Research Article

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Bifurcation analysis for a modified quasilinear equation with negative exponent

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Abstract: In this paper, we consider the following modified quasilinear problem:

$$\begin{cases} -\Delta u - \kappa u \Delta u^2 = \lambda a(x) u^{-\alpha} + b(x) u^{\beta} \ \ in\ \Omega, \\ \qquad u > 0 \ \ in\ \Omega, \qquad u = 0 \ \ on\ \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 3$, a, b are two bounded continuous functions, $\alpha > 0$, $1 < \beta \leq 22^{\star} - 1$ and $\lambda > 0$ is a bifurcation parameter. We use the framework of analytic bifurcation theory to obtain an analytic global unbounded path of solutions to the problem. Moreover, we get the direction of solution curve at the asmptotic point.

Keywords: Quasilinear equation, bifurcation analysis, negative exponent, singular term

MSC: 35B32, 35J10, 35J62, 35J75

1 Introduction

In this paper we consider the following modified quasilinear problem

$$\begin{cases} -\Delta u - \kappa u \Delta u^2 = \lambda a(x) u^{-\alpha} + b(x) u^{\beta} \ \ in\ \ \Omega, \\ \qquad u > 0 \ \ in\ \ \Omega, \qquad u = 0 \ \ on\ \ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $0 < a(x) \in \mathcal{C}(\overline{\Omega})$, $b(x) \in \mathcal{C}(\Omega) \cap L^{\infty}(\Omega)$ and may change sign, $\kappa > 0$ is a real constant, $\alpha > 0$, $1 < \beta \leq 22^{\star} - 1$, $2^{\star} = \frac{2N}{N-2}$, $\lambda > 0$ is a bifurcation parameter. Problem (1.1) is related to the standing wave solutions for the quasilinear Schrödinger equations

$$i\partial_t \psi = -\Delta\psi + \psi + \eta(|\psi|^2)\psi - \kappa\Delta\rho(|\psi|^2)\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $\psi = \psi(t, x)$, $\psi : \mathbb{R} \times \Omega \to \mathbb{C}$, $\kappa > 0$ is a real constant. Equation (1.2) has been applied extensively in many areas of physical phenomena, for the progress of this topic and the other modified Schrödinger equations one may refer to [3, 12, 13, 24–26, 29–32, 42, 44–46].

For $\kappa = 0$, problem (1.1) can be transformed into a semilinear one. In recent years, this type of equations has been studied extensively in both bounded and unbounded domains due to its wide applications in non-

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Newtonian fluids. For instance, Lazer and McKenna [27] studied the following semilinear problem

$$\begin{cases} -\Delta u = p(x)u^{-\gamma} & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{1.3}$$

For $p(x) > 0$ with some smoothness conditions, they showed that there exists a solution which is smooth on Ω and continuous on $\overline{\Omega}$, the Lazer-McKenna obstruction then was firstly presented: the equation has a H_0^1-solution if and only if $\gamma < 3$. For the power of 3, Sun and Zhang [40] provided an extension of the classical Lazer-McKenna obstruction and revealed the role of 3, they also gave a local description of the solution set. Lair and Shaker [28] proved that (1.3) has a unique weak H_0^1-solution on a bounded domain provided $\int_0^\varepsilon f(s)ds < \infty$ and $p(x) \in L^2(\Omega)$. For the regularity of (1.3), Gui and Lin [17] obtained the positive solutions that are Hölder-continuous up to the boundary and has even better regularity in some special cases. The problem was also studied by Ma and Wei [34] when $p(x) = -1$, they showed that the gradient estimates, L^1-estimates, global upper bounds, Liouville properties, classification of stable and finite Morse index solutions, and symmetry properties.

For the perturbed singular problem, Yang [43] considered the following problem with singular nonlinearity,

$$\begin{cases} -\Delta u = \lambda u^{-\gamma} + u^p & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{1.4}$$

For $0 < \gamma < 1 < p \le (N+2)/(N-2)$, Yang carried out a direct analysis in an H^1-neighborhood and proved that (1.4) has a solution which is a local minimiser with respect to the H^1-topology. Then the existence of the second solution was given by making use of Ekeland's variational principle. Arcoya and Moreno-Mérida [2] extended the results of [43] to all $\gamma > 0$ in subcritical case by establishing suitable approximated problems, they showed that there exists $\Lambda > 0$ such that (1.4) has two positive solutions for every $\lambda \in (0, \Lambda)$.

Apart from the existence and regularity of solutions for this type of equations, there are many researchers have obtained global bifurcation and local multiplicity results. For instance, the authors in [11] considered the following singular elliptic problem with exponential type growth in a bounded smooth domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} -\Delta u = \lambda(u^{-\delta} + h(u)e^{u^p}) & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where $1 \le p \le 2$, $0 < \delta < 1$ and $h(t)$ is a smooth "perturbation" of e^{t^p} as $t \to \infty$. For the radial case, they made a detailed study of the blow-up/convergence of the solution branch as it approaches to the asymptotic bifurcation point at infinity. For the critical case $p = 2$, they also interpreted all previous works on multiplicity in terms of the corresponding bifurcation diagrams and the asymptotic profile of large solutions along the branch at infinity. Later, Bougherara et al. [5] considered the following semilinear elliptic problem with a strong singular term in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ ($N \ge 2$),

$$\begin{cases} -\Delta u = \lambda(u^{-\delta} + f(u)) & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{1.5}$$

They improved the results of [11] to all $\delta > 0$, and obtained an analytic global unbounded path of solutions of (1.5) by using the framework of analytic bifurcation theory as developed in the work [4]. In two dimensions, for $0 < \delta < 1$ and certain classes of nonlinearities f with critical growth, it was shown that the existence of an analytic unbounded path of solutions of (1.5) whose Morse index is unbounded along the path and admits infinitely many turning points. Specially, for p-Laplacian differential operator, such bifurcation type results were obtained by Bai et al. [6], Papageorgiou, Rădulescu and Repovš [35, 36]. For more results about this type of equations, one may see [1, 8, 14, 18–20, 22, 38] and the references therein.

Motivated by the above papers and by the increasing interest on problems with singular nonlinearities, our main purpose in this paper is to investigate the analytic global bifurcation in the case of $\kappa = 1$ for (1.1). The quasilinear term $u\Delta u^2$ makes the problem much more complicated. By using a change of variables, the authors in [30] transformed the quasilinear Schrödinger equations into a new semilinear one and showed

that the existence of ground states of soliton-type solutions by variational method. Involving the quasilinear operator and singular nonlinearities in bounded domain, there are some results as well. For instance, the authors in [16] considered the following singular quasilinear problem for $\delta \in (0, 1)$ in a ball $\Omega \subset \mathbb{R}^N$,

$$\begin{cases} -\Delta u - u\Delta u^2 = \lambda u^3 - u - u^{-\delta} \ \ in \ \Omega, \\ u > 0 \ in \ \Omega, \qquad u = 0 \ on \ \partial\Omega. \end{cases} \tag{1.6}$$

They obtained the existence of solutions of (1.6) when λ belongs to a certain neighborhood of the first eigenvalue λ_1. Moameni and Offin [33] obtained the same results as in [16] by considering a more general class of equations. The authors in [37] considered the following class of singular quasilinear problem,

$$\begin{cases} -\Delta u - u\Delta u^2 = a(x)u^{-\delta} + h(x, u) \ \ in \ \Omega, \\ u > 0 \ in \ \Omega, \qquad u = 0 \ on \ \partial\Omega. \end{cases} \tag{1.7}$$

The function $a(x)$ is nonnegative, $\delta > 0$ is a constant and the nonlinearity $h(x, u)$ is continuous. They showed that the existence of a solution for the problem via sub-supersolution method when h has an arbitrary polynomial growth. For the second result, they showed that the existence of the second solution by applying the mountain pass theorem when h has subcritical growth. Recently, the authors in [39] studied problem (1.1) in the case of $0 < \alpha < 1$, they showed that the existence of a minimal solution as a minimum critical point of the energy functional, and then the second solution was also given by constrained critical point theory.

As far as we know in the literature there is little research on the bifurcation analysis of solutions to this type of quasilinear equation. The present paper is mainly consider the case of $\kappa = 1$, and regard (1.1) as a bifurcation problem with λ being the bifurcation parameter. There seem to be some difficulties to transform the quasilinear equation to a semilinear one by making a change of variables, such that the existence of solutions of (1.1) be equivalent to the existence of solutions to new transformed equation, and there holds similar properties. Inspired by [4], we use the framework of analytic bifurcation theory to obtain an analytic global unbounded path of solutions to the transformed equation and so to problem (1.1). Then the direction of the solution curve at the asmptotic point under some conditions is given by making use of local bifurcation theory.

The paper is organized as follows. In section 2, we state some preliminaries and main results including transforming quasilinear problem (1.1) into a semilinear elliptic one and give some definitions and lemmas. In section 3, we give the proof to the main results by using bifurcation theory to the transformed problem to obtain an global unbounded path of solutions, and the properties at turning point.

2 Preliminaries and main results

Taking into account the ideas of [30], we can use the change of variables $\omega = h^{-1}(u)$ to transform the quasilinear equation into a semilinear one with singularity at zero and superlinear at infinity, which h is defined by

$$h'(t) = (1 + 2|h(t)|^2)^{-1/2} \ \text{ for } t > 0, \quad h(-t) = -h(t) \ \text{ for } t \le 0.$$

Now, we list some properties of the function $h(t)$ as given in [41, 42].

Lemma 2.1. *Assume $h : \mathbb{R} \to \mathbb{R}$ is given as above, then there hold:*

(1) $h''(t) = -2h(t)(h'(t))^4, \ t > 0,$
(2) *h is unique, invertible, and $\mathcal{C}^\infty(\mathbb{R})$ – function,*
(3) $0 < h'(t) \le 1$ *for all* $t \in \mathbb{R}$,
(4) $|h(t)| \le |t|$ *for all* $t \in \mathbb{R}$,
(5) $\lim\limits_{t\to 0} \frac{h(t)}{t} = 1, \ \lim\limits_{t\to\infty} \frac{h(t)}{t} = 0$ *and* $\lim\limits_{t\to\infty} \frac{h(t)}{\sqrt{t}} = 2^{1/4}$,
(6) $|h(t)h'(t)| \le 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$,

(7) $h(t)/2 \le th'(t) \le h(t)$ *for all* $t \ge 0$,
(8) $|h(t)| \ge h(1)|t|$ *for* $|t| \le 1$, *and* $|h(t)| \ge h(1)|t|^{1/2}$ *for* $|t| \ge 1$,
(9) *the function* $h^{-\alpha}(t)h'(t)$ *is decreasing for* $t > 0$ *and* $\alpha > 0$,
(10) *the function* $h^{\beta}(t)h'(t)$ *is increasing for* $t > 0$ *and* $\beta \ge 1$.

Using Lemma 2.1, we can perform the changing of variables $\omega = h^{-1}(u)$ to infer that $u \in H^1_{loc}(\Omega)$ is a solution of (1.1) if and only if $\omega \in H^1_{loc}(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta\omega = [\lambda a(x)h(\omega)^{-\alpha} + b(x)h(\omega)^{\beta}]h'(\omega) \ \ in \ \ \Omega, \\ \qquad \omega > 0 \ \ in \ \ \Omega, \qquad \omega = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{2.1}$$

in the sense of the following definition.

Definition 2.2. *We say* $\omega \in H^1_{loc}(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$ *is a weak solution of* (2.1) *if* $\operatorname{ess\,inf}_K \omega > 0$ *for any compact set* $K \subset \Omega$, $(u-\varepsilon)^+ \in H^1_0(\Omega)$ *for any* $\varepsilon > 0$ *given, and*

$$\int_\Omega \nabla\omega\nabla\phi = \int_\Omega \lambda a(x)h(\omega)^{-\alpha}h'(\omega)\phi + b(x)h(\omega)^{\beta}h'(\omega)\phi, \ \ \text{for any } \phi \in \mathcal{C}^\infty_0(\Omega).$$

To state our main result, we denote the following set of all classical solutions to (2.1)

$$\mathcal{S} := \{\omega \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega}), \ \omega > 0 \text{ solves (2.1)}\}.$$

In the sequel, let φ_1 be the first positive eigenfunction for $-\Delta$ in $H^1_0(\Omega)$, $\|\varphi_1\|_{L^\infty(\Omega)} = 1$, we define

$$\phi_\alpha = \begin{cases} \varphi_1, \ \ 0 < \alpha < 1, \\ \varphi_1(-\log\varphi_1)^{\frac{1}{2}}, \ \ \alpha = 1, \\ \varphi_1^{\frac{2}{\alpha+1}}, \ \ \alpha > 1, \end{cases}$$

then

$$\mathcal{C}_\phi(\Omega) := \{\omega \in \mathcal{C}(\Omega) | \text{for some } C > 0, |\omega| \le C\phi(x), \ \forall x \in \Omega\}$$

defines a Banach space endowed with the norm

$$\|\omega\|_{\mathcal{C}_\phi(\Omega)} := \sup_{x\in\Omega}\Big|\frac{\omega}{\phi(x)}\Big|,$$

consequently,

$$\mathcal{C}^+_\phi(\Omega) := \{\omega \in \mathcal{C}_\phi(\Omega) \,\big|\, \inf_{x\in\Omega}\frac{\omega}{\phi(x)} > 0\}$$

is an open convex subset of $\mathcal{C}_\phi(\Omega)$.

Now, we define the following solution operator associated to (2.1):

$$F(\lambda, \omega) = \omega - (-\Delta)^{-1}[\lambda a(x)h(\omega)^{-\alpha} + b(x)h(\omega)^{\beta}]h'(\omega),$$

where $(\lambda, \omega) \in \mathbb{R}^+ \times \mathcal{C}^+_\phi(\Omega)$, $\lambda > 0$.

Remark 2.3. *Note that, for any* $\omega \in \mathcal{C}_0(\overline{\Omega})$ *solves equation* (2.1) *is indeed twice continuously differentiable in* Ω *by standard elliptic regularity, see for example [5][10].*

We recall some results about global analytic bifurcation theory that introduced in [7]. Let $\mathcal{X}$, $\mathcal{Y}$ be real Banach spaces, $\mathcal{U} \subset \mathbb{R} \times \mathcal{X}$ be an open set containing $(0, 0)$ in its closure and $F : \mathcal{U} \to \mathcal{Y}$ be an $\mathbb{R}$-analytic function. Define the solution set

$$\mathcal{S} = \{(\lambda, x) \in \mathcal{U} : \ F(\lambda, x) = 0\}$$

and the non-singular solution set

$$\mathcal{N} = \{(\lambda, x) \in \mathcal{S} : \ ker(\partial_x F(\lambda, x)) = 0)\}.$$

Definition 2.4. *A distinguished arc is a maximal connected subset of* $\mathcal{N}$.

Let us introduce the following assumptions:

(H1) Bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times \mathcal{X}$.
(H2) $\partial_x F(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in \mathcal{S}$.
(H3) There exists an analytic function $(\lambda, u) : (0, \varepsilon) \mapsto \mathcal{S}$ such that $\partial_x F(\lambda(s), u(s))$ is invertible for all $s \in (0, \varepsilon)$ and $\lim_{s \to 0^+} (\lambda(s), u(s)) = (0, 0)$.

Denote

$$\mathcal{A} := \{(\lambda(s), u(s)) : s > 0\}$$

and

$$\mathcal{A}^+ = \{(\lambda(s), u(s)) : s \in (0, \varepsilon)\}.$$

Evidently, $\mathcal{A}^+ \subset \mathcal{S}$. The function (λ, u) from $(0, \varepsilon)$ to $(0, \infty)$ in the $\mathbb{R}$-analytic case as follows.

Lemma 2.5. *Assume* $(H1) - (H3)$ *hold. Then* (λ, u) *can be extended as a continuous map (still called)* $(\lambda, u) : (0, \infty) \mapsto \mathcal{S}$ *with the following properties:*

(a) $\mathcal{A} \cap \mathcal{N}$ *is an at most countable union of distinct distinguished arcs* $\bigcup_{i=0}^{n} \mathcal{A}_i$, $n \le \infty$.
(b) $\mathcal{A}^+ \subset \mathcal{A}_0$.
(c) $\{s > 0 : ker(\partial_x F(\lambda(s), u(s))) \neq \{0\}\}$ *is a discrete set.*
(d) *At each of its points,* $\mathcal{A}$ *has a local analytic re-parameterization in the following sense: for each* $s^* \in (0, \infty)$, *there exists a continuous and injective map* $\rho^* : (-1, 1) \mapsto \mathbb{R}$ *such that* $\rho^*(0) = s^*$ *and the re-parametrisation*

$$(-1, 1) \ni t \mapsto \left(\lambda(\rho^*(t)), u(\rho^*(t))\right) \in \mathcal{A} \text{ is analytic.}$$

Furthermore, the map $s \mapsto \lambda(s)$ *is injective in a right neighborhood of* $s = 0$ *and for each* $s^* > 0$ *there exists* $\varepsilon^* > 0$ *such that* λ *is injective on* $[s^*, s^* + \varepsilon^*]$ *and* $[s^* - \varepsilon^*, s^*]$.
(e) *One of the following holds:*
  (i) $\|(\lambda(s), u(s))\|_{\mathbb{R} \times \mathcal{X}} \to \infty$ *as* $s \to \infty$,
  (ii) *the sequence* $\{(\lambda(s), u(s))\}$ *approaches the boundary of* $\mathcal{U}$ *as* $s \to \infty$,
  (iii) $\mathcal{A}$ *is the closed loop:*

$$\mathcal{A} = \{(\lambda(s), u(s)) : 0 \le s \le T, (\lambda(T), u(T)) = (0, 0) \text{ for some } T > 0\}.$$

In this case, chosing the smallest $T > 0$ *such that*

$$(\lambda(s + T), u(s + T)) = (\lambda(s), u(s)) \text{ for all } s \ge 0.$$

(f) *Suppose that* $\partial_x F(\lambda(s_1), u(s_1))$ *is invertible for some* $s_1 > 0$. *If* $(\lambda(s_1), u(s_1)) = (\lambda(s_2), u(s_2))$ *for some* $s_2 \neq s_1$, *then (e)(iii) occurs and* $|s_1 - s_2|$ *is an integer multiple of* T. *In particular, the map* $s \mapsto (\lambda(s), u(s))$ *is injective on* $[0, T)$.

From the definition of F, we immediately have the following Lemma.

Lemma 2.6. *Let* F *be given as above. Then*

(i) $\partial_\omega F(\lambda, \omega)v = v - \lambda(-\Delta)^{-1} a(x)[h(\omega)^{-\alpha} h'(\omega)]' v - (-\Delta)^{-1} b(x)[h(\omega)^{\beta} h'(\omega)]' v$,
(ii) $\partial_\lambda F(\lambda, \omega)v = -(-\Delta)^{-1} a(x) h(\omega)^{-\alpha} h'(\omega) v$,
(iii) $\partial_{\omega\omega} F(\lambda, \omega)(v, z) = -\lambda(-\Delta)^{-1} a(x)[h(\omega)^{-\alpha} h'(\omega)]'' vz - (-\Delta)^{-1} b(x)[h(\omega)^{\beta} h'(\omega)]'' vz$.

Now, we are ready to state our main results.

Theorem 2.7. *Assume that $\alpha > 0$, $1 < \beta \le 22^*$, and $b^+ \neq 0$. Then there exists $\Lambda \in (0, \infty)$ and an unbounded set $\mathcal{A} \subset (0, \Lambda] \times \mathcal{C}^+_{\phi_\alpha}(\Omega)$ of solutions to problem (2.1) which is globally parametrised by a continuous map $s \mapsto (\lambda(s), \omega(s))$ where $s \in (0, \infty)$ and $(\lambda(s), \omega(s)) \in \mathcal{A} \subset \mathcal{S}$. Furthermore, the path $\mathcal{A}$ has following properties:*

(i) $(\lambda(s), \omega(s)) \to (0, 0)$ *in* $\mathbb{R} \times \mathcal{C}_{\phi_\alpha}(\Omega)$ *as* $s \to 0^+$,
(ii) $\|\omega(s)\|_{\mathcal{C}_{\phi_\alpha}(\Omega)} \to \infty$ *as* $s \to \infty$,
(iii) $\{s \ge 0 : \partial_\omega F(\lambda(s), \omega(s))$ *is not invertible*$\}$ *is a discrete set,*
(iv) *the branch of minimal solutions* $\{(\lambda, \omega_\lambda) : 0 < \lambda < \Lambda\}$ *of* (2.1) *coincides with a path-connected portion of* $\mathcal{A}$ *which closure containing* $(0, 0)$*, furthermore, the minimal solution branch is parametrised by an analytic map,*
(v) $\mathcal{A}$ *has at least one asymptotic bifurcation point* $\Lambda_a \in [0, \Lambda]$,
(vi) *each point of* $\mathcal{A}$ *has a local analytic re-parametrization as follows: for each* $s^* \in (0, \infty)$*, there exists a continuous and injective map* $\rho^* : (-1, 1) \mapsto \mathbb{R}$ *such that* $\rho^*(0) = s^*$ *and the re-parametrisation*

$$(-1, 1) \ni t \mapsto (\lambda(\rho^*(t)), \omega(\rho^*(t))) \in \mathcal{A} \text{ is analytic}.$$

Moreover, the map $s \mapsto \lambda(s)$ *is injective in a right neighborhood of* $s = 0$ *and for each* $s^* > 0$ *there exists* $\varepsilon^* > 0$ *such that* λ *is injective on* $[s^*, s^* + \varepsilon^*]$ *and* $[s^* - \varepsilon^*, s^*]$.

Corollary 2.8. *In addition to assertions in Theorem 2.7, if* $(\Lambda, \omega_\Lambda) \in \mathcal{A}$ *for some* $\omega_\Lambda \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ *and* $\alpha \ge \sqrt{5} - 2$. *Then* $\mathcal{A}$ *turns to the left of* $\{\lambda = \Lambda\}$ *at the point* $(\Lambda, \omega_\Lambda) \in \mathcal{A}$.

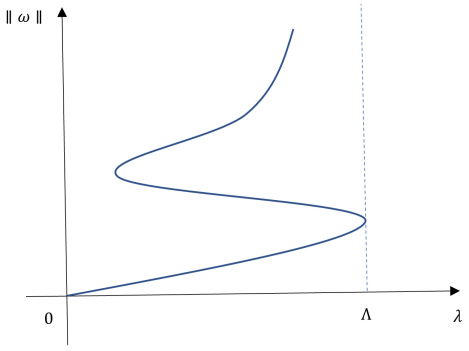

Fig. 2.1: Possible bifurcation branch

3 Local and global bifurcation analysis

In this section, we establish local and global bifurcation to problem (2.1). Firstly, we consider the properties of the linearised operator for the corresponding function F to the problem.

3.1 Analysis of the solution operator and linearised operator

Proposition 3.1. *Assume that the changing of variables h is defined section 2. Then the map $F : \mathbb{R} \times \mathcal{C}^+_{\phi_\alpha} \to \mathcal{C}_{\phi_\alpha}$ is well defined and analytic.*

Proof. We split the proof in three steps.

Step 1. $h(\omega) \in \mathcal{C}^+_{\phi_\alpha}$ for any $\omega \in \mathcal{C}^+_{\phi_\alpha}$. We just consider the case $0 < \alpha < 1$, because the case $\alpha \ge 1$ is similar. Since $\phi_\alpha = \varphi_1$, it follows from the properties of $h(t)$ and $\|\varphi_1\|_{L^\infty(\Omega)} = 1$, that there exist positive constants

C_1, C_2 depend on ω such that

$$0 < C_1 \le \frac{h(\omega)}{\varphi_1(x)} \le \frac{\omega}{\varphi_1(x)} \le C_2 < \infty, \ \forall x \in \Omega.$$

Then $h(\omega)^{-\alpha} \in \mathcal{C}^+_{\phi_\alpha^{-\alpha}}(\Omega)$ for $\omega \in \mathcal{C}^+_{\phi_\alpha}$. By the fact that $\lambda > 0$, $0 < a(x) \in \mathcal{C}(\overline{\Omega})$ and $h'(\omega) \in \mathcal{C}(\overline{\Omega})$, we conclude that $\lambda a(x) h(\omega)^{-\alpha} h'(\omega) \in \mathcal{C}^+_{\phi_\alpha^{-\alpha}}(\Omega)$.

Step 2. For any $\omega \in \mathcal{C}_{\phi_\alpha^{-\alpha}}(\Omega)$, by similar ideas made in the proofs of Proposition 2.3 in [5], we are able to obtain that $\omega \mapsto (-\Delta)^{-1}\omega \in \mathcal{C}_{\phi_\alpha}(\Omega)$ is a linear continuous map and hence analytic.

Step 3. Due to $\alpha > 0$, $1 < \beta \le 2^* - 1$ and the properties of $h(t)$, it is easy to see that $(-\Delta)^{-1} b(x) h(\omega)^\beta h'(\omega) \in \mathcal{C}_{\phi_\alpha}(\Omega)$ for any $\omega \in \mathcal{C}^+_{\phi_\alpha}$.

Hence, by the above three steps, we can obtain the result. □

Now, to show the existence of the analytic global path of solutions to F in $\mathbb{R}^+ \times \mathcal{C}^+_{\phi_\alpha}(\Omega)$, we consider the following problem:

$$\begin{cases} -\Delta w + kw = \lambda a(x) h(w)^{-\alpha} h'(w) + g(x) \quad in \ \Omega, \\ w = 0 \quad on \ \partial\Omega, \end{cases} \tag{3.1}$$

where $k \ge 0$ and $g(x)$ is a local Hölder continuous function in Ω.

To begin with, we have the following comparison principle.

Lemma 3.2. *Assume that there exist u and v satisfying the following inequalities in the weak sense,*

$$\begin{cases} -\Delta u \le \lambda a(x) h(u)^{-\alpha} h'(u) + g(x) - ku \quad in \ \Omega, \\ -\Delta v \ge \lambda a(x) h(v)^{-\alpha} h'(v) + g(x) - kv \quad in \ \Omega, \\ u \le v \ on \ \partial\Omega, \end{cases} \tag{3.2}$$

then there holds $u \le v$ in Ω.

Proof. Arguing by contradiction, assume that $\Omega_0 := \{x \in \Omega : u(x) > v(x)\} \ne \varnothing$. For fixed $\epsilon > 0$, let us define

$$u_\epsilon(x) = u(x) + \epsilon, \quad v_\epsilon(x) = v(x) + \epsilon$$

and

$$\varphi_\epsilon = \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{u_\epsilon}, \quad \psi_\epsilon = \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon}.$$

By pointwise limit, we get

$$\varphi_\epsilon \to \varphi := \frac{u^2 - v^2}{u}\chi_{\Omega_0}, \quad \psi_\epsilon \to \psi := \frac{u^2 - v^2}{v}\chi_{\Omega_0},$$

where χ_{Ω_0} represent the characteristic function of Ω_0.

By taking derivatives, we get

$$\nabla\varphi_\epsilon = \nabla u - 2\frac{v+\epsilon}{u+\epsilon}\nabla v + \frac{(v+\epsilon)^2}{(u+\epsilon)^2}\nabla u,$$

and

$$\nabla\psi_\epsilon = 2\frac{u+\epsilon}{v+\epsilon}\nabla u - \frac{(u+\epsilon)^2}{(v+\epsilon)^2}\nabla v - \nabla v$$

in Ω_0. From the above argument, we have that $\varphi_\epsilon, \psi_\epsilon \in H^1_{loc}(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$. So, by density arguments, we are able to test the first and second inequalities in (3.2) against φ_ϵ and ψ_ϵ, respectively, to obtain

$$\begin{aligned} \int_{\Omega_0} \left[\nabla u \nabla\varphi_\epsilon - \nabla v \nabla\psi_\epsilon\right] dx &= \int_{\Omega_0} \Big[|\nabla u|^2 - 2\frac{v+\epsilon}{u+\epsilon}\nabla u\nabla v + \frac{(v+\epsilon)^2}{(u+\epsilon)^2}|\nabla u|^2 \\ &\quad - 2\frac{u+\epsilon}{v+\epsilon}\nabla u\nabla v + \frac{(u+\epsilon)^2}{(v+\epsilon)^2}|\nabla v|^2 + |\nabla v|^2\Big] dx \\ &= \int_\Omega \Big[|\nabla u|^2 - 2\frac{v_\epsilon}{u_\epsilon}\nabla u\nabla v + \frac{v_\epsilon^2}{u_\epsilon^2}|\nabla u|^2 dx \\ &\quad - 2\frac{u_\epsilon}{v_\epsilon}\nabla u\nabla v + \frac{u_\epsilon^2}{v_\epsilon^2}|\nabla v|^2 + |\nabla v|^2\Big] dx. \end{aligned} \tag{3.3}$$

Now, set $W_1 := \nabla \ln u_\epsilon = \frac{\nabla u}{u_\epsilon}$ and $W_2 := \nabla \ln v_\epsilon = \frac{\nabla v}{v_\epsilon}$ in Ω_0, it follows from (3.3) and Lemma 4.2 of [23], that

$$\begin{aligned}
&\int_\Omega \big[\nabla u \nabla \varphi_\epsilon - \nabla v \nabla \psi_\epsilon\big] dx \\
&= \int_{\Omega_0} \big[|u_\epsilon^2|W_1|^2 - 2v_\epsilon^2 W_1 W_2 + v_\epsilon^2 + 2u_\epsilon^2 W_1 W_2 + u_\epsilon^2 |W_2|^2 + v_\epsilon^2 |W_2|^2\big] dx \\
&= \int_{\Omega_0} u_\epsilon^2 \big[|W_1|^2 - |W_2|^2 - 2W_2(W_1 - W_2)\big] + v_\epsilon^2 \big[|W_2|^2 - |W_1|^2 - 2V_1(V_2 - V_1)\big] dx \\
&\geq \int_{\Omega_0} (u_\epsilon^2 + v_\epsilon^2)|W_1 - W_2|^2 dx \geq 0.
\end{aligned} \tag{3.4}$$

On the other hand, since $a(x) > 0$ and $h(t)^{-\alpha} h'(t)$ is decreasing for $t > 0$ and $\alpha > 0$, we have

$$\begin{aligned}
&\int_\Omega \lambda a(x) \big[h(u)^{-\alpha} h'(u) \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{u_\epsilon} - h(v)^{-\alpha} h'(v) \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon}\big] dx \\
&= \int_{\Omega_0} \lambda a(x)(u_\epsilon^2 - v_\epsilon^2)^+ \big[h(u)^{-\alpha} h'(u) \frac{1}{u_\epsilon} - h(v)^{-\alpha} h'(v) \frac{1}{v_\epsilon}\big] dx \\
&\leq \int_{\Omega_0} \lambda a(x)(u_\epsilon^2 - v_\epsilon^2)^+ h(v)^{-\alpha} h'(v) \big(\frac{1}{u_\epsilon} - \frac{1}{v_\epsilon}\big) \\
&< 0.
\end{aligned} \tag{3.5}$$

Besides, we are able to obtain

$$\begin{aligned}
&\int_\Omega \big[k v \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{v_\epsilon} - k u \frac{(u_\epsilon^2 - v_\epsilon^2)^+}{u_\epsilon}\big] dx \\
&= \int_{\Omega_0} k(u_\epsilon^2 - v_\epsilon^2) \big(\frac{v}{v_\epsilon} - \frac{u}{u_\epsilon}\big) \\
&< 0.
\end{aligned}$$

This, together with (3.4) and (3.5), implies

$$0 \leq \int_\Omega \big[\nabla u \nabla \varphi_\epsilon - \nabla v \nabla \psi_\epsilon\big] dx < 0,$$

which is a contradiction. Thus, $\Omega_0 = \varnothing$, and hence the proof is completed. □

Lemma 3.3. *There exists a unique weak solution $w \in W_0^{1,q}(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$ for some $q > 1$ to problem* (3.1). *Furthermore, $w \in [c\phi_\alpha, \overline{\phi}]$ for $c > 0$ small enough if there exists $\overline{\phi} \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ which is a super-solution of* (3.1). *In particular, $w \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$.*

Proof. Given $k > 0$, $0 \leq g(x) \in L^\infty(\Omega)$, we consider the following approximated problem:

$$\begin{cases} -\Delta w + kw = \lambda a(x) h(w + \varepsilon)^{-\alpha} h'(w + \varepsilon) + g(x) \quad in\ \ \Omega, \\ w = 0 \quad on\ \ \partial\Omega. \end{cases} \tag{3.6}$$

It is easy to check that for $c > 0$ small enough, $\underline{\psi^\epsilon} = (c^{\frac{1+\alpha}{2}} \varphi_1 + \varepsilon^{\frac{1+\alpha}{2}})^{\frac{2}{1+\alpha}} - \varepsilon$ is a sub-solution of (3.6) for $\varepsilon > 0$. The unique positive solution $\overline{\psi}_\varepsilon \in H_0^1(\Omega)$ of

$$-\Delta \overline{\psi}_\varepsilon + k\overline{\psi}_\varepsilon = \lambda a(x) h(\varepsilon)^{-\alpha} h'(\varepsilon) + \|g(x)\|_{L^\infty(\Omega)}$$

is a super-solution of (3.6). Indeed, it follows from the monotonicity of $h(t)^{-\alpha} h'(t)$ that

$$\begin{aligned}
-\Delta \overline{\psi}_\varepsilon + k\overline{\psi}_\varepsilon =& \lambda a(x) h(\varepsilon)^{-\alpha} h'(\varepsilon) + \|g(x)\|_{L^\infty(\Omega)} \\
\geq& \lambda a(x)(h(\overline{\psi}_\varepsilon) + \varepsilon)^{-\alpha} h'(\overline{\psi}_\varepsilon) + g(x).
\end{aligned}$$

By comparison principle, it is obvious that $\underline{\psi^\epsilon} < \overline{\psi}_\varepsilon$. Then we obtain a solution $w_\varepsilon \in [\underline{\psi^\epsilon}, \overline{\psi}_\varepsilon]$ to (3.6) by standard arguments, and uniquely by the non-increasing nature of the right hand side in (3.6). Thus, we can

infer that w_ε is Hölder continuous on $\overline{\Omega}$ through elliptic regularity. In addition, $w_\varepsilon > 0$ in Ω by maximum principle.

Now we prove that w_ε is monotone as $\varepsilon \to 0^+$ by a comparison argument: let $0 < \varepsilon' < \varepsilon$, then we have

$$\begin{aligned}&-\Delta(w_{\varepsilon'} - w_\varepsilon) + k(w_{\varepsilon'} - w_\varepsilon)\\ &= \lambda a(x)\left[h(w_{\varepsilon'} + \varepsilon)^{-\alpha} h'(w_{\varepsilon'} + \varepsilon) - h(w_\varepsilon + \varepsilon)^{-\alpha} h'(w_\varepsilon + \varepsilon)\right].\end{aligned}$$

On the other hand, assume $x_0 = \arg\min_{\overline{\Omega}}(w_{\varepsilon'} - w_\varepsilon) \in \Omega$, and $(w_{\varepsilon'} - w_\varepsilon)(x_0) \le 0$, then it follows that

$$\begin{aligned}&-\Delta(w_{\varepsilon'} - w_\varepsilon)(x_0) + k(w_{\varepsilon'} - w_\varepsilon)(x_0)\\ &\quad -\lambda a(x)\left[h(w_{\varepsilon'}(x_0) + \varepsilon)^{-\alpha} h'(w_{\varepsilon'}(x_0) + \varepsilon) - h(w_\varepsilon(x_0) + \varepsilon)^{-\alpha} h'(w_\varepsilon(x_0) + \varepsilon)\right]\\ &< 0,\end{aligned}$$

which is a contradiction with the last equation. Thus we have $w_{\varepsilon'} > w_\varepsilon$ in Ω if $0 < \varepsilon' < \varepsilon$. Therefore, we obtain that

$$w = \lim_{\varepsilon \to 0^+} w_\varepsilon \ge c\phi_\alpha \text{ and } w \in \mathcal{C}_0(\overline{\Omega}) \tag{3.7}$$

and satisfies in the sense of distributions of (3.1).

Furthermore, $w \in W_0^{1,q}(\Omega)$ for some $q > 1$. Indeed, from $g(x) \in L^\infty(\Omega)$ and (3.7), it is easy to get that $g(x) + \lambda a(x) h(w)^{-\alpha} h'(w) \in L^1(\Omega, d(x, \partial\Omega)^s)$ for some $s < 1$. Then, it follows from Theorems 3 and 4 of [15], that $w \in W_0^{1,q}(\Omega)$ for some $q > 1$. Using the comparison principle again, we can derive that w is the unique weak solution of (3.1).

Now, assume that (3.1) has a super-solution $\overline{\phi} \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$. It is clear that $\overline{\phi}$ is also a super-solution of (3.6). Thus, for $c > 0$ small enough, we get $c\phi_\alpha \le \overline{\phi}$. Hence $\overline{\psi}_\varepsilon$ can be replaced by $\overline{\phi}$ and repeat the above argument to get a solution $w \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$. Thus we complete the proof. □

Since $0 < a(x) \in \mathcal{C}(\overline{\Omega})$ and $h(t)^{-\alpha} h'(t)$, $t > 0$, is decreasing, then $(h(t)^{-\alpha} h'(t))' \le 0$. From the idea of [5], let $m(x) := -a(x)(h(\omega)^{-\alpha} h'(\omega))'$, then we have $m(x) \ge 0$. Now, we consider the following problem

$$-\Delta v + m(x)v = m(x)z \text{ in } \Omega, \quad z \in \mathcal{C}_{\phi_\alpha}(\Omega).$$

We have the following result.

Lemma 3.4. *Assume that $m(x)$ is defined as above, $0 \le m(x) \le m_1 d(x, \partial\Omega)^{-2}$ for some positive constant m_1. Then, for given $z \in \mathcal{C}_{\phi_\alpha}(\Omega)$, there exists a unique $v \in \mathcal{C}_{\phi_\alpha}(\Omega)$ solves $-\Delta v + m(x)v = m(x)z$ in Ω. Furthermore, $\|v\|_{\mathcal{C}_{\phi_\alpha}(\Omega)} \le C\|mz\|_{\mathcal{C}^{-\alpha}_{\phi_\alpha}(\Omega)}$ for some constant $C > 0$ independent of z.*

To this end, we need the next Lemma.

Lemma 3.5. *If $m(x)$ is defined as above, then there exists positive constant m_1 such that $m(x) \le m_1 d(x, \partial\Omega)^{-2}$.*

Proof. Since $a(x) \in \mathcal{C}(\overline{\Omega})$ be bounded, it suffices to prove that there exists a positive constant C such that $-(h(\omega)^{-\alpha} h'(\omega))' \le Cd(x, \partial\Omega)^{-2}$ for all $\omega > 0$. In the following process, the C represents different positive constant. A direct calculation shows that

$$\begin{aligned}-(h(\omega)^{-\alpha} h'(\omega))' &= h(\omega)^{-\alpha-1}[\alpha(h'(\omega))^2 - h(\omega)h''(\omega)]\\ &= h(\omega)^{-\alpha-1}[\alpha(h'(\omega))^2 + 2h^2(\omega)(h'(\omega))^4]\\ &= h(\omega)^{-\alpha-1}(h'(\omega))^2[\alpha + 2h^2(\omega)(h'(\omega))^2].\end{aligned}$$

Note that $h(\omega)^{-\alpha-1} \sim d(x, \partial\Omega)^{-2}$ near $\partial\Omega$. Besides this, by Lemma 2.1-(3) and -(6), we have $(h'(\omega))^2 \le C$ and $[h(\omega)h'(\omega)]^2 \le C$ for all $\omega > 0$. Now, it is obvious that $-(h(\omega)^{-\alpha} h'(\omega))' \le Cd(x, \partial\Omega)^{-2}$ for all $\omega > 0$. Consequently, one can choose a positive constant m_1 such that $m(x) \le m_1 d(x, \partial\Omega)^{-2}$. □

Corollary 3.6. *Let $g(x) \in \mathcal{C}^{-\alpha}_{\phi_\alpha}(\Omega)$, $m(x) \in \mathcal{C}(\Omega)$ and $0 \le m(x) \le m_1 d(x, \partial\Omega)^{-2}$ for some positive constant m_1. If $v \in \mathcal{C}_{\phi_\alpha}(\Omega) \cap \mathcal{C}^2(\Omega)$ is a classical solution of*

$$-\Delta v + m(x)v = g(x) \text{ in } \Omega,$$

then $\|v\|_{\mathcal{C}_{\phi_\alpha}(\Omega)} \le c\|g\|_{\mathcal{C}^{-\alpha}_{\phi_\alpha}(\Omega)}$ for $c > 0$ independent of g.

Lemma 3.7. *The map $\partial_\omega F(\lambda, \omega)$ is Fredholm with index 0 for all $(\lambda, \omega) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\alpha}(\Omega)$.*

Proof. Rewrite $\partial_\omega F(\lambda, \omega) = I + A_\lambda + B$, where $A_\lambda v = -\lambda(-\Delta)^{-1}a(x)(h(\omega)^{-\alpha}h'(\omega))'v$, $Bv = -(-\Delta)^{-1}b(x)(h(\omega)^{\beta}h'(\omega))'v$.

Applying Lemma 3.4 with $m(x) = -\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))'$, which turns out that $I+A_\lambda$ is invertible on $\mathcal{C}_{\phi_\alpha}(\Omega)$. On the other hand, B is compact on $\mathcal{C}_{\phi_\alpha}(\Omega)$. Thus, $\partial_\omega F(\lambda, \omega)$ is Fredholm with index 0. □

3.2 Local and global bifurcation analysis

In this section, we shall show the existence of minimal solution to problem (2.1) for $\lambda \in (0, \Lambda)$, and then state that the full set of minimal solution can be parametrised by an analytic curve. Besides this, we shall illustrate some bifurcation results for $\lambda = \Lambda$, where

$$\Lambda := \sup\{\lambda > 0 : (2.1) \text{ has a weak solution}\}.$$

Lemma 3.8. *It holds $0 < \Lambda < \infty$ and (2.1) admits a minimal solution $\omega_\lambda \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ for all $0 < \lambda < \Lambda$ with $b^+(x) \neq 0$.*

Proof. Let $\underline{\phi}_\lambda = c\lambda^{\frac{1}{1+\alpha}}\phi_\alpha$, then we can check that $\underline{\phi}_\lambda$ is a sub-solution of (2.1) for all $\lambda > 0$ if $c > 0$ is chosen small enough. Next, we find a super-solution to (2.1). Consider the following problem

$$\begin{cases} -\Delta\omega = \overline{\lambda}a(x)h(\omega)^{-\alpha}h'(\omega) \ \ in \ \ \Omega, \\ \omega > 0 \ \ in \ \ \Omega, \quad \omega = 0 \ \ on \ \ \partial\Omega. \end{cases} \tag{3.8}$$

Since the conditions (g1) and (g2) of Theorem 2.2 in [10] are fulfilled by the nonlinear perturbation of (3.8), we conclude that (3.8) admits a unique solution $\psi_{\overline{\lambda}} \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ for any $\alpha > 0$. Let v solves

$$\begin{cases} -\Delta v = 1 \ \ in \ \ \Omega, \\ v > 0 \ \ in \ \ \Omega, \ \ v = 0 \ \ on \ \ \partial\Omega. \end{cases} \tag{3.9}$$

Define $\overline{\phi}_\lambda := \psi_{\overline{\lambda}} + Mv$. For an appropriate constant $M > 0$ and some $\overline{\lambda} > 0$, $\overline{\phi}_\lambda$ is a super-solution of (2.1) for $0 < \lambda < \overline{\lambda}$. Indeed, combining the monotonicity of $h(t)^{-\alpha}h'(t)$ with the properties of $h(t)$, we have

$$-\Delta\overline{\phi}_\lambda = \overline{\lambda}a(x)h(\psi_{\overline{\lambda}})^{-\alpha}h'(\psi_{\overline{\lambda}}) + M \ge \lambda a(x)h(\overline{\phi}_\lambda)^{-\alpha}h'(\overline{\phi}_\lambda) + b(x)h(\overline{\phi}_\lambda)^{\beta}h'(\overline{\phi}_\lambda),$$

if $\overline{\lambda} + M > \lambda + b(x)h(\overline{\phi}_\lambda)^{\beta}h'(\overline{\phi}_\lambda)$ sufficiently large. This also implies $\overline{\lambda} > 0$. Note that we can obtain $\overline{\phi}_\lambda \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$, then there holds $\liminf_{\lambda\to 0^+}\inf_{x\in\Omega}\frac{\overline{\phi}_\lambda}{\lambda^{\frac{1}{1+\alpha}}\phi_\alpha} > 0$. This implies for all $0 < \lambda < \overline{\lambda}$, there holds $\underline{\phi}_\lambda < \overline{\phi}_\lambda$ in Ω if choose $c > 0$ small enough.

Consider the following monotone iterative scheme for all $\lambda \in (0, \overline{\lambda})$,

$$\begin{cases} -\Delta\omega_n - \lambda a(x)h(\omega_n)^{-\alpha}h'(\omega_n) + k\omega_n = b(x)h(\omega_{n-1})^{\beta}h'(\omega_{n-1}) + k\omega_{n-1} \ \ in \ \ \Omega, \\ \omega_n = 0 \ \ on \ \ \partial\Omega, \end{cases} \tag{3.10}$$

with $\omega_0 = \underline{\phi}_\lambda$ and $k = k(\overline{\lambda}) > 0$ large enough such that $b(x)h(t)^{\beta}h'(t) + kt$ is non-decreasing on $[0, \|\overline{\phi}_{\overline{\lambda}}\|_{L^\infty(\Omega)}]$. From the comparison principle and Lemma 3.3, we can get the existence of ω_n, and $\underline{\phi}_\lambda < \omega_n < \overline{\phi}_\lambda$. It is easy

to check that $\underline{\phi}_\lambda$, $\overline{\phi}_\lambda$ are respectively sub and super-solution to (3.10). Furthermore, due to the monotonicity of the left side of (3.10), it follows that the monotonicity of the iterates, that is $\omega_n \geq \omega_{n-1}$. By Ascoli-Arzela Theorem, there exists ω_λ such that $\omega_n \to \omega_\lambda$ in $\mathcal{C}^+_{\phi_\alpha}(\Omega)$ as $n \to \infty$ and $\underline{\phi}_\lambda \leq \omega_\lambda \leq \overline{\phi}_\lambda$. That is, ω_λ is a minimal solution of (2.1) for $0 < \lambda < \overline{\lambda}$.

Now, we set

$$\Lambda := \sup\{\lambda > 0 : (2.1) \text{ has a weak solution}\}.$$

From the above argument, we have $\Lambda > 0$. We claim that $\Lambda < \infty$. In fact, taking φ_1 as the test function in (2.1), we obtain

$$\begin{aligned} &\int_\Omega (\lambda a(x)h(\omega)^{-\alpha}h'(\omega)\varphi_1 + b(x)h(\omega)^\beta h'(\omega)\varphi_1)dx \\ &= \int_\omega \varphi_1(-\Delta)\omega dx = \int_\omega \omega(-\Delta)\varphi_1 dx = \lambda_1 \int_\Omega \omega\varphi_1 dx. \end{aligned} \tag{3.11}$$

If we chose a $\lambda > 0$, large enough if necessary, such that $\lambda a(x)h(t)^{-\alpha}h'(t) + b(x)h(t)^\beta h'(t) > 2\lambda_1 t$ for all $t > 0$, which is a contradiction with (3.11). Thus $\Lambda < \infty$ must holds. Now, we can further get the existence of minimal solution $\omega_\lambda \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ for (2.1) with any $0 < \lambda < \Lambda$. In fact, taking $\underline{\phi}_\lambda = c\lambda^{\frac{1}{1+\alpha}}\phi_\alpha$ as a sub-solution, and $\overline{\phi}_{\lambda'}$, solves problem $(2.1)_{\lambda'}$, for appropriate $\lambda < \lambda' < \Lambda$ as a super-solution of (2.1). Then by the similar proceeding above, we can conclude that there exists a minimal solution $\omega_\lambda \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$ for all $0 < \lambda < \Lambda$, which completes the proof. □

Remark 3.9. *Suppose that there exists $M_0 > 0$ such that*

$$\lambda a(x)[h(t)^{-\alpha}h'(t)]' + b(x)[h(t)^\beta h'(t)]' \leq 0 \text{ in } (0, M_0),$$

if further choose λ_0 small enough such that $\sup_{0<\lambda<\lambda_0} \|\omega_\lambda\|_{\mathcal{C}(\overline{\Omega})} < M_0$. Then ω_λ is the unique solution in $(0, \lambda_0)\times\{\omega \in \mathcal{C}^+_0(\overline{\Omega}) : \|\omega\|_{\mathcal{C}(\overline{\Omega})} < M_0\}$.

Indeed, suppose $\widetilde{\omega}_\lambda$ is another solution and satisfies $\|\widetilde{\omega}_\lambda\|_{\mathcal{C}(\overline{\Omega})} < M_0$ with $\lambda < \lambda_0$. Let $\psi_\lambda = \omega_\lambda - \widetilde{\omega}_\lambda$, then ψ_λ solves

$$-\Delta\psi_\lambda - \left[\lambda a(x)(h(\xi_\lambda)^{-\alpha}h'(\xi_\lambda))' + b(x)(h(\xi_\lambda)^\beta h'(\xi_\lambda))'\right]\psi_\lambda = 0,$$

where ξ_λ lies between ω_λ and $\widetilde{\omega}_\lambda$. It is easy to see $\lambda a(x)(h(\xi_\lambda)^{-\alpha}h'(\xi_\lambda))' + b(x)(h(\xi_\lambda)^\beta h'(\xi_\lambda))' \leq 0$ and hence $\psi_\lambda \equiv 0$.

Lemma 3.10. *Let $\omega \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^+_{\phi_\alpha}(\Omega)$, $\lambda > 0$. If $\partial_\omega F(\lambda, \omega)\varphi = 0$ for some $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\alpha}(\Omega)$, then $\varphi \in H^1_0(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$ and is a H^1-weak solution for $-\Delta\varphi - [\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))' + b(x)(h(\omega)^\beta h'(\omega))']\varphi = 0$. Inversely, if $\varphi \in H^1_0(\Omega)$ is a non-negative H^1-weak solution for $-\Delta\varphi - [\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))' + b(x)(h(\omega)^\beta h'(\omega))']\varphi = \theta\varphi$ for some $\theta \in \mathbb{R}$, then $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$.*

Proof. For some $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\alpha}(\Omega)$, define the minimization problem $\inf_{\psi \in H^1_0(\Omega)} F(\psi)$, where

$$F(\psi) = \int_\Omega |\nabla\psi|^2 dx - \int_\Omega \lambda a(x)(h(\omega)^{-\alpha}h'(\omega))'\psi^2 dx - \int_\Omega b(x)(h(\omega)^\beta h'(\omega))'\varphi\psi dx.$$

By the properties of $h(t)$, $a(x)$ and $b(x)$, it is easy to show that the above functional is coercive and weakly lower semicontinuous on $H^1_0(\Omega)$. Then there exists a minimiser $\psi_0 \in H^1_0(\Omega)$ and is a non-trivial H^1-weak solution of

$$-\Delta\psi_0 - \lambda a(x)(h(\omega)^{-\alpha}h'(\omega))'\psi_0 = b(x)(h(\omega)^\beta h'(\omega))'\varphi.$$

By standard elliptic regularity, we have $\psi_0 \in \mathcal{C}^2(\Omega)$. Now, in H^1 -weak sense, let us take a comparison with the solution $\xi \in H^1_0(\Omega)$ of $-\Delta\xi = M$ in Ω, where $M = \sup_{\overline{\Omega}}|b(x)(h(\omega)^\beta h'(\omega))'\varphi|$, which infer that $\psi_0 \in \mathcal{C}_{\varphi_1}(\Omega)$. Then we get that $\varphi - \psi_0 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_0(\overline{\Omega})$ is a solution of

$$-\Delta(\varphi - \psi_0) - \lambda a(x)(h(\omega)^{-\alpha}h'(\omega))'(\varphi - \psi_0) = 0.$$

It follows from maximum principle that $\varphi \equiv \psi_0$. Hence $\varphi \in H_0^1(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$ and is a H^1-weak solution for $-\Delta\varphi - [\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))' + b(x)(h(\omega)^{\beta}h'(\omega))']\varphi = 0$.

Inversely, by similar ideas made in the proofs of Theorem 8.15 in [21], we can easily see that if $\varphi \in H_0^1(\Omega)$ is a non-negative H^1-weak solution of

$$-\Delta\varphi - [\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))' + b(x)(h(\omega)^{\beta}h'(\omega))']\varphi = \theta\varphi$$

for some $\theta \in \mathbb{R}$, then $\varphi \in \mathcal{C}_{\varphi_1}(\Omega) \cap \mathcal{C}^2(\Omega)$. □

Definition 3.11. *Let* $\Gamma := \{(\lambda, \omega_\lambda) : 0 < \lambda < \Lambda,\ \omega_\lambda \text{ is the minimal solution of } (2.1)\}$.

Lemma 3.12. *For any* $(\lambda, \omega_\lambda) \in \Gamma$, $\partial_\omega F(\lambda, \omega_\lambda)$ *is invertible, and further the full set of minimal solutions* Γ *is parametrised by an analytic map.*

Proof. We just consider the case of $\alpha > 1$, the case of $0 < \alpha \le 1$ is similar. Consider the following problem

$$\begin{cases} -\Delta v = \lambda a(x)h(v+\varepsilon)^{-\alpha}h'(v+\varepsilon) + b(x)h(v)^{\beta}h'(\omega) \quad in\ \Omega, \\ \qquad\qquad v > 0 \ in\ \Omega, \qquad v = 0 \ on\ \partial\Omega, \end{cases} \tag{3.12}$$

where $\varepsilon > 0$. Let $\psi_\varepsilon = (c^{\frac{\alpha+1}{2}}\varphi_1 + \varepsilon^{\frac{\alpha+1}{2}})^{\frac{2}{\alpha+1}} - \varepsilon$, then we can check that ψ_ε is a sub-solution of (3.12) if we chose $c = c_\lambda > 0$ small enough and $\lambda > 0$. On the other hand, we can find that ω_λ, obtained above, is a super-solution of (3.12). Then $\psi_\varepsilon \le \omega_\lambda$ and $\psi_\varepsilon \le \phi_\alpha$ can be guaranteed by restricting c_λ if necessary. Hence there exists a minimal solution v_λ^ε to (3.12) by using the method of monotone iteration again, which satisfies $\psi_\varepsilon \le v_\lambda^\varepsilon \le \omega_\lambda$.

Now, let us define

$$\begin{aligned} \Lambda_1(\lambda) = \inf_{\phi\in H_0^1(\Omega),\int_\Omega \phi^2=1} &\int_\Omega |\nabla\phi|^2 - \lambda a(x)[h(\omega_\lambda)^{-\alpha}h'(\omega_\lambda)]'\phi^2 dx \\ &- \int_\Omega b(x)[h(\omega_\lambda)^{\beta}h'(\omega_\lambda)]'\phi^2 dx \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \Lambda_1^\varepsilon(\lambda) = \inf_{\phi\in H_0^1(\Omega),\int_\Omega \phi^2=1} &\int_\Omega |\nabla\phi|^2 dx - \int_\Omega \lambda a(x)[h(v_\lambda^\varepsilon+\varepsilon)^{-\alpha}h'(v_\lambda^\varepsilon+\varepsilon)]'\phi^2 dx \\ &- \int_\Omega b(x)[h(v_\lambda^\varepsilon)^{\beta}h'(v_\lambda^\varepsilon)]'\phi^2 dx, \end{aligned} \tag{3.14}$$

where ω_λ is the minimal solution of (2.1), $\lambda \in (0, \Lambda)$.

Let $\varphi_\lambda^\varepsilon \in H_0^1(\Omega)$ be a nonnegative minimiser of (3.14). We can obtain $\Lambda_1^\varepsilon(\lambda) \ge 0$ for any $\lambda \in (0, \Lambda)$. Indeed, assume that $\Lambda_1^\varepsilon(\lambda) < 0$ for some $\varepsilon > 0$ and $\lambda \in (0, \Lambda)$. It is easy to verify that $v_\lambda^\varepsilon - \mu\varphi_\lambda^\varepsilon$ is a super-solution of (3.12) for $\mu > 0$ small enough. Using the method of monotone iteration again, we can conclude that there exists a solution to (3.12), $\widetilde{v}_\lambda$, such that $\psi_\varepsilon \le \widetilde{v}_\lambda \le v_\lambda^\varepsilon - \mu\varphi_\lambda^\varepsilon$, which is a contradiction with v_λ^ε is a minimal solution. It follows from $v_\lambda^\varepsilon \le \omega_\lambda$ and elliptic regularity, that there exists v_λ such that $v_\lambda^\varepsilon \to v_\lambda$ in $\mathcal{C}_{loc}(\Omega)$, that is, v_λ solves (2.1) and $v_\lambda \equiv \omega_\lambda$ by the minimality of ω_λ.

Furthermore, we can obtain $\Lambda_1(\lambda) \ge 0$ for any $\lambda \in (0, \Lambda)$. Indeed, note that $v_\lambda^\varepsilon + \varepsilon \ge \psi_\varepsilon + \varepsilon \ge c_\lambda\phi_\alpha$, $-(h(t+\varepsilon)^{-\alpha}h'(t+\varepsilon))' = h(t+\varepsilon)^{-\alpha-1}(h'(t+\varepsilon))^2[\alpha + 2h^2(t+\varepsilon)(h'(t+\varepsilon))^2]$. Applying Lemma 2.1 and Hardy's inequality, there exists $C > 0$ such that $2h^2(t+\varepsilon)(h'(t+\varepsilon))^2 \le C$ and

$$\int_\Omega h(v_\lambda+\varepsilon)^{-\alpha-1}\varphi_\lambda^2 \le c_\lambda^{-\alpha-1}\int_\Omega d^{-2}\varphi_\lambda^2 < \infty,$$

where φ_λ be the nonnegative minimiser of (3.13). Applying dominated convergence theorem, we have

$$\begin{aligned}\Lambda_1(\lambda) &= \int_\Omega |\nabla\varphi_\lambda|^2 dx + \lambda a(x)h(\omega_\lambda)^{-\alpha-1}(h'(\omega_\lambda))^2[\alpha + 2h^2(\omega_\lambda)(h'(\omega_\lambda))^2]\varphi_\lambda^2 dx \\ &\quad - \int_\Omega b(x)[h(\omega_\lambda)^\beta h'(\omega_\lambda)]'\varphi_\lambda^2 dx \\ &= \int_\Omega |\nabla\varphi_\lambda|^2 dx + \lambda a(x)h(v_\lambda^\varepsilon+\varepsilon)^{-\alpha-1}(h'(v_\lambda^\varepsilon+\varepsilon))^2[\alpha + 2h^2(v_\lambda^\varepsilon+\varepsilon)(h'(v_\lambda^\varepsilon+\varepsilon))^2]\varphi_\lambda^2 dx \\ &\quad - \int_\Omega b(x)[h(v_\lambda^\varepsilon)^\beta h'(v_\lambda^\varepsilon)]'\varphi_\lambda^2 dx + o_\varepsilon(1) \\ &\geq \Lambda_1^\varepsilon + o_\varepsilon(1),\end{aligned}$$

which yields $\Lambda_1(\lambda) \geq 0$ for any $0 < \lambda < \Lambda$.

Now we prove $\Lambda_1(\lambda) > 0$. Suppose there exists some $\lambda_0 \in (0, \Lambda)$ such that $\Lambda_1(\lambda_0) = 0$. Without loss of generality, we may assume that $\Lambda_1(\lambda) > 0$ for $\lambda_0 < \lambda < \Lambda$. Then we have

$$0 = \int_\Omega |\nabla\phi_{\lambda_0}|^2 dx - \lambda_0 \int_\Omega a(x)[h(\omega_{\lambda_0})^{-\alpha}h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx - \int_\Omega b(x)[h(\omega_{\lambda_0})^\beta h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx,$$

which implies

$$\lambda_0 \int_\Omega a(x)[h(\omega_{\lambda_0})^{-\alpha}h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx + \int_\Omega b(x)[h(\omega_{\lambda_0})^\beta h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx > 0.$$

For any $\lambda < \lambda_0$, combining $0 < a(x) \in \mathcal{C}(\overline{\Omega})$ with the monotonicity of $h(t)^{-\alpha}h'(t)$, we obtain

$$\int_\Omega |\nabla\phi_{\lambda_0}|^2 dx - \lambda \int_\Omega a(x)[h(\omega_{\lambda_0})^{-\alpha}h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx - \int_\Omega b(x)[h(\omega_{\lambda_0})^\beta h'(\omega_{\lambda_0})]'\phi_{\lambda_0}^2 dx < 0,$$

which implies $\Lambda_1(\lambda) < 0$ for $\lambda < \lambda_0$, a contradiction with $\Lambda_1(\lambda) \geq 0$ for all $\lambda \in (0, \Lambda)$. Therefore, $\Lambda_1(\lambda) > 0$ for all $0 < \lambda < \Lambda$.

Suppose $\partial_\omega F(\lambda, \omega_\lambda)$ is not invertible for some $\lambda \in (0, \Lambda)$. Then there exists $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\phi_\alpha}(\Omega)$ with $\int_\Omega \varphi^2 = 1$ satisfying

$$-\Delta\varphi - \lambda a(x)[h(\omega_\lambda)^{-\alpha}h'(\omega_\lambda)]'\varphi - b(x)[h(\omega_\lambda)^\beta h'(\omega_\lambda)]'\varphi = 0. \tag{3.15}$$

From Lemma 3.10, we have $\varphi \in H_0^1(\Omega)$ is a H^1-weak solution of (3.15), that is, $\Lambda_1(\lambda) = 0$, which is a contradiction with the above attained. Then we can apply the implicit function theorem at any $(\lambda, \omega_\lambda)$ for $\lambda \in (0, \Lambda)$ to obtain that the minimal solution branch is parametrised by an analytic map. □

Lemma 3.13. *There exist closed and bounded subsets of* $\mathcal{S} = \{(\lambda, \omega) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\alpha}(\Omega) : F(\lambda, \omega) = 0\}$ *are compact.*

Proof. Assume that $(\lambda, \omega) \in \mathcal{S}$ and ω solves (2.1). We claim that

$$\inf_{\mathcal{S}} \inf_{\Omega} \left(\frac{\omega}{\lambda^{\frac{1}{\alpha+1}}\phi_\alpha}\right) \geq c \tag{3.16}$$

for some positive constant c. Since $\omega \geq \omega_\lambda$, where ω_λ is obtained as above, then we can find a sufficiently small constant $c > 0$ such that $c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha$ is a sub-solution of (2.1) for $\lambda > 0$. Thus $\omega_\lambda \geq c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha$, which implies the claim is true.

Let $\mathcal{Q}$ be a closed and bounded subset of $\mathcal{S}$. Then there exists a constant $M > 0$ such that $\lambda + \|\omega\|_{\mathcal{C}_{\phi_\alpha}(\Omega)} \leq M$ for all $(\lambda, \omega) \in \mathcal{Q}$. It follows from (3.16) and the properties of $h(t)$, that there exists small enough c such that

$$|\Delta\omega| \leq \lambda a(x)h(c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha)^{-\alpha}h'(c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha) + |b|_\infty \sup_{[0,M]} h(\omega)^\beta h'(\omega). \tag{3.17}$$

Now, according to Lemma 2.1-(8), if $|c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha| \leq 1$, the another case is similar, then together with Lemma 2.1-(3), we can continue to calculate (3.17) as follows:

$$|\Delta\omega| \leq \lambda a(x)h(1)^{-\alpha}(c\lambda^{\frac{1}{\alpha+1}}\phi_\alpha)^{-\alpha} + |b|_\infty \sup_{[0,M]} h(\omega)^\beta h'(\omega)$$

$$\le \lambda^{\frac{1}{1+\alpha}} a(x)h(1)^{-\alpha}c^{-\alpha}\phi_\alpha^{-\alpha} + |b|_\infty \sup_{[0,M]} h(\omega)^\beta h'(\omega)$$
$$\le M^{\frac{1}{1+\alpha}} a(x)h(1)^{-\alpha}c^{-\alpha}\phi_\alpha^{-\alpha} + |b|_\infty \sup_{[0,M]} h(\omega)^\beta h'(\omega). \tag{3.18}$$

By the above estimate and applying Proposition 3.4 of [17], it is easy to see

$$\sup_{\mathcal{Q}} \|\omega\|_{\mathcal{C}^\tau(\overline{\Omega})} < \infty \text{ for some } \tau \in (0,1).$$

Let $\{(\lambda_n, \omega_n)\} \subset \mathcal{Q}$. Then there exists, up to a subsequence, $(\lambda_n, \omega_n) \to (\lambda_0, \omega_0)$ in $\mathcal{C}(\overline{\Omega})$. We claim that $(\lambda_n, \omega_n) \to (\lambda_0, \omega_0)$ in $\mathcal{C}_{\phi_\alpha}(\Omega)$. First, we claim $\lambda_0 \neq 0$. Otherwise, we have $\omega_{\lambda_n} \in H_0^1(\Omega)$ satisfies $\|\omega_{\lambda_n}\| \to 0$. By the above Lemma, $\omega_{\lambda_n} \to 0$ in $\mathcal{C}_{\phi_\alpha}(\Omega)$. By Lemma 3.8, we have $\omega_{\lambda_n} = \omega_n$ and implies $(\lambda_n, \omega_n) \to (0,0)$ in $\mathbb{R}^+ \times \mathcal{C}_{\phi_\alpha}(\Omega)$, which is a contradiction with (0,0) does not belonging to $\mathcal{Q}$. Thus $\lambda_0 > 0$. Then we have $-\Delta\omega_0 \in \mathcal{C}^+_{\phi_\alpha^{-\alpha}}(\Omega)$ by combining inequality (3.18) with bounded ω_n in $\mathcal{C}_{\phi_\alpha}(\Omega)$. Furthermore, by Lemma 3.4, it follows that $\omega_0 \in \mathcal{C}^+_{\phi_\alpha}(\Omega)$. Let $v_n = \omega_0 - \omega_n$, in virtue of (3.16), we have v_n solves

$$\begin{aligned} &-\Delta v_n + \big[-\lambda_0 a(x)(h(\xi_n)^{-\alpha}h'(\xi_n))' - b(x)(h(\zeta_n)^\beta h'(\zeta_n))' \big] v_n \\ &= (\lambda_0 - \lambda_n) a(x) h(\omega_n)^{-\alpha} h'(\omega_n) \\ &= o(1)\phi_\alpha^{-\alpha}, \end{aligned}$$

where ξ_n and ζ_n lie between ω_n and ω_0. Applying Corollary 3.6 with

$$m(x) = -\lambda_0 a(x)(h(\xi_n)^{-\alpha}h'(\xi_n))' - b(x)(h(\zeta_n)^\beta h'(\zeta_n))',$$

and meets the hypothesis what $m(x)$ needs if $b \ge 0$ small enough, and then we can obtain that $\omega_n \to \omega_0$ in $\mathcal{C}_{\phi_\alpha}(\Omega)$. This implies the Lemma holds. □

Remark 3.14. *It is clear that the above lemma remains true if $b(x) < 0$.*

Next we consider the bifurcation analysis at $\lambda = \Lambda$.

Lemma 3.15. *The solution of $F(\lambda, \omega) = 0$ near $(\Lambda, \omega_\Lambda)$ is described by a curve $(\lambda(s), \omega(s)) = (\Lambda + \tau(s), \omega_\Lambda + s\phi_\Lambda + x(s)$, where $s \to (\tau(s), x(s)) \in \mathbb{R} \times X$ is a continuously differentiable function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $x(0) = x'(0) = 0$. Furthermore, τ is of class $\mathcal{C}^2$ near 0 and $\tau''(0) < 0$ if $\alpha \ge \sqrt{5} - 2$ and $b^+ \neq 0$.*

Proof. From Lemma 2.6, we have the following function at $\lambda = \Lambda$:

$$\partial_\omega F(\Lambda, \omega_\Lambda) = I - \Lambda(-\Delta)^{-1} a(x)[h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda]' - (-\Delta)^{-1} b(x)[h(\omega_\Lambda)^\beta h'(\omega_\Lambda]'.$$

Then we have $\Lambda_1(\Lambda) \ge 0$ by the above obtained, and in fact $\Lambda_1(\Lambda) = 0$ by the implicit function theorem and (2.1) has no solution for $\lambda > \Lambda$. We now verify the conditions of the local bifurcation result of Cranduall-Rabinowitz [9]. From obtained above, the map $\partial_\omega F(\lambda, \omega)$ is Fredholm with index 0 for all $(\lambda, \omega) \in \mathbb{R}^+ \times \mathcal{C}^+_{\phi_\alpha}(\Omega)$, then we can easily get $\ker(\partial_\omega F(\Lambda, \omega_\Lambda))$ is one dimensional and spanned by φ_Λ which is the associated eigenfunction of Λ. We can also get $codimRange(\partial_\omega F(\Lambda, \omega_\Lambda)) = 1$. Now we claim that $\partial_\lambda F(\Lambda, \omega_\Lambda) \notin Range(\partial_\omega F(\Lambda, \omega_\Lambda))$, where $\partial_\lambda F(\lambda, \omega)v = -(-\Delta)^{-1} a(x)h(\omega)^{-\alpha}h'(\omega)v$. If it is not true, then there exists $v \in \mathcal{C}_0^1(\overline{\Omega})$ such that

$$\begin{aligned} &v - (-\Delta)^{-1}\big[\Lambda a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))'v - b(x)(h(\omega_\Lambda)^\beta h'(\omega_\Lambda))'v\big] \\ &= -(-\Delta)^{-1} a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda). \end{aligned}$$

By Lemma 3.10, we have $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}_{\varphi_1}(\Omega)$ solves

$$-\Delta v = \Lambda a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))'v - b(x)(h(\omega_\Lambda)^\beta h'(\omega_\Lambda))'v - a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda).$$

Multiplying the above equation by ϕ_Λ and integrating by parts, we obtain

$$-\int_\Omega a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda)\phi_\Lambda dx = 0,$$

which is a contradiction with the properties of $a(x)$ and $h(t)$. Thus the claim holds.

Now, let X be any complement of the span $\{\phi_\Lambda\}$ in $\mathcal{C}_0^1(\overline{\Omega})$. Define $\theta : \mathbb{R}\times\mathbb{R}\times X \to H_0^1(\Omega)$ with $\theta(s,\tau,x) = F(\Lambda+\tau, \omega_\Lambda + s\phi_\Lambda + x)$. It is easy to see that

$$\partial_{\tau,x}\theta(0,0,0) = (\partial_\lambda F(\Lambda,\omega_\Lambda), \partial_\omega F(\Lambda,\omega_\Lambda))$$

is an isomorphism from $\mathbb{R}\times X$ onto $\mathcal{C}^1(\overline{\Omega})\cap\mathcal{C}_0(\overline{\Omega})$. In fact, it follows from

$$codimRange(\partial_\omega F(\Lambda,\omega_\Lambda)) = 1 \text{ and } \partial_\lambda F(\Lambda,\omega_\Lambda) \notin Range(\partial_\omega F(\Lambda,\omega_\Lambda)),$$

that the map

$$(\tau, x) \mapsto \tau\partial_\lambda F(\Lambda,\omega_\Lambda) + \partial_\omega F(\Lambda,\omega_\Lambda)x \tag{3.19}$$

is injective for $(\tau,x)\in\mathbb{R}\times X$. Then, for $\epsilon > 0$, there exists a neighborhood V of 0 in $\mathbb{R}$ and a unique $\mathcal{C}^2$ function $g : (-\epsilon,\epsilon)\to V\times X$ so that $g(s) := (\tau(s), x(s))$ with $s\in(-\epsilon,\epsilon)$, $\tau(0)=0$, $x(0)=0$, $\theta(\tau(s),x(s)) = F(\Lambda+\tau(s), \omega_\Lambda + s\phi_\Lambda x(s)) = 0$. Differentiating the expression (3.19) with respect to s at $s=0$, we have

$$\tau'(0)\partial_\lambda F(\Lambda,\omega_\Lambda) + \partial_\omega F(\Lambda,\omega_\Lambda)(\phi_\Lambda + x'(0)) = 0.$$

Then $\tau'(0) = 0$ and $x'(0) = 0$ since $\partial_\omega F(\Lambda,\omega_\Lambda)\phi_\Lambda = 0$. Differentiating the above equation again with respect to s at $s=0$, we have

$$\tau''(0)\partial_\lambda F(\Lambda,\omega_\Lambda) + \partial_\omega F(\Lambda,\omega_\Lambda)x''(0) + \partial_{\omega\omega}F(\Lambda,\omega_\Lambda)(\phi_\Lambda,\phi_\Lambda) = 0.$$

Applying the dual product with $-\Delta\phi_\Lambda$, we obtain

$$\begin{aligned}
&\langle \tau''(0)\partial_\lambda F(\Lambda,\omega_\Lambda) + \partial_\omega F(\Lambda,\omega_\Lambda)x''(0) + \partial_{\omega\omega}F(\Lambda,\omega_\Lambda)(\phi_\Lambda,\phi_\Lambda), -\Delta\phi_\Lambda\rangle\\
&= \big\langle -\tau''(0)(-\Delta)^{-1}a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda) + \{x''(0) - \Lambda(-\Delta)^{-1}\\
&\quad [a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))'x''(0)] - (-\Delta)^{-1}[b(x)h(\omega_\Lambda)^{\beta}h'(\omega_\Lambda)x''(0)]\}\\
&\quad -\Lambda(-\Delta)^{-1}[a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))''\phi_\Lambda^2]\\
&\quad -(-\Delta)^{-1}[b(x)(h(\omega_\Lambda)^{\beta}h'(\omega_\Lambda))''\phi_\Lambda^2], -\Delta\phi_\Lambda\big\rangle\\
&= -\tau''(0)\langle (a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda), \phi_\Lambda\rangle + \langle x''(0), -\Delta\phi_\Lambda\\
&\quad -\Lambda a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))'\phi_\Lambda - b(x)(h(\omega_\Lambda)^{\beta}h'(\omega_\Lambda))'\phi_\Lambda\rangle\\
&\quad -\big\langle \Lambda a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))''\phi_\Lambda^2 + b(x)(h(\omega_\Lambda)^{\beta}h'(\omega_\Lambda))''\phi_\Lambda^2, \phi_\Lambda\big\rangle\\
&= -\tau''(0)\langle (a(x)h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda), \phi_\Lambda\rangle - \big\langle \Lambda a(x)(h(\omega_\Lambda)^{-\alpha}h'(\omega_\Lambda))''\phi_\Lambda^2\\
&\quad +b(x)(h(\omega_\Lambda)^{\beta}h'(\omega_\Lambda))''\phi_\Lambda^2, \phi_\Lambda\big\rangle\\
&= 0.
\end{aligned}$$

By computing, we can easily get $a(x)(h(\omega)^{-\alpha}h'(\omega))'' + b(x)(h(\omega)^{\beta}h'(\omega))'' > 0$ for $\alpha\ge\sqrt{5}-2$ and $b^+\neq 0$, and hence $\tau''(0) < 0$. □

Remark 3.16. *1. From the above results, we can infer that the direction of the solution curve at the neighborhood $\lambda=\Lambda$. That is, $\tau''(0) < 0$ means the curve of solution from right turns to left at $\{\lambda=\Lambda\}$.*

2. If $0<\alpha<1/3$ and $b<0$, there holds $\lambda a(x)(h(\omega)^{-\alpha}h'(\omega))'' + b(x)(h(\omega)^{\beta}h'(\omega))'' < 0$, and hence $\tau''(0) > 0$. The case of $\tau''(0) > 0$ needs further analysis.

3. The sign of $\tau''(0)$ is indefinite when $0<\alpha<\sqrt{5}-2$ and $b^+\neq 0$ or $1/3\le\alpha<1$ and $b<0$.

3.3 The proof of main results

Proof of Theorem 2.7. Let $\mathcal{U} = \mathbb{R}^+ \times \mathcal{X} = \mathbb{R}^+ \times \mathcal{C}_{\phi_\alpha}(\Omega)$ and the positive cone $\mathcal{W} = \mathcal{C}^+_{\phi_\alpha}(\Omega)$. Clearly $\mathcal{W}$ is open. Conditions (H1)-(H3) of Lemma 2.5 hold because of Lemma 3.13, Proposition 3.1, Lemma 3.7 and Lemma 3.12. In fact, by Lemma 3.12, we may fix $\mathcal{A}^+ = \{(\lambda(s), w(s)) : 0 < s < s_0\}$ for some $s_0 > 0$ be an analytic parametrisation which is one portion of minimal solution branch which given by $\{(\lambda, \omega_\lambda) \in \Gamma : 0 < \lambda < \lambda_0\}$. Then Lemma 2.5 holds. Next, we apply Lemma 2.5 to prove Theorem 2.7. It is clear that assertions (iii) and (vi) of Theorem 2.7 are true. From the definition of $\mathcal{A}$ and $\mathcal{A}^+$, assertion (i) easily obtained. Assertion (iv) can be get from Lemma 3.8 and Lemma 3.12.

It remains to prove assertion (ii), clearly, (v) is a consequence of (ii). In order to show assertion (ii), we only need verify the property (e)(i) of Lemma 2.5 occurring. If case (e)(ii) is occur, then there exists $(\lambda(s_n), \omega(s_n)) \to (0, \omega_0)$ as $s_n \to \infty$ in $\mathcal{U}$, where $(0, \omega_0)$ is a boundary point. Then $\Delta\omega(s_n) \to 0$ in $\mathcal{C}_{loc}(\Omega)$, i.e. $\omega_0 = 0$ and hence $\omega_0 \equiv 0$. By Lemma 3.8, it is easy to see that $\omega(s_n)$ is the minimal solution for all large s_n. However, the minimal solution arc $\mathcal{A}_0$ starting from $(0, 0)$ is isolated from other solutions, and hence, the distinguished arc corresponding to all large s coincide with $\mathcal{A}$, which is a contradiction with (a) of Lemma 2.5. By the same argument, we can rule out case (e)(iii). Hence case (e)(i) holds. Therefore, $\|\omega(s)\|_{\mathcal{C}_{\phi_\alpha(\Omega)}} \to \infty$ as $s \to \infty$ since (2.1) has no solution for $\lambda > \Lambda$. This completes the proof. □

Proof of Corollary 2.8. It follows from Lemma 3.15 and Lemma 2.5, that one can easily get the analytic path $\mathcal{A}$ turns to the left at the point $(\Lambda, \omega_\Lambda)$. □

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