Figueiredo and Razani *Boundary Value Problems* (2021) 2021:105
https://doi.org/10.1186/s13661-021-01580-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The sub-supersolution method for a nonhomogeneous elliptic equation involving Lebesgue generalized spaces

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Abstract

In this paper, a nonhomogeneous elliptic equation of the form

$$-\mathcal{A}(x,|u|_{L^{r(x)}})\,\mathrm{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u)$$
$$= f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}}$$

on a bounded domain Ω in $\mathbb{R}^N$ ($N > 1$) with C^2 boundary, with a Dirichlet boundary condition is considered. Using the sub-supersolution method, the existence of at least one positive weak solution is proved. As an application, the existence of at least one solution of a generalized version of the logistic equation and a sublinear equation are shown.

MSC: 35J91; 35J60; 35D30

Keywords: Nonlocal problem; $p(x)$-Laplacian; Sub-supersolution

1 Introduction

Partial differential equations involving the $p(x)$-Laplacian arise, for instance, in nonlinear elasticity, fluid mechanics, non-Newtonian fluids and image processing. Because of the broad set of applications, several studies related to the p-Laplacian, or in general the $p(x)$-Laplacian, operator have been reported (see for instance [4, 6, 16–18, 20–27, 29, 31, 32] and the references therein). One of the approaches to study the existence of solutions of elliptic partial differential equations is the sub-supersolution method. Some problems such as

$$\begin{cases} -\Delta_p u = |u|^{\alpha(x)}_{L^{q(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -a(\int_\Omega |u|^\gamma)\Delta u = f_\lambda(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and}$$

$$\begin{cases} -a(\int_\Omega |u|^q)\Delta u = h_1(x,u)f(\int_\Omega |u|^p) + h_2(x,u)g(\int_\Omega |u|^r) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

have been studied via the sub-supersolution method (see [1, 11, 33]). Also, one may refer to [5, 7, 8, 10, 12–14, 28] for other similar model problems.

Recently, the existence of solutions for nonlocal problems involving the $p(x)$-Laplacian operator

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}}) \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, has been studied [14, 15] via a new sub-supersolution method. In [14], the problem (1.1) for $p(x) \equiv 2$ (i.e., $-\Delta_{p(x)} = -\Delta$) is considered. They study the existence of a weak solution for three problems (the sublinear problem, the concave–convex problem and the logistic equation). Their arguments are mainly based on the existence of the first eigenvalue of the Laplacian operator $(-\Delta, H^1_0(\Omega))$. The $p(x)$-Laplacian operator, in general, has no first eigenvalue, that is, the infimum of the eigenvalues equals 0 (see [19]).

The lack of the existence of the first eigenvalue implies a considerable difficulty when dealing with boundary value problems involving the $p(x)$-Laplacian by using the sub-supersolution method. Papers that consider such problems by using the mentioned method are rare in the literature. Among such works we mention papers such as [2, 3, 24, 34].

In this paper, we are interested in the nonlocal problem

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}}) \operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with C^2 boundary, $|.|_{L^{m(x)}}$ is the norm of the space $L^{m(x)}(\Omega)$, $r, q, s, \alpha, \gamma : \Omega \to [0, \infty)$ are measurable functions and $\mathcal{A}, f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying certain conditions. To be more specific about the structure of the operator in (1.2), we deal with function $a : \mathbb{R}^+ \to \mathbb{R}^+$ of class C^1 satisfying the following conditions:

(a1) There exist constants $k_1, k_2, k_3, k_4 \geq 0$, $1 < p \leq l < N$ such that

$$k_1 t^p + k_2 t^l \leq a(t^p)t^p \leq k_3 t^p + k_4 t^l, \quad \text{for all } t \geq 0.$$

(a2) The function

$$t \mapsto A(t^p) \quad \text{is strictly convex,}$$

where $A(t) = \int_0^t a(s)\,ds$.

(a3) The function

$$t \mapsto a(t^p)t^{p-2} \quad \text{is increasing.}$$

Various operators occurring in applications are included as models for the boundary value problem (1.2), as one can see from the next examples.

Example 1.1 The following operators satisfying (a_1)–(a_3):

(i) If $a(t) = 1$, we obtain the p-Laplacian that is

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})\Delta_{p(x)}u = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k_1 = k_2 = k_3 = k_4 = 1$.

(ii) If $a(t) = 1 + t^{\frac{l-p}{p}}$ we obtain the (p,l)-Laplacian or $p\&l$-Laplacian with $1 < p \leq l < \infty$, that is

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})(\Delta_{p(x)}u + \Delta_{l(x)}u) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(iii) If $a(t) = 1 + \frac{t}{(1+t^2)^{\frac{1}{2}}}$ we obtain the p-Laplacian with $1 < p < \infty$, that is

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})\operatorname{div}((1 + \frac{|\nabla u|^p}{(1+|\nabla u|^{2p})^{\frac{1}{2}}})|\nabla u|^{p-2}\nabla u) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $l = p$, $k_1 + k_2 = 2$ and $k_3 + k_4 = 1$.

(iv) If $a(t) = (1 + \frac{1}{t^{\frac{2}{p}}})^{\frac{p-2}{2}}$ with $p \geq 2$ we obtain the generalized p-mean curvature operator, that is

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})\operatorname{div}((1 + |\nabla u|^2)^{\frac{p-2}{2}})\nabla u) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(v) If $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ with $p \geq 2$ we obtain

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})\operatorname{div}(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $l = p$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 2$.

(vi) If $a(t) = 1 + t^{\frac{l-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ with $p \geq 2$

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})(\Delta_p u + \Delta_l u + \operatorname{div}(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}})) \\ \quad = f(x,u)|\nabla u|^{\alpha(x)}_{L^{q(x)}} + g(x,u)|\nabla u|^{\gamma(x)}_{L^{s(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$.

The main aim of this paper is to prove the existence of a weak positive solution for (1.2) via the sub-supersolution method.

In the next section we present some preliminaries to construct a function space where the solution of (1.2) makes sense.

2 Function spaces

To study the solution of problem (1.2), we need to introduce a suitable function space, where the solution makes sense. To do this, we recall some facts about the known spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ (see [18, 30] and the references therein for more details).

Let $\Omega \subset IR^N$ ($N \geq 1$) be a bounded domain and

$$\mathcal{S}(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measurable}\}.$$

For $p \in L^\infty_+(\Omega)$, the generalized Lebesgue space and its norm are defined by

$$L^{p(x)}(\Omega) =: \left\{u \in \mathcal{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}, \quad \text{and}$$

$$|u|_{p(x)} := \inf\left\{\lambda > 0; \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

respectively. It is easy to see that the space $(L^{p(x)}(\Omega), |.|_{L^{p(x)}})$ is a Banach space.

Set

$$m^+ := \operatorname*{ess\,sup}_\Omega m(x) \quad \text{and} \quad m^- := \operatorname*{ess\,inf}_\Omega m(x),$$

where $m \in L^\infty(\Omega)$.

Proposition 2.1 *Let $\rho(u) := \int_\Omega |u|^{p(x)}\, dx$. For all $u, u_n \in L^{p(x)}(\Omega)$, $n \in \mathbb{N}$, the following assertions hold:*

(i) *Let $u \neq 0$ in $L^{p(x)}(\Omega)$, then $|u|_{L^{p(x)}} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$.*
(ii) *If $|u|_{L^{p(x)}} < 1 (= 1; > 1)$, then $\rho(u) < 1 (= 1; > 1)$.*
(iii) *If $|u|_{L^{p(x)}} > 1$, then $|u|^{p^-}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}}$.*
(iv) *If $|u|_{L^{p(x)}} < 1$, then $|u|^{p^+}_{L^{p(x)}} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}}$.*
(v) *$|u_n|_{L^{p(x)}} \to 0 \Leftrightarrow \rho(u_n) \to 0$, and $|u_n|_{L^{p(x)}} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.*

Theorem 2.2 *Let $p, q \in L^{\infty}_{+}(\Omega)$. The following statements hold*:

(i) *If $p^- > 1$ and $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ a.e. in Ω, then*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{L^{p(x)}}|v|_{L^{q(x)}}.$$

(ii) *If $q(x) \leq p(x)$, a.e. in Ω and $|\Omega| < \infty$, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

One can define the generalized Sobolev space

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p(x)}(\Omega), j = 1, \dots, N\right\},$$

with the norm $\|u\|_* = |u|_{L^{p(x)}} + \sum_{j=1}^N |\frac{\partial u}{\partial x_j}|_{L^{p(x)}}$, $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_*$.

Theorem 2.3 *If $p^- > 1$, then $W^{1,p(x)}(\Omega)$ is a Banach, separable and reflexive space.*

Proposition 2.4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider $p, q \in C(\overline{\Omega})$. Define the function $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $N \geq p(x)$. The following statements hold*:

(i) (*Poincaré inequality*) *If $p^- > 1$, then there is a constant $C > 0$ such that $|u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}}$ for all $u \in W_0^{1,p(x)}(\Omega)$.*

(ii) *If $p^-, q^- > 1$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

Note that $\|u\| := |\nabla u|_{L^{p(x)}}$ defines a norm in $W_0^{1,p(x)}(\Omega)$ that is equivalent to the norm $\|.\|_*$ (by (i) of Proposition 2.4).

Definition 2.5 Consider $u, v \in W^{1,p(x)}(\Omega)$. It is called

$$-\Delta_{p(x)} u \leq -\Delta_{p(x)} v,$$

if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi \leq \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$.

The following result is contained in [21, Lemma 2.2] and [16, Proposition 2.3].

Proposition 2.6 *Consider $u, v \in W^{1,p(x)}(\Omega)$. If $-\Delta_{p(x)} u \leq -\Delta_{p(x)} v$ and $u \leq v$ on $\partial\Omega$, (i.e., $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$), then $u \leq v$ in Ω. If $u, v \in C(\overline{\Omega})$ and $S = \{x \in \Omega : u(x) = v(x)\}$ is a compact set of Ω, then $S = \emptyset$.*

Lemma 2.7 ([16, Lemma 2.1]) *Let $\lambda > 0$ be the unique solution of problem*

$$\begin{cases} -\Delta_{p(x)} z_\lambda = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define $\rho_0 = \frac{p^-}{2|\Omega|^{\frac{1}{N}} C_0}$. *If* $\lambda \geq \rho_0$, *then* $|z_\lambda|_{L^\infty} \leq C^* M^{\frac{1}{p^- -1}}$ *and* $|z_\lambda|_{L^\infty} \leq C_* M^{\frac{1}{p^+ -1}}$ *if* $\lambda < \rho_0$. *Here*, C^* *and* C_* *are positive constants dependent only on* p^+, p^-, N, $|\Omega|$ *and* C_0, *where* C_0 *is the best constant of the embedding* $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$.

Regarding the function z_λ of the previous result, it follows from [17, Theorem 1.2] and [21, Theorem 1] that $z_\lambda \in C^1(\overline{\Omega})$ with $z_\lambda > 0$ in Ω.

3 Weak positive solution

In this section we prove the existence of a weak positive solution of problem (1.2), via the sub-supersolution method. In fact, we prove there exists $u \in [\underline{u}, \overline{u}]$ as the weak solution of (1.2), where $\underline{u}$ and $\overline{u}$ are subsolution and supersolution, respectively. To do this, we state the definition of a solution of the problem (1.2).

Definition 3.1 We say that $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is a (weak) solution of (1.2) if

$$\int_\Omega a\big(|\nabla u|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u \nabla \varphi = \int_\Omega \left(\frac{f(x,u)|u|_{L^{q(x)}}^{\alpha(x)}}{\mathcal{A}(x,|u|_{L^{r(x)}})} + \frac{g(x,u)|u|_{L^{s(x)}}^{\gamma(x)}}{\mathcal{A}(x,|u|_{L^{r(x)}})} \right)\varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

For $u, v \in \mathcal{S}(\Omega)$, we write $u \leq v$ if $u(x) \leq v(x)$ a.e. in Ω and

$$[u,v] := \big\{ w \in \mathcal{S}(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e. in } \Omega \big\}.$$

Definition 3.2 We say that $(\underline{u}, \overline{u})$ is a sub-supersolution pair for (1.2) if $\underline{u} \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $\overline{u} \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ are such that $\underline{u} \leq \overline{u}$, $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$ and if, for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, the following inequalities hold

$$\int_\Omega a\big(|\nabla \underline{u}|^{p(x)}\big)|\nabla \underline{u}|^{p(x)-2}\nabla u \nabla \varphi \leq \int_\Omega \left(\frac{f(x,\underline{u})|\underline{u}|_{L^{q(x)}}^{\alpha(x)}}{\mathcal{A}(x,|w|_{L^{r(x)}})} + \frac{g(x,\underline{u})|\underline{u}|_{L^{s(x)}}^{\gamma(x)}}{\mathcal{A}(x,|w|_{L^{r(x)}})} \right)\varphi \tag{3.1}$$

and

$$\int_\Omega a\big(|\nabla \overline{u}|^{p(x)}\big)|\nabla \overline{u}|^{p(x)-2}\nabla u \nabla \varphi \geq \int_\Omega \left(\frac{f(x,\overline{u})|\overline{u}|_{L^{q(x)}}^{\alpha(x)}}{\mathcal{A}(x,|w|_{L^{r(x)}})} + \frac{g(x,\overline{u})|\overline{u}|_{L^{s(x)}}^{\gamma(x)}}{\mathcal{A}(x,|w|_{L^{r(x)}})} \right)\varphi, \tag{3.2}$$

for all $w \in [\underline{u}, \overline{u}]$.

We will assume that the functions r, p, q, s, α and γ satisfy the following hypotheses:
(H_0) $p \in C^1(\overline{\Omega})$, $r, q, s \in L_+^\infty(\Omega)$, where

$$L_+^\infty(\Omega) = \big\{ m \in L^\infty(\Omega) \text{ with } \operatorname{ess\,inf} m(x) \geq 1 \big\},$$

and $\alpha, \gamma \in L^\infty(\Omega)$ satisfy

$$1 < p^- := \inf_\Omega p(x) \leq p^+ := \sup_\Omega p(x) < N \quad \text{and} \quad \alpha(x), \gamma(x) \geq 0 \quad \text{a.e. in } \Omega.$$

The main result of this section is to prove the existence of at least one solution of (1.2).

Theorem 3.3 *Suppose that* r, p, q, s, α *and* γ *satisfy* (H_0), $a:\mathbb{R}^+\to\mathbb{R}^+$ *is a* C^1 *function satisfying* (a1)–(a3), $(\underline{u},\overline{u})$ *is a pair of sub-supersolution for* (1.2) *with* $\underline{u}>0$ *a.e. in* Ω, $f(x,t),g(x,t)\geq 0$ *in* $\overline{\Omega}\times[0,|\overline{u}|_{L^\infty}]$ *are continuous functions and* $\mathcal{A}:\overline{\Omega}\times(0,\infty)\to\mathbb{R}$ *is continuous with* $\mathcal{A}(x,t)>0$ *in* $\overline{\Omega}\times[|\underline{u}|_{L^{r(x)}},|\overline{u}|_{L^{r(x)}}]$. *Then,* (1.2) *has at least one weak positive solution* $u\in[\underline{u},\overline{u}]$.

To prove this theorem, we need to prove some facts in the series of lemmas.
First, we study the existence and uniqueness of the solution

$$\begin{cases} -\operatorname{div}\bigl(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u\bigr)=G(v) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Lemma 3.4 *Let* $\Omega\subset\mathbb{R}^N$, $N\geq 2$, *be a smooth bounded domain and* $a:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ *be a* C^1 *function satisfying* (a_1), (a_2) *and* (a_3). *Assume* $G:L^{p(x)}(\Omega)\to L^{p'(x)}(\Omega)$ *is continuous and there exists* $K_0>0$ *such that* $|G(v)|\leq K_0$, *for all* $v\in L^{p(x)}(\Omega)$, *where* $p'(x)=\frac{p(x)}{p(x)-1}$. *Then, problem* (3.3) *has a unique solution* $u\in W_0^{1,l(x)}(\Omega)$.

Proof Consider the functional $\Im:W_0^{1,l(x)}(\Omega)\longrightarrow\mathbb{R}$ given by

$$\Im(u)=\frac{1}{p}\int_\Omega A\bigl(|\nabla u|^{p(x)}\bigr)\,dx-\int_\Omega G(v)u\,dx. \tag{3.4}$$

From (a_1) the functional (3.4) is well defined and thus $\Im\in C^1(W_0^{1,q}(\Omega),\mathbb{R})$. Also, $\Im$ is strictly convex and weakly lower semicontinuous by (a_2). Note that (a_1), $|G(v)|\leq K_0$ and Hölder's inequality imply

$$\Im(u)\geq\frac{k_1}{p^-}\|u\|^{p(x)}_{W_0^{1,p(x)}(\Omega)}+\frac{k_2}{l^-}\|u\|^{l(x)}_{W_0^{1,l(x)}(\Omega)}-K_0C\|u\|_{W_0^{1,l(x)}(\Omega)}$$

for some constant $C>0$ and all $u\in W_0^{1,l(x)}(\Omega)$ with $\rho(|\nabla u|)\geq 1$, which shows that $\Im$ is coercive. Hence, $\Im$ has a unique critical point (a global minimizer), which is the unique solution to (3.3). □

Lemma 3.5 *Under the hypotheses of Theorem* 3.3, *define the operator* $T:L^{p(x)}(\Omega)\to L^\infty(\Omega)$ *by*

$$(Tu)(x)=\begin{cases}\underline{u}(x) & \textit{if } u(x)\leq\underline{u}(x),\\ u(x) & \textit{if } \underline{u}(x)\leq u(x)\leq\overline{u}(x),\\ \overline{u}(x) & \textit{if } u(x)\geq\overline{u}(x).\end{cases}$$

where $\underline{u},\overline{u}\in L^\infty(\Omega)$ *and* $Tu\in[\underline{u},\overline{u}]$. *Moreover, let the operator* $H:[\underline{u},\overline{u}]\to L^{p'(x)}(\Omega)$ *be defined by*

$$H(v)(x)=\frac{f(x,v(x))|v|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|v|_{L^{r(x)}})}+\frac{g(x,v(x))|v|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|v|_{L^{r(x)}})},$$

where $p'(x) = \frac{p(x)}{p(x)-1}$ and $|.|_{L^{m(x)}}$ denotes the norm of $L^{m(x)}(\Omega)$. Then, the operators T, H and $u \mapsto HoT(u)$ are well defined and $u \mapsto HoT(u)$ is continuous.

Proof Similar to [14] one can show the operators H and $u \mapsto HoT(u)$ are well defined and $u \mapsto HoT(u)$ is continuous. □

Lemma 3.6 *Fix $v \in L^{p(x)}(\Omega)$ and define the operator $S : L^{p(x)}(\Omega) \to L^{p(x)}(\Omega)$, given by $S(v) = u$, where $u \in W_0^{1,p(x)}(\Omega)$ is the unique solution of (3.3). Then, S is compact and continuous.*

Proof Assume (v_n) is a bounded sequence in $L^{p(x)}(\Omega)$ and define $u_n := S(v_n)$, $n \in \mathbb{N}$. Then,

$$\int_\Omega a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi = \int_\Omega H(Tv_n)\varphi,$$

for all $n \in \mathbb{N}$ and $\varphi \in W_0^{1,p(x)}(\Omega)$. Consider the test function $\varphi = u_n$, by the inclusion $Tv_n \in [\underline{u}, \overline{u}]$, one can obtain

$$\int_\Omega a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)} \le K_0 \int_\Omega |u_n|,$$

for all $n \in \mathbb{N}$, where K_0 is an upper bound for HoT.

The embedding $L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and Poincaré's inequality show that

$$\int_\Omega a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)} \le C\|u_n\|,$$

for all $n \in \mathbb{N}$, where C is a constant that does not depend on $n \in \mathbb{N}$.

If $\|u_n\| > 1$, by Proposition 2.1 we have

$$\|u_n\|^{p^-} \le C\|u_n\|,$$

for all $n \in \mathbb{N}$, where the constant C does not depend on $n \in \mathbb{N}$. Therefore, the sequence (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, up to a subsequence, we have $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ for some $u \in W_0^{1,p(x)}(\Omega)$. Since the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is compact, we have $u_n \to u$ in $L^{p(x)}(\Omega)$. Therefore, S is a compact operator.

Now, we show that S is continuous. Assume (v_n) is a sequence in $L^{p(x)}(\Omega)$ with $v_n \to v$ in $L^{p(x)}(\Omega)$ for $v \in L^{p(x)}(\Omega)$. Define $u_n := S(v_n)$ and $u := S(v)$. Note that

$$\int_\Omega a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)}|\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi = \int_\Omega H(Tv_n)\varphi$$

and

$$\int_\Omega a\big(|\nabla u|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u \nabla\varphi = \int_\Omega H(Tv)\varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$. Such equations with $\varphi = u_n - u$ provide

$$\int_\Omega \big\langle a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)-2}\nabla u_n - a\big(|\nabla u|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u, \nabla(u_n - u)\big\rangle$$

$$= \int_\Omega \big[H(Tv_n) - H(Tv)\big](u_n - u).$$

Thus, the sequence (u_n) is bounded in $L^{p(x)}(\Omega)$ and by Hölder's inequality, we have

$$\left|\int_\Omega \big[H(Tv_n) - H(Tv)\big](u_n - u)\right| \le C\big|(HoT)(v_n) - (HoT)(v)\big|_{L^{p'(x)}},$$

where the constant C does not depend on $n \in \mathbb{N}$. The continuity of HoT shows

$$\int_\Omega \big\langle a\big(|\nabla u_n|^{p(x)}\big)|\nabla u_n|^{p(x)-2}\nabla u_n - a\big(|\nabla u|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u, \nabla(u_n - u)\big\rangle \to 0,$$

which implies the continuity of S. □

We recall a special case of the far-reaching Leray–Schauder theorem called Schaefer's Fixed Point Theorem.

Theorem 3.7 *Let S be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \varrho Sx \textit{ for some } 0 \le \varrho \le 1\}$$

is bounded. Then, S has a fixed point.

Lemma 3.8 *S has a fixed point in $L^{p(x)}(\Omega)$, i.e., there exists $u \in L^{p(x)}(\Omega)$ such that $S(u) = u$.*

Proof Since we can apply Theorem 3.7, we need to show that there exists $R > 0$ such that if $u = \varrho S(u)$ with $\varrho \in [0,1]$, then $|u|_{L^{p(x)}} < R$. In fact, if $\varrho = 0$, then $u = 0$. Suppose that $\varrho \neq 0$. In this case, we have $S(u) = \frac{u}{\varrho}$ and such an equality implies the identity

$$\int_\Omega a\left(\left|\left(\frac{u}{\varrho}\right)\right|^{p(x)}\right)\left|\nabla\left(\frac{u}{\varrho}\right)\right|^{p(x)-2}\nabla\left(\frac{u}{\varrho}\right)\nabla\varphi = \int_\Omega H(Tu)\varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$. Using the test function $\varphi = \frac{u}{\varrho}$ and by the embedding $L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega)$, we obtain

$$\int_\Omega a\left(\left|\left(\frac{u}{\varrho}\right)\right|^{p(x)}\right)\left|\nabla\left(\frac{u}{\varrho}\right)\right|^{p(x)} \le K_0 \int_\Omega \frac{|u|}{\varrho} \le \frac{C}{\varrho}|u|_{L^{p(x)}},$$

where $C > 0$ is a constant that does not depend on u and ϱ. If $|\nabla u|_{L^{p(x)}} > 1$, by Poincaré's inequality and Proposition 2.1, $|u|_{L^{p(x)}}^{p_- - 1} \le \varrho^{p^- - 1}C$, where C is a constant that does not depend on u and ϱ. □

Now, we can prove Theorem 3.3.

Proof Lemma 3.5 shows the operators H and $u \mapsto HoT(u)$ are well defined and $u \mapsto HoT(u)$ is continuous.

Fix $v \in L^{p(x)}(\Omega)$. Since $(HoT)(v) \in L^{\infty}(\Omega)$, by Lemma 3.4 problem (3.3) has a unique solution. Note that by Lemma 3.6 the operator $S : L^{p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact and continuous. Also, Lemma 3.8 shows that there exists a $u \in L^{p(x)}(\Omega)$ such that $u = S(u)$, then

$$\begin{aligned} &\int_\Omega a\big(|\nabla u|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u\nabla\varphi \\ &\quad= \int_\Omega\left(\frac{f(x,Tu)|Tu|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})} + \frac{g(x,Tu)|Tu|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\right)\varphi, \end{aligned} \tag{3.5}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

We claim that $u \in [\underline{u},\overline{u}]$. Considering $w = Tu$ in (3.1) and subtracting from (3.5), we obtain

$$\begin{aligned} &\int_\Omega \big\langle a\big(|\nabla \underline{u}|^{p(x)}\big)|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - a\big(|\nabla \underline{u}|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u, \nabla\varphi\big\rangle \\ &\quad\le \int_\Omega\left(\frac{f(x,\underline{u})|\underline{u}|^{\alpha(x)}_{L^{q(x)}} - f(x,Tu)|Tu|^{\alpha(x)}_{L^{q(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\right)\varphi \\ &\qquad+ \int_\Omega\left(\frac{g(x,\underline{u})|\underline{u}|^{\gamma(x)}_{L^{s(x)}} - g(x,Tu)|Tu|^{\gamma(x)}_{L^{s(x)}}}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\right)\varphi, \end{aligned}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \ge 0$.

Using the test function $\varphi := (\underline{u} - u)_+ = \max\{\underline{u} - u, 0\}$, and using that $f, g \ge 0$ in $[0, |\overline{u}|_{L^\infty}]$, $Tu = \underline{u}$ in $\{\underline{u} \ge u\} := \{x \in \Omega : \underline{u}(x) \ge u(x)\}$, we obtain

$$\begin{aligned} &\int_{\{\underline{u}\ge u\}} \big\langle a\big(|\nabla \underline{u}|^{p(x)}\big)|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - a\big(|\nabla \underline{u}|^{p(x)}\big)|\nabla u|^{p(x)-2}\nabla u, \nabla(\underline{u} - u)\big\rangle \\ &\quad\le \int_{\{\underline{u}\ge u\}} \frac{f(x,\underline{u})(|\underline{u}|^{\alpha(x)}_{L^{q(x)}} - |Tu|^{\alpha(x)}_{L^{q(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\varphi + \int_{\{\underline{u}\ge u\}} \frac{g(x,\underline{u})(|\underline{u}|^{\gamma(x)}_{L^{s(x)}} - |Tu|^{\gamma(x)}_{L^{s(x)}})}{\mathcal{A}(x,|Tu|_{L^{r(x)}})}\varphi \\ &\quad\le 0, \end{aligned}$$

which imply that $\underline{u} \le u$. A similar reasoning provides the inequality $u \le \overline{u}$ and the proof is complete. □

4 Applications

The main goal of this section is to apply Theorem 3.3 to some classes of nonlocal problems.

4.1 A generalization of the logistic equation

Here, we study a generalization of the classic logistic equation as follows:

$$\begin{cases} -\mathcal{A}(x,|u|_{L^{r(x)}})\operatorname{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = \lambda f(u)|u|^{\alpha(x)}_{L^{q(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

where the function $\mathcal{A}(x,t)$ satisfies

$$\mathcal{A}(x,0) \ge 0, \qquad \lim_{t\to 0^+}\mathcal{A}(x,t) = \infty \quad\text{and}\quad \lim_{t\to+\infty}\mathcal{A}(x,t) = \pm\infty.$$

We suppose that there exists a number $\theta > 0$ such that the function $f : [0,\infty) \to \mathbb{R}$ satisfies the conditions:

(f_1) $f \in C^0([0,\theta],\mathbb{R})$,

(f_2) $f(0) = f(\theta) = 0, f(t) > 0$ in $(0,\theta)$.

Problem (4.1) is a generalization of the problems studied in [9, 14]. The next result generalizes [14, Theorem 5].

Theorem 4.1 *Suppose that r, p, q, α satisfy* (H_0). *Assume $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying* (a1)–(a3), *f satisfies* (f_1), (f_2) *and $\mathcal{A}(x,t) > 0$ in $\overline{\Omega} \times (0, |\theta|_{L^{r(x)}}]$. Then, there exists $\lambda_0 > 0$ such that $\lambda \geq \lambda_0$,* (4.1) *has a positive solution $u_\lambda \in [0,\theta]$.*

Proof Consider the function $\widetilde{f}(t) = f(t)$ for $t \in [0,\varrho]$, and $\widetilde{f}(t) = 0$, for $t \in \mathbb{R} \setminus [0,\theta]$. The functional

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} A\bigl(|\nabla u|^{p(x)}\bigr)\,dx - \lambda \int_\Omega \widetilde{F}(u)\,dx, \quad u \in W_0^{1,p(x)}(\Omega),$$

where $A(t) = \int_0^t a(s)\,ds$ and $\widetilde{F}(t) = \int_0^t \widetilde{f}(s)\,ds$ is of class $C^1(W_0^{1,p(x)}(\Omega),\mathbb{R})$. Since $|\widetilde{f}(t)| \leq C$ for $t \in \mathbb{R}$, we have that J is coercive. Thus, J has a minimum z_λ, which is a weak solution of the problem

$$\begin{cases} -\operatorname{div}(a(|\nabla z|^{p(x)})|\nabla z|^{p(x)-2}\nabla z) = \lambda \widetilde{f}(z) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider a function $\varphi_0 \in W_0^{1,p(x)}(\Omega)$ such that $\widetilde{F}(\varphi_0) > 0$. Define $z_0 := z_{\widetilde{\lambda}_0}$, where $\widetilde{\lambda}_0 > 0$ satisfies

$$\int_\Omega \frac{1}{p(x)} A\bigl(|\nabla \varphi_0|^{p(x)}\bigr)\,dx < \widetilde{\lambda}_0 \int_\Omega \widetilde{F}(\varphi_0)\,dx.$$

Thus, $J_{\widetilde{\lambda}_0}(z_0) \leq J_{\widetilde{\lambda}_0}(\varphi_0) < 0$. Since $J_{\widetilde{\lambda}_0}(0) = 0$, we have $z_0 \not\equiv 0$. By [20, Theorem 4.1], we have $z_0 \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, and using [17, Theorem 1.2], we obtain that $z_0 \in C^{1,\alpha}(\overline{\Omega})$. Considering the test function $\varphi = z_0^- := \min\{z_0, 0\}$, we obtain $z_0 = z_0^+ \geq 0$. By Proposition 2.6, we have $z_0 > 0$.

Considering the test function $\varphi = (z_0 - \theta)^+ \in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned} &\int_\Omega a\bigl(|\nabla z_0|^{p(x)}\bigr)|\nabla z_0|^{p(x)-2}\nabla z_0 \nabla (z_0 - \theta)^+\,dx \\ &\quad = \widetilde{\lambda}_0 \int_{\{z_0 > \theta\}} \widetilde{f}(z_0)(z_0 - \theta)\,dx \\ &\quad = 0. \end{aligned}$$

Therefore

$$\int_{\{z_0>\theta\}} \bigl\langle a\bigl(|\nabla z_0|^{p(x)}\bigr)|\nabla z_0|^{p(x)-2}\nabla z_0 - a\bigl(|\nabla \theta|^{p(x)}\bigr)|\nabla \theta|^{p(x)-2}\nabla \theta, \nabla (z_0 - \theta)\bigr\rangle\,dx = 0,$$

which implies $(z_0 - \theta)_+ = 0$ in Ω. Thus, $0 < z_0 \leq \theta$.

Note that there is a constant $C > 0$ such that $|z_0|_{L^{q(x)}}^{\alpha(x)} \geq C$. Define

$$\begin{cases} \mathcal{A}_0 := \max\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [|z_0|_{L^{r(x)}}, |\theta|_{L^{r(x)}}]\} & \text{and} \\ \mu_0 := \frac{\mathcal{A}_0}{C}. \end{cases}$$

Then,

$$\begin{aligned} -\operatorname{div}\big(a\big(|\nabla z_0|^{p(x)}\big)|\nabla z_0|^{p(x)-2}\nabla z_0\big) &= \widetilde{\lambda}_0 f(z_0) \\ &= \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0\mu_0 f(z_0)|z_0|_{L^{q(x)}}^{\alpha(x)} \frac{\mathcal{A}_0}{\mu_0|z_0|_{L^{q(x)}}^{\alpha(x)}} \\ &\leq \frac{1}{\mathcal{A}_0}\widetilde{\lambda}_0\mu_0 f(z_0)|z_0|_{L^{q(x)}}^{\alpha(x)}. \end{aligned}$$

Thus, for each $\lambda \geq \widetilde{\lambda}_0\mu_0$ and $w \in [\varphi, \theta]$, we obtain

$$-\operatorname{div}\big(a\big(|\nabla z_0|^{p(x)}\big)|\nabla z_0|^{p(x)-2}\nabla z_0\big) \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})}\lambda f(z_0)|z_0|_{L^{q(x)}}^{\alpha(x)}.$$

Since $f(\theta) = 0$, it follows that (z_0, θ) is the sub-supersolution pair for (4.1) and the result is proved. □

4.2 A sublinear problem

Here, we use Theorem 3.3 to study the nonlocal problem

$$\begin{cases} -\mathcal{A}(x, |u|_{L^{r(x)}})(\Delta_{p(x)}u + \Delta u) = u^{\beta(x)}|u|_{L^{q(x)}}^{\alpha(x)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The above problem in the case $p(x) \equiv 2$, was considered recently in [14]. The result of this section generalizes [14, Theorem 3] and [15, Theorem 4.1].

Theorem 4.2 *Suppose that r, p, q, α satisfy (H_0), $\beta \in L^\infty(\Omega)$ is a nonnegative function, $\alpha^+ + \beta^+ < p^- - 1$ and $a_0 > 0$ is a positive constant. Assume one of the conditions holds:*
(A_1) $\mathcal{A}(x,t) \geq a_0$ in $\overline{\Omega} \times [0,\infty)$,
(A_2) $0 < \mathcal{A}(x,t) \leq a_0$ in $\overline{\Omega} \times (0,\infty)$, and $\lim_{t\to+\infty}\mathcal{A}(x,t) = a_\infty > 0$ uniformly in Ω.
Then, (4.2) has a positive solution.

Proof Suppose (A_1) holds, that is, $\mathcal{A}(x,t) \geq a_0$ in $\overline{\Omega} \times [0,+\infty)$. We will start by constructing $\overline{u}$. Let $\lambda > 0$ and $z_\lambda \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ be the unique solution of (2.1), where λ will be chosen later.

For $\lambda > 0$ sufficiently large, by Lemma 2.7 there is a constant $K > 1$ that does not depend on λ such that

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p^- - 1}} \quad \text{in } \Omega. \tag{4.3}$$

Since $\alpha^+ + \beta^+ < p^- - 1$, we can choose $\lambda > 1$ such that (4.3) occurs and

$$\frac{1}{a_0}K^{\beta^+}\lambda^{\frac{\alpha^+ + \beta^+}{p^- - 1}} \max\big\{|K|_{L^{q(x)}}^{\alpha^-}, |K|_{L^{q(x)}}^{\alpha^+}\big\} \leq \lambda. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\frac{1}{a_0} z_\lambda^{\beta(x)} |z_\lambda|_{L^{q(x)}}^{\alpha(x)} \leq \lambda.$$

Therefore,

$$\begin{cases} -(\Delta_{p(x)} z_\lambda + \Delta z_\lambda) \geq \frac{1}{\mathcal{A}(x,|w|_{L^{r(x)}})} z_\lambda^{\beta(x)} |z_\lambda|_{L^{q(x)}}^{\alpha(x)} & \text{in } \Omega, \\ z_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

for all $w \in L^\infty(\Omega)$.

Define $\mathcal{A}_\lambda := \max\{\mathcal{A}(x,t) : (x,t) \in \overline{\Omega} \times [0, |z_\lambda|_{L^{r(x)}}]\}$. We have

$$a_0 \leq \mathcal{A}\bigl(x, |w|_{L^{r(x)}}\bigr) \leq \mathcal{A}_\lambda \quad \text{in } \Omega,$$

for all $w \in [0, z_\lambda]$.

Now, we construct $\underline{u}$. Since $\partial\Omega$ is C^2, there is a constant $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$, where $d(x) := \operatorname{dist}(x, \partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}$. From [24, page 12], we have, for $\sigma \in (0,\delta)$ sufficiently small, the function $\phi = \phi(k,\sigma)$ defined by

$$\phi(x) = \begin{cases} e^{kd(x)} - 1 & \text{if } d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} (\frac{2\delta - t}{2\delta - \sigma})^{\frac{2}{p^- - 1}}\, dt & \text{if } \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} (\frac{2\delta - t}{2\delta - \sigma})^{\frac{2}{p^- - 1}}\, dt & \text{if } 2\delta \leq d(x), \end{cases}$$

belongs to $C_0^1(\overline{\Omega})$, where $k > 0$ is an arbitrary number, and

$$-\Delta_{p(x)}(\mu\phi) = \begin{cases} -k(k\mu e^{kd(x)})^{p(x)-1}[(p(x)-1) + (d(x) \\ \quad + \frac{\ln k\mu}{k})\nabla p(x) \nabla d(x) + \frac{\Delta d(x)}{k}] & \text{if } d(x) < \sigma, \\ \{\frac{1}{2\delta - \sigma} \frac{2(p(x)-1)}{p^- - 1} - (\frac{2\delta - d(x)}{2\delta - \sigma})[\ln k\mu e^{k\sigma} \\ \quad \times (\frac{2\delta - d(x)}{2\delta - \sigma})^{\frac{2}{p^- - 1}} \nabla p(x) \nabla d(x) + \Delta d(x)]\} \\ \quad \times (k\mu e^{k\sigma})^{p(x)-1} (\frac{2\delta - d(x)}{2\delta - \sigma})^{\frac{2(p(x)-1)}{p^- - 1} - 1} & \text{if } \sigma < d(x) < 2\delta, \\ 0 & \text{if } 2\delta < d(x), \end{cases}$$

for all $\mu > 0$.

From the above, one can write

$$-\Delta(\mu\phi) = \begin{cases} -k(k\mu e^{kd(x)})[1 + \frac{\Delta d(x)}{k}] & \text{if } d(x) < \sigma, \\ \{\frac{1}{2\delta - \sigma} 2 - (\frac{2\delta - d(x)}{2\delta - \sigma}) \Delta d(x)\} (k\mu e^{k\sigma}) (\frac{2\delta - d(x)}{2\delta - \sigma}) & \text{if } \sigma < d(x) < 2\delta, \\ 0 & \text{if } 2\delta < d(x), \end{cases}$$

for all $\mu > 0$.

Let $\sigma = \frac{1}{k} \ln 2^{\frac{1}{p^+}}$ and $\mu = e^{-ak}$, where $a = \frac{p^- - 1}{\max_{\overline{\Omega}} |\nabla p| + 1}$. Then, $e^{k\sigma} = 2^{\frac{1}{p^+}}$ and $k\mu \leq 1$ if $k > 0$ is sufficiently large. From [24, page 12], for $d(x) < \sigma$ or $2\delta < d(x)$ we obtain

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \leq 0 < \frac{1}{\mathcal{A}_\lambda} (\mu\phi)^{\beta(x)} |\mu\phi|_{L^{q(x)}}^{\alpha(x)}$$

and for $\sigma < d(x) < 2\delta$ we obtain

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le \tilde{C}(k\mu)^{p^--1}|\ln k\mu| + \tilde{C}(k\mu)|\ln k\mu|. \tag{4.5}$$

Since $\alpha^+ + \beta^+ < p^- - 1$, an application of L'Hospital's rule implies that

$$\lim_{k\to+\infty} \frac{\tilde{C}k^{p_--1} + \tilde{C}k}{e^{ak(p^--1-(\alpha^++\beta^+))}}\left|\ln \frac{k}{e^{ak}}\right| = 0. \tag{4.6}$$

If $\sigma \le d(x) < 2\delta$, we have $\phi(x) \ge 2^{\frac{1}{p^+}} - 1$ for all $k > 0$, because $e^{k\sigma} = 2^{\frac{1}{p^+}}$. Thus, there is a constant $C_0 > 0$ that does not depend on k such that $|\phi|^{\alpha(x)}_{L^{q(x)}} \ge C_0$ if $\sigma \le d(x) < 2\delta$. By (4.6), we can choose $k > 0$ large enough such that

$$\frac{C_1k^{p^--1} + \tilde{C}k}{e^{ak[(p^--1)-(\alpha^++\beta^+)]}}\left|\ln \frac{k}{e^{ak}}\right| \le \frac{C_0}{\mathcal{A}_\lambda}\bigl(2^{\frac{1}{p^+}} - 1\bigr)^{\beta^+}. \tag{4.7}$$

It is possible to choose $k > 0$ large such that $\mu\phi(x) \le 1$ for all $x \in \Omega$ satisfying $\sigma < d(x) < \delta$. Therefore, from (4.5) and (4.7), we have

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le \frac{1}{\mathcal{A}_\lambda}(\mu\phi)^{\beta(x)}|\mu\phi|^{\alpha(x)}_{L^{q(x)}} \quad \text{if } \sigma < d(x) < 2\delta,$$

for $k > 0$ large enough. Fix $k > 0$ satisfying the above property, and the inequality

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le 1.$$

For $\lambda > 1$, we have

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le -\Delta_{p(x)}(z_\lambda) - \Delta(z_\lambda).$$

Therefore, $\mu\phi \le z_\lambda$. The first part of the result is proved.

Now, suppose that $0 < \mathcal{A}(x,t) \le a_0$ in $\overline{\Omega} \times (0,\infty)$. Let δ, σ, μ, a, λ, z_λ and ϕ be as before. From the previous arguments, there exist $k > 0$ large enough and $\mu > 0$ sufficiently small such that

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le 1 \quad \text{and}$$
$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi)\nabla \le \frac{1}{a_0}(\mu\phi)^{\beta(x)}|\mu\phi|^{\alpha(x)}_{L^{q(x)}} \quad \text{in } \Omega.$$

In particular, for $w \in L^\infty(\Omega)$ with $\mu\phi \le w$, we have

$$-\Delta_{p(x)}(\mu\phi) - \Delta(\mu\phi) \le \frac{1}{\mathcal{A}(x,|w|_{L^{r(x)}})}(\mu\phi)^{\beta(x)}|\mu\phi|^{\alpha(x)}_{L^{q(x)}} \quad \text{in } \Omega. \tag{4.8}$$

Since $\lim_{t\to\infty}\mathcal{A}(x,t) = a_\infty > 0$ uniformly in Ω, there is a constant $a_1 > 0$ such that $\mathcal{A}(x,t) \ge \frac{a_\infty}{2}$ in $\overline{\Omega} \times (a_1,\infty)$. Define

$$\begin{cases} m_k := \min\{\mathcal{A}(x,t) : \overline{\Omega} \times [|\mu\phi|_{L^{r(x)}}, a_1]\} > 0 \quad \text{and} \\ \mathcal{A}_k := \min\{m_k, \frac{a_\infty}{2}\}. \end{cases}$$

Then, $\mathcal{A}(x,t) \ge \mathcal{A}_k$ in $\overline{\Omega} \times [|\mu\phi|_{L^{r(x)}},\infty)$.

Fix $k > 0$ satisfying (4.8). Let $\lambda > 1$ such that (4.3) occurs and

$$\frac{1}{\mathcal{A}_k} K^{\beta^+} \lambda^{\frac{\alpha^+ + \beta^+}{p^- - 1}} \max\left\{ |K|_{L^{q(x)}}^{\alpha^-}, |K|_{L^{q(x)}}^{\alpha^+} \right\} \leq \lambda,$$

where $K > 1$ is a constant that does not depend on k and λ (see Lemma 2.7). Thus, for all $w \in [\mu\phi, z_\lambda]$, we have

$$-\Delta_{p(x)}(z_\lambda) - \Delta(z_\lambda) \leq \frac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} z_\lambda^{\beta(x)} |z_\lambda|_{L^{q(x)}}^{\alpha(x)} \quad \text{in } \Omega.$$

From the weak comparison principle, we have $\mu\phi \leq z_\lambda$. Therefore, $(\mu\phi, z_\lambda)$ is a sub-supersolution pair for (4.2). $\square$

Acknowledgements
Not available.

Funding
Not available.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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Publisher's Note

Received: 22 October 2021 Accepted: 1 December 2021 Published online: 20 December 2021

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