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Non-standard bifurcation approach to nonlinear elliptic problems

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Dedicated to Julián on his 60th *birthday, who, together with Prof. Manuel Delgado, contributed to the development of the elliptic PDEs in Seville and Belém*

Abstract. *Bifurcation is a very useful method to prove the existence of positive solutions for nonlinear elliptic equations. The existence of an unbounded continuum of positive solutions emanating from zero or from infinity can be deduced in many problems. In this paper, we show the applicability of this method in some problems where the classical bifurcation results can not be directly applied.*

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1. Introduction

Consider a nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u + b(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $g : \mathbb{R} \mapsto [0, \infty)$ is a continuous map, $b \in C(\overline{\Omega})$ and λ is a real parameter.

The bifurcation method is one of the most well-known tools in order to study (nonnegative and nontrivial) solutions of (1). In fact, the bifurcation method provides the existence of an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times C^1_0(\overline{\Omega})$ of solutions of (1) emanating from the trivial solution at $\lambda = \lambda_1$, where λ_1 stands for the principal eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions, under the condition

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0 \tag{H_0}$$

see for instance [25] and [19].

In a similar way, if g verifies

$$\lim_{s\to+\infty}\frac{g(s)}{s}=0 \tag{H_∞}$$

then an unbounded continuum $\mathcal{C}_\infty$ of solutions of (1) emanates from infinity at $\lambda=\lambda_1$, [26]. In both cases, the results are similar if the limits are finite and not necessarily zero, see [4]. We point out that when (H_0) and (H_∞) are both satisfied, $\mathcal{C}_0$ and $\mathcal{C}_\infty$ do not have necessarily to coincide, see for instance [6].

We assume now that g verifies only (H_0) and not (H_∞). Then, the global behaviour of the continuum $\mathcal{C}_0$ depends strongly on g and the sign of b. Let us summarize the main results in this case. For that, we need to introduce some notation. Define the sets

$$\begin{aligned} B_+ &:= \{x\in\Omega : b(x)>0\},\\ B_- &:= \{x\in\Omega : b(x)<0\},\\ B_0 &= \operatorname{int}(\Omega\setminus(\overline{B_+}\cup\overline{B_-})), \end{aligned}$$

for which we will assume for simplicity that they are regular sets and that B_0 is also connected.

Given a subdomain $D\subset\Omega$, we denote by λ_1^D the principal eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions. Moreover, given $(\lambda,u)\in\mathcal{C}_0$ we define $\mathrm{Proj}_{\mathbf{R}}(\lambda,u)=\lambda$.

Finally, assume that there exists $p>1$ such that

$$\lim_{s\to+\infty}\frac{g(s)}{s^p}=g_0>0. \tag{S_∞}$$

Hence, when g verifies only (H_0) and not (H_∞), the main results can be summarized as follows:

1. If $b(x)\le b_1<0$ for all $x\in\overline{\Omega}$ for some $b_1\in\mathbb{R}$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_0)=(\lambda_1,+\infty)$ and as consequence there exists at least a positive solution for $\lambda>\lambda_1$.

2. If $b\le 0$, $b\ne 0$ in Ω and $B_0\ne\emptyset$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_0)=(\lambda_1,\lambda_1^{B_0})$. In this case, a bifurcation to infinity appears at $\lambda=\lambda_1^{B_0}$. Moreover, there exists at least a positive solution for $\lambda\in(\lambda_1,\lambda_1^{B_0})$.

3. Assume that b changes sign, (S_∞) and that $p<p^*$, for some $p^*\le(N+2)/(N-2)$. Then, $(-\infty,\lambda_1)\subset\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_0)\subset(-\infty,\overline{\lambda})$, for some $\overline{\lambda}<\infty$. In this case, there exists at least a positive solution for $\lambda<\lambda_1$.

There is a large literature on the above problem. Let us focus on those papers that mainly use the bifurcation technique to get the results. Thanks to the a priori bounds and the non-existence of positive solutions for $\lambda\le\lambda_1$, the case

$b(x) \leq b_1 < 0$ is the simplest one. For the case $b < 0$ and $B_0 \neq \emptyset$ we refer to [21] as a general reference, see also [3, 14, 15, 16, 23] and the references therein. For the case b changing sign, see for instance [2, 8, 22].

A similar study could be done if g verifies (H_∞) and not (H_0). However, in this case, the behaviour of $\mathcal{C}_\infty$ is less known in general. Let us focus on the particular case $g(u) = u^q$, $0 < q < 1$. Hence, we have the following results:

1. If $b \leq b_1 < 0$ for all $x \in \Omega$ for some $b_1 \in \mathbb{R}$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_\infty) = (\lambda_1, +\infty)$.
2. If $b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$ for some $b_0 \in \mathbb{R}$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_\infty) = (-\infty, \lambda_1)$.

See for instance [7, 12, 13, 24, 26].

In this paper, our main goal is to study the set of nonnegative and nontrivial solutions of (1) when conditions (H_0) and (H_∞) are not fullfilled. For that, we are going to study the following specific equation

$$\begin{cases} -\Delta u = \lambda u + b(x)(u^q + u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$0 < q < 1 < p$$

although most of the results obtained here are also true for more general set of functions.

Problem (2) can be included in a more general problem

$$\begin{cases} -\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

for a and b verifying several structural assumptions. Problem (3) has been analyzed in [1] when $b(x) = \gamma \geq 0$ under homogeneous Neumann boundary conditions. In [10] the author studied (3) when b changes sign and some further conditions on a and b. The author proved the existence of two nonnegative and nontrivial solutions when $\lambda < \lambda^*$ for some $\lambda^* \in \mathbb{R}$. First, the sub-supersolution method is used to prove the existence of a solution, which is a local minimum of the associated functional. Finally, using mainly the mountain pass theorem the existence of the second solution is shown. The case $\lambda = 0$ and $a(x) = \gamma c(x)$, regarding now γ as a real parameter, has been studied for many authors from the pioneering work [5], see for instance [9, 11, 18] and references therein.

We will study (2) for different conditions on b using bifurcation methods. In the first results, we deal with the case b changing sign and b negative, respectively. In both cases, we can not apply directly the bifurcation method, but we can consider a truncated problem where the bifurcation method can be applied and then use a compactness method. Our main results can be stated as follows (see Figure 1):

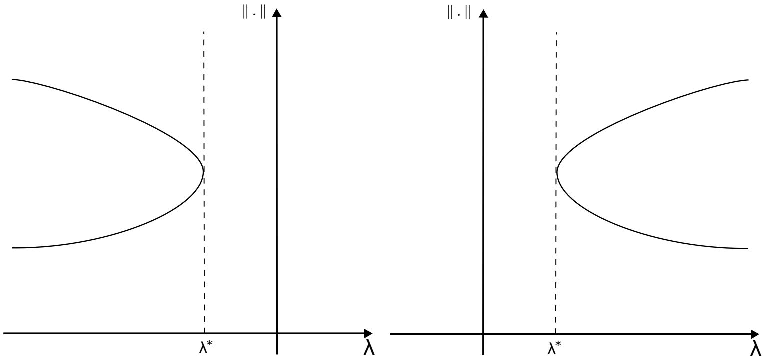

Figure 1: Minimal bifurcation diagrams of (2) in the cases b changing sign and b negative, respectively.

THEOREM 1.1. *Assume that* $0 < q < 1 < p$.

1. *Assume that b changes sign and that for x close ∂B_+,*

$$b^+(x) \approx [dist(x, \partial B_+)]^\gamma, \qquad \gamma \geq 0,$$

and

$$1 < p < \min\{(N+2)/(N-2), (N+1+\gamma)/(N-1)\}. \tag{4}$$

Then, there exists $\lambda^* \in \mathbb{R}$ such that for (2) possesses at least two non-negative and nontrivial solutions for $\lambda < \lambda^*$.

2. *Assume that $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$ and for some $b_1 \in \mathbb{R}$. Then, there exists $\lambda_* \in \mathbb{R}$ such that for (2) possesses at least two nonnegative and nontrivial solutions for $\lambda > \lambda_*$.*

Surprisingly, in the case $b \leq 0$ and $B_0 \neq \emptyset$, we obtain the existence of two continua bifurcating from the trivial solution and from infinity at the same point $\lambda = \lambda_1^{B_0}$. The main result is (see Figure 2):

THEOREM 1.2. *Assume that $b \leq 0$, $b \neq 0$ in Ω and $B_0 \neq \emptyset$. If $\lambda \leq \lambda_1$, (2) does not possess nonnegative and nontrivial solutions. Moreover:*

1. *From the trivial solution emanates at $\lambda = \lambda_1^{B_0}$ an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times L^\infty(\Omega)$ of nonnegative and nontrivial solutions of (2). Moreover, $\lambda_1^{B_0}$ is the unique bifurcation point from the trivial solution.*

2. *$\lambda = \lambda_1^{B_0}$ is a bifurcation point from infinity of nonnegative and nontrivial solutions, and it is the only one. Moreover, there exists an unbounded*

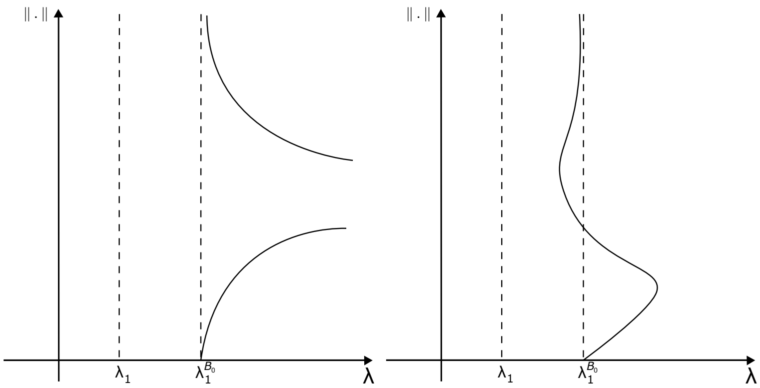

Figure 2: Minimal bifurcation diagrams of (2) when $b \leq 0$ and $B_0 \neq \emptyset$. In the first case both continua $\mathcal{C}_0$ and $\mathcal{C}_\infty$ are different; in the second one both coincide.

continuum $\mathcal{C}_\infty$ of nonnegative and nontrivial solutions of (2) such that

$$\mathcal{D}_\infty = \left\{ (\lambda, u) : u \neq 0, \left(\lambda, \frac{u}{\|u\|_\infty^2} \right) \in \mathcal{C}_\infty \right\} \cup \{(\lambda_1^{B_0}, 0)\}$$

is connected and unbounded.

In this case, we are not able to ascertain the global behaviour of these continua, mainly to the lack of the strong maximum principle in (2).

An outline of this work is as follows: Section 2 contains some properties of the principal eigenvalue of an elliptic problem. Section 3 is devoted to show the relative position between a family of supersolutions and a continuum of solutions of a nonlinear elliptic problem. In Section 3 we study in detail the truncated problems using the bifurcation method. In Sections 4 and 5 the main results are proved.

2. Eigenvalue problems

In this section we recall some useful properties of elliptic eigenvalue problems. Given a subdomain $D \subset \Omega$ we consider

$$\begin{cases} -\Delta u + m(x)u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{5}$$

where $m \in L^\infty(\Omega)$. The following result is well-known (see [20], where a detailed study of (5) and more general eigenvalue problems can be found)

LEMMA 2.1. *There exists a principal eigenvalue of (5), denoted by $\lambda_1^D(-\Delta+m)$. It is simple and isolated, and it is the only one whose eigenfunction associated can be chosen to be positive in D. If we denote by φ_1 a positive eigenfunction associated to $\lambda_1^D(-\Delta+m)$, then $\varphi_1 \in C^{1,\alpha}(\overline{D})$, $\alpha \in (0,1)$ and $\partial\varphi_1/\partial n < 0$ on ∂D where n is the outward unit vector normal to ∂D.*

Moreover, the following properties hold:

1. *Asume that m changes sing. Then $t \mapsto \lambda_1^D(-\Delta+tm)$ is continuous, concave and*
$$\lim_{t\to\pm\infty} \lambda_1^D(-\Delta+tm) = -\infty.$$
2. *Assume that $m(x) \le m_0 < 0$ for all $x \in D$. Then, $t \mapsto \lambda_1^D(-\Delta+tm)$ is continuous, decreasing and*
$$\lim_{t\to\pm\infty} \lambda_1^D(-\Delta+tm) = \mp\infty.$$

When $D = \Omega$, we omit the superscript and we denote $\lambda_1(-\Delta+m) = \lambda_1^\Omega(-\Delta+m)$. Moreover, when $m \equiv 0$ we simply write λ_1^D instead of $\lambda_1^D(-\Delta)$.

3. Relative position between a subcontinuum of solutions and a continuous family of supersolution

The main goal of this section is to generalize some results of [15]. Consider the general elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and locally Lipschitz in the second variable.

We define the positive cone in $C^1(\overline{\Omega})$

$$\mathcal{Q} := \{u \in C^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\},$$

whose interior and exterior are

$$\text{int}(\mathcal{Q}) = \{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\},$$

and

$$\text{ext}(\mathcal{Q}) = C^1(\overline{\Omega}) \setminus \mathcal{Q}.$$

We have the following result

Lemma 3.1. *Let $\overline{u} \in C^1(\overline{\Omega})$ be a supersolution of (6) with $\overline{u} > 0$ on $\partial\Omega$ and $u \in C_0^1(\overline{\Omega})$ a solution of (6). Then,*

$$\overline{u} - u \notin \partial\mathcal{Q}.$$

Proof. By contradiction assume that $\overline{u} - u \in \partial\mathcal{Q} = \overline{\mathcal{Q}} \setminus \mathrm{int}(\mathcal{Q})$. Then,

$$w(x) = \overline{u}(x) - u(x)$$

verifies that $w(x) \geq 0$ for all $x \in \Omega$ and $w(x_0) = 0$ for some $x_0 \in \Omega$. Observe that,

$$\begin{cases} -\Delta w + Mw \geq f(x, \overline{u}) + M\overline{u} - (f(x,u) + Mu) \geq 0 & \text{in } \Omega, \\ w > 0 & \text{on } \partial\Omega, \end{cases}$$

for some $M > 0$ large enough. The strong maximum principle asserts that $w(x) > 0$ for all $x \in \overline{\Omega}$. This is a contradiction. □

The main result of this section reads as follows:

Theorem 3.2. *Let $\mathcal{C}$ be a subcontinuum of solutions $\mathcal{C} \subset I \times C_0^1(\overline{\Omega})$ of (6), where $I \subset \mathbb{R}$ is a real interval. Let $U : I \mapsto C^1(\overline{\Omega})$ a continuous family of supersolutions of (6) with $U(\lambda) > 0$ on $\partial\Omega$. If for some $(\lambda_0, u_0) \in \mathcal{C}$, $u_0 \leq U(\lambda_0)$, then $u < U(\lambda)$ for all $(\lambda, u) \in \mathcal{C}$.*

Proof. Consider the continuous map $T : I \times C^1(\overline{\Omega}) \mapsto C^1(\overline{\Omega})$ given by

$$T(\lambda, u) := U(\lambda) - u. \tag{7}$$

Since T is continuous, then $T(\mathcal{C})$ is connected. By Lemma 3.1 we conclude that $T(\mathcal{C}) \cap \partial\mathcal{Q} = \emptyset$. Then, either $T(\mathcal{C})$ is completely inside $\mathrm{int}(\mathcal{Q})$ or completely outside. Since $T(\lambda_0, u_0) \in \mathrm{int}(\mathcal{Q})$, we deduce that $T(\mathcal{C}) \subset \mathrm{int}(\mathcal{Q})$. □

In fact, from the proof of Theorem 3.2, we obtain:

Corollary 3.3. *Let $\mathcal{C} \subset I \times C_0^1(\overline{\Omega})$ a subcontinuum of solutions of (6) and T the map defined in (7). Then, either*

1. *$T(\mathcal{C}) \subset \mathrm{int}(\mathcal{Q})$, and therefore, $u < U(\lambda)$, or*
2. *$T(\mathcal{C}) \subset \mathrm{ext}(\mathcal{Q})$.*

4. Study of the truncated problems

For $\delta > 0$ we define

$$f_\delta(s) := \begin{cases} \delta^{q-1}s & \text{if } s \in [0,\delta], \\ s^q & \text{if } s > \delta. \end{cases}$$

Let us consider now the truncated problem

$$\begin{cases} -\Delta u = \lambda u + b(x)(f_\delta(u) + u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

We point out that the nonlinear term is locally Lipschitz continuous in the second variable, and then by the strong maximum principle, any nonnegative and nontrivial solution of (8) is positive in all Ω.

4.1. b changes sign

First, we prove a non-existence result.

LEMMA 4.1. *Consider (λ, u) a positive solution of (8). Then*

$$\lambda \le \overline{\lambda} \qquad \textit{for some } \overline{\lambda} < \infty.$$

Moreover, if $B_0 \neq \emptyset$, then

$$\lambda \le \lambda_1^{B_0}.$$

Proof. Take a ball $B \subset B_+$ such that $b(x) \ge b_0 > 0$ for $x \in B$. Let φ_1^B be a positive eigenfunction associated to λ_1^B and consider

$$\varphi = \begin{cases} \varphi_1^B & \text{in } B, \\ 0 & \text{in } \Omega \setminus \overline{B}. \end{cases}$$

Since $\varphi \in H_0^1(\Omega)$, then on multiplying (8) by φ and using that $\partial\varphi_1^B/\partial n < 0$ on ∂B, we deduce that

$$0 \ge \int_B \left(\lambda - \lambda_1^B + b_0 \frac{f_\delta(u) + u^p}{u} \right) u\varphi_1^B,$$

which is a contradiction for λ large, for instance, for $\lambda \ge \lambda_1^B$.

Assume now that $B_0 \neq \emptyset$. Let $\varphi_1^{B_0}$ be a positive eigenfunction associated to $\lambda_1^{B_0}$ and consider

$$\varphi = \begin{cases} \varphi_1^{B_0} & \text{in } B_0, \\ 0 & \text{in } \Omega \setminus \overline{B_0}. \end{cases}$$

Now, we can follow the previous argument and conclude that

$$\lambda \le \lambda_1^{B_0}.$$

□

In the following theorem we show a priori bounds for the solutions of (8). In the first part, we obtain a priori bounds with respect to the parameter λ, and then, for a fix λ, with respect to δ. These results will be crucial in order to pass to the limit as $\delta \to 0$. For its proof, we will closely follow [2].

THEOREM 4.2. *Assume that for x close ∂B_+,*

$$b^+(x) \approx [dist(x, \partial B_+)]^\gamma, \qquad \gamma \geq 0,$$

and

$$1 < p < \min\{(N+2)/(N-2), (N+1+\gamma)/(N-1)\}. \tag{9}$$

1. *Then, for every bounded interval $\Lambda \subset \mathbb{R}$ there exists a positive constant M such that*

   $$\|u\|_\infty \leq M$$

   for any positive solution (λ, u) of (8), with $\lambda \in \Lambda$.

2. *Fix $\lambda \in \mathbb{R}$ and consider a sequence $\delta_n \to 0$. Denote by u_n a positive solution of (8). Then, there exists a positive constant $C > 0$ such that*

   $$\|u_n\|_\infty \leq C.$$

Proof. 1. This paragraph follows by Theorem 4.3 in [2].

2. In this case we can follow again the proof of Theorem 4.3 in [2], using a Gidas-Spruck argument [17] taking into account that

   $$f_\delta(u) \leq u^q.$$

□

We are ready to show the main result in this case (see Figure 3):

THEOREM 4.3. *Assume that b changes sign, $0 < q < 1 < p$ and p verifying (9). Then, there exists an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from $u \equiv 0$ at*

$$\lambda = \lambda_1(\delta) := \lambda_1(-\Delta - b(x)\delta^{q-1}).$$

For any $\delta < 1$.

1. *There exists $\lambda_1^+(\delta) < \lambda_1(\delta)$ such that (8) does not possess positive solution (λ, u) for $\lambda \leq \lambda_1^+(\delta)$ with $\|u\|_\infty \leq \delta$.*

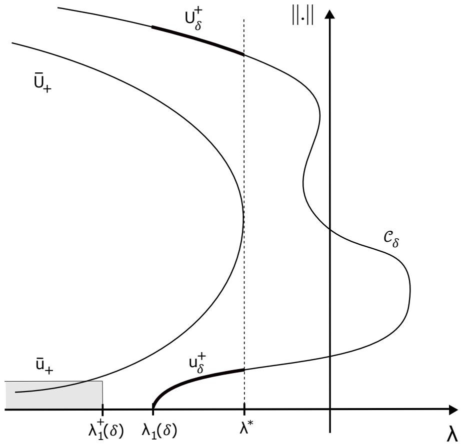

Figure 3: Minimal bifurcation diagram of (8) when b changes sign.

2. *There exist a real value $\lambda^* \in \mathbb{R}$ and two continuous families of supersolutions $\overline{u}_+, \overline{U}_+ : (-\infty, \lambda^*) \mapsto C^1(\overline{\Omega})$, all independent of δ, with $\overline{u}_+(\lambda) > 0, \overline{U}_+(\lambda) > 0$ on $\partial\Omega$. Moreover, $\overline{u}_+(\lambda) < \overline{U}_+(\lambda)$ for $\lambda < \lambda^*$ and $\overline{u}_+(\lambda^*) = \overline{U}_+(\lambda^*)$. Furthermore,*
$$\overline{u}_+(\lambda) \to 0 \quad \text{and} \quad \overline{U}_+(\lambda) \to +\infty \quad \text{in } L^\infty(\Omega) \text{ as } \lambda \to -\infty.$$
3. *For any $\lambda \in (\lambda_1(\delta), \lambda^*)$ there exist at least two solutions u_δ^+ and U_δ^+ of (8) with $(\lambda, u_\delta^+), (\lambda, U_\delta^+) \in \mathcal{C}_\delta$ such that*
$$\overline{u}_+(\lambda) - u_\delta^+ \in int(\mathcal{Q}) \quad \text{and} \quad \overline{U}_+(\lambda) - U_\delta^+ \in ext(\mathcal{Q}).$$

Proof. Since
$$\lim_{s\to 0^+} \frac{f_\delta(s)}{s} = \delta^{q-1},$$
it follows the existence of an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from the trivial solution at $\lambda = \lambda_1(-\Delta - b(x)\delta^{q-1}) = \lambda_1(\delta)$.

Thanks to Lemma 4.1 and the first paragraph of Theorem 4.2, we conclude the existence of $\lambda^+ \in \mathbb{R}$ such that
$$(-\infty, \lambda^+) \subset \mathrm{Proj}_{\mathbb{R}}(\mathcal{C}_\delta) \subset (-\infty, \overline{\lambda}). \tag{10}$$

For any $0 < \delta < 1$ consider a positive solution u of (8) such that $\|u\|_\infty \leq \delta < 1$. Observe that $u^p \leq u$ because $p > 1$. Then,
$$-\Delta u = \lambda u + b(x)(f_\delta(u) + u^p) = \lambda u + b(x)(\delta^{q-1}u + u^p) \leq u(\lambda + b_M(\delta^{q-1} + 1)),$$

where $b_M = \max_{x\in\overline{\Omega}} b(x)$, and hence

$$\lambda \geq \lambda_1(-\Delta - b_M(\delta^{q-1}+1)) = \lambda_1 - b_M(\delta^{q-1}+1).$$

It suffices to take

$$\lambda_1^+(\delta) := \lambda_1 - b_M(\delta^{q-1}+1).$$

We now build the families of supersolutions. Notice that $K>0$ is a supersolution of (8) if

$$0 \geq \lambda K + b(x)(f_\delta(K) + K^p).$$

Observe that

$$b(x)(f_\delta(K)+K^p) \leq b_M(\delta^{q-1}K\chi_{\{K\leq\delta\}}+K^q\chi_{\{K>\delta\}}+K^p) \leq b_M(\delta^q+K^q+K^p).$$

Using now that $\delta<1$, we have that K is supersolution of (8) if

$$b_M(K^{-1}+K^{q-1}+K^{p-1}) \leq -\lambda.$$

The function

$$h(x) := b_M(x^{-1}+x^{q-1}+x^{p-1})$$

attains a minimum at $x_{min}>0$, $h(x_{min}) = h_0 > 0$ and $h'(x)<0$ if $x<x_{min}$ while that $h'(x)>0$ if $x>x_{min}$. Then, taking $\lambda^* = -h_0$ for any $\lambda<\lambda^*$ there exist two positive constants K_i, $i=1,2$, such that $h(K_i) = -\lambda$, with $K_1<K_2$ and $K_1(\lambda)\to 0$ and $K_2(\lambda)\to+\infty$ as $\lambda\to-\infty$. Then, it suffices to take

$$\overline{u}_+(\lambda) = K_1(\lambda), \qquad \overline{U}_+(\lambda) = K_2(\lambda).$$

Now, we apply Theorem 3.2 with $I=(-\infty,\lambda*]$. By (10), the nonexistence of positive solutions with $\|u\|_\infty\leq\delta$ for $\lambda\leq\lambda_1^+(\delta)$, that $\mathcal{C}_\delta$ bifurcates at $\lambda=\lambda_1(\delta)$ and $\overline{u}_+(\lambda_1(\delta))>0$, it follows the existence of a positive solution u_δ^+ of (8) for any $\lambda\in(\lambda_1(\delta),\lambda^*]$ with $(\lambda,u_\delta^+)\in\mathcal{C}_\delta$ such that

$$\overline{u}_+(\lambda) - u_\delta^+ \in \operatorname{int}(\mathcal{Q}).$$

Moreover, we can conclude the existence of a positive solution of (8) for some $\lambda>\lambda^*$.

Now, we claim that there exists a subcontinuum $\mathcal{D}_\delta\subset\mathcal{C}_\delta$ such that

$$\overline{U}_+(\lambda) - U_\delta^+ \in \operatorname{ext}(\mathcal{Q}) \qquad (\lambda, U_\delta^+)\in\mathcal{D}_\delta, \quad \lambda\in(-\infty,\lambda^*]. \tag{11}$$

It is already known the existence of positive solutions $(\lambda,u)\in\mathcal{C}_\delta$ of (8) for all $\lambda\in(-\infty,\lambda^*]$. Moreover, it is not posible that $\overline{u}_+(\lambda)-u\in\operatorname{int}(\mathcal{Q})$ for all $(\lambda,u)\in\mathcal{C}_\delta$. Hence, there exists $(\lambda_0,u_0)\in\mathcal{C}_\delta$ such that $\overline{u}_+(\lambda_0)-u_0\in\operatorname{ext}(\mathcal{Q})$. Thus, from Corollary 3.3, there exists a subcontinuum $\mathcal{D}_\delta$ such that $\overline{u}_+(\lambda)-u\in\operatorname{ext}(\mathcal{Q})$ for all $(\lambda,u)\in\mathcal{D}_\delta$. Again, by Corollary 3.3, this subcontinuum has two possibilities, either

1. $\overline{U}_+(\lambda) - u \in \mathrm{ext}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, or
2. $\overline{U}_+(\lambda) - u \in \mathrm{int}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$.

We show that the second possibility is not possible, proving the claim (11). Indeed, if $\overline{U}_+(\lambda) - u \in \mathrm{int}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, since $\overline{u}_+(\lambda) - u \in \mathrm{ext}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, then we have for $\lambda = \lambda^*$ that

$$\overline{U}_+(\lambda^*) - u_{\lambda^*} \in \mathrm{int}(\mathcal{Q}), \qquad \overline{u}_+(\lambda^*) - u_{\lambda^*} \in \mathrm{ext}(\mathcal{Q}),$$

which is impossible because $\overline{U}_+(\lambda^*) = \overline{u}_+(\lambda^*)$. This completes the proof. □

4.2. $b(x) \le b_1 < 0$ for all $x \in \overline{\Omega}$

First, we show a necessary condition on λ for the existence of positive solution of (8).

LEMMA 4.4. *Assume $b(x) \le b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$ and consider (λ, u) a positive solution of (8). Then,*

$$\lambda \ge \lambda_1.$$

Proof. In this case, we have that $-\Delta u \le \lambda u$ in Ω, whence we deduce the result. □

With respect to the a priori bounds, we have:

LEMMA 4.5. *Assume $b(x) \le b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$ and consider (λ, u) a positive solution of (8). Then, there exists $C(\lambda) > 0$ such that*

$$\|u\|_\infty \le \max\{\delta, C(\lambda)\}. \tag{12}$$

Proof. Let $x_M \in \Omega$ be such that $u_M = u(x_M) = \max_{x \in \overline{\Omega}} u(x)$. Assume that $u_M > \delta$. Then,

$$\lambda u_M + b(x_M)(u_M^q + u_M^p) \ge 0,$$

and hence

$$-b_L(u_M^{q-1} + u_M^{p-1}) \le \lambda,$$

where $b_L = \min_{x \in \overline{\Omega}} b(x)$. This finishes the result. □

Our main result is the following (see Figure 4):

THEOREM 4.6. *Assume that $b(x) \le b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$. Then, there exists an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from $u \equiv 0$ at*

$$\lambda = \lambda_1(\delta) := \lambda_1(-\Delta - b(x)\delta^{q-1}).$$

Moreover, there exists δ_0 such that for $0 < \delta < \delta_0$, we have:

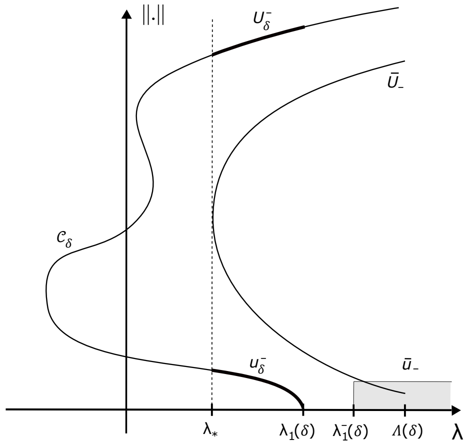

Figure 4: Minimal bifurcation diagram of (8) when b is negative.

1. *The existence of $\lambda_1^-(\delta) > \lambda_1(\delta)$ such that (8) does not possess positive solution (λ, u) for $\lambda \geq \lambda_1^-(\delta)$ with $\|u\|_\infty \leq \delta$.*
2. *There exist a real value $\lambda_* \in \mathbb{R}$, independent of δ, and two continuous families of supersolutions $\overline{u}_-, \overline{U}_- : [\lambda_*, \Lambda(\delta)] \mapsto C^1(\overline{\Omega})$ with $\overline{u}_-, \overline{U}_- > 0$ on $\partial\Omega$, where $\Lambda(\delta) = \delta^{q-1} + \delta^{p-1}$. Such families satisfy*

   $$\overline{u}_-(\lambda) < \overline{U}_-(\lambda) \quad \textit{for } \lambda \in [\lambda_*, \Lambda(\delta)] \textit{ and } \overline{u}_-(\lambda_*) = \overline{U}_-(\lambda_*), \overline{u}_-(\Lambda(\delta)) = \delta.$$

   Furthermore, $\overline{U}_-(\Lambda(\delta)) \to +\infty$ and $\overline{u}_-(\Lambda(\delta)) \to 0$ in $L^\infty(\Omega)$ as $\delta \to 0$.
3. *For $\lambda \in (\lambda_*, \lambda_1(\delta))$ there exist at least two solutions u_δ^- and U_δ^- of (8) such that*

   $$\overline{u}_-(\lambda) - u_\delta^- \in int(\mathcal{Q}) \quad \textit{and} \quad \overline{U_\delta^-} - U_\delta^- \in ext(\mathcal{Q}).$$

Proof. The proof is rather similar to the one of Theorem 4.3. We point out only the main differences.

Assume that $\|u\|_\infty \leq \delta$, then

$$-\Delta u = \lambda u + b(x)(\delta^{q-1}u + u^p) \geq \lambda u + b(x)(\delta^{q-1} + 1)u.$$

Therefore

$$\lambda \leq \lambda_1^-(\delta) := \lambda_1(-\Delta - b(x)(\delta^{q-1} + 1)).$$

Taking $K > 0$, we have that K is a supersolution of (8) provided that

$$h_\delta(K) := (\delta^{q-1}\chi_{\{K\leq\delta\}} + K^{q-1}\chi_{\{K>\delta\}} + K^{p-1}) \geq \frac{\lambda}{-b_M}.$$

Observe that $h_\delta(K)$ can be rewritten as

$$h_\delta(K) = \begin{cases} \delta^{q-1} + K^{p-1} & \text{if } K \le \delta, \\ K^{q-1} + K^{p-1} & \text{if } K > \delta. \end{cases}$$

A detailed study of $h_\delta(K)$ leads to the result. Indeed, since the infimum of the map $x \mapsto h(x) := x^{q-1} + x^{p-1}$ is attained in $x_{min} = ((p-1)/(1-q))^{1/(p-q)}$ and its value is $h(x_{min}) = h_0 > 0$, then, for δ small, x_{min} is also the minimum of $h_\delta(K)$. Then, for δ small, we have that the function h_δ has the following properties:

1. $x \in [0, \delta] \mapsto h_\delta(x) \in [\delta^{q-1}, \delta^{q-1} + \delta^{p-1}]$ is increasing.
2. $x \in [\delta, x_{min}] \mapsto h_\delta(x) \in [h_0, \delta^{q-1} + \delta^{p-1}]$ is decreasing.
3. $x \in [x_{min}, +\infty) \mapsto h_\delta(x) \in [h_0, +\infty)$ is increasing.

Hence, taking $\Lambda(\delta) = \delta^{q-1} + \delta^{p-1}$, for

$$\frac{\lambda}{-b_M} \in [h_0, \Lambda(\delta)],$$

there exist $K_1(\lambda) < K_2(\lambda)$ such that $h_\delta(K_i(\lambda)) = \frac{\lambda}{-b_M}$ with $\delta < K_1(\lambda) < K_2(\lambda)$. In fact, observe that in this region, $h_\delta(x) = x^{q-1} + x^{p-1}$, and therefore $K_i(\lambda)$ does not depend on δ. Moreover,

$$K_1(\lambda) \to \delta \quad \text{as } -\lambda/b_M \to \Lambda(\delta).$$

□

5. Proof Theorem 1.1

1. Let us fix $\lambda < \lambda^*$. By Lemma 2.1, $\lambda_1(\delta) \to -\infty$ as $\delta \to 0$. Hence, there exists δ_0 such that for $\delta \le \delta_0$ we have that $\lambda_1(\delta) < \lambda$. Then, $\lambda \in (\lambda_1(\delta), \lambda^*))$ and by Theorem 4.3 there exist two positive solutions, $u_\delta^+ < U_\delta^+$ of (8) for $\delta \le \delta_0$.

   On the other hand, thanks to the a priori bound given by the second paragraph of Theorem 4.2, we get that $\|U_\delta^+\|_\infty \le M$ for a constant M that does not depend on δ. Observe that

   $$f_\delta(U_\delta^+) \le (U_\delta^+)^q,$$

   and then $\{U_\delta^+\}$ is bounded in $W^{2,r}(\Omega)$ for any $r > 1$. Hence, we can pass to the limit and conclude that $U_\delta^+ \to U_0^+$ in $C^1(\overline{\Omega})$ as $\delta \to 0$, with U_0^+

a nonnegative solution of (2). Moreover, since $\overline{U}_+(\lambda) - U_\delta^+ \in \mathrm{ext}(\mathcal{Q})$ for all $\delta \leq \delta_0$, it follows the existence of $x_0 \in \Omega$ such that

$$U_0^+(x_0) \geq \overline{U}_+(\lambda)(x_0) > 0. \tag{13}$$

Hence, U_0^+ is a nonnegative and nontrivial solution of (2).

On the other hand, since $u_\delta^+ < \overline{u}_+(\lambda)$ we can conclude that $u_\delta^+ \to u_0^+ \geq 0$ in $C^1(\overline{\Omega})$ as $\delta \to 0$. We will prove that $u_0^+ \neq 0$. Assume by contradiction that $u_\delta^+ \to 0$ in $C^1(\overline{\Omega})$. Take a ball $B \subset B_+$ such that $b(x) \geq b_0 > 0$ in $\overline{B}$. Since λ is fixed, let us take M large enough such that

$$\lambda_1^B - \lambda \leq b_0 M.$$

For this M, let us take δ small such that $u_\delta^q \geq M u_\delta$ and

$$\lambda_1^B - \lambda \leq b_0 \min\{\delta^{q-1}, M\}.$$

On multiplying (8) by φ_1^B and integrating in B, we obtain

$$-\int_B \Delta u_\delta^+ \varphi_1^B = \lambda \int_B u_\delta^+ \varphi_1^B + \int_B b(x)(f_\delta(u_\delta^+) + (u_\delta^+)^p)\varphi_1^B.$$

Then,

$$\begin{aligned}\lambda_1^B \int_B u_\delta^+ \varphi_1^B + \int_{\partial B} \partial\varphi_1^B/\partial n u_\delta^+ > \lambda \int_B u_\delta^+ \varphi_1^B \\ + b_0 \int_B \big(\delta^{q-1} u_\delta^+ \chi_{\{u_\delta^+ \leq \delta\}} + M u_\delta^+ \chi_{\{u_\delta^+ > \delta\}}\big)\varphi_1^B.\end{aligned}$$

Using that $\partial\varphi_1^B/\partial n < 0$ on $\partial\Omega$, we conclude that

$$\lambda_1^B > \lambda + b_0 \min\{\delta^{q-1}, M\},$$

a contradiction. Hence, u_0^+ is a nontrivial and nonnegative solution of (2). Moreover, since

$$u_0^+ \leq \overline{u}_+(\lambda) < \overline{U}_+(\lambda),$$

and (13), it follows that $u_0^+ \neq U_0^+$. Thus, there exist at least two positive solutions of (2).

2. Assume that $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$. Let us fix $\lambda > \lambda_*$. Let us take $\delta_0 > 0$ small such that

$$\lambda < \min\{\lambda_1(\delta), \Lambda(\delta)\} \quad \text{for any } \delta \leq \delta_0.$$

Observe that this is possible thanks to the expression of $\Lambda(\delta)$ and Lemma 2.1.

Then, by Theorem 4.6 there exist two positive solutions $u_\delta^- < U_\delta^-$ of (8). With a similar argument to the one used in the first paragraph, we can show that $U_\delta^- \to U_0^-$ in $C^1(\overline{\Omega})$ as $\delta \to 0$, where U_0^- is a nonnegative solution of (2) and $U_0^- \neq 0$ in Ω.

On the other hand, we have that $u_\delta^- \to u_0^- \geq 0$ in $C^1(\overline{\Omega})$ as $\delta \to 0$ and $u_0^- \neq 0$. Indeed, arguing by contradiction, assume that $u_\delta^- \to 0$ in $C^1(\overline{\Omega})$. Then, for $M > 0$ we have that for $0 < \delta$ close to zero that $(u_\delta^-)^q \geq M u_\delta^-$. Hence,

$$\begin{aligned} -\Delta u_\delta^- &\leq \lambda u_\delta^- + b_M \big(\delta^{q-1} u_\delta^- \chi_{\{u_\delta^- \leq \delta\}} + (u_\delta^-)^q \chi_{\{u_\delta^- > \delta\}}\big) \\ &= (\lambda + b_M \min\{\delta^{q-1}, M\}) u_\delta^-, \end{aligned}$$

whence

$$\lambda_1 \leq \lambda + b_M \min\{\delta^{q-1}, M\},$$

again a contradiction for M large and δ sufficiently close to zero.

6. The case with bifurcation

Finally, we deal with the case $b \leq 0$, $b \neq 0$ in Ω and $B_0 \neq \emptyset$. For that, we will prove directly that from the trivial solution and from infinity emanate unbounded continua of nonnegative and nontrivial solutions of (2).

We will use the Leray-Schauder degree of K_λ in $B_\rho := \{u \in C(\overline{\Omega}) : \|u\|_\infty < \rho\}$, with respect to zero, denoted by $\deg(K_\lambda, B_\rho)$. The isolated index of u of K_λ is denoted by $i(K_\lambda, u)$. Let us define the map

$$K_\lambda : C_0(\overline{\Omega}) \to C_0(\overline{\Omega}); \qquad K_\lambda(u) := u - T(\lambda, u)$$

where

$$T(\lambda, u) := (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p)),$$

$u^+ := \max\{u, 0\}$, $C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \quad \text{on } \partial\Omega\}$ and $(-\Delta)^{-1}$ denotes the inverse of the laplacian-operator under homogeneous Dirichlet boundary conditions.

It is easy to show that u is nonnegative solution of (2) if and only if u is zero of the map K_λ. Moreover, by the standard regularization properties of T, T is a compact operator on $C_0(\overline{\Omega})$.

6.1. Bifurcation from zero

LEMMA 6.1. *If $\lambda < \lambda_1^{B_0}$, then $i(K_\lambda, 0) = 1$.*

Proof. Define the map $\mathcal{H}_1 : [0,1] \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ by

$$\mathcal{H}_1(t,u) = (-\Delta)^{-1}(t(\lambda u^+ + b(x)((u^+)^q + (u^+)^p))).$$

We show now that $\mathcal{H}_1$ is an admisible homotopy, for which it is sufficient to show that there exists $\gamma > 0$ such that

$$u \neq \mathcal{H}_1(t,u) \qquad \forall u \in \overline{B}_\gamma, u \neq 0 \text{ and } t \in [0,1].$$

Assume that there exist $u_n \in C_0(\overline{\Omega})\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$, such that

$$u_n = \mathcal{H}_1(t_n, u_n).$$

This is,

$$-\Delta u_n = t_n(\lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p)) \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

On multiplying the above equality by $u_n^- := \min\{u_n, 0\}$ and integrating in Ω, we infer that $u_n \geq 0$ in Ω.

Let us define

$$z_n = \frac{u_n}{\|u_n\|_2}.$$

Then, z_n verifies

$$-\Delta z_n = t_n(\lambda z_n + b(x)(\|u_n\|_2^{q-1} z_n^q + \|u_n\|_2^{p-1} z_n^p)) \text{ in } \Omega, \quad z_n = 0 \text{ on } \partial\Omega. \quad (14)$$

Since $b \leq 0$, on multiplying the above equality by z_n and integrating in Ω, we obtain that

$$\|z_n\|_{H_0^1} \leq C \qquad \text{for some } C > 0,$$

and hence, up a subsequence,

$$\begin{array}{ll} z_n \rightharpoonup z & \text{in } H_0^1(\Omega), \\ z_n \to z & \text{in } L^2(\Omega), \end{array}$$

for some $z \in H_0^1(\Omega)$, $z \geq 0$ and $\|z\|_2 = 1$.

Next, we show that

$$t_n \|u_n\|_2^{q-1} \to \infty. \quad (15)$$

Assume that for a subsequence $t_n\|u_n\|_2^{q-1} \to r^* \in [0,\infty)$. In such case, since $\|u_n\|_2 \to 0$ and $q < 1$ we obtain that $t_n \to 0$. Then, passing to the limit in (14), we obtain that

$$-\Delta z = r^* b(x) z^q \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega,$$

whence we deduce that $z = 0$, a contradiction.

We have that $z \equiv 0$ in $\Omega \setminus B_0$. Indeed, assume that $z(x) > 0$ in $D \subset \Omega \setminus B_0$. Take $\varphi \in C_c^\infty(D)$, then

$$-\int_D z_n \Delta\varphi = t_n \left(\lambda \int_D z_n \varphi + \|u_n\|_2^{q-1} \int_D b(x) z_n^q \varphi + \|u_n\|_2^{p-1} \int_D b(x) \varphi z_n^p \right).$$

Since $z_n \to z$ in $L^2(\Omega)$, we deduce that

$$\int_D b(x) z_n^q \varphi \to \int_D b(x) z^q \varphi < 0,$$

whence using (15)

$$-\int_D z_n \Delta\varphi \to -\infty,$$

a contradiction.

For any $\varphi \in H_0^1(B_0)$, prolongating this function by zero, and passing to the limit in (14), we get that

$$\int_{B_0} \nabla z \cdot \nabla\varphi = t^* \lambda \int_{B_0} z\varphi,$$

where $t_n \to t^* \in [0,1]$, and then

$$t^* \lambda = \lambda_1^{B_0}.$$

Hence, $\lambda \geq \lambda_1^{B_0}$, a contradiction.

Take $\epsilon \in (0, \delta]$, we have

$$\begin{aligned} i(K_\lambda, 0) &= \deg(K_\lambda, B_\epsilon) = \deg(I - \mathcal{H}_1(1,\cdot), B_\epsilon) \\ &= \deg(I - \mathcal{H}_1(0,\cdot), B_\epsilon) = \deg(I, B_\epsilon) = 1, \end{aligned}$$

where I denotes the identity map. The proof is complete.

□

LEMMA 6.2. *If $\lambda > \lambda_1^{B_0}$, then $i(K_\lambda, 0) = 0$.*

Proof. Let us take a positive and regular function $\varphi > 0$ in Ω. Let us define the map $\mathcal{H}_2 : [0,1] \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ by

$$\mathcal{H}_2(t,u) = (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p) + t\varphi).$$

We show now that $\mathcal{H}_2$ is an admisible homotopy, for which it is sufficient to prove that there exists $\gamma > 0$ such that

$$u \neq \mathcal{H}_2(t,u) \quad \forall u \in \overline{B}_\gamma, u \neq 0 \text{ and } t \in [0,1].$$

Assume that there exist $u_n \in C_0(\overline{\Omega})\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$, such that

$$u_n = \mathcal{H}_2(t_n, u_n).$$

This is,

$$-\Delta u_n = \lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p) + t_n\varphi \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Again, it can be shown that $u_n \geq 0$ in Ω. Let us define

$$z_n = \frac{u_n}{\|u_n\|_2}.$$

Hence, z_n verifies that

$$-\Delta z_n = \lambda z_n + b(x)(\|u_n\|_2^{q-1} z_n^q + \|u_n\|_2^{p-1} z_n^p) + \frac{t_n}{\|u_n\|_2}\varphi \quad \text{in } \Omega, \quad z_n = 0 \quad \text{on } \partial\Omega. \tag{16}$$

Now, on multiplying (16) by $\psi \in C_c^\infty(B_0)$, the formula of integration by parts gives

$$\frac{t_n}{\|u_n\|_2}\int_{B_0} \varphi\psi = -\lambda\int_{B_0} z_n\psi - \int_{B_0} z_n\Delta\psi.$$

Since $\|z_n\|_2 = 1$ it follows that

$$\frac{t_n}{\|u_n\|_2} \leq C \tag{17}$$

and then, for a subsequence, $t_n/\|u_n\|_2 \to t^* \geq 0$.

Since $\|z_n\|_2 = 1$ and $b \leq 0$, it follows from (16) and (17) that

$$\|z_n\|_{H_0^1} \leq C.$$

Arguing as in Lemma 6.1 we deduce that $z = 0$ in $\Omega \setminus B_0$. Moreover, passing to the limit in B_0 we conclude that

$$-\Delta z = \lambda z + t^*\varphi \quad \text{in } B_0, \quad z = 0 \quad \text{on } \partial B_0.$$

Since $t^* \geq 0$, we get that $\lambda \leq \lambda_1^{B_0}$ and a contradiction arises immediately.

Take $\epsilon \in (0, \gamma]$, we have that

$$\begin{aligned} i(K_\lambda, 0) &= \deg(K_\lambda, B_\epsilon) = \deg(I - \mathcal{H}_2(0,\cdot), B_\epsilon) \\ &= \deg(I - \mathcal{H}_2(1,\cdot), B_\epsilon) = 0. \end{aligned}$$

This last equality holds because we have proved that the equation

$$-\Delta u = \lambda u + b(x)(u^q + u^p) + \varphi$$

has not solution in $\overline{B}_\epsilon$.

□

6.2. Bifurcation from infinity

LEMMA 6.3. *Assume that* $\lambda < \lambda_1^{B_0}$. *Then, there exists* $R > 0$ *such that for any* $u \in C_0(\overline{\Omega})$ *with* $\|u\|_\infty \geq R$ *and for any* $t \in [0,1]$,

$$u \neq (-\Delta)^{-1}(t(\lambda u^+ + b(x)((u^+)^q + (u^+)^p))).$$

Proof. Assume by contradiction that there exist two sequences $\|u_n\|_\infty \to \infty$ and $t_n \in [0,1]$ such that

$$-\Delta u_n = t_n(\lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p)) \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Using elliptic regularity results, it is not hard to show that $\|u_n\|_2 \to \infty$. Now, the proof follows exactly as in Lemma 6.1, arguing now with $t_n\|u_n\|_\infty^{p-1}$ instead of $t_n\|u_n\|_\infty^{q-1}$. □

LEMMA 6.4. *Assume that* $\lambda > \lambda_1^{B_0}$ *and let* $\varphi \in C_0^1(\overline{\Omega})$, $\varphi > 0$ *in* Ω. *Then, there exists* $R > 0$ *such that for any* $u \in C_0(\overline{\Omega})$ *with* $\|u\|_\infty \geq R$ *and for any* $t \in [0,1]$,

$$u \neq (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p) + t\varphi).$$

Proof. In this case, the proof is rather similar to the proof of Lemma 6.2. □

Proof of Theorem 1.2. From Lemmas 6.1 and Lemma 6.2, it follows the existence of a continuum $\mathcal{C}_0$ of nonnegative and nontrivial solution of (2) emanating from the trivial solution at $\lambda = \lambda_1^{B_0}$. Moreover, it can be shown that this is the unique point of bifurcation form zero, and hence we can conclude that $\mathcal{C}_0$ is unbounded.

For the existence of $\mathcal{C}_\infty$ we perform the change of variable $z = u/\|u\|_\infty^2$ ($u \neq 0$). See, for instance [26] and [6]. Now, thanks to Lemmas 6.3 and 6.4, the existence of $\mathcal{C}_\infty$ can be deduced. We omit the details. □

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