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CIANI, Simone; FIGUEIREDO, Giovany M.; SUÁREZ, Antonio. Existence of positive eigenfunctions to an anisotropic elliptic operator via the sub-supersolution method. **Archiv der Mathematik**, v. 116, p. 85–95, 2021. DOI 10.1007/s00013-020-01518-4. Disponível em: <https://link.springer.com/article/10.1007/s00013-020-01518-4>. Acesso em: 03 nov. 2022.



Existence of positive eigenfunctions to an anisotropic elliptic operator via the sub-supersolution method

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Abstract. Using the sub-supersolution method, we study the existence of positive solutions for the anisotropic problem

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda u^{q-1} \quad (0.1)$$

where Ω is a bounded and regular domain of \mathbb{R}^N , $q > 1$, and $\lambda > 0$.

Mathematics Subject Classification. 35K65, 35B65, 35B45, 35K20.

Keywords. Anisotropic p -Laplacian, Positive solution, Sub-supersolution, Eigenvalues.

1. Introduction. In this paper, the main goal is to show the existence of positive solutions of the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda u^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $p_i > 1$, $i = 1, \dots, N$, $q > 1$, and λ is a real parameter. We will assume without loss of generality that the p_i are ordered increasingly, that is, $p_1 \leq \dots \leq p_N$.

There is a vast literature concerning anisotropic elliptic problems. We mention here only those references most strongly related to (1.1). First, in [9], it was proved that for $q < p_N$ and for any $\gamma > 0$, there exist $\lambda_\gamma > 0$ and a solution u_γ of (1.1) with $\|u_\gamma\|_p = \gamma$ and $\lambda = \lambda_\gamma$. As the authors themselves claim, from this result, the existence of solutions of (1.1) can not be deduced for a given λ . In [4], using mainly variational methods, it was proved that if $p_1 < q < p_N$, then there exist $0 < \lambda_* \leq \lambda^*$ such that:

- If $\lambda \leq \lambda_*$, (1.1) does not possess a positive solution.

- If $\lambda > \lambda^*$, (1.1) possesses at least one positive solution.

Finally, from the general results of [14, Corollary 1], we can deduce that for the case $1 < q < p_1$, with $p_1 \geq 2$, there exist $0 < \lambda_* < \lambda_{**}$ such that (1.1) possesses at least one solution for $\lambda \in (0, \lambda_*) \cup (\lambda_{**}, \infty)$.

In this paper, we complete and improve the above results. For that, we use the sub-supersolution method, see [1, 5], and [16] (see also [6–8], and references therein for the application of this method to problems with nonlinear reaction functions including singularities or critical exponent). The existence and regularity theory involving p -Laplacian type equations is nowadays mature enough to allow the discovery of new results by application to special cases suitably chosen. This method allows us not only to prove the existence of a solution, but also gives us lower and upper bounds of such a solution. Roughly speaking, we construct by hand sub- and supersolutions to the main equation, inspired by the usual way of determining energy estimates for these kind of equations. Specifically, our main result is the following.

Theorem 1.1.

1. Assume that $1 < q < p_1$. There exists a positive solution of (1.1) if and only if $\lambda > 0$.
2. Assume that $p_1 \leq q < p_N$. There exists $\Lambda > 0$ such that (1.1) does not possess positive solutions for $\lambda < \Lambda$ and (1.1) possesses at least one positive solution for $\lambda > \Lambda$.

An outline of the paper is the following: in Section 2, we recall some definitions and some properties of the eigenvalues and eigenfunctions of the classical p -Laplacian. Next, in Section 3, we enunciate the sub-supersolution method. Then in Section 4, we construct sub- and supersolutions by multiplication of powers of p -Laplacian eigenfunctions to be applied in the existence theorem.

2. Preliminary lemmas and setting. Consider $h(x, s) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Caratheodory function, i.e., measurable in x and continuous in the second variable s . Consider the anisotropic problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = h(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The natural framework to study (2.1) is the anisotropic Sobolev space $W_0^{1,\mathbf{p}}(\Omega)$, that is, the closure of $C_0^\infty(\Omega)$ under the anisotropic norm

$$\|u\|_{W^{1,\mathbf{p}}(\Omega)} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}$$

where $\frac{\partial u}{\partial x_i}$ denotes the i -th weak partial derivative of u . Recall that if we set

$$\sum_{i=1}^N \frac{1}{p_i} > 1, \quad p_i > 1 \quad \forall i = 1, \dots, N, \quad p^* := \frac{N}{\sum \frac{1}{p_i} - 1}, \quad p_\infty := \max\{p^*, p_N\}, \tag{2.2}$$

then for every $r \in [1, p_\infty]$, the embedding

$$W_0^{1,\mathbf{P}}(\Omega) \subset L^r(\Omega)$$

is continuous, and it is compact if $r < p_\infty$. More precisely, it holds the following directional Poincaré-type inequality for any $u \in C_c^1(\Omega)$ (see for instance [9])

$$\|u\|_r \leq \frac{d^i r}{2} \left\| \frac{\partial u}{\partial x_i} \right\|_r, \quad \forall r \geq 1, \quad d^i = \sup_{x,y \in \Omega} \langle x - y, e_i \rangle, \quad (2.3)$$

where $\{e_1, \dots, e_N\}$ denotes the canonical basis of \mathbb{R}^N .

The theory of embeddings of this kind of anisotropic Sobolev spaces is vast and we refer to [9] for the directional Poincaré-type inequality and to [12] for Sobolev and Morrey’s embeddings of the whole $W^{1,\mathbf{P}}(\Omega)$ space, obtained with an important geometric condition on the domain Ω , namely that it must be semi-rectangular. We briefly recall the definition for later use.

Definition 2.1. Let (p_1, p_2, \dots, p_N) be a vector of numbers $p_i > 1$, consisting of L distinct values. For $i \in \{1, \dots, N\}$, let N_i be the multiplicity of the values of p_i , with $\sum_{i=1}^L N_i = N$. A semi-rectangular domain is an open bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_L, \quad (2.4)$$

where $\Omega_i \subset \mathbb{R}^{N_i}$ are open bounded $C^{0,1}$ domains.

It is not the case that this semi-rectangular condition reflects in our construction of the solution: the existence of traces for this kind of functions is heavily depending on the geometry of the domain as shown in [12]. Regularity theory for anisotropic operators like the one defined by equation (1.1) is still a challenging open problem, see for example [3]. We also recall the following definition of a weak solution.

Definition 2.2. A function $u \in W^{1,\mathbf{P}}(\Omega)$ is defined to be a sub-(super-)solution of the problem (2.1) if $u \leq (\geq) 0$ in $\partial\Omega$ and $\forall 0 \leq \phi \in W_0^{1,\mathbf{P}}(\Omega)$, it satisfies

$$\int_{\Omega} \left[\sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} - h(x, u(x))\phi \right] dx \leq (\geq) 0. \quad (2.5)$$

Finally, a solution $u \in W_0^{1,\mathbf{P}}(\Omega)$ of (2.1) has to satisfy

$$\int_{\Omega} \left[\sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_i} - h(x, u(x))\phi \right] dx = 0 \quad \forall \phi \in W_0^{1,\mathbf{P}}(\Omega).$$

Now we introduce some well-known results concerning the eigenvalue problem for the p -Laplacian, that will be of crucial importance for the construction. Specifically, we consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.6)$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

The following result is well-known:

Lemma 2.1. *The eigenvalue problem (2.6) has a unique eigenvalue $\lambda = \lambda_1$ with the property of having a positive associated eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$, called principal eigenfunction. Moreover, λ_1 is simple, isolated, and is defined by*

$$\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla u|^p : u \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p dx = 1 \right\}.$$

Furthermore, $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ and $\partial\varphi_1/\partial n < 0$ on $\partial\Omega$, where n is the outward unit normal on $\partial\Omega$. Finally, for $N = 1$, we have that

$$|\nabla\varphi_1|^{p-2} \nabla\varphi_1 \in W^{1,2}(\Omega), \tag{2.7}$$

and in fact

$$-\Delta_p \varphi_1(x) = \lambda_1 \varphi_1(x) \quad \text{for a.e. } x \in \Omega.$$

Remark 2.1. The existence of λ_1 and the main properties of φ_1 are well-known, see [10, 13, 15]. Property (2.7) holds for $N = 1$, see for instance [11], and for $N \geq 2$ in some specific domains, for example, for Ω convex, see [2].

3. An existence sub-supersolution theorem. We start by stating an important theorem, for whose proof, we refer to [1, Theorem 5.1], that assures the existence of a solution between a sub- and a supersolution.

Theorem 3.1. *Consider problem (2.1) with $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Caratheodory function and bounded in x , and such that there exist $\underline{u}, \bar{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ subsolution and supersolution of (2.1) such that $\underline{u} \leq \bar{u}$. Then there exist a solution $u \in W_0^{1,p}(\Omega)$ of (2.1) such that*

$$\underline{u} \leq u \leq \bar{u}.$$

Proof. By continuity, the function h verifies condition (h_2) of [1]. This concludes the proof. □

4. Construction of sub- and supersolutions: proof of the main result. In this section, we prove Theorem 1.1. For that, we apply Theorem 3.1 to (1.1). Mainly, we construct the sub- and the supersolutions.

4.1. Subsolutions. Let us consider a rectangular bounded domain $U \subseteq \Omega$, i.e.,

$$U := \prod_{i=1}^N U_i, \quad \text{where } U_i = (a_i, b_i), \quad a_i, b_i \in \mathbb{R} \quad \forall i = 1, \dots, N.$$

Denote by $v_i = v_i(x_i)$ a positive principal eigenfunction of $-\Delta_{p_i}$ in U_i , that is,

$$\begin{cases} -\Delta_{p_i} v_i = \eta_i |v_i|^{p_i-2} v_i & \text{in } U_i, \\ v_i = 0 & \text{on } \partial U_i. \end{cases} \tag{4.1}$$

From Lemma 2.1, recall that, if n_i is the outward normal derivative to ∂U_i , we have

$$\frac{\partial v_i}{\partial n_i} < 0 \quad \text{on } \partial U_i. \tag{4.2}$$

Let us consider the function

$$\underline{u}(x) = \begin{cases} \epsilon \prod_{i=1}^N v_i^{\alpha_i}(x_i), & x \in U, \\ 0, & x \in \Omega \setminus \bar{U}, \end{cases} \tag{4.3}$$

where $\alpha_i > 1, i = 1, \dots, N$, and $\epsilon > 0$ will be chosen later.

Remark 4.1. We note that $\underline{u}(x) > 0$ in $\emptyset \neq U \subset \Omega$.

As every v_i is bounded, it is clear that $\underline{u} \in W^{1,p}(\Omega)$ and that $\underline{u}|_{\partial\Omega} = 0$. Hence, \underline{u} is a subsolution of (1.1) provided that

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx \leq \lambda \int_{\Omega} \underline{u}^{q-1} \phi dx \quad \forall \phi \in W_0^{1,p}(\Omega), \quad \phi \geq 0.$$

Observe that

$$\lambda \int_{\Omega} \underline{u}^{q-1} \phi dx = \lambda \epsilon^{q-1} \int_U \prod_{i=1}^N v_i^{\alpha_i(q-1)} \phi dx. \tag{4.4}$$

On the other hand, observe that

$$\frac{\partial \underline{u}}{\partial x_i} = \epsilon \alpha_i \left(\prod_{j \neq i} v_j^{\alpha_j} \right) v_i^{\alpha_i-1} \frac{\partial v_i}{\partial x_i} \quad \text{in } U_i.$$

Then, taking into account the positivity of $v_i, \forall i = 1, \dots, N$,

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx \\ &= \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left\{ \int_{U_i} \left[\epsilon \alpha_i \left(\prod_{j \neq i} v_j^{\alpha_j} \right) v_i^{\alpha_i-1} \right]^{p_i-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx_i \right\} d\hat{x}^i \end{aligned}$$

with the obvious notation for $d\hat{x}^i$. Next, by using an integration by parts argument and the Fubini-Tonelli theorem, we get

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx \\ &= - \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \int_{U_i} \frac{\partial}{\partial x_i} \left(v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \phi dx_i d\hat{x}^i \\ &+ \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \left\{ \int_{\partial U_i} v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial n_i} \phi dx_i \right\} d\hat{x}^i. \end{aligned}$$

The second term on the right can be discarded as $\frac{\partial v_i}{\partial n_i} < 0$ in ∂U_i , see (4.2). Considering that

$$\begin{aligned} & \frac{\partial}{\partial x_i} \left(v_i^{(\alpha_i-1)(p_i-1)} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \\ &= (\alpha_i - 1)(p_i - 1) v_i^{(\alpha_i-1)(p_i-1)-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + v_i^{(\alpha_i-1)(p_i-1)} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \end{aligned}$$

and, by construction, that

$$\frac{\partial}{\partial x_i} \left(\left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) = -\eta_i v_i^{p_i-1} \quad \text{in } U_i,$$

our subsolution condition is implied by the following inequality

$$\begin{aligned} & \sum_{i=1}^N \int_{\prod_{j \neq i} U_j} \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-1} \int_{U_i} \left\{ \left[(1 - \alpha_i)(p_i - 1) v_i^{(\alpha_i-1)(p_i-1)-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} \right. \right. \\ & \left. \left. + v_i^{(\alpha_i-1)(p_i-1)} \eta_i v_i^{p_i-1} \right] - \lambda \epsilon^{q-1} \left(\prod_{k=1}^N v_k^{\alpha_k(q-1)} \right) \right\} \phi \, dx_i d\hat{x}^i \leq 0. \end{aligned} \tag{4.5}$$

Let us require a condition on the pointwise integrand

$$\begin{aligned} \lambda &\geq \sum_{i=1}^N \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{(\alpha_i-1)(p_i-1)-1-\alpha_i(q-1)} \left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right] \\ &= \sum_{i=1}^N \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right]. \end{aligned}$$

Now we consider various cases.

- If $1 < q < p_1$, then, by choosing $\alpha_i > \frac{p_i}{(p_i-q)} > 1$ for all $i = 1, \dots, N$, and by letting $\epsilon \rightarrow 0^+$, we obtain that \underline{u} is a subsolution provided $\lambda > 0$.
- Assume that for some $i_0 \in \{1, \dots, N - 1\}$, we have $p_{i_0+1} > q \geq p_{i_0}$ or $q \geq p_N$. Then \underline{u} is a subsolution if

$$\lambda \geq \lambda_* := \max_U \mathcal{S}$$

where

$$\mathcal{S} = \sum_{i=1}^N \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$

We show that λ_* is finite. Observe that $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$ where

$$\mathcal{S}_0 = \sum_{i=1}^{i_0} \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$

and

$$\mathcal{S}_1 = \sum_{i=i_0+1}^N \left(\epsilon \alpha_i \prod_{j \neq i} v_j^{\alpha_j} \right)^{p_i-q} v_i^{\alpha_i(p_i-q)-p_i} \left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right].$$

We take $\alpha_i > p_i/(p_i - q)$ for $i = i_0 + 1, \dots, N$, so that \mathcal{S}_1 is finite.

On the other hand, if we take \mathcal{S}_0 under consideration, we observe that the behaviour next to ∂U is controlled: when $v_i \rightarrow 0^+$, then, as $\frac{\partial}{\partial n_i} v_i < 0$ on ∂U_i , we have that there exists $\delta > 0$ small enough such that the quantity

$$\left[(1 - \alpha_i)(p_i - 1) \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i} + \eta_i v_i^{p_i} \right] < 0 \quad \text{in } U_i^\delta \text{ for all } i = 1, \dots, i_0,$$

where

$$U_i^\delta := \{x_i \in U_i : \text{dist}(x_i, \partial U_i) < \delta\}.$$

Moreover, by continuity, \mathcal{S}_0 is bounded in $U - U_i^\delta$. Then \mathcal{S}_0 is bounded or negative in U and we can conclude that λ_* is finite.

4.2. Supersolutions. Since Ω is bounded, we can choose a domain U such that

$$\Omega \subset U = \prod_{i=1}^N U^i, \quad U^i = (a_i, b_i), \quad a_i, b_i \in \mathbb{R}.$$

Now for $M > 0$, we consider the function

$$\bar{u}(x) := M \prod_{i=1}^N v_i(x_i), \quad x \in \Omega,$$

where v_i are the first eigenfunctions to the p_i -Laplacian in U^i , whose first eigenvalue we denote by η^i . Observe that

$$\bar{u}_{\partial\Omega} > 0.$$

Then \bar{u} is a supersolution of (1.1) if for all $0 \leq \phi \in W_0^{1,\mathbf{P}}(\Omega)$ it holds

$$\lambda \int_{\Omega} M^{q-1} \prod_{i=1}^N v_i^{q-1} \phi \, dx \leq \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx.$$

It is clear that

$$\begin{aligned} & \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \bar{u}}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx \\ &= \int_{\Omega} \sum_{i=1}^N \left[M \left(\prod_{j \neq i} v_j \right) \right]^{p_i-1} \left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \frac{\partial \phi}{\partial x_i} \, dx \\ &= - \sum_{i=1}^N \int_{\Omega} \left(M \prod_{j \neq i} v_j \right)^{p_i-1} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v_i}{\partial x_i} \right|^{p_i-2} \frac{\partial v_i}{\partial x_i} \right) \phi \, dx \\ &= \sum_{i=1}^N \eta^i \int_{\Omega} \left(M \prod_{j=1}^N v_j \right)^{p_i-1} \phi \, dx. \end{aligned}$$

Thus we may ask for the strong condition

$$\lambda^* := \sum_{i=1}^N \eta^i \left(M \prod_{j=1}^N v_j \right)^{p_i - q} \geq \lambda. \tag{4.6}$$

Hence, if $1 < q < p_N$, by letting $M \rightarrow \infty$, we have that \bar{u} is a supersolution $\forall \lambda > 0$.

Proof of Theorem 1.1. If $\lambda \leq 0$, the maximum principle assures that there does not exist a positive solution of (1.1).

1. Assume $1 < q < p_1$. Fix $\lambda > 0$. Then we can choose $\epsilon > 0$ small and M large enough such that \underline{u}, \bar{u} are sub-supersolutions of (1.1) and $\underline{u} \leq \bar{u}$ in Ω . Theorem 3.1 assures the existence of a solution u of (1.1) such that $\underline{u} \leq u \leq \bar{u}$. This completes this case.
2. Assume $p_1 \leq q < p_N$. In this case, taking for example $\epsilon = 1$, we have that \underline{u} is a subsolution provided that $\lambda \geq \lambda_*$ for some λ_* . On the other hand, we can take M large such that \bar{u} is supersolution and $\underline{u} \leq \bar{u}$. Thus, there exists a positive solution for $\lambda \geq \lambda_*$.

Now, we define

$$\Lambda := \inf \{ \lambda : (1.1) \text{ possesses at least one positive solution} \}.$$

We have proved that $\Lambda < \infty$. For $p_1 < q < p_N$, in [4], it was proved that $0 < \Lambda$. We show now that this is also true for $q = p_1$. Indeed, let us now consider $q = p_1$ and let us multiply the equation (1.1) by u and integrate it on Ω to obtain

$$\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} = \lambda \|u\|_{p_1}^{p_1} = \lambda \int_{\Omega} |u|^{p_1} dx.$$

Now we use the embedding (2.3) on $r = p_1$ to get

$$\left(\frac{d^1 p_1}{2} \right)^{-p_1} \|u\|_{p_1}^{p_1} \leq \left\| \frac{\partial u}{\partial x_1} \right\|_{p_1}^{p_1} + \sum_{i=2}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} = \lambda \|u\|_{p_1}^{p_1},$$

and thus

$$\|u\|_{p_1}^{p_1} \left[\lambda - \left(\frac{2}{d^1 p_1} \right)^{p_1} \right] \geq 0.$$

But if $\lambda < \left(\frac{2}{d^1 p_1} \right)^{p_1}$ this quantity is negative and it implies $\|u\|_{p_1} = 0$. This inequality implies that $\Lambda \geq \left(\frac{2}{d^1 p_1} \right)^{p_1} > 0$ in order to have a nontrivial solution.

We prove now that for all $\lambda > \Lambda$, we have the existence of a positive solution. Indeed, fix $\lambda_0 > \Lambda$. Then, by definition of Λ , there exists $\mu \in (\Lambda, \lambda_0)$ and a positive solution, denoted by u_μ , of (1.1) for $\lambda = \mu$. Since $\mu < \lambda_0$, it is clear that u_μ is a subsolution of (1.1) for $\lambda = \lambda_0$. On the other hand, for M large, there exists a supersolution \bar{u} of (1.1) for $\lambda = \lambda_0$. Finally, thanks to regularity results, see for instance [1, Proposition 4.1], we have that $u_\mu \in L^\infty(\Omega)$. Hence, for M large, $u_\mu \leq \bar{u}$, and we can

conclude the existence of a positive solution for $\lambda = \lambda_0$. This completes the proof. □

Remark 4.2. Since our subsolution \underline{u} is strictly positive in U , we have by Theorem 3.1 that $u \geq \underline{u} > 0$ in a nonempty open set contained in Ω . In the case $p_1 \geq 2$, by [4, Corollary 4.4], we have $u > 0$ in Ω .

Remark 4.3. We comment a possible further generalization.

Let $x = (x_1, \dots, x_N)$, where $x_i \in \Omega_i \subset \mathbb{R}^{N_i}$, Ω_i being an open, bounded, and convex domain. Denote by ∇_{x_i} the gradient along the vector x_i and div_{x_i} its divergence, and let

$$\Delta_{p_i} u = \text{div}_{x_i} (|\nabla_{x_i} u|^{p_i-2} \nabla_{x_i} u)$$

be the p_i -Laplacian acting on the vector x_i . Problems of the kind of

$$\begin{cases} -\sum_{i=1}^N \Delta_{p_i} u = \lambda u^{q-1} & \text{in } \Omega = \prod \Omega_i, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

can be faced with the same technique, using properties of the p_i -Laplacian principal eigenfunctions and the integrability condition (2.7) recently obtained in [2].

Acknowledgements. The authors wish to thank professors A. Cianchi and V. Vespri for enlightening conversations about the problem. Moreover, the authors acknowledge IMUS (Mathematics Institute of the University of Seville), IEMath-GR (Mathematics Institute of the University of Granada), and University of Cadiz for supporting a PhD course from which the collaboration arose. S. Ciani is partially funded by INdAM group GNAMPA, A. Suárez by PGC2018-098308-B-I00 (MCI/AEI/FEDER, UE), and G.M. Figueiredo by CNPQ, CAPES, and FAPDF. Finally, we would like to acknowledge the anonymous referee for her/his constructive criticism which have increased the value of the present paper and stimulated curiosity towards new open problems.

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Received: 31 March 2020