Universidade de Brasília

A mini-max algorithm for semilinear elliptic problems

Daniel Raom Santiago Bezerra Costa da Silva

Advisor: PhD Liliane de Almeida Maia

Department of Mathematics
Universidade de Brasília

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À minha família de Natal. À minha família de Brasília.
À minha família do Recife. À minha família do Rio.

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Resumo

Estudamos um problema não linear elíptico geral em $\mathbb{R}^N$ e provamos, por meio de uma estrutura variacional do problema, a existência de uma solução *ground state* (de energia mínima), a qual é o mínimo do funcional associado ao problema restrito à variedade de Pohozaev. Este mínimo coincide com o nível do passo da montanha uma vez que o funcional associado possui a geometria necessária. Nós então propomos e implementamos um algoritmo numérico para encontrar as soluções *ground state* para uma ampla classe de problemas elípticos em $\mathbb{R}^N$, e fornecemos diversos exemplos para os quais este novo método pode ser aplicado.

Abstract

We study a general nonlinear elliptic problem in $\mathbb{R}^N$ and prove, by means of a variational structure of the problem, the existence of a ground state solution (of minimal energy), which is also the minimum of the functional associated to the problem constrained to the Pohozaev manifold. This minimum coincides the mountain pass level since the associated functional possesses the necessary geometry. We then propose and implement an algorithm to find the ground state solutions for a wide class of elliptic problems in $\mathbb{R}^N$, and provide several examples for which this new method can be applied.

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Introduction

The celebrated Mountain Pass Theorem of Ambrosetti and Rabinowitz [2] has been widely used in the past forty five years for finding weak solutions of semilinear elliptic problems as critical points of an associated functional. Solutions are found on the mini-max levels of the functional.

A numerical approach of this theorem was first introduced by Choi and McKenna in [10]. Their work showed that, when carefully implemented, the algorithm is globally convergent and leads to a solution with the required mountain pass property.

Later, Chen, Ni and Zhou in [9] observed that this algorithm may converge to a solution with morse index greater or equal to two, and not to the ground state mountain pass level. In order to circumvent this limitation, they created a new algorithm based on the fact that the minimum of the associated functional constrained to the Nehari manifold is equal to the mini-max level obtained by the Mountain Pass Theorem. This equivalence follows when the nonlinear terms in the equation are superquadratic [15, 26, 37]. For asymptotically linear problems this is not true in general. However, more recently, the ground state level was shown to be equal to the minimum of the functional restricted to the Pohozaev manifold (see Jeanjean and Tanaka [18]).

This work deals with the study of these results, starting with a general nonlinear elliptic problem in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), u \neq 0 \end{cases} \tag{1}$$

and then dealing with a more particular class of problems, such as

$$\begin{cases} -\Delta u + \lambda u = f(u) \quad \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), u \neq 0, \end{cases} \tag{2}$$

and using important results from Berestycki-Lions [6] and Jeanjean-Tanaka [18] to prove some important lemmas that will help us devise a mini-max algorithm for the visualization of ground state solutions.

We first define the problem in Chapter 1 by presenting a semilinear elliptic equation in $\mathbb{R}^N$ and impose a few conditions on the nonlinearity g. Then, we show our problem can be reformulated in a variational setting, and the associated functional is of class C^1.

In Chapter 2 we define what is known as the Pohozaev set and prove it is in fact a C^1 Hilbert manifold. We prove the existence of a least energy (or ground state) solution and show that it is also the solution for which the functional attains a minimum when constrained to the Pohozaev manifold. Furthermore, we show that the least energy level is also the mountain pass level, since our associated functional will possess a geometry suitable for applying the Mountain Pass Lemma of Ambrosetti-Rabinowitz [2].

In Chapter 3 we prove a few Lemmas that will help us devise the algorithm at the end of the Chapter.

In Chapter 4, with an algorithm in hands, we show how it was implemented and shall perform a few numerical experiments to assess if it possesses good, so to speak, numerical properties.

Finally, in Chapter 5 we present several applications for which our algorithm can be applied to solve problem (2). We end the Chapter and this work by making some concluding remarks about the algorithm and how it could be extended to more general problems.

The main idea of the algorithm is to project a nonzero function on the Pohozaev manifold of the associated functional, rather than on the Nehari manifold. The goal is to find a ground state solution of the problem by (constrained) minimization on this former manifold, something which has not yet been studied numerically in the literature. The approach of using the Nehari manifold in order to solve minimization problems has been widely studied both analytically and numerically (see, for example, [9] and [17]). For a given function $u \neq 0$, in general there exists a $t > 0$ such that tu lies on the Nehari manifold. There are several applications for which this approach is natural: for superlinear problems, with homogeneous nonlinearities for example, projections on Nehari are unique and hence we can guarantee results on existence and uniqueness of solutions. However, there are cases where this condition fails to be satisfied, and so we must consider another subset where we know its projections are always unique:

the Pohozaev manifold. Likewise, for a given function $u \neq 0$, there is a unique $t > 0$ such that $u(\cdot/t)$ lies on Pohozaev, and with this scaling, it might be appropriate to use this manifold when handling more general problems. From this perspective, we obtain, numerically, positive solutions for a semilinear problem and in particular, for superlinear and asymptotically linear problems, the latter not having been studied numerically anywhere on the literature.

Chapter 1

A semilinear elliptic problem in $\mathbb{R}^N$

1.1 Introduction

In this Chapter, we present a semilinear elliptic equation defined on a domain which is the whole $\mathbb{R}^N$, to be solved in a suitable function space. Because of the nonlinearity term, certain conditions must be imposed if we want to guarantee existence and desired regularity properties for the solution: those are provided in Section 1.2. It is a fortunate feature that nonlinear elliptic equations of such type sometimes possess a variational structure: if we have a differential equation $Lu = 0$, (L is a differential operator), by considering an associated functional $\phi : H \to \mathbb{R}$ in a certain space H with the property that its derivative is equal to Lu, we may then look for points of ϕ which are points of *minimum*, *maximum*, or *saddle-like* points, a.k.a. points of *minimax*. In Section 1.3, we show that the problem defined on Section 1.2 does indeed possess a variational setting. From such a configuration, we then proceed to demonstrate that the associated functional is well-defined and continuously differentiable, on Section 1.4. This is the first condition that must be verified in order to proceed with the search for critical points of the associated functional.

1.2 Defining the Problem

Consider the following general nonlinear elliptic problem:

$$\begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), u \neq 0. \end{cases} \tag{1.1}$$

We assume the following conditions on the nonlinearity g :

(g1) $g : \mathbb{R} \to \mathbb{R}$ is a continuous and odd function.

(g2) $-\infty < \liminf_{s \to 0} \frac{g(s)}{s} \leq \limsup_{s \to 0} \frac{g(s)}{s} = -\lambda < 0$ if $N \geq 3$.

(g3) When $N \geq 3$, $\lim_{s \to \infty} \frac{|g(s)|}{s^{\frac{N+2}{N-2}}} = 0$.

(g4) There exists $\zeta > 0$ such that $G(\zeta) = \int_0^\zeta g(s)ds > 0$.

Remark 1.1. When $N = 2$, other conditions must be imposed on the nonlinearity $g(s)$ other than (g2) and (g3). However, the results that will follow on Chapters 1 and 2 also hold for the case $N = 2$. Check [6] and [18] for proofs.

When considering nonlinearities of the type $g(u) = f(u) - \lambda u$, where $\lambda > 0$, in general one has $f(t) = |t|^p$, which is associated with the study of a light beam in what is known as a Kerr medium [1], and if $f(t)$ is asymptotically linear, for example $f(t) = \frac{|t|^3}{1 + |t|^2}$, the medium is saturable and might be better fitted to model the optical phenomenon. We will work with nonlinearities $g(u)$ in this form. In this case, problem (1.1) is simplified and so we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = f(u) \quad \text{in} \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), u \neq 0, \end{cases} \tag{1.2}$$

where $N \geq 3$ and $\lambda > 0$, and the following conditions on the nonlinearity term $f(u)$:

(f1) $f \in C^1[0, +\infty]$

(f2) f(u) = o(u) as $u \to 0$.

(f3) Either there is a positive constant $a < \lambda$ such that $\dfrac{f(u)}{u} \to a$ as $u \to +\infty$, or $\dfrac{f(u)}{u} \to +\infty$.

(f4) There exist positive constants c_1, c_2 such that

$$|f'(u)| \leq c_1 + c_2|u|^{p-2}, \tag{1.3}$$

with $2 < p < 2^* := \dfrac{2N}{N-2}$, if $N \geq 3$.

Remark 1.2. Condition (f1) says that f is continuously differentiable in $[0, +\infty]$. Condition (f2) means that $\lim_{u\to 0} \dfrac{f(u)}{u} = 0$. Condition (f3) states the growth conditions imposed on the nonlinearity f.

We use the following notations for the inner product and the norm in $H^1(\mathbb{R}^N)$:

$$(u,v)_{H^1} := \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) \quad \text{and} \quad \|u\|_{H^1,\lambda} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda u^2) \right)^{1/2}. \tag{1.4}$$

Remark 1.3. Observe that the norm $\|u\|^2_{H^1,\lambda}$ defined in 1.4 is equivalent to the norm

$$\|u\|^2_{H^1} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \right)^{1/2}.$$

1.3 Variational Formulation

In the calculus of variations, a class of nonlinear problems is said to have a variational formulation if the nonlinear differential operator of the PDE is the "derivative" of a certain energy functional. The energy functional is also known as the lagrangian and the associated PDE, the Euler-Lagrange Equation [16]. This approach allows us to look for critical points of the associated functional instead of directly solving the PDE. This method will be made clear in the sequel.

From our semilinear elliptic problem, multiply the equation by a test function $v \in C_c^\infty(\mathbb{R}^N)$ and integrate over $\mathbb{R}^N$ to get:

$$\int_{\mathbb{R}^N} (-\Delta u) v \, dx + \int_{\mathbb{R}^N} \lambda uv \, dx = \int_{\mathbb{R}^N} fv \, dx \tag{1.5}$$

Now consider the Divergence theorem (see A.1) on a domain $\Omega \subset \mathbb{R}^N$:

$$\int_{\Omega} (\Delta u) v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma - \int_{\Omega} \nabla u \nabla v \, dx, \tag{1.6}$$

where $d\sigma$ is the surface measure of $\partial\Omega$. Then

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \lambda \int_{\mathbb{R}^N} uv \, dx = \int_{\mathbb{R}^N} fv \, dx \quad \forall v \in C_c^{\infty}(\mathbb{R}^N) \tag{1.7}$$

If we now consider $\Omega = \mathbb{R}^N$ and consider the "boundary condition" $|u(x)| \to 0$ as $|x| \to \infty$, then since $C_c(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, expression (1.6) also holds for all $v \in H^1(\mathbb{R}^N)$.

It is known that for the case of linear elliptic equations that possess a variational setting, i.e. equations that are the Euler-Lagrange equations of certain functionals, the Riesz Representation theorem (see A.11) suffices for obtaining the variational formulation of such problems. In the case of general partial differential equations in divergence form, a more general result is needed. In the Calculus of Variations, this is known as the Lax-Milgram Theorem, which we state below. Let us first recall the following definitions:

Definition 1.1. A bilinear form $B(.,.)$ is a function $B : V \times V \to K$ that is linear in each argument, where V is a vector space and K is a field.

Remark 1.4. For the Lax-Milgram Theorem, the bilinear form is defined on a Hilbert Space and the field is that of the real numbers. This means one takes $V = H, K = \mathbb{R}$ on the definition above. Note also that throughout this work, the Hilbert space H is actually the function space $H^1(\mathbb{R}^N)$.

Definition 1.2. A bilinear form $B(.,.)$ is called bounded if there exists a constant $\alpha > 0$ such that

$$|B(u,v)| \leq \alpha \|x\| \|y\| \quad \forall u, v \in V.$$

Definition 1.3. A bilinear form $B(.,.)$ is called coercive if there exists a number $\beta > 0$ such that

$$\beta \|u\|^2 \leq B(u,u) \quad \forall u \in V.$$

Theorem 1.5. (Lax-Milgram) Let $B(.,.)$ be a bounded, coercive bilinear form defined on a Hilbert space H. Then for every bounded linear functional $f : H \to \mathbb{R}$ ($f \in H^*$), there exists a unique element $u \in H$ such that

$$B(u, v) = \langle f, v\rangle \quad \forall v \in H.$$

This abstract principle is the basis for preparing the ground for the existence of a weak solution to our original problem.

Remark 1.6. As we stated in the beggining of this section, for linear partial differential equations, the bilinear form is symmetric, and so the Riesz Representation Theorem (see A.11) directly applies. The Lax-Milgram Theorem is a stronger result for the more general cases.

In our case, the bilinear form $B : H^1(\mathbb{R}^N) \times \mathbb{R}^N \to \mathbb{R}$ associated to the divergence form elliptic operator L, given by $Lu = \Delta u + \lambda u$, $\lambda > 0$, is

$$B(u, v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \lambda \int_{\mathbb{R}^N} uv \, dx \quad u, v \in H^1(\mathbb{R}^N). \tag{1.8}$$

In the following two Lemmas, we shall prove that the bilinear mapping $B(.,.)$ defined by (1.8) is bounded and coercive. After defining an appropriate bounded linear functional, we will be ready to apply the Lax-Milgram theorem.

Lemma 1.1. There exists a constant $\alpha > 0$ such that $|B(u, v)| \leq \alpha \|u\|_{H^1} \|v\|_{H^1}$.

Proof. Note that

$$\begin{aligned} |B(u, v)| &= \left| \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \lambda \int_{\mathbb{R}^N} uv \, dx \right| \\ &= (u, v)_{H^1} \\ &\leq \|u\|_{H^1} \|v\|_{H^1}, \end{aligned}$$

from the Cauchy-Schwarz inequality. This proves $B(.,.)$ is a bounded bilinear form. □

Lemma 1.2. There exists a constant $\beta > 0$ such that $\beta \|u\|^2_{H^1} \leq B(u, u)$.

Proof. Putting $u = v$ in Equation (1.8), we obtain $|B(u, u)| = \|u\|_{H^1}$ and so there certainly exists $\beta > 0$ such that

$$\beta \|u\|_{H^1} = \beta |B(u, u)| \leq B(u, u).$$

Any $0 < \beta < 1$ should suffice. This proves coercivity. □

Furthermore, consider the functional $F : H^1(\mathbb{R}^N) \to \mathbb{R}^N$ given by $F(u) = \int_{\mathbb{R}^N} fv\,dx$ $\forall u \in H^1(\mathbb{R}^N)$. If we fix $f \in L^2(\mathbb{R}^N)$ and set $\langle f, v\rangle := (f,v)_{L^2} = \int_{\mathbb{R}^N} fv\,dx$. This is a bounded linear functional defined on $L^2(\mathbb{R}^N)$, and thus on $H^1(\mathbb{R}^N)$. Also, since $B(.,.)$ is a bounded, coercive bilinear form defined on the Hilbert space $H^1(\mathbb{R}^N)$, we are in position to apply the Lax-Milgram Theorem. Then, there must exist a unique element $u \in H^1(\mathbb{R}^N)$ such that

$$B(u,v) = \langle f, v\rangle \quad \forall v \in H^1(\mathbb{R}^N). \tag{1.9}$$

Consequently, since u satisfies (1.9), it is a weak solution of the problem.

From this weak formulation of the problem, we shall make the following definitions.

Let $F(u) = \int_0^u f(t)dt$ and the functional associated to this problem in $H^1(\mathbb{R}^N)$ be $I : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda u^2)dx - \int_{\mathbb{R}^N} F(u)dx. \tag{1.10}$$

In the next section we shall prove that the functional defined in (1.10) is well-defined and is, in fact, of class C^1. In fact, we will prove that the same statement is valid in the more general setting when considering $g(u)$ with conditions (g1)-(g4), that is, the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx - \int_{\mathbb{R}^N} G(u)dx \tag{1.11}$$

is well-defined and is of class C^1. After that, since $I'(u)\phi$ is well-defined, we may use the variational setting and proceed to work with the functional, as weak solutions of problem are precisely the critical points of I, i.e. $I'(u) = 0$.

1.4 The associated functional is of Class C^1

We would like to prove that the functional associated to our original equation is of class C^1, that is, $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ as defined in 1.10 is a continuously differentiable functional. In order to do so, we start from the definitions of the Fréchet derivative and the Gatêaux derivative (see A.5 and A.2), and then use a proposition due to Schwartz (see A.12) that characterizes that a functional $\Phi : X \to \mathbb{R}$ is of class C^1 if, and only if, for every $u \in X$ the Gatêaux derivative $D\Phi(u) : X \to \mathbb{R}$ exists, is a bounded linear operator and the differential operator $D\Phi : X \to X'$ is continuous. Also, the Fréchet derivative and the Gatêaux derivative coincide.

We say a functional $\Phi : X \to \mathbb{R}$ is of class $C^1(A, \mathbb{R})$ if and only if the Fréchet derivative (see A.5) of Φ exists for every point of $u \in A \subset X$ and the application $\Phi' : A \to X'$ is continuous.

Let us first consider the following auxiliary functionals $\Phi, \Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$:

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \tag{1.12}$$

$$\Psi(u) = \int_{\mathbb{R}^N} G(u) dx, \tag{1.13}$$

and also $\hat{\Phi}, \hat{\Psi} : H^1(\mathbb{R}^N) \to \mathbb{R}$:

$$\hat{\Phi}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx \tag{1.14}$$

$$\hat{\Psi}(u) = \int_{\mathbb{R}^N} F(u) dx, \tag{1.15}$$

We will show that these functionals are all continuously differentiable, that is, of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Proposition 1.1. $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$, as defined in (1.12), is a well-defined functional of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Proof. We would like to show that, given $u, v \in H^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$, the limit

$$\lim_{t \to 0} \frac{\Phi(u + tv) - \Phi(u)}{t}$$

exists, which is the Gatêaux derivative $D\Phi(u)$. We shall also prove that the Gatêaux derivative (see A.2) is also linear and bounded, and that the differential operator $D\Phi : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is continuous. Then, from Lemma one gets that Φ is a class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$ functional.

Very well: let $u, v \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Then

$$
\begin{aligned}
\frac{\Phi(u+tv)-\Phi(u)}{t} &= \frac{1}{2t}\left[\int_{\mathbb{R}^N} |\nabla(u+tv)|^2 dx - \int_{\mathbb{R}^N} |\nabla u|^2\right] \\
&= \frac{1}{2t}\left[\int_{\mathbb{R}^N} |\nabla + t\nabla v|^2 dx - \int_{\mathbb{R}^N} |\nabla u|^2\right] \\
&= \frac{1}{2t}\left[\langle \nabla u + t\nabla v, \nabla u + t\nabla v\rangle - \int_{\mathbb{R}^N} |\nabla u|^2\right] \\
&= \frac{1}{2t}\left[\int_{\mathbb{R}^N} \langle \nabla u, \nabla u\rangle + 2t\langle \nabla u, \nabla v\rangle - t^2\langle \nabla v, \nabla v\rangle dx - \int_{\mathbb{R}^N} |\nabla u|^2 dx\right] \\
&= \frac{1}{2t}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) + \frac{2t}{2t}\int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle dx - \frac{t^2}{2t}\int_{\mathbb{R}^N} |\nabla v|^2 dx \\
&= \int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle dx - \frac{t}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx
\end{aligned}
$$

and thus

$$
\lim_{t\to 0} \frac{\Phi(u+tv)-\Phi(u)}{t} = \lim_{t\to 0} \int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle dx - \frac{t}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx = \int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle dx. \quad (1.16)
$$

This means that the Gatêaux derivative $D\Phi(u)$ exists for every $u \in H^1(\mathbb{R}^N)$ and is given by the limit 1.16. Also, note that given $u, v \in H^1(\mathbb{R}^N)$,

$$
|D\Phi(u)v| = |\langle u, v\rangle_H| \leq \|u\|\|v\|. \quad (1.17)
$$

Thus the Gatêaux derivative $D\Phi(u) : H^1(\mathbb{R}^N) \to \mathbb{R}^N$ not only exists for all $u \in H^1(\mathbb{R}^N)$, but it is also linear and bounded for every $u \in H^1(\mathbb{R}^N$.

We now affirm that the differential operator $D\Phi : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ is continuous. Consider a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Note that

$$
\begin{aligned}
\|D\Phi(u_n) - D\Phi(u)\|_{H^{-1}} &= \sup_{\|v\|\leq 1} |D\Phi(u_n)v - D\Phi(u)v| \\
&= \sup_{\|v\|\leq 1} \left|\int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v\rangle dx - \int_{\mathbb{R}^N} \langle \nabla u, \nabla v\rangle dx\right|
\end{aligned}
$$

$$= \sup_{\|v\|\leq 1} \left| \int_{\mathbb{R}^N} \langle \nabla(u_n - u), \nabla v \rangle dx \right|$$
$$= \sup_{\|v\|\leq 1} |D\Phi(u_n - u)v|$$
$$= \sup_{\|v\|\leq 1} \|u_n - u\|\|v\|.$$

Now, since $\|v\| \leq 1$ and $\|u_n - u\| \to 0$ because $u_n \to u$ in $H^1(\mathbb{R}^N)$, then

$$\|D\Phi(u_n) - D\Phi(u)\|_{H^{-1}} \to 0$$

as $n \to 0$. Thus $D\Phi(u_n) \to D\Phi(u)$ in H^{-1} and this proves the differential operator

$$D\Phi : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$$

is continuous. Now applying Lemma, we conclude that Φ is a class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$ functional. □

Remark 1.7. Note that, by the fact that a certain functional P is of class C^1 $\iff$ the Fréchet derivative of P, given by P' exists for every point $u \in H^1(\mathbb{R}^N)$ and the application $P' : H^1((\mathbb{R}^N) \to \mathbb{R}$ is continuous, we obtain the existence of the Fréchet derivative for the functional Φ defined in (1.12). Also, the Fréchet derivative P' and the Gatêaux derivative DP coincide.

Proposition 1.2. $\Psi(u) = \int_{\mathbb{R}^N} G(u)dx$, as defined in (1.13), is a well-defined functional of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Proof. First, we will show that the functional

$$\Psi(u) = \int_{\mathbb{R}^N} G(u)dx$$

is in fact well defined. In order to show that, note that $\int_{\mathbb{R}^N} G(u)dx \in \mathbb{R}$ and thus $G(u) \in L^1(\mathbb{R}^N)$ - this is neeed if one wants the integral to make sense at all. Furthermore, $u \in H^1(\mathbb{R}^N)$, and so (see A.18).

We now show that Ψ is of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$, and for that we will divide the proof in two steps, on the lines of [6]:

(i) for $u, v \in H^1(\mathbb{R}^N)$,

$$\left|\frac{1}{t}\hat{\Psi}(u+tv) - \hat{\Psi}(u) - t\int_{\mathbb{R}^N} g(u)v\right| \to 0 \quad \text{as} \quad t \to 0, \quad t > 0.$$

(ii) if $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$, then

$$\sup_{v\in H^1(\mathbb{R}^N), \|v\|_{H^1(\mathbb{R}^N)}\leq 1} \left|\int_{\mathbb{R}^N} (g(u_n) - g(u))v dx\right| \to 0 \quad \text{as} \quad n \to \infty.$$

For (i), note that

$$\begin{aligned}\left|\frac{1}{t}\Psi(u+tv) - \Psi(u) - t\int_{\mathbb{R}^N} g(u)v\right| &= \left|\frac{1}{t}\int_{\mathbb{R}^N} G(u+tv)dx - \int_{\mathbb{R}^N} G(u)dx - t\int_{\mathbb{R}^N} g(u)vdx\right| \\ &\leq \int_{\mathbb{R}^N} |G(u+tv) - G(u) - tg(u)v)\frac{1}{t}|dx.\end{aligned}$$

Now, it holds that

$$\begin{aligned}\left|\frac{1}{t}(G(u+tv - G(u) - tg(u)v))\right| &\leq \left(\sup_{t\in[0,1]} |g(u+tv)| + |g(u)|\right)|v| \\ &= (C|u| + C|v| + C|u|^l + C|v|^l)|v|,\end{aligned}$$

this because $g(s) \leq C|s| + C|s|^l$, $s \in \mathbb{R}, l = 2^* - 1 = \dfrac{2N}{N-2} - 1$. By putting $h = (C|u| + C|v| + C|u|^l + C|v|^l)|v|$, we then get

$$\left|\frac{1}{t}(G(u+tv - G(u) - tg(u)v))\right| \leq \int_{\mathbb{R}^N} hdx,$$

and note that by the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, one gets $h \in L^1_+(\mathbb{R}^N)$.

For (ii), note that from the proof of (i), we have $u_n \to u$ in $L^{l+1} = L^{2^*}$ since $u_n \to u$ in $H^1(\mathbb{R}^N)$ strongly and the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Thus, passing to a subsequence if necessary, there exists $\hat{u} \in L^{2^*}_+(\mathbb{R}^N), \tilde{u} \in L^2_+(\mathbb{R}^N)$ such that

$$|u|, |u_n| \leq \hat{u} \quad \text{almost everywhere in} \quad \mathbb{R}^N,$$

$$|u|, |u_n| \leq \tilde{u} \quad \text{almost everywhere in} \quad \mathbb{R}^N.$$

Then, for any $R > 0$, one has that

$$\sup_{\|u\|_{H^1(\mathbb{R}^N)}\le 1} \left| \int_{|x|\ge R} (g(u_n) - g(u))v dx \right| \le C\|\tilde{u}\|_{L^2(|x|\ge R)} \left(\sup_{\|v\|_{H^1(\mathbb{R}^N)}\le 1} \|v\|_{L^2(|x|\ge R)} \right) + C\|\hat{u}\|^l_{L^{l+1}(|x|\ge R)} \left(\sup_{\|v\|_{H^1(\mathbb{R}^N)}\le 1} \|v\|_{L^{l+1}(|x|\ge R)} \right).$$

Hence

$$\sup_{v\in H^1(\mathbb{R}^N), \|v\|_{H^1(\mathbb{R}^N)}\le 1} \left| \int_{|x|\ge R} (g(u_n) - g(u))v dx \right| \le C\|\tilde{u}\|_{L^2(|x|\ge R)} + C\|\hat{u}\|^l_{L^{l+1}(|x|\ge R)}.$$

Now, since, $\tilde{u} \in L^2(\mathbb{R}^N)$ and $\hat{u} \in L^{l+1}(\mathbb{R}^N)$, there exists $R_0 > 0$ such that, given $\epsilon > 0$, one has

$$\sup_{v\in H^1(\mathbb{R}^N), \|v\|_{H^1(\mathbb{R}^N)}\le 1} \left| \int_{|x|\ge R} (g(u_n) - g(u))v dx \right| \le \epsilon,$$

as desired.

Thus, from (i) and (ii), we may conclude that the auxiliary functional Ψ is well-defined and of class C^1, which concludes the proof. □

Remark 1.8. Since Φ and Ψ are both class C^1 functionals, as seen on Propositions 1.1 and 1.2, we may then compose these auxiliary functionals to define a new functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$, given by

$$I(u) = \Phi(u) - \Psi(u),$$

which is precisely the associated functional of our problem. I is automatically well-defined and of class C^1.

Proposition 1.3. $\hat{\Phi} : H^1(\mathbb{R}^N) \to \mathbb{R}$, as defined in (1.14), is a well-defined functional of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Proposition 1.4. $\hat{\Psi}(u) = \int_{\mathbb{R}^N} G(u)dx$, as defined in (1.15), is a well-defined functional of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Remark 1.9. Note that, for $G(u) = F(u) - \dfrac{\lambda}{2}u^2$, the functional

$$\begin{aligned}\hat{I}(u) &= \hat{\Phi}(u) - \hat{\Psi}(u)\\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2\,dx - \int_{\mathbb{R}^N} F(u)\,dx.\end{aligned}$$

is automatically well-defined and is of class $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. This comes from the fact that conditions (g1)-(g4) are in fact more general than conditions (f1)-(f4).

To conclude this Chapter, we recall that after having the problem defined, we formulated it in a variational form. This translates the problem to one in critical point theory, namely that of minimization of a certain functional. For this, we used the Lax-Milgram theorem to show that weak solutions of our problem are critical points of the associated functional. Now, instead of trying to solve a nonlinear PDE, which is hard in general, we have the quest to look for minima of this functional. As we will see in the Chapter 2, this functional is such that it is not bounded below and so instead of looking for minima, a method of constrained minimization on an appropriate space will be key to helping us understand what type of critical points we will find.

Chapter 2

A ground state solution

2.1 Preliminaries

In Chapter 1, we were able to successfully show that one can translate our original semilinear elliptic problem into a variational one: the critical points of the associated functional are precisely the weak solutions of the problem. This is a standard approach in critical point theory, for which the natural next step is to minimize the functional on the function space $H^1(\mathbb{R}^N)$.

In this Chapter, we will first show that our functional is neither bounded above or below, and so direct minimization procedures fail. However, in Section 2.4 we will provide a natural constraint known as the Pohozaev set, which will help us better look for solutions to our problem through constrained minimization. With this approach, we follow a result due to Berestycki and Lions [6] in Section 2.3 and prove the existence of a non trivial solution, which is the ground state solution - it has the least energy level among all possible solutions. . In Section 2.4, we show that our functional possesses what is known as a mountain pass geometry. This property will allows us to minimaximize over a suitable class of functions, and the corresponding critical level we will find is called the mountain pass level. We also prove a result that identifies the mountain pass solution with the ground state, that is, the least energy solution. This last result was proved by Jeanjean and Tanaka [18], and will be the heart of the algorithm we will propose later on.

First, consider the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} G(u)\,dx. \tag{2.1}$$

This functional is not bounded due to the presence of the gradient term. It is also not bounded from below, since from condition there exists $v \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(v) > 0$. By the scale change one gets that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} G(v)\,dx = \frac{1}{2}\left(1 - \frac{N-2}{N}\right)\int_{\mathbb{R}^N} |\nabla v|^2\,dx = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx > 0, \tag{2.2}$$

thus

$$I(v_\sigma) = \frac{\sigma^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \sigma^N \int_{\mathbb{R}^N} G(v)\,dx. \tag{2.3}$$

Since $\int_{\mathbb{R}^N} G(v)\,dx > 0$ then we conclude $I(v_\sigma) \to -\infty$ as $\sigma \to +\infty$.

2.2 Pohozaev set

In this section we define what is called the Pohozaev set. We show that it is in fact a manifold in Lemma 2.1 and, as we will see in Lemma 2.2, it is a natural constraint for the associated functional: every critical point of I constrained to $\mathcal{P}$ is also a critical point of the unconstrained functional I. We then show the existence of a solution in Section 2.3 via a method of constrained minimization on a suitable set of functions, and show that it is also a least energy solution. In fact, in Lemma we show that this set has a one-to-one correspondence to the Pohozaev manifold. With this result, we show the minimum of the functional constrained to the Pohozaev set has the least energy level of the associated functional among all possible solutions, in Lemma 2.3.

Definition 2.1. The **Pohozaev set** $\mathcal{P}$ is the set

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus 0 : J(u) = 0\}, \tag{2.4}$$

where the functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$J(u) := (N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx - 2N \int_{\mathbb{R}^N} G(u) dx. \tag{2.5}$$

2.2.1 A Class C^1 manifold

We would now like to prove a few properties of the Pohozaev set in the next Lemma. The definition of manifold of class C^1 used in the Lemma is the following, and can be found in Chapter 6 of reference [3]:

Definition 2.2. Let X be a Hilbert space and $\mathcal{T}$ a set of indices. A topological space M is said to be a **Hilbert manifold of class** C^k modelled on X if there exist both an open covering $\{U_i\}_{i\in\mathcal{T}}$ of M and a family $\psi_i : U_i \to X$ of mappings such that the following conditions hold:

- $V_i = \psi(U_i)$ is open in X and ψ_i is a homeomorphism from U_i to V_i;

- $\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ is of class C^1(continuously differentiable).

Lemma 2.1. Let the functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ be defined as in (2.5). Then

(a) $\{u \equiv 0\}$ is an isolated point of $J^{-1}(\{0\})$, the inverse image of the set $\{0\}$ by the functional J;

(b) There exists $\sigma > 0$ such that $||u|| > \sigma$, for all $u \in \mathcal{P}$;

(c) $\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : J(u) = 0\}$ is closed;

(d) $\mathcal{P}$ is a manifold of class C^1;

(e) There exist positive constants ρ and δ such that $I(u) \geq \rho$ and $|||\nabla u||_2 \geq \delta$, for all $u \in \mathcal{P}$.

Proof. (a) From $g(u) = f(u) - \lambda u$ and $G(u) = \displaystyle\int_0^u g(t)dt$, we have:

$$\begin{aligned}
G(u) &= \int_0^u (f(t) - \lambda t)dt \\
&= \int_0^u f(t)dt - \int_0^u \lambda t dt \\
&= F(u) - F(0) - \lambda \frac{t^2}{2}\bigg|_0^u \\
&= F(u) - \frac{\lambda}{2}u^2,
\end{aligned}$$

since $g(0) = 0 \Rightarrow f(0) = 0 \Rightarrow F(0) = 0$ and we may rewrite $J(u)$ as

$$\begin{aligned} J(u) &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - N\int_{\mathbb{R}^N}\left(F(u) - \frac{\lambda}{2}u^2\right)dx \\ &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - N\int_{\mathbb{R}^N} F(u) + \frac{N}{2}\lambda u^2. \end{aligned}$$

But clearly $N > N-2 \iff \dfrac{N}{2} > \dfrac{N-2}{2}$, and so we get

$$\begin{aligned} J(u) &> \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - N\int_{\mathbb{R}^N} F(u) + \frac{N-2}{2}\lambda u^2 dx \Rightarrow \\ J(u) &> \frac{N-2}{2}\left[\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx\right] - N\int_{\mathbb{R}^N} F(u)dx. \end{aligned}$$

But $\|u\|_{H^1,\lambda} = \displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx$, and so

$$J(u) \geq \frac{N-2}{2}\|u\|_\lambda^2 - N\int_{\mathbb{R}^N} F(u)dx. \tag{2.6}$$

Now, since $F(s) = \displaystyle\int_0^s f(t)dt \geq 0$ and $\displaystyle\lim_{s\to 0}\frac{f(s)}{s} = 0$ (from condition (f_2)), we get

$$\lim_{s\to 0}\frac{F(s)}{s^2} = 0$$

(by L'Hôpital's rule), or equivalently: given $\epsilon > 0$ there exists $\delta > 0$ such that $|s| < \delta$ implies

$$|F(s)| < \frac{\epsilon}{2}|s|^2. \tag{2.7}$$

Also, from condition (f_3), there either exists a positive constant $a > \lambda$ such that

$$(i)\quad \lim_{s\to+\infty}\frac{f(s)}{s} = a,$$

or it holds that

$$(ii)\quad \lim_{s\to+\infty}\frac{f(s)}{s} = +\infty.$$

Let us separate these two cases:

(i) In this case, let us notice that from condition (f4), upon integration we get

$$|f(u)| \leq \epsilon|u| + |u|^{p-1}, \tag{2.8}$$

with $p-1 < 2^*$. Then also

$$|F(u)| \leq \epsilon\frac{|u|^2}{2} + \frac{1}{p}|u|^p\,dx, \tag{2.9}$$

after we which we can affirm there exists a constant $C(p) > 0$ with $2 \leq p \leq 2^*$ such that

$$|F(s)| \leq \frac{\epsilon}{2}|u|^2 + C(p)\,|s|^p\,. \tag{2.10}$$

Now, by the Gagliardo-Nirenberg-Sobolev inequality (see A.15) and the estimates (2.7) and (2.10), one can start back from inequality (2.6) and perform the following further estimates:

$$\begin{aligned}
J(u) &> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - N\int_{\mathbb{R}^N} F(u)dx \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - N\int_{\mathbb{R}^N} \frac{\epsilon}{2}\,|u|^2 + C(p)\,|u|^p\,dx \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - \frac{N\epsilon}{2\lambda}dx\int_{\mathbb{R}^N} \lambda u^2 dx - NC(p)\int_{\mathbb{R}^N} |u|^p dx \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - \frac{N\epsilon}{2\lambda}\int_{\mathbb{R}^N} \lambda u^2 dx - NC(p)\|u\|^p_p \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - \frac{N\epsilon}{2\lambda}\int_{\mathbb{R}^N} \lambda u^2 dx - NC_1(p)\|\nabla u\|^2_2 \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - \frac{N\epsilon}{2\lambda}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda u^2\right)dx - NC_1(p)\left(\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx\right) \\
&= \frac{1}{2}\left(N-2-\frac{N\epsilon}{\lambda}\right)\|u\|^2_\lambda - NC_1(p)\|u\|^2_{H^1,\lambda} \\
&\geq \frac{1}{2}\left(\frac{\lambda\,(N-2)-N\epsilon}{\lambda}\right)\|u\|^2_\lambda - NC_1(p)\|u\|^p_{H^1,\lambda}.
\end{aligned}$$

First, denote $C := C_1(p)$ for simplification. Then, by taking $\epsilon > 0$ sufficiently small and $0 < \rho < 1$ such that $\lambda(N-2) - N\epsilon > 0$ and $\rho^p < \frac{1}{4NC}\left(N - 2 - \frac{N\epsilon}{\lambda}\right)\rho^2$, one has that $\|u\|_{H^1,\lambda} = \rho \Rightarrow$

$$
\begin{aligned}
J(u) &> \frac{1}{2}\left(\frac{\lambda\,(N-2) - N\epsilon}{\lambda}\right)\rho^2 - NC\rho^p \\
&> \frac{1}{2}\left(\frac{\lambda\,(N-2) - N\epsilon}{\lambda}\right)\rho^2 - NC\left[\frac{1}{4NC}\left(N - 2 - \frac{N\epsilon}{\lambda}\right)\rho^2\right] \\
&= \frac{1}{2}\left(\frac{\lambda\,(N-2) - N\epsilon}{\lambda}\right)\rho^2 - \frac{1}{4}\left(\frac{\lambda(N-2) - N\epsilon}{\lambda}\right)\rho^2 \\
&= \frac{1}{2}\left(\frac{\lambda\,(N-2) - N\epsilon}{\lambda}\right)\rho^2 \\
&> 0.
\end{aligned}
$$

Therefore, $J(u) > 0$ and so $u = 0$ is isolated. The claim is proved for Case (i).

(ii) For this case, note first that as in Case (i), we have the condition $\lim_{s\to 0} \frac{f(s)}{s} = 0$ upon our nonlinearity f, which again by L'Hôpital's rule renders $\lim_{s\to 0} \frac{F(s)}{s^2} = 0$, and so given $\epsilon > 0$, there exists $\delta > 0$ such that $|s| < \delta \Rightarrow$

$$
\left|\frac{F(s)}{s^2} - 0\right| < \epsilon \Rightarrow |F(s)| < \epsilon\,|s|^2\,. \tag{2.11}
$$

Also, from the second condition in (f_3), we have that $\lim_{s\to 0} \frac{f(s)}{s} = +\infty$ implies (L'Hôpital's rule) $\lim_{s\to 0} \frac{F(s)}{s^2} = +\infty$, or equivalently: given $\epsilon > 0$ there exists $\delta > 0$ such that $|s| > \delta \Rightarrow$

$$
\frac{|s|}{\delta} < 1 \Rightarrow \left(\frac{|s|}{\delta}\right)^2 > \left(\frac{|s|}{\delta}\right)^p. \tag{2.12}
$$

Putting together the conditions $|s| < \delta$ and $|s| > \delta$, we use the inequalities obtained in (2.11) and (2.12) to get that, for all $|s| > 0$:

$$
\begin{aligned}
|F(s)| &< \epsilon |s|^2 + \frac{1}{\delta^p} |s|^p \\
&= \epsilon\delta^2 \left(\frac{|s|}{\delta}\right)^2 + \frac{1}{\delta^p} |s|^p \\
&< \epsilon \frac{\delta^2}{\delta^p} |s|^p + \frac{1}{\delta^p} |s|^p \\
&= a_1(\delta,\epsilon) |s|^p + a_2(\delta) |s|^p \\
&= C_1(\delta,\epsilon) |s|^p . \qquad (2.13)
\end{aligned}
$$

Then, given $\epsilon > 0$ and $2 \le p \le 2^*$, estimate (2.13) holds true for some $C(p) > 0$:

$$|F(s)| < C(p) |s|^p . \qquad (2.14)$$

Now, by the Sobolev embedding $H^1\left(\mathbb{R}^N\right) \hookrightarrow L^{\frac{2N}{N-2}}$ and the estimate (2.13), one has that, following from (2.6):

$$
\begin{aligned}
J(u) &\ge \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - N\int_{\mathbb{R}^N} F(u)dx \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - N\int_{\mathbb{R}^N} C(p)|u|^p dx \\
&= \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC(p)\|u\|^p_{L^p} \\
&> \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC(p)\|\nabla u\|^2_{L^2} \\
&\ge \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC(p)S\left(\|\nabla u\|^2_{L^2} + \lambda\|u\|^2_{L^2}\right) \\
&= \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC_1(p)\left(\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx\right) \\
&= \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC_1(p)\|u\|^2_{H^1,\lambda} \\
&\ge \frac{N-2}{2}\|u\|^2_{H^1,\lambda} - NC_1(p)\|u\|^p_{H^1,\lambda}.
\end{aligned}
$$

Before proceeding with the calculations, let us put $C := C_1(p)$ for simplification. Then, by taking ϵ sufficiently small and $0 < \rho < 1$ such that $\frac{N-2}{2} > 0$, which holds trivially since $N \ge 3 \iff \frac{N-2}{2} \ge \frac{1}{2} > 0$, and $\rho^p < \frac{1}{4NC}(N-2)\rho^2$, we will have that $\|u\|_{H^1,\lambda} = \rho \Rightarrow$

$$\begin{aligned}
J(u) &> \frac{N-2}{2}\rho^2 - NC\rho^p \\
&> \frac{N-2}{2}\rho^2 - NC\left(\frac{1}{4NC}(N-2)\rho^2\right) \\
&= \frac{N-2}{2}\rho^2 - \frac{1}{4}(N-2)\rho^2 \\
&= \left(\frac{1}{2}-\frac{1}{4}\right)(N-2)\rho^2 \\
&= \frac{1}{2}\frac{N-2}{2}\rho^2 \\
&> 0.
\end{aligned}$$

Thus $J(u) > 0$. The claim is proved for Case (ii) and so $\{u \equiv 0\}$ is indeed an isolated point of $J^{-1}(\{0\})$.

(b) Since $\{u \equiv 0\}$ is an isolated point of $J^{-1}(0)$, there must be a ball $||u|| \leq \sigma$ which does not intersect $\mathcal{P}$ for some $\sigma > 0$. Then, for such a $\sigma > 0$ it holds $||u|| > \sigma$ for all $u \in \mathcal{P}$.

(c) Note that $J(u)$ is a class C^1 functional (thus continuous) and so $\mathcal{P} \cup \{0\} = J^{-1}(\{0\})$ is a closed subset, for it is the inverse image of a closed set by a continuous functional. Moreover, $\{u \equiv 0\}$ is an isolated point in $J^{-1}(\{0\})$ and we conclude that $\mathcal{P}$ is a closed set.

(d) Let us consider the derivative of $J(u)$ applied in u:

$$\langle J'(u), u\rangle = J'(u)u = (N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx - N\int_{\mathbb{R}^N}\left(f(u)u - \lambda u^2\right)dx \tag{2.15}$$

Since $u \in \mathcal{P}$, it follows that

$$(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 dx = 2N\int_{\mathbb{R}^N} G(u)dx.$$

Substituting this last expression back into the first term of Equation (2.15), we have:

$$\begin{aligned}
J'(u)u &= 2N\int_{\mathbb{R}^N} G(u)dx - N\int_{\mathbb{R}^N}\left(f(u)u-\lambda u^2\right)dx \\
&= 2N\int_{\mathbb{R}^N}\left(F(u)-\frac{\lambda}{2}u^2\right)dx - N\int_{\mathbb{R}^N}\left(f(u)u-\lambda u^2\right)dx \\
&= 2N\int_{\mathbb{R}^N} F(u)dx - N\int_{\mathbb{R}^N}(f(u)u)dx - N\lambda\int_{\mathbb{R}^N}u^2dx + N\lambda\int_{\mathbb{R}^N}u^2dx \\
&= 2N\int_{\mathbb{R}^N}\left(F(u)-\frac{1}{2}f(u)u\right)dx.
\end{aligned}$$

From condition (f_5), $\frac{1}{2}f(u)u - F(u) > 0 \;\; \forall u \in \mathbb{R}^+ \setminus \{0\}$, and so $J'(u)u < 0 \;\; u \in \mathcal{P}$. This shows $\mathcal{P}$ is a class C^1 manifold.

(e) Finally, we shall prove the last item. If $u \in \mathcal{P}$, it holds

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^2\,dx &= \frac{2N}{N-2}\int_{\mathbb{R}^N} G(u)dx \\
&= \frac{2N}{N-2}\int_{\mathbb{R}^N}\left(F(u)-\lambda\frac{u^2}{2}\right)dx \\
&= \frac{2N}{N-2}\int_{\mathbb{R}^N}F(u)dx - \frac{\lambda N}{N-2}\int_{\mathbb{R}^N}u^2dx \\
&= 2^*\int_{\mathbb{R}^N}F(u)dx - \frac{\lambda N}{N-2}\int_{\mathbb{R}^N}u^2\,dx
\end{aligned} \tag{2.16}$$

From the assumptions (f_1), (f_2), (f_4), given $\epsilon > 0$. there exists $C_1 = C_1(\epsilon) > 0$ such that

$$|F(u)| \leq \frac{\epsilon}{2}u^2 + C_1|u|^p, \;\; 2 < p < 2^*. \tag{2.17}$$

Using inequality (2.17) with $\epsilon < 1/2$ and the Gagliardo-Nirenberg-Sobolev inequality (see A.15) , we perform the following estimates in (2.16):

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &\le 2^* \int_{\mathbb{R}^N} \left(\frac{\epsilon}{2}u^2 + C_1|u|^p\right)\,dx - \frac{\lambda N}{N-2}\int_{\mathbb{R}^N} u^2\,dx \Rightarrow \\ ||\nabla u||_2^2 &\le 2^*\left(\frac{\epsilon}{2}||u||_2^2 + C_1||u||_{2^*}^{2^*}\right) - \frac{\lambda N}{N-2}||u||_2^2 \\ &= 2^*\lambda\epsilon\|u\|_2^2 - \frac{1}{2}\frac{2N}{N-2}\lambda\|u\|_2^2 + 2^*C_1\|u\|_{2^*}^{2^*} \\ &= 2^*\lambda\epsilon\|u\|_2^2 - 2^*\frac{\lambda}{2}\|u\|_2^2 + 2^*C_1\|u\|_{2^*}^{2^*} \\ &\le 2^*\lambda\left(\epsilon - \frac{1}{2}\right)\|u\|_2^2 + 2^*C_1S^{-1}\|\nabla u\|_2^{2^*},\end{aligned}$$

with $S := S(2, N)$. Also, one certainly has $\left(\epsilon - \frac{1}{2}\right) < 0$ from our choice for ϵ, then:

$$\begin{aligned}||\nabla u||_2^2 &< 2^*C_1S^{-1}\|\nabla u\|_2^{2^*} \Rightarrow \\ ||\nabla u||_2^{2-2^*} &< 2^*C_1S^{-1} \Rightarrow \\ ||\nabla u||_2 &< \left(2^*C_1S^{-1}\right)^{\frac{1}{2-2^*}}.\end{aligned} \tag{2.18}$$

Inverting inequality (2.18) and putting $\delta := \left(\frac{S}{2^*C_1}\right)^{\frac{1}{2^*-2}}$, note that we get

$$\delta < \|\nabla u\|_2. \tag{2.19}$$

We are left to check whether the ρ given in (e) actually exists. Dividing (2.16) by 2^* and substituting into the functional $I(u)$ gives us:

$$\begin{aligned}\frac{1}{2^*}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &= \int_{\mathbb{R}^N} F(u)\,dx - \frac{\lambda N}{N-2}\frac{N-2}{2N}\int_{\mathbb{R}^N} u^2\,dx \Rightarrow \\ -\frac{1}{2^*}\int_{\mathbb{R}^N} |\nabla u|^2\,dx &= -\int_{\mathbb{R}^N} F(u) + \frac{\lambda}{2}u^2\,dx,\end{aligned}$$

and so

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} u^2\, dx - \int_{\mathbb{R}^N} F(u)\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |\nabla u|^2\, dx \\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} |\nabla u|^2\, dx \\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|\nabla u\|_2^2 \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\delta := \rho. \qquad (2.20)
\end{aligned}$$

From this we conclude there exists ρ such that $I(u) \geq \rho\ \forall u \in \mathcal{P}$. This concludes the proof of (e) and thus, of Lemma 2.1.

□

2.2.2 A natural constraint

This next Lemma shows that the Pohozaev manifold $\mathcal{P}$ is a natural constraint to the associated functional. Since our variational method looks for critical points in order to find solutions of the semilinear elliptic PDE, this result helps us in the sense that constrained minimization on this subset of the function space $H^1(\mathbb{R}^N)$ will allow us to minimaximize on the right function space, since on $H^1(\mathbb{R}^N)$, the associated functional is neither bounded above nor below, as we have seen previously, and so one can not proceed with direct minimization/maximization procedures on $H^1(\mathbb{R}^N)$.

Lemma 2.2. A function $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ is a critical point of $I \iff$ u is a critical point of I restricted to $\mathcal{P}$.

Proof. Let $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ be a critical point of the functional $I|_{\mathcal{P}}$, i.e. of I restricted to the Pohozaev manifold. By the Lagrange multiplier theorem on Banach spaces (see A.6), let us consider $I : H^1(\mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$ and $J : H^1(\mathbb{R}^{\mathbb{N}}) \to \mathbb{R}$. I is a class C^1 Fréchet differentiable functional which has a local extremum $I(u) = \min\{I(v) : v \in B_r(x_0), J(v) = 0\}$ at the regular point u, and so there must exist an element $\eta \in \mathbb{R}$ such that the lagrangian functional L, given by

$$L(v) = I(v) + \langle J(v), \eta\rangle \quad \forall v \in H^1(\mathbb{R}^N),$$

is stationary at u, i.e.

$$I'(u) + \langle J'(u), \eta \rangle = 0 \Rightarrow I'(u) + \eta J'(u) = 0. \tag{2.21}$$

Applying u on Equation (2.21), we get:

$$\langle I'(u), u \rangle + \langle \eta J'(u), u \rangle = 0. \tag{2.22}$$

Since $N \geq 3$, this is equivalent to

$$0 = \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 \, dx - \int_{\mathbb{R}^N} f(u)u \, dx \tag{2.23}$$

$$+ \eta \left(2(N-2) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2N \int_{\mathbb{R}^N} (f(u)u - \lambda u^2) \, dx \right). \tag{2.24}$$

The expression (2.23) yields the following Euler-Lagrange equation:

$$-\Delta u + \lambda u - f(u) + 2\eta \left(-(N-2)\Delta u + \lambda N u - N f(u) \right) = 0,$$

which, after rearrangement of the terms, becomes

$$-(1 + 2\eta(N-2))\Delta u + \lambda(1 + 2\eta N)u = (1 + 2\eta N) f(u). \tag{2.25}$$

Note that Equation (2.25) has the Pohozaev manifold associated with it, given by $\mathcal{H}^{-1}(\{0\})$, where

$$\mathcal{H}(u) := \frac{(1 + 2\eta(N-2))(N-2)}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - N \int_{\mathbb{R}^N} \left((1 + 2\eta N) F(u) - \lambda \frac{(1 + 2\eta N)}{2} u^2 \right) dx.$$

We may then rewrite $\mathcal{H}$ as:

$$\mathcal{H}(u) = \frac{(1 + 2\eta(N-2))(N-2)}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - N(1 + 2\eta N) \int_{\mathbb{R}^N} \left(F(u) - \lambda \frac{u^2}{2} \right) dx. \tag{2.26}$$

However, since $u \in \mathcal{P}$, then $J(u) = 0$, and thus, by (2.26),

$$\mathcal{H}(u) = -2\eta(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \, .$$

Finally, since u is a solution of Equation (2.25) it satisfies $\mathcal{H}(u) = 0$. Thus, we obtain

$$\eta(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 0 \, .$$

To conclude, from $N \geq 3$ and $\int_{\mathbb{R}^N} |\nabla u|^2 dx > 0$, we have $\eta = 0$. This means that Equation (2.21) given by the Lagrange Multiplier Theorem is actually $I'(u) = 0$ and so u is a critical point of the unconstrained functional I. This finishes the proof of the Lemma.

□

2.3 Existence of a least energy solution

In the present Section, we shall prove the existence of a solution to our problem 1.1. This is done by a constrained minimization technique, rather than directly looking for critical points of the associated energy functional I. This approach was first introduced by Coleman, Glazer and Martin [11], the likes of which inspired the tackling of the problem by Berestycki and Lions [6]. More precisely, we consider the constrained minimization problem below:

$$\text{minimize the set } S = \{\|\nabla u\|_2^2; u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u)\, dx = 1\}. \tag{2.27}$$

After showing the existence of a nontrivial solution to problem 1.1, we show there exists a bijection between this set (2.27) and the Pohozaev manifold in Lemma 2.4. Such a Lemma will be crucial to later show in Lemma 2.2 that $m = \inf_{u \in \mathcal{P}} I(u) = \inf_{u \in S} I(u)$ - this means that the least energy solution found by means of Theorem 2.1 is the minimum of the functional constrained to the Pohozaev manifold.

Theorem 2.1. Suppose $N \geq 3$ and let the nonlinearity g satisfy conditions (g1)-(g4) defined on Chapter 1. Under these hypotheses, the constrained minimization problem (2.27) has a solution $u \in H^1(\mathbb{R}^N)$. Also, there exists a Lagrange multiplier $\theta > 0$ such that u satisfies, at least in the distribution sense, the equation

$$-\Delta u = \theta g(u) \quad \text{in} \quad \mathbb{R}^N. \tag{2.28}$$

Then $u_{\sqrt{\theta}}$ is a solution of the original problem (1.1).

Proof. We divide our proof into 5 steps:

Step 1. S is a non-empty set

Step 2. Selection of an adequate minimizing sequence

Step 3. Estimates for u_n

Step 4. Passage to the limit

Step 5. Conclusion.

Step 1. S is a non-empty set.

Consider the open ball $B_1(0)$ and, for a given $0 < \epsilon < 1$, define $A_\epsilon = B_{1+\epsilon}(0) \setminus B_1(0)$.

From hypothesis (g1) on the nonlinearity g, there exists a $\zeta > 0$ such that $G(\zeta) > 0$. For $R > 1$, define the auxiliary function u_ζ, where

$$u_\zeta = \begin{cases} \zeta, & x \in B_1(0), \\ 0, & x \in B^c_{1+\epsilon}(0). \end{cases} \tag{2.29}$$

Then u_ζ clearly belongs to $H^1(\mathbb{R}^N)$. Also, $u_\zeta(x) = u_\zeta(|x|)$ is a continuous, non increasing function of $r = |x|$. Since $0 < u_\zeta(x) < \zeta$ by construction, then $|G(u_\zeta(x)| < C$. Furthermore,

$$\int_{\mathbb{R}^N} G(u_\zeta) dx = \int_{B_{1+\epsilon}(0)} G(u_\zeta dx = \int_{B_1(0)} G(u_\zeta) dx + \int_{A_\epsilon} G(u_\zeta) dx, \tag{2.30}$$

$$\int_{B_1(0)} G(u_\zeta) dx = G(\zeta) meas(B_1(0)) > 0, \quad \text{and} \tag{2.31}$$

$$\left| \int_{A_\epsilon} G(u_\zeta) dx \right| \leq \int_{A_\epsilon} |G(u_\zeta)| \, dx \leq C meas(A_\epsilon) = C\epsilon, \tag{2.32}$$

where $meas(\Omega)$ denotes the Lebesgue measure of the set Ω.

From expression (2.32), we get that

$$-C\epsilon \leq -\int_{A_\epsilon} |G(u_\zeta)| \, dx \leq \left| \int_{A_\epsilon} G(u_\zeta) dx \right|. \tag{2.33}$$

Thus, going back to expression (2.30) by using expressions (2.31) and (2.33), one gets:

$$\int_{\mathbb{R}^N} G(u_\zeta)dx = \int_{B_2(0)} G(u_\zeta)dx \geq G(\zeta)meas(B_1(0)) - C\epsilon > 0. \tag{2.34}$$

Note that we can similarly take the balls with increasing radii $B_R(0)$ and define the auxiliary functions u_{ζ_R} for each $R > 1$:

$u_{\zeta_R} =$

$$\begin{cases} \zeta, & x \in B_R(0), \\ 0, & x \in B^c_{R+\epsilon}(0). \end{cases} \tag{2.35}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} G(u_{\zeta_R})dx = \int_{B_{R+1}(0)} G(u_{\zeta_R})dx \\ \geq G(\zeta)meas(B_R(0)) - |B_{R+1}(0) - B_R(0)| \left(\max_{s\in(0,\zeta]} |G(s)| \right) \quad (2.36) \\ > 0. \quad (2.37) \end{aligned}$$

Hence there exist constants $C, C^{'} > 0$ such that:

$$\int_{\mathbb{R}^N} G(u)dx \geq CR^N - C^{'}R^{N-1}, \tag{2.38}$$

and so by taking $R > 0$ large enough, $\int_{\mathbb{R}^N} G(u_{\zeta_R})dx > 0$. If we rename $u := u_{\zeta_R}$ and introduce a scale change on ζ_R, namely $u_t(x) = u\left(\frac{x}{t}\right)$, we have

$$\int_{\mathbb{R}^N} G(u_t)dx = t^N \int_{\mathbb{R}^N} G(u)dx. \tag{2.39}$$

Thus, for an appropriate choice of $t > 0$, we get $\int_{\mathbb{R}^N} G(u_t)dx = 1$. Since this happens for some $u \in H^1(\mathbb{R}^N)$, we conclude that the set S is not empty.

Step 2. Selection of an adequate minimizing sequence

First, there exists a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(u_n)dx = 1$ and

$$\lim_{n\to\infty} \|\nabla u_n\|_2^2 = T = \inf\{\|\nabla v\|_2^2; v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u_n)dx = 1\} \geq 0.$$

Let u_n^* denote the Schwarz spherical rearrangement of $|u_n|$ (see A.3). u_n^* is also known as the Schwarz symmetrized function of u_n. We will replace (u_n) for the Schwarz spherical rearrangement (u_n^*) we will assume from now on that, for all n, u_n is nonnegative, spherically symmetric and nonincreasing with $r = |x|$.

Step 3. Estimates for u_n

We will now show that $\|u_n\|_{H^1(\mathbb{R}^N)}$ is bounded. For $s \geq 0$, define the following functions:

$$g_1(s) = (g(s) + ms)^+ = \max(g(s) + ms, 0),$$
$$g_2(s) = g_1(s) - g(s).$$

The notation $a^+ = \max(a, 0)$ stands for the positive part of a.

Then, extend g_1 and g_2 as odd functions for $s \leq 0$. This gives $g = g_1 - g_2$, with $g_1, g_2 \geq 0$ on $\mathbb{R}^+$, and also

$$g_1(s) = o(s) \quad \text{as} \quad s \to 0. \tag{2.40}$$
$$\lim_{s\to\infty} \frac{g_1(s)}{s^l} = 0, \quad \text{where} \quad l = \frac{N+2}{N-2}; \tag{2.41}$$
$$g_2(s) \geq ms \quad \forall s \geq 0 \tag{2.42}$$

Let $G_i(\tau) = \int_0^\tau g_i(s)ds, i = 1, 2$. From (2.40), (2.41) and (2.42), we see that for a given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$g_1(s) \leq C_\epsilon s^l + \epsilon g_2(s), \quad \forall s \geq 0. \tag{2.43}$$

Upon integration of Equation (2.43), one gets

$$G_1(s) \leq C_\epsilon |s|^{l+1} + \epsilon G_2(s), \quad \forall s \in \mathbb{R}. \tag{2.44}$$

Now, since $\|\nabla u_n\|_2^2 \to T$, then $\|\nabla u_n\|_2$ is bounded, which in turn implies by the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ A.13 that

$$\|u_n\|_{2^*} \leq C, \tag{2.45}$$

where $2^* = l + 1 = \dfrac{2N}{N-2}$.

Thus, if we write $\int_{\mathbb{R}^N} G(u_n) = 1$ in the form

$$\int_{\mathbb{R}^N} G_1(u_n)dx = \int_{\mathbb{R}^N} G_2(u_n)dx + 1 \tag{2.46}$$

and using inequalities (2.44) and (2.45), we arrive at

$$\int_{\mathbb{R}^N} G_2(u_n)dx + 1 \leq C_\epsilon \int_{\mathbb{R}^N} |u_n|^{l+1}dx + \epsilon \int_{\mathbb{R}^N} G_2(u_n)dx \tag{2.47}$$

$$= C_\epsilon \|u_n\|_{l+1}^{l+1} + \epsilon \int_{\mathbb{R}^N} G_2(u_n)dx \tag{2.48}$$

$$= C_\epsilon C + \epsilon \int_{\mathbb{R}^N} G_2(u_n)dx. \tag{2.49}$$

In consequence, note that since inequality (2.44) holds for any given $\epsilon > 0$, we may choose $\epsilon = 1/2$ in it, and so

$$\begin{aligned} &\int_{\mathbb{R}^N} G_2(u_n)dx + 1 \leq \bar{C} + \frac{1}{2}\int_{\mathbb{R}^N} G_2(u_n)dx \Rightarrow \\ &\frac{1}{2}\int_{\mathbb{R}^N} G_2(u_n)dx + 1 \leq \bar{C} \Rightarrow \\ &\int_{\mathbb{R}^N} G_2(u_n)dx \leq \frac{\bar{C}-1}{2} \leq \bar{C}, \end{aligned} \tag{2.50}$$

where we put $\bar{C} = C_\epsilon C$.

Finally, upon integrating inequality (2.42) from 0 to τ one readily obtains

$$G_2(\tau) \geq \frac{m}{2}\tau^2. \tag{2.51}$$

Then, after integrating Equation (2.51) on the whole space $\mathbb{R}^N$ and using inequality (2.50), we arrive at

$$\frac{m}{2}\int_{\mathbb{R}^N} u_n^2 dx \leq \int_{\mathbb{R}^N} G_2(u_n)dx \leq \bar{C}. \tag{2.52}$$

Thus $\|u_n\|_2^2 \leq \dfrac{2}{m}\bar{C}$, which implies $\|u_n\|_{H^1(\mathbb{R}^N)}$ to be bounded (recall that the boundedness of $\|\nabla u_n\|_2$ was obtained above in Step 2.) Furthermore, by making use of Hölder's inequality (see A.5) with conjugate exponents $q = q' = 2$, we have that since $u_n \in L^q(\mathbb{R}^N) = L^{q'}(\mathbb{R}^N) = L^2(\mathbb{R}^N)$, then $u_n^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |u_n^2| dx \leq \|u_n\|_2 \|u_n\|_2. \tag{2.53}$$

Thus by the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $2 \leq p \leq 2^*$ (see A.13) we have $\|u_n\|_{L^p(\mathbb{R}^N)} \leq C$ for any $2 \leq p \leq 2^*$.

Step 4. Passage to the limit

First, note that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n, because u_n is radial, non decreasing and also bounded in $L^2(\mathbb{R}^N)$, and thus $|u_n(x)| \leq C|x|^{\frac{-N}{2}}, \quad x \in \mathbb{R}^N$, with C independent of n. This is the Radial Lemma of Strauss (see A.17).

Now, since (u_n) is bounded in $H^1(\mathbb{R}^N)$, then we may extract a subsequence of (u_n), namely (u_{n_k}), such that it converges weakly in $H^1(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$ to a function $u \in H^1(\mathbb{R}^N)$, that is, $u_{n_k} \rightharpoonup u$. For simplification purposes, this subsequence will be renamed as u_n. Observe also that $u \in H^1(\mathbb{R}^N)$ is spherically symmetric and increasing with $r = |x|$.

Let us define $Q(s) = s^2 + |s|^{2^*-1}$. Since $g_1(s) = o(s)$ as $s \to 0$ and $\lim\limits_{s\infty} \dfrac{g_1(s)}{s^{2^*-1}} = 0$, then note that upon integration we obtain

$$\lim_{s\to 0} \frac{G_1(s)}{s^2} = 0 \quad \text{and} \lim_{s\to\infty} \frac{G_1(s)}{s^2} = 0.$$

Then, it holds that

$$\lim_{s\to\infty} \frac{G_1(s)}{Q(s)} = \lim_{s\to 0} \frac{G_1(s)}{s^2 + |s|^{2^*}} = 0.$$

Also,

$$\sup_n \int_{\mathbb{R}^N} Q(u_n)dx < +\infty,$$
$$G_1(u_n) \to G_1(u) \quad \text{as} \quad n \to \infty \quad ,$$
$$u_n(x) \to 0 \quad \text{as} \quad |x| \to +\infty \quad \text{uniformly in n.} \tag{2.54}$$

Therefore, if we suppose (w_n) is a sequence of functions $\mathbb{R}^N \to \mathbb{R}$ belonging to a bounded set of $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, $1 < p < q < +\infty$, and suppose also that w_n converges strongly to some w almost everywhere in $\mathbb{R}^N$, and that w_n as well as w satisfies (2.54). Then by Fatou Lemma (see A.3), $w \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, and applying the compactness lemma of Strauss (see A.16) with $P(s) = |s|^r, Q(s) = |s|^p + |s|^q, u_n = w_n - w$, we conclude that $w_n \to w$, i.e. converges strongly, in $L^r(\mathbb{R}^N)$ for all $r \in (p, q)$.

Thus

$$\int_{\mathbb{R}^N} G_1(u_n)dx \to \int_{\mathbb{R}^N} G_1(u)dx \quad \text{as} \quad n \to \infty. \tag{2.55}$$

Now, $(G_2(u_n))\,dx$ is a sequence of functions in L^1 that satisfy

(a) $\forall n$, $G_2(u_n) \geq 0$ almost everywhere

(b) $\sup_n \int_{\mathbb{R}^N} G_2(u_n) < \infty.$

For almost all $x \in \mathbb{R}^N$ we set $G_2(u) = \liminf G_2(u_n)$, so by Fatou Lemma (A.3), $G_2(u) \in L^1$ and

$$\int_{\mathbb{R}^N} G_2(u)dx \leq \liminf \int_{\mathbb{R}^N} G_2(u_n)dx.$$

Now, by writing $\int_{\mathbb{R}^N} G_1(u_n)dx = \int_{\mathbb{R}^N} G_2(u_n)dx + 1$ and taking the limit, we have that

$$\lim \int_{\mathbb{R}^N} G_1(u_n)dx \leq \liminf \int_{\mathbb{R}^N} G_2(u_n)dx + 1 \leq \lim \int_{\mathbb{R}^N} G_2(u_n)dx + 1 \to \tag{2.56}$$

$$\int_{\mathbb{R}^N} G_1(u)dx \leq \int_{\mathbb{R}^N} G_2(u)dx + 1. \tag{2.57}$$

Thus, from inequality (2.57) and from $G(u) = G_1(u) - G_2(u)$, we have that

$$\int_{\mathbb{R}^N} G_(u)dx \leq 1.$$

We also know that $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq \lim |\nabla u_n|^2 = T$. Suppose, by contradiction, that $\int_{\mathbb{R}^N} G(u)dx > 1$. Then, by performing a scale change $u_t(x) = u\left(\frac{x}{t}\right)$, we have

$$\int_{\mathbb{R}^N} |\nabla u_t|^2 \, dx = t^{N-2} \int_{\mathbb{R}^N} |\nabla u| \, dx \leq t^{N-2} T.$$

It also holds that $\int_{\mathbb{R}^N} |\nabla u_t|^2 \, dx \geq T$, by the very definition of T given earlier. This means that

$$T \leq \int_{\mathbb{R}^N} |\nabla u_t|^2 \, dx \leq t^{N-2} T \Rightarrow T = 0. \tag{2.58}$$

Thus $\int_{\mathbb{R}^N} |\nabla u| \, dx = 0$, which in turn implies $u = 0$, which contradicts $\int_{\mathbb{R}^N} G(u)dx > 0$. Then it must be that

$$\int_{\mathbb{R}^N} G(u)dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = T > 0. \tag{2.59}$$

$u \in H^1(\mathbb{R}^N)$ is a solution of the minimization problem.

Step 5. Conclusion Note that the functionals $P = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ and $V = \int_{\mathbb{R}^N} G(u)dx$ are both C^1 functionals well-defined on $H^1(\mathbb{R}^N)$ (see Lemma of Chapter 1, Section 4) and so there exists a Lagrange multiplier θ such that $\frac{1}{2}P'(u) = \theta V'(u)$.

We affirm that it must be $\theta > 0$. Certainly $\theta \neq 0$ since $\theta = 0 \Rightarrow u = 0$. Suppose, by contradiction, that < 0. Note that $V'(u) \neq 0$, otherwise $V'(u) = 0 \iff g(u) = 0 \iff u = 0 \iff V(u) = 0$, a contradiction because $V(u) = 1$.

Thus, consider a function $w \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\langle V'(u), w \rangle = \int_{\mathbb{R}^N} g(u)wdx > 0.$$

Since $V(u + \epsilon w) \approx V(u) + \epsilon < V'(u), w >$ and $P(u + \epsilon w) \approx P(u) + 2\epsilon\theta < V'(u), w >$ for $\epsilon \to 0$ and $\theta > 0$, one can find $\epsilon > 0$ sufficiently small such that $u + \epsilon w$ satisfies $V(v) > V(u) = 1$ and $P(v) < P(u) = T$. Again, by a scale change, there exists t_0,

$0 < t_0 < 1$ such that $V(v_{t_0}) = 1$ and $P(v_{t_0}) < T$, which is an absurd. Hence $\theta > 0$. Thus $u \in H^1(\mathbb{R}^N)$ satisfies the equation $-\Delta u = \theta g(u)$ in $\mathbb{R}^N$, and so $u\left(\frac{\cdot}{\sqrt{\theta}}\right) = u_{\sqrt{\theta}}$ is a solution to the problem.

□

Lemma 2.3. Let u denote a solution of 1.1. Then for any solution v of 1.1 one has $0 < I(u) \leq I(v)$.

Proof. First, by g we mean the truncated g, that is $\tilde{g}$. Let $\overline{u}$ be the solution of the constrained minimization problem obtained before in Theorem 2.6, so that
$\int_{\mathbb{R}^N} G(\overline{u})\,dx = 1$ and $\|\overline{u}\|_2^2 = \min\{\|\nabla u\|_2^2; u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u)\,dx = 1\}$.
Then, there exists $\theta > 0$ such that $-\Delta\overline{u} = \theta g(\overline{u})$ in $\mathbb{R}^N$, and u is defined by $u = \overline{u}_{\sqrt{\theta}}$.

By Pohozaev identity (see A.9) one has

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx = N\int_{\mathbb{R}^N} G(u)\,dx \Rightarrow \int_{\mathbb{R}^N} |\nabla u|^2\,dx = \frac{2N}{N-2}\int_{\mathbb{R}^N} G(u)\,dx.$$

From the scale change relations,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx = \theta^{\frac{N-2}{2}}\int_{\mathbb{R}^N} |\nabla\overline{u}|^2\,dx$$

and

$$\int_{\mathbb{R}^N} G(u)\,dx = \theta^{N/2}\int_{\mathbb{R}^N} G(\overline{u})\,dx = \theta^{N/2}.$$

Again by Pohozaev identity, we get:

$$\begin{aligned}\frac{N-2}{2}\theta^{\frac{N-2}{2}}\int_{\mathbb{R}^N} |\nabla\overline{u}|^2\,dx = N\theta^{N/2} \Rightarrow \\ \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla\overline{u}|^2\,dx = \theta^{\frac{N}{2}-\left(\frac{N-2}{2}\right)} \Rightarrow \\ \theta = \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla\overline{u}|^2\,dx.\end{aligned}$$

We now affirm that the action, that is, the energy functional, of a solution to problem 1.1 has the expression $I(u) = \frac{1}{N}\|\nabla u\|_2^2$. Surely, from Pohozaev identity, one has

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} G(u)\,dx \\
&= \frac{1}{2}\|\nabla u\|_2^2 - \frac{N-2}{2N}\|\nabla u\|_2^2 \\
&= \frac{1}{2}\left(1 - \frac{N-2}{N}\right)\|\nabla u\|_2^2 \\
&= \frac{1}{2}\left(\frac{N-(N-2)}{N}\right)\|\nabla u\|_2^2 \\
&= \frac{1}{N}\|\nabla u\|_2^2 > 0. \qquad (2.60)
\end{aligned}
$$

Thus, using $\theta = \dfrac{N-2}{2N}\|\nabla \overline{u}\|_2^2$, $I(u)$ becomes:

$$
I(u) = \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} (\|\nabla \overline{u}\|_2^2)^{N/2}. \qquad (2.61)
$$

Now let v denote another solution to our original problem. Once more, by Pohozaev identity:

$$
\int_{\mathbb{R}^N} |\nabla v|^2\,dx = \frac{2N}{N-2}\int_{\mathbb{R}^N} G(v)\,dx.
$$

Let $\sigma > 0$ be such that $\int_{\mathbb{R}^N} G(v_\sigma)\,dx = 1$, that is, $\sigma = \left(\int_{\mathbb{R}^N} G(v_\sigma)\,dx\right)^{-1/N}$, or from Pohozaev identity,

$$
\sigma = \left(\frac{N-2}{2N}\right)^{-1/N}\left(\int_{\mathbb{R}^N} G(v)\,dx\right)^{-1/N}
$$

Let us express $I(v)$ in terms of $\|\nabla v_\sigma\|_2^2$. On the one hand, we already have expression (2.60). On the other hand, $\|\nabla v_\sigma\|_2^2 = \sigma^{N-2}\|\nabla v\|_2^2$, so that

$$
\|\nabla v_\sigma\|_2^2 = \left(\frac{N-2}{2N}\right)^{-\frac{(N-2)}{N}} (\|\nabla v\|_2^2)^{2/N}.
$$

Hence

$$
I(v) = \frac{1}{N}\|\nabla v\|_2^2 = \frac{1}{N}\left(\frac{N-2}{2N}\right)^{\frac{(N-2)}{N}} \left(\nabla v_\sigma\|_2^2\right)^{N/2} \qquad (2.62)
$$

Since $\overline{u}$ solves the constrained minimization problem and $\int_{\mathbb{R}^N} G(v_\sigma)\,dx = 1$, we have

$$\int_{\mathbb{R}^N} G(v_\sigma)\,dx \geq \int_{\mathbb{R}^N} G(\overline{u})\,dx.$$

Using this inequality together with expressions (2.61) and (2.62) we conclude $I(v) \geq I(u)$, as desired.

□

Lemma 2.4. Consider the set $S = \{u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} G(u)dx = 1\}$. There exists a one-to-one correspondence between S and $\mathcal{P}$, namely there is a map $\Phi : S \to \mathcal{P}$ given by

$$\Phi(u)(x) = u\left(\frac{x}{t_u}\right), \tag{2.63}$$

where $t_u = \sqrt{\dfrac{N-2}{2N}}\|\nabla u\|_2$.

Proof. First, let us show that ϕ is a function of S in $\mathcal{P}$. Given $u \in S$, one has by definition that $\int_{\mathbb{R}^N} G(u)dx = 1$. But then

$$\begin{aligned}
J(u_t) &= \frac{N-2}{2}\int_{\mathbb{R}^N}\left|\nabla u\left(\frac{x}{t_u}\right)\right|^2 dx \\
&= \frac{N-2}{2}\int_{\mathbb{R}^N} t_u^{N-2}\left|\nabla u\left(x\right)\right|^2 dx \\
&= \frac{N-2}{2}\left(\left(\frac{N-2}{2N}\right)^{1/2}\|\nabla u\|_2\right)^{N-2}\|\nabla u\|_2^2 \\
&= \frac{N-2}{2}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}}\|\nabla u\|_2^{N-2}\|\nabla u\|_2^2 \\
&= \frac{N-2}{2}\frac{N}{N}\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}}\|\nabla u\|_2^N \\
&= N\left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}+1}\|\nabla u\|_2^N \\
&= N\left(\frac{N-2}{2N}\right)^{\frac{N}{2}}\|\nabla u\|_2^N.
\end{aligned}$$

But

$$t_u^N = \left[\left(\frac{N-2}{2N}^{1/2}\|\nabla u\|_2\right)\right]^N = \left(\frac{N-2}{2N}\right)^{N/2}\|\nabla u\|_2^N,$$

and so

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N}\left|\nabla u\left(\frac{x}{t_u}\right)\right|dx &= N\left(\frac{N-2}{2N}\right)^{N/2}\|\nabla u\|_2^N\\ &= Nt_u^N\\ &= Nt_u^N\int_{\mathbb{R}^N}G(u(x))dx\\ &= N\int_{\mathbb{R}^N}G\left(u\left(\frac{x}{t_u}\right)\right)dx.\end{aligned}$$

Thus $u\left(\frac{x}{t_u}\right)\in\mathcal{P}$, since

$$\frac{N-2}{2}\int_{\mathbb{R}^N}\left|\nabla u\left(\frac{x}{t_u}\right)\right|dx = N\int_{\mathbb{R}^N}G\left(u\left(\frac{x}{t_u}\right)\right)dx.$$

We now prove that Φ is onto (surjective). For that, take $\hat{u}(x)\in\mathcal{P}$. We would like to show there exists a change of variables $x=\alpha y$ such that $\int_{\mathbb{R}^N}G(\hat{u}(\alpha y))dy=1$, since this would imply $\hat{u}(\alpha y)\in S$. We have $\hat{u}\in\mathcal{P}$ by hypothesis, and so

$$\int_{\mathbb{R}^N}G(\hat{u}(x))dx=\frac{N-2}{2N}\|\nabla u\|_2^2. \tag{2.64}$$

Since the existence of such change of variables $x=\alpha y$ implies that

$$1=\int_{\mathbb{R}^N}G(\hat{u}(\alpha y))dy=\frac{1}{\alpha^N}G(\hat{u}(x))dx,$$

then by Equation (A.19) this happens if, and only if, $\alpha^N=\dfrac{N-2}{2}\|\nabla\hat{u}\|_2^2$.

Let us then take $\hat{u}(\alpha x)\in S$, and see that we can get t_u in terms of α :

$$t_u=\sqrt{\frac{N-2}{2N}}\|\nabla\hat{u}(\alpha\cdot)\|_2=\sqrt{\frac{N-2}{2N}}\alpha^{\frac{2-N}{2}}\|\nabla\hat{u}(\cdot)\|_2.$$

Note that this gives us

$$\frac{\alpha}{t_u}=\frac{\alpha}{\sqrt{\frac{N-2}{2N}}\alpha^{\frac{2-N}{2}}\|\nabla\hat{u}(\cdot)\|_2}=\frac{\alpha^{N/2}}{\sqrt{\frac{N-2}{2N}}\|\nabla\hat{u}(\cdot)\|_2}=\frac{\alpha^{N/2}}{\alpha^{N/2}}=1.$$

Thus $\Phi(\hat{u}(\alpha x))=\hat{u}(x)$, and this shows that Φ is onto.

For the injectivity, we need to show that given $u, v \in S$ with $\Phi(u) = \Phi(v)$, one can obtain $u = v$. By assumption, let $u, v \in S$ with $\Phi(u) = \Phi(v)$, and so $u\left(\frac{x}{t_u}\right) = v\left(\frac{x}{t_v}\right)$, $\forall x \in \mathbb{R}^N$. Note that

$$
\begin{aligned}
\|\nabla\Phi(u)\|_2 &= \int_{\mathbb{R}^N} |\nabla\Phi(u)|^2 dx \\
&= \int_{\mathbb{R}^N} \left|\nabla u\left(\frac{x}{t_u}\right)\right|^2 dx \\
&= t_u^{N-2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \\
&= t_u^{N-2} \|\nabla u\|_2^2. \qquad (2.65)
\end{aligned}
$$

Similarly,

$$
\|\nabla\Phi(v)\|_2 = t_v^{N-2}\|v\|_2^2. \qquad (2.66)
$$

But $\Phi(u) = \Phi(v)$, and so putting together (2.65) and (2.66) we get

$$
\begin{aligned}
\|\nabla\Phi(u)\|_2 = \|\nabla\Phi(u)\|_2 &\iff t_u^{N-2}\|\nabla u\|_2^2 = t_v^{N-2}\|\nabla v\|_2^2 \\
&\iff \left[\left(\frac{N-2}{2N}\right)^{1/2} \|\nabla u\|_2\right]^{N-2} \|\nabla u\|_2^2 = \left[\left(\frac{N-2}{2N}\right)^{1/2} \|\nabla u\|_2\right]^{N-2} \|\nabla v\|_2^2 \\
&\iff \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^{N-2+2} = \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla v\|_2^{N-2+2} \\
&\iff \|\nabla u\|_2^N = \|\nabla v\|_2^N \\
&\iff \|\nabla u\|_2 = \|\nabla v\|_2. \qquad (2.67)
\end{aligned}
$$

From Equation (2.67) one obtains that

$$
t_u = \left(\frac{N-2}{2N}\right)^{1/2} \|\nabla u\|_2 = \left(\frac{N-2}{2N}\right)^{1/2} \|\nabla v\|_2 = t_v,
$$

thus

$$
u\left(\frac{x}{t_u}\right) = v\left(\frac{x}{t_v}\right) \Rightarrow u\left(\frac{x}{t_u}\right) = v\left(\frac{x}{t_u}\right) \quad \forall x \in \mathbb{R}^N. \qquad (2.68)
$$

Putting $y = \dfrac{x}{t_u}$ on expression (2.68), we conclude that $u = v$. This proves that Φ is one-to-one.

We now have a one-to-one correspondence between the sets S and $\mathcal{P}$, namely the bijection Φ.

$\square$

With Lemma 2.4 in hand, we may now prove that the infimum of the associated functional I constrained to the Pohozaev manifold P is actually the same as the infimum of this same functional constrained to the set S (see Theorem 2.1).

Theorem 2.2.

$$m = \inf_{u \in \mathcal{P}} I(u) = \inf_{u \in S} I(u) \tag{2.69}$$

Proof. First observe we have the following expressions if we start from the definition of the scaling t_u as given by Lemma 2.4:

$$\begin{aligned}
t_u &= \sqrt{\frac{N-2}{2N}} \|\nabla u\|_2 \Rightarrow \\
t_u^{N-2} &= \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^{N-2} \Rightarrow \\
\frac{1}{2} t_u^{N-2} &= \frac{(N-2)^{\frac{N-2}{2}}}{2^{\frac{N-2}{2}} N^{\frac{N-2}{2}}} \frac{1}{2} \|\nabla u\|_2^{N-2} \Rightarrow \\
\frac{1}{2} t_u^{N-2} \|\nabla u\|_2^2 &= \frac{(N-2)^{\frac{N-2}{2}}}{2^{\frac{N}{2}} N^{\frac{N}{2}-1}} \|\nabla u\|_2^N \\
&= \frac{(N-2)^{\frac{N}{2}}}{2^{\frac{N}{2}} N^{\frac{N}{2}}} \frac{(N-2)^{-1}}{N^{-1}} \|\nabla u\|_2^N \\
&= \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \frac{N}{N-2} \|\nabla u\|_2^N .
\end{aligned} \tag{2.70}$$

Also, note that

$$t_u^N = \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \|\nabla u\|_2^N . \tag{2.71}$$

Now, given $u \in S$, then clearly $t_u^N \int_{\mathbb{R}^N} G(u) z, dx = t_u^N$ since $\int_{\mathbb{R}^N} G(u)\, dx = 1$, and we have $I(\Phi_{t_u}(u)) \in \mathcal{P}$. Thus, by using Equations (2.70) and (2.71), one gets

$$
\begin{aligned}
I(\Phi(u)) &= \frac{1}{2} t_u^{N-2} \|\nabla u\|_2^2 - t_u^N \int_{\mathbb{R}^N} G(u) dx \\
&= \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \frac{N}{N-2} \|\nabla u\|_2^N - \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \|\nabla u\|_2 \\
&= \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \|\nabla u\|_2^N \left(\frac{N}{N-2} - 1\right) \\
&= \left(\frac{N-2}{2N}\right)^{\frac{N}{2}} \|\nabla u\|_2^N \left(\frac{2}{N-2}\right) \\
&= \frac{(N-2)^{\frac{N}{2}} - 1}{2^{\frac{N}{2}-1} N^{\frac{N}{2}} - 1 + 1} \|\nabla u\|_2^N \\
&= \frac{1}{N} \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N.
\end{aligned} \tag{2.72}
$$

From Equation (2.72), we have

$$
\inf_{u \in \mathcal{P}} I(u) = \inf_{u \in S} I(\Phi_{t_u}(u)) = \inf_{u \in S} \frac{1}{N} \left(\frac{N-2}{2N}\right)^{\frac{N-2}{2}} \|\nabla u\|_2^N. \tag{2.73}
$$

From Theorem 2.1, we know that $\inf_{u \in S} \|\nabla u\|_2^2$ is actually achieved and the corresponding u_t is a least energy solution. Thus, from the one-to-one correspondence between S and $\mathcal{P}$ given in Lemma 2.4, we conclude $m = \inf_{u \in S} I(u) = \inf_{u \in \mathcal{P}} I(u)$, as desired.

□

Remark 2.3. Observe that by Theorem 2.1, the infima referred to in Lemma 2.2 are actually achieved, and so they are actually (constrained) minima.

2.4 A Mountain Pass Geometry

In this section, we shall prove that the functional possesses a mountain pass geometry and this will render us a well-defined critical value $b > 0$ of the associated functional, also known as the mountain pass value. This is done in Lemmas 2.5 and 2.6. Also, with the levels b and m in hand, we will prove that $b = m$, and so the mountain pass value gives the least energy level. This last result was first done in the work of Jeanjean and Tanaka [18].

Our goal is to prove that our energy functional has a property known as a mountain pass geometry. We define what this means below.

Remark 2.4. This is an important checkpoint in this work since from this point on we shall consider a nonlinearity of the type $g(s) = f(s) - \lambda s$. We have seen earlier in Section 1.4 that, under these conditions, $I(u)$ is well-defined on $H^1(\mathbb{R}^N)$ and of class C^1, since it satisfies conditions (g1)-(g4). Whenever possible, we still considered a general nonlinearity in some parts of the proofs that will follow in the next Lemmas. Nevertheless, from Jeanjean and Tanaka [18], one guarantees that the associated functional to problem possesses a mountain pass geometry for any g satisfying conditions (g1)-(g4).

Definition 2.3. The functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ is said to possess a **mountain pass geometry** if it satisfies the following three conditions:

1. I(0) = 0.
2. There exist $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \rho \ \forall \|u\|_{H^1} = \rho$.
3. There exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1} > \rho$ and $I(u_0) < 0$.

We will need to recall the following conditions, first defined in Chapter 1. They are conditions (f1), (f2), (g3) and (g4).

(a) f is continuously differentiable on the set $[0, +\infty)$

(b) $\lim_{s \to 0} \frac{f(s)}{s} = 0$

(c) $\lim_{s \to +\infty} \frac{g(s)}{s^{\frac{N+2}{N-2}}} = 0.$

(d) There exists $\zeta > 0$ such that $G(\zeta) = \int_0^\zeta g(\tau) d\tau > 0.$

Remark 2.5. Note that, from item (e) of Lemma 2.2, one automatically gets that the second property of the mountain pass geometry holds for $u \in \mathcal{P}$.

Lemma 2.5. 1. and 2. hold for the associated functional of the problem.

Proof. Let us start by proving 1.:

We have that $I : H^1(\mathbb{R}^N) \to \mathbb{R}^N$ is a functional given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u),$$

where $G(u) = \int_0^u g(\tau)d\tau$. Note that

$$\begin{aligned} I(0) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(0)|^2 - \int_{\mathbb{R}^N} G(0) \\ &= 0 - \int_{\mathbb{R}^N}\int_0^0 g(\tau)d\tau \\ &= 0. \end{aligned}$$

This proves 1.

As for property 2., from hypothesis (a), for a given $\epsilon > 0$, there exists $\delta > 0$ such that $|s| < \delta \Rightarrow$

$$|f(s)| \leq \epsilon\,|s|\,. \tag{2.74}$$

Also, from hypothesis (b), for a given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ and $\delta > 0$ such that $|s| > \delta \Rightarrow$

$$\left|\frac{g(s)}{s^{\frac{N+2}{N-2}}}\right| = \frac{|f(s) - \lambda s|}{|s|^{\frac{N+2}{N-2}}} \leq \frac{|f(s)|}{|s|^{\frac{N+2}{N-2}}} + \frac{\lambda\,|s|}{|s|^{\frac{N+2}{N-2}}},$$

where the last inequality is the triangle inequality. Thus

$$|f(s)| < \epsilon|s|^{\frac{N+2}{N-2}} - \lambda\,|s|\,, \tag{2.75}$$

and so gathering estimates (2.74) and (2.75), we first multiply both inequalities by a -1 factor in order to reverse the inequalities and get

$$-\,|f(s)| \geq \epsilon|s|^{\frac{N+2}{N-2}} - \lambda\,|s| - \epsilon\,|s|\,, \tag{2.76}$$

and thus -$|f(s)| \geq -C_\epsilon\,|s|^{\frac{N+2}{N-2}} - (\lambda + \epsilon)\,|s| \quad \forall s > 0$, where upon integrating (2.75) from 0 to s and recalling f is an odd function, one finds:

$$\int_0^s -|f(\tau)|\,d\tau \geq \int_0^s -(\lambda+\epsilon)\,|\tau|\,d\tau - C_\epsilon \int_0^s |\tau|^{\frac{N+2}{N-2}}\,d\tau \Rightarrow$$
$$-\int_0^s |f(\tau)|\,d\tau \geq -(\lambda+\epsilon)\frac{\tau^2}{2}\Big|_0^s - C_\epsilon \frac{\tau^{\frac{N+2}{N-2}+1}}{\frac{N+2}{N-2}+1}\Bigg|_0^s \Rightarrow$$
$$-F(s) \geq -(\lambda+\epsilon)\frac{s^2}{2} - C_\epsilon \frac{s^{\frac{2N}{N-2}}}{\frac{2N}{N-2}} \Rightarrow$$
$$-F(s) \geq -(\lambda+\epsilon)\frac{s^2}{2} - C_\epsilon \frac{s^{2^*}}{2^*},$$

from where we get the estimate

$$-F(s) \geq -(\lambda+\epsilon)\frac{s^2}{2} - C_\epsilon' s^{2^*} \quad \forall s > 0. \tag{2.77}$$

Now we use the Gagliardo-Nirenberg-Sobolev inequality A.15 that states that if $1 \leq p < N$, then $W^{1,p}\left(\mathbb{R}^N\right) \subset L^{p^*}(\mathbb{R}^N)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ and there exists a constant $C = C(p,N)$ such that $\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^2} \quad \forall u \in H^1(\mathbb{R}^N)$.

In our case, $p = 2$ and the embedding we are looking at is the continuous injection $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and there exists a constant $C = C(2,N)$ such that

$$\|u\|_{L^{\frac{2N}{N-2}}} \leq C\|\nabla u\|_{L^2} \quad \forall u \in H^1(\mathbb{R}^N).$$

Then it follows that, for a constant $C_\epsilon'' > 0$,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} G(u)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \int_{\mathbb{R}^N} (\lambda-\epsilon)\,\frac{u^2}{2} - C_\epsilon''\,|u|^{\frac{2N}{N-2}}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|\,dx + \frac{(\lambda-\epsilon)}{2}\int_{\mathbb{R}^N} \frac{u^2}{2}dx - C_\epsilon'' \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx \\
&\geq \frac{1}{2}\min\{1,(\lambda-\epsilon)\}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\,dx - C_\epsilon'' \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx \\
&= \frac{1}{2}\min\{1,(\lambda-\epsilon)\}\|u\|_{H^1}^2 - C_\epsilon'' \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx.
\end{aligned} \tag{2.78}$$

Let us work on the second term on the inequality given in (2.78). First, note that

$$\|u\|_{\frac{2N}{N-2}} \leq C\|u\|_{H^1},$$

which comes from the continuous injection $H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. Then one has the following estimates, after exchanging $\dfrac{2N}{N-2}$ for 2^*, which corresponds to the critical exponent:

$$\begin{aligned}
\|u\|_{2^*}^{2^*} &\leq C^{2^*}\|\nabla u\|_2^{2^*} \\
&= C^{2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{\frac{1}{2}\frac{2N}{N-2}} \\
&\leq C^{2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\, dx\right)^{\frac{1}{2}\frac{2N}{N-2}} \\
&= C^{2^*}\|u\|_{H^1}^{2^*},
\end{aligned}$$

with $C = C(2, N)$. Then $-\|u\|_{2^*}^{2^*} \geq -C^{2^*}\|\nabla u\|_2^{2^*}$. Going back to the inequality obtained in (2.78), we get

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\min\{1, (\lambda-\epsilon)\}\|u\|_{H^1}^2 - C_\epsilon''\|u\|_{2^*}^{2^*} \\
&\geq \frac{1}{2}\min\{1, (\lambda-\epsilon)\}\|u\|_{H^1}^2 - C\|u\|_{H^1}^{2^*} \quad \forall u \in H^1(\mathbb{R}^N),
\end{aligned}$$

where we have put $C := C_\epsilon'' C^{2^*}$.

Finally, by taking ϵ sufficiently small and $0 < \rho < 1$ such that $\lambda - \epsilon > 0$ and $\rho^{2^*} < \dfrac{1}{4C}\min\{1, (\lambda-\epsilon)\}$, one has that $\|u\|_{H^1} = \rho \Rightarrow$

$$\begin{aligned}
I(u) &\geq \frac{1}{2}\min\{1, (\lambda-\epsilon)\}\rho^2 - C\rho^{2^*} \quad \forall u \in H^1(\mathbb{R}^N) \\
&> \frac{1}{2}\min\{1, (\lambda-\epsilon)\}\rho^2 - C\frac{1}{4C}\min\{1, (\lambda-\epsilon)\}\rho^2 \\
&= \frac{1}{4}\min\{1, (\lambda-\epsilon)\}\rho^2.
\end{aligned}$$

By taking $0 < \rho < 1$ sufficiently small, we can see that $I(u) \geq \delta \quad \forall \|u\|_{H^1} = \rho$ Also, for all $0 < \|u\|_{H^1(\mathbb{R}^N)} \leq \rho$, it holds that $I(u) > 0$.

$\square$

Remark 2.6. Note that properties 1 and 2 together gives us that $I(0) \geq 0$ for all $0 \leq \|u\|_{H^1} \leq \rho$.

Now, let us prove 3. Consider the set

$$\Gamma = \{\gamma(t) \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right); \gamma(0) = 0, I(\gamma(1)) < 0\}. \tag{2.79}$$

This is the set of all the continuous maps $[0,1] \to H^1\left(\mathbb{R}^N\right)$. We already have that $I(0) = 0$ and $I(u) > 0$ for all $0 < \|u\|_{H^1} \leq \rho$. If one can prove that Γ is not empty, there exists a $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1} > \rho$ and $I(u) < 0$ and thus condition 3 is satisfied. After that, we are done.

Let us then prove that, in fact, Γ is not empty. We restate it in the form of the following Lemma:

Lemma 2.6. (3. also holds for the associated functional) Under the assumptions (a), (b), (c) and (d), there exists a path $\gamma \in \Gamma$ satisfying the following properties, where $\omega(x)$ is a given ground state/least energy solution to our original problem:

(A) $\omega \in \gamma([0,1])$.

(B) $\max_{t\in[0,1]} I(\gamma(t)) = m$.

Proof. We shall, by construction of a path $\gamma \in \Gamma$, first find a curve $\gamma(t) : [0,L] \to H^1\left(\mathbb{R}^N\right)$ such that

$$\gamma(0) = 0, I(\gamma(L)) < 0,$$
$$\omega \in \gamma([0,L]),$$
$$\max_{t\in[0,L]} I(\gamma(t)) = m,$$

and after performing a scale change in t, obtain the sought path $\gamma \in \Gamma$.

With that in mind, let

$$\gamma(t)(x) = \begin{cases} \omega\left(\frac{x}{t}\right), & t > 0 \\ 0, & t = 0. \end{cases}$$

From this definition, we can calculate:

$$\begin{aligned} \|\gamma(t)\|_{H^1}^2 &= \int_{\mathbb{R}^N} |\nabla\gamma(t)|^2 + |\gamma(t)|^2 \, dx \\ &= \int_{\mathbb{R}^N} \left|\nabla\omega\left(\frac{x}{t}\right)\right|^2 + \left|\omega\left(\frac{x}{t}\right)\right|^2 dx. \end{aligned}$$

We now perform a change of variables $z = \dfrac{x}{t}$, and so

$$\begin{aligned} \|\gamma(t)\|_{H^1}^2 &= \int_{\mathbb{R}^N} \left(\frac{|\nabla\omega(z)|^2}{t^2} + |\omega(z)|^2\right) t^N dz \\ &= \int_{\mathbb{R}^N} |\nabla\omega(z)|^2 \frac{t^N}{t^2} dz + \int_{\mathbb{R}^N} |\omega(z)|^2 t^N dz \\ &= t^{N-2} \int_{\mathbb{R}^N} |\nabla\omega(z)|^2 \, dz + t^N \int_{\mathbb{R}^N} |\omega(z)|^2 \, dz \\ &= t^{N-2}\|\nabla\omega\|_2^2 + t^N\|\omega\|_2^2. \end{aligned} \tag{2.80}$$

Also, for the functional I, it holds that:

$$\begin{aligned} I(\gamma(t)) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(\gamma(t))|^2 \, dx - \int_{\mathbb{R}^N} G(\gamma(t)) dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla\omega\left(\frac{x}{t}\right)\right|^2 dx - \int_{\mathbb{R}^N} G\left(\omega\left(\frac{x}{t}\right)\right) dx \\ &= \frac{t^{N-2}}{2}\|\nabla\omega\|_2^2 - t^N \int_{\mathbb{R}^N} G(\omega) dx. \end{aligned}$$

From Pohozaev identity, $\dfrac{N-2}{2}\displaystyle\int_{\mathbb{R}^N} |\nabla\omega|^2 \, dx = N \int_{\mathbb{R}^N} G(\omega) dx$, and since $\|\nabla\omega\|_2^2 > 0$, then $\displaystyle\int_{\mathbb{R}^N} G(\omega) dx > 0$, and so

$$\begin{aligned}\frac{d}{dt}I(\gamma(t)) &= \frac{d}{dt}\left(\frac{t^{N-2}}{2}\|\nabla\omega\|_2^2 - t^N\int_{\mathbb{R}^N}G(\omega)dx\right)\\ &= \frac{d}{dt}(t^{N-2})\frac{1}{2}\|\nabla\omega\|_2^2 - \frac{d}{dt}(t^N)\int_{\mathbb{R}^N}G(\omega)dx\\ &= (N-2)t^{N-3}\frac{1}{2}\|\nabla\omega_2^2 - Nt^{N-1}\int_{\mathbb{R}^N}G(\omega)dx,\end{aligned}$$

and now we use Pohozaev identity on the first term of this last expression to get:

$$\begin{aligned}\frac{d}{dt}I(\gamma(t)) &= Nt^{N-3}\int_{\mathbb{R}^N}G(\omega)dx - Nt^{N-1}\int_{\mathbb{R}^N}G(\omega)dx\\ &= Nt^{N-1}\int_{\mathbb{R}^N}G(\omega)dx\left(t^{-2}-1\right).\end{aligned} \tag{2.81}$$

Then, if $t \in (0,1)$, we have $t^{-2} - 1 = \dfrac{1}{t^2} - 1 > 0$, whereas $\int_{\mathbb{R}^N}G(\omega)dx > 0$ implies that $Nt^{N-1} > 0$, we get from (2.81) that $\dfrac{d}{dt}\left(I(\gamma(t))\right) > 0$.

On the other hand, if $t > 1$, then $\dfrac{1}{t^2} - 1 < 0$ and so we can see from (2.81) that $\dfrac{d}{dt}\left(I(\gamma(t))\right) < 0$ in this case. This means that there exists $m > 0$ such that $\max_{t\in[0,L]} I(\gamma(t)) = m$. Furthermore, note that

$$\begin{aligned}\frac{d}{dt}I(\gamma(t)) = 0 &\iff Nt^{N-1}\int_{\mathbb{R}^N}G(\omega)\left(t^{-2}-1\right)dx\\ &\iff t^{-2}-1=0\\ &\iff \frac{1}{t^2}=1\\ &\iff t=1.\end{aligned} \tag{2.82}$$

For $t = 1$, $I(\gamma(t)) = \dfrac{t^{N-2}}{2}\|\nabla\omega\|_2^2 - t^N\int_{\mathbb{R}^N}G(\omega)dx$ becomes

$$\begin{aligned}I(\gamma(1)) &= \frac{1}{2}\|\nabla\omega\|_2^2 - \int_{\mathbb{R}^N}G(\omega)dx\\ &= I(\omega).\end{aligned} \tag{2.83}$$

For $t = 0$, by definition we have $\gamma(0) = 0$, and so $I(\gamma(0)) = I(0) = 0$ as we have seen by property 1. For a sufficiently large $L > 1$, note that

$$I(\gamma(L)) = \frac{L^{N-2}}{2}\|\nabla\omega\|_2^2 - L^N \int_{\mathbb{R}^N} G(\omega)dx < 0, \tag{2.84}$$

since the second term dominates for large L.

Therefore we have $I(\gamma(L)) < 0$ for some L>1 (sufficiently large). Thus so far, we can guarantee the existence of a curve $\gamma(t) : [0, L] \to H^1(\mathbb{R}^N)$ such that $\gamma(0) = 0$ and $I(\gamma(L)) < 0$.

We shall now prove that $\omega \in \gamma([0, L])$ and $\max_{t\in[0,L]} I(\gamma(t)) = m$. First, recall that $\frac{d}{dt}I(\gamma(t)) = 0 \iff t = 1$ from (2.82). This gives us $\max_{t\in[0,L]} I(\gamma(t)) = I(\gamma(1)) = I(\omega)$. Thus, if ω is a least energy solution of

$$\begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

then we would like to show that $\omega \in \gamma([0, L])$. But for $t = 1$, $I(\gamma(t)) = I(\gamma(1)) = I(\omega)$, i.e. there exists a $t \in [0, L]$ such that $\omega \in \gamma([0, L])$.

Furthermore, since the critical points of I are precisely the solutions of the problem, we have from (2.82) that $I(\gamma(t))$ takes its maximum value at $t = 1$, and so

$$\max_{t\in[0,L]} I(\gamma(t)) = I(\omega) = m. \tag{2.85}$$

Now let us perform a scale change, that is, consider the map $\alpha : [0, 1] \to [0, L]$, $s \to \alpha(s) = sL$. First, compose the paths α and γ by constructing this new path $\gamma \circ \alpha : [0, 1] \to H^1(\mathbb{R}^N)$. This new path $\gamma' : \gamma \circ \alpha$ is such that $\gamma'(t) = \gamma(\alpha(s))$. Then we proceed with the following calculations:

$$
\begin{aligned}
I(\gamma(\alpha(s))) &= \frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla\omega\left(\frac{x}{\alpha(s)}\right)\right|^2 dx - \int_{\mathbb{R}^N} G\left(\omega\left(\frac{x}{\alpha(s)}\right)\right) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla\omega\left(\frac{x}{sL}\right)\right|^2 dx - \int_{\mathbb{R}^N} G\left(\omega\left(\frac{x}{sL}\right)\right) dx \\
&= \frac{(sL)^{N-2}}{2}\|\nabla\omega\|_2^2 - (sL)^N \int_{\mathbb{R}^N} G(\omega)\, dx \\
&= L^{N-2}\frac{s^{N-2}}{2}\|\nabla\omega\|_2^2 - L^N s^N \int_{\mathbb{R}^N} G(\omega) dx.
\end{aligned}
$$

Also, we get

$$
\frac{d}{ds}I(\gamma(\alpha(s))) = \frac{L^{N-2}}{2}\|\omega\|_2^2(N-2)s^{N-3} - NL^N s^{N-1}\int_{\mathbb{R}^N} G(\omega)dx. \tag{2.86}
$$

From Pohozaev identity, $\dfrac{N-2}{2}\|\nabla\omega\|_2^2 = N\displaystyle\int_{\mathbb{R}^N} G(\omega)dx$, then

$$
\begin{aligned}
\frac{d}{ds}I(\gamma^{'}(\alpha(s))) &= \frac{N-2}{2}\|\nabla u\|_2^2 L^{N-2}s^{N-3} - NL^N s^{N-1}\int_{\mathbb{R}^N} G(\omega)dx \\
&= N\int_{\mathbb{R}^N} G(\omega)L^{N-2}s^{N-3} - NL^N s^{N-1}\int_{\mathbb{R}^N} G(\omega)dx \\
&= NL^N s^{N-1}\int G(\omega)\left(L^{-2}s^{-2}-1\right)dx,
\end{aligned}
$$

from where

$$
\begin{aligned}
\frac{d}{ds}I(\gamma^{'}(s)) = 0 &\iff NL^N s^{N-1}\int G(\omega)\left(L^{-2}s^{-2}-1\right)dx = 0 \\
&\iff L^{-2}s^{-2} - 1 = 0 \\
&\iff (Ls)^2 = 1 \\
&\iff Ls = 1 \\
&\iff s = \frac{1}{L},
\end{aligned}
$$

which is a critical point.

If $s \in \left[0, \frac{1}{L}\right)$, then $\frac{d}{ds}I(\gamma^{'}(s)) > 0$ because $L^{-2}s^{-2} - 1 \Rightarrow s^2L^2 < \frac{1}{L^2}L^2 = 1 \Rightarrow \frac{1}{s^2L^2} > 1 \implies L^{-2}s^{-2} - 1 > 0$.

Also, if $s \in \left(\frac{1}{L}, 1\right]$, then $\frac{d}{ds} I(\gamma^{'}(s)) < 0$ because $s^2L^2 > \left(\frac{1}{L}\right)^2 L^2 = 1$, and so $s^2L^2 > 1 \Rightarrow \frac{1}{s^2L^2} < 1 \Rightarrow L^{-2}s^{-2} - 1 < 0$.

It is important to see that, for $s = \frac{1}{L}$, we get

$$
\begin{aligned}
I(\gamma^{'}\left(\frac{1}{L}\right)) &= L^{N-2}\frac{\left(\frac{1}{L}\right)^{N-2}}{2}\|\nabla\omega\|_2^2 - L^N\left(\frac{1}{L}\right)^N \int_{\mathbb{R}^N} G(\omega)dx \\
&= \frac{L^{N-2}/L^{N-2}}{2}\|\nabla\omega\|_2^2 - \frac{L^N}{L^N}\int_{\mathbb{R}^N} G(\omega)dx \\
&= \frac{1}{2}\|\nabla\omega\|_2^2 - \int_{\mathbb{R}^N} G(\omega)dx \\
&= I(\omega).
\end{aligned}
$$

Then we have found a path $\gamma^{'} \in \Gamma$ such that $\omega \in \gamma^{'}([0,1])$ and $\max_{s\in[0,1]} I(\gamma^{'}(s)) = I(\omega)$, where after renaming, is the path desired in the Lemma.

Finally, this means that Γ is not empty, and so the functional I possesses the mountain pass geometry and the Lemma is proved.

□

2.5 The mountain pass level is the least energy level

Remark 2.7. From this result, this means that the mountain pass value

$$
b = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \tag{2.87}
$$

is well defined and $b > 0$ is a critical value of I.

We now prove the following Lemma:

Lemma 2.7. $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$ for all $\gamma \in \Gamma$.

Proof: Let

$$
\begin{aligned}
J(u) &= \frac{N-2}{2}\|\nabla u\|_2^2 - N\int_{\mathbb{R}^N} G(u)dx \\
&= NI(u) - \|\nabla u\|_2^2.
\end{aligned}
$$

We saw that there exists $\rho > 0$ such that

$$0 < \|u\|_{H^1} \leq \rho \Rightarrow J(u) > 0.$$

For any $\gamma \in \Gamma$ we have $\gamma(0) = 0$ and $J(\gamma(1)) \leq NI(\gamma(1)) < 0$.
Thus there exists $t_0 \in (0,1)$ such that

$$\|\gamma(t_0)_{H^1}\| > \rho,$$
$$J(\gamma(t_0) = 0.$$

Since $\gamma(t_0) \in \gamma([0,1]) \cap \mathcal{P}$ we conclude $\gamma([0,1]) \cap \mathcal{P} \neq \emptyset$. □

Lemma 2.8. *$m \geq b$.*

Proof. Recall that the mountain pass value

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \tag{2.88}$$

is well defined and $b > 0$ is a critical value of I. Also, if $m = \inf\{I(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\}\}$ is the least energy level, the corresponding solution v is such that there exists a path γ, as we have shown, satisfying $\max_{t \in [0,1]} I(\gamma(t)) = I(v) = m$. From the very definition of b, one gets

$$m = I(v) \geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = b \geq \delta > 0.$$

Thus $m \geq b$.

□

Lemma 2.9. *$m \leq b$.*

Proof. From Lemma 2.7 and since $m = \inf_{u \in \mathcal{P}} I(u) = \min_{u \in \mathcal{P}} I(u)$,

$$m \leq I(\gamma(t_\gamma)) \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = b \leq \max_{t \in [0,1]} I(\gamma(t)),$$

and so $m \leq b$. □

Remark 2.8. From Lemmas 2.8 and 2.9, one concludes that $b = m$. This means that the mountain pass level b gives the least energy level m of the corresponding least energy solution v.

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) = \inf_{u\in\mathcal{P}} I(u). \tag{2.89}$$

This next Lemma will prove to be essential in the construction of the algorithm in Chapter 3. It shows that, under the condition $\int_{\mathbb{R}^N} G(u) > 0$, any function $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ can be projected on the Pohozaev manifold, and such a projection is actually unique.

Lemma 2.10. For each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ with $\int_{\mathbb{R}^N} G(u) > 0$ there exists a unique real number $t_0 > 0$ such that $u\left(\frac{\cdot}{t_0}\right) \in \mathcal{P}$ and $I\left(u\left(\frac{\cdot}{t_0}\right)\right)$ is the maximum of the function $t \to I\left(u\left(\frac{\cdot}{t}\right)\right), t > 0$.

Proof. We first define the auxiliary function $h : (0, \infty) \to \mathbb{R}$, $t \to h(t) := I\left(u\left(\frac{\cdot}{t}\right)\right)$.

We have seen previously that

$$I\left(u\left(\frac{\cdot}{t}\right)\right) = \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\lambda t^N}{2}\int_{\mathbb{R}^N} u^2 - t^N \int_{\mathbb{R}^N} F(u).$$

Thus, for $N \geq 3$ which is our case, we have that $h'(t) = 0 \iff$

$$t^{N-3}\left((N-2)\int_{\mathbb{R}^N} |\nabla u|^2 + 2Nt^2\int_{\mathbb{R}^N}\left[\frac{\lambda}{2}u^2 - F(u)\right]\right) = 0.$$

Therefore we have either that $t = 0$ or

$$t^2 = \frac{(N-2)\int_{\mathbb{R}^N} |\nabla u|^2}{2N\int_{\mathbb{R}^N}\left[\frac{-\lambda}{2}u^2 + F(u)\right]} = \frac{(N-2)\int_{\mathbb{R}^N} |\nabla u|^2}{2N\int_{\mathbb{R}^N} G(u)}. \tag{2.90}$$

Then the Lemma is proved.

□

We may now conclude this Chapter. We found there exists a nontrivial solution via constrained minimization on a set S which is in bijection with the Pohozaev manifold P. We then proved that this solution is such that its energy functional is the minimum constrained to P, which is attained. Finally, this minimum is the same as the critical level obtained from the Mountain Pass Lemma of Ambrosetti and Rabinowitz [2], also known as the mountain pass level.

Chapter 3

A mini-max algorithm

3.1 Preliminaries

In this Chapter we prove some results that will help us devise an algorithm which will converge to the ground state solution. The algorithm is at the end of this Chapter.

In Section 3.2 we define the direction of gradient descent and translate finding this function to solving a linear Poisson equation. In Section 3.3 we show that our minimizing sequence is bounded in $H^1(\mathbb{R}^N)$. In Section 3.4 we show that a Palais-Smale sequence going to the mountain pass level for our associated functional I constrained to the Pohozaev manifold is also a Palais-Smale sequence for I in the space $H^1(\mathbb{R}^N)$.

3.2 The Steepest Descent direction

The steepest descent direction at $w_1 \in H^1(\mathbb{R}^N)$ corresponds to finding $\hat{v} \in H^1(\mathbb{R}^N)$ with $\|\hat{v}\| = 1$ such that

$$\frac{I(w_1 + \varepsilon\hat{v}) - I(w_1)}{\varepsilon} \tag{3.1}$$

is as negative as possible as $\varepsilon \to 0$. This is equivalent to finding the minimum of the Fréchet derivative at w_1 on $\hat{v}$, i.e. $I'(w_1)\hat{v}$, subject to the constraint $\|\hat{v}\| = 1$.

The device for solving numerically the steepest descent direction can be found by means of a linear equation detailed by J. Horák in [17]. For the sake of completeness we recall it here. Introducing the Lagrange Multiplier μ, we therefore look for the unconstrained minimum of the functional $L : H^1(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$L(\hat{v}) := I'(w_1)\hat{v} + \mu \int_{\mathbb{R}^N} \nabla\hat{v}\cdot\nabla\hat{v} + \hat{v}\cdot\hat{v}\, dx,$$

or, equivalently,

$$L(\hat{v}) := \int_{\mathbb{R}^N} \nabla w_1 \cdot \nabla\hat{v} + \lambda w_1 \hat{v} - f(w_1)\hat{v} + \mu(|\nabla\hat{v}|^2 + |\hat{v}|^2)\, dx.$$

The Fréchet derivative of L exists and is given by

$$L'(\hat{v})\phi = \int_{\mathbb{R}^N} \nabla w_1 \cdot \nabla\phi + \lambda w_1\phi - f(w_1)\phi + 2\mu(\nabla\hat{v}\cdot\nabla\phi + \hat{v}\phi)\, dx,$$

for any $\phi \in H^1(\mathbb{R}^N)$. Hence, $L'(\hat{v}) = 0$ corresponds to a weak solution $\hat{v} \in H^1(\mathbb{R}^N)$ of the linear equation:

$$2\mu(\Delta\hat{v} - \hat{v}) = -\Delta w_1 + \lambda w_1 - f(w_1)\,. \tag{3.2}$$

Note that, despite the fact that the Lagrange multiplier μ is not known, we can still find a solution for (3.2). If we set $\tilde{v} := 2\mu\hat{v}$, equation (3.2) can be written in terms of $\tilde{v}$ as:

$$(\Delta\tilde{v} - \tilde{v}) = -\Delta w_1 + \lambda w_1 - f(w_1), \tag{3.3}$$

which can be readily solved for $\tilde{v}$. Now, if the condition $\|\hat{v}\| = 1$ is to be satisfied, we must have:

$$\left(\left[\int_{\mathbb{R}^N} |\nabla\hat{v}|^2 + |\hat{v}|^2\right]^{1/2}\right)^2 = \int_{\mathbb{R}^N} \left|\nabla\left(\frac{\tilde{v}}{2\mu}\right)\right|^2 + \left|\frac{\tilde{v}}{2\mu}\right|^2 = 1, \tag{3.4}$$

which renders

$$\mu = \frac{\|\tilde{v}\|}{2}. \tag{3.5}$$

Thus, by taking μ as in (3.5), we are able to determine $\hat{v}$ from $\tilde{v}$.

This next Lemma will help us, in a sense, descend on the topology of the functional in *Step 4* of the algorithm.

Lemma 3.1. Let v and w in $H^1(\mathbb{R}^N)\setminus\{0\}$ and $\alpha\in\mathbb{R}$ such that $\int_{\mathbb{R}^N} G(w+\alpha v)>0$, $\Phi(\alpha):=I(w+\alpha v)$, $\gamma(\alpha):=(w+\alpha v)\left(\frac{\cdot}{t(\alpha)}\right)\in\mathcal{P}$ and $\Psi(\alpha):=I(\gamma(\alpha))$. If $t(\alpha)$ is bounded from below by a positive constant, then $\lim_{\alpha\to\pm\infty} I(\gamma(\alpha))=+\infty$ and $\min I(\gamma(\alpha))=I(\gamma(\hat\alpha))$ is attained for some $\hat\alpha\in\mathbb{R}$. Otherwise, if $\lim_{\alpha_j\to+\infty} t(\alpha_j)=0$ on a subsequence $(\alpha_j)_{j\in\mathbb{N}}$ and there exists $\delta>0$ such that $\Phi'(\alpha)<0$, for $0<\alpha<\delta$, then either:

(i) there is $\hat\alpha>0$ which is a point of local minimum of $I(\gamma(\alpha))$ or

(ii) $I(w+\alpha v)\le I\left((w+\alpha v)\left(\frac{\cdot}{t(\alpha)}\right)\right)<I(w)$ for $\alpha>0$.

Proof. First, let us note that, by Lemma 2.10, $\gamma(\alpha)$ is well-defined in an open subset of $\mathbb{R}$. It holds that

$$\begin{aligned} J(u) &= (N-2)\int_{\mathbb{R}^N}|\nabla u|^2\,dx-2N\int_{\mathbb{R}^N}G(u)\,dx \\ &= 2N\left(I(u)-\frac{1}{N}\|\nabla u\|_2^2\right). \end{aligned}$$

If $u\in\mathcal{P}$, then

$$\|\nabla u\|_2^2=NI(u). \tag{3.6}$$

Putting $u=\gamma(\alpha)$ and assuming α ranges on an unbounded subset of $\mathbb{R}$, we have two possibilities, either $t_\alpha\ge\bar t>0$, for some positive constant $\bar t$, hence $\left\|\nabla(w+\alpha v)\left(\frac{\cdot}{t_\alpha}\right)\right\|_2^2=t_\alpha^{N-2}||\nabla(w+\alpha v)||_2^2\to+\infty$ as $\alpha\to\pm\infty$, and so, by (3.6) it follows that $\lim_{\alpha\to\pm\infty} I(\gamma(\alpha))=+\infty$. The minimum is attained because I and $\gamma(\alpha)$ are continuous.

Otherwise, up to a subsequence, $t_\alpha\to 0$ as $\alpha\to+\infty$ or $\alpha\to-\infty$. By assumption the first case holds and also $\Phi'(\alpha)<0$, so that $I(w+\alpha v)<I(w)$, for $0<\alpha<\delta$. Since $I\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$, $\gamma\in C^1(\mathbb{R})$ and $\Phi'(\alpha)=I'(w+\alpha v)v<0$, then $\Psi'(\alpha)=I'(\gamma(\alpha))\gamma'(\alpha)<0$, for α positive and sufficiently small, because $||(w+\alpha v)-\gamma(\alpha)||\to 0$ as $\alpha\to 0$. If Ψ' changes sign, then there is $\hat\alpha$ such that $I(\gamma(\hat\alpha))$ is a local minimum. However, if $\Psi'(\alpha)$ does not change sign for $\alpha>0$, then by Lemma 2.10, $\Psi'(\alpha)<0$ and $\Psi(0)=I(w)$ we obtain

$$I(w+\alpha v)\le I\left(w+\alpha v\left(\frac{\cdot}{t_\alpha}\right)\right)<I(w). \tag{3.7}$$

□

Remark 3.1. Note that in case $t_\alpha \to 0$ as $\alpha \to -\infty$, one may repeat the previous proof exchanging v for $-v$ and α for $-\alpha$.

We antecipate that this previous lemma is going to be applied choosing $v = -\nabla I(w)$, the steepest descent direction of I at w (see Section 3).

3.3 A bounded minimizing sequence

Notice that by case (ii) of Lemma 3.1, $\Psi(\alpha) = I(\gamma(\alpha)) > m_{\mathcal{P}}$ for all $\alpha > 0$, and $\Psi'(\alpha) < 0$, which lead to $I(\gamma(\alpha)) \to b \geq m_{\mathcal{P}}$ as $\alpha \to +\infty$. Before proceeding to the next section, we shall investigate more deeply what happens in this case. For this purpose, we prove the boundedness of the minimizing sequence $(u_k) \subset \mathcal{P}$.

Lemma 3.2. If $(u_k) \subset \mathcal{P}$, $k \in \mathbb{N}$, and $I(u_k) \to b$, then (u_k) is bounded in $H^1(\mathbb{R}^N)$.

Proof. By hypothesis, $u_k \in \mathcal{P}$ and $I(u_k) \to b$. By Lemma 2.5, there exist positive constants ρ and δ such that $I(u) \geq \rho$ and $\|\nabla u\|_2 \geq \delta$ for all $u \in \mathcal{P}$. Better yet, from the proof of Lemma 2.5, one gets

$$c + 1 \geq I(u_k) \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \|\nabla u_k\|_2 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \delta \geq \rho,$$

hence

$$C_1 < \|\nabla u_k\|_2 < C_2, \tag{3.8}$$

with $C_1, C_2 > 0$.

Now, from conditions $(f1), (f2), (f3)$ we have that, given $\epsilon > 0$, there exists $C_3 = C_3(\epsilon) > 0$ such that

$$|F(u)| \leq \frac{\epsilon}{2} u^2 + C_3 |u|^p, \quad 2 < p < 2^*. \tag{3.9}$$

Also, since $u_k \in \mathcal{P}$, then

$$\int_{\mathbb{R}^N} |\nabla u_k|^2 dx = 2^* \int_{\mathbb{R}^N} F(u_k) dx - \frac{\lambda N}{N-2} \int_{\mathbb{R}^N} u_k^2 dx. \tag{3.10}$$

By rearranging Equation (3.10), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u_k|^2 dx &= \frac{2N}{N-2}\int_{\mathbb{R}^N} F(u_k)dx - \frac{\lambda N}{N-2}\int_{\mathbb{R}^N} u_k^2 dx \Rightarrow \\
\int_{\mathbb{R}^N} |\nabla u_k|^2 dx &= \frac{2N}{N-2}\left(\int_{\mathbb{R}^N} F(u_k)dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} u_k^2 dx\right) \Rightarrow \\
\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u_k|^2 dx &= \int_{\mathbb{R}^N} F(u_k)dx - \frac{\lambda}{2} u_k^2 dx \Rightarrow \\
\frac{\lambda}{2}\|u_k\|_2^2 &= \int_{\mathbb{R}^N} F(u_k)dx - \frac{N-2}{2N}\|\nabla u_k\|_2^2. \qquad (3.11)
\end{aligned}$$

Now, using inequality (3.9) on Equation (3.11):

$$\begin{aligned}
\frac{\lambda}{2}\|u_k\|_2^2 &= \int_{\mathbb{R}^N} F(u_k)dx - \frac{N-2}{2N}\|\nabla u_k\|_2^2 \Rightarrow \\
&\leq \frac{\epsilon}{2}\int_{\mathbb{R}^N} u_k^2 dx + C_1\int_{\mathbb{R}^N} |u_k|^p - \frac{N-2}{2N}\|\nabla u_k\|_2^2 \\
&= \frac{\epsilon}{2}\|u_k\|_2^2 + C_1\|u_k\|_p^p - \frac{N-2}{2N}\|\nabla u_k\|_2^2. \qquad (3.12)
\end{aligned}$$

Let us estimate the term $\|u_k\|_p^p$ on Equation (3.12) by using an interpolation inequality between L^p spaces. Note that $2 \leq p \leq 2^*$, so that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{(1-\theta)}{2^*}$$

for some $0 < \theta < 1$.

It is easy to see that for $\theta = 0$, one gets $p = 2^*$, and for $\theta = 1$, one gets $p = 2$.

Since $u_k \in L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, then

$$\begin{aligned}
\|u_k\|_p^p &= \int_{\mathbb{R}^N} |u_k|^p dx \\
&= \int_{\mathbb{R}^N} |u_k|^{2\theta}|u_k|^{(1-\theta)2^*} dx. \qquad (3.13)
\end{aligned}$$

From Holder's inequality (see A.5) with conjugate exponents θ and $1-\theta$, we get from Equation (3.13):

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{R}^N} |u_k|^{2\theta \frac{1}{\theta}}\right)^{\theta \frac{2}{2}} \left(\int_{\mathbb{R}^N} |u_k|^{2^*(1-\theta)\frac{1}{1-\theta}}\right)^{(1-\theta)\frac{2^*}{2^*}} \\
&= \|u_k\|_2^{2\theta} \|u_k\|_{2^*}^{2^*(1-\theta)}. \qquad (3.14)
\end{aligned}
$$

We now estimate expression (3.14) by using Young inequality (A.19) with $p = \dfrac{1}{\theta}$, $q = \dfrac{1}{1-\theta}$ and putting $a = \|u_k\|_2^{2\theta}, b = \|u_k\|_{2^*}^{2^*(1-\theta)}$, where $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ thus becomes

$$
\begin{aligned}
\|u_k\|_2^{2\theta} \|u_k\|_{2^*}^{2^*(1-\theta)} &\leq \frac{\|u_k\|_2^{2\theta p}}{p} + \frac{\|u_k\|_{2^*}^{2^*(1-\theta)} q}{q} \\
&= \frac{\|u_k\|_2^{2\theta \frac{1}{\theta}}}{\frac{1}{\theta}} + \frac{\|u_k\|_{2^*}^{2^*(1-\theta)\frac{1}{1-\theta}}}{\frac{1}{1-\theta}} \\
&= \theta \|u_k\|_2^2 + (1-\theta)\|u_k\|_{2^*}^{2^*} \\
&\leq \|u_k\|_2^2 + \|u_k\|_{2^*}^{2^*} \qquad (3.15)
\end{aligned}
$$

where the last inequality comes from the fact that $\theta + (1-\theta) = 1$ and $0 \leq \theta \leq 1$.

The estimate for $\|u\|_p^p$ then finally comes from putting together expressions (3.13), (3.14) and (3.15):

$$
\|u_k\|_p^p \leq \|u_k\|_2^2 + \|u_k\|_{2^*}^{2^*}. \qquad (3.16)
$$

We now use the Gagliardo-Nirenberg-Sobolev inequality (A.15) to estimate the term $\|u_k\|_{2^*}^{2^*}$:

$$
\|u_k\|_{2^*}^{2*} \leq C_4 \|\nabla u_k\|_p^{2^*}, \quad 1 \leq p < N. \qquad (3.17)
$$

In particular, this holds for $p = 2$ because $N \geq 3$.

We may now finally go back to expression (3.12) and use the estimate for $\|u_k\|_p^p$ (given in (3.12)):

$$
\frac{\epsilon}{2}\|u_k\|_2^2 + C_1\|u_k\|_p^p - \frac{N-2}{2N}\|u_k\|_2^2 \leq \frac{\epsilon}{2}\|u_k\|_2^2 + C_1(\|u_k\|_2^2 + \|u_k\|_{2^*}^{2^*}) - \frac{N-2}{2N}\|\nabla u_k\|_2^2 \quad (3.18)
$$

Using estimate (3.17):

$$
\begin{aligned}
&\frac{\epsilon}{2}\|u_k\|_2^2 + C_1(\|u_k\|_2^2 + \|u_k\|_{2^*}^{2^*}) - \frac{N-2}{2N}\|u_k\|_{2^*}^{2^*} \\
&\leq \frac{\epsilon}{2}\|u_k\|_2^2 + C_1(\|u_k\|_2^2 + C_4\|u_k\|_{2^*}^{2^*}) - \frac{N-2}{2N}\|u_k\|_{2^*}^{2^*}
\end{aligned} \tag{3.19}
$$

From the bound (3.8), one gets

$$
0 < C_1^{2^*} < \|\nabla u_k\|_2^{2^*} < C_2^{2^*} \tag{3.20}
$$

and

$$
0 < C_1^2 < \|\nabla u_k\|_2^2 < C_2^2. \tag{3.21}
$$

Using bounds (3.20) and (3.21) for $\|u_k\|_2^{2^*}$ and $\|\nabla\|_2^2$ back on expression (3.19), we obtain:

$$
\begin{aligned}
&\leq \frac{\epsilon}{2}\|u_k\|_2^2 + C_1(\|u_k\|_2^2 + C_4 C_2^{2^*}) - \frac{N-2}{2N}\|\nabla u_k\|_2^2 \\
&= \left(\frac{\epsilon}{2} + C_1\right)\|u_k\|_2^2 + C_1 C_4 C_2^{2^*} - \frac{N-2}{2N}C_2^2 \\
&= C_\epsilon \|u_k\|_2^2 + C_5 - C_6.
\end{aligned} \tag{3.22}
$$

Thus, from estimate (3.22), one finally obtains that

$$
\begin{aligned}
\frac{\lambda}{2}\|u_k\|_2^2 &\leq C_\epsilon + C_5 + C_6 \Rightarrow \\
\left(\frac{\lambda}{2} - C_\epsilon\right)\|u_k\|_2^2 &\leq C_5 - C_6 \Rightarrow \\
\|u_k\|_2^2 &\leq \frac{C_5 - C_6}{\frac{\lambda}{2} - C_\epsilon},
\end{aligned} \tag{3.23}
$$

and so by putting $M = \dfrac{C_5 - C_6}{\frac{\lambda}{2} - C_\epsilon}$ on inequality (3.23), we arrive at

$$
\|u_k\|_2^2 \leq M, \quad \forall k \geq 1. \tag{3.24}
$$

This estimate (3.24) shows that (u_k) is bounded in $L^2(\mathbb{R}^N)$. Also, from (3.8) , we have that (∇u_k) is bounded in $L^2(\mathbb{R}^N)$. Then (u_k) is a bounded sequence in $H^1(\mathbb{R}^N)$. This concludes the proof.

□

3.4 A weakly convergent Palais-Smale sequence

We now show that if one now has a Palais-Smale sequence A.5, which must be a bounded minimizing sequence $(u_k) \subset \mathcal{P}$ by Lemma 3.2, it is also a Palais-Smale sequence for the functional I in $H^1(\mathbb{R}^N)$.

Lemma 3.3. Let $(u_k) \subset \mathcal{P}$ be a $(PS)_b$ sequence for I constrained to $\mathcal{P}$ with $b = m_{\mathcal{P}}$, then (u_k) is a $(PS)_b$ sequence for I (free).

Proof. Note that (u_k) is a minimizing sequence of I on $\mathcal{P}$, i. e. $I(u_k) \to m_{\mathcal{P}}$ and $I'|_{\mathcal{P}}(u_k) \to 0$ as $k \to \infty$. By Lemma 6, (u_k) is bounded. Let us now prove that $I'(u_k) \to 0$, as $k \to \infty$. Suppose by contradiction that this is not the case. Then, there exists $\sigma > 0$ and a subsequence $\left(u_{k_j}\right)$ with

$$\|I'(u_{k_j})\| > \sigma, \qquad \text{for all } j \geq 1 \text{ large.}$$

We claim that there exists a positive constant C such that

$$|I'(u_{k_j})(\varphi) - I'(v)(\varphi)| \leq C\|u_{k_j} - v\|\|\varphi\|, \quad \text{for all } j \geq 1 \text{ and any } v, \varphi \in H^1(\mathbb{R}^N). \tag{3.25}$$

Indeed, first note that, for every $w, \varphi, \psi \in H^1(\mathbb{R}^N)$, we have

$$I''(w)(\varphi, \psi) = \int \nabla\varphi\nabla\psi + \lambda \int \varphi\psi - \int f'(w)\varphi\psi. \tag{3.26}$$

Also, by the Mean Value Theorem, for any $u, v \in H^1(\mathbb{R}^N)$ and $\varphi \in H^1(\mathbb{R}^N)$, there exists $\xi \in (0, 1)$ with

$$I'(v)(\varphi) - I'(u)(\varphi) = I''(u + \xi(v - u))(\varphi, v - u).$$

Therefore, for all $j \geq 1$ we find $\xi_j \in (0,1)$ such that from formula (3.26) we obtain by taking into account hypothesis (f_4) and using Hölder inequality (A.5)

$$\begin{aligned} I'(u_{k_j})(\varphi) - I'(v)(\varphi) &= I''(u_{k_j} + \xi_j(v - u_{k_j}))(\varphi, v - u_{k_j}) \\ &= \int \nabla\varphi\nabla(v - u_{k_j}) + \lambda \int \varphi(v_j - u_j) \\ &- \int f'(u_{k_j} + \xi_j(v - u_{k_j}))\varphi(v - u_{k_j}) \\ &\leq C\|\varphi\|\|v - u_{k_j}\| + \int \Big| a_1 + a_2|u_{k_j} + \xi_j(v - u_{k_j})|^{p-2} \Big| |\varphi||v - u_{k_j}| \\ &\leq C\|\varphi\|\|v - u_{k_j}\|. \end{aligned}$$

In turn, taking the supremum over the $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\| \leq 1$, we get

$$\|I'(v) - I'(u_{k_j})\|_{H^{-1}} \leq C\|v - u_{k_j}\|,$$

for all $j \geq 1$ and any $v \in H^1(\mathbb{R}^N)$, which concludes the verification of the claim. Therefore, if $\|u_{k_j} - v\| < \tilde{\delta}/C := 2\delta$, then we have $\|I'(u_{k_j}) - I'(v)\| < \tilde{\delta}$, for all $v \in H^1(\mathbb{R}^N)$ and $j \geq 1$. This yields, $\sigma - \tilde{\delta} < \|I'(u_{k_j})\| - \tilde{\delta} < \|I'(v)\|$, for all $j \geq 1$ large. For $\tilde{\delta} \in (0, \sigma)$, we have $\tilde{\sigma} := \sigma - \tilde{\delta} > 0$ and

$$\forall v \in H^1(\mathbb{R}^N): \quad v \in B_{2\delta}(u_{k_j}) \implies \|I'(v)\| > \tilde{\sigma}.$$

Let us now set $\varepsilon := \min\{p/2, \tilde{\sigma}\delta/8\}$ and $S := \{u_{(k_j)\}}$. Then, by virtue of [37, Lemma 2.3], there is a deformation $\eta : [0,1] \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ at the level b, such that

$$\eta(1, I^{b+\varepsilon} \cap \mathcal{P}) \subset I^{b-\varepsilon}, \qquad I(\eta(1,u)) \leq I(u), \qquad \text{for all } u \in H^1(\mathbb{R}^N).$$

For j large enough, since u_{k_j} is minimizing for b, we have

$$\max_{t>0} I(u_{k_j}(\cdot/t)) = I(u_{k_j}) < b + \varepsilon. \tag{3.27}$$

Then, by the properties of the deformation η we can infer that

$$\max_{t>0} I(\eta(1, u_{k_j}(\cdot/t)) < b - \varepsilon.$$

On the other hand, for j and L fixed large, $\gamma(t) := \eta(1, u_{k_j}(\cdot/Lt))$ is a path in Γ since

$$I(\gamma(1)) = I(\eta(1, u_{k_j}(\cdot/L))) \leq I(u_{k_j}(\cdot/L)) = \frac{L^{N-2}}{2}\int_{\mathbb{R}^N} |(\nabla u_{k_j}|^2 - L^N \int_{\mathbb{R}^N} \left(F(u_{k_j}) - \lambda \frac{u_{k_j}^2}{2}\right)$$
$$= \frac{L^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla u_{k_j}|^2 - L^N \int_{\mathbb{R}^N} G(u_{k_j}) < 0, \quad \text{for } L \to \infty.$$

Hence, we deduce that

$$c \leq \max_{t\in[0,1]} I(\eta(1, u_{k_j}(\cdot/Lt)) = \max_{t>0} I(\eta(1, u_{k_j}(\cdot/t)) < b - \varepsilon < b,$$

contradicting that fact that $b = m_{\mathcal{P}} = c$. The lemma is then proved. □

3.5 A strongly convergent Palais-Smale sequence

A $(PS)_b$ sequence for I is bounded, therefore converges weakly in the Hilbert space, but it is not strongly convergent in general unless there is more information about compactness for the functional. This can be obtained, for instance, if the function space has some symmetry properties, as it is going to be seen in the next lemma.

Lemma 3.4. Let the functional $I : H^1_{rad}(\mathbb{R}^N) \to \mathbb{R}$ and $(u_k) \subset \mathcal{P} \subset H^1_{rad}(\mathbb{R}^N)$ be a $(PS)_b$ sequence for I with $b = m_{\mathcal{P}}$. Then, up to a subsequence, $u_k \to u \in \mathcal{P}$, strongly in $H^1(\mathbb{R}^N)$.

Proof. Since $(u_k) \subset \mathcal{P}$ is a $(PS)_b$ sequence for I, then $I(u_k) \to b$ and $I'(u_k) \to 0$ in $H^{-1}_{rad}(\mathbb{R}^N)$. By Lemma 3.2, (u_k) is bounded, hence using the Sobolev compact embeddings of $H^1_{rad}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, $2 < p < 2^*$, it follows that up to subsequences,

i) u_k converges weakly to $\bar{u}$,

ii) u_k converges to $\bar{u}$ in $L^p(\mathbb{R}^N)$, $2 < p < 2^*$,

iii) $u_k(x)$ converges to $\bar{u}(x)$, pointwise, a. e. in $\mathbb{R}^N$.

Now it is standard to show that $u_k \to u$, strongly in $H^1(\mathbb{R}^N)$. Observe that

$$\|u_k - \bar{u}\|^2 = \langle I'(u_k) - I'(\bar{u}), u_k - \bar{u}\rangle + \int_{\mathbb{R}^N} (f(u_k) - f(\bar{u}))(u_k - \bar{u}).$$

By Lemma 3.3, $I'(u_k) \to 0$, (u_k) is bounded and (i), it clearly follows that

$$\langle I'(u_k) - I'(\bar{u}), u_k - \bar{u} \rangle \to 0.$$

On the other hand, (ii), (iii) and the hypotheses (f_2) and (f_4) on f, as in [33] (Theorem 5), imply that $f(u_k) \to f(\bar{u})$ strongly in $L^{p'}(\mathbb{R}^N)$, for $p^{-1} + p'^{-1} = 1$, yielding

$$\left| \int_{\mathbb{R}^N} (f(u_k) - f(\bar{u}))(u_k - \bar{u}) \right| \to 0\,, \quad k \to \infty.$$

This completes the proof of the lemma.

□

3.6 A constructive algorithm

The general approach in solving numerically the proposed problem is the following: we restate the problem in a variational formulation on a Hilbert space with a constraint that defines the Pohozaev manifold $\mathcal{P}$ and then use the steepest descent method allied with projections on $\mathcal{P}$ to find minima of the functional I constrained to the direction found by the former. By iterating such a process, we arrive at the minimum of I constrained to $\mathcal{P}$, which is the ground state solution obtained by the Mountain Pass Lemma [2]. The formulated algorithm is derived from the rigorous theoretical results aforementioned and it converges to the positive ground state solution of problem (1.1). The main idea is to descend along paths projected on the Pohozaev manifold $\mathcal{P}$, which are precisely $\gamma(t) = u\left(\frac{\cdot}{t}\right)$ constructed from Lemma 2.10.

In the work of Choi and McKenna [10], a constructive form of the Mountain Pass Lemma of Ambrosetti and Rabinowitz [2], first formulated by Aubin and Ekeland [5], was implemented numerically by allying the finite element method with a method of steepest descent. This was done by starting with a local minimum and connecting it with a path to a point e with $I(e) \leq 0$ of lower altitude (Theorem A.8), finding the maximum of I along this path, then deforming it in such a way as to make the maximum along the path decrease as fast as possible and, finally, if that maximum turns out to have been a critical point, they stop, or else, repeat this process. They apply the algorithm in a rectangle to a homogeneous superlinear nonlinearity of type u^p, $1 < p < 2^*$, but this algorithm has been applied to problems with no symmetry assumptions, even on unbounded domains. A few years later, Ding, Costa and Chen [13] devised an algorithm for finding sign-changing solutions by constructing a link from a given critical point which would then lead to a new one, and they compute these solutions for bounded domains both with and without symmetry, for odd nonlinearities. Theoretical studies by Ding and Ni [15], a couple of years after [5], showed that a solution to the more general problem 1.1 with the nonlinearity obeying a monotonicity condition on a bounded domain (but also in $\mathbb{R}^{\mathbb{N}}$) exists by a constrained minimization argument on the so called Nehari manifold. Then, Chen, Ni and Zhou [9] used this approach to adapt the preceding algorithms and solve for more general bounded domains with projections on Nehari manifold. However, the limitation of this idea is that a unique projection is required in order to apply the constrained minimization problem successfully which, in turn, depends on the monotonicity assumption. We weaken this condition by, rather than constraining the problem to the Nehari manifold, following with the clever idea of projections on the Pohozaev manifold since, for problem (1.2) with the imposed conditions (f1)-(f4) and (g1), they are guaranteed to be unique. This extends our framework to more general problems, including nonhomogeneous superlinear problems in unbounded domains, as well as nonhomogeneous asymptotically linear problems.

Subsequently, Horák in [17] exploited numerical minimization under a general constraint and applied his algorithm of the Constrained Steepest Descent Method (CSDM) to find Mountain Pass solutions via constrained minimization on the Nehari manifold. Moreover, a second algorithm called Constrained Mountain Pass Algorithm (CMPA) is presented in [17], which enables to obtain critical points on the Nehari manifold of higher energy levels. In our work, the proposed algorithm is in analogy with CSDM for ground state solutions, applied here to the Pohozaev manifold, with one main difference: in [17] the orthogonal projection of the gradient to the tangent space to the manifold is used, whereas here a different kind of projection is employed (Lemma 3.1), as we always reproject on $\mathcal{P}$ as we descend along the steepest descent direction. Furthermore, our scheme is convenient since it repeatedly depends on taking a function $w \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(w)\,dx > 0$ and finding the maximum of the associated functional restricted to $\mathcal{P}$ by means of a direct formula using the parameter t in Lemma 2.10, and hence this approach is expected to lighten the computational cost at this step. Finally, by virtue of Theorem 2.2 the ground state solution corresponds to the minimum on the Pohozaev manifold, and so our algorithm shows to be fitted in finding this minimal action solution since it converges to the unique positive ground state solution starting from a non-zero initial guess $w_0 \in H^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} G(w_0) > 0$.

The general idea of the new algorithm is made clear in the sequel:

Step 1. Take an initial guess $w_0 \in H^1(\mathbb{R}^N)$ such that $w_0 \neq 0$ and $\int_{\mathbb{R}^N} G(w_0) > 0$, under the assumption that 0 is a local minimum of I, since I has the Mountain Pass geometry;

Step 2. Find $t_* > 0$ by means of Equation 2.90 such that

$$I\left(w_0\left(\frac{\cdot}{t_*}\right)\right) = \max_{t>0} I\left(w_0\left(\frac{\cdot}{t}\right)\right), \tag{3.28}$$

and set $w_1 = w_0\left(\frac{\cdot}{t_*}\right)$. This is possible because $\int_{\mathbb{R}^N} G(w_0) > 0$ and hence one can use Lemma 2.10;

Step 3. Find the steepest descent direction $\hat{v} \in H^1(\mathbb{R}^N)$ at $w_1 \in H^1(\mathbb{R}^N)$, from (3.3) and (3.5), obtaining $\tilde{v} = -\nabla I(w_1)$. If $\|\tilde{v}\| < \varepsilon$, then output and stop.

Step 4. For $0 < \alpha_0$ small, there exists $t(\alpha_0)$ such that $(w_1 + \alpha_0 \hat{v})\left(\frac{\cdot}{t(\alpha_0)}\right) \in \mathcal{P}$. Fix a large $K \in \mathbb{N}$ and iterate $\alpha_k := k\alpha_0$, for $k \in \mathbb{N}$ and $(w_1 + \alpha_k \hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right) \in \mathcal{P}$, that is, we descend along the steepest descent direction. In view of Lemma 3.1, we can either find α_{k_0} for $k = k_0 \leq K$ such that

$$I\left((w_1 + \alpha_{k_0}\hat{v})\left(\frac{\cdot}{t(\alpha_{k_0})}\right)\right) = \min_{\alpha_k} I\left((w_1 + \alpha_k \hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right)\right)$$

or such a minimum is not attained after K iterations and

$$I(w_1 + \alpha_k \hat{v}) \leq I\left(w_1 + \alpha_k \hat{v}\left(\frac{\cdot}{t(\alpha_k)}\right)\right) < I(w_1).$$

In the latter case, let $1 \leq k_0 \leq K$ be the largest k such that $\int_{\mathbb{R}^N} G(w_1 + k_0\alpha_0\hat{v}) > 0$. In either case, proceed to Step 5.

Step 5. Redefine $w_0 := w_1 + k_0\alpha_0\hat{v}$. Go to Step 2.

It is important to point out that the repetition of *Step 2* in the algorithm produces a sequence (w_{1j}) such that $I'|_{\mathcal{P}}(w_{1j}) \to 0$ as $j \to +\infty$ by the steepest descent argument. If $I(w_{1j}) \to b = m_{\mathcal{P}}$, when working in the space of radial functions $H^1_{rad}(\mathbb{R}^N)$ for the applications in Chapter 5, by Lemmas 3.3 and 3.4 the algorithm will extract a strongly convergent subsequence, which converges to a positive ground state solution. Considering nonlinearities $g(u) := f(u) - \lambda u$ which satisfy, for instance, the assumptions by Serrin and Tang [29] (Theorem 1), that if, for some $\beta > 0$,

(i) g is continuous on $[0, +\infty)$, with $g(u) \leq 0$ in $[0, \beta)$ and $g(u) > 0$ for $u > \beta$;

(ii) $g \in C^1(\beta, +\infty)$, with $ug'(u)/g(u)$ nonincreasing on $(\beta, +\infty)$,

it is known that the radial positive ground state is unique. Hence, the aforementioned subsequence (w_{1j}) would converge to the positive ground state solution and the algorithm will terminate successfully. On the other hand, if *Step 2* results in a constrained $(PS)_b$ sequence at a higher energy level $b > m_{\mathcal{P}}$, then in some iteration the sequence would start to present a sign-changing function $w_{1\bar{j}}$. At this point, one should go back to *Step 1* and take another initial guess w_0, but now satisfying $I(w_0) < I(w_{1\bar{j}})$. Nevertheless, we emphasize the fact that, at least for the applications presented on Chapter 5, this second case did not occur in our simulations.

Chapter 4

Implementation and Numerical experiments

4.1 Preliminaries

From the algorithm 3.6, one has to be careful in the implementation of the new method. This Chapter deals with this assessment (Section 4.3) and with numerical experiments necessary to validate our new algorithm (Section 4.4) and guarantee its convergence and robustness (Section 4.5).

4.2 Radial symmetry

The algorithm presented in the previous section is applicable for general nonlinearities, which satisfy the hypotheses stated in the introduction and can be applied to problems with no symmetry assumptions, provided one works in a scenario to regain compactness in $\mathbb{R}^{\mathbb{N}}$. However, for the sake of simplicity, we are going to implement for nonlinearities which satisfy conditions that imply that the ground state solution is radially symmetric.

4.2.1 Radial symmetry

Since $f \in \mathcal{C}^1(\mathbb{R})$ is odd and f satisfies (f1) - (f4), a classical result of Berestycki and Lions [6] establishes the existence of a ground state solution $\omega \in \mathcal{C}^2(\mathbb{R}^N)$ to the problem (1.2), which is positive, radially symmetric and decreasing in the radial direction (see Theorem 1 in [6]). In fact, by a result from Li and Ni [20], if $g'(0) \leq 0$ then any positive solution of (1.1) is, up to a translation, radially symmetric (see Theorem 1 in [20]). Moreover, this radial positive solution is unique when extra hypotheses are satisfied (see Serrin and Tang [29]). Therefore, we are going to restrict ourselves to the $H^1_{rad}(\mathbb{R}^N)$, the subspace of radial functions of $H^1(\mathbb{R}^N)$, without loss of generality. Since the functions are all radially symmetric, the integrals are calculated in the real line by a change from cartesian to spherical variables, with $u(r, \theta, \phi) = u(r)$. Moreover, all the partial differential equations involved are transformed into ordinary differential equations in the radius variable. Since we are working on $\mathbb{R}^3$, the problem is reduced to:

$$\begin{cases} -u''(r) - \dfrac{2}{r}u'(r) + \lambda u(r) = f(u(r)), \ r > 0, \\ u(r) \to 0, \ r \to +\infty, \\ u'(0) = 0. \end{cases} \tag{4.1}$$

Moreover, our functional I, projected on $\mathcal{P}$ depends on:

$$h(t) := I\left(u\left(\frac{\cdot}{t}\right)\right) = 4\pi\left(\frac{t^2}{2}\int_0^{+\infty}|u'|^2r^2dr + \frac{\lambda t^3}{2}\int_0^{+\infty}|u|^2r^2dr - t^3\int_0^{+\infty}F(u)r^2dr\right)$$

and its derivative is given by:

$$h'(t) := I'\left(u\left(\frac{\cdot}{t}\right)\right) = 4\pi\left(t\int_0^{+\infty}|u'|^2r^2dr + 3\frac{\lambda t^2}{2}\int_0^{+\infty}|u|^2r^2dr - 3t^2\int_0^{+\infty}F(u)r^2dr\right).$$

Therefore, the value of t that projects u on $\mathcal{P}$ is directly given by $h'(t) = 0$:

$$t^2 = \frac{\displaystyle\int_0^{+\infty}|u'|^2r^2dr}{6\displaystyle\int_0^{+\infty}\left[-\frac{\lambda}{2}|u|^2 + F(u)\right]r^2dr} = \frac{\displaystyle\int_0^{+\infty}|u'|^2r^2dr}{6\displaystyle\int_0^{+\infty}G(u)r^2dr}. \tag{4.2}$$

4.3 Numerical Implementation

4.3.1 Discretisation and numerical methods

We start by noting that the algorithm presented in Section 3.6 does not involve solving directly (4.1) and, therefore, it does not need to be discretised or treated numerically otherwise. The parts of the algorithm that need to be treated numerically are the calculations of the functional $I(u)$ and of the projection parameter t, which involve the calculation of integrals, and the calculation of the steepest descent direction, which is given by the Poisson problem in equation (3.2). We will describe briefly below how these were implemented.

The steepest descent direction, given by the solution of (3.2), can be found by first solving for $\tilde{v}$ in (3.3), which can be written in terms of a radially symmetric problem, that is:

$$-\tilde{v}''(r) - \frac{2}{r}\tilde{v}'(r) + \tilde{v}(r) = w_1''(r) + \frac{2}{r}w_1'(r) - \lambda w_1(r) + f(w_1), \tag{4.3}$$

with w_1 given from *Step 2*, and with boundary conditions given by

$$\tilde{v}(r) \to 0,\ r \to +\infty, \quad \text{and} \quad \tilde{v}'(0) = 0. \tag{4.4}$$

We use second order centered finite differences (see [7] and [4]) to discretise (4.3) on the interval $\Omega = [0, R^*]$. We define the mesh $\mathcal{M} = \{r_0, r_1, \dots, r_M\}$ as the set of the $M+1$ points $r_i \in \Omega$ that are used in the discretisation of the interval Ω. These points are defined as $r_i = i\Delta r$, $i = 0, 1, \dots, M$, where $\Delta r = R^*/M$ is the space step. Defining $\tilde{v}_i = \tilde{v}(r_i)$, and similarly with w_1, we obtain the discretised version of (4.3) as:

$$\alpha\tilde{v}_{i+1} + \beta\tilde{v}_i + \gamma\tilde{v}_{i-1} = \alpha' w_{1i+1} + \beta' w_{1i} + \gamma' w_{1i-1} + f(w_{1i}), \tag{4.5}$$

with

$$\alpha = \frac{1}{\Delta r^2} + \frac{1}{r_i\Delta r},\ \beta = -\left(\frac{2}{\Delta r^2} + 1\right),\ \gamma = \frac{1}{\Delta r^2} - \frac{1}{r_i\Delta r} \tag{4.6}$$

and

$$\alpha' = -\alpha,\ \beta' = \frac{2}{\Delta r^2} + \lambda,\ \gamma' = -\gamma. \tag{4.7}$$

We now observe that (4.5) is a linear system of $M+1$ equations in terms of the steepest descent function $\tilde{v}_i$, which is solved by an SOR method [38] with relaxation parameter chosen as 1.9.

Note that the boundary conditions of (4.3), given in (4.4), also have to be discretised. The first boundary condition in (4.4) is taken to be $\tilde{v}_M = 0$, where $\tilde{v}_M = \tilde{v}(R^*)$, with R^* large enough so that $\tilde{v}_M = 0$ is a good approximation of $\tilde{v}(r) \to 0$ as $r \to +\infty$. We discuss the influence of the choices of R^* in Subsection 4.5.2. The second boundary condition in (4.4) is discretised using a second order forward finite difference, which gives $\tilde{v}_0 = \dfrac{4v_1 - v_2}{3}$.

The integrals involved in the algorithm 3.6 were evaluated using a standard trapezoidal rule,

$$\int_0^{R^*} h(r)dr = \left(\frac{h(0)+h(R^*)}{2} + \sum_{i=1}^{M-1} h(r_i)\right)\Delta r + \mathcal{O}(\Delta r^3), \tag{4.8}$$

for a function $h(r)$. Note that the truncation error in this approximation is $\mathcal{O}(\Delta r^3)$.

Finally, we note from Section 3.2 that the steepest descent function $\tilde{v}$ has to be normalised in order to obtain $\hat{v}$, and therefore the solution obtained in (4.5) has to be divided by 2μ, as detailed in (3.5) and mentioned in *Step 3*, so that we can control with α_k how much we descend along the steepest descent direction. However, we must keep track of the actual value of the norm of the steepest descent function found, since we need it to assess the convergence of the algorithm, as stated on the *Step 3* in Section 3.6.

4.3.2 Pohozaev projection step

Given an initial guess $w_0 \in H^1_{rad}$, one can verify that $\int_{\mathbb{R}^N} G(w_0) > 0$, which is done by calculating this integral using the trapezoidal rule on the interval Ω. First, by Lemma 2.10, we calculate t_* by solving for t in (4.2). Then, we need to construct the function w_1 from the relation $w_1(\cdot) = w_0\left(\frac{\cdot}{t_*}\right)$. However, we note that w_1 is calculated from w_0 evaluated at rescaled points which, unless $t_* = 1$, are not the ones available in the mesh $\mathcal{M}$. We could estimate the value of w_0 at the rescaled points via an interpolation procedure, but this would be an unnecessary source of error to the numerical algorithm. Instead, we choose to rescale the mesh $\mathcal{M}$ by rescaling all its points, i.e., $\mathcal{M} = \{\tilde{r_0}, \tilde{r}_1, \ldots, \tilde{r}_M\}$, with $\tilde{r_i}$ given by $\tilde{r_i} = r_i t_*$. The function w_1 can now be simply constructed by setting $w_1(\tilde{r_i}) = w_0(r_i)$ for $i = 0, 1, \ldots, M$. Note that, although this procedure does not introduce the additional error of interpolation procedures to the numerical algorithm, the interval Ω, the mesh $\mathcal{M}$ and the space step Δr have to be recalculated at each projection step.

Moreover, in *Step 4*, projections of the line $w_1 + \alpha_k\hat{v}$ with varying α_k are calculated for $t(\alpha_k)$ by, again, solving for t in (4.2). Note that this is done in the same setting as *Step 2*.

4.3.3 Descending on Pohozaev manifold

First, evaluate the functional I on $w_1 \in \mathcal{P}$. We consider a given α_0 (typically we choose $\alpha_0 = 10^{-1}$) and we evaluate $I\left((w_1 + \alpha_k\hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right)\right)$, for increasing integers k, until we find $k = \bar{k}$ such that

$$I\left((w_1 + \alpha_{\bar{k}}\hat{v})\left(\frac{\cdot}{t(\alpha_{\bar{k}})}\right)\right) > I\left(w_1 + \alpha_{\bar{k}-1}\hat{v})\left(\frac{\cdot}{t(\alpha_{\bar{k}-1})}\right)\right). \tag{4.9}$$

When this $\bar{k}$ is found, we redefine $w_1^{new} := w_1 + \alpha_{\bar{k}-1}\hat{v}$ and project it on $\mathcal{P}$. We then take $\alpha_0 \leftarrow \alpha_0/10$ and repeat the procedure until, with the desired accuracy, we reach the minimum of I along the steepest descent direction $\hat{v}$. Typically, this is done until $\alpha_0 = \alpha_{min} = 10^{-10}$. It should be noted that, depending on the local topology of $I(u)$, the algorithm might identify a local minimum for which, after the refinement of α_0 takes place, we have both

$$I\left((w_1 + \alpha_k\hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right)\right) > I\left((w_1 + \alpha_{k-1}\hat{v})\left(\frac{\cdot}{t(\alpha_{k-1})}\right)\right) \tag{4.10}$$

and

$$I\left((w_1 + \alpha_k\hat{v})\left(\frac{\cdot}{t(\alpha_k)}\right)\right) > I\left(w_1 + \alpha_{k+1}\hat{v})\left(\frac{\cdot}{t(\alpha_{k+1})}\right)\right). \tag{4.11}$$

In fact, the algorithm has found a local maximum instead. The strategy in this case is to choose the function that gives the minimum on the right hand side of equations (4.10) and (4.11), and set it as w_1^{new}. The descent procedure would then carry on as described in (4.9).

Remark 4.1. We recall here that our algorithm is not exempt from finding solutions other than the ground state. If the second case in *Step 4* leads to a sequence of functions for which the associated energy functional I asymptotes a constant value, then *Step 3* may give a steepest descent direction for which its norm goes to zero, and so we may find a critical point w_c. Radially symmetric critical points of higher energy levels do exist (see [34], [6]). In our applications where the ground state is positive radially symmetric (and unique), it suffices to check if w_c changes sign or not. At this point, we check if w_c is a positive function and, in case it is not, we then return to *Step 1* by taking an initial guess w_0 such that $I(w_0) < I(w_c)$ in order to proceed with the search for the ground state solution.

4.4 Validation

In order to verify that the implementation of our new algorithm is correct, we compare the result with the solution found via a different method (the midpoint method with Richardson extrapolation implemented in Maple 2018). The singularity of the equation is dealt with by assuming that the boundary conditions are defined as $u = \varepsilon_1$ at $r = 100$, at which point we expect that u is sufficiently close to zero, and $u'(\varepsilon_2) = \varepsilon_3$, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10^{-35}$. The results are plotted in Figure 4.1. We observe a very good agreement with the result obtained by the aforementioned method.

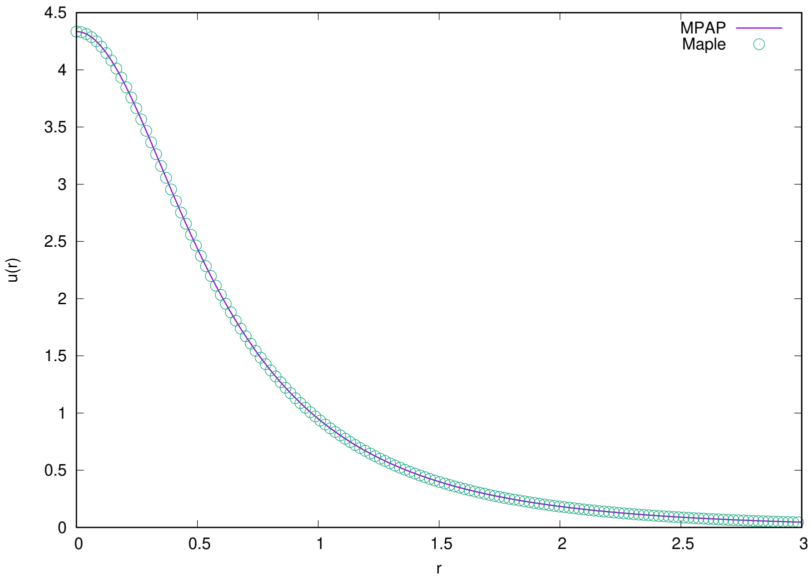

Figure 4.1 Comparison between the results given by the mini-max algorithm (heavy line) and Maple (circles) for $f(u) = u^3$ and $\lambda = 1.0$, with standard set of parameters.

4.5 Convergence, dependence on the initial guess and other numerical experiments

4.5.1 Convergence

We now assess the influence of the discretisation size Δr on the results. Figure 4.2 shows a comparison of the solution given by the algorithm presented in Section 3.6 for several values of Δr. On Figure 4.2 (left), we plot the solution obtained for different mesh sizes, corresponding to Δr ranging from 0.02 to 0.0004, and we observe that no significant differences on the profile of the solution can be noticed. However, we do note that there is a difference on the tail of the solution when Δr changes. Nevertheless, the differences are minor and due to the fact that the final length of Ω is actually calculated by the algorithm during the projection step, and will change depending on M and on the initial value of R^*. This will be discussed further on Section 4.5.2. Figure 4.2 (right) indicates that this phenomenon does not compromise significantly the value of $I(u)$ for sufficiently large M. For $M > 400$, the differences among the solutions are negligible and the differences between consecutive curves and the values of $I(u)$ become smaller and smaller as M grows.

4.5.2 Initial size of the domain

The point where the boundary condition at infinity is imposed at the beginning of the simulations defines the size R^* of the domain Ω in which we define w_0. We have to choose R^* sufficiently large, so that the numerical boundary condition is as realistic as possible, since we are looking for solution in $H^1_{rad}(\mathbb{R}^N)$. The effects of the choice of R^* on the final results is assessed by measuring $||\tilde{v}||$ at the end of the simulations for different values of R^*. The results, shown in Figure (4.3), indicate that the smaller R^*, the larger the final $||\tilde{v}||$ will be. As R^* increases, we observe that $||\tilde{v}||$ decays as ${R^*}^{-2}$ until it reaches a plateau at around $R^* = 10$. For larger values of R^*, there is no significant change on the final value of $||\tilde{v}||$. This indicates that, for each choice of Δr, there is a minimum critical value of R^* that must be chosen in order to achieve the best possible value of $||\tilde{v}||$ in the end of the simulations. Even further, for two different choices of Δr, as R^* increases, we observe that the plateau is reached for the same R^*.

4.5.3 Robustness

Finally, we compare the results obtained by the standard choice of numerical parameters of our algorithm with a coarser mesh in which we have also reduced the values of α_{min} to 10^{-2} and the tolerance for the SOR algorithm tol_{SOR} to the determination of the steepest descent direction also to 10^{-2}. We observe very good agreement between the results, that is, the overall profile of the solution in the coarser approximation reproduces the shape of the refined solution, with the exception of the values close to $r = 0$. In fact, the value of the functional $I(u)$ is only overestimated by around 0.1% when the solution found by using the coarser parameters is used. This indicates that the algorithm

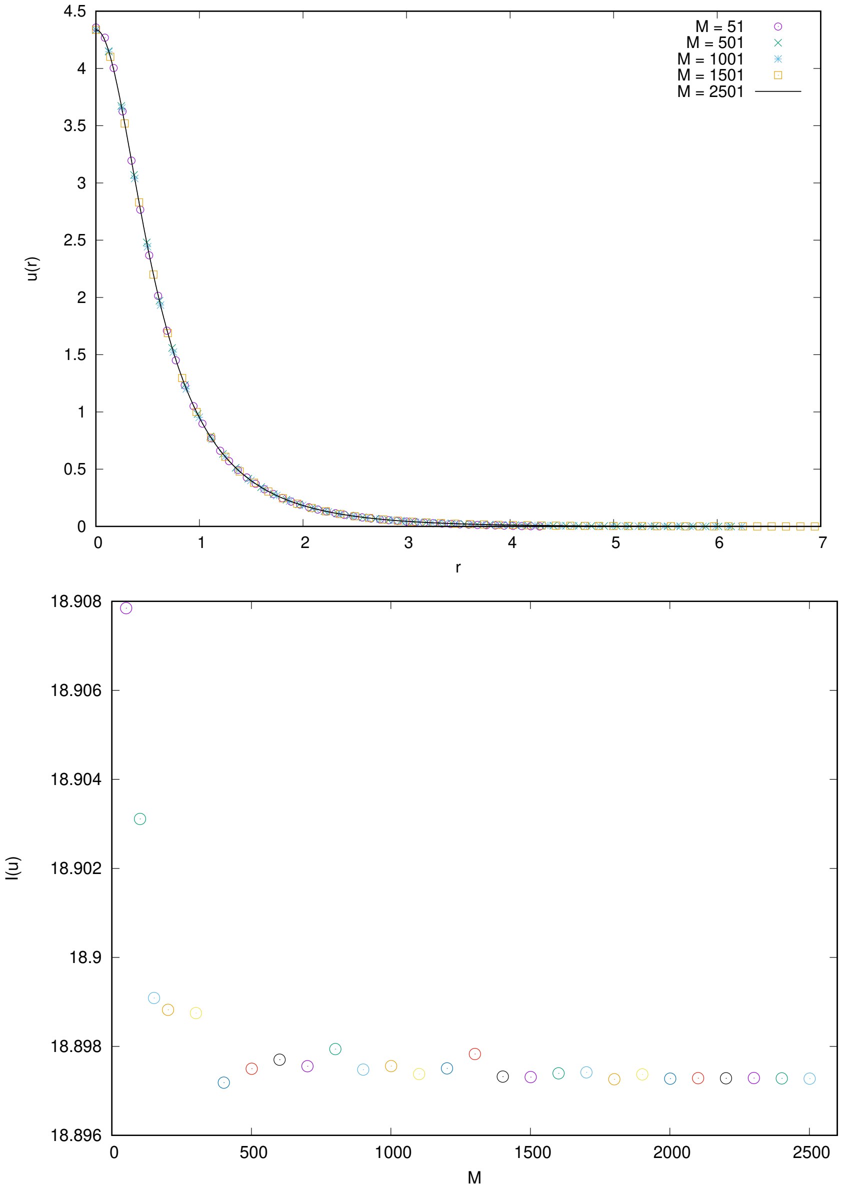

Figure 4.2 Comparison of the solution obtained by MMAP for different values of M (top) and the values of $I(u)$ (bottom).

is very robust and converges to the desired function even with very limited computational resources.

Remark 4.2. Solving the Poisson equation (4.3) is expected to be the most computationally expensive part of our algorithm and so, a parameter which must be given a good amount of significance is the tolerance for the convergence of the SOR. Since we are unaware of the local topology of the functional I, our initial guess w_0 from *Step 1* might have high energy or be far from $\mathcal{P}$. Being so, at first, the tolerance tol_{SOR} on

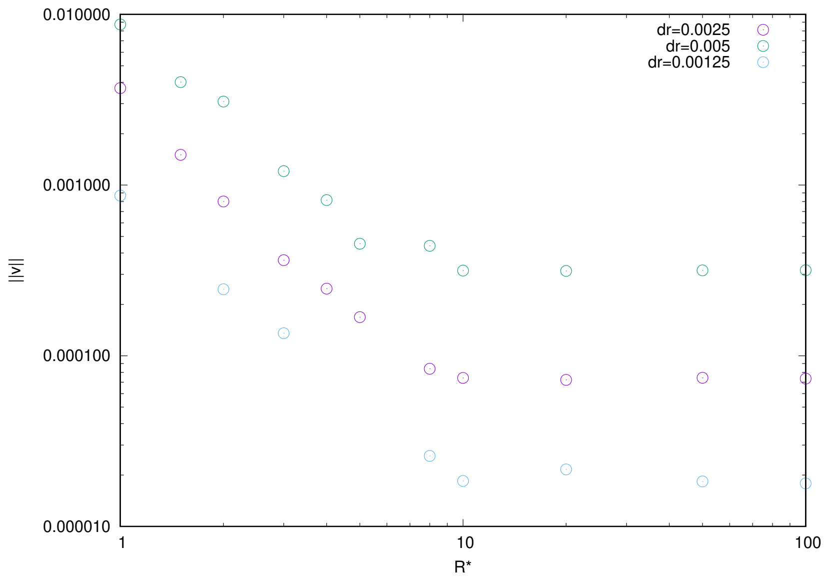

Figure 4.3 Influence of the initial size R^* of the domain Ω versus the norm of the steepest descent direction obtained on the last iteration, for $\Delta r = 0.00125; 0.0025; 0.005$. The decay of $||v||$ to the minimum value obtained for large R^* is roughly given by R^{*-2}.

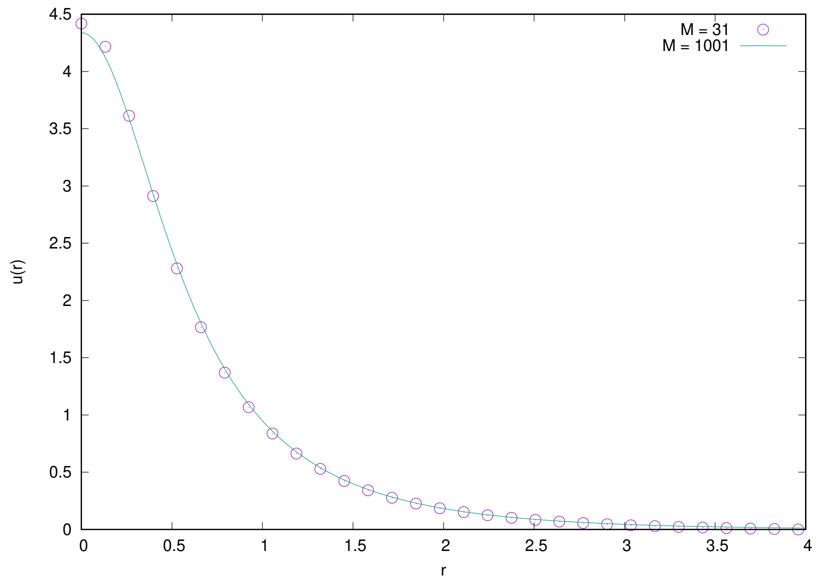

Figure 4.4 Comparison between the results given by the standard set of parameters (heavy line), and by the coarser mesh (circles), in which $M = 31$, $\alpha_{min} = 10^{-2}$ and $tol_{SOR} = 10^{-2}$.

the calculation of the steepest descent direction might be relaxed but, once we get close enough to the sought mini-max solution, this parameter must be refined.

Remark 4.3. For the choice of the initial guess w_0 in *Step 1*, by the fact that in order to project w_0 on the Pohozaev manifold it is sufficient that the restriction $\int_{\mathbb{R}^N} G(w_0)\, dx > 0$ is satisfied, such a condition is mild compared to the initial guesses in the other algorithms in the literature since, according to Lemma 2.10, any suitable $w_0 \in H^1(\mathbb{R}^N)\setminus\{0\}$ can be chosen.

Also, if one takes a function with exponential decay and calibrates the gradient term and the nonlinearity term to construct an initial that has a high enough energy functional level, this means the algorithm can 'descend' the initial guess along the iterations to converge to the ground state solution. In contrast, choosing a suitable initial guess, but that has low energy, the algorithm is likely to search forever in lower levels.

Remark 4.4. We note that *Step 4*, which involves the reprojection to $\mathcal{P}$ of the functions obtained during the descent stage of the algorithm, can be relaxed to a less computationally intensive version if we choose to perform the reprojections every N_r steps. In fact, we have run several tests for N_r ranging from 2 to 100 and no noticeable changes were observed neither on the shape of the solution nor on the value of $I(u)$ for the case $f(u) = u^3$, $\lambda = 1.0$ with the standard set of parameters.

Chapter 5

Applications

5.1 Preliminaries

The aim of this Chapter is to provide the reader with examples of nonlinearities f such that the problem 1.2 can be tackled numerically by the algorithm proposed in Section 3.6.

In Section 5.2 we use the algorithm to solve problem 1.2 for the case where f is superlinear. In Section 5.3 we deal with an asymptotically linear f and in Section 5.4 we consider the case where $I(tu)$ has two maxima for $t > 0$, and so previous algorithms in the literature could not find a solution due to the lack of a unique projection.

5.2 Superlinear Problems

5.2.1 The case $f(u) = u^3$ in $\mathbb{R}^3$

For superlinear nonlinearities $|u|^p$, $1 < p < 2^* - 1$, the algorithms proposed prior to this work were able to tackle problem (1.2), which can also be managed by our algorithm. We can, apart from the validations performed in the previous section, assess its precision in calculating the maximum of the solution, which is attained in the origin, by recalling that the positive solution is radially symmetric and decreasing in the radial direction. Simple calculations show that

$$u_\lambda(r) = \lambda^{\frac{1}{p-1}} u_1(\sqrt{\lambda}\, r), \tag{5.1}$$

is the positive solution of problem (1.2), with $f(u) = u^3$, where u_1 is the positive solution with $\lambda = 1.0$.

In Table 5.1 we present the maximum heights $u(0)$ for several values of λ, obtained by our algorithm. On the other hand, assuming that the height of u_1 is given by our algorithm, that is, $u_1(0) = 4.33691$, we calculate $u_\lambda(0)$ for $\lambda = 0.1, 0.5, 2.0, 3.0$ using (5.1). The comparison of the heights $u(0)$ obtained numerically and the height $u_\lambda(0)$ obtained by (5.1) gives an error that is less than 0.1%. Figure 5.1 shows the profiles of the solutions of problem 1.2 obtained by the algorithm for those values of λ.

Table 5.1 Results for $u(0)$ for the case $f(u) = u^3$ obtained for different values of λ. In this table, we present the value of the norm of the steepest descent $||v||$ at the end of the calculations, of $I(u)$ for the solution and the relative error of $u(0)$ with respect to the theoretical value $u_\lambda(0)$ in (5.1).

λ	$u(0)$	$\|\|v\|\|$	$I(u)$	error
0.1	1.37148	$5.6 \cdot 10^{-4}$	5.97615	$< 0.1\%$
0.5	3.06678	$4.0 \cdot 10^{-4}$	13.36246	$< 0.1\%$
1.0	4.33691	$6.0 \cdot 10^{-4}$	18.89734	–
2.0	6.13321	$7.7 \cdot 10^{-4}$	26.72488	$< 0.1\%$
3.0	7.51153	$9.3 \cdot 10^{-4}$	32.73110	$< 0.1\%$

5.3 Aymptotically linear Problems with a monotonicity condition

5.3.1 The case $f(u) = \dfrac{u^3}{1 + su^2}$ in $\mathbb{R}^3$

The asymptotically linear problems $\dfrac{|u|^p}{1 + s|u|^{p-1}}$, $1 < p < 2^* - 1$, $0 < \lambda s < 1$, satisfy the monotonicity condition $f(u)/u$ increasing for $u > 0$ and so, could be handled by the algorithms in [9] - since projections on the Nehari manifold rely on this hypothesis - but were not attempted. Using the devised algorithm, we have found the ground state solution in the case $f(u) = \dfrac{u^3}{1 + su^2}$. Figure 5.2 shows the solution for this nonlinearity with $\lambda = 1.0$ and $s = 0.5$. For reference purposes, we include on Table 5.2 the values of $u(0)$ for the positive solution u. Also, Figure 5.3 shows the descending energy of the functional from the initial guess w_0, here chosen as $100\, e^{-10\, r^2}$, to the solution. For the other values of λ and s, the initial guesses we chose were similar, and of the form

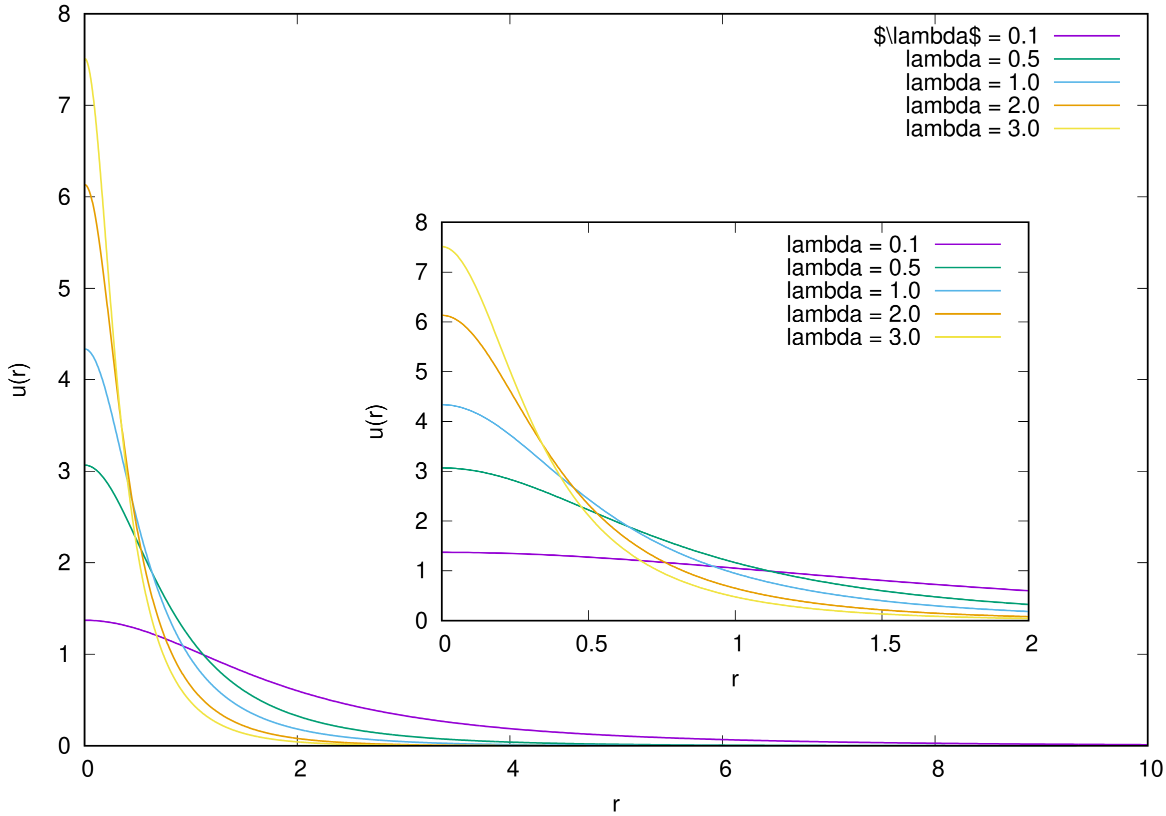

Figure 5.1 Profile of solutions for $f(u) = u^3$ for different values of λ.

$a\exp(-br^2)$, with a ranging from 4.5 to 100.0 and b ranging from 2 to 10. For validation purposes, we present on Table 5.3 a list of values of the solution found for $\lambda = 1.0$ and $s = 0.5$.

5.4 Other examples

The real improvements of our algorithm compared to others in the literature are presented in the next two examples. In order to obtain the positive ground state solution of (1.2), if the nonlinear term $f(u)$ satisfies conditions (f1)-(f4) our algorithm is applicable and gives the correct solution, whereas other existing algorithms cannot be applied either because it requires unique projections on the Nehari manifold [9] or because superquadratic conditions on the nonlinearity f are assumed [10].

Table 5.2 Values of $u(0)$ obtained for the case $f(u) = \dfrac{u^3}{1+su^2}$ for several combinations of λ and s. Note that we can only obtain solutions when $\lambda s < 1$. $M = 3501$.

		λ					
		0.1	0.3	0.5	0.7	1.0	5.0
	0.1	1.33183	2.23513	2.84300	3.34310	3.99690	12.61528
	0.3	1.29034	2.18677	2.87000	3.51098	4.50062	–
s	0.5	1.27125	2.22308	3.05319	3.94794	5.64139	–
	0.7	1.26344	2.29849	3.33592	4.65516	8.08286	–
	1.0	1.26374	2.46503	3.98912	6.76196	–	–
	5.0	1.78424	–	–	–	–	–

Table 5.3 Values of $u(r)$ for $f(u) = \dfrac{u^3}{1+su^2}$ for several r with $\lambda = 1.0$, $s = 0.5$.

r	$u(r)$	r	$u(r)$	r	$u(r)$	r	$u(r)$
0.000	5.64139	2.004	2.99197	5.005	0.11309	8.007	1.86676×10^{-3}
0.100	5.63348	2.205	2.58907	5.207	8.88979	8.208	8.34267×10^{-4}
0.201	5.60837	2.608	1.84032	5.601	5.56388×10^{-2}	8.300	3.91292×10^{-4}
0.302	5.56672	3.002	1.23610	6.003	3.45536×10^{-2}	8.351	1.55421×10^{-4}
0.402	5.50879	3.203	0.98899	6.204	2.72241×10^{-2}	8.376	3.87317×10^{-5}
0.604	5.34578	3.605	0.61708	6.607	1.68230×10^{-2}	8.384	0.000000
1.006	4.84857	4.007	0.37890	7.000	1.03282×10^{-2}		
1.199	4.54191	4.201	0.29952	7.202	7.93701×10^{-3}		
1.601	3.80120	4.603	0.18367	7.604	4.38170×10^{-3}		

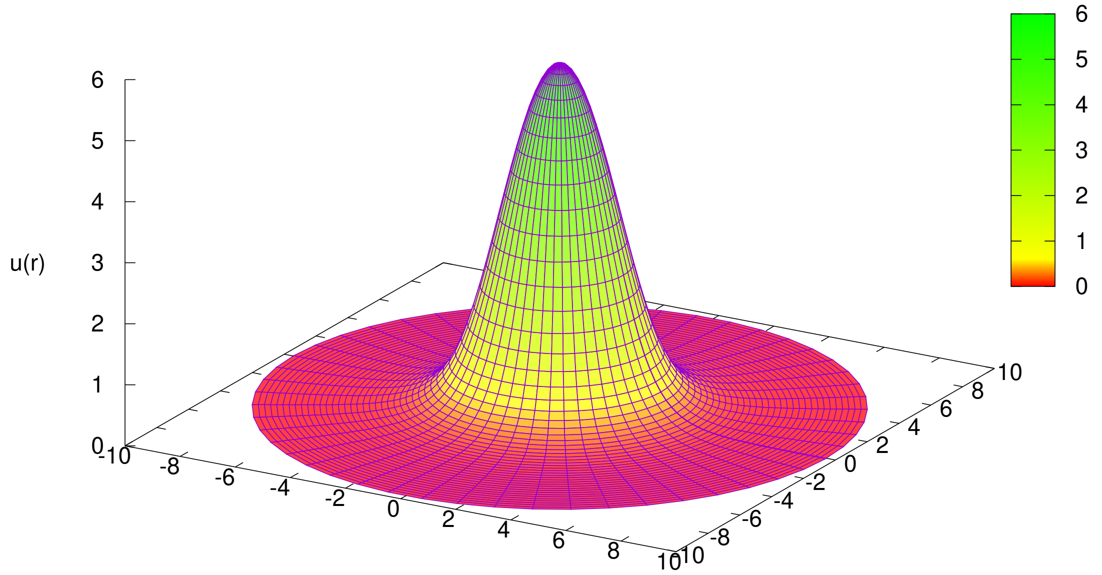

Figure 5.2 Surface plot of solution for $f(u) = \dfrac{u^3}{1+su^2}$ with $\lambda = 1.0$, $s = 0.5$. $u(0) = 5.64139$, $I(u) = 161.92929$, $||v|| = 2.5 \times 10^{-4}$.

5.4.1 Example where $I(tu)$ has two maxima for $t > 0$

This example illustrates a situation where the functional I evaluated in the direction tu, for $t \in \mathbb{R}$, has at least two maximum values at t_1 and t_2, for instance, and hence the algorithm MPA developed by Chen, Ni and Zhou in [9], which takes the unique projection on the Nehari manifold on the direction of the vector u (*Step 3*), does not work.

Choosing $F(u) = Bu^3 - Cu^4 + Du^5$ in (1.10), and so

$$f(u) = 3Bu^2 - 4Cu^3 + 5Du^4, \tag{5.2}$$

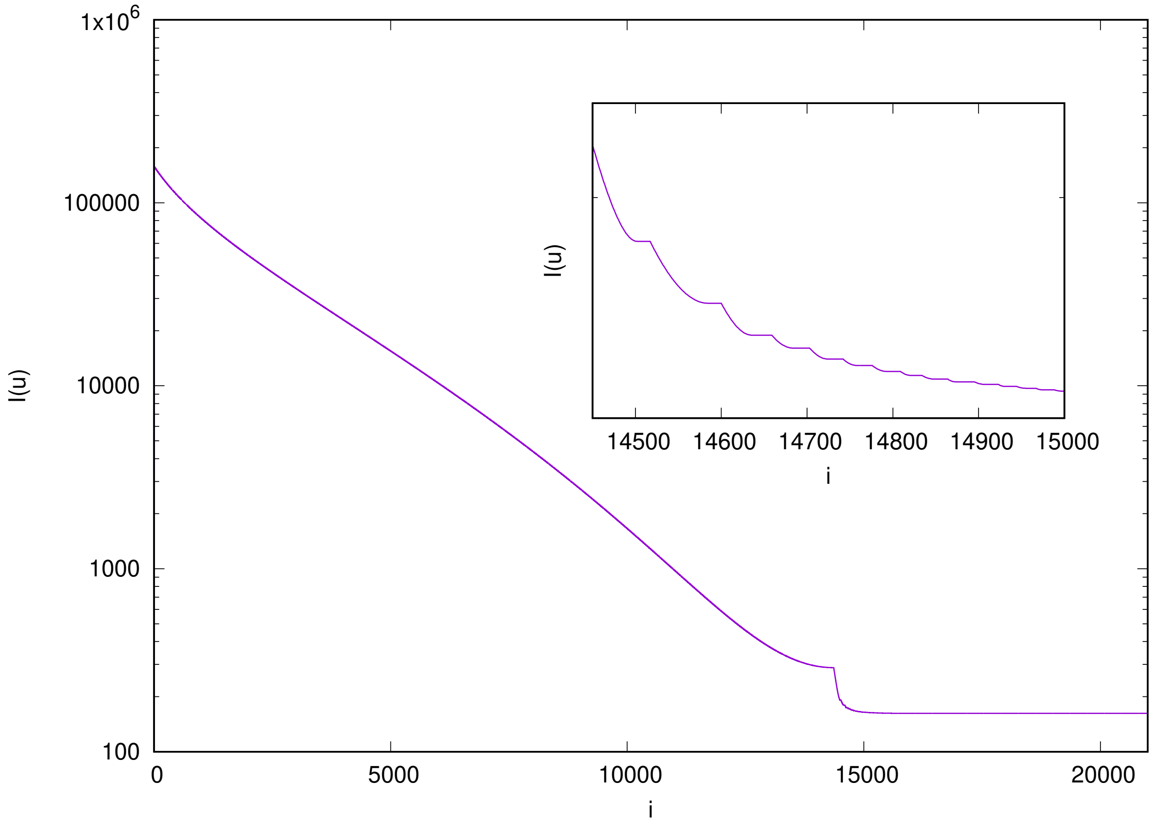

Figure 5.3 Descending energy of the associated functional along the iterations of the algorithm, for $f(u) = \dfrac{u^3}{1+su^2}$ with $\lambda = 1.0$, $s = 0.5$. Logarithmic scale on the y axis.

with $\lambda = 1$, and taking

$$u(r) = \begin{cases} \dfrac{1}{\sqrt{4\pi}} \quad , & |r| \leq R \\ \dfrac{1}{\sqrt{4\pi}} \quad e^{-|R-r|}, & |r| \geq R \end{cases}$$

with $R \approx 3.075$, $A = \dfrac{||u||^2}{2}$ and positive constants B, C and D such that

$$\begin{aligned} I(tu) &= t^2 \frac{||u||}{2} - \int F(tu) \\ &= t^2 A - Bt^3 \int u^3 + Ct^4 \int u^4 - Dt^5 \int u^5 \\ &= -t^5 + (5+\sqrt{5})t^4 - 2(4+\sqrt{5})t^3 + 4(1+\sqrt{5})t^2 \end{aligned}$$

gives rise to an example for $I(tu)$ having two maxima. Figure 5.4 shows $u(r)$ and $I(tu)$. Those two maxima are given by $I(r_1) = I(r_2) = \dfrac{128}{25\sqrt{5}}$. The profile of the solution for problem 1.2 with $f(u)$ as in (5.2), with $\lambda = 3.0$ solved by our mini-max algorilthm is shown in Figure 5.5.

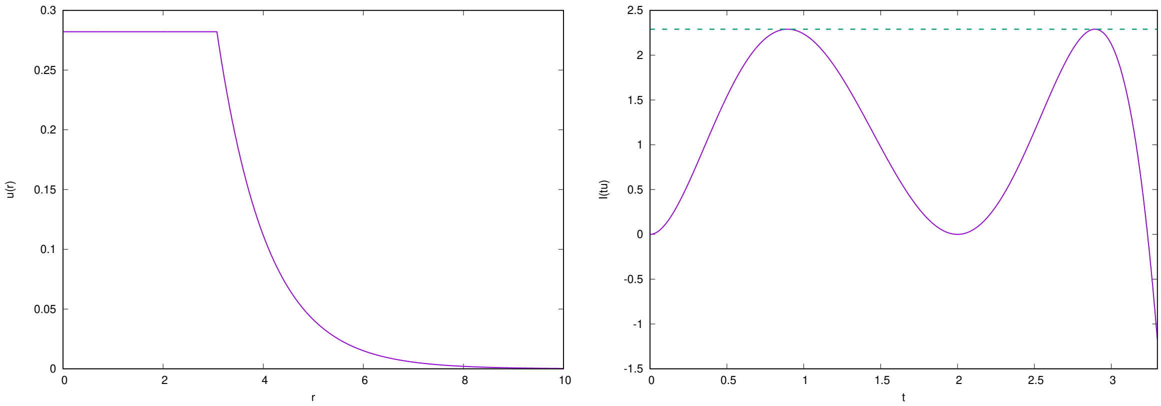

Figure 5.4 An example of function u (left) for which $I(tu)$ has two maxima (right), with $f(u)$ as in (5.2).

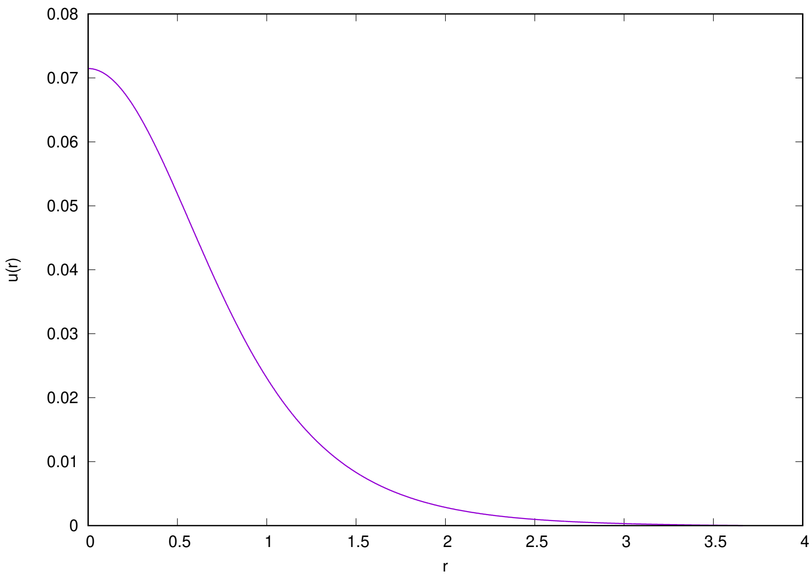

Figure 5.5 Profile of solution for the nonlinearity (5.2) for $\lambda = 3.0$.

5.5 Concluding remarks

The algorithm presented in this paper is based in a novel approach of finding a critical point of a functional associated to the Euler equation, which may model Physical problems, by constrained minimization method in the appropriate Pohozaev manifold. The main advantage is that it can tackle asymptotically linear as well as superlinear problems with no assumption of monotonicity on $f(u)/u$. This improves previous results by solving for those problems already studied and complementing with new problems which could not be treated by the preceding algorithms in the literature.

Remark 5.1. It is important to observe that the algorithm can be applied to general nonlinearities $g(u)$ as long as it satisfies the conditions (g1)-(g4) and provided the associated functional possesses a mountain pass geometry, hence one can hope to visualize ground state solutions for a wide class of elliptic problems in $\mathbb{R}^N$.

Remark 5.2. The example

$$f(u) = \frac{u^7 - \frac{5}{2}u^5 + 2u^3}{1 + u^6}, \tag{5.3}$$

shown in Figure 5.6 (left), does not satisfy the monotonicity condition of $f(u)/u$, shown in Figure 5.6 (right), increasing in the variable u, for $u > 0$. However, taking $0 < \lambda < 1$, projections on the Pohozaev manifold can be performed, hence algorithm 3.6 can be applied.

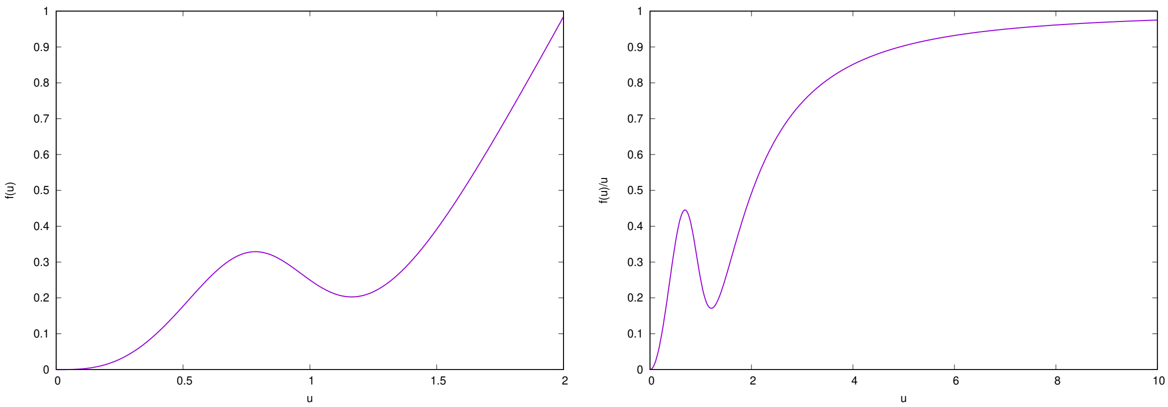

Figure 5.6 Example of a nonlinearity for which the monotonicity condition does not hold. $f(u)$ (left) and $f(u)/u$ (right).

Remark 5.3. This algorithm can also be extended to tackle numerically weakly coupled elliptic systems, such as

$$\begin{cases} -\Delta u + u = |u|^{2q-2}u + b|v|^q|u|^{q-2}v \quad \text{in} \quad \mathbb{R}^N \\ -\Delta v + \omega^2 v = |v|^{2q-2}v + b|u|^q|v|^{q-2}v \quad \text{in} \quad \mathbb{R}^N \\ u(x) \to 0, v(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{5.4}$$

where $\omega, b > 0$ are constants and q is such that $2 < q < 2^* = \dfrac{2N}{N-2}$ if $N = 3$, and $2^* = +\infty$ if $N = 1, 2$. This problem was studied by Maia-Montefusco-Pellacci in [21].

When $q = 2$, the system becomes

$$\begin{cases} -\Delta u + u = |u|^2 u + b|v|^2 u \quad \text{in} \quad \mathbb{R}^N \\ -\Delta v + \omega^2 v = |v|^2 v + b|u|^2 v \quad \text{in} \quad \mathbb{R}^N \\ u(x) \to 0, v(x) \to 0 \quad \text{as} \quad |x| \to \infty, \end{cases} \tag{5.5}$$

In Figure 5.7 (left), one has fixed $b = 30$ with varying ω. In Figure 5.7 (right), one has fixed $\omega = 2$ with varying b.

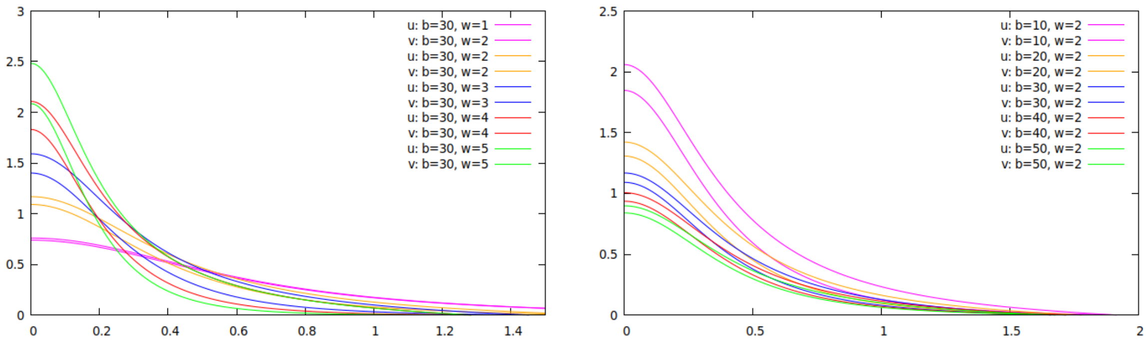

Figure 5.7 The ground state solutions (u,v) for a few pairs of the (b, ω) parameters.

Remark 5.4. Finally, the theoretical backing of the algorithm 3.6 is the variational method where the associated functional I is defined on the Hilbert space $H^1(\mathbb{R}^N)$, which is continuously embedded in $L^{2^*}(\mathbb{R}^N)$. Hence, critical and supercritical nonlinear terms, $\lim_{u \to +\infty} f(u)/u^p = +\infty$, with $p \geq 2^* - 1$, cannot be accessed.

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Appendix A

Auxiliary results

A.1 Divergence theorem

A reference can be found in [16].

Theorem A.1. Consider U a bounded open subset of $\mathbb{R}^N$ and with boundary ∂U of class C^1.

We have

$$\int_U div\, u\, dx = \int_{\partial U} u \cdot \nu\, dS \tag{A.1}$$

for each vector field $u \in C^1(\overline{U}, \mathbb{R}^N)$.

A.2 Dual Space of H^1

A reference can be found in [16] (Section 5.9.)

Definition A.1. We denote by $H^{-1}(\mathbb{R}^N)$ the dual space to $H^1(\mathbb{R}^N)$. f belongs to $H^{-1}(\mathbb{R}^N)$ if f is a bounded linear functional on $H^1(\mathbb{R}^N)$.

A.3 Ekeland Variational Principle

A reference can be found in [34] (Chapter 1, Section 5).

Theorem A.2. Let M be a complete metric space endowed with a metric d and let $E : M \to \mathbb{R} \bigcup +\infty$ be lower semicontinuous, bounded from below, and not equal to ∞.

Then for any $\epsilon, \delta > 0$ and any $u \in M$ with

$$E(u) \leq \inf_M E + \epsilon,$$

there is an element $v \in M$ strictly minimizing the functional

$$E(w) + \frac{\epsilon}{\delta} d(v, w).$$

Moreover, one has that

$$E(u) \leq E(v), \quad d(u, v) \leq \delta.$$

A.4 Fatou Lemma

See [16] for a reference.

Theorem A.3. Let $\Omega \subset \mathbb{R}^N$ be a Lebesgue-measurable set and (f_n) a sequence of Lebesgue-measurable, non-negative functions defined on Ω. Then

$$\int_\Omega \liminf_{n\to\infty} f_n \, dx \leq \liminf_{n\to\infty} \int_\Omega f_n \, dx.$$

A.5 Hölder Inequality

See [16] for a reference.

Theorem A.4. Given $1 \leq p \leq +\infty$, let $u \in L^p(\Omega)$ and $v \in L^{p'}$, where p' is the conjugate exponent of p, i.e. it is such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ (and we set $p' = +\infty$ when $p = 1$.) Then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)},$$

A.6 Fréchet differentiability

A reference can be found in [3].

Theorem A.5. Consider X, Y Banach Spaces and let $L(X, Y)$ denote the space of linear continuous maps from X to Y.

We say that $f : U \to Y$ is Fréchet differentiable at $u \in U \subset X$ with derivative $f' \in L(X,Y)$ if $f(u+h) = f(u) + \langle f'(u), h\rangle + o(\|h\|)$, as $h \to 0$.

Furthermore, f is said Fréchet differentiable on U if it is Fréchet differentiable at every point $u \in U$.

The limit

$$\lim_{\epsilon\to 0} \frac{f(u+\epsilon h) - f(u)}{\epsilon} \quad .=. \quad \langle f'(u), h\rangle$$

is the **Fréchet derivative** of f at $u \in X$.

A.7 Gatêaux differentiability

See [36] for a reference.

Definition A.2. Let X be a Banach space and considrer a functional $Phi : X \to \mathbb{R}$. We say Φ is **Gatêaux differentiable** at a point $u \in X$ when there exists a linear functional T_0 such that

$$\lim_{t\to 0} \frac{\Phi(u+tv) - \Phi(u) - T_0 v}{t} \quad \forall\, v \in X. \tag{A.2}$$

When such a linear functional exists, it is unique and T_0 is called the **Gatêaux derivative** of Φ at point $u \in X$. We denote it by $D\Phi(u)$.

A.8 Lagrange Multiplier Theorem on Banach Spaces

See [12] for a reference.

Theorem A.6. Consider Z a Banach space and X^* its dual. If the continuously differentiable functional f has a local extremum under the constraint $H(x) = \theta$ at the regular point x_0, then there exists an element $z_0^* \in Z^*$ such that the Lagrangian functional

$$L(x) = f(x) + \langle H(x), z_o^*\rangle$$

is such that

$$f'(x) + \langle z_0^*, H'(x_0)\rangle = \theta.$$

A.9 Lebesgue Dominated Convergence Theorem

A reference can be found in [27].

Theorem A.7. Let $\Omega \subset \mathbb{R}^N$ a Lebsegue-measurable set and (f_n) a sequence of Lebesgue-measurable, integrable functions defined on Ω. Suppose there exists a Lebesgue-measurable function $f : \Omega \to \mathbb{R}$ such that

$$f_n(x) \to f(x), \text{a.e. in } \Omega,$$

and suppose there exists an integrable function $g : \Omega \to \mathbb{R}$ such that

$$|f_n| \leq g, \text{a.e. in } \Omega, \forall n \in \mathbb{N}.$$

Then f is integrable and

$$\int_\Omega f\,dx = \lim_{n\to\infty} \int_\Omega f_n\,dx.$$

A.10 Mountain Pass Lemma of Ambrosetti-Rabinowitz

A reference can be found in [16], [22] and [2].

Theorem A.8. Assume $I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ such that, $I(0) = 0$ and

(I_1) there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$, and

(I_2) there exists an $e \in H^1(\mathbb{R}^N) \setminus \overline{B_\rho(0)}$ and $I(e) \leq 0$. Define

$$c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I(u), \tag{A.3}$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) | \gamma(0) = 0, \gamma(1) = e\}.$$

Then, if I satisfies $(PS)_c$, the level c is a critical level of I, i.e, there exists $u \in H^1(\mathbb{R}^N)$ such that $I(u) = c$ and $I'(u) = 0$.

A.11 Pohozaev identity

A proof of the following result can be found in many references, such as [6] and [25]. Nevertheless, we prove it here in a detailed manner.

Theorem A.9. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and let

$$G(t) = \int_0^t g(s)\, ds.$$

Let u satisfy

$$\begin{cases} -\Delta u = g(u) \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{A.4}$$

Assume furthermore that $u \in L^\infty_{loc}(\mathbb{R}^N)$, $\nabla u \in L^2(\mathbb{R}^N)$ and $G(u) \in L^1(\mathbb{R}^N)$.
Then

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u)\, dx. \tag{A.5}$$

Proof. First, because $u \in L^\infty_{loc}(\mathbb{R}^N)$, from the partial differential equation $-\Delta u = g(u)$ we get $u \in L^p_{loc}(\mathbb{R}^N)$. Since the laplacian belongs to $L^p_{loc}(\mathbb{R}^N)$, then $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ for any $1 \leq p < \infty$.

Now multiply (A.4) by $x \cdot \nabla u$ and compute

$$0 = (-\Delta u + g(u))\,(x \cdot \nabla u) \tag{A.6}$$

By Green's Formula, if $\Omega \subset \mathbb{R}^N$ is an open, bounded, smooth set and $u \in C^2(\overline{\Omega})$, $v \in C^1(\overline{\Omega})$, then

$$\int_\Omega (\Delta u) \cdot v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \nu d\sigma - \int_\Omega \nabla u \cdot \nabla v dx, \tag{A.7}$$

where $\nu = \nu(x)$ is the outward vector normal to $\partial\Omega$ at x, $\dfrac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu(x)$ and σ is the surface measure on $\partial\Omega$. Applying Green's Formula to (A.6), we get

$$-\int_\Omega (\Delta u)\,(x \cdot \nabla u)\, dx = \int_\Omega \nabla u \cdot \nabla\,(x \cdot \nabla u)\, dx - \int_\Omega \frac{\partial u}{\partial \nu}\,(x \cdot \nabla u)\, d\sigma \tag{A.8}$$

As for the other term in (A.6), we may treat it in an easier manner :

$$g\left(u\right)\left(x\cdot\nabla u\right)=x\cdot\nabla\left(G\left(u\right)\right)=\nabla\left(\frac{1}{2}\left|x\right|^{2}\right)\cdot\nabla\left(G\left(u\right)\right),$$

where we apply Green's Formula:

$$\int_{\Omega}g\left(u\right)\left(x\cdot\nabla u\right)dx=\int_{\Omega}\nabla\left(\frac{1}{2}\left|x\right|^{2}\right)\cdot\nabla\left(G\left(u\right)\right)dx \tag{A.9}$$

$$=\int_{\partial\Omega}G\left(u\right)\frac{\partial}{\partial\nu}\left(\frac{1}{2}\left|x\right|^{2}\right)d\sigma-N\int_{\Omega}G\left(u\right)dx \tag{A.10}$$

Since $G(u(x))=G(0)=0$ in $\partial\Omega$,

$$\int_{\Omega}g\left(u\right)\left(x\cdot\nabla u\right)dx=-N\int_{\Omega}G\left(u\right)dx. \tag{A.11}$$

Now, we shall treat the Equation (A.8) to better understand its terms:

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(x\cdot\nabla u\right)=\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)&=\sum_{i=1}^{N}\left(\frac{\partial x_i}{\partial x_j}\frac{\partial u}{\partial x_i}+x_i\frac{\partial^2 u}{\partial x_j\partial x_i}\right)\\&=\sum_{i=1}^{N}\left(\delta_{ij}\frac{\partial u}{\partial x_i}+x_i\frac{\partial^2 u}{\partial x_j\partial x_i}\right)\\&=\sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_j\partial x_i}x_i+\frac{\partial u}{\partial x_j}\right)\end{aligned} \tag{A.12}$$

We can use this expression to compute $\nabla u\cdot\nabla\left(x\cdot\nabla u\right)$:

$$\begin{aligned}\nabla u\cdot\nabla\left(x\cdot\nabla u\right)&=\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N}\right)\cdot\left[\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_N}\right)\left(\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)\right]\\&=\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_N}\right)\cdot\left[\frac{\partial}{\partial x_1}\left(\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right),\dots,\frac{\partial}{\partial x_N}\left(\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)\right],\end{aligned}$$

where we use (A.12) to get:

$$\begin{aligned}
\nabla u \cdot \nabla (x \cdot \nabla u) &= \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right) \cdot \left[\sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_i \partial x_1} x_1 + \frac{\partial u}{\partial x_1}\right), \ldots, \sum \left(\frac{\partial^2 u}{\partial x_i \partial x_N} x_N + \frac{\partial u}{\partial x_N}\right)\right] \\
&= \frac{\partial u}{\partial x_1} \sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_i \partial x_1} x_1 + \frac{\partial u}{\partial x_1}\right) + \ldots + \frac{\partial u}{\partial x_N} \sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_i \partial x_N} x_N + \frac{\partial u}{\partial x_N}\right) \\
&= \sum_{j=1}^{N} \frac{\partial u}{\partial x_j} \sum_{i=1}^{N}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} x_i + \frac{\partial u}{\partial x_j}\right) \\
&= \sum_{j=1}^{N} \frac{\partial u}{\partial x_j} \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i \partial x_j} x_i + \sum_{j=1}^{N} \frac{\partial u}{\partial x_j} \sum_{i=1}^{N} \frac{\partial u}{\partial x_j} \\
&= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j}\right)^2\right) x_i + \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j},
\end{aligned}$$

where on the first term of this last expression we used the identity

$$\frac{\partial}{\partial x_i}\left(\frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j}\right)^2\right) = 2\frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) = \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

As for the second term, we simply use that

$$\sum_{j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} = |\nabla u|^2,$$

and so we get

$$\begin{aligned}
\nabla u \cdot \nabla (x \cdot \nabla u) &= \frac{1}{2}\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N}\left(\frac{\partial u}{\partial x_j}\right)^2\right) x_i + |\nabla u|^2 \\
&= \frac{1}{2} x \cdot \nabla \left(|\nabla u|^2\right) + |\nabla u|^2 .
\end{aligned}$$

Then, by using the identity $x = \nabla\left(\frac{1}{2}|x|^2\right)$, we are then left with:

$$\nabla u \cdot \nabla (x \cdot \nabla u) = \frac{1}{2}\nabla\left(\frac{1}{2}|x|^2\right) \cdot \nabla\left(|\nabla u|^2\right) + |\nabla u|^2 . \tag{A.13}$$

Now, following from (A.8), by putting $u = \frac{1}{2}|x|^2$, $v = |\nabla u|^2$ and integrating Equation (A.13) we will have:

$$\int_\Omega \nabla u \cdot \nabla (\nabla u \cdot x)\, dx = \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Omega \nabla \left(\frac{1}{2}|x|^2\right) \cdot \nabla \left(|\nabla u|^2\right) dx \tag{A.14}$$

$$= \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\left(\int_{\partial\Omega} \frac{\partial}{\partial v}\left(\frac{1}{2}|x|^2\right)|\nabla u|^2 \, d\sigma - \int_\Omega \Delta\left(\frac{1}{2}|x|^2\right)|\nabla u|^2 \, dx\right)$$

But

$$\Delta\left(\frac{1}{2}|x|^2\right) = \nabla \cdot \nabla \left(\frac{1}{2}|x|^2\right) = \nabla \cdot \nabla \left(\frac{1}{2}\langle x, x\rangle^2\right) = \nabla \cdot \left(2 \cdot \frac{1}{2}\nabla \left(\langle x, x\rangle\right)\right) = N,$$

and

$$\frac{\partial}{\partial \nu}\left(\frac{1}{2}|x|^2\right) = \frac{\partial}{\partial \nu}\left(\frac{1}{2}\left(\langle x, x\rangle\right)^2\right) = \frac{\partial}{\partial \nu}\left(\langle x, x\rangle\right) = \langle \frac{\partial x}{\partial \nu}, \frac{\partial x}{\partial \nu}\rangle = \nu(x) \cdot x,$$

and so

$$\int_\Omega \nabla u \cdot \nabla (\nabla u \cdot x)\, dx = \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial \nu}\left(\frac{1}{2}|x|^2\right)|\nabla u|^2 \, d\sigma - \frac{N}{2}\int_\Omega |\nabla u|^2 dx$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla (\nabla u \cdot x)\, dx = \frac{2-N}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial \nu}\left(\frac{1}{2}|x|^2\right)|\nabla u|^2 \, d\sigma. \tag{A.15}$$

Finally, substituting (A.15) in (A.8) gives:

$$-\int_\Omega \Delta u\,(x \cdot \nabla u)\, dx = \frac{2-N}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial \nu}\left(\frac{1}{2}|x|^2\right)|\nabla u|^2 \, d\sigma - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u\,(x \cdot \nabla u)\, d\sigma \tag{A.16}$$

In order to deal with the boundary terms, we note that $u = 0$ on $\partial\Omega$, and then we have $\nabla u(x) = \dfrac{\partial u(x)}{\partial \nu}\nu(x)$ for every $x \in \partial\Omega$, so that $|\nabla u| = \left|\dfrac{\partial u}{\partial \nu}\right|$ and $\nabla u \cdot x = \dfrac{\partial u}{\partial \nu}\nu(x) \cdot x$ on $\partial\Omega$. With this information, we may simplify this last expression to:

$$-\int_{\Omega} \Delta u\left(x\cdot\nabla u\right)dx = \frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}dx+\frac{1}{2}\int_{\partial\Omega}\frac{\partial}{\partial\nu}\left(\frac{1}{2}\left|x\right|^{2}\right)\left|\nabla u\right|^{2}d\sigma-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\frac{\partial u}{\partial\nu}\nu\left(x\right)\cdot x\right)d\sigma \tag{A.17}$$

$$\frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}d\sigma\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial u}{\partial\nu}\right|^{2}d\sigma=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\nabla u\left(x\right)\cdot x\right)d\sigma$$

$$\Rightarrow \frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}d\sigma\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial u}{\partial\nu}\right|^{2}d\sigma=\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\left(\frac{\partial u}{\partial\nu}\nu\left(x\right)\cdot x\right)d\sigma$$

$$\Rightarrow \frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}d\sigma=\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial u}{\partial\nu}\right|^{2}d\sigma$$

$$-\int_{\Omega}\Delta u\left(x\cdot\nabla u\right)dx=\frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}dx-\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial u}{\partial\nu}\right|^{2}d\sigma. \tag{A.18}$$

Finally, putting (A.18) and (A.11) together:

$$\int_{\Omega}\Delta u\left(x\cdot\nabla u\right)dx-\int_{\Omega}g\left(u\right)\left(x\cdot\nabla u\right)dx=\frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}dx+N\int_{\Omega}G\left(u\right)dx-\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial\nu}{\partial v}\right|^{2}d\sigma$$

$$\Rightarrow\int_{\Omega}\left(-\Delta u-g\left(u\right)\right)\left(x\cdot\nabla u\right)dx=\frac{2-N}{2}\int_{\Omega}\left|\nabla u\right|^{2}dx+N\int_{\Omega}G\left(u\right)dx-\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial\nu}{\partial v}\right|^{2}d\sigma$$

$$\Rightarrow\frac{N-2}{2}\int_{\Omega}\left|\nabla u\right|^{2}dx-N\int_{\Omega}G\left(u\right)=-\frac{1}{2}\int_{\partial\Omega}\nu\left(x\right)\cdot x\left|\frac{\partial u}{\partial\nu}\right|^{2}d\sigma, \tag{A.19}$$

which is Pohozaev identity (for a bounded domain). We can then proceed with the calculations to obtain the Pohozaev identity in the case where the domain is actually the whole $\mathbb{R}^N$. For this, consider the bounded domain to be the open ball with radius R centered at the origin, that is, $\Omega = B_R(0)$. We then rewrite (A.19):

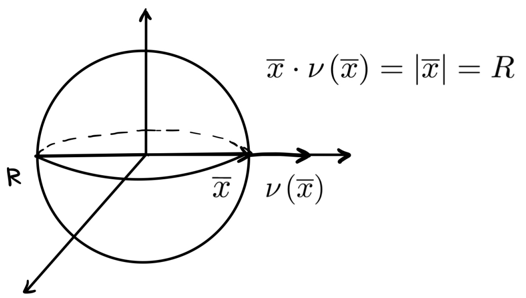

Figure A.1 The inner product of the position and the normal vector on the open ball $B_R(0)$.

$$\frac{N-2}{2}\int_{B_R(0)} |\nabla u|^2 \, dx - N\int_{B_R(0)} G(u) = -\frac{1}{2}\int_{\partial B_R(0)} \nu(x)\cdot x \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma \tag{A.20}$$

Now, note that on the open ball $B_R(0)$, we have that $x \cdot \nu(x) = |x| = R$, which can be easily seen from the schematics in Figure A.1.

With this in mind, we substitute this back on our calculations on Equation (A.20):

$$\frac{N-2}{2}\int_{B_R(0)} |\nabla u|^2 \, dx - N\int_{B_R(0)} G(u)\, dx = -\frac{1}{2}R\int_{\partial B_R(0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma$$

$$\Rightarrow \int_{B_R(0)} |\nabla u|^2 \, dx - \frac{2N}{N-2}\int_{B_R(0)} G(u)\, dx = -\frac{1}{2}\frac{2R}{N-2}\int_{\partial B_R(0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma$$

$$\Rightarrow \int_{B_R(0)} |\nabla u|^2 \, dx - \frac{2N}{N-2}\int_{B_R(0)} G(u)\, dx = -\frac{2R}{N-2}\left(\frac{1}{2}\int_{\partial B_R(0)} \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma\right). \tag{A.21}$$

From here, since we know $|\nabla u| = \left(\frac{\partial u}{\partial v}\right)$, then we may rewrite the identity (A.21):

$$\int_{B_R(0)} |\nabla u|^2 dx - \frac{2N}{N-2}\int_{B_R(0)} G(u) dx = \frac{-2R}{N-2}\left(\frac{1}{2}\int_{\partial B_R(0)} |\nabla u|^2 d\sigma + \int_{\partial B_R(0)} G(u) d\sigma\right) \tag{A.22}$$

We will now show that, at least for one suitably chosen sequence $R_n \to +\infty$, the right hand side of the Equation (A.22) above converges to zero. In order to do so, let us note that since $|\nabla u| \in L^2(\mathbb{R}^N)$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx &= \int_0^{+\infty} \int_{\partial B_R(0)} |\nabla u\,(r,\theta)|^2 \, dS_r r^{N-1} dr \\
&= \int_0^{+\infty} r^{N-2} r \int_{\partial B_R(0)} |\nabla u\,(r,\theta)|^2 \, dS_r dr \\
&= \int_0^{+\infty} r^{N-2} \int_{\partial B_R(0)} |\nabla u\,(r,\theta)|^2 \, r dS_r dr \\
&= \int_0^{+\infty} r^{N-2} \int_{\partial B_R(0)} |\nabla u\,(r,\theta)\,|^2 x \cdot \eta dS_r dr < \infty.
\end{aligned}
$$

Now, we will suppose that there does not exist a sequence $r_n \to \infty$ such that

$$r_n \int_{\partial B_R(0)} \frac{1}{2} |\nabla u\,(r,\theta)|^2 \, dS_r \to 0.$$

Then it must hold

$$\liminf_{r \to \infty} \int_{\partial B_R(0)} \frac{1}{2} |\nabla u\,(r,\theta)|^2 \, dS_r = k > 0.$$

By putting

$$\zeta\,(r) = r \int_{\partial B_R(0)} \frac{1}{2} |\nabla u\,(r,\theta)|^2 \, dS_r > 0,$$

we have

$$\int_0^{+\infty} r^{N-2} \zeta\,(r)\, dr > \int_{R_0}^{+\infty} r^{N-2} \zeta\,(r)\, dr > k \int_{R_0}^{+\infty} r^{N-2} dr = +\infty,$$

a contradiction, since $|\nabla u| \in L^2(\mathbb{R}^N)$. Then there exists a sequence $r_n \to \infty$ such that

$$r_n \int_{\partial B_R(0)} \frac{1}{2} |\nabla u\,(r,\theta)|^2 \, dS_r \to 0.$$

In a similar manner, since we have

$$\int_{\mathbb{R}^n} \frac{1}{2} |\nabla u|^2 + G\,(u)\, dx = \int_0^{+\infty} \left(\int_{\partial B_R(0)} \frac{1}{2} |\nabla u|^2 + G\,(u)\, ds \right) dr < +\infty,$$

then there exists a sequence $r_n \to \infty$ as $n \to \infty$ such that

$$r_n \int_{\partial B_R(0)} \frac{1}{2} |\nabla u\,(r,\theta)|^2 + G\,(u\,(r,\theta))\, dS_r \to 0,$$

because if

$$\liminf_{r \to +\infty} r \int_{\partial B_R(0)} |\nabla u(r,\theta)|^2 + G(u(r,\theta))\, dS_r = \alpha$$

were true, then it would be that

$$\int_{\partial B_{R(0)}} \frac{1}{2} |\nabla u|^2 + G(u)\, dS \notin L^1(0,+\infty),$$

a contradiction.

Thus, there exists a sequence $r_n \to \infty$ as $n \to +\infty$ such that the right hand side of (A.22) converges to zero.

Then, from the fact that

$$\int_{B_{R(0)}} |\nabla u|^2\, dx \to \int_{\mathbb{R}^N} |\nabla u|^2\, dx$$

and

$$\int_{B_{R(0)}} G(u) dx \to \int_{\mathbb{R}^N} G(u) dx,$$

by putting $R = r_n$ and $r_n \to \infty$, we derive the Pohozaev identity for the case where the domain is $\mathbb{R}^N$:

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u)\, dx. \tag{A.23}$$

To conclude, note that in our calculations we proved the Pohozaev identity for the case of a bounded domain Ω, where we then took in particular the open ball centered at the origin with radius R, i.e. $\Omega = B_R(0)$. By using an argument of growing the radii R_n of these balls, after showing that the boundary terms go to zero when $R_n \to +\infty$, we can pass to the limit and find the identity for the case where $\Omega = \mathbb{R}^N$.

This concludes the proof.

□

A.12 Principle of Symmetric Criticality

It is due to Palais. One can find a rerefence on Ambrosetti, Malchiodi [3].

Theorem A.10. Suppose that the topological group G acts on a Hilbert space X through isometries and let $\Phi \in C^1(x, \mathbb{R})$ be $G-$invariant $\Rightarrow$ any critical point of Φ on $Fix(G) = \{u \in X : gu = u, \forall g \in G|u \quad \text{is a critical point of} J\ \}$ on X.

Remark: in our case, the topological group G is the rotation group $SO(N)$, where N is the dimension. Our solutions $u \in H^1(\mathbb{R}^N)$ are invariant under the action of this group.

A.13 Riesz Representation Theorem

One can find a reference for this important theorem in [16].

Let H be a Hilbert space endowed with inner product $(.,.)_H$. Let H^* denote its dual space, that is, the collection of all bounded linear functionals $f : H \to \mathbb{R}$ on H. If $u \in H$, $f \in H^*$, we denote $\langle f, u \rangle$ for the real number $f(u)$.

Theorem A.11. For each $f \in H^*$ there exists a unique element $v \in H$ such that

$$\langle f, v \rangle = (u, v)_H \quad \forall f \in H^*.$$

A.14 Schwartz Lemma

See [28] for a reference.

Theorem A.12. Let X be a Banach space. A functional $\Phi : X \to \mathbb{R}$ is of class C^1 if, and only if, the following two conditions hold:

1. for all $u \in X$, the Gatêaux derivative $D\Phi(u); X \to \mathbb{R}$ exists and is a bounded linear operator;

2. the differential operator $D\Phi : X \to X^*$ is continuous.

Here, X^* is the dual space of X. Also, the Gatêaux and the Fréchet derivative coincide.

A.15 Schwarz symmetrization

A reference can be found in [6].

Definition A.3. Consider $f \in L^1(\mathbb{R}^N)$. Then f^*, the **Schwarz symmetrized function** of f, is a radial, nonincreasing (in r) and measurable function such that, given $\alpha > 0$,

$$m\{f^* \geq \alpha\} = m\{|f| \geq \alpha\},$$

where m denotes the Lebesgue measure.

One has that

$$\int_{\mathbb{R}^N} F(f)\, dx = \int_{\mathbb{R}^N} F(f*)\, dx$$

for every continuous function F such that $F(f)$ is integrable.

A.16 Sobolev embeddings

References can all be found in [16].

If W and V are normed spaces such that $W \subset V$, we say that W is continuously embedded in V when the inclusion application $i : W \to V$, given by $i(x) = x$, is continuous. If this application is also compact, we say that W is compactly embedded in V.

A.16.1 Continuous embedding theorems

Theorem A.13. Let $\Omega \subset \mathbb{R}^N$ be an open set with smooth boundary. Then the following embeddings are continuous:

1. $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $N \geq 3$ and $2 \leq p \leq 2^* = \dfrac{2N}{N-2}$

2. $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $N = 1, 2$ and $2 \leq p < \infty$.

A.16.2 Compact embedding theorems

Theorem A.14. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded subset with smooth boundary. Then the following embeddings are compact:

1. $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $N \geq 3$ and $2 \leq p < 2^* = \dfrac{2N}{N-2}$

2. $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $N = 1, 2$ and $2 \leq p < \infty$.

Thus one can see that the for $N \geq 3$, the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ with $p = 2^*$ is not compact, and for $N = 1, 2$ one has that all the continuous embeddings $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \leq p < \infty$ are compact.

A.16.3 Gagliardo-Nirenberg-Sobolev inequality

Theorem A.15. If $1 \leq p < N$. Then there exists a constant C, depending only on p and N, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)} \quad \forall\, u \in C_c^1(\mathbb{R}^N), \tag{A.24}$$

where N is the dimension, and p^* is the Sobolev conjugate of p, that is,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}, \quad p > p^*.$$

A.16.4 Palais-Smale Compactness Condition

A reference can be found in [16].

Definition A.4. A functional $I \in C^1(H; \mathbb{R})$, where H is a Hilbert space, is said to satisfy the Palais-Smale compactness condition if each sequence $(u_k)_{k=1}^{\infty} \subset H$ such that

- $(I(u_k)_{k=1}\infty$ is bounded and

- $I'(u_k) \to 0$ in H

is precompact in H.

Definition A.5. The sequence $(u_k)_{k=1}^{\infty} \subset H$ is called a Palais-Smale sequence

A.17 Strauss Compactness Lemma

A reference can be found in [6] and [33].

Theorem A.16. Consider $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying

$$\frac{P(s)}{Q(s)} \to 0 \quad as \quad |s| \to +\infty.$$

Let (u_n) be a sequence of measurable functions mapping $\mathbb{R}^N$ to $\mathbb{R}$ such that

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| \, dx < +\infty$$

and

$$P(u_n(x)) \to v(x) \quad \text{a.e.} \quad in \mathbb{R}^N, \quad \text{as} \quad n \to +\infty.$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n(x)) - v(x)| \, dx \to 0 \quad \text{as} \quad n \to +\infty.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \to 0 \quad \text{as} \quad s \to 0,$$

and also assume that $u_n(x) \to 0$ as $|x| \to +\infty$, uniformly with respect to n, then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

A.18 Strauss Radial Lemma

A reference can also be found in [6] and [33]. This next Lemma the uniform decay at infinity some radial functions present.

Theorem A.17. Consider $N \geq 2$. Every radial function $u \in H^1(\mathbb{R}^N)$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$ and such that

$$|U(x)| \leq C_N |x|^{(1-N)/2} \|u\|_{H^1(\mathbb{R}^N)} \quad \text{for} \quad |x| \geq \alpha_N,$$

where the constants C_N and α_N depend only on the dimension.

A.19 Vainberg Theorem

The following result can be found in [6] and [8].

Theorem A.18. Let $\Omega \subset \mathbb{R}^N$ be a Lebesgue-measurable set, (f_n) a sequence of functions such that $f_n \in L^p(\Omega)\, \forall n \in \mathbb{N}$ and $f \in L^p(\Omega)$ such that $\|f_n - f\|_{L^p(\Omega)} \to 0$ as $n \to \infty$. Then there exist a function $h \in L^p(\Omega)$ and a subsequence (f_{n_k}) such that

1. $f_{n_k}(x) \to f(x)$ a.e. in Ω.

2. $|f_{n_k}(x)| \leq h(x)$ a.e. in Ω for all $k \in \mathbb{N}$.

A.20 Young Inequality

See [16] for a reference.

Theorem A.19. Let $1 < p, q < \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

for $a, b > 0$.