Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Estimates of eigenvalues of an elliptic differential system in divergence form

by

Marcio Costa Araújo Filho

Brasília

2021

Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Estimates of eigenvalues of an elliptic differential system in divergence form

by

Marcio Costa Araújo Filho[1]

Advisor: Prof. Dr. Changyu Xia
Co-Advisor: Prof. Dr. José Nazareno Vieira Gomes

Thesis presented to the Graduate Program of the Department of Mathematics of the Universidade de Brasília as part of the requisites to obtain the degree of Ph.D. IN MATHEMATICS.

Brasília

2021

[1]The author had financial support from CAPES and CNPq during the elaboration of this work.

Universidade de Brasília

Instituto de Ciências Exatas

Departamento de Matemática

ESTIMATES OF EIGENVALUES OF AN ELLIPTIC DIFFERENTIAL SYSTEM IN DIVERGENCE FORM

por

Marcio Costa Araújo Filho *

Tese apresentada ao Departamento de Matemática da Universidadede Brasília, como parte dos requisitos para obtenção do grau de

DOUTOR EM MATEMÁTICA

Brasília, 19 de novembro de 2021.

Comissão Examinadora:

Prof. Dr. JOÃO PAULO DOS SANTOS - MAT/UnB (Presidente)

Prof. Dr. PEDRO ROITMAN - MAT/UnB (Membro)

Prof. Dr. DETANG ZHOU - UFF (Membro)

Prof. Dr. LEVI ROSA ADRIANO - UFG (Membro)

* O autor foi bolsista da CAPES/FAPERO durante a elaboração desta Tese.

RELATÓRIO DE DEFESA DE PÓS-GRADUAÇÃO
GRADUATE PROGRAM DEFENSE REPORT

Informações/ Attention:

Documento deverá ser assinado por/ This document must be signed by:

- Presidente da Comissão Examinadora/ The Chair of the Examining Board
- Membros participantes da Comissão Examinadora/ Members of the Examining Board
- Coordenador(a) do Curso/ Coordinator of the Program
- Discente/ Student

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Levi Rosa Adriano**, **Usuário Externo**, em 23/11/2021, às 16:15, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Marcio Costa Araújo Filho**, **Usuário Externo**, em 23/11/2021, às 18:04, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Joao Paulo dos Santos**, **Professor(a) de Magistério Superior do Departamento de Matemática do Instituto de Ciências Exatas**, em 24/11/2021, às 10:56, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Pedro Roitman**, **Professor(a) de Magistério Superior do Departamento de Matemática do Instituto de Ciências Exatas**, em 26/11/2021, às 17:19, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Detang Zhou**, **Usuário Externo**, em 27/11/2021, às 18:46, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

Ficha catalográfica elaborada automaticamente,
com os dados fornecidos pelo(a) autor(a)

Ce

Costa Araújo Filho, Marcio
Estimates of eigenvalues of an elliptic differential system in divergence form / Marcio Costa Araújo Filho; orientador Changyu Xia; co-orientador José Nazareno Vieira Gomes. -- Brasília, 2021.
77 p.

Tese (Doutorado - Doutorado em Matemática) -- Universidade de Brasília, 2021.

1. Eigenvalue problems. 2. Estimate of eigenvalues. 3. Elliptic differential system. 4. Gaussian soliton. 5. Rigidity results. I. Xia, Changyu, orient. II. Nazareno Vieira Gomes, José, co-orient. III. Título.

Dedico este trabalho aos meus pais Marcio e Maria Lúcia,
à minha querida esposa Raquel,
ao meu amado filho Felipe,
e aos meus irmãos Manoel, Moabe e Mateus.

“Não fui eu que ordenei a você que seja forte e corajoso? Não tenha medo e se sinta acovardado, porque Javé seu Deus, vai estar com você por onde você andar.”(Josué 1:9)

“Ele dá ânimo ao cansado e recupera as forças do enfraquecido. Até os jovens se afadigam e cansam, e os moços também tropeçam e caem. Mas, os que esperam em Javé renovam suas forças, criam asas como águias. Correm e não se afadigam, podem andar que não se cansam.”(Isaías 40:29-31)

Agradecimentos

Agradeço primeiramente a Deus, em nome de Jesus, pelo dom da vida e pela saúde para poder concluir esse projeto.

Aos meus pais, Marcio e Maria Lúcia, por serem meu espelho, minha motivação e meus maiores incentivadores, mesmo sendo um casal de lavradores/agricultores semianalfabetos, sempre acreditaram na força da educação e fizeram o possível (e às vezes o impossível) para que seus quatro filhos pudessem estudar.

Agradeço também de todo o coração, à minha querida esposa Raquel pelo seu companheirismo e apoio em todos esses anos que estamos juntos, desde a época da graduação, por ter sido tão compreensiva em abdicar muito de nosso tempo para que eu pudesse finalizar esse projeto profissional. Você têm sido a alegria e apoio para o meu presente e a motivação e inspiração para o meu futuro.

Aos meus três irmãos, Manoel, Moabe e Mateus, pelo apoio e pela confiança a mim depositados. De maneira especial, ao Manoel por sempre me incentivar nos estudos, pelo apoio dado durante os seis anos que me concedeu moradia em Rio Branco-Acre, juntamente com sua esposa Rozineide a quem também devo minha gratidão, tempo esse que finalizei o ensino médio, conclui a graduação e fiz boa parte do mestrado em matemática.

Ao meu orientador, Prof. Dr. Xia Changyu, por ter aceitado me orientar, por suas sugestões e orientações durante toda a etapa do meu doutorado.

Ao meu coorientador, Prof. Dr. José Nazareno Vieira Gomes, agradeço imensamente por todos os conselhos, sugestões, motivações, conversas, conhecimentos transmitidos e, principalmente, por acreditar em mim e em meu trabalho. O senhor, sem dúvidas, é a peça fundamental em toda essa minha fase de pesquisa, e em específico, para a conclusão dessa tese de doutorado. Se hoje entendo um pouco do que é a pesquisa em matemática, devo isso ao senhor. Meus sinceros agradecimentos e que Deus o abençoe sempre.

Ao Prof. Dr. João Paulo dos Santos por ter aceitado prontamente ser o presidente de minha banca de qualificação de Geometria, quando meu orientador estava afastado para o pós-doutorado. E também por aceitar ser o presidente da banca de defesa desta tese. Obrigado pela confiança, o senhor com certeza é um dos exemplos de excelente profissional da Educação Superior que levarei comigo como inspiração.

Agradeço também aos membros da banca por terem aceitado avaliar este trabalho e pelas preciosas sugestões e correções que o tornaram melhor.

Aos professores do PPGM-UnB com quem pude aprender um pouco de matemática. Bem como, me aproximar e compreender a matemática de ponta a nível nacional e internacional. Em especial, gostaria de citar aqui as(os) professores que contribuíram, direta ou indiretamente, com o meu doutorado: Prof. Dr. Carlos Alberto dos Santos; Prof. Dr. Giovany Figueiredo; Profa. Dra. Keti Tenenblat; Profa. Dra. Liliane Maia; Prof. Dr. Marcelo Furtado; Prof. Dr. Noraí Rocco; Prof. Dr. Pedro Roitman; Prof. Dr. Raimundo Bastos; Prof. Dr. Ricardo Ruviário; Prof. Dr. Tarcísio Silva e Prof. Dr. Willian Silva.

Aos funcionários das secretarias de graduação e de pós-graduação da Matemática da

UnB por sempre nos atenderem com muita competência.

Aos meus colegas de doutorado e mestrado do PPGM-UnB, que contribuíram direta ou indiretamente com minha formação, com muitos momentos de descontração, mas com certeza com muito aprendizado e cooperação mútua. Mesmo correndo o risco de esquecer alguém, gostaria de citar os nomes: Alancooc, Alex, Carlos, Diego, Fábio, Geraldo, Geovane, Gustavo, Felipe Kelmer, Felipe Quintino, Hector Andrés, Laena, Léticia, Lucas Brito, Lucas Loyer, Lucas, Nathália, Paulo, Rômulo, Segundo Manuel, Túlio e Welinton. Em especial ao time do Norte: Cleber Pereira (AC), João Batista (AM/AC) Josean Alves (AC), Mateus de Souza (AC) e Vinicius Bandeira (AM), quando essa galera se juntava não tinha pra ninguém, como por exemplo, no futebol do Mat e nas prosas da copa.

Não poderia deixar de agradecer à Universidade Federal de Rondônia-UNIR, em particular, ao Departamento de Matemática e Estatística do Campus de Ji-Paraná, pela licença concedida para cursar este doutorado. Em especial, aos professores Marlos Albuquerque e Emerson Ribeiro pela amizade e apoio durante esse processo.

Aos colegas da UFAC que contribuíram para o meu início na matemática, tanto na graduação quanto no mestrado, alguns dos quais foram meus professores e outros colegas e amigos no mestrado. Em especial: Clebes Brandão, José Ivan, José Roberto, Marcos Aurélio, Sérgio Brazil e Tadeu.

- À CAPES e à FAPERO, pelo apoio financeiro durante a elaboração deste trabalho.

Resumo

Nesta tese calculamos estimativas universais de autovalores de um sistema de equações diferenciais elípticas acoplado na forma divergente em um domínio limitado no espaço Euclidiano. Como aplicação, mostramos um interessante caso de rigidez de desigualdades de autovalores do Laplaciano, mais precisamente, consideramos uma família enumerável de domínios limitados no soliton Gaussiano contrátil, que torna o comportamento de estimativas conhecidas dos autovalores do Laplaciano invariante por uma perturbação de primeira ordem deste operador. Também tratamos do soliton Gaussiano expansivo em dois cenários diferentes. Finalizamos com o caso especial de tensores livres de divergência, o qual está diretamente relacionado ao operador de Cheng-Yau.

Palavras-chave: Problemas de autovalores; Estimativas de autovalores; Sistema diferencial elíptico; Soliton Gaussiano; Resultados de Rigidez.

Abstract

In this thesis we compute universal estimates of eigenvalues of a coupled system of elliptic differential equations in divergence form on a bounded domain in Euclidean space. As an application, we show an interesting case of rigidity inequalities of the eigenvalues of the Laplacian, more precisely, we consider a countable family of bounded domains in Gaussian shrinking soliton that makes the behavior of known estimates of the eigenvalues of the Laplacian invariant by a first-order perturbation of the Laplacian. We also address the Gaussian expanding soliton case in two different settings. We finish with the special case of divergence-free tensors which is closely related to the Cheng-Yau operator.

Keywords: Eigenvalue problems; Estimate of eigenvalues; Elliptic differential system; Gaussian soliton; Rigidity results.

Contents

Introduction

In this thesis, we study an eigenvalue problem that involve a second-order elliptic operator in divergence form. It is an eigenvalue problem for a coupled system of second-order elliptic differential equations on a bounded domain in Euclidean spaces. We will be more precise in the next paragraph where we present such problem. For this, let us consider a symmetric positive definite $(1,1)$-tensor T and a smooth function η on an n-dimensional Euclidean space $\mathbb{R}^n$, so that we can define a second-order elliptic differential operator $\mathscr{L}$ in the (η, T)-divergence form, as follows

$$\mathscr{L}f := \mathrm{div}_\eta(T(\nabla f)) = \mathrm{div}(T(\nabla f)) - \langle \nabla\eta, T(\nabla f)\rangle,$$

where div stands for the divergence operator and ∇ for the gradient operator.

We study the eigenvalue estimates for an operator which is a second-order perturbation of $\mathscr{L}$. More precisely, let $\mathbb{R}^n$ be the n-dimensional Euclidean space with its canonical metric $\langle , \rangle$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We compute universal estimates of the eigenvalues of the coupled system of second-order elliptic differential equations, namely:

$$\begin{cases} \mathscr{L}\mathbf{u} + \alpha\nabla(\mathrm{div}_\eta \mathbf{u}) & = & -\sigma\mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, the constant α is non-negative and $\mathscr{L}\mathbf{u} = (\mathscr{L}u^1, \mathscr{L}u^2, \ldots, \mathscr{L}u^n)$.

Notice that, since Ω is bounded in the mentioned problem, there exist two positive real constants ε and δ, such that $\varepsilon I \leq T \leq \delta I$, where I is the $(1,1)$-tensor identity on $\mathbb{R}^n$.

Problem 1 is partial differential equations (PDE) with the Dirichlet boundary condition. It is known that PDE's play a fundamental role not only from a mathematical point of view, but also in the description and modeling of many physical and probabilistic phenomena. Such equations appear, for example, in Laplace's equation, in Helmholtz's equation, linear transport equation, in Liouville's equation, Kolmogorov's equation and Schrödinger's equation. In a differential geometry context, an interesting example of PDE appears in the equation of minimal surfaces, see e.g. [14, 17]. In particular, Schrödinger's equation is a central equation in quantum mechanics. For instance, the eigenvalues of

Schrödinger's equation corresponding to the allowed energy levels of the quantum system, and the gap between them is the gap between the energy levels. These eigenvalues are related to the Hamiltonian operator that appears in Schrödinger's equation. Indeed, this equation is an eigenvalue problem for the Hamiltonian operator where the eigenvalues are the (allowed) total energies.

So, the analysis of the sequence of the eigenvalues of elliptic differential operators in divergence forms in bounded domains is an interesting topic in both mathematics and physics. In particular, problems linking the shape of a domain to the spectrum of an operator are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that they involve different fields of mathematics such as spectral theory, Riemannian geometry, and partial differential equations. Not only the literature about this subject is already very rich, but also it is not unlikely that operators in divergence forms may play a fundamental role in the understanding of countless physical facts.

We now present a brief overview of research related to Problem 1. When η is a constant and T is the identity operator I on $\mathbb{R}^n$, Problem 1 becomes

$$\begin{cases} \Delta \mathbf{u} + \alpha \nabla(\operatorname{div} \mathbf{u}) = -\sigma \mathbf{u} & \text{in} \quad \Omega, \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where $\Delta \mathbf{u} = (\Delta u^1, \ldots, \Delta u^n)$ and Δ is the Laplacian operator on $C^\infty(\Omega)$. The operator $\Delta + \alpha\nabla\mathrm{div}$ is known as Lamé's operator. In the 3-dimensional case it shows up in the elasticity theory, more precisely, in this case Problem 2 for $\alpha = \frac{\lambda+\mu}{\mu}$ describes the behaviour of the elastic vibration, where λ and μ are the positive constants of Lamé and $\mathbf{u} = (u^1, u^2, u^3)$ denotes the elastic displacement vector, see Pleijel [30] or Kawohl and Sweers [21].

In 1985, for Problem 2, Levine and Protter [23] proved

$$\sum_{i=1}^{k} \sigma_i \geq \frac{4\pi^2 n}{n+2} \frac{k^{1+2/n}}{(V\omega_{n-1})^{2/n}}, \quad \text{for} \quad k = 1, 2, \ldots,$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere. Furthermore, Hook [20] studied universal inequalities for eigenvalues of Problem 2 and proved that

$$\sum_{i=1}^{k} \frac{\sigma_i}{\sigma_{k+1} - \sigma_i} \geq \frac{n^2 k}{4(n+\alpha)}, \quad \text{for} \quad k = 1, 2, \ldots. \tag{3}$$

Livitin and Parnovski [24] obtained

$$\sigma_{k+1} - \sigma_k \leq \frac{\max\{4 + \alpha^2; (n+2)\alpha + 8\}}{n+\alpha} \frac{1}{k} \sum_{i=1}^{k} \sigma_i, \quad \text{for} \quad k = 1, 2, \ldots.$$

Cheng and Yang [9] proved the following universal inequality of Yang type:

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i) \leq \frac{2\sqrt{n+\alpha}}{n}\Big\{\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^{\frac{1}{2}}\sigma_i\Big\}^{\frac{1}{2}}. \tag{4}$$

Also in [9], the authors gave the following estimate for a lower order eigenvalues of the Problem 2

$$\frac{\sigma_2+\sigma_3+\cdots+\sigma_{n+1}}{\sigma_1} \leq n+4(1+\alpha). \tag{5}$$

Recently, Chen et al. [7] proved the following

$$\sigma_{k+1}-\sigma_k \leq \frac{4(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\sigma_i. \tag{6}$$

When η is not necessarily constant and $T = I$, Problem 1 is rewritten as

$$\begin{cases} \Delta_\eta \mathbf{u} + \alpha\nabla(\mathrm{div}_\eta \mathbf{u}) &= -\sigma\mathbf{u} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where $\Delta_\eta \mathbf{u} = (\Delta_\eta u^1, \ldots, \Delta_\eta u^n)$ and $\Delta_\eta = \mathrm{div}_\eta \nabla$ is the drifted Laplacian operator on $C^\infty(\Omega)$. Du and Bezerra in [13] obtained the following estimates for the eigenvalues of Problem 7

$$\sigma_{k+1}-\sigma_k \leq \frac{4(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i + C_0), \tag{8}$$

where $C_0 = \sup_\Omega\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\}$. These authors, also get the following estimate for lower order eigenvalues of Problem 7

$$\sum_{i=1}^{n}(\sigma_{i+1}-\sigma_1) \leq 4(1+\alpha)(\sigma_1 + C_0). \tag{9}$$

Notice that Inequality (8) generalizes Inequality (6), whereas Inequality (9) generalizes Inequality (5).

This thesis is divided into two main chapters. In Chapter 1 we establish the conventions, definitions, and tools needed for all the rest of the work. For example, in Section 1.2 we give a brief review of some background material on tensors. In Section 1.3, we also present some properties of the operator $\mathscr{L}$ such as its relation with the Cheng-Yau operator. In the last two sections of the chapter, we present some properties of the problem in question as well as we list some results that will be useful in the next chapter.

In Chapter 2 we compute inequalities for eigenvalues of Problem 1. In Section 2.1, we begin by presenting two main results, in the more general settings of Problem 1.

We observe that Lemma 1 below is the key tool in this chapter, which is known in the literature in particular cases. For instance, in a particular case, it was used by Chen et al. [7] to prove Inequality (4).

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 1 and $\mathbf{u}_i$ be a normalized vector-valued eigenfunction corresponding to σ_i. Then, for any $f \in C^2(\Omega) \cap C^1(\partial\Omega)$ and any positive constant B, we obtain*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\Big\{(1-B)\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2dm - B\alpha\int_\Omega |\nabla f\cdot \mathbf{u}_i|^2dm\Big\}$$
$$\leq \frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\|T(\nabla f,\nabla \mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\|^2,$$

where $T(\nabla f,\nabla \mathbf{u}_i)=(T(\nabla f,\nabla u^1),\ldots,T(\nabla f,\nabla u^n))$ and $\mathbf{u}_i=(u_i^1,\ldots,u_i^n)$.

From Lemma 1 we obtain our first theorem, which is a quadratic inequality of σ_{k+1}.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 1. For any positive integer k, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big\{\Big[(\sigma_i-\alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2)^{\frac{1}{2}}+\frac{T_0}{2\sqrt{\delta}}\Big]^2+\frac{C_0}{\delta}\Big\},$$

where

$$C_0=\sup_\Omega\Big\{\frac{1}{2}\mathrm{div}(T^2(\nabla\eta))-\frac{1}{4}|T(\nabla\eta)|^2\Big\}+\frac{\delta}{2}T_0\eta_0, \tag{10}$$

$T_0=\sup_\Omega|\mathrm{tr}(\nabla T)|$ and $\eta_0=\sup_\Omega|\nabla\eta|$.

We observe that the quadratic estimate in Theorem 1 is the most appropriate inequality for the applications of our results in the Chapter 2. In particular, the constant C_0 in (10) has a crucial importance for us.

In Section 2.1, we also give an estimate for the sum of lower order eigenvalues in terms of the first eigenvalue and its correspondent eigenfunction.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 1, for $i=1,\ldots,n$, and $\mathbf{u}_1$ be a normalized eigenfunction corresponding to the first eigenvalue. Then, for any positive integer k, we have*

$$\sum_{i=1}^{n}(\sigma_{i+1}-\sigma_1) \leq \frac{4\delta(\delta+\alpha)}{\varepsilon^2}\Big\{\Big[(\sigma_1-\alpha\|\mathrm{div}_\eta \mathbf{u}_1\|^2)^{\frac{1}{2}}+\frac{T_0}{2\sqrt{\delta}}\Big]^2+\frac{C_0}{\delta}\Big\},$$

where C_0 is given by (10).

We note that, since $\alpha \geq 0$, from Theorem 2 we obtain immediately Inequality (9), and consequently Inequality (5), see Corollary 2.

In Subsection 2.2.1, we obtain applications for the case where the operator $\mathscr{L}$ becomes the drifted Laplacian Δ_η, that is, for Problem 7. The first one is the following quadratic inequality in σ_{k+1} obtained immediately from Theorem 1.

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 7. For any positive integer k, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2+C_0), \tag{11}$$

where $C_0 = \sup_\Omega\left\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\right\}$. Moreover, $\sigma_i - \alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2 + C_0 > 0$, for $i = 1, \ldots, k$.

The following corollary is an immediate consequence of Theorem 2.

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 7, for $i = 1, \ldots, n$, and $\mathbf{u}_1$ be a normalized eigenfunction corresponding to the first eigenvalue. For any positive integer k, we get*

$$\sum_{i=1}^{n}(\sigma_{i+1}-\sigma_1) \leq 4(1+\alpha)(\sigma_1+D_1), \tag{12}$$

where $D_1 = -\alpha\|\mathrm{div}_\eta \mathbf{u}_1\|^2 + C_0$ and $C_0 = \sup_\Omega\left\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\right\}$.

Let us consider

$$D_0 = -\alpha \min_{j=1,\ldots,k}\|\mathrm{div}_\eta \mathbf{u}_j\|^2 + C_0, \tag{13}$$

so that, from (11), we obtain

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i+D_0). \tag{14}$$

Notice that $\sigma_i + D_0 > 0$, and we immediately recover the following inequality:

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i, \tag{15}$$

which has been obtained by Chen et al. [7, Corollary 1.2] for Problem 2. Indeed, it follows from (13) and (14), since $\alpha \geq 0$ and we can take η to be a constant. Moreover, Inequality (15) implies Inequality (4), whereas Inequality (4) implies Inequality (3). However, we highlight that Inequality (14) provides an estimate for the eigenvalues of Problem 2 which is better than Inequality (15). Besides, note that Inequality (12) is better than Inequality (9) in Du and Bezerra [13]; whereas Inequality (14) is better than Inequality (1.3) again in [13].

Since Inequality (14) is a quadratic inequality for σ_{k+1}, solving it we obtain an upper bounded for σ_{k+1} and the gap of consecutive eigenvalues of Problem 7.

Corollary 3. *Under the same setup as in Corollary 2.2.1, we have*

$$\begin{aligned}\sigma_{k+1}+D_0 \leq & \Big(1+\frac{2(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)+\Big[\Big(\frac{2(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2 \\ & -\Big(1+\frac{4(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}\end{aligned}$$

and

$$\sigma_{k+1}-\sigma_k \leq 2\Big[\Big(\frac{2(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2-\Big(1+\frac{4(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}, \tag{16}$$

where D_0 is given by (13).

We emphasize that Inequality (16) strengthens inequalities (6) and (8), in the sense that they are easily obtained from Inequality (16).

Again from (14) and by applying the recursion formula of Cheng and Yang, we obtain the following corollary.

Corollary 4. *Under the same setup as in Corollary 2.2.1, we have*

$$\sigma_{k+1}+D_0 \leq \Big(1+\frac{4(n+\alpha)}{n^2}\Big)k^{\frac{2(n+\alpha)}{n^2}}(\sigma_1+D_0), \tag{17}$$

where D_0 is given by (13).

From the classical Weyl's asymptotic formula for the eigenvalues [38], we know that estimate (17) is optimal in the sense of the order on k.

Notice that the appearance of the constant C_0 in the previous results is natural, since we did not impose any restriction on the function η. We highlight that this constant has an interesting geometric interpretation, actually, it can be obtained as the supremum of the scalar curvature on the warped product $\Omega \times \mathbb{S}^1$ with respect a rescaling the warped metric $g = g_0 + e^{-\eta}d\theta^2$, where g_0 stands for the canonical metric in the domain $\Omega \subset \mathbb{R}^n$ and $d\theta^2$ is the canonical metric of the unit sphere $\mathbb{S}^1$.

So, we ask the following natural question:

Question 1. *Under which conditions the inequalities for the eigenvalues obtained in the previous corollaries do not depend on the constant C_0 for a nontrivial function η?*

We give an answer to this question in Corollary 5 considering annular domains in the Gaussian shrinking soliton. First, we recall that the gradient Ricci soliton (M^n, g, η) is characterized by $Ric + \nabla^2\eta = \lambda g$, for some constant λ, where $Ric + \nabla^2\eta$ is called the Bakry-Emery Ricci tensor. For $\lambda = 0, \lambda > 0$ and $\lambda < 0$, the gradient Ricci soliton is called steady, shrinking, and expanding, respectively. Then, let us consider the countable family of bounded domains $\{\Omega_l\}_{l=1}^{\infty}$ in Gaussian shrinking or expanding soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$ given by

$$\Omega_l = \mathbb{B}(r_l) - \bar{\mathbb{B}}(\sqrt{2n/|\lambda|}) = \Big\{x \in \mathbb{R}^n; \frac{2n}{|\lambda|} < |x|^2 < r_l^2\Big\}, \tag{18}$$

where $r_l > \sqrt{2n/|\lambda|}$ is a rational number, and $\mathbb{B}(r)$ stands for the open ball of radius r centered at the origin in $\mathbb{R}^n$.

Corollary 5 (Non-dependence of η). *Let us consider the family of domains $\{\Omega_l\}_{l=1}^{\infty}$ given by (18) in Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$. Let σ_i be the i-th eigenvalue of the drifted Laplacian Δ_η on real-valued functions, with drifting function $\eta(x) = \frac{\lambda}{2}|x|^2$, on each Ω_l with Dirichlet boundary condition. Then,*

$$\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)\sigma_i, \tag{19}$$

$$\sigma_{k+1} \leq \Big(1 + \frac{2}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\sigma_i + \Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 - \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}, \tag{20}$$

$$\sigma_{k+1} - \sigma_k \leq 2\Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 - \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}},$$

$$\sigma_{k+1} \leq \Big(1 + \frac{4}{n}\Big)k^{\frac{2}{n}}\sigma_1 \tag{21}$$

and

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq 4\sigma_1. \tag{22}$$

We highlight that Inequalities (19), (20), (21) and (22) have the same behavior as the known estimates of the eigenvalues of the Laplacian, see [8, Inequality (1.7)](or in the proof of [39, Theorem 1]), [8, Inequality (1.8)], [8, Corollary 2.1] and [3, Inequality (6.2)], respectively. Therefore, the countable family of bounded in Gaussian shrinking soliton given by (18) makes the behavior of known estimates of the Laplacian invariant by a first-order perturbation of the Laplacian.

In the next two corollaries we will apply our results, in the case of identity tensor, to the Gaussian expanding soliton.

Corollary 6. *Let $\mathbb{B}(r)$ be the open ball of radius r centered at the origin in Gaussian expanding soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$. Let σ_1 be the first eigenvalue of the drifted Laplacian*

Δ_η on real-valued functions, with drifting function $\eta(x) = \frac{\lambda}{2}|x|^2$, on $\mathbb{B}(r)$ with Dirichlet boundary condition. Then, we have

$$\sigma_1 \geq \frac{\pi^2 n}{64r^2} - \frac{\lambda n}{2},$$

and the next estimate for the sum of lower order eigenvalues σ_i of Δ_η in terms of the first eigenvalue

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq 4(\sigma_1 + \frac{\lambda n}{2}).$$

Corollary 7. *Let us consider the family of domains $\{\Omega_l\}_{l=1}^{\infty}$ given by (18) in Gaussian expanding soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$. Let σ_i be the i-th eigenvalue of the drifted Laplacian Δ_η on real-valued functions, with drifting function $\eta(x) = \frac{\lambda}{2}|x|^2$, on each Ω_l with Dirichlet boundary condition. Then, it is valid the following estimate for the sum of lower order eigenvalues of Δ_η in terms of the first eigenvalue:*

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq 4(\sigma_1 + \lambda n).$$

In Subsection 2.2.2 we generalize the main results of Subsection 2.2.1. For this, let us consider the case where the tensor T is divergence-free, that is, $\text{div}T = 0$. Divergence-free tensors often appear in physical facts, for instance, dynamic fluids, see Serre [36]. We highlight that Serre's work deals with divergence-free positive definite symmetric tensors and fluid dynamics. We can notice that, when T is divergence-free (see Eq. (1.13)), the operator $\mathscr{L}$ becomes

$$\mathscr{L}f = \Box f - \langle \nabla\eta, T(\nabla f)\rangle, \tag{23}$$

where $\Box$ is the operator introduced by Cheng and Yau [11] which arise from the study of complete hypersurfaces of constant scalar curvature in space forms. Therefore, Eq. (23) is a first-order perturbation of the Cheng-Yau operator, and it defines a *drifted Cheng-Yau operator* which we denote by $\Box_\eta$ with a drifting function η. Furthermore, in this case, our Problem 1 becomes

$$\begin{cases} \Box_\eta \mathbf{u} + \alpha\nabla(\text{div}_\eta \mathbf{u}) &= -\sigma \mathbf{u} \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{24}$$

where $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, the constant α is non-negative and $\Box_\eta \mathbf{u} = (\Box_\eta u^1, \Box_\eta u^2, \ldots, \Box_\eta u^n)$.

Now, also from Theorems 1 and 2 we immediately obtain the next two corollaries for Problem 24.

Corollary 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 24. For any positive integer k, we get*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2+\frac{C_0}{\delta}),$$

where $C_0 = \sup_\Omega\left\{\frac{1}{2}\mathrm{div}(T^2(\nabla\eta)) - \frac{1}{4}|T(\nabla\eta)|^2\right\}$.

Corollary 9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 2.22, for $i = 1, \ldots, n$, and $\mathbf{u}_1$ be a normalized eigenfunction corresponding to the first eigenvalue. Then, we get*

$$\sum_{i=1}^{n}(\sigma_{i+1}-\sigma_1) \leq \frac{4\delta(\delta+\alpha)}{\varepsilon^2}(\sigma_1+D_1),$$

where $D_1 = -\alpha\|\mathrm{div}_\eta\mathbf{u}_1\|^2 + \frac{C_0}{\delta}$.

Also in Subsection 2.2.2, from Corollary 8 and following the steps of the proof of Corollary 3, we obtain the inequalities.

Corollary 10. *Under the same setup as in Corollary 8, and by defining* $D_0 = \frac{C_0}{\delta} - \alpha\min_{j=1,\ldots,k}\|\mathrm{div}_\eta\mathbf{u}_j\|^2$, *we have*

$$\begin{aligned}\sigma_{k+1}+D_0 \leq &\Big(1+\frac{2\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\Big)\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0) + \Big[\Big(\frac{2\delta(n\delta+\alpha)}{\varepsilon^2n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2 \\ &- \Big(1+\frac{4\delta(n\delta+\alpha)}{\varepsilon^2n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}\end{aligned}$$

and

$$\sigma_{k+1}-\sigma_k \leq 2\Big[\Big(\frac{2\delta(n\delta+\alpha)}{\varepsilon^2n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2 - \Big(1+\frac{4\delta(n\delta+\alpha)}{\varepsilon^2n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}.$$

Again from Corollary 8 and by applying the recursion formula of Cheng and Yang [8], we obtain the next corollary.

Corollary 11. *Under the same setup as in Corollary 10, we have*

$$\sigma_{k+1}+D_0 \leq \Big(1+\frac{4\delta(\delta n+\alpha)}{\varepsilon^2n^2}\Big)k^{\frac{2\delta(n\delta+\alpha)}{\varepsilon^2n^2}}(\sigma_1+D_0).$$

In Section 2.3, in the more general setting, we obtain the gap of consecutive eigenvalues of Problem 2.1. In fact, since $\alpha \geq 0$, immediately from Theorem 1, we obtain

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big[\Big(\sqrt{\sigma_i}+\frac{1}{2\sqrt{\delta}}T_0\Big)^2+\frac{C_0}{\delta}\Big]. \tag{25}$$

Hence, from Inequality (25) we get the following estimate.

Corollary 12. *Under the same setup as in Theorem 1, we get*

$$\sigma_{k+1} \leq \varsigma_k + \sqrt{\varsigma_k^2 - \vartheta_k}$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1} - \sigma_k \leq 2\sqrt{\varsigma_k^2 - \vartheta_k},$$

where

$$\varsigma_k = \frac{1}{k}\Big\{ \sum_{i=1}^{k} \sigma_i \quad + \tfrac{2\delta(n\delta+\alpha)}{n^2\varepsilon^2} \textstyle\sum_{i=1}^{k} \Big[\Big(\sqrt{\sigma_i} + \tfrac{1}{2\sqrt{\delta}} T_0 \Big)^2 + \tfrac{C_0}{\delta} \Big] \Big\}$$

and

$$\vartheta_k = \frac{1}{k}\Big\{ \sum_{i=1}^{k} \sigma_i^2 \quad + \tfrac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2} \textstyle\sum_{i=1}^{k} \sigma_i \Big[\Big(\sqrt{\sigma_i} + \tfrac{1}{2\sqrt{\delta}} T_0 \Big)^2 + \tfrac{C_0}{\delta} \Big] \Big\}.$$

From Corollary 2.3.1, using Chebyshev's inequality we have.

Corollary 13. *Under the same setup as in Theorem 2.1.1, we get*

$$\sigma_{k+1} \leq \frac{1}{k} \sum_{i=1}^{k} \sigma_i + \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2} \frac{1}{k} \sum_{i=1}^{k} \Big[\Big(\sqrt{\sigma_i} + \frac{1}{2\sqrt{\delta}} T_0 \Big)^2 + \frac{C_0}{\delta} \Big] \tag{26}$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1} - \sigma_k \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2} \frac{1}{k} \sum_{i=1}^{k} \Big[\Big(\sqrt{\sigma_i} + \frac{1}{2\sqrt{\delta}} T_0 \Big)^2 + \frac{C_0}{\delta} \Big]. \tag{27}$$

Chapter 1

Elliptic differential system in divergence form

In this chapter, we establish the tools, notations, conventions, and definitions needed for all the rest of the work. We start the chapter by presenting definitions and properties about differential operators in Section 1.1, e.g., the Laplacian operator. Next, we present the main properties regarding tensors in Section 1.2. In Section 1.3, we will see the motivation for operator $\mathscr{L}$ and present some important information such as its relation with the Cheng-Yau operator. In the last two sections of this chapter, we see important tools for the course of the thesis, for instance, the recursive formula of Cheng and Yang. We would like to emphasize that throughout the text we are working with bounded domains.

1.1 Gradient, Hessian and Laplacian

In this section we give definitions of the operators: gradient, Hessian and Laplacian. The interested reader is referred to the book by Chavel [6] for more details.

Let us consider a Riemannian manifold $(M, \langle , \rangle)$ so that we define the following operators.

Definition 1.1.1. *Given a real-valued $C^k, k \geq 1$, function f on M, we define the gradient of f, denoted by ∇f, to be the vector field on M for which*

$$\langle \nabla f, X \rangle = X(f),$$

for all $X \in \mathfrak{X}(M)$.

One has for all functions f, h on M

i) $\nabla(f + h) = \nabla f + \nabla h$;

ii) $\nabla(fh) = h\nabla f + f\nabla h$.

Definition 1.1.2. *Given a $C^k, k \geq 1$, vector field X on M, define the real-valued function the divergence of $X, \mathrm{div}X$, by*

$$(\mathrm{div}X)(p) = trace\{Y \to \nabla_Y X\},$$

where $Y \in T_pM$ and ∇ is the Levi-Civita connection of M.

The divergence of X is a C^{k-1} function on M, and for the function f, and vector fields X, Y on M, we have the followings properties

i) $\mathrm{div}(X+Y) = \mathrm{div}X + \mathrm{div}Y$;

ii) $\mathrm{div}(fX) = f(\mathrm{div}X) + \langle \nabla f, X\rangle$.

Another important operator is the Hessian, its definition is given below.

Definition 1.1.3. *Given a real-valued $C^k, k \geq 2$, function f on M, we define the Hessian of f, denoted by $\nabla^2 f$, by*

$$\nabla^2 f(X,Y) = \langle \nabla_X \nabla f, Y\rangle, \quad \textit{for all} \quad X, Y \in \mathfrak{X}(M).$$

Now we define the Laplacian operator of real-valued functions on M.

Definition 1.1.4. *For any $C^k, k \geq 2$, function f on M we define the function the Laplacian of f, denoted by Δf, by*

$$\Delta f = \mathrm{div}(\nabla f).$$

One has that $\Delta f \in C^{k-2}$, and from divergence properties, the Laplacian satisfies

i) $\Delta(f+h) = \Delta f + \Delta h$;

ii) $\mathrm{div}(h\nabla f) = h\Delta f + \langle \nabla h, \nabla f\rangle$;

iii) $\Delta(fh) = h\Delta f + 2\langle \nabla f, \nabla h\rangle + f\Delta h$.

To finish this section, we would like to observe the following relation between Hessian and Laplacian

$$\Delta f = \mathrm{tr}(\nabla^2 f) \quad \text{for all} \quad f \in C^k, k \geq 2.$$

1.2 Tensors in Riemannian manifolds

Let us start this section with the tensor definition and its properties. This is the fundamental concept for understanding this work. The interested reader is referred to the books by Petersen [29] and Lee [22] for further results about tensors.

Definition 1.2.1. *A $(1, r)$-tensor in a Riemannian manifold $(M, \langle , \rangle)$ is a multilinear map*

$$T : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{(r-factors)} \longrightarrow \mathfrak{X}(M)$$

over the ring $C^\infty(M)$ of differential maps in M. Moreover, a $(0, r)$-tensor is defined in an analogous way just by changing the counter-domain to the ring $C^\infty(M)$ of differential maps in M. Formally

$$T(Y_1, \ldots, fX + \ell Y, \ldots, Y_r) = fT(Y_1, \ldots, X, \ldots, Y_r) + \ell T(Y_1, \ldots, Y, \ldots, Y_r),$$

for all $X, Y \in \mathfrak{X}(M)$ and $f, \ell \in C^\infty(M)$.

We can identify a $(0, r)$-tensor T with a $(1, r + 1)$-tensor which we will still indicate by T through the Riemannian metric $\langle , \rangle$, as follow

$$\langle T(X_1, \ldots, X_{r-1}), X_r\rangle = T(X_1, \ldots, X_r).$$

In particular, the metric tensor $\langle , \rangle$ is identified with the identity $(1, 1)$-tensor I in $\mathfrak{X}(M)$.

Another important definition is the definition of a covariant derivative of a tensor.

Definition 1.2.2. *The covariant derivative of an $(1, r)$-tensor T is an $(1, r + 1)$-tensor ∇T given by*

$$\nabla T(X, Y_1, \ldots, Y_r) = \nabla_X(T(Y_1, \ldots, Y_r)) - T(\nabla_X Y_1, \ldots, Y_r) - \cdots - T(Y_1, \ldots, \nabla_X Y_r).$$

For each $X \in \mathfrak{X}(M)$ we can define the covariant derivative $\nabla_X T$ of T as a tensor of the same order of T given by

$$\nabla_X T(Y_1, \ldots, Y_r) := \nabla T(X, Y_1, \ldots, Y_r).$$

Analogously the covariant derivative of the a $(0, r)$-tensor is a $(0, r + 1)$-tensor given by (1.2.2). The tensor T is parallel when $\nabla T \equiv 0$.

Another important concept is the divergence of a tensor.

Definition 1.2.3. *We define the divergence of a $(1, r)$-tensor T in $(M^n, \langle , \rangle)$ as the $(0, r)$-tensor given by*

$$(\mathrm{div}T)(v_1, \ldots, v_r)(p) = \mathrm{tr}\{w \mapsto (\nabla_w T)(v_1, \ldots, v_r)(p)\},$$

where $p \in M^n$, $(v_1, \ldots, v_r) \in T_pM \times \cdots \times T_pM$ and tr denote the trace. Moreover, we say that the tensor T is divergence-free when $\mathrm{div}T = 0$.

Given an n-dimensional Riemannian manifold $(M, \langle , \rangle)$, to each $X \in \mathfrak{X}(M)$ we associate the $(0,1)$-tensor $X^\flat : \mathfrak{X}(M) \to C^\infty(M)$, given by

$$X^\flat(Y) = \langle X, Y \rangle \quad \text{for all} \quad Y \in \mathfrak{X}(M).$$

We denote by $^\sharp : \mathfrak{X}^*(M) \to \mathfrak{X}(M)$ the inverse of the mapping $^\flat \ : \ \mathfrak{X}(M) \ \to \ \mathfrak{X}^*(M)$, called the musical isomorphism.

Now we recall the *Hilbert-Schmidt inner product*, for this, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis in T_pM, S and T be $(1,1)$-tensor with adjoints S^* and T^*, respectively. The *Hilbert-Schmidt inner product* is given by

$$\langle T, S \rangle := \mathrm{tr}(TS^*) = \sum_{i=1}^{n} \langle TS^*(e_i), e_i \rangle = \sum_{i=1}^{n} \langle S^*(e_i), T^*(e_i) \rangle = \sum_{i=1}^{n} \langle T(e_i), S(e_i) \rangle.$$

Let T be a symmetric and positive definite $(1,1)$-tensor in a Riemannian manifold M^n, let us define the vector field $\mathrm{tr}(\nabla T) \in \mathfrak{X}(M)$ by

$$\mathrm{tr}(\nabla T) := \sum_{i=1}^{n} (\nabla T)(e_i, e_i) = \sum_{i=1}^{n} \Big(\nabla_{e_i} T(e_i) - T(\nabla_{e_i} e_i) \Big), \tag{1.1}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame in $p \in M$.

Since T is symmetric, $\nabla_X T$ is also symmetric for each $X \in \mathfrak{X}(M)$, that is,

$$\langle (\nabla_X T)Y, Z \rangle = \langle Y, (\nabla_X T)Z \rangle, \quad \forall Y, Z \in \mathfrak{X}(M). \tag{1.2}$$

In fact, for $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\begin{aligned} \langle (\nabla_X T)Y, Z \rangle &= \langle \nabla_X T(Y) - T(\nabla_X Y), Z \rangle \\ &= X\langle T(Y), Z \rangle - \langle T(Y), \nabla_X Z \rangle - \langle \nabla_X Y, T(Z) \rangle \\ &= X\langle Y, T(Z) \rangle - \langle Y, T(\nabla_X Z) \rangle - \langle \nabla_X Y, T(Z) \rangle \\ &= \langle Y, \nabla_X T(Z) - T(\nabla_X Z) \rangle = \langle Y, (\nabla_X T)Z \rangle. \end{aligned}$$

Using this fact we get the following lemma.

Lemma 1.2.1. *Let T be a $(1,1)$-tensor symmetric in a Riemannian manifold M^n. If T is divergence-free, then* $\mathrm{tr}(\nabla T) = 0$.

Proof. Fix $p \in M$ and let $\{e_1, \ldots, e_n\}$ be a local orthonormal geodesic frame in an open

$\Omega \subset M$ containing p. Since T is divergence-free, that is, $\text{div}T = 0$, from (1.2) we have

$$0 = \text{div}T(v) = \sum_{i=1}^{n} \langle (\nabla_{e_i} T)(v), e_i \rangle = \sum_{i=1}^{n} \langle v, (\nabla_{e_i} T)(e_i) \rangle$$
$$= \langle v, \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i) \rangle, \quad \text{for all} \quad v \in \mathfrak{X}(\Omega),$$

hence $\sum_{i=1}^{n} (\nabla_{e_i} T)(e_i) = 0$. Therefore, from (1.1)

$$\text{tr}(\nabla T) = \sum_{i=1}^{n} (\nabla T)(e_i, e_i) = \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i) = 0.$$

□

Lemma 1.2.2. *Let T be a $(1,1)$-tensor symmetric and positive definite in a Riemannian manifold M^n. If $\varepsilon I \leq T \leq \delta I$, for some positive real numbers ε and δ, then*

$$\varepsilon \langle T(X), X \rangle \leq |T(X)|^2 \leq \delta \langle T(X), X \rangle \quad \textit{for all} \quad X \in \mathfrak{X}(M). \tag{1.3}$$

In particular, we obtain

$$\varepsilon^2 |\nabla \eta|^2 \leq |T(\nabla \eta)|^2 \leq \delta^2 |\nabla \eta|^2, \tag{1.4}$$

for some function $\eta \in C^\infty(M)$.

Proof. By hypotheses, we have

$$\varepsilon |X|^2 \leq \langle T(X), X \rangle \leq \delta |X|^2 \quad \text{for all} \quad X \in \mathfrak{X}(M). \tag{1.5}$$

Since T is symmetric and positive definite there exists a local frame $\{e_i\}_{i=1}^n$ such that $T(e_i) = \gamma_i e_i$ with $\gamma_i > 0$ for all $1 \leq i \leq n$. From (1.5) we get

$$\varepsilon \leq \gamma_i = \langle T(e_i), e_i \rangle \leq \delta \quad \text{for all} \quad 1 \leq i \leq n. \tag{1.6}$$

Let $X \in \mathfrak{X}(M)$, which can be expressed in terms of this frame as $X = \sum_{i=1}^n a_i e_i$, from (1.6) we obtain

$$|T(X)|^2 = \langle T(\sum_{i=1}^{n} a_i e_i), T(\sum_{j=1}^{n} a_j e_j) \rangle = \sum_{i=1}^{n} \gamma_i^2 a_i^2.$$

Now note that

$$\varepsilon \sum_{i=1}^{n} \gamma_i a_i^2 \leq \sum_{i=1}^{n} \gamma_i \gamma_i a_i^2 \leq \delta \sum_{i=1}^{n} \gamma_i a_i^2$$

which is enough to complete the proof of the lemma. □

1.3 Operator $\mathscr{L}$ in the divergence form and its properties

In this section, the manifold $(M^n, \langle , \rangle)$ is assumed to be complete and the bounded domain $\Omega \subset M$ is assumed to be connected (otherwise the problem decomposes into a finite number of independent subproblems) and with smooth boundary $\partial\Omega$. The section is brief and serves to set the stage, introducing some basic notation and describing what is meant by the properties of a $(1,1)$-tensor on bounded domains.

Throughout the thesis, we will be constantly using the identification of a $(0,2)$-tensor $T : \mathfrak{X}(\Omega) \times \mathfrak{X}(\Omega) \to C^\infty(\Omega)$ with its associated $(1,1)$-tensor $T : \mathfrak{X}(\Omega) \to \mathfrak{X}(\Omega)$ by the equation

$$\langle T(X), Y \rangle = T(X, Y).$$

In particular, the tensor $\langle , \rangle$ will be identified with the identity I in $\mathfrak{X}(\Omega)$.

We would like to extend the definition of the divergence as follows.

Definition 1.3.1. *For each $X \in \mathfrak{X}(M)$ and a fixed function $\eta \in C^\infty(M)$, let us define the η-divergence of X as follows*

$$\mathrm{div}_\eta X := e^\eta \mathrm{div}(e^{-\eta} X) = \mathrm{div} X - \langle \nabla\eta, X \rangle.$$

Therefore, we can see from the previous definition and the usual properties of divergence of vector fields that

$$\mathrm{div}_\eta(fX) = f\mathrm{div}_\eta X + \langle \nabla f, X \rangle \quad \text{and} \quad \mathrm{div}(e^{-\eta} X) = e^{-\eta}\mathrm{div}_\eta X, \tag{1.7}$$

for all $f \in C^\infty(M)$.

We can define a second-order elliptic differential operator $\mathscr{L}$ in the (η, T)-divergence form as follows:

Definition 1.3.2. *Let T be a symmetric and positive definite $(1,1)$-tensor on a Riemannian manifold $(M, \langle , \rangle)$. Let us define the (η, T)-divergence operator by*

$$\mathscr{L} f := \mathrm{div}_\eta(T(\nabla f)) = \mathrm{div}(T(\nabla f)) - \langle \nabla\eta, T(\nabla f) \rangle, \tag{1.8}$$

for all $f \in C^\infty(M)$, where div *stands for the divergence operator and ∇ for the gradient operator.*

Now, we see some information and properties of the operator $\mathscr{L}$. Notice that the (η, T)-divergence form of $\mathscr{L}$ in Eq. (1.8) on Ω allows us to check that the divergence theorem remains true in the form

$$\int_\Omega \mathrm{div}_\eta X dm = \int_{\partial\Omega} \langle X, \nu \rangle d\mu. \tag{1.9}$$

In particular, for $X = T(\nabla f)$,

$$\int_\Omega \mathscr{L} f dm = \int_{\partial\Omega} T(\nabla f, \nu) d\mu,$$

where $dm = e^{-\eta} d\Omega$ and $d\mu = e^{-\eta} d\partial\Omega$ are the weight volume form on Ω and the volume form on the boundary $\partial\Omega$ induced by the outward unit normal vector ν on $\partial\Omega$, respectively. Thus, the integration by parts formula is given by

$$\int_\Omega \ell \mathscr{L} f dm = -\int_\Omega T(\nabla \ell, \nabla f) dm + \int_{\partial\Omega} \ell T(\nabla f, \nu) d\mu, \tag{1.10}$$

for all $\ell, f \in C^\infty(\Omega)$. Hence, $\mathscr{L}$ is a formally self-adjoint operator in the Hilbert space of all functions in $L^2(\Omega, dm)$ that vanish on $\partial\Omega$, with inner product given by (1.10).

Moreover, since $\mathscr{L} f := \mathrm{div}_\eta(T(\nabla f))$, for all real-valued functions $f, \ell \in C^\infty(\Omega)$ it is immediate from the properties of div_η and the symmetry of T that

$$\mathscr{L}(fh) = f\mathscr{L}h + 2T(\nabla f, \nabla h) + h\mathscr{L}f. \tag{1.11}$$

In fact, from (1.7) we have

$$\begin{aligned}
\mathscr{L}(fh) &= \mathrm{div}_\eta(T(\nabla(fh))) = \mathrm{div}_\eta(T(f\nabla h + h\nabla f)) \\
&= \mathrm{div}_\eta T(f\nabla h) + \mathrm{div}_\eta(T(h\nabla f)) \\
&= f\mathrm{div}_\eta(T(\nabla h)) + \langle \nabla f, T(\nabla h)\rangle + h\mathrm{div}_\eta(T(\nabla f)) + \langle \nabla h, T(\nabla f)\rangle \\
&= f\mathscr{L} + 2T(\nabla f, \nabla h) + h\mathscr{L}f.
\end{aligned}$$

Moreover, we can see that the operator $\mathscr{L}$ appears as the trace of a $(1,1)$-tensor on a Riemannian manifold M^n. In fact, let us consider the $(1,1)$-tensor

$$\tau_{\eta,f} := \nabla T(\nabla f) - \frac{1}{n} T(\nabla f, \nabla \eta) I,$$

we have

$$
\begin{aligned}
\operatorname{tr}(\tau_{\eta,f}) &= \sum_{i=1}^{n}\langle \nabla T(\nabla f)(e_i) - \frac{1}{n}T(\nabla f,\nabla\eta)e_i, e_i\rangle\\
&= \sum_{i=1}^{n}\langle \nabla_{e_i} T(\nabla f), e_i\rangle - \frac{1}{n}T(\nabla f,\nabla\eta)\sum_{i=1}^{n}\langle e_i, e_i\rangle\\
&= \operatorname{div} T(\nabla f) - T(\nabla f,\nabla\eta) = \mathscr{L} f.
\end{aligned}
$$

Hence, we obtain

$$\overset{\circ}{\tau}_{\eta,f} = \tau_{\eta,f} - \frac{\mathscr{L} f}{n} I,$$

and

$$|\overset{\circ}{\tau}_{\eta,f} f|^2 \geq \frac{(\mathscr{L} f)^2}{n}.$$

Cheng and Yau [11] introduced a differential operator appropriate for the study of complete hypersurfaces of constant scalar curvature in space forms, namely

$$\Box f = \operatorname{tr}(\nabla^2 f \circ T) = \langle \nabla^2 f, T\rangle,$$

where $f \in C^\infty(M)$ and T is a symmetric $(1,1)$-tensor. In fact, with a careful study of this operator, using a divergence-free tensor, Cheng and Yau obtained remarkable rigidity results for such hypersurfaces. For instance, for Euclidean space, they proved that the only complete and non-compact hypersurfaces with non-negative constant normalized scalar curvature and non-negative sectional curvature are the generalized cylinders, for more details see [11].

It is worth mentioning here the paper by Gomes and Miranda [15] from which we know some geometric motivations to work with the operator $\mathscr{L}$ in the (η, T)-divergence form in bounded domains in Riemannian manifolds. They showed that it appears as the trace of a $(1,1)$-tensor on a Riemannian manifold M, and computed a Bochner-type formula for it. They also observed a relation between operator $\mathscr{L}$ and operator $\Box$, see Eq. (1.13). Such a relation follows from

$$\operatorname{div}_\eta(T(h\nabla f)) = h\langle \operatorname{div}_\eta T, \nabla f\rangle + h\langle \nabla^2 f, T\rangle + T(\nabla h, \nabla f), \tag{1.12}$$

where $\operatorname{div}_\eta T := \operatorname{div} T - d\eta \circ T$ is the η-divergence of a symmetric tensor T, and $d\eta \circ T = \langle \nabla\eta, T(\cdot)\rangle = T(\nabla\eta, \cdot)$. Its proof follows immediately from the properties of divergence operator and definition of the Hilbert-Schmidt inner product, for a complete proof see Gomes [16], for the η =constant case, and Mesquita [31] for η non-constant case. In particular, this formula has been used by Gomes to obtain a characterization of the Euclidean sphere. He also highlights that it has already been used in the literature for several cases, the interested reader can consult Rosenberg [35], Obata and Yano [26] or

Yano's book [40].

From (1.12) we get the promised relation

$$\mathscr{L} f = \Box f + \langle \mathrm{div}_\eta T, \nabla f\rangle = \Box f + \langle \mathrm{div} T, \nabla f\rangle - \langle \nabla\eta, T(\nabla f)\rangle. \tag{1.13}$$

Using Eq. (1.13), Gomes and Miranda [15] gave a *Bochner-type formula* for the operator $\mathscr{L}$. This relation shows that the operator $\mathscr{L}$ can be seen as a first-order perturbation of the operator $\Box$ of Cheng and Yang in the case where the tensor is divergence-free, see Section 2.2.2.

1.4 Basic concepts from an elliptic differential system in divergence form

In this section we recall some information necessary for our work regarding the elliptic differential operator $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$ with Dirichlet boundary condition, such as some information about spectral theory of these operators and the Rayleigh quotient. Initially, we would like to emphasize that the concepts and properties in Sections 1.2 and 1.3 remains true when the complete Riemannian manifold M^n is $\mathbb{R}^n$.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with its canonical metric $\langle , \rangle$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Recall that we are interested to compute universal estimates of the eigenvalues of the following problem:

$$\begin{cases} \mathscr{L}\mathbf{u} + \alpha\nabla(\mathrm{div}_\eta \mathbf{u}) & = & -\sigma\mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{1.14}$$

where $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, the constant α is non-negative and $\mathscr{L}\mathbf{u} = (\mathscr{L}u^1, \mathscr{L}u^2, \ldots, \mathscr{L}u^n)$.

For a vector-valued function $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ from Ω to $\mathbb{R}^n$, we define

$$\nabla\mathbf{u} = (\nabla u^1, \ldots, \nabla u^n).$$

Now, we are going to some definitions for T, and since there is no risk of confusion, we are using the same notation T for both definitions. Let X be a vector field on Ω, we define

$$T(\nabla\mathbf{u}) = (T(\nabla u^1), \ldots, T(\nabla u^n)),$$

and

$$
\begin{aligned}
T(X,\nabla\mathbf{u}) &= (\langle T(X),\nabla u^1\rangle,\dots,\langle T(X),\nabla u^n\rangle)\\
&= (\langle X,T(\nabla u^1)\rangle,\dots,\langle X,T(\nabla u^n)\rangle)\\
&= (T(X,\nabla u^1),\dots,T(X,\nabla u^n)). \qquad (1.15)
\end{aligned}
$$

Furthermore, from (1.7) we have

$$\mathrm{div}_\eta((\mathrm{div}_\eta\mathbf{u})\mathbf{v}) = \mathrm{div}_\eta\mathbf{u}\mathrm{div}_\eta\mathbf{v} + \mathbf{v}\cdot\nabla(\mathrm{div}_\eta\mathbf{u}).$$

Hence, from (1.9) we obtain

$$\int_\Omega \mathbf{v}\cdot\nabla(\mathrm{div}_\eta\mathbf{u})dm = -\int_\Omega \mathrm{div}_\eta\mathbf{u}\mathrm{div}_\eta\mathbf{v}dm, \qquad (1.16)$$

for all vector-valued function $\mathbf{u} = (u^1,u^2,\dots,u^n)$ and $\mathbf{v} = (v^1,v^2,\dots,v^n)$ both from Ω to $\mathbb{R}^n$, with $\mathbf{v}$ vanishing on $\partial\Omega$.

Now, using (1.11), the next equation is well understood for a vector-valued function $\mathbf{u}$ and a real-valued function $f\in C^\infty(\Omega)$

$$
\begin{aligned}
\mathscr{L}(f\mathbf{u}) &= (\mathscr{L}(fu^1),\dots,\mathscr{L}(fu^n))\\
&= (f\mathscr{L}u^1 + 2\langle T(\nabla f),\nabla u^1\rangle + \mathscr{L}(f)u^1,\dots,f\mathscr{L}u^n + 2\langle T(\nabla f),\nabla u^n\rangle + \mathscr{L}(f)u^n)\\
&= f(\mathscr{L}u^1,\dots,\mathscr{L}u^n) + 2(\langle T(\nabla f),\nabla u^n\rangle,\dots,\langle T(\nabla f),\nabla u^n\rangle) + \mathscr{L}(f)(u^1,\dots,u^n)\\
&= f\mathscr{L}\mathbf{u} + 2T(\nabla f,\nabla\mathbf{u}) + \mathscr{L}f\mathbf{u}. \qquad (1.17)
\end{aligned}
$$

Let us denote by $\mathbb{L}^2(\Omega,dm)$ the space of vector-valued functions with the inner product

$$\langle\mathbf{u},\mathbf{v}\rangle_{\mathbb{L}^2} = \langle u^1,v^1\rangle_{L^2} + \cdots + \langle u^n,v^n\rangle_{L^2},$$

where $\mathbf{u} = (u^1,\dots,u^n)$, $\mathbf{v} = (v^1,\dots,v^n)$ are vector-valued functions in $\Omega\in\mathbb{R}^n$ and $\langle u^i,v^i\rangle_{L^2} = \int_\Omega u_iv_idm$, $1\le i\le n$, is the inner product in $L^2(\Omega,dm)$. Besides, we are using the classical norms: $|\mathbf{u}|^2 = \sum_{i=1}^n(u^i)^2$ and $\|\mathbf{u}\|^2_{\mathbb{L}^2} = \int_\Omega|\mathbf{u}|^2dm$.

Lemma 1.4.1. *For the operator* $\mathscr{L}+\alpha\nabla\mathrm{div}_\eta$ *the following identity applies*

$$\langle\mathscr{L}\mathbf{u}+\alpha\nabla(\mathrm{div}_\eta\mathbf{u}),\mathbf{v}\rangle\rangle_{\mathbb{L}^2} = \langle\mathbf{u},\mathscr{L}\mathbf{v}+\alpha\nabla(\mathrm{div}_\eta\mathbf{v})\rangle_{\mathbb{L}^2},$$

where $\mathbf{u} = (u^1,\dots,u^n)$ *and* $\mathbf{v} = (v^1,\dots,v^n)$ *are vector-valued functions that vanish on* $\partial\Omega$.

Proof. In fact, from (1.10) we have

$$\begin{aligned}
\int_\Omega \mathscr{L}\mathbf{u}\cdot\mathbf{v}dm &= +\int_\Omega \mathscr{L}u^1v^1dm + \cdots + \int_\Omega \mathscr{L}u^nv^ndm \\
&= -\int_\Omega T(\nabla u^1, \nabla v^1)dm - \cdots - \int_\Omega T(\nabla u^n, \nabla v^n)dm \\
&= -\int_\Omega T(\nabla v^1, \nabla u^1)dm - \cdots - \int_\Omega T(\nabla v^n, \nabla u^n)dm \\
&= \int_\Omega \mathscr{L}v^1u^1dm + \cdots + \int_\Omega \mathscr{L}v_nu^ndm \\
&= \int_\Omega \mathbf{u}\cdot\mathscr{L}\mathbf{v}dm. \qquad (1.18)
\end{aligned}$$

And from (1.16) we have

$$\int_\Omega \nabla(\mathrm{div}_\eta\mathbf{u})\cdot\mathbf{v}dm = \int_\Omega \mathbf{u}\cdot\nabla(\mathrm{div}_\eta\mathbf{v})dm. \qquad (1.19)$$

Therefore, from (1.18) and (1.19) we get

$$\begin{aligned}
\langle\mathscr{L}\mathbf{u}+\alpha\nabla(\mathrm{div}_\eta\mathbf{u}),\mathbf{v}\rangle_{\mathbb{L}^2} &= \int_\Omega(\mathscr{L}\mathbf{u}+\alpha\nabla(\mathrm{div}_\eta\mathbf{u}))\cdot\mathbf{v}dm \\
&= \int_\Omega \mathscr{L}\mathbf{u}\cdot\mathbf{v}dm + \alpha\int_\Omega \nabla(\mathrm{div}_\eta\mathbf{u})\cdot\mathbf{v}dm \\
&= \int_\Omega \mathbf{u}\cdot\mathscr{L}\mathbf{v}dm + \alpha\int_\Omega \mathbf{u}\cdot\nabla(\mathrm{div}_\eta\mathbf{v})dm \\
&= \int_\Omega \mathbf{u}\cdot(\mathscr{L}\mathbf{v}+\alpha\nabla(\mathrm{div}_\eta\mathbf{v}))dm \\
&= \langle\mathbf{u},\mathscr{L}\mathbf{v}+\alpha\nabla(\mathrm{div}_\eta\mathbf{v})\rangle_{\mathbb{L}^2},
\end{aligned}$$

and we conclude the proof of the lemma. □

Lemma 1.4.1 says that $\mathscr{L}+\alpha\nabla\mathrm{div}_\eta$ is a formally self-adjoint operator in the Hilbert space $\mathbb{L}^2(\Omega, dm)$ of all vector-valued functions that vanish on $\partial\Omega$. Therefore, the eigenvalue problem 2.1 has a real and discrete spectrum $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \to \infty$, where each σ_i is repeated according to its multiplicity and its positively is ensure by Lemma 1.4.2. Eigenspaces belonging to distinct eigenvalues are orthogonal in $\mathbb{L}^2(\Omega, dm)$, which is the direct sum of all the eigenspaces. We refer to the dimension of each eigenspace as the multiplicity of the eigenvalue.

Lemma 1.4.2. *All eigenvalues of Problem 1.14 are positive.*

Proof. Let $\mathbf{u} = (u^1, \ldots, u^n)$ be an eigenfunction with corresponding eigenvalue σ. Then, from (1.10) we have

$$\begin{aligned} -\int_\Omega \mathbf{u}\cdot \mathscr{L}\mathbf{u}dm &= -\int_\Omega u^1\mathscr{L}u^1 dm - \cdots - \int_\Omega u^n \mathscr{L}u^n dm \\ &= \int_\Omega T(\nabla u^1, \nabla u^1)dm + \cdots + \int_\Omega T(\nabla u^n, \nabla u^n)dm \\ &= \int_\Omega T(\nabla \mathbf{u}, \nabla \mathbf{u})dm, \end{aligned} \tag{1.20}$$

Moreover, from (1.16) we get

$$-\int_\Omega \mathbf{u}\cdot\nabla(\mathrm{div}_\eta \mathbf{u})dm = \int_\Omega (\mathrm{div}_\eta \mathbf{u})^2 dm = \|\mathrm{div}_n \mathbf{u}\|^2. \tag{1.21}$$

Hence, from (1.14), (1.20) and (1.21) we obtain

$$\sigma \int_\Omega \mathbf{u}^2 dm = \int_\Omega T(\nabla \mathbf{u}, \nabla \mathbf{u})dm + \|\mathrm{div}_n \mathbf{u}\|^2 \geq \varepsilon \int_\Omega |\nabla \mathbf{u}|^2 dm.$$

Note that this is sufficient to conclude that all eigenvalues of Problem 1.14 are positive. □

We now adapt the Rayleigh quotient for our operator, the interested reader is encouraged to consult about it on Chavel [6] and Olver [27].

Definition 1.4.1. *The Rayleigh quotient of* $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$ *is defined as*

$$R(\mathbf{u}) = \frac{\langle -(\mathscr{L} + \alpha\nabla\mathrm{div}_\eta)\mathbf{u}, \mathbf{u}\rangle_{\mathbb{L}^2}}{\|\mathbf{u}\|^2_{\mathbb{L}^2}}.$$

Theorem 1.4.1. *The minimum value of Rayleigh quotient of* $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$,

$$\sigma_1 = \min\{R(\mathbf{u}) \quad \textit{with} \quad \mathbf{u} \neq 0 \quad \textit{and} \quad \mathbf{u}|_{\partial\Omega} = 0\},$$

is the smallest eigenvalue of the operator $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$. *Moreover, any* $0 \neq \mathbf{u}_1 \in \mathbb{L}^2(\Omega, dm)$ *that achieves this minimum value, is an associated eigenfunction, that is,* $-(\mathscr{L} + \alpha\nabla\mathrm{div}_\eta)\mathbf{u}_1 = \sigma_1\mathbf{u}_1$.

Proof. See [27, Theorem 9.42]. □

One of the most important results for obtaining our theorems in Chapter 2 is the following characterization of the eigenvalues of our problem.

Theorem 1.4.2. *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ *be the eigenfunctions corresponding to the first* $n-1$ *eigenvalues* $0 < \sigma_1 \leq \cdots \leq \sigma_{n-1}$ *of* $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$. *Let*

$$V_{n-1} = \{\mathbf{v} \in \mathbb{L}^2(\Omega, dm) \quad \textit{with} \quad \mathbf{v}|_{\partial\Omega} = 0; \langle \mathbf{v}, \mathbf{u}_1\rangle_{\mathbb{L}^2} = \cdots = \langle \mathbf{v}, \mathbf{u}_{n-1}\rangle_{\mathbb{L}^2} = 0\}$$

be the set of vector-valued functions that are orthogonal to the indicated eigenfunctions. Then the minimum value of the Rayleigh quotient function restricted to the subspace V_{n-1} is the n-th eigenvalue of $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$, that is,

$$\begin{aligned}\sigma_n &= \min\{R(\mathbf{v}); 0 \neq \mathbf{v} \in V_{n-1}\} \\ &= \min_{0\neq\mathbf{v}\in V_{n-1}} -\frac{\int_\Omega \mathbf{v}\cdot(\mathscr{L}\mathbf{v} + \alpha\nabla(\mathrm{div}_\eta\mathbf{v}))dm}{\int_\Omega |\mathbf{v}|^2 dm}.\end{aligned} \tag{1.22}$$

Proof. See [27, Theorem 9.43]. □

The interested reader can also see a version of the previous theorem in Chavel [6].

1.5 Auxiliary results

In this section, we present some known results from the literature which are related to our results.

The first one, is the inequality of real numbers known as the *recursion formula of Cheng and Yang* which has been used by many researchers to obtain estimates of eigenvalues, for instance, by Chen et al. [7, Corollary 1.4]. We would like to emphasize that the recursive formula in Lemma 1.5.1 was initially proved by Cheng and Yang [9, Theorem 2.1] for $t = n$ a natural number. Since n being a natural or real number does not influence the proof, later in Cheng and Yang [10, Theorem 2.1], they rewritten this formula for any positive real number t to serve their purposes of studying the Pólya conjecture for the case of complete Riemannian manifolds.

Lemma 1.5.1 (Cheng and Yang [10]). *Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ be non-negative real numbers satisfying*

$$\sum_{i=1}^{k}(\mu_{k+1} - \mu_i)^2 \leq \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1} - \mu_i).$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2, \quad F_k = (1 + \frac{2}{t})G_k^2 - T_k.$$

Then, we have

$$F_{k+1} \leq C(t,k)\Big(\frac{k+1}{k}\Big)^{\frac{4}{t}}F_k,$$

where t is any positive real number and

$$C(t,k) = 1 - \frac{1}{3t}\Big(\frac{k}{k+1}\Big)^{\frac{4}{t}}\frac{(1+\frac{2}{t})(1+\frac{4}{t})}{(k+1)^3} < 1.$$

Proof. The proof is the same as that made by Cheng and Yang [9, Theorem 2.1] replacing the natural number n by the real number t. For a complete and more accessible proof, see Miranda [32, Lemma 2.4] where she used $t = \frac{n}{c}$ for some positive real number c. $\square$

As an application of the previous lemma and following the same steps by Cheng and Yang [9, Corollary 2.1] we have the next corollary.

Corollary 1.5.1. *Under the same setup of Lemma 1.5.1, we have*

$$\mu_{k+1} \leq \Big(1 + \frac{4}{t}\Big) k^{\frac{2}{t}} \mu_1.$$

Proof. The proof is the same as in [9, Corollary 2.1]. For a complete and detailed proof, see [32, Corollary 2.1]. $\square$

The next lemma often appears in similar configurations when looking for universal inequalities for eigenvalues of elliptic problems, it is based on the Yang inequality for Laplacian eigenvalues. Actually, this lemma is a more general version of such Yang's result [39], and follows the same steps as its proof. A more detailed proof can be found in Miranda [32, Theorem 2.1]. For the sake of completeness, we are going to prove it in a more general setting.

Lemma 1.5.2. *Let $v_1 \leq v_2 \leq \cdots \leq v_{k+1}$ be an non-negative real numbers satisfying*

$$\sum_{i=1}^{k} (v_{k+1} - v_i)^2 \leq \kappa_0 \sum_{i=1}^{k} (v_{k+1} - v_i) v_i. \tag{1.23}$$

for some positive real number κ_0. Then, we have

$$v_{k+1} \leq \Big(1 + \kappa_0\Big) \frac{1}{k} \sum_{i=1}^{k} v_i + \Big[\Big(\frac{\kappa_0}{2} \frac{1}{k} \sum_{i=1}^{k} v_i \Big)^2 - (1 + \kappa_0) \frac{1}{k} \sum_{j=1}^{k} \Big(v_j - \frac{1}{k} \sum_{i=1}^{k} v_i \Big)^2 \Big]^{\frac{1}{2}}, \tag{1.24}$$

and

$$v_{k+1} - v_k \leq 2 \Big[\Big(\frac{\kappa_0}{2} \frac{1}{k} \sum_{i=1}^{k} v_i \Big)^2 - (1 + \kappa_0) \frac{1}{k} \sum_{j=1}^{k} \Big(v_j - \frac{1}{k} \sum_{i=1}^{k} v_i \Big)^2 \Big]^{\frac{1}{2}}. \tag{1.25}$$

Proof. Notice that (1.23) is a quadratic inequality of v_{k+1}, that is,

$$\mathcal{Q}(v_{k+1}) = k v_{k+1}^2 - v_{k+1} \Big(2 + \kappa_0\Big) \sum_{i=1}^{k} v_i + \Big(1 + \kappa_0\Big) \sum_{i=1}^{k} v_i^2 \leq 0,$$

hence the discriminant of $\mathcal{Q}(v_{k+1})$ satisfies

$$\mathcal{D} = \Big(2 + \kappa_0\Big)^2 \Big(\sum_{i=1}^{k} v_i \Big)^2 - 4k \Big(1 + \kappa_0\Big) \sum_{i=1}^{k} v_i^2 \geq 0. \tag{1.26}$$

Therefore, let r_1 and r_2 be the smaller and the biggest root of $\mathcal{Q}(v_{k+1})$, respectively. Since $\mathcal{Q}(v_{k+1}) \leq 0$ and $\mathcal{D} \geq 0$ we must have $r_1 \leq v_{k+1} \leq r_2$, therefore

$$v_{k+1} \leq r_2 = \frac{1}{2k}\Big(2 + \kappa_0\Big) \sum_{i=1}^{k} v_i + \frac{1}{2k}\sqrt{\mathcal{D}}.$$

Substituting (1.26) into the previous inequality, we get

$$\begin{aligned} v_{k+1} \leq & \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i + \Big[\Big(\frac{1}{k} + \frac{\kappa_0}{2k}\Big)^2 \Big(\sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big) \sum_{i=1}^{k} v_i^2\Big]^{\frac{1}{2}} \qquad (1.27) \\ = & \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i + \Big[\Big(\frac{\kappa_0}{2k} \sum_{i=1}^{k} v_i\Big)^2 + \frac{1}{k^2}\Big(1 + \kappa_0\Big)\Big(\sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big) \sum_{i=1}^{k} v_i^2\Big]^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} v_{k+1} \leq & \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i + \Big[\Big(\frac{\kappa_0}{2k} \sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big)\Big(\sum_{i=1}^{k} v_i^2 - \frac{1}{k}\Big(\sum_{i=1}^{k} v_i\Big)^2\Big)\Big]^{\frac{1}{2}} \\ = & \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i \\ & + \Big[\Big(\frac{\kappa_0}{2k} \sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big)\Big(\sum_{i=1}^{k} v_i^2 - \frac{2}{k}\Big(\sum_{i=1}^{k} v_i\Big)^2 + \frac{1}{k}\Big(\sum_{i=1}^{k} v_i\Big)^2\Big)\Big]^{\frac{1}{2}} \\ = & \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i \\ & + \Big[\Big(\frac{\kappa_0}{2k} \sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big)\Big(\sum_{i=1}^{k} v_i^2 - \frac{2}{k} \sum_{i,j=1}^{k} v_i v_j + \frac{1}{k}\Big(\sum_{i=1}^{k} v_i\Big)^2\Big)\Big]^{\frac{1}{2}}. \end{aligned}$$

Consequently, we conclude that

$$v_{k+1} \leq \frac{1}{k}\Big(1 + \frac{\kappa_0}{2}\Big) \sum_{i=1}^{k} v_i + \Big[\Big(\frac{\kappa_0}{2k} \sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1 + \kappa_0\Big) \sum_{j=1}^{k} \Big(v_j - \frac{1}{k} \sum_{i=1}^{k} v_i\Big)^2\Big]^{\frac{1}{2}},$$

which proves (2.13). Now, notice that Inequality (1.23) also holds if we replace the integer k with $k-1$, that is,

$$\sum_{i=1}^{k-1} (v_k - v_i)^2 \leq \kappa_0 \sum_{i=1}^{k-1} (v_k - v_i) v_i,$$

or equivalently,

$$\sum_{i=1}^{k} (v_k - v_i)^2 \leq \kappa_0 \sum_{i=1}^{k} (v_k - v_i) v_i.$$

Therefore, v_k satisfies the same quadratic inequality and we have

$$v_k \geq r_1 = \frac{1}{k}\Big(1+\frac{\kappa_0}{2}\Big)\sum_{i=1}^{k} v_i - \Big[\Big(\frac{\kappa_0}{2k}\sum_{i=1}^{k} v_i\Big)^2 - \frac{1}{k}\Big(1+\kappa_0\Big)\sum_{j=1}^{k}\Big(v_j - \frac{1}{k}\sum_{i=1}^{k} v_i\Big)^2\Big]^{\frac{1}{2}}.$$

From (1.24) and the previous inequality we get (1.25) and complete the proof of Corollary 2.2.3. □

Corollary 1.5.2. *Under the same setup as Corollary 2.2.3, Inequality* (1.24) *implies*

$$v_{k+1} \leq (1+\kappa_0)\frac{1}{k}\sum_{i=1}^{k} v_i.$$

Proof. In fact, (1.24) is equivalently to (1.27), and since $-k\sum_{i=1}^{k} v_i^2 \leq -(\sum_{i=1}^{k} v_i)^2$, from (1.27) we get

$$\begin{aligned} v_{k+1} &\leq \frac{1}{k}\Big(1+\frac{\kappa_0}{2}\Big)\sum_{i=1}^{k} v_i + \Big[\Big(\frac{1}{k}+\frac{\kappa_0}{2k}\Big)^2\Big(\sum_{i=1}^{k} v_i\Big)^2 - \frac{2}{k}\Big(\frac{1}{2k}+\frac{\kappa_0}{2k}\Big)k\sum_{i=1}^{k} v_i^2\Big]^{\frac{1}{2}} \\ &\leq \frac{1}{k}\Big(1+\frac{\kappa_0}{2}\Big)\sum_{i=1}^{k} v_i + \Big[\Big(\frac{1}{k}+\frac{\kappa_0}{2k}\Big)^2\Big(\sum_{i=1}^{k} v_i\Big)^2 - \frac{2}{k}\Big(\frac{1}{2k}+\frac{\kappa_0}{2k}\Big)\Big(\sum_{i=1}^{k} v_i\Big)^2\Big]^{\frac{1}{2}}. \end{aligned}$$

Moreover, we notice that

$$\begin{aligned} \Big(\frac{1}{k}+\frac{\kappa_0}{2k}\Big)^2 - \frac{2}{k}\Big(\frac{1}{2k}+\frac{\kappa_0}{2k}\Big) &= \Big(\frac{1}{2k}+\frac{\kappa_0}{2k}+\frac{1}{2k}\Big)^2 - \frac{2}{k}\Big(\frac{1}{2k}+\frac{\kappa_0}{2k}\Big) \\ &= \Big(\frac{1}{2k}+\frac{\kappa_0}{2k}-\frac{1}{2k}\Big)^2 = \Big(\frac{\kappa_0}{2k}\Big)^2. \end{aligned}$$

Therefore, we obtain

$$v_{k+1} \leq \frac{1}{k}\Big(1+\frac{\kappa_0}{2}\Big)\sum_{i=1}^{k} v_i + \Big[\Big(\frac{\kappa_0}{2k}\Big)^2\Big(\sum_{i=1}^{k} v_i\Big)^2\Big]^{\frac{1}{2}} = \Big[\Big(1+\frac{\kappa_0}{2}\Big)+\frac{\kappa_0}{2}\Big]\frac{1}{k}\sum_{i=1}^{k} v_i,$$

which completes the proof. □

For an n-dimensional complete Riemannian manifold $(M^n, \langle,\rangle)$ isometrically immersed in $\mathbb{R}^m$ let us denote by $\mathcal{A}$ the second fundamental form and so $\mathbf{H} = \frac{1}{n}\mathrm{tr}(\mathcal{A})$ is the mean curvature vector. We can associate with a symmetric $(1,1)$-tensor T the following normal vector field:

$$\begin{aligned}\mathbf{H}_T &= \frac{1}{n}\sum_{i,j=1}^{n} T(e_i,e_j)\mathcal{A}(e_i,e_j)\\ &= \frac{1}{n}\sum_{i=1}^{n}\mathcal{A}(T(e_i),e_i)\\ &:= \frac{1}{n}\mathrm{tr}(\mathcal{A}\circ T),\end{aligned}$$

where $\{e_1, e_2, \dots, e_n\}$ is a local orthonormal frame of TM and $\mathbf{H}_T$ is called the generalized mean curvature vector. The definition of the generalized mean curvature vector had been considered by Grosjean [18] and Roth [33, 34] to get upper bound for the first positive eigenvalue of the operator $\mathscr{L}$.

The next lemma is a rewrite of some equalities obtained by Gomes and Miranda [15] and plays an important role in obtaining our Lemma 2.4.2.

Lemma 1.5.3 (Gomes and Miranda)**.** *Let Ω be a domain of an n-dimensional complete Riemannian manifold M isometrically immersed in $\mathbb{R}^m$, T be a $(1,1)$-tensor symmetric on Ω, and $x = (x_1, \dots, x_m)$ be the position vector of the immersion of M in $\mathbb{R}^m$, then*

$$\sum_{\ell=1}^{m} T(\nabla x_\ell, \nabla x_\ell) = \mathrm{tr}(T),$$

$$\mathrm{div}_\eta(T(\nabla x)) = n\mathbf{H}_T(x) + \mathrm{tr}(\nabla T)(x) - T(\nabla\eta)(x), \tag{1.28}$$

and, consequently

$$\sum_{\ell=1}^{m} (\mathscr{L}x_\ell)^2 = n^2|\mathbf{H}_T|^2 + |\mathrm{tr}(\nabla T) - T(\nabla\eta)|^2,$$

where $\mathrm{div}_\eta(T(\nabla x)) := (\mathrm{div}_\eta(T(\nabla x_1)), \dots, \mathrm{div}_\eta(T(\nabla x_m)))$ *and* $\mathbf{H}_T$ *is the generalized mean curvature vector of immersion. In particular, for $M = \mathbb{R}^m$ we get*

$$\sum_{\ell=1}^{m} (\mathscr{L}x_\ell)^2 = |\mathrm{tr}(\nabla T) - T(\nabla\eta)|^2. \tag{1.29}$$

Proof. The proof of this lemma is the same presented by Gomes and Miranda [15, Eq. 3.23], with a slight modification. In fact, just replace $\sum_{i=1}^{n}\mathcal{A}(T(e_i),e_i) = n\mathbf{H}_T$ in their proof. □

The following theorem is due to Ma and Liu [25, Theorem 3] and it will be used in one of our results in Chapter 2.2. They considered the eigenvalue problem for the drifted

Laplacian in convex domains of Euclidean space and showed that the first eigenfunction for this problem is convex. Moreover, using this and a gradient estimate, they also showed a lower bound for the gap between the first and the second eigenvalue.

Theorem 1.5.1 (Ma and Liu [25]). *Assume that η is a smooth concave function on the closure of the bounded convex domain $\Omega \subset \mathbb{R}^n$. Assume that,*

$$f = \frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2,$$

is concave on $\bar{\Omega}$. Let λ_1 and λ_2 be the first two eigenvalues non-zero of problem:

$$\begin{cases} -\Delta_\eta u = \lambda u & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

Then, we have

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2},$$

where d is diameter of Ω.

Proof. See [25, Theorem 3]. □

Lemma 1.5.4 (Chebyshev's Inequality [1]). *Let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ two arbitrary sets of real numbers such that either $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$ or $a_1 \geq a_2 \geq \cdots \geq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_k$. Then*

$$\Big(\frac{1}{k}\sum_{i=1}^{k} a_i\Big)\Big(\frac{1}{k}\sum_{i=1}^{k} b_i\Big) \leq \frac{1}{k}\sum_{i=1}^{k} a_i b_i.$$

The equality holds if, and only if, either $a_1 = a_2 = \ldots = a_k$ or $b_1 = b_2 = \ldots = b_k$.

We would like to observe that a generalized version of Chebyshev's Inequality can be founded in Hardy et al. [19, p. 43].

Chapter 2

Estimates of eigenvalues of an elliptic differential system in divergence form

In this chapter, we prove some estimates of eigenvalues of an elliptic differential system in divergence form. We begin by presenting the main results and some corollaries. Next, we give applications of our main results to particular cases, for example, to the Gaussian shrinking soliton. In the last section we prove our results. This chapter is a joint work with professor José N. V. Gomes [2].

2.1 Main results

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with its canonical metric $\langle , \rangle$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let us consider a symmetric positive definite $(1,1)$-tensor T on $\mathbb{R}^n$ and a function $\eta \in C^2(\mathbb{R}^n)$, so that we can define a second-order elliptic differential operator $\mathscr{L}$ in the (η, T)-divergence form as in Definition 1.3.2.

In this chapter, we address the eigenvalue problem for an operator which is a second-order perturbation of $\mathscr{L}$. More precisely, we compute universal estimates of the eigenvalues of the coupled system of second-order elliptic differential equations, namely:

$$\begin{cases} \mathscr{L}\mathbf{u} + \alpha\nabla(\mathrm{div}_\eta \mathbf{u}) & = & -\sigma\mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, the constant α is non-negative and $\mathscr{L}\mathbf{u} = (\mathscr{L}u^1, \mathscr{L}u^2, \ldots, \mathscr{L}u^n)$.

We proved in Section 1.3 that $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$ is a formally self-adjoint operator in the Hilbert space of all vector-valued functions that vanish on $\partial\Omega$. Let us consider the sequence

$$0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_k \le \cdots \to \infty, \tag{2.2}$$

of the eigenvalue problem (2.1), where each σ_i is repeated according to its multiplicity.

A special case of Problem 2.1 occurs when T is divergence-free, see Problem 2.22. For the sake of convenience, we address this case in Section 2.2.2. Some results from Problems 2.3 and 2.4 below are particular cases of this section. However, these two latter problems still remain prototype for us. In the next two paragraphs, we make brief comments about them.

When η is a constant and T is the identity operator I on $\mathbb{R}^n$, Problem 2.1 becomes

$$\begin{cases} \Delta \mathbf{u} + \alpha \nabla(\text{div } \mathbf{u}) & = & -\sigma \mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2.3}$$

where $\Delta \mathbf{u} = (\Delta u^1, \ldots, \Delta u^n)$ and Δ is the Laplacian operator on $C^\infty(\Omega)$. The operator $\Delta + \alpha \nabla \text{div}$ is known as Lamé's operator. In the 3-dimensional case it appears in the elasticity theory and α is determined by the positive constants of Lamé, so the assumption $\alpha \ge 0$ is justified. For further details on this issue, the interested reader can consult Pleijel [30] or Kawohl and Sweers [21]. It is worth mentioning here the works of Levine and Protter [23], Livitin and Parnovski [24], Hook [20], Cheng and Yang [9] and Chen et al. [7] in which we can find some interesting estimates of the eigenvalues of Problem 2.3. We will be more precise later when we discuss the three latter papers.

When η is not necessarily constant and $T = I$, Problem 2.1 is rewritten as

$$\begin{cases} \Delta_\eta \mathbf{u} + \alpha \nabla(\text{div}_\eta \mathbf{u}) & = & -\sigma \mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2.4}$$

where $\Delta_\eta \mathbf{u} = (\Delta_\eta u^1, \ldots, \Delta_\eta u^n)$ and $\Delta_\eta = \text{div}_\eta \nabla$ is the drifted Laplacian operator on $C^\infty(\Omega)$. The drifted Laplacian as well as the Bakry-Emery Ricci tensor $Ric + \nabla^2 \eta$ are the most appropriate geometric objects to study the smooth metric measure spaces $(M^n, g, e^{-\eta} dvol_g)$. In particular, the Bakry-Emery Ricci tensor has been especially studied in the theory of Ricci solitons, since a gradient Ricci soliton (M^n, g, η) is characterized by $Ric + \nabla^2 \eta = \lambda g$, for some constant λ.

In Corollary 2.2.5, we show an interesting case of rigidity inequalities of eigenvalues of the Laplacian in a countable family of bounded domains in Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$ by taking a specific isoparametric function as being the drifting function η, see Remarks 2.2.2 and 2.2.3. We address the Gaussian expanding soliton case in Corollaries 2.2.6 and 2.2.7.

Our proofs will be facilitated by analyzing the more general setting in which the function η is not necessarily constant and T is not necessarily the identity. In this case, we prove an universal quadratic estimate for the eigenvalues of Problem 2.1, which is an essential tool to obtain some of our estimates.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 2.1. For any positive integer k, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big\{\Big[(\sigma_i-\alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2)^{\frac{1}{2}}+\frac{T_0}{2\sqrt{\delta}}\Big]^2+\frac{C_0}{\delta}\Big\},$$

where

$$C_0 = \sup_{\Omega}\Big\{\frac{1}{2}\mathrm{div}(T^2(\nabla\eta)) - \frac{1}{4}|T(\nabla\eta)|^2\Big\} + \frac{\delta}{2}T_0\eta_0, \tag{2.5}$$

$T_0 = \sup_\Omega |\mathrm{tr}(\nabla T)|$ *and* $\eta_0 = \sup_\Omega |\nabla\eta|$.

Remark 2.1.1. *Notice that the constant C_0 in Eq.* (2.5) *has been appropriately defined such that* $\Big[(\sigma_i-\alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2)^{\frac{1}{2}}+\frac{T_0}{2\sqrt{\delta}}\Big]^2+\frac{C_0}{\delta} > 0$*, for $i = 1,\ldots,k$.*

We identify the quadratic estimate in Theorem 2.1.1 as the most appropriate inequality for the applications of our results. In particular, the constant C_0 in (2.5) has a crucial importance for us.

Theorem 2.1.1 is an extension for $\mathscr{L}+\alpha\nabla\mathrm{div}_\eta$ on vector-valued functions of the well-known Yang's estimate of the eigenvalues of the Laplacian on real-valued functions. Its proof is motivated by the corresponding results for the Laplacian on real-valued functions case by Yang [39], for $\Delta+\alpha\nabla\mathrm{div}$ on vector-valued functions case by Chen et al. [7, Theorem 1.1], and for $\mathscr{L}$ on real-valued functions case by Gomes and Miranda [15].

We also prove an estimate for the sum of lower order eigenvalues in terms of the first eigenvalue and its correspondent eigenfunction.

Theorem 2.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 2.1, for $i = 1,\ldots,n$, and $\mathbf{u}_1$ be a normalized eigenfunction corresponding to the first eigenvalue. Then, we get*

$$\sum_{i=1}^{n}(\sigma_{i+1}-\sigma_1) \leq \frac{4\delta(\delta+\alpha)}{\varepsilon^2}\Big\{\Big[(\sigma_1-\alpha\|\mathrm{div}_\eta \mathbf{u}_1\|^2)^{\frac{1}{2}}+\frac{T_0}{2\sqrt{\delta}}\Big]^2+\frac{C_0}{\delta}\Big\},$$

where C_0 is given by (2.5)*.*

Theorem 2.1.2 is an extension for $\mathscr{L}+\alpha\nabla\mathrm{div}_\eta$ on vector-valued functions of a stronger result obtained by Cheng and Yang [9, Theorem (1.2)] for lower order eigenvalues of Problem 2.3. Its proof is motivated by the corresponding results for $\Delta+\alpha\nabla\mathrm{div}$ case in [9, Theorem 1.2] as well as for $\Delta_\eta+\alpha\nabla\mathrm{div}_\eta$ case in [13, Theorem 1.3].

2.2 Applications of the main results

We start by presenting applications of our results for the case in which the tensor T is divergence-free. In Section 2.2.1, we get inequalities estimates Problem 2.4. Next, we prove an interesting case of non-dependence of η in this case.

2.2.1 Identity tensor case

We begin this section by defining a known class of the functions which is closely related to our applications. A nonconstant smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is called transnormal function if

$$|\nabla f|^2 = b(f), \tag{2.6}$$

for some smooth function b on the range of f in $\mathbb{R}$. The function f is called an isoparametric function if it moreover satisfies

$$\Delta f = a(f), \tag{2.7}$$

for some continuous function a on the range of f in $\mathbb{R}$.

Eq. (2.6) implies that the level set hypersurfaces of f are parallel hypersurfaces and it follows from Eq. (2.7) that these hypersurfaces have constant mean curvature. Isoparametric functions appear in the isoparametric hypersurfaces theory (i.e., has constant principal curvatures) systematically developed by Cartan [5] on space forms. Wang [37] considered the problem of extending this theory to a general Riemannian manifold and studied some properties of (2.6) and (2.7) more closely. Notice that isoparametric functions exist on a large class of spaces (e.g. symmetric spaces) other than space forms. Currently, new examples of isoparametric functions on Riemannian manifolds have been discovered, for instance, the potential function of any noncompact gradient Ricci soliton (M^n, g, η) with constant scalar curvature R is an isoparametric function, since we can assume that η (after a possible rescaling) satisfies $|\nabla\eta|^2 = 2\lambda\eta - R$ and $\Delta\eta = \lambda n - R$, see, e.g., Chow et al. [12]. In particular, the potential function of the Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$ is an isoparametric function, see Example 2.2.1. This latter fact and a brief analysis of the constant C_0 in (2.5) were the main motivations to consider the isoparametric function $\eta(x) = \frac{\lambda}{2}|x|^2$ to give some applications of our results. The quadratic estimate below is a basic result for it.

Corollary 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 2.4. For any positive integer k, we have*

$$\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)(\sigma_i - \alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2 + C_0), \tag{2.8}$$

where $C_0 = \sup_\Omega \left\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\right\}$*. Moreover,* $\sigma_i - \alpha\|\mathrm{div}_\eta \mathbf{u}_i\|^2 + C_0 > 0$*, for* $i = 1, \ldots, k$*.*

Proof. In Problem 2.4 we must have $T = I$. Then, we get $\varepsilon = \delta = 1$ and $T_0 = 0$. Hence, the result of the corollary follows from Theorem 2.1.1. □

The following corollary is an immediate consequence of Theorem 2.1.2.

Corollary 2.2.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* σ_i *be the i-th eigenvalue of Problem 2.4, for* $i = 1, \ldots, n$*, and by* $\mathbf{u}_1$ *a normalized eigenfunction corresponding to the first eigenvalue. Then, for any positive integer k, we have*

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq 4(1+\alpha)(\sigma_1 + D_1), \tag{2.9}$$

where $D_1 = -\alpha\|\mathrm{div}_\eta \mathbf{u}_1\|^2 + C_0$ *and* $C_0 = \sup_\Omega \left\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\right\}$*.*

Proof. In fact, in this case we also have $\varepsilon = \delta = 1$ and $T_0 = 0$. Hence, the result of the corollary follows from Theorem 2.1.2. □

Notice that the appearance of the constant C_0 is natural, since we did not impose any restriction on the function η. We highlight that this constant has an unexpected geometric interpretation. Indeed, let us consider the warped metric $g = g_0 + e^{-\eta}d\theta^2$ on the product $\Omega \times \mathbb{S}^1$, where g_0 stands for the canonical metric in the domain $\Omega \subset \mathbb{R}^n$, whereas $d\theta^2$ is the canonical metric of the unit sphere $\mathbb{S}^1$, so that the scalar curvature of g is given by $\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2$. Hence, by rescaling the previous warped metric g we can obtain C_0 as the supremum of the scalar curvature on the warped product $\Omega \times \mathbb{S}^1$ with this new metric. Moreover, we ask the following natural question:

Question 2.2.1. *Under which conditions the inequalities for the eigenvalues obtained from* (2.8) *and* (2.9) *do not depend on the constant* C_0 *for a nontrivial function* η*?*

We give an answer to this question by using a specific family of domains in Gaussian shrinking soliton. More precisely, we consider a countable family of bounded domains in $\mathbb{R}^n$ that makes the behavior of known estimates of eigenvalues of the Laplacian invariant by a first-order perturbation of the Laplacian, see Corollary 2.2.5.

Coming back to Corollary 2.2.1, we define

$$D_0 = -\alpha \min_{j=1,\ldots,k} \|\mathrm{div}_\eta \mathbf{u}_j\|^2 + C_0, \tag{2.10}$$

so that, from (2.8), we get

$$\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)(\sigma_i + D_0). \tag{2.11}$$

Notice that $\sigma_i + D_0 > 0$.

Now, as mentioned in Section 2.1, we immediately recover the following inequality:

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sigma_i, \tag{2.12}$$

which has been obtained by Chen et al. [7, Corollary 1.2] for Problem 2.3. Indeed, it follows from (2.10) and (2.11), since $\alpha \geq 0$ and we can take η to be a constant. Moreover, Inequality (2.12) implies Theorem 1.1 in Cheng and Yang [9], whereas [9, Theorem 1.1] implies Theorem 10 in Hook [20]. However, we highlight that Inequality (2.11) provides an estimate for the eigenvalues of Problem 2.3 which is better than Inequality (2.12).

In the case of Problem 2.4, we can see that Inequality (2.9) is better than Inequality (1.7) in Du and Bezerra [13]; whereas Inequality (2.11) is better than Inequality (1.3) again in [13].

Besides, from Inequality (2.11) and following the steps of the proof of [15, Theorem 3], we obtain the inequalities:

Corollary 2.2.3. *Under the same setup as in Corollary 2.2.1, we have*

$$\begin{aligned}\sigma_{k+1}+D_0 \leq & \Big(1+\frac{2(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)+\Big[\Big(\frac{2(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2 \\ & -\Big(1+\frac{4(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}\end{aligned} \tag{2.13}$$

and

$$\sigma_{k+1}-\sigma_k \leq 2\Big[\Big(\frac{2(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i+D_0)\Big)^2-\Big(1+\frac{4(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j-\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}, \tag{2.14}$$

where D_0 is given by (2.10).

Proof. Notice that $\sigma_{k+1}-\sigma_i = \sigma_{k+1}+D_0-\sigma_i-D_0$. Let $v_i = \sigma_i + D_0$, then from (2.11) we have

$$\sum_{i=1}^{k}(v_{k+1}-v_i)^2 \leq \frac{4(n+\alpha)}{n^2}\sum_{i=1}^{k}(v_{k+1}-v_i)v_i. \tag{2.15}$$

By (2.2), we can see that $v_1 \leq v_2 \leq \cdots v_{k+1}$. Setting $\kappa_0 = \frac{4(n+\alpha)}{n^2}$ into (2.15), we will be in Lemma 1.5.2 settings, and so we complete the proof using that lemma. □

Again from (2.11) and by applying the recursion formula of Cheng and Yang (see Corollary 1.5.1), we obtain the following corollary.

Corollary 2.2.4. *Under the same setup as in Corollary 2.2.1, we have*

$$\sigma_{k+1} + D_0 \leq \Big(1 + \frac{4(n+\alpha)}{n^2}\Big) k^{\frac{2(n+\alpha)}{n^2}} (\sigma_1 + D_0), \tag{2.16}$$

where D_0 is given by (2.10).

Proof. Notice that the recursion formula by Cheng and Yang [8, Corollary 2.1] remains true for any positive real number, see Corollary 1.5.1. In particular, it holds for $t = \bar{n} = \frac{n^2}{n+\alpha}$, then from (2.15) we can apply Corollary 1.5.1 to obtain immediately (2.16). $\square$

From the classical Weyl's asymptotic formula for the eigenvalues [38], we know that estimate (2.16) is optimal in the sense of the order on k.

Remark 2.2.1. *If $D_0 = 0$, then the inequalities of eigenvalues* (2.11) *and* (2.16) *have the same behavior as the known estimates of the eigenvalues of $\Delta + \alpha\nabla\mathrm{div}$, see Inequality* (2.12) *and Chen et al. [7, Corollary 1.4], respectively. In the same way, from Corollary 1.5.2, the inequalities of eigenvalues* (2.13) *and* (2.14) *imply*

$$\sigma_{k+1} \leq \Big(1 + \frac{4(n+\alpha)}{n^2}\Big)\frac{1}{k}\sum_{i=1}^{k}\sigma_i \quad \text{and} \quad \sigma_{k+1} - \sigma_k \leq \frac{4(n+\alpha)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\sigma_i,$$

which have the same behavior as the inequalities of the eigenvalue of $\Delta + \alpha\nabla\mathrm{div}$ obtained by Chen et al. [7, Corollary 1.3].

If $D_1 = 0$, then the inequality of eigenvalues (2.9) *has the same behavior as the known estimate of the eigenvalues of $\Delta + \alpha\nabla\mathrm{div}$ proved by Cheng and Yang [9, Theorem 1.2].*

Remark 2.2.2. *For $\alpha = 0$ case, if $C_0 = 0$ for some function η (possibly radial or isoparametric), then the inequalities* (2.9), (2.11), (2.13) *and* (2.16) *have the same behavior as the known estimates of the eigenvalues of the Laplacian, see [3, Inequality (6.2)], [39, Theorem 1], [8, Inequality (1.8)] and [8, Corollary 2.1], respectively. We highlight that [3, Inequality (6.2)] was first obtained by Payne et al. [28] in the two-dimensional case.*

Example 2.2.1 below is a special case of $C_0 = 0$. To see this, let us consider an isoparametric function $\eta(x) = \frac{\lambda}{2}(x_1^2 + \cdots + x_k^2)$ on $\mathbb{R}^n$, where λ is any nonzero real number, k an integer with $0 < k \leq n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It is easy to verify that

$$|\nabla\eta|^2 = 2\lambda\eta \ \text{ and } \ \Delta\eta = \lambda k.$$

In particular, if $k = n$, the function $\eta(x) = \frac{\lambda}{2}|x|^2$ is the potential function of the Gaussian shrinking ($\lambda > 0$) or expanding ($\lambda < 0$) soliton on $\mathbb{R}^n$. We now take $\eta(x) = \frac{\lambda}{2}|x|^2$ into the equation of C_0 in Corollary 2.2.1, so that,

$$C_0 = \sup_{\Omega}\Big\{\frac{\lambda n}{2} - \frac{\lambda^2}{4}|x|^2\Big\}. \tag{2.17}$$

With these considerations in mind, we write the next two examples.

Example 2.2.1. *Let us consider the family of bounded domains* $\{\Omega_l\}_{l=1}^{\infty}$ *in Gaussian shrinking or expanding soliton* $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$ *given by*

$$\Omega_l = \mathbb{B}(r_l) - \bar{\mathbb{B}}(\sqrt{2n/|\lambda|}) = \Big\{x \in \mathbb{R}^n; \frac{2n}{|\lambda|} < |x|^2 < r_l^2\Big\},$$

where $r_l > \sqrt{2n/|\lambda|}$ *is a rational number, and* $\mathbb{B}(r)$ *stands for the open ball of radius* r *centered at the origin in* $\mathbb{R}^n$*. So,*

$$\min_{\bar{\Omega}_l} |x|^2 = \frac{2n}{|\lambda|}, \quad \textit{for all} \quad l = 1, 2, \ldots.$$

(a) Shrinking case:

$$C_0 = \frac{\lambda}{2}\sup_{\Omega_l}\Big\{n - \frac{\lambda}{2}|x|^2\Big\} = \frac{\lambda}{2}\Big(n - \frac{\lambda}{2}\min_{\bar{\Omega}_l}|x|^2\Big) = 0, \quad \textit{for all} \quad l = 1, 2, \ldots.$$

(b) Expanding case:

$$C_0 = \frac{\lambda}{2}\inf_{\Omega_l}\Big\{n - \frac{\lambda}{2}|x|^2\Big\} = \frac{\lambda}{2}\Big(n - \frac{\lambda}{2}\min_{\bar{\Omega}_l}|x|^2\Big) = \lambda n, \quad \textit{for all} \quad l = 1, 2, \ldots.$$

Example 2.2.2. *Let us consider the domain* Ω *to be the open ball* $\mathbb{B}(r)$ *of radius* r *centered at the origin in Gaussian shrinking or expanding soliton* $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$*. From Eq. (2.17), we easily see that* $C_0 = \lambda n/2$ *for both shrinking and expanding case.*

We are now in the position to give the interesting applications that we had promised.

Corollary 2.2.5 (Non-dependence of η). *Let us consider the family of domains* $\{\Omega_l\}_{l=1}^{\infty}$ *given by Example 2.2.1 in Gaussian shrinking soliton* $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$*. Let* σ_i *be the* i*-th eigenvalue of the drifted Laplacian* Δ_η *on real-valued functions, with drifting function* $\eta(x) = \frac{\lambda}{2}|x|^2$*, on each* Ω_l *with Dirichlet boundary condition. Then,*

$$\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)\sigma_i,$$

$$\sigma_{k+1} \leq \Big(1 + \frac{2}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\sigma_i + \Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 - \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}},$$

$$\sigma_{k+1} - \sigma_k \leq 2\Big[\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 - \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}},$$

$$\sigma_{k+1} \leq \Big(1 + \frac{4}{n}\Big) k^{\frac{2}{n}} \sigma_1$$

and

$$\sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) \leq 4\sigma_1.$$

Proof. We start by taking $\alpha = 0$ as in Problem 2.4. Next, we note that the constant $C_0 = 0$ for the shrinking case. So, the required inequalities follows as an immediate application of the inequalities (2.11), (2.13), (2.14), (2.16) and (2.9), respectively. □

Remark 2.2.3. *Notice that Corollary 2.2.5 can be regarded as rigidity inequalities (see Remark 2.2.2) on the family of bounded domains $\{\Omega_l\}_{l=1}^{\infty}$ given by Example 2.2.1 in Gaussian shrinking soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$.*

Now, we will address the expanding case.

Corollary 2.2.6. *Let $\mathbb{B}(r)$ be the open ball of radius r centered at the origin in Gaussian expanding soliton $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$. Let σ_1 be the first eigenvalue of the drifted Laplacian Δ_η on real-valued functions, with drifting function $\eta(x) = \frac{\lambda}{2}|x|^2$, on $\mathbb{B}(r)$ with Dirichlet boundary condition. Then, we have*

$$\sigma_1 \geq \frac{\pi^2 n}{64 r^2} - \frac{\lambda n}{2},$$

and the next estimate for the sum of lower order eigenvalues σ_i of Δ_η in terms of the first eigenvalue

$$\sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) \leq 4(\sigma_1 + \frac{\lambda n}{2}). \tag{2.18}$$

Proof. Let σ_1 and σ_2 be the first and second eigenvalues of the drifted Laplacian Δ_η on real-valued functions on $\mathbb{B}(r)$ with Dirichlet boundary condition, respectively. If $\lambda < 0$, then both $\eta = \frac{\lambda}{2}|x|^2$ and $f = \frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2$ are concave functions on the closure of the convex domain $\mathbb{B}(r)$. Thus, we can apply Theorem 1.5.1 by Ma and Liu [25] to obtain

$$\sigma_2 - \sigma_1 \geq \frac{\pi^2}{16 r^2}. \tag{2.19}$$

On the other hand,we can use (2.14) or (2.16), for $\alpha = 0$, to get

$$\sigma_2 - \sigma_1 \leq \frac{4}{n}\sigma_1 + 2\lambda. \tag{2.20}$$

Combining (2.19) and (2.20), we conclude that

$$\sigma_1 \geq \frac{\pi^2 n}{64 r^2} - \frac{\lambda n}{2}.$$

Moreover, we can use (2.9), for $\alpha = 0$, to obtain (2.18). □

Corollary 2.2.7. *Let us consider the family of domains* $\{\Omega_l\}_{l=1}^{\infty}$ *given by Example 2.2.1 in the Gaussian expanding soliton* $(\mathbb{R}^n, \delta_{ij}, \frac{\lambda}{2}|x|^2)$*. Let* σ_i *be the* i*-th eigenvalue of the drifted Laplacian* Δ_η *on real-valued functions, with drifting function* $\eta(x) = \frac{\lambda}{2}|x|^2$*, on each* Ω_l *with Dirichlet boundary condition. Then, the following estimate holds for the sum of lower order eigenvalues of* Δ_η *in terms of the first eigenvalue:*

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq 4(\sigma_1 + \lambda n).$$

Proof. We can use (2.9), for $\alpha = 0$, to deduce the required estimate. □

Remark 2.2.4. *A final remark is in order. We observe that Corollaries 2.2.1 and 2.2.2 can be obtained from Corollaries 2.2.8 and 2.2.9, respectively. Whereas Corollaries 2.2.3 and 2.2.4 can be obtained from Corollaries 2.2.10 and 2.2.11, respectively. However, as we already mentioned before, they have been a prototype for us.*

2.2.2 Divergence-free tensors case

This section is a generalization of some results of Section 2.2.1. Here, we are assuming the tensor T to be *divergence-free*, i.e., $\mathrm{div}T = 0$. Divergence-free tensors often arise from physical facts. We can find some of them in fluid dynamics, for instance, in the study of: compressible gas; rarefied gas; steady/self-similar flows and relativistic gas dynamics, see e.g. Serre [36]. We highlight that Serre's work deals with divergence-free positive definite symmetric tensors and fluid dynamics.

Example 2.2.3. *Let* f *be a smooth function on a Riemannian manifold* $(M, \langle,\rangle)$ *and define*

$$T_f := -df \otimes df + \frac{|\nabla f|^2}{2}\langle,\rangle.$$

When $\Delta f = 0$*, the symmetric tensor* T_f *is divergence-free. In fact, using the well-known fact (see e.g. Barros and Gomes [4, Lemma 2])*

$$\mathrm{div}(df \otimes df) = \Delta f df + d\frac{|\nabla f|^2}{2}$$

and since $\Delta f = 0$*, we get*

$$\mathrm{div}T_f := -\mathrm{div}(df \otimes df) + d\frac{|\nabla f|^2}{2} = 0.$$

In $\mathbb{R}^n - \{0\}$ *we can take* $f(x) = \ln |x|^2$*.*

Example 2.2.4. *Let* $(M^n, \langle,\rangle)$ *be an* $n(\geq 3)$*-dimensional Einstein manifold, that is,* $\mathrm{Ric} = \frac{R_0}{n}\langle,\rangle$ *where* Ric *is the Ricci tensor and* $R_0 = \mathrm{tr}(\mathrm{Ric})$ *is the scalar curvature, which* *by Schur's lemma it must be constant. If $R_0 > 0$, then* Ric *is a tensor symmetric and positive definite which is divergence-free, since in general* divRic $= \frac{dR_0}{n}$. *Moreover, if $R_0 < 0$ the Einstein tensor*

$$E = \mathrm{Ric} - \frac{R_0}{2}\langle , \rangle$$

is a tensor symmetric and positive definite which is divergence-free.

For divergence-free tensors, from Eq. (1.13), the operator $\mathscr{L}$ can be decomposed as follows

$$\mathscr{L} f = \Box f - \langle \nabla \eta, T(\nabla f) \rangle, \tag{2.21}$$

where $\Box$ is the operator introduced by Cheng and Yau [11], namely:

$$\Box f = \mathrm{tr}(\nabla^2 f \circ T) = \langle \nabla^2 f, T \rangle.$$

Cheng-Yau operator arises from the study of complete hypersurfaces of constant scalar curvature in space forms. For more details, the reader can be consult Gomes and Miranda [15].

Eq. (2.21) is a first-order perturbation of the Cheng-Yau operator, and it defines a *drifted Cheng-Yau operator* which we denote by $\Box_\eta$ with a drifting function η.

We now turn our attention to the main problem of this paper. Since T is divergence-free, the coupled system of second-order elliptic differential equations (2.1) becomes

$$\begin{cases} \Box_\eta \mathbf{u} + \alpha \nabla(\mathrm{div}_\eta \mathbf{u}) & = & -\sigma \mathbf{u} & \text{in} & \Omega, \\ \mathbf{u} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2.22}$$

where $\mathbf{u} = (u^1, u^2, \ldots, u^n)$ is a vector-valued function from Ω to $\mathbb{R}^n$, the constant α is non-negative and $\Box_\eta \mathbf{u} := (\Box_\eta u^1, \Box_\eta u^2, \ldots, \Box_\eta u^n)$. Moreover, we have $\mathrm{tr}(\nabla T) = 0$, because T is divergence-free. Thus, the constant C_0 in (2.5) becomes

$$C_0 = \sup_{\Omega} \left\{ \frac{1}{2} \mathrm{div}(T^2(\nabla \eta)) - \frac{1}{4} |T(\nabla \eta)|^2 \right\}.$$

Hence, from Theorems 2.1.1 and 2.1.2 we immediately obtain the next two corollaries.

Corollary 2.2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $\mathbf{u}_i$ be a normalized eigenfunction corresponding to i-th eigenvalue σ_i of Problem 2.22. For any positive integer k, we get*

$$\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leq \frac{4\delta(n\delta + \alpha)}{n^2 \varepsilon^2} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)(\sigma_i - \alpha \|\mathrm{div}_\eta \mathbf{u}_i\|^2 + \frac{C_0}{\delta}).$$

Proof. Since T is divergence-free, from Lemma 1.2.1 we have $\mathrm{tr}(\nabla T) = 0$ and so $T_0 = 0$ which, combining with Theorem 2.1.1 completes the proof. $\Box$

Corollary 2.2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 2.22, for $i = 1, \ldots, n$, and $\mathbf{u}_1$ be a normalized eigenfunction corresponding to the first eigenvalue. For some positive integer k, we get*

$$\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1) \leq \frac{4\delta(\delta+\alpha)}{\varepsilon^2}(\sigma_1 + D_1),$$

where $D_1 = -\alpha\|\mathrm{div}_\eta \mathbf{u}_1\|^2 + \frac{C_0}{\delta}$.

Proof. Since T is divergence-free, from Lemma 1.2.1 we have $\mathrm{tr}(\nabla T) = 0$ and so $T_0 = 0$ which, combining with Theorem 2.1.2 implies the result. □

Now, from Corollary 2.2.8 and following the same steps of the proof of Corollary 2.2.3, we obtain the estimates.

Corollary 2.2.10. *Under the same setup as in Corollary 2.2.8, defining*

$$D_0 = -\alpha \min_{j=1,\ldots,k} \|\mathrm{div}_\eta \mathbf{u}_j\|^2 + \frac{C_0}{\delta},$$

we have

$$\begin{aligned}\sigma_{k+1} + D_0 \leq & \Big(1 + \frac{2\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\Big)\frac{1}{k}\sum_{i=1}^{k}(\sigma_i + D_0) + \Big[\Big(\frac{2\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i + D_0)\Big)^2 \\ & - \Big(1 + \frac{4\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}\end{aligned}$$

and

$$\sigma_{k+1} - \sigma_k \leq 2\Big[\Big(\frac{2\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\frac{1}{k}\sum_{i=1}^{k}(\sigma_i + D_0)\Big)^2 - \Big(1 + \frac{4\delta(n\delta+\alpha)}{\varepsilon^2 n^2}\Big)\frac{1}{k}\sum_{j=1}^{k}\Big(\sigma_j - \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2\Big]^{\frac{1}{2}}.$$

Proof. To proof this corollary it is enough to consider $v_i = \sigma_i + D_0$ and $k_0 = \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}$ into Lemma 1.5.2. □

Again from Corollary 2.2.8 and by applying the recursion formula of Cheng and Yang [8], we obtain the next corollary.

Corollary 2.2.11. *Under the same setup as in Corollary 2.2.10, we have*

$$\sigma_{k+1} + D_0 \leq \Big(1 + \frac{4\delta(\delta n+\alpha)}{\varepsilon^2 n^2}\Big)k^{\frac{2\delta(n\delta+\alpha)}{\varepsilon^2 n^2}}(\sigma_1 + D_0).$$

Proof. It is sufficient take $\mu_i = \sigma_i + D_0$ and $t = \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}$ into Corollary 1.5.1. □

2.3 Gap of consecutive eigenvalues in the general setting

In this section we obtain the gap of consecutive eigenvalues of Problem 2.1 in the more general setting.

Since $\alpha \geq 0$ we obtain immediately from Theorem 2.1.1 that

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2 \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big[\Big(\sqrt{\sigma_i}+\frac{1}{2\sqrt{\delta}}T_0\Big)^2+\frac{C_0}{\delta}\Big]. \tag{2.23}$$

Inequality (2.23) is a quadratic inequality of σ_{k+1} and solving it we derive the following estimate for the upper bound of σ_{k+1} in terms of the first k eigenvalues and the gap of consecutive eigenvalues.

Corollary 2.3.1. *Under the same setup as in Theorem 2.1.1, we get*

$$\sigma_{k+1} \leq \varsigma_k + \sqrt{\varsigma_k^2 - \vartheta_k} \tag{2.24}$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1} - \sigma_k \leq 2\sqrt{\varsigma_k^2 - \vartheta_k}, \tag{2.25}$$

where

$$\varsigma_k = \frac{1}{k}\Bigg\{\sum_{i=1}^{k}\sigma_i + \frac{2\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}\Big[\Big(\sqrt{\sigma_i}+\frac{1}{2\sqrt{\delta}}T_0\Big)^2+\frac{C_0}{\delta}\Big]\Bigg\}$$

and

$$\vartheta_k = \frac{1}{k}\Bigg\{\sum_{i=1}^{k}\sigma_i^2 + \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}\sigma_i\Big[\Big(\sqrt{\sigma_i}+\frac{1}{2\sqrt{\delta}}T_0\Big)^2+\frac{C_0}{\delta}\Big]\Bigg\}.$$

Proof. To simplify the notation, let $\Lambda_i = \Big(\sqrt{\sigma_i}+\frac{1}{2\sqrt{\delta}}T_0\Big)^2+\frac{C_0}{\delta}$ and $\alpha_0 = \frac{4(n\delta+\alpha)}{n^2\varepsilon^2}$, then from Inequality (2.23) we have

$$\mathcal{P}(\sigma_{k+1}) = \sigma_{k+1}^2 - \Big(\frac{2}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)\sigma_{k+1} + \frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 + \frac{\alpha_0}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i \leq 0.$$

Hence the discriminant of $\mathcal{P}(v_{k+1})$ satisfies

$$\mathcal{D} = \Big(\frac{2}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2 - 4\Big(\frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 + \frac{\alpha_0}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i\Big) \geq 0.$$

Since $\mathcal{P}(\sigma_{k+1}) \leq 0$ we have $R_1 \leq \sigma_{k+1} \leq R_2$, where R_1 and R_2 are the roots of $\mathcal{P}$, respectively. Thus,

$$\begin{aligned}\sigma_{k+1} \leq R_2 &= \frac{1}{2}\Big(\frac{2}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big) + \frac{1}{2}\sqrt{\mathcal{D}} \\ &= \frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{2k}\sum_{i=1}^{k}\Lambda_i + \sqrt{\frac{\mathcal{D}}{4}},\end{aligned}$$

therefore, substituting α_0, Λ_i and $\mathcal{D}$ into the previous inequality we obtain (2.24).

On the other hand, replacing the integer k with $k-1$ into Theorem 2.1.1 we can see that σ_k satisfies the same quadratic inequality, that is, we get $\mathcal{P}(\sigma_k) \leq 0$, then by the same arguments, we have

$$\begin{aligned}\sigma_k \geq R_1 &= \frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{2k}\sum_{i=1}^{k}\Lambda_i - \sqrt{\frac{\mathcal{D}}{4}} \\ &= \varsigma_k - \sqrt{\varsigma_k^2 - \vartheta_k}.\end{aligned} \tag{2.26}$$

Thus, from (2.24) and (2.26) we obtain (2.25) and conclude the proof of the corollary. □

The next result is a consequence of Corollary 2.3.1.

Corollary 2.3.2. *Under the same setup as in Theorem 2.1.1, we get*

$$\sigma_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\frac{1}{k}\sum_{i=1}^{k}\Big[\Big(\sqrt{\sigma_i} + \frac{1}{2\sqrt{\delta}}T_0\Big)^2 + \frac{C_0}{\delta}\Big] \tag{2.27}$$

and the gap of any consecutive eigenvalues

$$\sigma_{k+1} - \sigma_k \leq \frac{4\delta(n\delta+\alpha)}{n^2\varepsilon^2}\frac{1}{k}\sum_{i=1}^{k}\Big[\Big(\sqrt{\sigma_i} + \frac{1}{2\sqrt{\delta}}T_0\Big)^2 + \frac{C_0}{\delta}\Big]. \tag{2.28}$$

Proof. Using the notation $\Lambda_i = \Big(\sqrt{\sigma_i} + \frac{1}{2\sqrt{\delta}}T_0\Big)^2 + \frac{C_0}{\delta}$ and $\alpha_0 = \frac{2(n\delta+\alpha)}{n^2\varepsilon^2}$ in Corollary 2.3.1. From Chebyshev's inequality, see Lemma 1.5.4, since $\sigma_1 \leq \cdots \leq \sigma_k$ and $\Lambda_1 \leq \cdots \leq \Lambda_k$, we known that

$$\Big(\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 \leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 \quad \text{and} \quad \Big(\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)\Big(\frac{1}{k}\sum_{i=1}^{k}\Lambda_i\Big) \leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i.$$

Hence, we have

$$\begin{aligned}
\varsigma_k^2 &= \Big(\frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2 \\
&= \Big(\frac{1}{k}\sum_{i=1}^{k}\sigma_i\Big)^2 + \frac{2\alpha_0}{k^2}\sum_{i=1}^{k}\sigma_i\sum_{i=1}^{k}\Lambda_i + \Big(\frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2 \\
&\leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 + \frac{2\alpha_0}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i + \Big(\frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2.
\end{aligned}$$

So, we get

$$\begin{aligned}
\varsigma_k^2 - \vartheta_k &\leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 + \frac{2\alpha_0}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i + \Big(\frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2 - \frac{1}{k}\sum_{i=1}^{k}\sigma_i^2 - \frac{2\alpha_0}{k}\sum_{i=1}^{k}\sigma_i\Lambda_i \\
&= \Big(\frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i\Big)^2.
\end{aligned}$$

Therefore, from (2.24) we obtain

$$\begin{aligned}
\sigma_{k+1} &\leq \varsigma_k + \sqrt{\varsigma_k^2 - \vartheta_k} \\
&\leq \frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i + \frac{\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i \\
&= \frac{1}{k}\sum_{i=1}^{k}\sigma_i + \frac{2\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i,
\end{aligned}$$

which is enough to prove (2.27). Now, from (2.25) we have

$$\begin{aligned}
\sigma_{k+1} - \sigma_i &\leq 2\sqrt{\varsigma_k^2 - \vartheta_k} \\
&\leq \frac{2\alpha_0}{k}\sum_{i=1}^{k}\Lambda_i,
\end{aligned}$$

which is enough to prove (2.28). □

Remark 2.3.1. *We would like to observe that Inequality* (2.27) *generalizes [13, Inequality (1.4)], as well as generalizes [9, Inequality (1.6)]. Moreover, Inequality* (2.28) *generalizes [13, Inequality (1.5)].*

2.4 Proof of the main results

2.4.1 Three technical lemmas

In order to prove our first theorem in this chapter, we need three technical lemmas. The first one is motivated by the corresponding results to Problem 2.3 proven by Chen et al. [7, Lemma 2.1] and to Problem 2.4 proven by Du and Bezerra [13, Lemma 2.1]. Here, we follow the steps of the proof of Lemma 2.1 in [7] with appropriate adaptations for $\mathscr{L} + \alpha\nabla\mathrm{div}_\eta$.

Lemma 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, σ_i be the i-th eigenvalue of Problem 2.1 and $\mathbf{u}_i$ be a normalized vector-valued eigenfunction corresponding to σ_i. Then, for any $f \in C^2(\Omega) \cap C^1(\partial\Omega)$ and any positive constant B, we obtain*

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\Big\{(1-B)\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2 dm - B\alpha\int_\Omega |\nabla f\cdot\mathbf{u}_i|^2 dm\Big\}$$
$$\leq \frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\|^2.$$

Proof. Let $\mathbf{u}_i$ be a normalized vector-valued eigenfunction corresponding to σ_i, i.e., it satisfies

$$\begin{cases} \mathscr{L}\mathbf{u}_i + \alpha\nabla(\mathrm{div}_\eta \mathbf{u}_i) = -\sigma_i\mathbf{u}_i & \text{in} \quad \Omega, \\ \mathbf{u}_i = 0 & \text{on} \quad \partial\Omega, \\ \int_\Omega \mathbf{u}_i\cdot\mathbf{u}_j dm = \delta_{ij} & \text{for any} \quad i,j. \end{cases} \tag{2.29}$$

Since σ_{k+1} is the minimum value of the Rayleigh quotient (1.22) (or see [27, Theorem 9.43]), we must have

$$\sigma_{k+1} \leq -\frac{\int_\Omega \mathbf{v}\cdot(\mathscr{L}\mathbf{v}+\alpha\nabla(\mathrm{div}_\eta\mathbf{v}))dm}{\int_\Omega |\mathbf{v}|^2 dm}, \tag{2.30}$$

for any nonzero vector-valued function $\mathbf{v} : \Omega \to \mathbb{R}^n$ satisfying

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega \mathbf{v}\cdot\mathbf{u}_i dm = 0, \quad \text{for any,} \quad i = 1,\ldots,k.$$

Let us denote by $a_{ij} = \int_\Omega f\mathbf{u}_i\cdot\mathbf{u}_j dm = a_{ji}$ to consider the vector-valued functions $\mathbf{v}_i$ given by

$$\mathbf{v}_i = f\mathbf{u}_i - \sum_{j=1}^{k} a_{ij}\mathbf{u}_j, \tag{2.31}$$

so that

$$\mathbf{v}_i|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega \mathbf{u}_j\cdot\mathbf{v}_i dm = 0, \quad \text{for any} \quad i,j = 1,\ldots,k. \tag{2.32}$$

Then, we can take $\mathbf{v} = \mathbf{v}_i$ in (2.30) and use formula (1.16) to obtain

$$\sigma_{k+1}\|\mathbf{v}_i\|^2 \leq \int_\Omega \Big(-\mathbf{v}_i \cdot \mathscr{L}\mathbf{v}_i + \alpha(\mathrm{div}_\eta \mathbf{v}_i)^2 \Big) dm. \tag{2.33}$$

From (2.31) and (1.17), we get

$$\begin{aligned}\mathscr{L}\mathbf{v}_i =& f\mathscr{L}\mathbf{u}_i + 2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L}f\mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathscr{L}\mathbf{u}_j \\ =& f(-\sigma_i \mathbf{u}_i - \alpha\nabla(\mathrm{div}_\eta \mathbf{u}_i)) + 2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L}f\mathbf{u}_i \\ & - \sum_{j=1}^k a_{ij}(-\sigma_j \mathbf{u}_j - \alpha\nabla(\mathrm{div}_\eta \mathbf{u}_j)),\end{aligned}$$

that is,

$$\begin{aligned}\mathscr{L}\mathbf{v}_i =& -\sigma_i f\mathbf{u}_i + \sum_{j=1}^k a_{ij}\sigma_j \mathbf{u}_j + 2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L}f\mathbf{u}_i \\ & - \alpha f\nabla(\mathrm{div}_\eta \mathbf{u}_i) + \alpha\sum_{j=1}^k a_{ij}\nabla(\mathrm{div}_\eta \mathbf{u}_j).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_\Omega -\mathbf{v}_i \cdot \mathscr{L}\mathbf{v}_i dm =& \sigma_i\|\mathbf{v}_i\|^2 - \int_\Omega \mathbf{v}_i \cdot (2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L}f\mathbf{u}_i)dm \\ & + \alpha\Big(\int_\Omega f\mathbf{v}_i \cdot \nabla(\mathrm{div}_\eta \mathbf{u}_i)dm - \sum_{j=1}^k a_{ij}\int_\Omega \mathbf{v_i} \cdot \nabla(\mathrm{div}_\eta \mathbf{u}_j)dm\Big).\end{aligned} \tag{2.34}$$

From (1.7) and (1.16)

$$\int_\Omega f\mathbf{v}_i \cdot \nabla(\mathrm{div}_\eta \mathbf{u}_i)dm = -\int_\Omega f\mathrm{div}_\eta \mathbf{u}_i \mathrm{div}_\eta \mathbf{v}_i dm - \int_\Omega \mathrm{div}_\eta \mathbf{u}_i \nabla f \cdot \mathbf{v}_i dm.$$

But, from (2.31)

$$-f\mathrm{div}_\eta \mathbf{u}_i = -\mathrm{div}_\eta \mathbf{v}_i + \nabla f \cdot \mathbf{u}_i - \sum_{j=1}^k a_{ij}\mathrm{div}_\eta \mathbf{u}_j,$$

then

$$\begin{aligned}\int_\Omega f\mathbf{v}_i\cdot\nabla(\mathrm{div}_\eta\mathbf{u}_i)dm =& -\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i)^2dm+\int_\Omega\mathrm{div}_\eta\mathbf{v}_i\nabla f\cdot\mathbf{u}_idm\\ &-\sum_{j=1}^k a_{ij}\int_\Omega\mathrm{div}_\eta\mathbf{u}_j\mathrm{div}_\eta\mathbf{v}_idm-\int_\Omega\mathrm{div}_\eta\mathbf{u}_i\nabla f\cdot\mathbf{v}_idm\\ =& -\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i)^2dm+\int_\Omega\mathrm{div}_\eta\mathbf{v}_i\nabla f\cdot\mathbf{u}_idm\\ &+\sum_{j=1}^k a_{ij}\int_\Omega\mathbf{v}_i\cdot\nabla(\mathrm{div}_\eta\mathbf{u}_j)dm-\int_\Omega\mathrm{div}_\eta\mathbf{u}_i\nabla f\cdot\mathbf{v}_idm.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_\Omega f\mathbf{v}_i\cdot\nabla(\mathrm{div}_\eta\mathbf{u}_i)dm-\sum_{j=1}^k a_{ij}\int_\Omega\mathbf{v}_i\cdot\nabla(\mathrm{div}_\eta\mathbf{u}_j)dm\\ &=-\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i)^2dm+\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i\nabla f\cdot\mathbf{u}_i-\mathrm{div}_\eta\mathbf{u}_i\nabla f\cdot\mathbf{v}_i)dm\\ &=-\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i)^2dm-\int_\Omega(\nabla(\nabla f\cdot\mathbf{u}_i)+\mathrm{div}_\eta\mathbf{u}_i\nabla f)\cdot\mathbf{v}_idm.\end{aligned}\tag{2.35}$$

So, replacing (2.35) into (2.34), we obtain

$$\begin{aligned}-\sigma_i\|\mathbf{v}_i\|^2=&\int_\Omega\mathbf{v}_i\cdot\mathscr{L}\mathbf{v}_idm-\int_\Omega\mathbf{v}_i\cdot(2T(\nabla f,\nabla\mathbf{u}_i)+\mathscr{L}f\mathbf{u}_i)dm\\ &-\alpha\int_\Omega(\mathrm{div}_\eta\mathbf{v}_i)^2dm-\alpha\int_\Omega(\nabla(\nabla f\cdot\mathbf{u}_i)+\mathrm{div}_\eta\mathbf{u}_i\nabla f)\cdot\mathbf{v}_idm.\end{aligned}\tag{2.36}$$

Hence, from (2.33) and (2.36), we have

$$\begin{aligned}(\sigma_{k+1}-\sigma_i)\|\mathbf{v}_i\|^2\le&-\int_\Omega(2T(\nabla f,\nabla\mathbf{u}_i)+\mathscr{L}f\mathbf{u}_i)\cdot\mathbf{v}_idm\\ &-\alpha\int_\Omega(\nabla(\nabla f\cdot\mathbf{u}_i)+\mathrm{div}_\eta\mathbf{u}_i\nabla f)\cdot\mathbf{v}_idm.\end{aligned}\tag{2.37}$$

Claim 1. *Using integration by parts formula* (1.10) *and* (2.31)*, we get*

$$\int_\Omega(2T(\nabla f,\nabla\mathbf{u}_i)+\mathscr{L}f\mathbf{u}_i)\cdot\mathbf{v}_idm=-\int_\Omega|\mathbf{u}_i|^2T(\nabla f,\nabla f)dm-2\sum_{j=1}^k a_{ij}b_{ij},\tag{2.38}$$

where

$$b_{ij}=\int_\Omega\Big(T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big)\cdot\mathbf{u}_jdm=-b_{ji}.\tag{2.39}$$

Furthermore, we also have the following

Claim 2. *By straightforward computation from* (1.7), (1.9) *and* (2.31), *we have*

$$\int_\Omega (\nabla(\nabla f\cdot \mathbf{u}_i) + \mathrm{div}_\eta \mathbf{u}_i \nabla f)\cdot \mathbf{v}_i dm$$
$$= \sum_{j=1}^k a_{ij}\int_\Omega (\nabla f\cdot \mathbf{u}_i \mathrm{div}_\eta \mathbf{u}_j - \mathrm{div}_\eta \mathbf{u}_i \nabla f\cdot \mathbf{u}_j) dm - \int_\Omega |\nabla f\cdot \mathbf{u}_i|^2 dm. \tag{2.40}$$

Putting

$$w_i = -\int_\Omega (2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f\mathbf{u}_i)\cdot \mathbf{v}_i dm - \alpha \int_\Omega \Big(\nabla(\nabla f\cdot \mathbf{u}_i) + \mathrm{div}_\eta \mathbf{u}_i \nabla f\Big)\cdot \mathbf{v}_i dm, \tag{2.41}$$

we have, from (2.37) and (2.41)

$$(\sigma_{k+1} - \sigma_i)\|\mathbf{v}_i\|^2 \le w_i. \tag{2.42}$$

Furthermore, from (2.38) and (2.40)

$$w_i = \int_\Omega |\mathbf{u}_i|^2 T(\nabla f, \nabla f) dm + 2\sum_{j=1}^k a_{ij} b_{ij}$$
$$- \alpha \sum_{j=1}^k a_{ij} \int_\Omega (\nabla f\cdot \mathbf{u}_i \mathrm{div}_\eta \mathbf{u}_j - \mathrm{div}_\eta \mathbf{u}_i \nabla f\cdot \mathbf{u}_j) dm + \alpha \int_\Omega |\nabla f\cdot \mathbf{u}_i|^2 dm. \tag{2.43}$$

Since $\mathscr{L}$ is self-adjoint (see (1.18)), from (2.29) and (2.39),

$$\begin{aligned}
2b_{ij} =& \int_\Omega \Big(2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f\mathbf{u}_i\Big)\cdot \mathbf{u}_j dm = \int_\Omega \Big(\mathscr{L}(f\mathbf{u}_i) - f\mathscr{L}\mathbf{u}_i\Big)\cdot \mathbf{u}_j dm \\
=& \int_\Omega \mathscr{L}(f\mathbf{u}_i)\cdot \mathbf{u}_j dm + \int_\Omega (-\mathscr{L}\mathbf{u}_i)\cdot (f\mathbf{u}_j) dm \\
=& \int_\Omega f\mathbf{u}_i \cdot \mathscr{L}\mathbf{u}_j dm + \int_\Omega (-\mathscr{L}\mathbf{u}_i)\cdot (f\mathbf{u}_j) dm \\
=& -\int_\Omega (f\mathbf{u}_i)\cdot (\sigma_j \mathbf{u}_j + \alpha \nabla \mathrm{div}_\eta \mathbf{u}_j) dm + \int_\Omega (f\mathbf{u}_j)\cdot (\sigma_i \mathbf{u}_i + \alpha \nabla \mathrm{div}_\eta \mathbf{u}_i) dm \\
=& \sigma_i \int_\Omega f\mathbf{u}_i\cdot \mathbf{u}_j dm - \sigma_j \int_\Omega f\mathbf{u}_i \cdot \mathbf{u}_j dm + \alpha \int_\Omega \big((f\mathbf{u}_j)\cdot \nabla \mathrm{div}_\eta \mathbf{u}_i - (f\mathbf{u}_i)\cdot \nabla \mathrm{div}_\eta \mathbf{u}_j\big) dm \\
=& (\sigma_i - \sigma_j) a_{ij} + \alpha \int_\Omega \big((f\mathbf{u}_j)\cdot \nabla \mathrm{div}_\eta \mathbf{u}_i - (f\mathbf{u}_i)\cdot \nabla \mathrm{div}_\eta \mathbf{u}_j\big) dm.
\end{aligned} \tag{2.44}$$

From, (1.7) and (1.9) (or immediately from (1.16)) we have

$$\int_\Omega (f\mathbf{u}_j)\cdot \nabla \mathrm{div}_\eta \mathbf{u}_i dm = -\int_\Omega \mathrm{div}_\eta \mathbf{u}_i \mathrm{div}_\eta \mathbf{u}_j dm - \int_\Omega \mathrm{div}_\eta \mathbf{u}_i \nabla f\cdot \mathbf{u}_j dm,$$

and

$$\int_\Omega (f\mathbf{u}_i)\cdot\nabla \mathrm{div}_\eta \mathbf{u}_j dm = -\int_\Omega \mathrm{div}_\eta \mathbf{u}_j \mathrm{div}_\eta \mathbf{u}_i dm - \int_\Omega \mathrm{div}_\eta \mathbf{u}_j \nabla f\cdot \mathbf{u}_i dm.$$

Therefore, substituting the two previous equalities into (2.44), we get

$$2b_{ij} = (\sigma_i-\sigma_j)a_{ij} + \alpha\int_\Omega (\nabla f\cdot\mathbf{u}_i \mathrm{div}_\eta \mathbf{u}_j - \mathrm{div}_\eta \mathbf{u}_i \nabla f\cdot\mathbf{u}_j)dm,$$

then

$$2\sum_{j=1}^k a_{ij}b_{ij} = \sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2 + \alpha\sum_{j=1}^k a_{ij}\int_\Omega (\nabla f\cdot\mathbf{u}_i \mathrm{div}_\eta \mathbf{u}_j - \mathrm{div}_\eta \mathbf{u}_i \nabla f\cdot\mathbf{u}_j)dm. \quad (2.45)$$

Thus, combining (2.43) and (2.45) we obtain

$$w_i = \int_\Omega |\mathbf{u}_i|^2 T(\nabla f,\nabla f)dm + \sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2 + \alpha\int_\Omega |\nabla f\cdot\mathbf{u}_i|^2 dm. \quad (2.46)$$

For any constant $B>0$, from (2.32), (2.38) and the inequality of Cauchy-Schwarz, we infer

$$\begin{aligned}
&(\sigma_{k+1}-\sigma_i)^2\Big(\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2 dm + 2\sum_{j=1}^k a_{ij}b_{ij}\Big)\\
=&(\sigma_{k+1}-\sigma_i)^2\Big\{-2\int_\Omega\Big(T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i-\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big)\cdot\mathbf{v}_i dm\Big\}\\
\leq&2(\sigma_{k+1}-\sigma_i)^2\|\mathbf{v}_i\|\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i-\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big\|\\
\leq&(\sigma_{k+1}-\sigma_i)^3 B\|\mathbf{v}_i\|^2+\frac{\sigma_{k+1}-\sigma_i}{B}\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i-\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big\|^2. \quad (2.47)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i-\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big\|^2\\
=&\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-2\sum_{j=1}^k b_{ij}\int_\Omega\Big(T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big)\cdot\mathbf{u}_j dm+\Big\|\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big\|^2\\
=&\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-2\sum_{j=1}^k b_{ij}^2+\sum_{j=1}^k b_{ij}^2=\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-\sum_{j=1}^k b_{ij}^2
\end{aligned}$$

hence, using the previous equality, from (2.47), (2.42) and (2.46) , we obtain

$$
\begin{aligned}
&(\sigma_{k+1}-\sigma_i)^2\Big(\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2dm+2\sum_{j=1}^k a_{ij}b_{ij}\Big)\\
&\quad\le(\sigma_{k+1}-\sigma_i)^2Bw_i+\frac{\sigma_{k+1}-\sigma_i}{B}\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i-\sum_{j=1}^k b_{ij}\mathbf{u}_j\Big\|^2\\
&\quad\le(\sigma_{k+1}-\sigma_i)^2B\Big(\int_\Omega|\mathbf{u}_i|^2T(\nabla f,\nabla f)dm+\sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2+\alpha\int_\Omega|\nabla f\cdot\mathbf{u}_i|^2dm\Big)\\
&\qquad+\frac{\sigma_{k+1}-\sigma_i}{B}\Big(\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-\sum_{j=1}^k b_{ij}^2\Big).
\end{aligned}
$$

Summing over i from 1 to k, we obtain

$$
\begin{aligned}
&\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2\Big(\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2dm+2\sum_{j=1}^k a_{ij}b_{ij}\Big)\\
&\quad\le\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2B\Big(\int_\Omega|\mathbf{u}_i|^2T(\nabla f,\nabla f)dm+\sum_{j=1}^k(\sigma_i-\sigma_j)a_{ij}^2+\alpha\int_\Omega|\nabla f\cdot\mathbf{u}_i|^2dm\Big)\\
&\qquad+\sum_{i=1}^k\frac{\sigma_{k+1}-\sigma_i}{B}\Big(\Big\|T(\nabla f,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-\sum_{j=1}^k b_{ij}^2\Big). \qquad (2.48)
\end{aligned}
$$

Since $a_{ij}=a_{ji}$ and $b_{ij}=-b_{ji}$, we have

$$
\begin{aligned}
2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)^2a_{ij}b_{ij}=&2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j+\sigma_j-\sigma_i)a_{ij}b_{ij}\\
=&2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j)a_{ij}b_{ij}-2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)a_{ij}b_{ij}\\
=&-2\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)a_{ij}b_{ij},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)^2(\sigma_i-\sigma_j)a_{ij}^2=\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j+\sigma_j-\sigma_i)(\sigma_i-\sigma_j)a_{ij}^2\\
&\quad=\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j)(\sigma_i-\sigma_j)a_{ij}^2+\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_j-\sigma_i)(\sigma_i-\sigma_j)a_{ij}^2\\
&\quad=-\sum_{i,j=1}^k(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)^2a_{ij}^2,
\end{aligned}
$$

where we have used the facts

$$\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j)a_{ij}b_{ij}=0 \quad \text{and} \quad \sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_{k+1}-\sigma_j)(\sigma_i-\sigma_j)a_{ij}^2=0.$$

Therefore, from (2.48), we get

$$\begin{aligned}
&\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\Big((1-B)\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2dm - B\alpha\int_\Omega |\nabla f\cdot \mathbf{u}_i|^2dm\Big)\\
&\le \sum_{i=1}^{k}\frac{\sigma_{k+1}-\sigma_i}{B}\Big\|T(\nabla f,\nabla \mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2 + 2\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)a_{ij}b_{ij}\\
&\quad -\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)(\sigma_i-\sigma_j)^2a_{ij}^2B-\sum_{i,j=1}^{k}\frac{\sigma_{k+1}-\sigma_i}{B}b_{ij}^2\\
&=\sum_{i=1}^{k}\frac{\sigma_{k+1}-\sigma_i}{B}\Big\|T(\nabla f,\nabla \mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2\\
&\quad -\frac{1}{B}\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)\Big[(\sigma_i-\sigma_j)^2a_{ij}^2B^2-2(\sigma_i-\sigma_j)a_{ij}Bb_{ij}+b_{ij}^2\Big]\\
&=\sum_{i=1}^{k}\frac{\sigma_{k+1}-\sigma_i}{B}\Big\|T(\nabla f,\nabla \mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2-\frac{1}{B}\sum_{i,j=1}^{k}(\sigma_{k+1}-\sigma_i)\Big[(\sigma_i-\sigma_j)a_{ij}^2B-b_{ij}\Big]^2,
\end{aligned}$$

since $\sigma_{k+1}-\sigma_i\ge 0$ and $B>0$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\Big((1-B)\int_\Omega T(\nabla f,\nabla f)|\mathbf{u}_i|^2dm - B\alpha\int_\Omega |\nabla f\cdot \mathbf{u}_i|^2dm\Big)\\
&\le \frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big\|T(\nabla f,\nabla \mathbf{u}_i)+\frac{1}{2}\mathscr{L}f\mathbf{u}_i\Big\|^2.
\end{aligned}$$

This finishes the proof of Lemma 2.4.1. □

2.4.2 Proof of Claims 1 and 2

Here, we prove the Claims 1 and 2 as mentioned in the proof of Lemma 2.4.1.

Proof of Claim 1

Proof. From definition of $\mathbf{v}_i$ we have

$$\begin{aligned}\int_\Omega (2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f \mathbf{u}_i) \cdot \mathbf{v}_i dm = & \int_\Omega (2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f \mathbf{u}_i) \cdot (f\mathbf{u}_i) dm \\ & - 2\sum_{j=1}^k a_{ij} \int_\Omega \Big(T(\nabla f, \nabla \mathbf{u}_i) + \frac{1}{2}\mathscr{L} f \mathbf{u}_i \Big) \cdot \mathbf{u}_j dm \\ = & \int_\Omega (2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f \mathbf{u}_i) \cdot (f\mathbf{u}_i) dm - 2\sum_{j=1}^k a_{ij} b_{ij}.\end{aligned} \tag{2.49}$$

Moreover, notice that

$$\begin{aligned}\int_\Omega \big(2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f \mathbf{u}_i\big) \cdot (f\mathbf{u}_i) dm = & 2\int_\Omega T(\nabla f, \nabla u_i^1) f u_i^1 dm + \int_\Omega f (u_i^1)^2 \mathscr{L} f dm + \cdots \\ & + 2\int_\Omega T(\nabla f, \nabla u_i^1) f u_i^n dm + \int_\Omega f(u_i^n)^2 \mathscr{L} f dm,\end{aligned} \tag{2.50}$$

since $\mathbf{u}_i|_{\partial\Omega} = 0$ from integration by parts (1.10), for all $1 \leq k \leq n$ we get

$$\begin{aligned}\int_\Omega f(u_i^k)^2 \mathscr{L} f dm = & -\int_\Omega \langle \nabla\big(f(u_i^k)^2\big), T(\nabla f)\rangle dm \\ = & -\int_\Omega (u_i^k)^2 \langle \nabla f, T(\nabla f)\rangle dm - 2\int_\Omega f u_i^k \langle \nabla u_i^k, T(\nabla f)\rangle dm \\ = & -\int_\Omega (u_i^k)^2 T(\nabla f, \nabla f) dm - 2\int_\Omega f u_i^k T(\nabla u_i^k, \nabla f) dm,\end{aligned}$$

and substituting the previous equality into (2.50) we obtain

$$\begin{aligned}\int_\Omega & \big(2T(\nabla f, \nabla \mathbf{u}_i) + \mathscr{L} f \mathbf{u}_i\big) \cdot (f\mathbf{u}_i) dm \\ & = 2\int_\Omega T(\nabla f, \nabla u_i^1) f u_i^1 dm - \int_\Omega (u_i^1)^2 T(\nabla f, \nabla f) dm \\ & \quad - 2\int_\Omega f u_i^1 T(\nabla u_i^1, \nabla f) dm + \cdots + 2\int_\Omega T(\nabla f, \nabla u_i^n) f u_i^n dm \\ & \quad - \int_\Omega (u_i^n)^2 T(\nabla f, \nabla f) dm - 2\int_\Omega f u_i^n \langle \nabla u_i^n, T(\nabla f)\rangle dm \\ & = -\int_\Omega (u_i^1)^2 T(\nabla f, \nabla f) dm - \cdots - \int_\Omega (u_i^n)^2 T(\nabla f, \nabla f)\rangle dm \\ & = -\int_\Omega |\mathbf{u}_i|^2 T(\nabla f, \nabla f) dm.\end{aligned} \tag{2.51}$$

Therefore, from (2.49) and (2.51) we obtain (2.38) and conclude the proof. □

Proof of Claim 2

Proof. From definition of $\mathbf{v}_i$ we obtain

$$\int_\Omega \big(\nabla(\nabla f\cdot \mathbf{u}_i)+\mathrm{div}_\eta \mathbf{u}_i\nabla f\big)\cdot \mathbf{v}_i dm = \int_\Omega (\nabla(\nabla f\cdot \mathbf{u}_i)+\mathrm{div}_\eta \mathbf{u}_i\nabla f)\cdot \big(\mathbf{u}_i f-\sum_{j=1}^k a_{ij}\mathbf{u}_j\big)dm$$
$$=-\sum_{j=1}^k a_{ij}\int_\Omega \big(\nabla(\nabla f\cdot \mathbf{u}_i)+\mathrm{div}_\eta \mathbf{u}_i\nabla f\big)\cdot \mathbf{u}_j dm$$
$$+\int_\Omega \big(\nabla(\nabla f\cdot \mathbf{u}_i)+\mathrm{div}_\eta \mathbf{u}_i\nabla f\big)\cdot (f\mathbf{u}_i)dm. \quad (2.52)$$

From (1.7) we have

$$\mathrm{div}_\eta\big((\nabla f\cdot \mathbf{u}_i)\mathbf{u}_j\big)=(\nabla f\cdot \mathbf{u}_i)\mathrm{div}_\eta \mathbf{u}_j+\nabla(\nabla f\cdot \mathbf{u}_i)\cdot \mathbf{u}_j,$$

and

$$\begin{aligned}\mathrm{div}_\eta\big((\nabla f\cdot \mathbf{u}_i)(f\mathbf{u}_i)\big)&=(\nabla f\cdot \mathbf{u}_i)\mathrm{div}_\eta(f\mathbf{u}_i)+\nabla(\nabla f\cdot \mathbf{u}_i)\cdot(f\mathbf{u}_i)\\&=(\nabla f\cdot \mathbf{u}_i)f\mathrm{div}_\eta \mathbf{u}_i+|\nabla f\cdot \mathbf{u}_i|^2+\nabla(\nabla f\cdot \mathbf{u}_i)\cdot(f\mathbf{u}_i),\end{aligned}$$

hence, from the previous equalities and (1.9) we get

$$\int_\Omega \nabla(\nabla f\cdot \mathbf{u}_i)\cdot \mathbf{u}_j dm=-\int_\Omega (\nabla f\cdot \mathbf{u}_i)\mathrm{div}_\eta \mathbf{u}_j dm, \quad (2.53)$$

and

$$\int_\Omega \nabla(\nabla f\cdot \mathbf{u}_i)\cdot(f\mathbf{u}_i)dm=-\int_\Omega(\nabla f\cdot \mathbf{u}_i)f\mathrm{div}_\eta \mathbf{u}_i dm-\int_\Omega|\nabla f\cdot \mathbf{u}_i|^2dm. \quad (2.54)$$

Substituting (2.53) and (2.54) into (2.52) we complete the proof. □

The proof of the next lemma follows the steps of the proof of Proposition 2 in Gomes and Miranda [15] with appropriate adaptations for vector-valued functions from Ω to $\mathbb{R}^n$.

Lemma 2.4.2. *Let Ω be a bounded domain in Euclidean space $\mathbb{R}^n$, σ_i be the i-th eigenvalue of Problem 2.1 and $\mathbf{u}_i$ be a normalized vector-valued eigenfunction corresponding to σ_i. Then, for some positive real numbers ε and δ, we get*

$$\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)^2\le \frac{4(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^k(\sigma_{k+1}-\sigma_i)\Big\{\frac14\int_\Omega|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)-T(\nabla\eta)|^2dm$$
$$+\int_\Omega \mathbf{u}_i\cdot\Big(T(\mathrm{tr}(\nabla T),\nabla \mathbf{u}_i)-T(T(\nabla\eta),\nabla\mathbf{u}_i)\Big)dm+\|T(\nabla \mathbf{u}_i)\|^2\Big\}.$$

Proof. Let $\{x_\beta\}_{\beta=1}^n$ be the coordinate functions of $\mathbb{R}^n$, then taking $f=x_\beta$ in Lemma 2.4.1

and summing over β from 1 to n, we get

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\sum_{\beta=1}^{n}\left\{(1-B)\int_\Omega T(\nabla x_\beta,\nabla x_\beta)|\mathbf{u}_i|^2dm-B\alpha\int_\Omega|\nabla x_\beta\cdot\mathbf{u}_i|^2dm\right\}$$
$$\leq\frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\sum_{\beta=1}^{n}\left\|T(\nabla x_\beta,\nabla\mathbf{u}_i)+\frac{1}{2}\mathscr{L}x_\beta\mathbf{u}_i\right\|^2$$
$$=\frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\int_\Omega\sum_{\beta=1}^{n}\left|T(\nabla x_\beta,\nabla\mathbf{u}_i)+\frac{1}{2}\mathrm{div}_\eta(T(\nabla x_\beta))\mathbf{u}_i\right|^2dm.$$

Therefore,

$$\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\sum_{\beta=1}^{n}\left\{(1-B)\int_\Omega T(\nabla x_\beta,\nabla x_\beta)|\mathbf{u}_i|^2dm-B\alpha\int_\Omega|\nabla x_\beta\cdot\mathbf{u}_i|^2dm\right\}$$
$$\leq\frac{1}{B}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\int_\Omega\sum_{\beta=1}^{n}\left\{\frac{|\mathbf{u}_i|^2}{4}(\mathrm{div}_\eta(T(\nabla x_\beta)))^2\right.$$
$$\left.+\mathbf{u}_i\cdot\Big(\mathrm{div}_\eta(T(\nabla x_\beta))T(\nabla x_\beta,\nabla\mathbf{u}_i)\Big)+|T(\nabla x_\beta,\nabla\mathbf{u}_i)|^2\right\}dm. \tag{2.55}$$

By straightforward computation, we have

$$\sum_{\beta=1}^{n}T(\nabla x_\beta,\nabla x_\beta)=\sum_{\beta=1}^{n}\langle T(e_\beta),e_\beta\rangle=\mathrm{tr}(T)\quad\text{and}\quad\sum_{\beta=1}^{n}|\nabla x_\beta\cdot\mathbf{u}_i|^2=|\mathbf{u}_i|^2.$$

Similarly to the calculations in [15, Eq. (3.23)] (see Eq. (1.29) in Lemma (1.5.3)) we obtain

$$\sum_{\beta=1}^{n}(\mathrm{div}_\eta(T(\nabla x_\beta)))^2=\sum_{\beta=1}^{n}(\mathscr{L}x_\beta)^2=|\mathrm{tr}(\nabla T)-T(\nabla\eta)|^2.$$

From (1.28) (see [15, Eq. (3.24)]), for all $k=1,\dots,n$, we get

$$\begin{aligned}\sum_{\beta=1}^{n}\mathrm{div}_\eta(T(\nabla x_\beta))T(\nabla x_\beta,\nabla u_i^k)=&\sum_{\beta=1}^{n}\mathrm{div}_\eta(T(\nabla x_\beta))T(\nabla u_i^k)(x_\beta)\\=&\langle\mathrm{div}_\eta(T(\nabla x)),T(\nabla u_i^k)\rangle\\=&\langle\mathrm{tr}(\nabla T)-T(\nabla\eta),T(\nabla u_i^k)\rangle\\=&\langle\mathrm{tr}(\nabla T),T(\nabla u_i^k)\rangle-\langle T(\nabla\eta),T(\nabla u_i^k)\rangle,\end{aligned}$$

where $\operatorname{div}_\eta(T(\nabla x)) := (\operatorname{div}_\eta(T(\nabla x_1)), \ldots, \operatorname{div}_\eta(T(\nabla x_n)))$. Then, using (1.15), we obtain

$$
\begin{aligned}
&\sum_{\beta=1}^{n} \operatorname{div}_\eta(T(\nabla x_\beta))T(\nabla x_\beta, \nabla \mathbf{u}_i) = \\
&= \Big(\sum_{\beta=1}^{n} \operatorname{div}_\eta(T(\nabla x_\beta))T(\nabla x_\beta, \nabla u_i^1), \ldots, \sum_{\beta=1}^{n} \operatorname{div}_\eta(T(\nabla x_\beta))T(\nabla x_\beta, \nabla u_i^n) \Big) \\
&= (\langle \operatorname{tr}(\nabla T), T(\nabla u_i^1)\rangle - \langle T(\nabla \eta), T(\nabla u_i^1)\rangle, \ldots, \langle \operatorname{tr}(\nabla T), T(\nabla u_i^n)\rangle - \langle T(\nabla \eta), T(\nabla u_i^n)\rangle) \\
&= (\langle \operatorname{tr}(\nabla T), T(\nabla u_i^1)\rangle, \ldots, \langle \operatorname{tr}(\nabla T), T(\nabla u_i^n)\rangle) - (\langle T(\nabla \eta), T(\nabla u_i^1)\rangle, \ldots, \langle T(\nabla \eta), T(\nabla u_i^n)\rangle) \\
&= T(\operatorname{tr}(\nabla T), \nabla \mathbf{u}_i) - T(T(\nabla \eta), \nabla \mathbf{u}_i).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\sum_{\beta=1}^{n} |T(\nabla x_\beta, \nabla \mathbf{u}_i)|^2 &= \sum_{\beta=1}^{n} |T(e_\beta, \nabla \mathbf{u}_i)|^2 = \sum_{\beta=1}^{n} |\big(\langle e_\beta, T(\nabla u_i^1)\rangle, \ldots, \langle e_\beta, T(\nabla u_i^n)\rangle\big)|^2 \\
&= \sum_{\beta=1}^{n} \langle e_\beta, T(\nabla u_i^1)\rangle^2 + \cdots + \sum_{\beta=1}^{n} \langle e_\beta, T(\nabla u_i^n)\rangle^2 = \sum_{j=1}^{n} |T(\nabla u_i^j)|^2 = |T(\nabla \mathbf{u}_i)|^2.
\end{aligned}
$$

Remembering that $\|T(\nabla \mathbf{u}_i)\|^2 = \int_\Omega |T(\nabla \mathbf{u}_i)|^2 dm$ and substituting the previous equalities into (2.55), we get

$$
\begin{aligned}
&\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \Big[(1 - B) \int_\Omega \operatorname{tr}(T) |\mathbf{u}_i|^2 dm - B\alpha \Big] \\
&\quad \leq \frac{1}{B} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) \Bigg\{ \frac{1}{4} \int_\Omega |\mathbf{u}_i|^2 |\operatorname{tr}(\nabla T) - T(\nabla \eta)|^2 dm \\
&\qquad + \int_\Omega \mathbf{u}_i \cdot \Big(T(\operatorname{tr}(\nabla T), \nabla \mathbf{u}_i) - T(T(\nabla \eta), \nabla \mathbf{u}_i) \Big) dm + \|T(\nabla \mathbf{u}_i)\|^2 \Bigg\}. \qquad (2.56)
\end{aligned}
$$

Since $\varepsilon I \leq T \leq \delta I$, then $n\varepsilon \leq \operatorname{tr}(T) \leq n\delta$. Hence from (2.56)

$$
\begin{aligned}
&\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 [n\varepsilon - (n\delta + \alpha)B] \\
&\quad \leq \frac{1}{B} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) \Bigg\{ \frac{1}{4} \int_\Omega |\mathbf{u}_i|^2 |\operatorname{tr}(\nabla T) - T(\nabla \eta)|^2 dm \\
&\qquad + \int_\Omega \mathbf{u}_i \cdot \Big(T(\operatorname{tr}(\nabla T), \nabla \mathbf{u}_i) - T(T(\nabla \eta), \nabla \mathbf{u}_i) \Big) dm + \|T(\nabla \mathbf{u}_i)\|^2 \Bigg\}. \qquad (2.57)
\end{aligned}
$$

Let us consider

$$
\begin{aligned}
M_i =& \frac{1}{4}\int_\Omega |\mathbf{u}_i|^2 |\mathrm{tr}(\nabla T) - T(\nabla\eta)|^2 dm \\
&+ \int_\Omega \mathbf{u}_i \cdot (T(\mathrm{tr}(\nabla T), \nabla \mathbf{u}_i) - T(T(\nabla\eta), \nabla\mathbf{u}_i))dm + \|T(\nabla\mathbf{u}_i)\|^2,
\end{aligned}
$$

so that, from (2.57), we have

$$
n\varepsilon B \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 - (n\delta + \alpha) B^2 \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leq \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) M_i. \tag{2.58}
$$

Furthermore, since B is arbitrary positive constant, putting

$$
B = \left\{ \frac{\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)}{(n\delta + \alpha)\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2} M_i \right\}^{\frac{1}{2}}
$$

into (2.58), we obtain

$$
n\varepsilon \left\{ \frac{\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)}{(n\delta + \alpha)\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2} M_i \right\}^{\frac{1}{2}} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leq 2 \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) M_i,
$$

and if we square the previous equality we get

$$
\frac{n^2\varepsilon^2}{n\delta + \alpha} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leq 4 \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) M_i,
$$

that is enough to complete the proof of Lemma 2.4.2. □

With these considerations in mind, we can rewrite the previous lemma in a more convenient way for us.

Lemma 2.4.3. *Under the same setup as in Lemma 2.4.2, we get*

$$
\begin{aligned}
\sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i)^2 \leq& \frac{4(n\delta + \alpha)}{n^2\varepsilon^2} \sum_{i=1}^{k} (\sigma_{k+1} - \sigma_i) \Big\{ \|T(\nabla\mathbf{u}_i)\|^2 + C \\
&+ \frac{1}{4}\int_\Omega |\mathbf{u}_i|^2 \langle \mathrm{tr}(\nabla T), \mathrm{tr}(\nabla T) - 2T(\nabla\eta)\rangle dm + \int_\Omega \mathbf{u}_i \cdot T(\mathrm{tr}(\nabla T), \nabla\mathbf{u}_i) dm \Big\},
\end{aligned}
$$

where $C = \sup_\Omega \left\{ \frac{1}{2}\mathrm{div}(T^2(\nabla\eta)) - \frac{1}{4}|T(\nabla\eta)|^2 \right\}$ *has been chosen such that the term on the right-hand side must be positive.*

Proof. We make use of Lemma 2.4.2. For this, we must notice that

$$
|\mathrm{tr}(\nabla T) - T(\nabla\eta)|^2 = |\mathrm{tr}(\nabla T)|^2 - 2\langle \mathrm{tr}(\nabla T), T(\nabla\eta)\rangle + |T(\nabla\eta)|^2,
$$

hence

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)-T(\nabla\eta)|^2dm+\int_\Omega \mathbf{u}_i\cdot(T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)-T(T(\nabla\eta),\nabla\mathbf{u}_i)))dm\\
=&\frac{1}{4}\int_\Omega |\mathbf{u}_i|^2|T(\nabla\eta)|^2dm-\int_\Omega \mathbf{u}_i\cdot T(T(\nabla\eta),\nabla\mathbf{u}_i)dm+\frac{1}{4}\int_\Omega|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)|^2dm\\
&-\frac{1}{2}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),T(\nabla\eta)\rangle dm+\int_\Omega \mathbf{u}_i\cdot T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)dm. \qquad (2.59)
\end{aligned}$$

Since $\mathbf{u}_i|_{\partial\Omega}=0$ by Eq. (1.15) and the divergence theorem, we have

$$\begin{aligned}
&-\int_\Omega \mathbf{u}_i\cdot T(T(\nabla\eta),\nabla\mathbf{u}_i)dm\\
&=-\int_\Omega u_i^1\langle T^2(\nabla\eta),\nabla u_i^1\rangle dm-\cdots-\int_\Omega u_i^n\langle T^2(\nabla\eta),\nabla u_i^n\rangle dm\\
&=-\frac{1}{2}\int_\Omega\langle T^2(\nabla\eta),\nabla(u_i^1)^2\rangle dm-\cdots-\frac{1}{2}\int_\Omega\langle T^2(\nabla\eta),\nabla(u_i^n)^2\rangle dm\\
&=\frac{1}{2}\int_\Omega(u_i^1)^2\mathrm{div}_\eta(T^2(\nabla\eta))dm+\cdots+\frac{1}{2}\int_\Omega(u_i^n)^2\mathrm{div}_\eta(T^2(\nabla\eta))dm\\
&=\frac{1}{2}\int_\Omega|\mathbf{u}_i|^2\mathrm{div}_\eta(T^2(\nabla\eta))dm.
\end{aligned}$$

Substituting the previous equation in Eq. (2.59), we get

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)-T(\nabla\eta)|^2dm+\int_\Omega \mathbf{u}_i\cdot(T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)-T(T(\nabla\eta),\nabla\mathbf{u}_i))dm\\
=&\int_\Omega|\mathbf{u}_i|^2\Big(\frac{1}{4}|T(\nabla\eta)|^2+\frac{1}{2}\mathrm{div}_\eta(T^2(\nabla\eta))\Big)dm+\frac{1}{4}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),\mathrm{tr}(\nabla T)\rangle dm\\
&-\frac{1}{2}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),T(\nabla\eta)\rangle dm+\int_\Omega \mathbf{u}_i\cdot T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)dm\\
=&\int_\Omega|\mathbf{u}_i|^2\Big(\frac{1}{2}\mathrm{div}(T^2(\nabla\eta))-\frac{1}{4}|T(\nabla\eta)|^2\Big)dm+\frac{1}{4}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),\mathrm{tr}(\nabla T)\rangle dm\\
&-\frac{1}{2}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),T(\nabla\eta)\rangle dm+\int_\Omega \mathbf{u}_i\cdot T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)dm,
\end{aligned}$$

where we are using in the last equality that $\mathrm{div}_\eta(T^2(\nabla\eta))=\mathrm{div}(T^2(\nabla\eta))-|T(\nabla\eta)|^2$. By setting $C=\sup_\Omega\left\{\frac{1}{2}\mathrm{div}(T^2(\nabla\eta))-\frac{1}{4}|T(\nabla\eta)|^2\right\}$ in the previous equality, we have

$$\begin{aligned}
&\frac{1}{4}\int_\Omega |\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)-T(\nabla\eta)|^2dm+\int_\Omega \mathbf{u}_i\cdot(T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)-T(T(\nabla\eta),\nabla\mathbf{u}_i)))dm\\
&\le C+\frac{1}{4}\int_\Omega|\mathbf{u}_i|^2\langle\mathrm{tr}(\nabla T),\mathrm{tr}(\nabla T)-2T(\nabla\eta)\rangle dm+\int_\Omega \mathbf{u}_i\cdot T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)dm. \qquad (2.60)
\end{aligned}$$

Replacing Inequality (2.60) into Lemma 2.4.2, we complete the proof of Lemma 2.4.3. □

Now, we are in a position to give the proof of the two theorems of this thesis. For this, let us make use of the results from the previous section.

2.4.3 Proof of Theorem 2.1.1

Proof. The proof is a consequence of Lemma 2.4.3. We begin by computing

$$\begin{aligned}\frac{1}{4}\int_{\Omega}|\mathbf{u}_i|^2\langle \mathrm{tr}(\nabla T), \mathrm{tr}(\nabla T)-2T(\nabla\eta)\rangle dm =&\frac{1}{4}\int_{\Omega}|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)|^2 dm\\ &-\frac{1}{2}\int_{\Omega}|\mathbf{u}_i|^2\langle \mathrm{tr}(\nabla T), T(\nabla\eta)\rangle dm.\end{aligned}$$

Since $T_0=\sup_{\Omega}|\mathrm{tr}(\nabla T)|$ and $\eta_0=\sup_{\Omega}|\nabla\eta|$, we have

$$\frac{1}{4}\int_{\Omega}|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)|^2 dm\leq\frac{1}{4}T_0^2\int_{\Omega}|\mathbf{u}_i|^2 dm=\frac{1}{4}T_0^2,$$

and using (1.4) we get

$$\begin{aligned}-\frac{1}{2}\int_{\Omega}|\mathbf{u}_i|^2\langle \mathrm{tr}(\nabla T), T(\nabla\eta)\rangle dm &\leq \frac{1}{2}\int_{\Omega}|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)||T(\nabla\eta)|dm\\ &\leq\frac{\delta}{2}\int_{\Omega}|\mathbf{u}_i|^2|\mathrm{tr}(\nabla T)||\nabla\eta|dm\\ &\leq\frac{\delta}{2}T_0\eta_0.\end{aligned}$$

Then,

$$\frac{1}{4}\int_{\Omega}|\mathbf{u}_i|^2\langle \mathrm{tr}(\nabla T), \mathrm{tr}(\nabla T)-2T(\nabla\eta)\rangle dm\leq\frac{1}{4}T_0^2+\frac{\delta}{2}T_0\eta_0. \tag{2.61}$$

Furthermore,

$$\begin{aligned}\int_{\Omega}\mathbf{u}_i\cdot T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)dm &\leq\Big(\int_{\Omega}|\mathbf{u}_i|^2dm\Big)^{\frac{1}{2}}\Big(\int_{\Omega}|T(\mathrm{tr}(\nabla T),\nabla\mathbf{u}_i)|^2dm\Big)^{\frac{1}{2}}\\ &\leq T_0\Big(\int_{\Omega}|T(\nabla\mathbf{u}_i)|^2dm\Big)^{\frac{1}{2}}=T_0\|T(\nabla\mathbf{u}_i)\|.\end{aligned} \tag{2.62}$$

Substituting (2.61) and (2.62) into Lemma 2.4.3, we obtain

$$\begin{aligned}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)^2\leq&\frac{4(n\delta+\alpha)}{n^2\varepsilon^2}\sum_{i=1}^{k}(\sigma_{k+1}-\sigma_i)\Big\{\|T(\nabla\mathbf{u}_i)\|^2+\frac{1}{4}T_0^2+T_0\|T(\nabla\mathbf{u}_i)\|\\ &+\frac{\delta}{2}T_0\eta_0+C\Big\}.\end{aligned}$$

Moreover, from the proof of Lemma 2.4.2 and Lemma 2.4.3, we can see that

$$
\begin{aligned}
0 < \sum_{\beta=1}^{n} \left\| T(\nabla x_\beta, \nabla \mathbf{u}_i) + \frac{1}{2}\mathscr{L} x_\beta \mathbf{u}_i \right\|^2 \leq & \Big\{ \|T(\nabla \mathbf{u}_i)\|^2 + \frac{1}{4}T_0^2 + T_0\|T(\nabla \mathbf{u}_i)\| \\
& + \frac{\delta}{2}T_0\eta_0 + C \Big\} \\
= & \Big(\|T(\nabla \mathbf{u}_i)\| + \frac{1}{2}T_0 \Big)^2 + C_0,
\end{aligned}
$$

where $C_0 = C + \frac{\delta}{2}T_0\eta_0$ and $\{x_\beta\}_{\beta=1}^n$ are the canonical coordinate functions of $\mathbb{R}^n$. Thus, we get

$$
\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4(n\delta + \alpha)}{n^2\varepsilon^2} \sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i) \Big\{ \Big(\|T(\nabla \mathbf{u}_i)\| + \frac{1}{2}T_0 \Big)^2 + C_0 \Big\}. \tag{2.63}
$$

From (2.1), (1.10) and (1.16) we obtain

$$
\sigma_i = \int_\Omega T(\nabla \mathbf{u}_i) \cdot \nabla \mathbf{u}_i dm + \alpha \|\mathrm{div}_\eta \mathbf{u}_i\|^2.
$$

Since there exist positive real numbers ε and δ such that $\varepsilon I \leq T \leq \delta I$, from the previous inequality and (1.3), we get

$$
\|T(\nabla \mathbf{u}_i)\|^2 \leq \delta \int_\Omega T(\nabla \mathbf{u}_i) \cdot \nabla \mathbf{u}_i dm = \delta(\sigma_i - \alpha \|\mathrm{div}_\eta \mathbf{u}_i\|^2). \tag{2.64}
$$

Therefore, from (2.63) and (2.64) we obtain

$$
\sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i)^2 \leq \frac{4(n\delta + \alpha)}{n^2\varepsilon^2} \sum_{i=1}^{k}(\sigma_{k+1} - \sigma_i) \Big\{ \Big[\sqrt{\delta}(\sigma_i - \alpha \|\mathrm{div}_\eta \mathbf{u}_i\|^2)^{\frac{1}{2}} + T_0 \Big]^2 + C_0 \Big\},
$$

which is enough to complete the proof. □

2.4.4 Proof of Theorem 2.1.2

Proof. Let $\{x_\beta\}_{\beta=1}^n$ be the standard coordinate functions of $\mathbb{R}^n$. Let us consider the matrix $D = (d_{ij})_{n\times n}$ where

$$
d_{ij} := \int_\Omega x_i \mathbf{u}_1 \cdot \mathbf{u}_{j+1} dm.
$$

Using the orthogonalization of Gram and Schmidt, we know that there exists an upper triangle matrix $R = (r_{ij})_{n\times n}$ and an orthogonal matrix $S = (s_{ij})_{n\times n}$ such that $R = SD$, namely

$$
r_{ij} = \sum_{k=1}^{n} s_{ik} d_{kj} = \sum_{k=1}^{n} s_{ik} \int_\Omega x_k \mathbf{u}_1 \cdot \mathbf{u}_{j+1} dm = \int_\Omega \Big(\sum_{k=1}^{n} s_{ik} x_k \Big) \mathbf{u}_1 \cdot \mathbf{u}_{j+1} dm = 0,
$$

for $1 \leq j < i \leq n$. Putting $y_i = \sum_{k=1}^{n} s_{ik} x_k$, we have

$$\int_\Omega y_i \mathbf{u}_1 \cdot \mathbf{u}_{j+1} dm = 0 \quad \text{for} \quad 1 \leq j < i \leq n.$$

Let us denote by $a_i = \int_\Omega y_i |\mathbf{u}_1|^2 dm$ to consider the vector-valued functions $\mathbf{w}_i$ given by

$$\mathbf{w}_i = (y_i - a_i)\mathbf{u}_1, \tag{2.65}$$

so that

$$\mathbf{w}_i|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega \mathbf{w}_i \cdot \mathbf{u}_{j+1} dm = 0, \quad \text{for any} \quad j = 1, \ldots, i-1.$$

Then, we can take $\mathbf{v} = \mathbf{w}_i$ in (2.30) and to use formula (1.16) to obtain

$$\sigma_{i+1} \|\mathbf{w}_i\|^2 \leq \int_\Omega (-\mathbf{w}_i \cdot \mathscr{L}\mathbf{w}_i + \alpha(\mathrm{div}_\eta \mathbf{w}_i)^2) dm. \tag{2.66}$$

Using (1.17) we get

$$\begin{aligned}
&- \int_\Omega \mathbf{w}_i \cdot \mathscr{L}\mathbf{w}_i dm = - \int_\Omega \mathbf{w}_i \cdot [(y_i - a_i)\mathscr{L}\mathbf{u}_1 + \mathbf{u}_1 \mathscr{L} y_i + 2T(\nabla y_i, \nabla \mathbf{u}_1)] dm \\
&= - \int_\Omega \mathbf{w}_i \cdot [(y_i - a_i)(-\sigma_1 \mathbf{u}_1 - \alpha \nabla \mathrm{div}_\eta \mathbf{u}_1) + \mathbf{u}_1 \mathscr{L} y_i + 2T(\nabla y_i, \nabla \mathbf{u}_1)] dm \\
&= \sigma_1 \|\mathbf{w}_i\|^2 + \alpha \int_\Omega (y_i - a_i)\mathbf{w}_i \cdot \nabla \mathrm{div}_\eta \mathbf{u}_1 dm - \int_\Omega \mathbf{w}_i \cdot (\mathbf{u}_1 \mathscr{L} y_i + 2T(\nabla y_i, \nabla \mathbf{u}_1)) dm.
\end{aligned} \tag{2.67}$$

Using (1.7) and (1.9), by a computation analogous to (2.35), we obtain

$$\begin{aligned}
\alpha \int_\Omega (y_i - a_i)\mathbf{w}_i \cdot \nabla \mathrm{div}_\eta \mathbf{u}_1 dm = & - \alpha \int_\Omega (\mathrm{div}_\eta \mathbf{w}_i)^2 dm \\
& - \alpha \int_\Omega (\nabla(\nabla y_i \cdot \mathbf{u}_1) + \mathrm{div}_\eta \mathbf{u}_1 \nabla y_i) \cdot \mathbf{w}_i dm.
\end{aligned}$$

Substituting the previous equality into (2.67), we get

$$\begin{aligned}
\int_\Omega (-\mathbf{w}_i \cdot \mathscr{L}\mathbf{w}_i + \alpha(\mathrm{div}_\eta \mathbf{w}_i)^2) dm = & \sigma_1 \|\mathbf{w}_i\|^2 - \alpha \int_\Omega (\nabla(\nabla y_i \cdot \mathbf{u}_1) + \mathrm{div}_\eta \mathbf{u}_1 \nabla y_i) \cdot \mathbf{w}_i dm \\
& - \int_\Omega \mathbf{w}_i \cdot (\mathbf{u}_1 \mathscr{L} y_i + 2T(\nabla y_i, \nabla \mathbf{u}_1)) dm.
\end{aligned} \tag{2.68}$$

Replacing (2.68) into (2.66), we have

$$(\sigma_{i+1}-\sigma_1)\|\mathbf{w}_i\|^2 \leq -\int_\Omega \mathbf{w}_i\cdot(\mathbf{u}_1\mathscr{L}y_i+2T(\nabla y_i,\nabla\mathbf{u}_1))dm \\ -\alpha\int_\Omega \mathbf{w}_i\cdot(\nabla(\nabla y_i\cdot\mathbf{u}_1)+\mathrm{div}_\eta\mathbf{u}_1\nabla y_i)dm. \tag{2.69}$$

By a straightforward computation, we have, from (1.7), (1.9), (1.10) and (2.65),

$$-\int_\Omega \mathbf{w}_i\cdot(\mathbf{u}_1\mathscr{L}y_i+2T(\nabla y_i,\nabla\mathbf{u}_1))dm=\int_\Omega |\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm. \tag{2.70}$$

$$-\alpha\int_\Omega \mathbf{w}_i\cdot(\nabla(\nabla y_i\cdot\mathbf{u}_1)+\mathrm{div}_\eta\mathbf{u}_1\nabla y_i)dm=\alpha\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2dm. \tag{2.71}$$

Therefore, substituting (2.70) and (2.71) into (2.69) we obtain

$$(\sigma_{i+1}-\sigma_1)\|\mathbf{w}_i\|^2\leq\int_\Omega|\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm+\alpha\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2dm. \tag{2.72}$$

From (2.70), for any constant $B>0$, we infer

$$\begin{aligned}(\sigma_{i+1}-\sigma_1)&\int_\Omega|\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm\\ &=(\sigma_{i+1}-\sigma_1)\Big\{-2\int_\Omega\mathbf{w}_i\cdot\Big(\frac{1}{2}\mathbf{u}_1\mathscr{L}y_i+T(\nabla y_i,\nabla\mathbf{u}_1)\Big)dm\Big\}\\ &\leq2(\sigma_{i+1}-\sigma_1)\|\mathbf{w}_i\|\Big\|\frac{1}{2}\mathbf{u}_1\mathscr{L}y_i+T(\nabla y_i,\nabla\mathbf{u}_1)\Big\|\\ &\leq B(\sigma_{i+1}-\sigma_1)^2\|\mathbf{w}_i\|^2+\frac{1}{B}\Big\|\frac{1}{2}\mathbf{u}_1\mathscr{L}y_i+T(\nabla y_i,\nabla\mathbf{u}_1)\Big\|^2,\end{aligned}$$

hence using (2.72) and the previous inequality we get

$$\begin{aligned}(\sigma_{i+1}-\sigma_1)&\int_\Omega|\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm\\ &\leq B(\sigma_{i+1}-\sigma_1)\Big(\int_\Omega|\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm+\alpha\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2dm\Big)\\ &\quad+\frac{1}{B}\Big\|\frac{1}{2}\mathbf{u}_1\mathscr{L}y_i+T(\nabla y_i,\nabla\mathbf{u}_1)\Big\|^2.\end{aligned} \tag{2.73}$$

Summing over i from 1 to n in (2.73), we conclude that

$$\begin{aligned}&\sum_{i=1}^n(\sigma_{i+1}-\sigma_1)(1-B)\int_\Omega|\mathbf{u}_1|^2T(\nabla y_i,\nabla y_i)dm\\ &\quad\leq B\alpha\sum_{i=1}^n(\sigma_{i+1}-\sigma_1)\int_\Omega|\nabla y_i\cdot\mathbf{u}_1|^2dm+\frac{1}{B}\sum_{i=1}^n\Big\|\frac{1}{2}\mathbf{u}_1\mathscr{L}y_i+T(\nabla y_i,\nabla\mathbf{u}_1)\Big\|^2.\end{aligned} \tag{2.74}$$

From the definition of y_i and the fact that S is an orthogonal matrix, we know that $\{y_i\}_{i=1}^n$ are also the coordinate functions in $\mathbb{R}^n$. Then, as in the proof of Theorem 2.1.1, we can also get

$$0 < \sum_{i=1}^{n} \left\| \frac{1}{2}\mathbf{u}_1 \mathscr{L} y_i + T(\nabla y_i, \nabla \mathbf{u}_1) \right\|^2 \leq \Big(\|T(\nabla \mathbf{u}_1)\| + \frac{1}{2}T_0 \Big)^2 + C_0,$$

where C_0 is given by Eq. (2.5). Using (2.74) and $\varepsilon I \leq T \leq \delta I$, we obtain

$$\sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1)(\varepsilon - B(\delta + \alpha)) \leq \frac{1}{B}\Big\{ (\|T(\nabla \mathbf{u}_1)\| + \frac{1}{2}T_0)^2 + C_0 \Big\}. \tag{2.75}$$

Let us consider

$$N_1 = (\|T(\nabla \mathbf{u}_1)\| + \frac{1}{2}T_0)^2 + C_0,$$

so that, from (2.75) we have

$$\varepsilon B \sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) - B^2(\delta + \alpha) \sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) \leq N_1. \tag{2.76}$$

Since B is an arbitrary positive constant, we can take

$$B = \left\{ \frac{N_1}{(\delta + \alpha)\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1)} \right\}^{\frac{1}{2}}$$

into (2.76) and therefore we get

$$\varepsilon \left\{ \frac{N_1}{(\delta + \alpha)\sum_{i=1}^{n}(\sigma_{i+1} - \sigma_1)} \right\}^{\frac{1}{2}} \sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) \leq 2N_1. \tag{2.77}$$

And if we square both sides of Inequality (2.77), we obtain

$$\sum_{i=1}^{n} (\sigma_{i+1} - \sigma_1) \leq \frac{4(\delta + \alpha)}{\varepsilon^2} N_1 \tag{2.78}$$

$$= \frac{4(\delta + \alpha)}{\varepsilon^2} \Big\{ (\|T(\nabla \mathbf{u}_1)\| + \frac{1}{2}T_0)^2 + C_0 \Big\}. \tag{2.79}$$

We can take $i = 1$ in Inequality (2.64) and replace in (2.78) to obtain Theorem 2.1.2. □

Concluding remarks and future work

In this thesis, we studied eigenvalue estimates of a second-order elliptic problem. This problem is a generalization of the Lamé system, see Chapter 2. We rely on known techniques to make our proofs, many of which required adaptations to tensor theory. We would like to mention that the system of second-order elliptic differential equations in (2.1) is uncoupled for $\alpha = 0$. In particular, when $T = I$ and η is constant, it becomes the Laplacian problem, in this case, we recover known estimates in the literature, see Remark 2.2.2.

We observe the influence of the potential function η on the eigenvalue estimates. This occurs by means of $C_0 = \sup_\Omega\{\frac{1}{2}\Delta\eta - \frac{1}{4}|\nabla\eta|^2\}$. As we mentioned in p. 31, C_0 has a natural geometric interpretation in some cases. We show how such an influence happens, e.g., we obtain a class of domains in Gaussian shrinking soliton in which our estimates do not depend on the potential function of this soliton, answering the question proposed in Question 2.2.1, see Corollary 2.2.5.

It is also interesting to observe the relation between operator $\mathscr{L}$ and Cheng-Yau operator $\square$ when the tensor T is divergence-free, it was done in Eq. 2.21. In fact, Eq. 2.21 says that operator $\mathscr{L}$ is a first-order perturbation of Cheng-Yau operator $\square$, because this we call it the drifted Cheng-Yau operator. Furthermore, in Subsection 2.2.2 we obtain results that involve the drifted Cheng-Yau operator, e.g., an estimate of the type of the first Yang inequality, which was not considered previously in the literature.

As future work, we plan to get eigenvalues estimates for second-order elliptic problems for bounded domains in a complete Riemannian manifold isometrically immersed in a Euclidean space, as well as eigenvalue estimates for fourth-order elliptic problems.

Bibliography

[1] 11 - Algebraic Inequalities, Editor(s): Alan Jeffrey, Daniel Zwillinger, I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products (Seventh Edition), Academic Press, 2007, Pages 1059-1061.

[2] M. C. Araújo Filho and J. N. Gomes, Estimates of eigenvalues of an elliptic differential systemin divergence form. arXiv e-prints, arXiv: 2011.13507, 2020.

[3] M. S. Ashbaugh and R. D. Benguria, More bounds on eigenvalue ratios for Dirichlet Laplacians in n dimensions, SIAM J. Math. Anal. 24 (6) (1993) 1622-1651.

[4] A. Barros, J. N. V. Gomes, Triviality of Compact m-Quasi-Einstein Manifolds, Results Math. 71 (2017), 241–250.

[5] É. Cartan. Familles de surfaces isoparametriques dans les espaces à courbure constante. Ann. Mat. Pura Appl. IV. Ser. 17 (1938) 177-191.

[6] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, New York, 1984.

[7] D. Chen, Q.M. Cheng, Q. Wang and C. Xia, On eigenvalues of a system of elliptic equations and of the biharmonic operator, J. Math. Anal. Appl. 387 (2012) 1146-1159.

[8] Q. M. Cheng, H.C. Yang, Bounds on eigenvalues of Dirichlet Laplacian, Math. Ann. 337 (2007) 159–175.

[9] Q. M. Cheng, H.C. Yang, Universal inequalities for eigenvalues of a system of elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A 139, no. 2 (2009) 273–285.

[10] Q. M. Cheng, H.C. Yang, Estimates for eigenvalues on Riemannian manifolds. J. Differential Equations 247 (2009) 2270–2281.

[11] S.Y. Cheng, S.T. Yau, Hypersurfaces with constant scalar curvature, Math. Ann. 225 (1977) 195-204.

[12] B. Chow et al., The Ricci Flow: Techniques and Applications Part I: Geometric Aspects. Mathematical Surveys and Monographs, vol. 135, AMS, Providence, RI, 2007.

[13] F. Du, A.C. Bezerra, Estimates for eigenvalues of a system of elliptic equations with drift and of bi-drifting laplacian, Comm. Pure Appl. Anal. 6 (2) (2017) 475-491.

[14] L. Evans, Partial differential equations. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp. ISBN: 0-8218-0772-2

[15] J. N. V. Gomes and J.F.R. Miranda, Eigenvalue estimates for a class of elliptic differential operators in divergence form, Nonlinear Anal. 176 (2018) 1-19.

[16] J. N. Gomes, Rigidez de superfícies de contato e caracterização de variedades riemannianas munidas de um campo conforme ou de alguma métrica especial, Tese de doutorado. Universidade Federal do Ceará, Fortaleza, (2012).

[17] D. S. Grebenkor, B.T. Naguyen, Geometrical structure of Laplacian eigenfunctions, SIAM Rev. 55 (4) (2013) 601–667.

[18] J. F. Grosjean, Extrinsic upper bounds for the first eigenvalue of elliptic operators. Hokkaido Math. J. 33(2) (2004) 319–339.

[19] G. H. Hardy, J.E. Littlewood, and G. Pólya, Inequalities (second edition), Cambridge University Press, Cambridge, 1952, MR 0944909 (89d:26016), Zbl 0634.26008.

[20] S. M. Hook, Domain independent upper bounds for eigenvalues of elliptic operator, Trans. Amer. Math. Soc. 318 (1990) 615–642.

[21] B. Kawohl, G. Sweers, Remarks on eigenvalues and eigenfunctions of a special elliptic system, Z. Angew. Math. Phys. 38 (1987) 730–740.

[22] Lee, J. M. Introduction to smooth manifolds, Springer-Verlag, New York, 2003.

[23] H. A. Levine, M.H. Protter, Unrestricted lower bounds for eigenvalues of elliptic equations and systems of equations with applications to problems in elasticity, Math. Methods Appl. Sci. 7 (1985) 210–222.

[24] M. Levitin, L. Parnovski, Commutators, spectral trance identities, and universal estimates for eigenvalues, J. Funct. Anal. 192 (2002) 425–445.

[25] L. Ma, B. Liu, Convexity of the first eigenfunction of the drifting Laplacian operator and its applications, New York J. Math. 14 (2008) 393-401.

[26] M. Obata, K. Yano, Conformal changes of Riemannian metrics. J. Di. Geom., v. 4, p. 53-72, 1970.

[27] P. J. Olver, Introduction to Partial Differential Equations, Undergraduate Texts in Mathematics, Springer, DOI 10.1007/978-3-319-02099-0.

[28] L. E. Payne, G. Pólya, H.F. Weinberger, On the ratio of consecutive eigenvalues. J. Math. Phys. 35 (1956) 289– 298.

[29] P. Petersen, Riemannian Geometry, Springer-Verlag, New York, 1998.

[30] A. Pleijel, Proprietés asymptotique des fonctions fondamentales du problems des vibrations dans un corps élastique, Ark. Mat. Astron. Fys. 26 (1939) 1-9.

[31] R. R. Mesquita, Fórmulas variacionais tipo Hadamard para os autovalores do η-laplaciano e aplicações, Tese de Doutorado. Universidade Federal do Amazonas, Manaus, (2014).

[32] J. F. R. Miranda, Uma nova forma aberta do princípio do máximo fraco e estimativas de autovalores para uma classe de operadores diferenciais elípticos, Tese (Doutorado em Matemática), Universidade Federal do Amazonas, Manaus, 2015.

[33] J. Roth, General Reilly-Type inequalities for submanifolds of weighted Euclidean spaces, Colloquium Mathematicae 144(1) (2016) 127-136.

[34] J. Roth, Reilly-type inequalities for Paneitz and Steklov eigenvalues. Potential Anal. 53 (3) (2020) 773–798.

[35] H. Rosenberg, Hypersurfaces of constant curvature in space forms. Bull. Sc. Math., v. 117, p. 217-239, 1993.

[36] D. Serre, Divergence-free positive symmetric tensors and fluid dynamics, Ann. I.H.Poincaré – AN35, 1209–1234 (2018)

[37] Q. M. Wang, Isoparametric functions on Riemannian manifolds I, Math. Ann. 277 (1987) 639-646.

[38] H. Weyl, Über die asymptotische Verteilung der Eigenwerte, Nachr. König. Gesellschaft. Wiss. Göttingen, Mathe-Phys. Klass (1911) 110-117.

[39] H. C. Yang, An estimate of the difference between consecutive eigenvalues, preprint IC/91/60 of ICTP, Trieste, 1991.

[40] K. Yano, Integral formulas in Riemannian geometry. New York: Marcel Dekker, 1970.