 we have from Corollary 1.4 of [34] that M is $f-$non-parabolic. We will show all ends of M must be $f-$non-parabolic. Suppose E is a parabolic end of M, first, we claim both $|f|$ and $|\nabla f|$ are bounded in E. Indeed, consider the function

$$u = 1 + f_0 + f$$

where $f_0 = \max_M f$, whose existence is guaranteed by Theorem 1.D, then clearly $u \geq 1$ and, from (1.14),

$$|\nabla u|^2 = |\nabla f|^2 \leq 4\lambda(f - f_0) = -4\lambda(u-1) < -4\lambda u, \tag{2.28}$$

and, from both (1.13) and (1.14),

$$\begin{aligned} \Delta_f u =& \Delta u - \langle \nabla f, \nabla u\rangle = -\Delta f + |\nabla f|^2 = -n\lambda + \frac{n-2}{2(n-1)}R + |\nabla f|^2 \\ \geq& -n\lambda + (n-2)\lambda + 2\lambda(f - f_0) = -2\lambda + 2\lambda(f-f_0) = -2\lambda(1 + f_0 - f) \\ =& -2\lambda u. \end{aligned} \tag{2.29}$$

Let $w = \mathrm{e}^{-\frac{1}{2}u}$, then, from the inequalities above,

$$\begin{aligned} \Delta_f w =& \Delta w - \langle \nabla f, \nabla w\rangle = \Delta\left(\mathrm{e}^{-\frac{1}{2}u}\right) - \left\langle \nabla f, \nabla\left(\mathrm{e}^{-\frac{1}{2}u}\right)\right\rangle \\ =& -\frac{1}{2}\mathrm{e}^{-\frac{1}{2}u}\Delta u + \frac{1}{4}\mathrm{e}^{-\frac{1}{2}u}|\nabla u|^2 + \frac{1}{2}\mathrm{e}^{-\frac{1}{2}u}\langle \nabla f, \nabla u\rangle \\ =& -\frac{1}{2}\mathrm{e}^{-\frac{1}{2}u}\Delta_f u + \frac{1}{4}\mathrm{e}^{-\frac{1}{2}u}|\nabla u|^2 \\ \leq& \mathrm{e}^{-\frac{1}{2}u}\lambda u - \mathrm{e}^{-\frac{1}{2}u}\lambda u \\ =& 0. \end{aligned}$$

If f is unbounded, so should be u, this would imply w is a positive $f-$superharmonic function on E which achieves its infimum at infinity of E, which implies E is an $f-$non-parabolic end, therefore, f must be bounded on E and, in view of (1.14), so must be $|\nabla f|$, as we claimed above.

According to Munteanu and Wang in [41], this implies there are constants $C_1, C_2 > 0$ such that the weighted volume of the unitary ball is bounded below by

$$V_f(B_x(1)) \geq C_1 e^{-C_2 r(x)} \tag{2.30}$$

for any $x \in E$, furthermore, C_1 and C_2 are independent of x. Next, we show that this leads to a contradiction.

Notice that, in particular, we have from Lemma 2.3 that first eigenvalue $\mu_1(E)$ is positive, thus, we can apply Lemma 2.6 to conclude weighted volume of E is finite and, together with the boundedness of u we have

$$\frac{1}{R}\int\limits_{E(R)} u^{2k} e^{-2\sqrt{-2\lambda k+\mu_1(E)}\rho} e^{-f} \leq \frac{C}{R}\int\limits_{M} \mathrm{e}^{-f} \to 0 \text{ as } R \to \infty$$

for any fixed $k \geq 1$. On the other hand, from (2.28) and (2.29), it is also true that

$$\Delta_f u^k = k u^{k-1}\Delta_f u + k(k+1)u^{k-2}|\nabla u|^2 \geq -2\lambda k u^k$$

for any $k \geq 1$. Then, by making $c_0 = -2\lambda k$ in Lemma 2.4, we conclude

$$\int\limits_{E\setminus E(\rho)} u^{2k} e^{-f} \leq C(a) e^{-2\sqrt{-2\lambda k+\mu_1(E)}(\rho-\rho_0)} \int\limits_{E(\rho_0)\setminus E(\rho_0-1)} u^{2k} e^{-f}, \tag{2.31}$$

for any $k \geq 1$ and $a = \sqrt{-2\lambda k + \mu_1(E)}$. Let

$$C(k) = C(a) \int\limits_{E(\rho_0)\setminus E(\rho_0-1)} u^{2k} e^{-f},$$

thus, $C(k)$ is independent of ρ and, since $u \geq 1$, we have

$$V_f(B_x(1)) \leq \int\limits_{E\setminus E(\rho)} e^{-f} \leq C(k) e^{-2\sqrt{-2\lambda k+\mu_1(E)}(\rho-\rho_0)}$$

for any x such that $V_f(B_x(1)) \subset E\setminus E(\rho)$. Combined with (2.30), this implies for ρ big enough that

$$C_1 e^{-C_2\rho} \leq C(k) e^{-2\sqrt{-2\lambda k+\mu_1(E)}(\rho-\rho_0)},$$

that is,
$$C_3 e^{\left(2\sqrt{-2\lambda k+\mu_1(E)}-C_2\right)\rho} \leq C(k),$$
setting k big enough to have $2\sqrt{-2\lambda k+\mu_1(E)}-C_2>0$ and making $\rho\to\infty$ we arrive to a contradiction. Thus, from the discussion above, all ends of M must be $f-$non-parabolic.

Next, we show M must either have exactly one end or be the rigid Ricci soliton of Theorem 2.J.

Suppose M has at least two ends. Then, by [32] (Theorem 1.G here), there exists a positive non-constant $f-$harmonic function h such that $h<1$ and
$$\int_M |\nabla h|^2 \mathrm{e}^{-f} < \infty. \tag{2.32}$$

Given $\Delta_f h = 0$, we have from the Bochner formula (1.10), Schouten soliton equation (1.12) and Kato's inequality that
$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 =& |\nabla^2 h|^2 + \mathrm{Ric}_f(\nabla h,\nabla h)\\ \geq& |\nabla|\nabla h||^2 + \left(\frac{R}{2(n-1)}+\lambda\right)|\nabla h|^2,\end{aligned}$$
and, since $\frac{1}{2}\Delta_f|\nabla h|^2 = |\nabla h|\Delta_f|\nabla h| + |\nabla|\nabla h||^2$, we conclude
$$\Delta_f|\nabla h| \geq \left(\frac{R}{2(n-1)}+\lambda\right)|\nabla h|. \tag{2.33}$$
By taking an arbitrary cut-off function ϕ and plugging $|\nabla h|\phi$ in Poincaré's inequality from Lemma 2.3, we get
$$\begin{aligned}\int_M \sigma(|\nabla h|\phi)^2\mathrm{e}^{-f} \leq& \int_M |\nabla(|\nabla h|\phi)|^2\mathrm{e}^{-f}\\ =& \int_M \langle \nabla|\nabla h|\phi + |\nabla h|\nabla\phi, \nabla|\nabla h|\phi + |\nabla h|\nabla\phi,\rangle \mathrm{e}^{-f}\\ =& \int_M \left(|\nabla|\nabla h||^2\phi^2 + 2|\nabla h|\phi\langle\nabla|\nabla h|,\nabla\phi\rangle + |\nabla h|^2|\nabla\phi|^2\right)\mathrm{e}^{-f}.\end{aligned} \tag{2.34}$$

From Green's identity (1.9) we know

$$-\int_M |\nabla h|(\Delta_f|\nabla h|)\phi^2 e^{-f} = \int_M \langle \nabla|\nabla h|, \nabla(|\nabla h|\phi^2)\rangle e^{-f}$$
$$= \int_M \langle \nabla|\nabla h|, \nabla|\nabla h|\phi^2 + 2\phi|\nabla h|\nabla\phi\rangle e^{-f}. \tag{2.35}$$

Since $|\nabla h|$ has finite energy, we can pick ϕ such that $|\nabla\phi| \to 0$ and $\int_M |\nabla h|^2|\nabla\phi|e^{-f} \to 0$, thus, combining (2.35) with equations (2.33) and (2.34) we conclude

$$\int_M \sigma(|\nabla h|\phi)^2 e^{-f} \leq -\int_M |\nabla h|(\Delta_f|\nabla h|)\phi^2 e^{-f} + \int_M |\nabla h|^2|\nabla\phi|^2 e^{-f}$$
$$\leq -\int_M \left(\frac{R}{2(n-1)} + \lambda\right)|\nabla h|^2\phi^2 e^{-f},$$

Recalling $\sigma = \frac{n-2}{2(n-1)}R - n\lambda$, inequality above implies

$$0 \geq \int_M \left(\frac{n-2}{2(n-1)}R - n\lambda + \frac{R}{2(n-1)} + \lambda\right)(|\nabla h|\phi)^2 e^{-f}$$
$$= \int_M \left(\frac{1}{2}R - (n-1)\lambda\right)(|\nabla h|\phi)^2 e^{-f} \geq 0,$$

where we used (1.13) for the latest inequality. Since h is non-constant and $\phi \neq 0$, this imply $R = 2(n-1)\lambda$, so the Schouten soliton equation is equivalent to

$$\mathrm{Ric} + \nabla^2 f = 2\lambda g$$

in M, that is, M is a gradient Ricci soliton with constant scalar curvature $R = 2(n-1)\lambda$. By making $\lambda_0 = 2\lambda$ in Theorem 2.J, we have M must be isometric to the product $\mathbb{R}^1 \times N^{n-1}$ and the proof is complete.

□

Chapter

3

Ends of shrinking $\rho-$Einstein solitons

In this chapter, we will show that under a boundedness hypothesis over the scalar curvature R, shrinking gradient $\rho-$Einstein solitons must be connected at infinity. Here, we will additionally assume that $\rho \in \left[0, \frac{1}{2(n-1)}\right)$. The case $\rho = 0$ (that is, when M is a gradient Ricci soliton) was originally proved by Munteanu and Wang in their work [44]. Our final result here, covers their result as a particular case.

3.1 $\varphi-$Non-parabolicity of all ends

Recall a complete manifold (M, g, f, λ) is said to be a gradient $\rho-$Einstein soliton if

$$\mathrm{Ric} + \nabla^2 f = (\rho R + \lambda) g.$$

Throughout this chapter, we will consider on M the weight e^{φ} with

$$\varphi = -af \tag{3.1}$$

and $a > 0$ a fixed constant. The Bakry–Emery Ricci tensor associated with to this new weighted smooth metric measure space is given by

$$\mathrm{Ric}_{\varphi} = \mathrm{Ric} + \nabla^2 \varphi$$

which, from the definition of $\rho-$Einstein soliton, means

$$\mathrm{Ric}_{\varphi} = (\rho R + \lambda) g - (a+1) \nabla^2 f.$$

In order to count the number of ends of shrinking $\rho-$Einstein gradient solitons, we will show next that, under certain conditions, $(M, g, \lambda, \rho, e^{\varphi})$ has only $\varphi-$non-parabolic ends (Theorem 3.1). The proof of our result will be based on Munteanu and Wang's proof in [43] of the following result.

Theorem 3.K. *[43] Let (M, g, f) be a gradient shrinking Kähler Ricci soliton. Then (M, g) has only one end.*

As we are going to base our proof on the one of Theorem 3.K, we present now a sketch of such proof given by Munteanu and Wang. The proof can be divided into in five main steps:

- **Step 1.** Proof starts by supposing the existence of a $\varphi-$parabolic end E, which implies the existence of a $\varphi-$harmonic function $h \geq 1$ such that $u = 1$ in ∂E and that diverges to infinity (see (3.22)). By manipulating such a function, the authors prove that
$$\int\limits_{B_x(1)} |\nabla \ln h|^2 \leq C_0 e^{-\frac{a}{4}r(x)^2 + cr(x)}. \tag{3.2}$$

- **Step 2.** Setting $v = \ln h$ and $\sigma = |\nabla v|^2$ with the aim of transforming (3.2) into a point-wise inequality, the authors prove a differential inequality in terms only of σ, $\nabla\sigma$ and f, namely
$$(p-2)\int\limits_M \sigma^{p-2}|\nabla\sigma|^2\phi^2 \leq C(n)(a+1)^2 p\int\limits_M f\sigma^p\phi^2 + C(n)(a+1)\int\limits_M \sigma^p|\nabla\phi|^2 \tag{3.3}$$
for p large enough, a constant $C(n)$ independent of p and for any cut-off function ϕ with support on the unit ball $B_{x_0}(1)$, for a fixed point $x_0 \in E$.

- **Step 3.** Combining (3.3) with Theorem 1.A, authors apply Nash-Moser theory to conclude σ satisfies the mean value inequality
$$\sigma(x) \leq Ce^{c(n)r(x)} \int\limits_{B_x(1)} \sigma,$$
where $r(x) = d(x_0, x)$.

- **Step 4.** Integral inequality found in **Step 3**, combined with 3.2, leads to conclude h must be bounded, which is a contradiction, implying that all ends on M must be $\varphi-$non-parabolic.

- **Step 5.** Finally, the authors use the Kähler geometry of M to show that the assumption of more than one $\varphi-$non-parabolic end leads quickly to a contradiction, concluding M must be connected at infinity.

Remark 3.1. *It is worth mentioning that Kähler hypothesis of Theorem 3.K is only used on the last step to conclude connectedness at infinity and first four steps are still valid for non-Kähler shrinking gradient Ricci solitons.*

In our next result, we prove that all ends on a shrinking gradient $\rho-$Einstein soliton with non-negative bounded scalar curvature are $\varphi-$non-parabolic. Its proof is based on the first steps of the aforementioned Theorem 3.K and, as Steps 1 and 2 above depend on the soliton structure of M, we will focus mainly on proving that analogous results are valid for $\rho-$Einstein solitons. This way, we set the conditions for the application of **Step 3** from [43], which depends only on the fact of M being a smooth metric measure space with $\mathrm{Ric}_f \geq 0$.

Theorem 3.1. *Let (M, g, f, λ) be a shrinking gradient $\rho-$Einstein soliton with scalar curvature $0 \leq R \leq K$ for some positive constant K and $\rho \geq 0$. Then all ends of M are $\varphi-$non-parabolic.*

Proof: Recall that an end E of $(M, g, \lambda, \rho, e^{\varphi})$ is said to be $\varphi-$non-parabolic if there exists a positive Green's function for the weighted Laplacian

$$\Delta_{\varphi} u := \Delta u - \langle \nabla \varphi, \nabla u \rangle$$

satisfying the Neumann boundary conditions on ∂E. Otherwise, it is called $\varphi-$parabolic. We now show that M does not admit any $\varphi-$parabolic ends.

Suppose E is a $\varphi-$parabolic end of M. Then, from Theorem 1.H, there exists a proper $\varphi-$harmonic function h on the end such that,

$$h \geq 1 \text{ on } E, \;\; h = 1 \text{ on } \partial E, \;\; \lim_{x \to E(\infty)} h(x) = \infty, \;\text{ and }\; \Delta_{\varphi} h = 0. \tag{3.4}$$

Our goal is to show that (3.4) leads to a contradiction, which implies that all ends of (M, g) are $\varphi-$non-parabolic.

For $t > 1$ and $1 < b < c$, we define the sets

$$l(t) := \{x \in E : h(x) = t\},$$
$$L(b, c) := \{x \in E : b < h(x) < c\}.$$

From (3.4) we know $l(t)$ and $\overline{L(b,c)}$ are compact. Since on the level set $h=t$ we have the unit normal $\nu=\nabla h/|\nabla h|$ we get by the divergence theorem that

$$\begin{aligned} 0 &= \int_{L(b,c)} (\Delta_\varphi h)e^{-\varphi} = \int_{\partial L(b,c)} (\partial_\nu h)e^{-\varphi} \\ &= \int_{\partial L(b,c)} \left\langle \frac{\nabla h}{|\nabla h|}, \nabla h \right\rangle e^{-\varphi} = \int_{l(c)} |\nabla h| e^{-\varphi} - \int_{l(b)} |\nabla h| e^{-\varphi}, \end{aligned}$$

which implies that the weighted integral over the set $l(t)$ of $|\nabla h|$ is independent of t. From the co-area formula (1.1) we have

$$\begin{aligned} \int_E |\nabla \ln h|^2 e^{-\varphi} &= \int_{L(1,\infty)} |\nabla \ln h|^2 e^{-\varphi} = \int_{L(1,\infty)} \frac{|\nabla h|^2}{h^2} e^{-\varphi} \\ &= \int_1^\infty \left(\int_{l(t)} \frac{|\nabla h|}{h^2} e^{-\varphi} \right) dt = \int_1^\infty \frac{1}{t^2} e^{-\varphi} dt \int_{l(t_0)} |\nabla h| e^{-\varphi} \\ &= C_0 < \infty. \end{aligned}$$

For any $x \in E$ such that $B_x(1) \subset E$, we have

$$\left(\min_{x \in B_x(1)} e^{-\varphi} \right) \int_{B_x(1)} |\nabla \ln h|^2 e^{-\varphi} \leq \int_{B_x(1)} |\nabla \ln h|^2 e^{-\varphi} e^{-\varphi} \leq C_0, \tag{3.5}$$

if $e^{-\varphi}$ attains its minimum in $\tilde{x}$ over $\overline{B_x(1)}$, then from Proposition 1.3.1

$$\min_{x \in \overline{B_x(1)}} e^{-\varphi} = e^{af(\tilde{x})} \geq e^{a(\frac{1}{4} r(\tilde{x}) - \alpha)^2} \geq e^{\frac{a}{16} r(\tilde{x})^2 - \frac{a\alpha}{2} r(\tilde{x})}$$

where $r(x) = d(x, x_0)$ from a fixed point $x_0 \in M$. By noticing that for any $x \in B_x(1)$ we have $r(x)+1 \geq r(\tilde{x}) \geq r(x)-1$, we have that

$$\min_{x \in B_x(1)} e^{-\varphi} \geq e^{\frac{a}{16}(r(x)-1)^2 - \frac{a\alpha}{2}(r(x)+1)},$$

which together with (3.5) implies

$$\int_{B_x(1)} |\nabla \ln h|^2 \leq C_0 e^{-\frac{a}{16}(r(x)-1)^2 + \frac{a\alpha}{2}(r(x)+1)}. \tag{3.6}$$

Notice equation (3.6) is analogous to that found in Step 1 of the proof of Theorem 3.K.

We will work now in showing the function $\sigma := |\nabla \ln h|^2$ satisfies an inequality analogous to (3.3). Let

$$v := \ln h,$$

then $\nabla v = \nabla h/h$, and, since h is $\varphi-$harmonic,

$$\begin{aligned}\Delta v =& \sum_i e_i(e_i(\ln h)) = \sum_i e_i\left(\frac{e_i(h)}{h}\right) = \frac{\Delta h}{h} - \frac{|\nabla h|^2}{h^2} \\ =& -\frac{a\langle \nabla h, \nabla f\rangle}{h} - \frac{|\nabla h|^2}{h^2} = -a\langle \nabla v, \nabla f\rangle - |\nabla v|^2.\end{aligned} \tag{3.7}$$

By applying the Bochner formula on v, and the $\rho-$Einstein equation, we get

$$\begin{aligned}\frac{1}{2}\Delta|\nabla v|^2 =& |\nabla^2 v|^2 + \langle \nabla \Delta v, \nabla v\rangle + \mathrm{Ric}(\nabla v, \nabla v) \\ =& |\nabla^2 v|^2 - a\langle \nabla\langle \nabla v, \nabla f\rangle, \nabla v\rangle - \langle \nabla|\nabla v|^2, \nabla v\rangle + (\rho R + \lambda)|\nabla v|^2 - \nabla^2 f(\nabla v, \nabla v).\end{aligned} \tag{3.8}$$

Notice that

$$\begin{aligned}\langle \nabla\langle \nabla v, \nabla f\rangle, \nabla v\rangle =& \left\langle \nabla\left(\sum_i f_i v_i\right), \nabla v\right\rangle = \left\langle \sum_{i,j}(f_{ij}v_i + f_i v_{ij})\, e_j, \sum_j v_j e_j\right\rangle \\ =& \sum_{ij}(f_{ij}v_i v_j + v_{ij} f_i v_j) = \nabla^2 f(\nabla v, \nabla v) + \nabla^2 v(\nabla f, \nabla v),\end{aligned} \tag{3.9}$$

and, from Theorem 1.C and Young's inequality,

$$\begin{aligned}a\nabla^2 v(\nabla f, \nabla v) \leq& a|\nabla^2 v||\nabla f||\nabla v| \leq \frac{1}{2}|\nabla^2 v|^2 + \frac{a^2}{2}|\nabla f|^2|\nabla v|^2 \\ \leq& \frac{1}{2}|\nabla^2 v|^2 + \frac{a^2(\delta f + \epsilon)}{2}|\nabla v|^2.\end{aligned} \tag{3.10}$$

On the other hand, from Schwarz inequality, (3.7) and Young's inequality,

$$\begin{aligned}|\nabla^2 v|^2 \geq& \frac{(\Delta v)^2}{n} = \frac{1}{n}\left(a\langle \nabla v, \nabla f\rangle + |\nabla v|^2\right)^2 = \frac{1}{n}\left(a^2\langle \nabla v, \nabla f\rangle^2 + 2a\langle \nabla v, \nabla f\rangle|\nabla v|^2 + |\nabla v|^4\right) \\ \geq& \frac{1}{n}\left(2a\langle \nabla v, \nabla f\rangle|\nabla v|^2 + |\nabla v|^4\right) \geq \frac{1}{n}\left(|\nabla v|^4 - 2a|\nabla v||\nabla f||\nabla v|^2\right) \\ \geq& \frac{1}{n}\left(|\nabla v|^4 - \frac{1}{2}(2a|\nabla v||\nabla f|)^2 - \frac{1}{2}|\nabla v|^4\right) = \frac{1}{n}\left(\frac{1}{2}|\nabla v|^4 - 2a^2|\nabla v|^2|\nabla f|^2\right) \\ \geq& \frac{1}{2n}|\nabla v|^4 - a^2|\nabla v|^2|\nabla f|^2 \geq \frac{1}{2n}|\nabla v|^4 - a^2(\delta f + \epsilon)|\nabla v|^2.\end{aligned} \tag{3.11}$$

By plugging (3.9), (3.10) and (3.11) into (3.8) we get

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla v|^2 =&|\nabla^2 v|^2 - a\langle\nabla\langle\nabla v,\nabla f\rangle,\nabla v\rangle - \langle\nabla|\nabla v|^2,\nabla v\rangle + (\rho R+\lambda)|\nabla v|^2 - \nabla^2 f(\nabla v,\nabla v)\\
=&|\nabla^2 v|^2 - a\left(\nabla^2 f(\nabla v,\nabla v) + \nabla^2 v(\nabla f,\nabla v)\right) - \langle\nabla|\nabla v|^2,\nabla v\rangle\\
&+ (\rho R+\lambda)|\nabla v|^2 - \nabla^2 f(\nabla v,\nabla v)\\
\geq&|\nabla^2 v|^2 - a\nabla^2 f(\nabla v,\nabla v) - \frac{1}{2}|\nabla^2 v|^2 - \frac{a^2(\delta f+\epsilon)}{2}|\nabla v|^2 - \langle\nabla|\nabla v|^2,\nabla v\rangle\\
&+ (\rho R+\lambda)|\nabla v|^2 - \nabla^2 f(\nabla v,\nabla v)\\
=&\frac{1}{2}|\nabla^2 v|^2 - \langle\nabla|\nabla v|^2,\nabla v\rangle + \left(\rho R+\lambda-\frac{a^2(\delta f+\epsilon)}{2}\right)|\nabla v|^2 - (a+1)\nabla^2 f(\nabla v,\nabla v)\\
\geq&\frac{1}{2}\left(\frac{1}{2n}|\nabla v|^4 - a^2(\delta f+\epsilon)|\nabla v|^2\right) - \langle\nabla|\nabla v|^2,\nabla v\rangle + \left(\rho R+\lambda-\frac{a^2(\delta f+\epsilon)}{2}\right)|\nabla v|^2\\
&- (a+1)\nabla^2 f(\nabla v,\nabla v)\\
=&\frac{1}{4n}|\nabla v|^4 - \langle\nabla|\nabla v|^2,\nabla v\rangle + \left(\rho R+\lambda-a^2(\delta f+\epsilon)\right)|\nabla v|^2 - (a+1)\nabla^2 f(\nabla v,\nabla v)
\end{aligned} \tag{3.12}$$

By making $\sigma = |\nabla v|^2$, inequality (3.12) can be rewritten using Einstein summation for the term $\nabla^2 f(\nabla v,\nabla v)$ as

$$\sigma^2 \leq 4n\left(a^2(\delta f+\epsilon) - \rho R - \lambda\right)\sigma + 4n\langle\nabla\sigma,\nabla v\rangle + 4n(a+1)f_{ij}v_iv_j + 2n\Delta\sigma. \tag{3.13}$$

Consider ϕ a cut-off function over M and $p>0$ large enough depending only on n. Then, multiplying inequality (3.13) by $\sigma^{p-1}\phi^2$, and integrating over M we get

$$\begin{aligned}
\int_M \sigma^{p+1}\phi^2 \leq 4n\int_M \left(\rho R+\lambda-a^2(\delta f+\epsilon)\right)\sigma^p\phi^2 + 4n\int_M \langle\nabla\sigma,\nabla v\rangle\sigma^{p-1}\phi^2\\
+ 4n(a+1)\int_M f_{ij}v_iv_j\sigma^{p-1}\phi^2 + 2n\int_M \sigma^{p-1}(\Delta\sigma)\phi^2.
\end{aligned} \tag{3.14}$$

Now, we proceed to bound the right side of (3.14) in terms of only f, σ, ϕ and $\nabla\phi$.

By noticing that

$$\int_M(\sigma^p\phi^2)\Delta v = -\int_M\langle\nabla(\sigma^p\phi^2),\nabla v\rangle = -\int_M\langle\nabla\sigma^p,\nabla v\rangle\phi^2 - \int_M\langle\nabla\phi^2,\nabla v\rangle\sigma^p,$$

we get that the integral on the second term on the right-hand side of (3.14) can be bounded using Young's inequality, (3.7) and (3.10) as follows,

$$\begin{aligned}\int_M \langle\nabla\sigma,\nabla v\rangle\sigma^{p-1}\phi^2 =&\frac{1}{p}\int_M \langle\nabla\sigma^p,\nabla v\rangle\phi^2 = -\frac{1}{p}\int_M (\sigma^p\phi^2)\Delta v - \frac{1}{p}\int_M \langle\nabla\phi^2,\nabla v\rangle\sigma^p\\ =&\frac{1}{p}\int_M \sigma^p\left(a\langle\nabla v,\nabla f\rangle+\sigma\right)\phi^2 - \frac{1}{p}\int_M \langle\nabla\phi^2,\nabla v\rangle\sigma^p\\ \leq&\frac{a}{p}\int_M \sigma^p\langle\nabla v,\nabla f\rangle\phi^2 + \frac{1}{p}\int_M \sigma^{p+1}\phi^2 - \frac{2}{p}\int_M \phi\langle\nabla\phi,\nabla v\rangle\sigma^p\\ \leq&\frac{a}{p}\int_M \sigma^p|\nabla v||\nabla f|\phi^2 + \frac{1}{p}\int_M \sigma^{p+1}\phi^2 + \frac{2}{p}\int_M |\nabla\phi||\nabla v|\sigma^p\phi\\ \leq&\frac{a^2}{2p}\int_M \sigma^p|\nabla f|^2\phi^2 + \frac{1}{2p}\int_M \sigma^{p+1}\phi^2 + \frac{1}{p}\int_M \sigma^{p+1}\phi^2 + \frac{2}{p}\int_M |\nabla\phi|^2\sigma^p\\ &+\frac{1}{2p}\int_M \phi^2\sigma^{p+1}\\ \leq&\frac{a^2}{2p}\int_M \sigma^p(\delta f+\epsilon)\phi^2 + \frac{2}{p}\int_M \sigma^{p+1}\phi^2 + \frac{2}{p}\int_M |\nabla\phi|^2\sigma^p.\end{aligned} \tag{3.15}$$

Regarding the last term on the right side of (3.14), by integrating by parts we get

$$\begin{aligned}\int_M \sigma^{p-1}(\Delta\sigma)\phi^2 =& -\int_M \langle\nabla(\sigma^{p-1}\phi^2),\nabla\sigma\rangle\\ =& -(p-1)\int_M \sigma^{p-2}\langle\nabla\sigma,\nabla\sigma\rangle\phi^2 - 2\int_M \sigma^{p-1}\langle\nabla\phi,\nabla\sigma\rangle\phi,\end{aligned}$$

second term of the right-hand side of the equation above is bounded by

$$-2\int_M \sigma^{p-1}\langle\nabla\phi,\nabla\sigma\rangle\phi \leq 2\int_M \sigma^{p-2}|\nabla\phi||\nabla\sigma|\sigma\phi \leq \int_M \sigma^{p-2}\left(|\nabla\sigma|^2\phi^2+|\nabla\phi|^2\sigma^2\right).$$

Thus, for p big enough

$$\begin{aligned}\int_M \sigma^{p-1}(\Delta\sigma)\phi^2 \leq& -(p-2)\int_M \sigma^{p-2}|\nabla\sigma|^2\phi^2 + \int_M \sigma^p|\nabla\phi|^2\\ \leq& -\frac{p}{2}\int_M \sigma^{p-2}|\nabla\sigma|^2\phi^2 + \int_M \sigma^p|\nabla\phi|^2.\end{aligned} \tag{3.16}$$

Now we find an estimate for the third term on the right side of (3.14). First, notice that for, any j,

$$\left(f_i v_i v_j \sigma^{p-1}\phi^2\right)_j = f_{ij} v_i v_j \sigma^{p-1}\phi^2 + f_i v_{ij} v_j \sigma^{p-1}\phi^2 + f_i v_i v_{jj}\sigma^{p-1}\phi^2 + (p-1) f_i v_i v_j \sigma^{p-2}\sigma_j \phi^2 + 2 f_i v_i v_j \sigma^{p-1}\phi\phi_j.$$

Thus, by the divergence theorem, we have

$$\begin{aligned}\int_M f_{ij} v_i v_j \sigma^{p-1}\phi^2 = &-\int_M f_i v_{ij} v_j \sigma^{p-1}\phi^2 - \int_M f_i v_i v_{jj}\sigma^{p-1}\phi^2 \\ &- (p-1)\int_M f_i v_i v_j \sigma_j \sigma^{p-2}\phi^2 - 2\int_M f_i v_i v_j \sigma^{p-1}\phi\phi_j.\end{aligned} \tag{3.17}$$

Each term in the equation above can be estimated as follows. For the first term, we apply Green's identity getting

$$\begin{aligned}-\int_M f_i v_{ij} v_j \sigma^{p-1}\phi^2 &= -\frac{1}{2}\int_M \left\langle \nabla|\nabla v|^2, \nabla f\right\rangle \sigma^{p-1}\phi^2 \\ &= -\frac{1}{2}\int_M \left\langle \sigma^{p-1}\nabla\sigma, \nabla f\right\rangle \phi^2 \\ &= -\frac{1}{2p}\int_M \langle \nabla\sigma^p, \nabla f\rangle \phi^2 \\ &= -\frac{1}{2p}\left(\int_M \langle\nabla(\sigma^p\phi^2), \nabla f\rangle - \int_M \langle \nabla\phi^2 \nabla f\rangle \sigma^p\right) \\ &= \frac{1}{2p}\int_M (\Delta f)\sigma^p\phi^2 + \frac{1}{2p}\int_M \langle\nabla\phi^2\nabla f\rangle\sigma^p \\ &\le \frac{1}{2p}\int_M (\Delta f)\sigma^p\phi^2 + \frac{1}{p}\int_M |\nabla\phi||\nabla f|\phi\sigma^p \\ &\le \frac{1}{2p}\int_M (\Delta f)\sigma^p\phi^2 + \int_M |\nabla\phi|^2\sigma^p + \int_M (\epsilon f + \delta)\phi^2\sigma^p.\end{aligned}$$

For the second term, we have

$$
\begin{aligned}
-\int_M f_i v_i v_{jj}\sigma^{p-1}\phi^2 &= -\int_M \langle\nabla v,\nabla f\rangle(\Delta v)\sigma^{p-1}\phi^2 = \int_M \left(a\langle\nabla v,\nabla f\rangle+\sigma\right)\langle\nabla v,\nabla f\rangle\sigma^{p-1}\phi^2\\
&= a\int_M \langle\nabla v,\nabla f\rangle^2\sigma^{p-1}\phi^2 + \int_M \langle\nabla v,\nabla f\rangle\sigma^p\phi^2\\
&\le a\int_M |\nabla f|^2\sigma^p\phi^2 + \int_M |\nabla v||\nabla f|\sigma^p\phi^2\\
&\le a\int_M (\delta f+\epsilon)\sigma^p\phi^2 + \int_M \left(\frac{np(a+1)}{2}|\nabla f|^2 + \frac{1}{2np(a+1)}|\nabla v|^2\right)\sigma^p\phi^2\\
&\le np(a+1)\int_M (\delta f+\epsilon)\sigma^p\phi^2 + \frac{1}{np(a+1)}\int_M \sigma^{p+1}\phi^2.
\end{aligned}
$$

Third term on (3.17) can be bounded by

$$
\begin{aligned}
-(p-1)\int_M f_i v_i v_j\sigma_j\sigma^{p-2}\phi^2 \le& p\int_M |\nabla f|\sigma|\nabla\sigma|\sigma^{p-2}\phi^2\\
\le& p\int_M \left(2n(a+1)|\nabla f|^2\sigma^2 + \frac{|\nabla\sigma|^2}{8n(a+1)}\right)\sigma^{p-2}\phi^2\\
\le& 2pn(a+1)\int_M (\delta f+\epsilon)\sigma^p\phi^2 + \frac{p}{8n(a+1)}\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2.
\end{aligned}
$$

Finally, we can estimate the last term on the right-hand side of (3.17) by

$$
-2\int_M f_i v_i v_j\sigma^{p-1}\phi\phi_j \le 2\int_M \left(\sigma|\nabla f||\nabla\phi|\phi\right)\sigma^{p-1} \le \int_M (\delta f+\epsilon)\sigma^p\phi^2 + \int_M |\nabla\phi|^2\sigma^p.
$$

Plugging these four estimates in (3.17), we get

$$
\begin{aligned}
\int_M f_{ij}v_i v_j\sigma^{p-1}\phi^2 \le& \frac{1}{2p}\int_M (\Delta f)\sigma^p\phi^2 + \int_M |\nabla\phi|^2\sigma^p + \int_M (\epsilon f+\delta)\phi^2\sigma^p + np(a+1)\int_M (\delta f+\epsilon)\sigma^p\phi^2\\
&+ \frac{1}{np(a+1)}\int_M \sigma^{p+1}\phi^2 + 2pn(a+1)\int_M (\delta f+\epsilon)\sigma^p\phi^2\\
&+ \frac{p}{8n(a+1)}\int_M |\nabla\sigma|^2\sigma^{p-2}\phi^2 + \int_M (\delta f+\epsilon)\sigma^p\phi^2 + \int_M |\nabla\phi|^2\sigma^p,
\end{aligned}
$$

which implies

$$\begin{aligned}4n(a+1)\int_M f_{ij}v_iv_j\sigma^{p-1}\phi^2 &\le \frac{2n(a+1)}{p}\int_M(\Delta f)\sigma^p\phi^2+4n(a+1)\int_M|\nabla\phi|^2\sigma^p\\&+4n(a+1)\int_M(\epsilon f+\delta)\phi^2\sigma^p+4n^2(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2\\&+\frac{4}{p}\int_M\sigma^{p+1}\phi^2+8n^2(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2+\frac{p}{2}\int_M|\nabla\sigma|^2\sigma^{p-2}\phi^2\\&+4n(a+1)\int_M(\delta f+\epsilon)\sigma^p\phi^2+4n(a+1)\int_M\sigma^p|\nabla\phi|^2\\&\le 2(a+1)\int_M(\Delta f)\sigma^p\phi^2+C_1(n)(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2\\&+\frac{4}{p}\int_M\sigma^{p+1}\phi^2+\frac{p}{2}\int_M|\nabla\sigma|^2\sigma^{p-2}\phi^2+C_2(n)(a+1)\int_M\sigma^p|\nabla\phi|^2,\end{aligned} \tag{3.18}$$

for $C_1(n)$ and $C_2(n)$ constants depending only on n. By plugging (3.15), (3.16) and (3.18) into (3.14) we get

$$\begin{aligned}\int_M\sigma^{p+1}\phi^2 &\le 4n\int_M\left(a^2(\delta f+\epsilon)-\rho R-\lambda\right)\sigma^p\phi^2+4n\left(\frac{a^2}{2p}\int_M(\delta f+\epsilon)\sigma^p\phi^2+\frac{2}{p}\int_M\sigma^{p+1}\phi^2\right.\\&\left.+\frac{2}{p}\int_M|\nabla\phi|^2\sigma^p\right)+\left(2(a+1)\int_M(\Delta f)\sigma^p\phi^2+C_1(n)(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2\right.\\&\left.+\frac{4}{p}\int_M\sigma^{p+1}\phi^2+\frac{p}{2}\int_M|\nabla\sigma|^2\sigma^{p-2}\phi^2+C_2(n)(a+1)\int_M\sigma^p|\nabla\phi|^2\right)\\&+2n\left(-\frac{p}{2}\int_M\sigma^{p-2}|\nabla\sigma|^2\phi^2+\int_M\sigma^p|\nabla\phi|^2\right).\end{aligned}$$

Thus, taking p big enough such that $1-\frac{8n-4}{p}\geq 0$ we have

$$
\begin{aligned}
0 \leq & \left(1-\frac{8n}{p}-\frac{4}{p}\right)\int_M \sigma^{p+1}\phi^2 \\
\leq & \, 4n\int_M \left(a^2(\delta f+\epsilon)-\rho R-\lambda\right)\sigma^p\phi^2+2a^2\int_M(\delta f+\epsilon)\sigma^p\phi^2+8\int_M|\nabla\phi|^2\sigma^p \\
& +C_1(n)(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2+\frac{p}{2}\int_M|\nabla\sigma|^2\sigma^{p-2}\phi^2 \\
& +C_2(n)(a+1)\int_M\sigma^p|\nabla\phi|^2-np\int_M\sigma^{p-2}|\nabla\sigma|^2\phi^2+2n\int_M\sigma^p|\nabla\phi|^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\left(np-\frac{p}{2}\right)\int_M\sigma^{p-2}|\nabla\sigma|^2\phi^2 \leq & \; 4n\int_M\left(a^2(\delta f+\epsilon)-\rho R-\lambda\right)\sigma^p\phi^2 \\
& +2(a+1)\int_M(\Delta f)\sigma^p\phi^2+2a^2\int_M(\delta f+\epsilon)\sigma^p\phi^2+8\int_M|\nabla\phi|^2\sigma^p \\
& +C_1(n)(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2+C_2(n)(a+1)\int_M|\nabla\phi|^2\sigma^p+2n\int_M|\nabla\phi|^2\sigma^p.
\end{aligned} \tag{3.19}
$$

Notice that taking the trace of the $\rho-$Einstein soliton equation and recalling $0\leq R\leq K$ and $0\leq\rho\leq 1/2(n-1)$ we have

$$
\begin{aligned}
2(a+1)\Delta f-4n(\rho R+\lambda) =& 2n(a+1)(\rho R+\lambda)-4n(\rho R+\lambda) \\
=& 2n(a-1)(\rho R+\lambda) \\
\leq & 2n(a-1)C_3
\end{aligned}
$$

for some constant C_3 clearly independent of p; by plugging this into (3.19) one gets

$$
\begin{aligned}
\left(np-\frac{p}{2}\right)\int_M\sigma^{p-2}|\nabla\sigma|^2\phi^2 \leq & \; 4na^2\int_M(\delta f+\epsilon)\sigma^p\phi^2+2n(a-1)C_3\int_M\sigma^p\phi^2 \\
& +2a^2\int_M(\delta f+\epsilon)\sigma^p\phi^2+8\int_M|\nabla\phi|^2\sigma^p+C_1(n)(a+1)^2p\int_M(\delta f+\epsilon)\sigma^p\phi^2 \\
& +C_2(n)(a+1)\int_M|\nabla\phi|^2\sigma^p+2n\int_M|\nabla\phi|^2\sigma^p.
\end{aligned}
$$

Finally, given $a+1 \leq (a+1)^2$ and $a^2 \leq (a+1)^2$ we enlarge some terms on the right and merge similar integral terms to conclude

$$\frac{p}{2}\int_M \sigma^{p-2}|\nabla\sigma|^2\phi^2 \leq C(n)(a+1)^2 p\int_M (\delta f + C)\,\sigma^p\phi^2 + C(n)(a+1)\int_M \sigma^p|\nabla\phi|^2 \tag{3.20}$$

for some constants $c(n)$ and C independent of p and for any cut-off function ϕ with support on the unit ball $B_{x_0}(1)$.

Expression (3.20) is analogous to that one found by Munteanu and Wang on their proof of Theorem 3.K, (see equation (3.3)). Furthermore, as $\rho R + \lambda \geq 0$, it is clear Ric_f is non-negative, thus, we are on the conditions of Theorem 1.A and **Step 3** of the proof of Theorem 3.K, and by the arguments therein, we can conclude

$$\sigma(x) \leq Ce^{c(n)r(x)}\int_{B_x(1)} \sigma$$

for any $x \in E$. Combined with (3.6) this implies

$$|\nabla \ln h(x)| \leq \sqrt{\sigma} \leq C_1 e^{\left(c(n)\frac{r(x)}{2} - \frac{a}{32}(r(x)-1)^2 + \frac{a\alpha}{4}(r(x)+1)\right)} = Ce^{\left(c_2 r(x) - \frac{a}{32}r^2(x)\right)} \tag{3.21}$$

for constants C and c_2 that do not depend on $r(x)$. Let $\gamma : [0,\infty) \to M$ be an arbitrary normalized minimizing geodesic on M with $\gamma(0) = x_0$, then, for $x = \gamma(t) \in \gamma$ we have that $r(x) = t$, and from (3.21)

$$\int_\gamma |\nabla \ln h(x)|dx = \int_0^\infty |\nabla \ln h(\gamma(t))|dt \leq C\int_0^\infty e^{\left(c_2 t - \frac{a}{32}t^2\right)}dt \leq K_0$$

for some finite positive constant K_0. Therefore, for an arbitrary $\tilde{x} \in E$ we can take γ as being the normalized minimizing geodesic from x_0 such that $\tilde{x} = \gamma(t_0)$ for some $t_0 > 0$, and from inequality above we have

$$\begin{aligned} K_0 &\geq \int_0^{t_0} |\nabla \ln h(\gamma(t))|dt \geq \int_0^{t_0} \langle \nabla \ln h(\gamma(t)), \gamma'(t)\rangle dt = \int_0^{t_0} \frac{d}{dt}\left(\ln h(\gamma(t))\right)dt \\ &= \ln h(\gamma(t_0)) - \ln h(\gamma(0)) = \ln h(\tilde{x}) - \ln h(x_0), \end{aligned}$$

then

$$h(\tilde{x}) \leq e^{K_1}$$

for some constant K_1 depending only on x_0, by making $\tilde{x} \to E(\infty)$ we have $h(\tilde{x})$ is bounded, contradicting (3.4). Thus, M must have only $\varphi-$non-parabolic ends as wanted to prove.

□

3.2 Bound on the scalar curvature and connectedness at infinity

We now focus on showing that gradient $\rho-$Einstein solitons must have only one end if we assume in addition an appropriate bound on the scalar curvature. First, assuming the existence of more than one end and that the scalar curvature is non-negative and bounded, we establish in Lemmas 3.2-3.5 the integrability of several functions, which culminate in a key integral inequality, stated in Lemma 3.6. A particular case of this inequality is used to obtain a contradiction under a suitable bound on R, proving the main result of this section, Theorem 3.7.

Take an end E_1 of M and denote by $E_2 = M \backslash E_1$ another end of M such that $M = E_1 \cup E_2$. Since both E_1 and E_2 must be $\varphi-$non-parabolic, from Theorem 1.G there is a non-constant $\varphi-$harmonic function u in M such that

$$\begin{gathered} \Delta_\varphi u = \Delta u + a\langle \nabla f, \nabla u\rangle = 0, \\ 0 < u < 1 \text{ on } M, \\ \inf_{E_1} u = 0 \text{ and } \sup_{E_2} u = 1. \end{gathered} \tag{3.22}$$

Moreover (see e.g. [30–32, 44]), u is obtained as the limit of a sequence of $\varphi-$harmonic functions u_i defined on geodesic balls $B(x_0, r_i)$ of radii r_i such that

$$\begin{aligned} u_i &= 0 \text{ on } \partial B(x_0, r_i) \cap E_1, \\ u_i &= 1 \text{ on } \partial B(x_0, r_i) \cap E_2. \end{aligned} \tag{3.23}$$

As mentioned in [44], it is also true that

$$\int_M |\nabla u|^2 e^{-\varphi} = \int_M |\nabla u|^2 e^{af} < \infty. \tag{3.24}$$

We now show some estimates involving u, its derivatives, and the geometry of M, under the assumption of M being a gradient shrinking $\rho-$Einstein soliton with non-negative bounded scalar curvature $R \leq K$, for some constant K. Throughout the proofs, we will tacitly use $\rho \geq 0$.

Remark 3.2. *As the aim of these estimates is to show the finiteness of integral terms, different constants will be noted indistinctly as C without further clarifications.*

Lemma 3.2. *For any constant $b < a$, we have*

$$\int_M |\nabla u|^2 e^{2bf} < \infty.$$

Proof: From (3.24), it suffices to proof the inequality for $b > a/2$. First, notice that

$$\begin{aligned} \Delta_{-2bf} e^{-bf} =& \Delta e^{-bf} - \left\langle \nabla(-2bf), \nabla e^{-bf} \right\rangle \\ =& - b\Delta f e^{-bf} - b^2 |\nabla f|^2 e^{-bf} \\ =& - \left(b\Delta f + b^2 |\nabla f|^2 \right) e^{-bf}, \end{aligned} \tag{3.25}$$

then, by Proposition 2.2.2, the Poincaré type inequality

$$\int_M \left(b\Delta f + b^2 |\nabla f|^2 \right) \phi^2 e^{2bf} \leq \int_M |\nabla \phi|^2 e^{2bf} \tag{3.26}$$

holds for any $\phi \in C_0^\infty(M)$.

Denote by $E_1(x_0, r) = E_1 \cap B_{x_0}(r)$, set $r_0 > 0$ and let

$$\psi(x) = \begin{cases} 0 & \text{on } M \backslash (E_1 \backslash E_1(x_0, r_0)), \\ r(x) - r_0 & \text{on } E_1(x_0, r_0 + 1) \backslash E_1(x_0, r_0), \\ 1 & \text{on } E_1 \backslash E_1(x_0, r_0 + 1). \end{cases}$$

Let $\phi = \psi u_i$ for u_i as given in (3.23), then $\phi \in C_0^\infty(M)$ and from inequality (3.26) we get

$$\int_M \left(b\Delta f + b^2 |\nabla f|^2 \right) \psi^2 u_i^2 e^{2bf} \leq \int_M |\nabla(\psi u_i)|^2 e^{2bf}. \tag{3.27}$$

Expanding the right-hand side of inequality above we have from Green's identity

$$\begin{aligned}
\int_M |\nabla(\psi u_i)|^2 e^{2bf} &= \int_M \langle u_i\nabla\psi + \psi\nabla u_i, u_i\nabla\psi + \psi\nabla u_i\rangle e^{2bf} \\
&= \int_M \left(u_i^2|\nabla\psi|^2 + 2u_i\psi\langle\nabla u_i, \nabla\psi\rangle + \psi^2|\nabla u_i|^2\right) e^{2bf} \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} + \frac{1}{2}\int_M \langle\nabla u_i^2, \nabla\psi^2\rangle e^{2bf} + \int_M \psi^2|\nabla u_i|^2 e^{2bf} \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - \frac{1}{2}\int_M \psi^2\Delta_{(-2bf)} u_i^2 e^{2bf} + \int_M \psi^2|\nabla u_i|^2 e^{2bf} \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - \int_M \psi^2\left(u_i\Delta_{(-2bf)}u_i + |\nabla u_i|^2\right) e^{2bf} + \int_M \psi^2|\nabla u_i|^2 e^{2bf} \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - \int_M \psi^2 u_i\Delta_{(-2bf)}u_i e^{2bf}.
\end{aligned}$$

Since u_i are $\varphi-$harmonic we know $\Delta u_i = -a\langle\nabla f, \nabla u_i\rangle$, then

$$\begin{aligned}
\int_M |\nabla(\psi u_i)|^2 e^{2bf} &= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - \int_M \psi^2 u_i(\Delta u_i + 2b\langle\nabla f, \nabla u_i\rangle) e^{2bf} \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - (2b-a)\int_M \psi^2 u_i\langle\nabla f, \nabla u_i\rangle e^{2bf} \qquad (3.28) \\
&= \int_M u_i^2|\nabla\psi|^2 e^{2bf} - \left(b - \frac{a}{2}\right)\int_M \psi^2\langle\nabla f, \nabla u_i^2\rangle e^{2bf}.
\end{aligned}$$

By noticing

$$\begin{aligned}\int_M \psi^2\langle\nabla f,\nabla u_i^2\rangle e^{2bf} &= \int_M \langle\nabla(\psi^2 f e^{2bf}),\nabla u_i^2\rangle - \int_M f\langle\nabla(\psi^2 e^{2bf}),\nabla u_i^2\rangle \\ &= -\int_M u_i^2\Delta(\psi^2 f e^{2bf}) - \int_M \langle\nabla(\psi^2 e^{2bf}),\nabla(f u_i^2)\rangle + \int_M u_i^2\langle\nabla(\psi^2 e^{2bf}),\nabla f\rangle \\ &= -\int_M u_i^2\left[\Delta(\psi^2 f)e^{2bf} + \psi^2 f\Delta e^{2bf} + 2\langle\nabla(\psi^2 f),\nabla e^{2bf}\rangle\right] \\ &\quad + \int_M u_i^2 f\Delta(\psi^2 e^{2bf}) + \int_M u_i^2\langle\nabla\psi^2,\nabla f\rangle e^{2bf} + 2b\int_M u_i^2\psi^2|\nabla f|^2 e^{2bf} \\ &= -\int_M u_i^2\left[f\Delta\psi^2 + \psi^2\Delta f + 2\langle\nabla\psi^2,\nabla f\rangle + \psi^2 f(2b\Delta f + 4b^2|\nabla f|^2)\right. \\ &\qquad\left. + 4bf\langle\nabla\psi^2,\nabla f\rangle + 4b\psi^2|\nabla f|^2\right]e^{2bf} \\ &\quad + \int_M u_i^2 f\left[\Delta\psi^2 + \psi^2 2b\Delta f + 4b^2\psi^2|\nabla f|^2 + 4b\langle\nabla\psi^2,\nabla f\rangle\right]e^{2bf} \\ &\quad + \int_M u_i^2\langle\nabla\psi^2,\nabla f\rangle e^{2bf} + 2b\int_M u_i^2\psi^2|\nabla f|^2 e^{2bf} \\ &= -\int_M u_i^2\left[\psi^2\Delta f + \langle\nabla\psi^2,\nabla f\rangle + 2b\psi^2|\nabla f|^2\right]e^{2bf},\end{aligned}$$

we conclude from (3.28) that

$$\begin{aligned}\int_M |\nabla(\psi u_i)|^2 e^{2bf} = \int_M u_i^2|\nabla\psi|^2 e^{2bf} + \left(b-\frac{a}{2}\right)\int_M u_i^2\langle\nabla f,\nabla\psi^2\rangle e^{2bf} \\ + \left(b-\frac{a}{2}\right)\int_M u_i^2\left(\psi^2\Delta f + 2b\psi^2|\nabla f|^2\right)e^{2bf}.\end{aligned} \tag{3.29}$$

Since both $\nabla\psi$ and u_i are bounded, and $\nabla\psi$ has support only on the annulus $E_1(x_0,r_0+1)\backslash E_1(x_0,r_0)$, first and second integral on the right side of (3.29) are finite. Thus, given the convergence of the u_i's, there is a constant $C>0$ independent of i, such that

$$\int_M |\nabla(\psi u_i)|^2 e^{2bf} \leq \left(b-\frac{a}{2}\right)\int_M \left(\Delta f + 2b|\nabla f|^2\right)u_i^2\psi^2 e^{2bf} + C. \tag{3.30}$$

Combining with (3.27), we conclude that

$$b\int_M \left(\Delta f + b|\nabla f|^2\right) u_i^2\psi^2 e^{2bf} \leq \left(b - \frac{a}{2}\right)\int_M \left(\Delta f + 2b|\nabla f|^2\right) u_i^2\psi^2 e^{2bf} + C,$$

this is,

$$\int_M \left(\frac{a}{2}\Delta f + \left(ab - b^2\right)|\nabla f|^2\right) u_i^2\psi^2 e^{2bf} \leq C.$$

Applying Theorems 1.B and 1.C, inequality above implies

$$\int_M \left(\frac{a}{2}[(n\rho - 1)R + n\lambda] + \left(ab - b^2\right)(\alpha f - \beta)\right) u_i^2\psi^2 e^{2bf} \leq C,$$

since R is bounded, this means there are constants C_1, C_2 independent of i and depending only on a, b, λ and K such that

$$\left(ab - b^2\right)\int_M \left(\alpha f + C_1\right) u_i^2\psi^2 e^{2bf} \leq C_2,$$

which up to a translation implies that

$$\int_{E_1} f u_i^2 e^{2bf} \leq C$$

for a possible new suitable finite constant C; from (3.30) we know

$$\int_{E_1} |\nabla u_i|^2 e^{2bf} \leq C' \int_{E_1} \left(\delta f + C''\right) e^{2bf} + C.$$

By combining these last two inequalities, we can conclude that

$$\int_{E_1} |\nabla u_i|^2 e^{2bf} \leq C.$$

Which proves the lemma over E_1 for u_i. Proceeding analogously, similar estimate holds on E_2 as well (given it is, at most, also a $\varphi-$non-parabolic end), thus, by making $i \to \infty$, the Lemma follows.

□

For the next Lemmas, we will consider the cut-off function

$$\phi(x) = \begin{cases} 1 & \text{on } D(T), \\ T + 1 - f(x) & \text{on } D(T+1)\backslash D(T), \\ 0 & \text{on } M\backslash D(T+1), \end{cases} \tag{3.31}$$

where $D(T) = \{x \in M : f(x) \leq T\}$.

We now show the square norm of the hessian of u is also integrable.

Lemma 3.3. *For any constant $b < a$, we have*

$$\int_M |u_{ij}|^2 e^{2bf} < \infty.$$

Proof: Consider ϕ as given by (3.31). From the divergence theorem, we know

$$0 = \int_M \left(u_{ij}u_i\phi^2 e^{2bf}\right)_j = \int_M u_{ijj}u_i\phi^2 e^{2bf} + \int_M |u_{ij}|^2\phi^2 e^{2bf} + 2\int_M u_{ij}u_i\phi_j\phi e^{2bf} + 2b\int_M u_{ij}u_i\phi^2 f_j e^{2bf},$$

this is,

$$\int_M |u_{ij}|^2\phi^2 e^{2bf} = -\int_M u_{ijj}u_i\phi^2 e^{2bf} - 2b\int_M u_{ij}u_i\phi^2 f_j e^{2bf} - 2\int_M u_{ij}u_i\phi_j\phi e^{2bf}. \tag{3.32}$$

By Young's inequality, second term on the right side can be estimated as

$$\begin{aligned} -2b\int_M u_{ij}u_i\phi^2 f_j e^{2bf} \leq & 2b\int_M |u_{ij}||\nabla u||\nabla f|\phi^2 e^{2bf} \\ \leq & \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + 4b^2\int_M |\nabla u|^2|\nabla f|^2\phi^2 e^{2bf} \\ \leq & \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + 2b^2\int_M |\nabla u|^2|\nabla f|^4\phi^2 e^{2bf} + 2b^2\int_M |\nabla u|^2\phi^2 e^{2bf}. \end{aligned} \tag{3.33}$$

Since $|\nabla f|^2 \leq (\epsilon f + \delta)$ and $b < a$, there is $y > 0$ such that $b + y < a$ and

$$\begin{aligned}\int_M |\nabla u|^2 |\nabla f|^4 e^{2bf} \leq & C_1 \int_M |\nabla u|^2 f^2 e^{2bf} + C_2 \int_M |\nabla u|^2 f e^{2bf} + C_3 \int_M |\nabla u|^2 e^{2bf} \\ = & \frac{C_1}{y^2} \int_M |\nabla u|^2 (yf)^2 e^{2bf} + \frac{C_2}{y} \int_M |\nabla u|^2 (yf) e^{2bf} + C_3 \int_M |\nabla u|^2 e^{2bf} \\ \leq & \frac{C_1}{y^2} \int_M |\nabla u|^2 e^{2yf} e^{2bf} + \frac{C_2}{y} \int_M |\nabla u|^2 e^{yf} e^{2bf} + C_3 \int_M |\nabla u|^2 e^{2bf} \\ = & \frac{C_1}{y^2} \int_M |\nabla u|^2 e^{2(y+b)f} + \frac{C_2}{y} \int_M |\nabla u|^2 e^{2(y/2+b)f} + C_3 \int_M |\nabla u|^2 e^{2bf},\end{aligned}$$

which, together with Lemma 3.2, implies

$$\int_M |\nabla u|^2 |\nabla f|^4 e^{2bf} < \infty. \tag{3.34}$$

In view of this and (3.33), we conclude

$$-2b \int_M u_{ij} u_i \phi^2 f_j e^{2bf} \leq \frac{1}{4} \int_M |u_{ij}|^2 \phi^2 e^{2bf} + C, \tag{3.35}$$

for a constant C independent of T. Analogously, last term on the right-hand side of (3.32) can be estimated by

$$\begin{aligned}-2 \int_M u_{ij} u_i \phi_j \phi e^{2bf} \leq & 2 \int_M |u_{ij}| |\nabla u| |\nabla \phi| \phi e^{2bf} \\ \leq & \frac{1}{4} \int_M |u_{ij}|^2 \phi^2 e^{2bf} + 4 \int_M |\nabla u|^2 |\nabla f|^2 e^{2bf} \\ \leq & \frac{1}{4} \int_M |u_{ij}|^2 \phi^2 e^{2bf} + 2 \int_M |\nabla u|^2 e^{2bf} + 2 \int_M |\nabla u|^2 |\nabla f|^4 e^{2bf} \\ \leq & \frac{1}{4} \int_M |u_{ij}|^2 \phi^2 e^{2bf} + C.\end{aligned} \tag{3.36}$$

Plugging (3.35) and (3.36) into (3.32) we get

$$\int_M |u_{ij}|^2 \phi^2 e^{2bf} \leq -2 \int_M u_{ijj} u_i \phi^2 e^{2bf} + C.$$

Given

$$\Delta|\nabla u|^2 = 2u_i u_{ijj} + 2|u_{ij}|^2,$$

it follows from the Bochner formula that

$$\begin{aligned}
\int_M |u_{ij}|^2\phi^2 e^{2bf} \leq& -2\int_M u_{ijj}u_i\phi^2 e^{2bf} + C\\
=& 2\int_M |u_{ij}|^2\phi^2 e^{2bf} - \int_M \Delta|\nabla u|^2\phi^2 e^{2bf} + C\\
=& 2\int_M |u_{ij}|^2\phi^2 e^{2bf} - 2\int_M \big(\langle\nabla\Delta u, \nabla u\rangle + |u_{ij}|^2 + \mathrm{Ric}(\nabla u, \nabla u)\big)\,\phi^2 e^{2bf} + C\\
=& -2\int_M \langle\nabla(\Delta u), \nabla u\rangle\phi^2 e^{2bf} - 2\int_M \mathrm{Ric}(\nabla u, \nabla u)\phi^2 e^{2bf} + C.
\end{aligned}$$

Since

$$\nabla(\Delta u\phi^2 e^{2bf}) = \nabla\Delta u\phi^2 e^{2bf} + 2\Delta u\phi\nabla\phi e^{2bf} + 2b\Delta u\phi^2\nabla f e^{2bf},$$

and

$$\Delta u = -a\langle\nabla u, \nabla f\rangle,$$

we have from Green's identity that

$$\begin{aligned}
\int_M \langle\nabla(\Delta u), \nabla u\rangle\phi^2 e^{2bf} =& \int_M \big\langle\nabla(\Delta u\phi^2 e^{2bf}), \nabla u\big\rangle - 2b\int_M \Delta u\,\langle\nabla f, \nabla u\rangle\,\phi^2 e^{2bf}\\
&- \int_M \Delta u\,\big\langle\nabla\phi^2, \nabla u\big\rangle\, e^{2bf}\\
=& -\int_M (\Delta u)^2\phi^2 e^{2bf} - 2b\int_M \Delta u\,\langle\nabla f, \nabla u\rangle\,\phi^2 e^{2bf} - \int_M \Delta u\,\big\langle\nabla\phi^2, \nabla u\big\rangle\, e^{2bf}\\
=& -a^2\int_M \langle\nabla u, \nabla f\rangle^2\phi^2 e^{2bf} + a\int_M \langle\nabla u, \nabla f\rangle\,\big\langle\nabla\phi^2, \nabla u\big\rangle\, e^{2bf} + 2ab\int_M \langle\nabla u, \nabla f\rangle^2\phi^2 e^{2bf}\\
=& (2ab - a^2)\int_M \langle\nabla u, \nabla f\rangle^2\phi^2 e^{2bf} + a\int_M \langle\nabla u, \nabla f\rangle\,\big\langle\nabla\phi^2, \nabla u\big\rangle\, e^{2bf}\\
\leq& |2ab - a^2|\int_M |\nabla u|^2|\nabla f|^2\phi^2 e^{2bf} + 2a\int_M |\nabla u|^2|\nabla f||\nabla\phi|\phi e^{2bf}\\
\leq& \big(|2ab - a^2| + 2a\big)\int_M |\nabla u|^2|\nabla f|^2 e^{2bf},
\end{aligned}$$

which from Lemma 3.2 and (3.34) implies

$$\left|\int_M \langle\nabla(\Delta u), \nabla u\rangle \phi^2 e^{2bf}\right| \leq C, \tag{3.37}$$

therefore

$$\int_M |u_{ij}|^2 \phi^2 e^{2bf} \leq 2\left|\int_M \mathrm{Ric}(\nabla u, \nabla u)\phi^2 e^{2bf}\right| + C. \tag{3.38}$$

From the $\rho-$Einstein soliton equation, and Lemma 3.2,

$$\begin{aligned}\left|\int_M \mathrm{Ric}(\nabla u, \nabla u)\phi^2 e^{2bf}\right| &= \left|\int_M \left[(\rho R + \lambda)|\nabla u|^2 - f_{ij}u_iu_j\right]\phi^2 e^{2bf}\right| \\ &\leq (\rho R+\lambda)\left|\int_M |\nabla u|^2\phi^2 e^{2bf}\right| + \left|\int_M f_{ij}u_iu_j\phi^2 e^{2bf}\right| \\ &\leq \left|\int_M f_{ij}u_iu_j\phi^2 e^{2bf}\right| + C.\end{aligned} \tag{3.39}$$

However, $\Delta u = -a\langle\nabla f, \nabla u\rangle$, and we can write

$$\langle\nabla(\Delta u), \nabla u\rangle = -\,a\,\langle\nabla\langle\nabla f, \nabla u\rangle, \nabla u\rangle = -au_{ji}u_if_j - af_{ji}u_iu_j,$$

hence, in view of (3.34), (3.37) and Lemma 3.2,

$$\begin{aligned}\left|\int_M f_{ij}u_iu_j\phi^2 e^{2bf}\right| &\leq \left|\int_M u_{ij}u_if_j\phi^2 e^{2bf}\right| + \frac{1}{a}\left|\int_M \langle\nabla(\Delta u), \nabla u\rangle\phi^2 e^{2bf}\right| \\ &\leq \left|\int_M u_{ij}u_if_j\phi^2 e^{2bf}\right| + C \leq \int_M |u_{ij}||\nabla u||\nabla f|\phi^2 e^{2bf} + C \\ &\leq \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + \int_M |\nabla u|^2|\nabla f|^2\phi^2 e^{2bf} + C \\ &\leq \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + \frac{1}{2}\int_M |\nabla u|^2|\nabla f|^4 e^{2bf} + \frac{1}{2}\int_M |\nabla u|^2 e^{2bf} + C \\ &\leq \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + C,\end{aligned}$$

together with (3.39), this implies

$$\left|\int_M \mathrm{Ric}(\nabla u, \nabla u)\phi^2 e^{2bf}\right| \leq \frac{1}{4}\int_M |u_{ij}|^2\phi^2 e^{2bf} + C. \tag{3.40}$$

By combining (3.38) and (3.40) we conclude

$$\int_M |u_{ij}|^2\phi^2 e^{2bf} < \infty,$$

which concludes the proof. □

The next estimate also holds.

Lemma 3.4. *For any constant $b < a$ we have*

$$\int_M |\mathrm{Ric}|^2|\nabla u|^2 e^{2bf} < \infty.$$

Proof: Consider ϕ as given by (3.31). From Theorem 1.B, we know

$$(1-2(n-1)\rho)\Delta R = \langle\nabla R, \nabla f\rangle + 2(\rho R^2 - |Ric|^2 + \lambda R),$$

then, letting $A = 1-2(n-1)\rho$, we get from Green's identity and Lemma (3.2) that

$$\begin{aligned}
2\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2 e^{2bf} =& -A\int_M \Delta R|\nabla u|^2\phi^2 e^{2bf} + \int_M \langle\nabla R, \nabla f\rangle|\nabla u|^2\phi^2 e^{2bf} \\
&+ 2\int_M R(\rho R + \lambda)|\nabla u|^2\phi^2 e^{2bf} \\
\leq& A\int_M \left\langle \nabla R, \nabla\left(|\nabla u|^2\phi^2 e^{2bf}\right)\right\rangle + \int_M \langle\nabla R, \nabla f\rangle|\nabla u|^2\phi^2 e^{2bf} + C \\
=& A\int_M \left\langle \nabla R, \nabla|\nabla u|^2\phi^2 + \nabla\phi^2|\nabla u|^2 + 2b\nabla f|\nabla u|^2\phi^2\right\rangle e^{2bf} \\
&+ \int_M \langle\nabla R, \nabla f\rangle|\nabla u|^2\phi^2 e^{2bf} + C \\
=& 2A\int_M (u_{ij}u_iR_j)\,\phi^2 e^{2bf} + A\int_M \left\langle \nabla R, \nabla\phi^2\right\rangle|\nabla u|^2 e^{2bf} \\
&+ (2Ab+1)\int_M \langle\nabla R, \nabla f\rangle\,|\nabla u|^2\phi^2 e^{2bf} + C.
\end{aligned} \tag{3.41}$$

If $\rho \neq \frac{1}{2(n-1)}$, we have $A \neq 0$ and (3.41) can be estimated as follows:

Since $A\nabla R = 2\mathrm{Ric}(\nabla f)$, we have from Young's inequality that

$$2A\left(u_{ij}u_iR_j\right) \leq 4|u_{ij}||\nabla u||\nabla f||\mathrm{Ric}| \leq \frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2 + 16|u_{ij}|^2|\nabla f|^2.$$

Analogously,

$$\begin{aligned} A\left\langle \nabla R, \nabla\phi^2\right\rangle|\nabla u|^2 =& 4\mathrm{Ric}(\nabla f,\nabla\phi)\phi|\nabla u|^2 \leq 4|\mathrm{Ric}||\nabla f|^2|\nabla u|^2\phi \\ \leq& \frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2\phi^2 + 16|\nabla u|^2|\nabla f|^2, \end{aligned}$$

and

$$\begin{aligned} (2Ab+1)\left\langle \nabla R,\nabla f\right\rangle|\nabla u|^2 =& \frac{2}{A}(2Ab+1)\mathrm{Ric}(\nabla f,\nabla f)|\nabla u|^2 \\ \leq& \frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2 + \frac{4(2Ab+1)^2}{A^2}|\nabla u|^2|\nabla f|^4. \end{aligned}$$

Therefore, we get from (3.41)

$$\begin{aligned} 2\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2e^{2bf} \leq& \int_M \left(\frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2 + 16|u_{ij}|^2|\nabla f|^2\right)\phi^2e^{2bf} \\ &+ \int_M \left(\frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2\phi^2 + 16|\nabla u|^2|\nabla f|^2\right)e^{2bf} \\ &+ \int_M \left(\frac{1}{4}|\mathrm{Ric}|^2|\nabla u|^2 + \frac{4(2Ab+1)^2}{A^2}|\nabla u|^2|\nabla f|^4\right)\phi^2e^{2bf} + C, \end{aligned} \tag{3.42}$$

or equivalently,

$$\begin{aligned} \frac{5}{4}\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2e^{2bf} \leq& 16\int_M |u_{ij}|^2|\nabla f|^2\phi^2e^{2bf} + 16\int_M |\nabla u|^2|\nabla f|^2e^{2bf} \\ &+ \frac{4(2Ab+1)^2}{A^2}\int_M |\nabla u|^2|\nabla f|^4\phi^2e^{2bf} + C. \end{aligned}$$

Recalling Lemmas 3.2 and 3.3 and equation (3.34), we conclude

$$\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2e^{2bf} < \infty,$$

as we wanted to prove. If $A=0$, then (3.41) becomes

$$2\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2 e^{2bf} \leq \int_M \langle \nabla R, \nabla f\rangle |\nabla u|^2\phi^2 e^{2bf} + C,$$

which can be estimated as

$$\begin{aligned}\int_M \langle \nabla R, \nabla f\rangle |\nabla u|^2\phi^2 e^{2bf} &= \int_M \left\langle \nabla\left(R|\nabla u|^2\phi^2 e^{2bf}\right), \nabla f\right\rangle - \int_M R\left\langle \nabla\left(|\nabla u|^2\phi^2 e^{2bf}\right), \nabla f\right\rangle \\ &= -\int_M R\Delta f|\nabla u|^2\phi^2 e^{2bf} - 2\int_M R|\nabla u|\left\langle \nabla|\nabla u|, \nabla f\right\rangle \phi^2 e^{2bf} \\ &\quad -\int_M R|\nabla u|^2\left\langle \nabla\phi^2, \nabla f\right\rangle e^{2bf} - 2b\int_M R|\nabla u|^2|\nabla f|^2\phi^2 e^{2bf}.\end{aligned} \tag{3.43}$$

Thus, from Theorem 1.B and Young's and Kato's inequalities, we have

$$\begin{aligned}\int_M \langle \nabla R, \nabla f\rangle |\nabla u|^2\phi^2 e^{2bf} \leq &\int_M \left|\left(\frac{n}{2(n-1)} - 1\right) R + n\lambda\right| R|\nabla u|^2\phi^2 e^{2bf} \\ &+2\int_M R|\nabla u||\nabla|\nabla u|||\nabla f|\phi^2 e^{2bf} + \int_M R|\nabla u|^2|\nabla\phi^2||\nabla f|e^{2bf} + C \\ \leq &C_1\int_M |\nabla u|^2\phi^2 e^{2bf} + C_2\int_M |\nabla u|^2|\nabla f|^2\phi^2 e^{2bf} + \int_M |\nabla|\nabla u||^2\phi^2 e^{2bf} \\ &+ C_3\int_M |\nabla u|^2|\nabla f|^2\phi e^{2bf} + C \\ \leq &C_1\int_M |\nabla u|^2\phi^2 e^{2bf} + C_2\int_M |\nabla u|^2|\nabla f|^2\phi^2 e^{2bf} + \int_M |u_{ij}|^2\phi^2 e^{2bf} \\ &+ C_4\int_M |\nabla u|^2\phi^2 e^{2bf} + C_5\int_M |\nabla u|^2|\nabla f|^4 e^{2bf} + C.\end{aligned}$$

Applying Lemmas 3.2, 3.3 and equation (3.34), we get

$$\int_M |\mathrm{Ric}|^2|\nabla u|^2\phi^2 e^{2bf} \leq \frac{1}{2}\int_M \langle \nabla R, \nabla f\rangle |\nabla u|^2\phi^2 e^{2bf} + C < \infty.$$

Which concludes the proof of the Lemma.

$\square$

As a consequence of the lemmas above, we get the following estimate:

Lemma 3.5. *For any constant $b < a$*

$$\int_M |\nabla u| e^{bf} + \int_M |u_{ij}| e^{bf} + \int_M |\mathrm{Ric}||\nabla u| e^{bf} < \infty.$$

Proof: According to Lemmas 3.2, 3.3 and 3.4, by making $\bar{b} = \frac{1}{2}(a+b) < a$, we have

$$\int_M |\nabla u|^2 e^{2\bar{b}f} < \infty, \quad \int_M |u_{ij}|^2 e^{2\bar{b}f} < \infty \quad \text{and} \quad \int_M |\mathrm{Ric}|^2 |\nabla u|^2 e^{2\bar{b}f} < \infty.$$

Now, since $b - a < 0$, we get from the Cauchy-Schwarz inequality that

$$\left(\int_M |\nabla u| e^{bf}\right)^2 = \left(\int_M |\nabla u| e^{\bar{b}f} \cdot e^{(\bar{b}-a)f}\right)^2 \leq \left(\int_M |\nabla u|^2 e^{2\bar{b}f}\right)\left(\int_M e^{(b-a)f}\right) < \infty,$$

analogously, we can easily check that

$$\left(\int_M |u_{ij}| e^{bf}\right)^2 < \infty \quad \text{and} \quad \left(\int_M |\mathrm{Ric}||\nabla u| e^{bf}\right)^2 < \infty,$$

by taking square roots and adding up these inequalities the Lemma is proven.

□

Finally, prior to our main result, we also need to prove the next inequality.

Lemma 3.6. *For all $b < a$,*

$$\begin{aligned}
2\int_M |\mathrm{Ric}|^2 |\nabla u| e^{bf} \leq & -((2b-a)A+1)\int_M R\langle \nabla|\nabla u|, \nabla f\rangle e^{bf} \\
& - \int_M R\left((Ab+1)(\Delta f) - (aA+2)(\rho R + \lambda)\right)|\nabla u| e^{bf} \\
& - (Ab^2+b)\int_M R|\nabla f|^2 |\nabla u| e^{bf} - (a+1)A\int_M \mathrm{Ric}(\nabla u, \nabla u) R |\nabla u|^{-1} e^{bf},
\end{aligned}$$

where $A = 1 - 2(n-1)\rho$.

Proof: Consider ϕ as given by (3.31). From the identity

$$(1 - 2(n-1)\rho)\Delta R = \langle \nabla R, \nabla f\rangle + 2(\rho R^2 - |Ric|^2 + \lambda R),$$

we have

$$\begin{aligned}2\int_M |\mathrm{Ric}|^2|\nabla u|\phi^2 e^{bf} =& -A\int_M \Delta R|\nabla u|\phi^2 e^{bf} + \int_M \langle\nabla R,\nabla f\rangle|\nabla u|\phi^2 e^{bf}\\ &+2\int_M R(\rho R+\lambda)|\nabla u|\phi^2 e^{2bf}.\end{aligned}\tag{3.44}$$

From Green's identity, second integral on the right-hand side can be written as

$$\begin{aligned}\int_M \langle\nabla R,\nabla f\rangle|\nabla u|\phi^2 e^{bf} =& \int_M \langle\nabla f,\nabla\left(R|\nabla u|\phi^2 e^{bf}\right)\rangle - \int_M R\langle\nabla f,\nabla\left(|\nabla u|\phi^2 e^{bf}\right)\rangle\\ =& -\int_M (\Delta f)R|\nabla u|\phi^2 e^{bf} - \int_M R\langle\nabla f,\nabla|\nabla u|\rangle\phi^2 e^{bf}\\ &-\int_M R\langle\nabla f,\nabla\phi^2\rangle|\nabla u|e^{bf} - b\int_M R|\nabla f|^2|\nabla u|\phi^2 e^{bf},\end{aligned}$$

by plugging this into (3.44), we get

$$\begin{aligned}2\int_M |\mathrm{Ric}|^2|\nabla u|\phi^2 e^{bf} =& -A\int_M \Delta R|\nabla u|\phi^2 e^{bf} - \int_M (\Delta f)R|\nabla u|\phi^2 e^{bf}\\ &-\int_M R\langle\nabla f,\nabla|\nabla u|\rangle\phi^2 e^{bf} - \int_M R\langle\nabla f,\nabla\phi^2\rangle|\nabla u|e^{bf}\\ &-b\int_M R|\nabla f|^2|\nabla u|\phi^2 e^{bf} + 2\int_M R(\rho R+\lambda)|\nabla u|\phi^2 e^{2bf}.\end{aligned}\tag{3.45}$$

On the one hand, from the Bochner formula, $\varphi-$harmonicity of u and the soliton equation, we have

$$\begin{aligned}\frac{1}{2}\Delta|\nabla u|^2 =& |u_{ij}|^2 + \langle\nabla\Delta u,\nabla u\rangle + \mathrm{Ric}(\nabla u,\nabla u)\\ =& |u_{ij}|^2 - a\left\langle\nabla\langle\nabla u,\nabla f\rangle,\nabla u\right\rangle + \mathrm{Ric}(\nabla u,\nabla u)\\ =& |u_{ij}|^2 - a u_{ij}u_i f_j - a f_{ij}u_i u_j + \mathrm{Ric}(\nabla u,\nabla u)\\ =& |u_{ij}|^2 - a u_{ij}u_i f_j - a(\rho R+\lambda)|\nabla u|^2 + a\mathrm{Ric}(\nabla u,\nabla u) + \mathrm{Ric}(\nabla u,\nabla u)\\ =& |u_{ij}|^2 - \frac{a}{2}\langle\nabla|\nabla u|^2,\nabla f\rangle - a(\rho R+\lambda)|\nabla u|^2 + (a+1)\mathrm{Ric}(\nabla u,\nabla u)\\ =& |u_{ij}|^2 - a|\nabla u|\langle\nabla|\nabla u|,\nabla f\rangle - a(\rho R+\lambda)|\nabla u|^2 + (a+1)\mathrm{Ric}(\nabla u,\nabla u).\end{aligned}\tag{3.46}$$

On the other hand, Kato's inequality implies

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla u|\Delta|\nabla u| + |\nabla|\nabla u||^2 \leq |\nabla u|\Delta|\nabla u| + |u_{ij}|^2,$$

therefore, it follows from (3.46) that

$$\Delta|\nabla u| \geq -a\langle\nabla|\nabla u|, \nabla f\rangle - a(\rho R + \lambda)|\nabla u| + (a+1)\mathrm{Ric}(\nabla u, \nabla u)|\nabla u|^{-1}.$$

Thus,

$$\begin{aligned}\Delta\left(|\nabla u|e^{bf}\right) =& \Delta|\nabla u|e^{bf} + |\nabla u|\Delta e^{bf} + 2\langle\nabla|\nabla u|, \nabla e^{bf}\rangle \\ \geq& -a\langle\nabla|\nabla u|, \nabla f\rangle e^{bf} - a(\rho R+\lambda)|\nabla u|e^{bf} + (a+1)\mathrm{Ric}(\nabla u, \nabla u)|\nabla u|^{-1}e^{bf} \\ &+ b|\nabla u|\left(\Delta f\right)e^{bf} + b^2|\nabla u||\nabla f|^2 e^{bf} + 2b\langle\nabla|\nabla u|, \nabla f\rangle e^{bf} \\ =& (2b-a)\langle\nabla|\nabla u|, \nabla f\rangle e^{bf} + \left(b(\Delta f) + b^2|\nabla f|^2 - a(\rho R+\lambda)\right)|\nabla u|e^{bf} \\ &+ (a+1)\mathrm{Ric}(\nabla u, \nabla u)|\nabla u|^{-1}e^{bf}.\end{aligned} \tag{3.47}$$

From Green's identity,

$$\begin{aligned}-\int_M (\Delta R)|\nabla u|\phi^2 e^{bf} =& \int_M \langle\nabla R, \nabla\left(|\nabla u|\phi^2 e^{bf}\right)\rangle \\ =& \int_M \langle\nabla R, \nabla\left(|\nabla u|e^{bf}\right)\rangle\phi^2 + \int_M \langle\nabla R, \nabla\phi^2\rangle|\nabla u|e^{bf} \\ =& \int_M \langle\nabla(R\phi^2), \nabla\left(|\nabla u|e^{bf}\right)\rangle - \int_M R\langle\nabla\phi^2, \nabla\left(|\nabla u|e^{bf}\right)\rangle + \int_M \langle\nabla R, \nabla\phi^2\rangle|\nabla u|e^{bf} \\ =& -\int_M R\Delta\left(|\nabla u|e^{bf}\right)\phi^2 - \int_M R\langle\nabla\phi^2, \nabla\left(|\nabla u|e^{bf}\right)\rangle + \int_M \langle\nabla R, \nabla\phi^2\rangle|\nabla u|e^{bf};\end{aligned}$$

combined with (3.47), this becomes

$$\begin{aligned}
-\int_M (\Delta R)|\nabla u|\phi^2 e^{bf} \leq &-(2b-a)\int_M R\langle \nabla|\nabla u|, \nabla f\rangle e^{bf}\phi^2 \\
&-\int_M R\left(b(\Delta f)+b^2|\nabla f|^2 - a(\rho R+\lambda)\right)|\nabla u| e^{bf}\phi^2 \\
&-(a+1)\int_M \mathrm{Ric}(\nabla u, \nabla u) R|\nabla u|^{-1} e^{bf}\phi^2 \\
&-\int_M R\langle \nabla \phi^2, \nabla\left(|\nabla u| e^{bf}\right)\rangle + \int_M \langle \nabla R, \nabla \phi^2\rangle |\nabla u| e^{bf}.
\end{aligned}$$

By plugging the inequality above into (3.45), we get

$$\begin{aligned}
2\int_M |\mathrm{Ric}|^2|\nabla u|\phi^2 e^{bf} \leq &-(2b-a)A\int_M R\langle \nabla|\nabla u|, \nabla f\rangle e^{bf}\phi^2 \\
&-A\int_M R\left(b(\Delta f)+b^2|\nabla f|^2 - a(\rho R+\lambda)\right)|\nabla u| e^{bf}\phi^2 \\
&-(a+1)A\int_M \mathrm{Ric}(\nabla u, \nabla u) R|\nabla u|^{-1} e^{bf}\phi^2 \\
&-A\int_M R\langle \nabla \phi^2, \nabla\left(|\nabla u| e^{bf}\right)\rangle + A\int_M \langle \nabla R, \nabla \phi^2\rangle |\nabla u| e^{bf} \\
&-\int_M (\Delta f) R|\nabla u|\phi^2 e^{bf} - \int_M R\langle \nabla f, \nabla|\nabla u|\rangle \phi^2 e^{bf} \\
&-\int_M R\langle \nabla f, \nabla \phi^2\rangle |\nabla u| e^{bf} - b\int_M R|\nabla f|^2|\nabla u|\phi^2 e^{bf} \\
&+2\int_M R(\rho R+\lambda)|\nabla u|\phi^2 e^{2bf},
\end{aligned}$$

this is,

$$
\begin{aligned}
2\int_M |\mathrm{Ric}|^2|\nabla u|\phi^2 e^{bf} \le &-((2b-a)A+1)\int_M R\langle\nabla|\nabla u|,\nabla f\rangle e^{bf}\phi^2\\
&-\int_M R\left((Ab+1)(\Delta f)-aA(\rho R+\lambda)-2(\rho R+\lambda)\right)|\nabla u|e^{bf}\phi^2\\
&-(Ab^2+b)\int_M R|\nabla f|^2|\nabla u|\phi^2 e^{bf}-(a+1)A\int_M \mathrm{Ric}(\nabla u,\nabla u)R|\nabla u|^{-1}e^{bf}\phi^2\\
&-A\int_M R\langle\nabla\phi^2,\nabla\left(|\nabla u|e^{bf}\right)\rangle-\int_M R\langle\nabla f,\nabla\phi^2\rangle|\nabla u|e^{bf}\\
&+A\int_M \langle\nabla R,\nabla\phi^2\rangle|\nabla u|e^{bf}.
\end{aligned}
\tag{3.48}
$$

Regarding the last three terms on inequality above, notice in first place that we can take (3.34) and proceed like in Lemma 3.5 to conclude

$$\int_M |\nabla f|^2|\nabla u|e^{bf}<\infty. \tag{3.49}$$

Next, from Kato's inequality,

$$
\begin{aligned}
\int_M R\langle\nabla\phi^2,\nabla\left(|\nabla u|e^{bf}\right)\rangle =&\int_M R\langle\nabla\phi^2,\nabla|\nabla u|\rangle e^{bf}+b\int_M R\langle\nabla\phi^2,\nabla f\rangle|\nabla u|e^{bf}\\
\le& C_1\int_{D(T+1)\setminus D(T)}|\nabla\phi||\nabla|\nabla u||e^{bf}+C_2\int_{D(T+1)\setminus D(T)}|\nabla f|^2|\nabla u|e^{bf}\\
\le& C_1\int_{D(T+1)\setminus D(T)}|\nabla\phi||u_{ij}|e^{bf}+C_2\int_{D(T+1)\setminus D(T)}|\nabla f|^2|\nabla u|e^{bf}.
\end{aligned}
$$

From Lemma 3.5 and (3.49), we know both integrals on the right-hand side above vanish as $T\to\infty$, then, we conclude that

$$-A\int_M R\langle\nabla\phi^2,\nabla\left(|\nabla u|e^{bf}\right)\rangle-\int_M R\langle\nabla f,\nabla\phi^2\rangle|\nabla u|e^{bf}\to 0 \text{ as } T\to\infty.$$

Analogously, given $A\nabla R=2\mathrm{Ric}(\nabla f)$, we can use Young's inequality to estimate the last term on (3.48) by

$$\begin{aligned} A\int_M \langle \nabla R, \nabla\phi^2\rangle|\nabla u|e^{bf} \leq & 2\int_{D(T+1)\setminus D(T)} \mathrm{Ric}(\nabla f,\nabla f)|\nabla u|e^{bf} \\ \leq & 2\int_{D(T+1)\setminus D(T)} |\mathrm{Ric}||\nabla f|^2|\nabla u|e^{bf} \\ \leq & \int_{D(T+1)\setminus D(T)} |\mathrm{Ric}|^2 e^{bf} + \int_{D(T+1)\setminus D(T)} |\nabla f|^4|\nabla u|^2 e^{bf}, \end{aligned}$$

and from Lemma 3.5 and (3.34) we infer

$$A\int_M \langle \nabla R, \nabla\phi^2\rangle|\nabla u|e^{bf} \to 0 \ \text{ as } \ T\to\infty.$$

In view of this, by making $T\to\infty$ in (3.48), we conclude

$$\begin{aligned} 2\int_M |\mathrm{Ric}|^2|\nabla u|e^{bf} \leq & -((2b-a)A+1)\int_M R\langle\nabla|\nabla u|,\nabla f\rangle e^{bf} \\ & -\int_M R\left((Ab+1)(\Delta f) - aA(\rho R+\lambda) - 2(\rho R+\lambda)\right)|\nabla u|e^{bf} \\ & -(Ab^2+b)\int_M R|\nabla f|^2|\nabla u|e^{bf} - (a+1)A\int_M \mathrm{Ric}(\nabla u,\nabla u)R|\nabla u|^{-1}e^{bf}, \end{aligned} \tag{3.50}$$

as we wanted to prove.

□

We are now ready to prove the main and final result of this chapter:

Theorem 3.7. *Let (M^n, g, f, λ) be a shrinking gradient $\rho-$Einstein soliton with $n\geq 4$, non-negative scalar curvature satisfying*

$$R \leq \frac{2n(n-3)\lambda}{3(n-3)-(3n^2-12n+5)\rho}, \tag{3.51}$$

and such that $\rho\in\left[0, \frac{1}{2(n-1)}\right)$. Then M has only one end.

Proof: Assume by contradiction that M has two ends. Then there is a $\varphi-$harmonic function u satisfying (3.22), (3.23) and (3.24). As the scalar curvature is bounded and, according to Theorem 1.C and Proposition 1.3.1, the potential function grows quadratically, we can apply Lemma 3.6.

Set $a = A^{-1} = (1-2(n-1)\rho)^{-1}$ and $b = 0$ in (3.50) to get

$$2\int_M |\mathrm{Ric}|^2|\nabla u| \leq -\int_M R\left(\Delta f - 3(\rho R + \lambda)\right)|\nabla u| - (1+A)\int_M \mathrm{Ric}(\nabla u, \nabla u) R|\nabla u|^{-1}. \tag{3.52}$$

On the other hand, for any constant $\gamma > 0$

$$\begin{aligned}-(1+A)\mathrm{Ric}(\nabla u, \nabla u)R|\nabla u|^{-1} =& -(1+A)R_{ij}u_iu_j|\nabla u|^{-1}R\\ =& -(1+A)(R_{ij}-\gamma R g_{ij})u_iu_j|\nabla u|^{-1}R - |1+A|\gamma|\nabla u|R^2.\end{aligned}$$

First term on the right-hand side of equation above can be estimated as

$$\begin{aligned}-(R_{ij}-\gamma R g_{ij})u_iu_j|\nabla u|^{-1}R \leq& |R_{ij}-\gamma R g_{ij}||\nabla u|R\\ \leq& |\nabla u|\left(|R_{ij}-\gamma R g_{ij}|^2 + \frac{1}{4}R^2\right)\\ =& |R_{ij}-\gamma R g_{ij}|^2|\nabla u| + \frac{1}{4}|\nabla u|R^2\\ =& \left(|\mathrm{Ric}|^2 + R^2\gamma^2 n - 2\gamma R^2\right)|\nabla u| + \frac{1}{4}|\nabla u|R^2\\ =& |\mathrm{Ric}|^2 + \left(\gamma^2 n - 2\gamma + \frac{1}{4}\right)|\nabla u|R^2,\end{aligned}$$

which implies

$$-(1+A)\mathrm{Ric}(\nabla u, \nabla u)R|\nabla u|^{-1} \leq (1+A)\left(|\mathrm{Ric}|^2|\nabla u| + \left(\gamma^2 n - 3\gamma + \frac{1}{4}\right)|\nabla u|R^2\right). \tag{3.53}$$

As $\gamma^2 n - 3\gamma + \frac{1}{4}$ attains its minimum when $\gamma = \frac{3}{2n}$, we can optimize (3.53) to conclude

$$-(1+A)\mathrm{Ric}(\nabla u, \nabla u)R|\nabla u|^{-1} \leq (1+A)|\mathrm{Ric}|^2|\nabla u| + (1+A)\left(\frac{n-9}{4n}\right)|\nabla u|R^2. \tag{3.54}$$

Plugging this into (3.52), we get

$$2\int_M |\mathrm{Ric}|^2|\nabla u| \leq -\int_M R\left(\Delta f - 3(\rho R+\lambda)\right)|\nabla u| + (1+A)\int_M |\mathrm{Ric}|^2|\nabla u| + (1+A)\left(\frac{n-9}{4n}\right)\int_M |\nabla u|R^2,$$

this is,

$$(1-A)\int_M |\mathrm{Ric}|^2|\nabla u| \leq -\int_M R\left(\Delta f - 3(\rho R+\lambda)\right)|\nabla u| + (1+A)\left(\frac{n-9}{4n}\right)\int_M |\nabla u|R^2. \tag{3.55}$$

Since $\Delta f = (n\rho-1)R + n\lambda$, and $|\mathrm{Ric}|^2 \geq \frac{R^2}{n}$, we conclude from (3.55) that

$$\begin{aligned}0 \leq &-\int_M R\left((n\rho-1)R+n\lambda-3(\rho R+\lambda)\right)|\nabla u| + \left(\frac{(1+A)(n-9)}{4n} - \frac{1-A}{n}\right)\int_M |\nabla u|R^2\\ =&-(n-3)\lambda\int_M R|\nabla u| + \left[\frac{(1+A)(n-9)}{4n} - \frac{1-A}{n} + 3\rho - (n\rho-1)\right]\int_M |\nabla u|R^2,\end{aligned}$$

to wit

$$(n-3)\lambda\int_M R|\nabla u| \leq \left[\frac{(1-(n-1)\rho)(n-9)}{2n} - \frac{2(n-1)\rho}{n} + (3-n)\rho + 1\right]\int_M R^2|\nabla u|,$$

and consequently

$$\frac{2n(n-3)\lambda}{3(n-3)-(3n^2-12n+5)\rho}\int_M R|\nabla u| \leq \int_M R^2|\nabla u|. \tag{3.56}$$

It is worth noticing that $3(n-3)-(3n^2-12n+5)\rho > 0$ for all n as $0 \leq \rho < \frac{1}{2(n-1)}$. Together with the hypothesis over R, this implies

$$R = \frac{2n(n-3)\lambda}{3(n-3)-(3n^2-12n+5)\rho},$$

and all inequalities above are in fact equalities, in particular, from equality on Young's inequality, we must have

$$\left|\mathrm{Ric} - \frac{3}{2n}Rg\right| = \frac{1}{2}R.$$

Thus,

$$\left|\mathrm{Ric} - \frac{3}{2n}Rg\right|^2 = |\mathrm{Ric}|^2 - \frac{3}{n}R^2 + \frac{9}{4n}R^2 = \frac{1}{4}R^2,$$

which implies

$$\begin{aligned} |\mathrm{Ric}|^2 &= \left(\frac{n+3}{4n}\right)\frac{4n^2(n-3)^2\lambda^2}{(3(n-3)-(3n^2-12n+5)\rho)^2} \\ &= \frac{n(n+3)(n-3)^2\lambda^2}{(3(n-3)-(3n^2-12n+5)\rho)^2}. \end{aligned} \tag{3.57}$$

On the other hand, since R is proven to be constant, $\Delta R = \nabla R = 0$, then we have from Theorem 1.B that

$$|\mathrm{Ric}|^2 = \rho R^2 + \lambda R,$$

combined with (3.57) and the value of R, this means

$$\begin{aligned} \frac{n(n+3)(n-3)^2\lambda^2}{(3(n-3)-(3n^2-12n+5)\rho)^2} =& \rho\left(\frac{2n(n-3)\lambda}{3(n-3)-(3n^2-12n+5)\rho}\right)^2 \\ &+ \frac{2n(n-3)\lambda^2}{3(n-3)-(3n^2-12n+5)\rho}. \end{aligned}$$

Since the numerators are not null, $\lambda > 0$ and $n \geq 4$, expression above can be simplified to

$$(n-3)(n+3) = 4n\rho(n-3) + 6(n-3) - 2(n^2-12n+5)\rho,$$

which after a couple of computations leads to

$$n^2 - 6n + 9 = -2\rho(n^2 - 6n + 5). \tag{3.58}$$

Recalling again $n \geq 4$, we can analyze (3.58) for three cases: If $n = 4$, then from hypothesis $\rho < \frac{1}{2(n-1)} = \frac{1}{6}$, but making $n = 4$ in (3.58) one gets $\rho = \frac{1}{6}$, which is impossible. In case $n = 5$, a simple substitution into (3.58) implies $4 = 0$, again impossible. Finally, for any $n \geq 6$, both $n^2 - 6n + 9$ and $n^2 - 6n + 5$ are positive, an this way (3.58) implies

$$\rho = -\frac{n^2-6n+5}{2(n^2-6n+9)} < 0,$$

contradicting once again the hypothesis of $\rho \in \left[0, \frac{1}{2(n-1)}\right)$.

In any case, (3.58) leads to a contradiction for every $n \geq 4$. This means such φ−harmonic function u, satisfying (3.50) and (3.52), shall not exist. Thus, M must have only one end, and the theorem is proven.

□

Remark 3.3. *We firmly believe that non-negativity hypothesis of R can be removed from the statement of Theorem 3.7. This is motivated by the fact that scalar curvature is proven to be non-negative for shrinking and steady gradient Ricci solitons [18] and by the results of Catino* et al. *on compact Ricci-Bourguignon flow solutions [11, Thm. 4.1] and solitons with complete vector field ∇f, which suggest that an extension of the non-negativity result in [11] for non-compact $\rho-$Einstein solitons with $\rho < \frac{1}{2(n-1)}$ is plausible. We expect to elaborate further on this idea in upcoming works.*

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