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por

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Abstract

In this thesis we study the non-abelian q-tensor square, denoted by $G \otimes^q G$, of a group G and the group $\nu^q(G)$, a certain extension of $G \otimes^q G$ by $G \times G$, where q is a non-negative integer. Our interest is to understand the behaviour of these groups and other relevant sections of $\nu^q(G)$ under the assumption that G belongs to certain families of finite p-groups, where p is a prime number. We prove that if G is a powerful or potent p-group, then all terms of the lower central series and of the derived series of $\nu^q(G)$ are also powerful or potent, respectively, with the only exception of the whole group itself. For the q-tensor square we also get these results under the additional assumption that $\exp(G)$ divides q. Moreover, we establish some bounds for the exponent of these group constructions in each case, based on the exponent of the group argument G. For instance, if G is a powerful p-group we prove that $\exp(G \otimes^q G)$ divides $\exp(G)$ if p is odd or if $p = 2$ and either q is odd or 4 divides q, and $\exp(G \otimes^q G)$ divides $2\exp(G)$ if $p = 2$ and 4 does not divide q. In the potent's family we give a bound for the $\exp(\nu^q(G))$ in terms of the $\exp(G)$. In particular, we find an upper bound for $\exp(\nu^q(G))$ in terms of $\exp(\nu^q(G/N))$ and $\exp(N)$ when G admits some specific normal subgroup N. Our results extend some existing bounds found in the literature for the particular case $q = 0$.

Key-Words and Phrases: Finite p-groups; q-tensor square of groups; potent and powerful p-groups.

Resumo

Nesta tese estudamos o quadrado q-tensorial não-abeliano, denotado por $G \otimes^q G$, de um grupo G e o grupo $\nu^q(G)$, uma certa extensão de $G \otimes^q G$ por $G \times G$, sendo q um inteiro não negativo. Nosso interesse é entender o comportamento desses grupos e outras relevantes seções de $\nu^q(G)$ sob a suposição de que G pertence a certas famílias de p-grupos finitos, onde p é um número primo. Provamos que se G é um p-grupo *powerful* ou *potent*, então todos os termos da série central inferior e da série derivada de $\nu^q(G)$ são também *powerful* ou *potent*, respectivamente, com a única exceção o próprio grupo todo. Para o quadrado q-tensorial também obtemos esses resultados sob a suposição adicional de que $\exp(G)$ divide q. Além disso, estabelecemos algumas cotas para o expoente dessas construções de grupos em cada caso com base no expoente do grupo argumento G. Por exemplo, se G é um p-grupo *powerful*, provamos que $\exp(G \otimes^q G)$ divide $\exp(G)$, se p é ímpar ou se $p = 2$ e q é ímpar ou 4 divide q, e $\exp(G \otimes^q G)$ divide $2\exp(G)$, se $p = 2$ e 4 não divide q. Na família dos *potents*, damos uma cota para o $\exp(\nu^q(G))$ em termos do $\exp(G)$. Em particular, encontramos uma cota superior para o $\exp(\nu^q(G))$ em termos de $\exp(\nu^q(G/N))$ e de $\exp(N)$ quando G admite algum subgrupo normal específico N. Nossos resultados estendem algumas cotas existentes para o caso particular $q = 0$ encontradas na literatura.

Palavras-chave: p-grupos finitos; quadrado q-tensorial de grupos; p-grupos *potent* e *powerful*.

Notation

$\lceil r \rceil$	The smallest integer greater than or equal to r.
$\lfloor r \rfloor$	The greatest integer less than or equal to r.
x^y	$y^{-1}xy$.
$[x, y]$	$x^{-1}y^{-1}xy$.
$[x_1, \ldots, x_n]$	$[[x_1, \ldots, x_{n-1}], x_n]$.
$H \leq G$	H is a subgroup of the group G.
$\|G\|$	Order of the group G.
$\langle X \rangle$	Subgroup generated by the set X.
$[H_1, H_2]$	$\langle [x, y] \mid x \in H_1, y \in H_2 \rangle$.
$[H_1, \ldots, H_n]$	$[[H_1, \ldots, H_{n-1}], H_n]$.
$[H_{1,k} H_2]$	$[H_1, H_2, \ldots, H_2]$, H_2 appears k times.
$d(G)$	Minimum number of generators of the group G.
$\exp(G)$	Exponent of the group G.
$N_G(H)$	Normalizer of the subgroup H in the group G.
$Z(G)$	Center of the group G.
$\|G : H\|$	Index of the subgroup H in the group G.
$N \trianglelefteq G$	N is a normal subgroup of the group G.
G/N	Quotient group of the group G by the normal subgroup N.
$\Phi(G)$	Frattini subgroup of the group G.
$\gamma_n(G)$	n-th term of the lower central series of the group G.
$[G, G] = G'$	Derived subgroup of the group G.

$G^{(n)}$	n-th term of the derived series of the group G.
G^{p^i}	Subgroup generated by the set of the p^i-th powers of the elements of the group G.
$G^{\{p^i\}}$	Set of the p^i-th powers of the elements of the group G.
$\Omega_i(G)$	Subgroup generated by the elements of the group G of order less than or equal to p^i.
$\Omega_{\{i\}}(G)$	Set of the elements of the group G of order less than or equal to p^i.
$G \cong H$	Isomophism between the groups G and H.
$G \times H$	Direct product of the groups G and H.
$G * H$	Free product of the groups G and H.
$G \otimes H$	Non-abelian tensor product of the groups G and H.
$G \otimes^q H$	Non-abelian q-tensor product of the groups G and H.
$G \wedge G$	Non-abelian exterior square of the group G.
$M(G)$	The Schur Multiplier of the group G.

Contents

Introduction

In this thesis we focus on the study of certain structural and closure properties of the group $\nu^q(G)$ and some of its relevant sections, when G is a finite group belonging to certain families, where q is a non-negative integer. Introducing the group $\nu^q(G)$ was motivated by its relationship with the so called non-abelian q-tensor square $G \otimes^q G$ of G, which in turn is a modular version of the non-abelian tensor square $G \otimes G$, a particular case of the non-abelian tensor product of groups first introduced by Brown and Loday in their seminal paper [14]. In order to establish our work environment, we will define all of these concepts below.

To this end, let G and H be groups each of which acts upon the other (on the right)

$$\begin{array}{rcl} G \times H & \to & G \\ (g,h) & \mapsto & g^h \end{array} \quad \text{and} \quad \begin{array}{rcl} H \times G & \to & H \\ (h,g) & \mapsto & h^g \end{array}$$

in a compatible way, that is, for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g^{(h^{g_1})} = ((g^{g_1^{-1}})^h)^{g_1} \ \text{ and } \ h^{(g^{h_1})} = ((h^{h_1^{-1}})^g)^{h_1},$$

where both G and H acts on themselves by conjugation. In this situation, the *non-abelian tensor product* $G \otimes H$, as defined by Brown and Loday in [14], is the group generated by the symbols $g \otimes h$, where $g \in G$ and $h \in H$, subject to the defining relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

for all $g, g_1 \in G$, $h, h_1 \in H$.

Still in [14], the authors gave a topological significance for the non-abelian tensor product. They used it to describe the third relative homotopy group of a triad as a non-abelian tensor product of the second relative homotopy groups of appropriate subspaces.

More specifically, [14, Corollary 3.2], the third triad homotopy group is given by

$$\pi_3(X; A, B) \cong \pi_2(A, C) \otimes \pi_2(B, C)$$

where X is a pointed space and $\{A, B\}$ is an open cover of X such that A, B and $C = A \cap B$ are connected and (A, C), (B, C) are 1-connected (see also [7]).

When $G = H$ and all actions are by conjugation in G, then the group $G \otimes G$ is called the *non-abelian tensor square* of G. There are important invariants of the group G, such as the Schur Multiplier, $M(G)$, and the non-abelian exterior square, $G \wedge G$, among others, that appear as sections of $G \otimes G$.

In [44], Rocco derived some properties of the non-abelian tensor square of a group G via its embedding in a larger group, $\nu(G)$, defined as follows. Let G and G^φ be groups, isomorphic via $\varphi : G \to G^\varphi$, $g \mapsto g^\varphi$ for all $g \in G$. Then $\nu(G)$ is defined as (see also [20])

$$\nu(G) = \left\langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall g, h, k \in G \right\rangle. \tag{1}$$

Here and in the sequel, for arbitrary elements $x, y \in G$ we write x^y to mean the conjugate $y^{-1}xy$ of x by y; then the commutator of x and y is $[x, y] = x^{-1}y^{-1}xy$.

It is a well known fact (see [44]) that the subgroup $\Upsilon(G) = [G, G^\varphi]$ of $\nu(G)$ is isomorphic to the non-abelian tensor square $G \otimes G$. The above mentioned modular version of the operator ν was considered in [15], where for any non-negative integer q the authors introduced and studied the group $\nu^q(G)$, which in turn is an extension of the q-tensor square $G \otimes^q G$, first defined by Conduché and Rodriguez-Fernandez in [16] (see also [21], [12]).

To define the group $\nu^q(G)$, let q be a positive integer and let $\mathcal{K} = \{\widehat{k} \mid k \in G\}$ be a set of symbols, one for each element of the group G. For $q = 0$ we set $\mathcal{K}$ equal to the empty set. Write $F(\mathcal{K})$ for the free group on $\mathcal{K}$ and consider the free product $\nu(G) * F(\mathcal{K})$. Denote by J the normal closure in $\nu(G) * F(\mathcal{K})$ of the following elements:

$$g^{-1}\widehat{k}g(\widehat{k^g})^{-1} \tag{2}$$

$$(g^\varphi)^{-1}\widehat{k}(g^\varphi)(\widehat{k^g})^{-1} \tag{3}$$

$$(\widehat{k})^{-1}[g, h^\varphi]\widehat{k}[g^{k^q}, (h^{k^q})^\varphi]^{-1} \tag{4}$$

$$(\widehat{k})^{-1}\widehat{kk_1}(\widehat{k_1})^{-1}\prod_{i=1}^{q-1}[k, (k_1^{-i})^\varphi]^{-k^{q-1-i}} \tag{5}$$

$$[\widehat{k}, \widehat{k_1}][k^q, (k_1^q)^\varphi]^{-1} \tag{6}$$

$$\widehat{[g,h]}[g,h^{\varphi}]^{-q} \tag{7}$$

for all $\widehat{k}, \widehat{k_1} \in \mathcal{K}$ and $g, h \in G$.

According to [15], the group $\nu^q(G)$ is then defined to be the factor group

$$\nu^q(G) := (\nu(G) * F(\mathcal{K}))/J. \tag{8}$$

Notice that for $q = 0$ the set of relations (2) to (7) is empty; in this case we have $\nu^0(G) = (\nu(G) * F(\mathcal{K}))/J \cong \nu(G)$.

We denote by K the subgroup of $\nu^q(G)$ generated by the images of $\mathcal{K}$ in $\nu^q(G)$. By relation (4) we see that K normalizes the subgroup $[G, G^{\varphi}]$ in $\nu^q(G)$; therefore, $\Upsilon^q(G) := [G, G^{\varphi}]K$ is a normal subgroup of $\nu^q(G)$, K is normal in $\nu^q(G)$ by (2) and (3). Then we have

$$\nu^q(G) = (\Upsilon^q(G) \cdot G) \cdot G^{\varphi}, \tag{9}$$

where the dots indicate (internal) semidirect products. It is known (see for instance [15, Proposition 2.9]) that the group $\Upsilon^q(G)$ is naturally isomorphic to the q-tensor square $G \otimes^q G$, for all $q \geq 0$.

We are interested in studying the behavior of $\Upsilon^q(G)$ and $\nu^q(G)$, as well as other relevant sections of $\nu^q(G)$, under the hypothesis that G belongs to certain families of finite p-groups (where p is a prime number), for instance if G is powerful, potent or of maximal class. Moreover, one of our purposes is to find upper bounds for the exponents of these constructions based on the exponent of the argument G.

Recall that a finite p-group G is said to be *powerful* if $p > 2$ and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$. We can define a more general class of p-groups. We call a finite p-group *potent* if $p > 2$ and $\gamma_{p-1}(G) \leq G^p$, or $p = 2$ and $G' \leq G^4$. Note that the family of potent p-groups contains all powerful p-groups. A subgroup N of G is *potently embedded* in G if $[N, G] \leq N^4$ for $p = 2$, or $[N,_{p-2} G] \leq N^p$ for p odd (N is *powerfully embedded* in G if $[N, G] \leq N^4$ for $p = 2$, or $[N, G] \leq N^p$ for p odd).

In [38], Moravec proved that if G is a powerful p-group, then the non-abelian tensor square $[G, G^{\varphi}]$ and the derived subgroup $\nu(G)'$ are powerfully embedded in $\nu(G)$. Moreover, the exponent $\exp(\nu(G)')$ divides $\exp(G)$, which gives an upper bound for the exponent of the non-abelian tensor square. Our first contributions in this thesis are extending these results to $\nu^q(G)$ and to the q-tensor square $G \otimes^q G$, $q \geq 0$, now considering the families of powerful and potent p-groups. The next two theorems, about powerful p-groups, were published in [25].

Theorem 1. *Let G be a powerful finite p-group.*

(i) If $i \geq 2$, then $\gamma_i(\nu^q(G))$ is powerfully embedded in $\nu^q(G)$.

(ii) If $i \geq 1$, then $\nu^q(G)^{(i)}$ is powerfully embedded in $\nu^q(G)$.

(iii) Assume that $\exp(G)$ divides q. Then $\Upsilon^q(G)$ is powerfully embedded in $\nu^q(G)$.

Notice that, as usual we write $\gamma_i(G)$ for the i-th term of the lower central series and $G^{(i)}$ for the i-th term of the derived series of a group G, for all $i \geq 1$.

Theorem 2. *Let G be a powerful finite p-group.*

(i) If p is odd, then $\exp(\Upsilon^q(G))$ divides $\exp(G)$.

(ii) If $p = 2$ and either q is odd or 4 divides q, then $\exp(\Upsilon^q(G))$ divides $\exp(G)$.

(iii) If $p = 2$ and 4 does not divide q, then $\exp(\Upsilon^q(G))$ divides $2\exp(G)$.

It is important to note that by bounding the exponent of $\Upsilon^q(G)$ we are also bounding the exponents of the q-exterior square $G \wedge^q G$ and of the second homology group $H_2(G, \mathbb{Z}_q)$, both sections of $\Upsilon^q(G)$. For finite groups G and $q = 0$ it is known that $H_2(G, \mathbb{Z})$ is isomorphic to the so-called Schur multiplier $M(G)$, an invariant of the group G introduced by I. Schur early in the last century when he was studying projective representations of finite groups (see Section 1.5). There is a longstanding conjecture, attributed to Schur himself, that the exponent of $M(G)$ divides the exponent of G. This is true for groups of exponent 2 or 3, but in 1974 Bayes, Kautsky and Wamsley [8] gave an example of a group G of order 2^{68} with exponent 4, where $M(G)$ has exponent 8. However, the conjecture remained open for odd exponent groups until the last year, when Michael Vaughan-Lee [52] exhibited an example of a group of order 5^{4122} and exponent 5 with Schur multiplier of exponent 25, and an example of a group of order 3^{11983} and exponent 9 such that the Schur multiplier has exponent 27. Currently, the most realistic conjecture is that the exponent of $M(G)$ divides $(\exp G)^2$ (see Moravec[40]). However, based on the examples mentioned above, there are those who conjecture that for finite p-groups G, the exponent of $M(G)$ divides $p\exp(G)$ (see Thomas [51] for a survey on this conjecture).

In [5], in collaboration with Bastos, de Melo and Nunes, we studied $\exp(\nu(G))$ and the terms of the lower central series of the $\nu(G)$, when G is a potent p-group. The next two results generalizes these part for the group $\nu^q(G)$, where q is a non-negative integer.

Theorem 3. *Let p be a prime, q a positive integer and G a potent finite p-group.*

(i) If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\nu^q(G))$ is potently embedded in $\nu^q(G)$.

(ii) *Moreover, if exp(G) divides q, then the q-tensor square $\Upsilon^q(G)$ is potently embedded in $\nu^q(G)$.*

(iii) *The i-th term of the derived series $\nu^q(G)^{(i)}$ is a potent p-group, for $i \geq 1$.*

Still considering potent p-groups, we have the following

Theorem 4. *Let G be a potent finite p-group and suppose that $\exp(G) = p^e$.*

(i) *If $p \geq 3$ or 4 divides q, then $\exp(\nu^q(G))$ divides p^{e+1}.*

(ii) *If $p = 2$ and either q is odd or 4 divides q, then $\exp(\nu^q(G))$ divides 2^{e+2}*

(iii) *If $p = 2$ and 4 does not divide q, then $\exp(\nu^q(G))$ divides 2^{e+3}.*

A p-group of order p^n with $n > 3$ is said to be of maximal class if it has nilpotency class $n - 1$. It is possible to prove that all p-groups of maximal class contains a potent maximal subgroup or a maximal subgroup of class at most $p - 1$. In order to prove some bounds for the exponent of $\nu^q(G)$ when G is a p-group of maximal class, firstly we give a bound for this exponent in terms of some specific normal subgroups of G. The next three results are generalizations of the some theorems of article [4].

Theorem 5. *Let p be a prime and N be a normal subgroup of the p-group G. Suppose that $\gamma_p(N) = 1$ or that N is a potent subgroup of G.*

(i) *If $p \geq 3$, then $\exp(\nu^q(G))$ divides $p \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$.*

(ii) *If $p = 2$ and either q is odd or 4 divides q, then $\exp(\nu^q(G))$ divides $4 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$.*

(iii) *If $p = 2$ and 4 does not divide q, then $\exp(\nu^q(G))$ divides $8 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$.*

In [13], Brown, Johnson and Robertson described the non-abelian tensor square of 2-groups of maximal class. In particular, if G is a 2-group of maximal class, then $\exp([G, G^{\varphi}])$ divides $\exp(G)$ (cf. [13, Propositions 13–15]). Consequently, $\exp(\nu(G))$ divides $\exp(G)^2$. In [39], Moravec proved that if G is a p-group of maximal class, then $\exp(M(G))$ divides $\exp(G)$. In this way, as a consequence of the last theorem we give a bound for the exponent of $\nu^q(G)$ in terms of the exponent of G.

Corollary 1. *Let G be a p-group of maximal class.*

(i) *If $p \geq 3$ or 4 divides q, then $\exp(\nu^q(G))$ divides $p^2 \cdot \exp(G)$.*

(ii) *If $p = 2$ and either q is odd or 4 divides q, then $\exp(\nu^q(G))$ divides $4 \cdot 2 \cdot \exp(G)$.*

(iii) If $p = 2$ and 4 does not divide q, then $\exp(\nu^q(G))$ divides $2 \cdot 4^2 \cdot \exp(G)$.

In 2008, Fernández-Alcober, González-Sánches and Jaikin-Zapirain (see [22]) defined a finite p-group such that it has some *power-commutator condition*. This kind of conditions involves inclusions in certain power-commutator subgroup (that is, a subgroup formed by taking commutators and powers in any order). For example, inclusions satisfied for powerful and potent finite p-group, $\gamma_2(G) \leq G^4$, $\gamma_2(G) \leq G^p$ or $\gamma_{p-1}(G) \leq G^p$, are conditions of this type.

The *coclass* of a finite p-group G of order p^n and nilpotency class c is defined to be $r(G) = n - c$. Groups of maximal class have coclass equal to 1. These two families of finite p-groups satisfies a power-commutator condition, namely, $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$, where m and s are positive integers such that $m \geq s$.

In some sense the next result can be viewed as an extension of the result proved by Moravec (in [38]) and by Bastos et al. (in [5]), now considering the above equality as an assumption.

Theorem 6. *Let G be a p-group, with p a prime. Let m and s be positive integers such that $m \geq s$ and suppose that $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$. Consider $q \geq 0$. Then*

(i) If $i > m$, then $\gamma_{i+s+1}(\nu^q(G)) = \gamma_{i+1}(\nu^q(G))^p$.

(ii) If p is odd, then $\exp(\gamma_{m+1}(\nu^q(G))) = \exp(\gamma_m(G))$.

(iii) If $p = 2$ and $\gamma_m(G)$ is powerful, then $\exp(\gamma_{m+1}(\nu^q(G))) = \exp(\gamma_m(G))$.

This thesis is a compilation of the our articles [25], [5] and [4]. Moreover, it is organised in five chapters in addition to this introduction. In the first chapter we present notations and definitions that are used in the whole work. Moreover, we define and establish the main properties of the families of finite p-groups studied in this work. In the second chapter we define all group constructions investigated in the sequel, exploring their main properties and some of their most relevant sections. We also describe the evolution of these concepts over the years and the importance of the results involving these theories. In the third chapter we present some results obtained in [5] and [4] for the groups $\nu(G)$ and $\chi(G)$, where G is finite p-group belonging to the families of powerful, potent, p-groups of maximal class or coclass r. In the fourth chapter we prove some technical results, used in the proofs of the results for the construction $\nu^q(G)$ described above. The latter are proved in the fifth chapter. Moreover, in this last chapter we give an example to show that some of the considered properties of a finite p-groups are not in general inherited by the group $\nu^q(G)$ for all positive integer q.

Chapter 1

Preliminaries

In this chapter we present some basic concepts and results of Group Theory that will be used in our work. We omit their proofs, but the corresponding references are given.

1.1 Commutators and Commutator Subgroup

Let x, y be elements of a group G. The conjugate of x by y is the element $x^y = y^{-1}xy$; the commutator of x and y is defined to be $[x, y] = x^{-1}y^{-1}xy$ $(= x^{-1}x^y)$. Observe that x and y commute if, and only if, $[x, y] = 1$. For $n \geq 2$ and elements $x_1, x_2, \ldots, x_n$, the *simple commutator of weight* n is defined recursively by $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$ and by convention $[x_1] = x_1$. Moreover, for $n \in \mathbb{N}$ we will use $[x,\ _n\ y]$ to indicate the commutator $[x, y, \ldots, y]$, where y appears n times.

The next result brings together some properties and identities involving commutators.

Theorem 1.1.1. *[42, 5.1.5] Let x, y and z be elements of the group G. Then*

(i) $[y, x] = [x, y]^{-1}$.

(ii) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$.

(iii) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$.

(iv) $[x, y^{-1}] = [y, x]^{y^{-1}}$.

(v) $[x^{-1}, y] = [y, x]^{x^{-1}}$.

(vi) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y] = 1$ *(Hall-Witt's Identity).*

Similarly, it is possible to define the commutator subgroup of two nonempty subsets, H and K, of the group G by

$$[H,K] = \langle [h,k] \mid h \in H,\ k \in K \rangle.$$

It is clear that $[H,K] = [K,H]$. In the particular case when $H = G = K$ we have the derived subgroup of the group G, denoted by G'.

Recursively for $n \geq 2$ and nonempty subsets $H_1, \ldots, H_n$ of G we define $[H_1, \ldots, H_n] = [[H_1, \ldots, H_{n-1}], H_n]$. Besides that we have the notation $[H,\ _n\ K] = [H, K, \ldots, K]$, where K appears n times.

Theorem 1.1.2. *[26, Theorem 2.1] Let G be a group and H, K, L subgroups of G. Then:*

(i) *$H \leq N_G(K)$ if and only if $[H,K] \leq K$.*

(ii) *$K \trianglelefteq G$ and G/K abelian if and only if $[G,G] \leq K$.*

(iii) *If $K \leq H$ are each normal in G, then $H/K \leq Z(G/K)$ if and only if $[H,G] \leq K$.*

(iv) *If H, K, L are normal in G, then $[HK, L] = [H,L][K,L]$.*

(v) *If ϕ is an endomorphism of G, then $[H,K]^\phi = [H^\phi, K^\phi]$. In particular, $[H,K]$ is normal in G if both H and K so are.*

The next result is known as Three Subgroups Lemma.

Lemma 1.1.3. *[42, 5.1.10]* ***(Three subgroups lemma)*** *Let H, K and L be subgroups of a group G. If two of the commutator subgroups $[H,K,L]$, $[K,L,H]$ and $[L,H,K]$ are contained in a normal subgroup of G, then so is the third. In particular, if $[H,K,L] = 1 = [K,L,H]$, then $[L,H,K] = 1$.*

A particular case of this result is when L and $H = M = K$ are normal subgroups of G, so $[L,M,M]$ is a normal subgroup too. Since $[M,L,M] = [L,M,M]$, by the Three Subgroups Lemma it holds that $[M,M,L] \leq [L,M,M]$.

1.2 Solvable and Nilpotent Groups

A series (of finite length) in a group G is a finite sequence of subgroups including 1 and G such that each member of the sequence is a normal subgroup of its successor: thus a series can be written in the following way:

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G.$$

The G_i are the terms of the series and the quotient groups G_{i+1}/G_i are the factors of the series. If all G_i are distinct, the integer n is called the length of the series.

Since the normality is not a transitive relation, the G_i need not be normal subgroups of G. A subgroup which is a term of at least one series is said to be subnormal in G. Thus H is subnormal in G if and only if there exist distinct subgroups $H_0 = H, H_1, \ldots, H_n = G$ such that $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$; the latter we call a series between H and G.

If the terms of a series are normal in G we shall speak of a normal series.

Definition 1.2.1. *A group G is said to be soluble (or solvable) if it has an abelian series, by which we mean a series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$ in which each factor G_{i+1}/G_i is abelian. If G is a soluble group, the length of a shortest abelian series in G is called the derived length of G.*

By the last definition we have that G has derived length 0 if and only if it has order 1. Also the groups with derived length at most 1 are just the abelian groups. A soluble group with derived length at most 2 is said to be metabelian. The first example of a non-abelian soluble group is the symmetric group S_3. The next two results contain some of the most elementary properties of soluble groups.

Proposition 1.2.2. *[42, 5.1.1] The class of soluble groups is closed with respect to the formation of subgroups, images, and extensions of its members.*

Proposition 1.2.3. *[42, 5.1.2] The product of two normal soluble subgroups of a group is soluble.*

Another concept based on a series of the group G is that of nilpotency. The precise definition and some of their properties follow.

Definition 1.2.4. *A group G is called nilpotent if it has a normal central series, that is, a normal series $1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$ such that G_{i+1}/G_i is contained in the center of G/G_i for all i. If G is nilpotent, the length of a shortest central series of G is the nilpotency class of G and it will be denote by $cl(G)$.*

Observe that if the first quotient G_1/G_0 is contained in the center of G/G_0 then it means that $1 \neq G_1 \leq Z(G)$, where $Z(G)$ is the center of the group. Therefore, we can conclude that the nilpotent groups have non-trivial center. Moreover, the non-trivial abelian groups are nilpotent groups of class 1. The trivial group has class 0.

Examples of nilpotent groups can be obtained through subgroups, quotients and finite direct product, as the next results show us.

Proposition 1.2.5. *[42, 5.1.4] The class of nilpotent groups is closed under the formation of subgroups, images, and finite direct products.*

Remember that G' is the derived subgroup of the group G, which is generated by all commutators in G. Repeating the process of formation of the derived subgroups we have a descending sequence of characteristic subgroups (that is, each term of this series is invariant by all automorphisms of G) in the following way

$$G = G^{(0)} \geq G^{(1)} = G' \geq G^{(2)} \geq \cdots$$

where $G^{(n+1)} = (G^{(n)})'$. Not necessarily this series reach 1 or even terminate, but in any way it is called the derived series of the group G. Observe that each factor $G^{(n)}/G^{(n+1)}$ is an abelian group and the first factor, G/G', is the largest abelian quotient group of G, so it is known as the abelianization of G and some times denoted by G^{ab}.

Beyond these series we have two more: a descending series (the lower series) and an ascending series (the upper series) of the group G. The lower central series is constructed by repeatedly commuting with the group G, that is, it is defined by

$$G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \gamma_3(G) \geq \cdots$$

where $\gamma_n(G) = [\gamma_{n-1}(G), G]$. It is easy to verify that each term of this series is characteristic in G and satisfies $\gamma_{i+1}(G) \leq \gamma_i(G)$; this means that the series is central in G. Just like the derived series the lower central series does not in general reach 1. Moreover, the first two terms of both series are equal.

Now, the upper central series is

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \zeta_2(G) \leq \cdots$$

where $\zeta_{n+1}(G)/\zeta_n(G)$ is defined as the center of $G/\zeta_n(G)$. Each term of this series is a characteristic subgroup of G and $\zeta_1(G) = Z(G)$ is the center of G. This series need not reach G, but in the finite case this series terminates with a subgroup called the hypercenter.

Supposing the group G is nilpotent, we have the main properties of these series described in the next result.

Theorem 1.2.6. *[42, 5.1.9] Let $1 = G_0 \leq G_1 \leq \ldots \leq G_n = G$ be a central series in of nilpotent group G. Then:*

(i) $\gamma_i(G) \leq G_{n-i+1}$, so that $\gamma_{n+1}(G) = 1$.

(ii) $G_i \leq \zeta_i(G)$*, so that* $\zeta_n(G) = G$*.*

(iii) *The nilpotency class of* G = *the length of the upper central series* = *the length of the lower central series.*

As application of the Three Subgroups Lemma we have the following result, which gives us some relations between the last two series defined above.

Proposition 1.2.7. *[42, 5.1.11] Let* G *be any group and let* i *and* j *be positive integers.*

(i) $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$*.*

(ii) $\gamma_i(\gamma_j(G)) \leq \gamma_{ij}(G)$*.*

(iii) $[\gamma_i(G), \zeta_j(G)] \leq \zeta_{j-i}(G)$ *if* $j \geq i$*.*

(iv) $\zeta_i(G/\zeta_j(G)) = \zeta_{i+j}(G)/\zeta_j(G)$*.*

A property of the lower central series very useful in our work concernes the weight and the kind of commutators which generate a term of this series. This property is given by the second item of the next result.

Lemma 1.2.8. *[32, Lemma 3.6] In any group* G*, for every* $k \in \mathbb{N}$*,*

(i) $\gamma_k(G)$ *contains all commutators of weights* $\geq k$ *in the elements of* G*.*

(ii) $\gamma_k(G)$ *is generated by the simple commutators of weight* k *in the elements of* G*.*

(iii) If $G = \langle M \rangle$*, then* $\gamma_k(G)$ *is generated by the simple commutators of weight* $\geq k$ *in the elements* $m^{\pm 1} \in M$*.*

Since our main interest is about some constructions using finite p-groups, which are nilpotent, we will finish this section giving some characterizations of finite nilpotent groups.

Theorem 1.2.9. *[42, 5.2.4] Let* G *be a finite group. Then the following properties are equivalent:*

(i) G *is nilpotent;*

(ii) Every subgroup of G *is subnormal.*

(iii) G *satisfies the normalizer condition, that is, every proper subgroup is properly contained in its normalizer.*

(iv) Every maximal subgroup of G *is normal.*

(v) G *is the direct product of its Sylow's subgroups.*

1.3 Free Groups and Presentations

A group G is said to be generated by a subset X if each of its elements can be expressed as a product of elements of the set X or their inverses, that is, as a product of members of $X^{\pm} = \{x, x^{-1} \mid x \in X\}$. Such a product is called a word in G. Moreover, a relation in G is an equation between two words in the group.

Consider the set R of the relations that holds in G. We say that G is presented by X and R if every relation that holds in G is a consequence of R. The precise definition is based on the concept of free group, which is given in the next definition.

Definition 1.3.1. *A group F is said to be free on a subset $X \subseteq F$ if, given any group G and any map $\theta : X \to G$, there is a unique homomorphism $\theta' : F \to G$ extending θ, that is, having the property that $\theta'(x) = \theta(x)$, for all $x \in X$, or that the diagram*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\theta"] & F \arrow[dl, dashed, "\exists!\ \theta'"] \\ G & \end{tikzcd}$$

is commutative. Then X is called a basis of F and $|X|$ is the rank of F, written $r(F)$.

We can obtain a concept of free abelian group replacing the word "group" by "abelian group" in the previous definition. It is possible to construct a free group on any subset X, which show us that the existence of this group. Moreover, the rank of a free group is well-defined, as the next result shows.

Proposition 1.3.2. *[42, 2.1.4] If F_1 is free on X_1 and F_2 is free on X_2 and if $|X_1| = |X_2|$, then $F_1 \cong F_2$.*

The fundamental importance of free groups in the group theory is that every group is an image of a free group, as in the following result.

Proposition 1.3.3. *[29, Proposition 4] Every group is isomorphic to a factor group of some free group.*

Free groups have the property that their subgroups are also free. The Nielsen-Schreier theorem's shows us this fact and it provides the rank of this subgroup depending on the rank of G and the finite index of this subgroup in the group.

Theorem 1.3.4. *[29, Theorem 1] (Nielsen-Schreier) Let F be a free group and H a subgroup of F. Then H is free. Moreover, if $|F : H| = n$ and $r(F) = r$ are both finite, then $r(H) = (r-1)n + 1$.*

Definition 1.3.5. *The normal closure $\overline{X}$ of any nonempty subset X of any group G is the intersection of all the normal subgroups of G which contain X.*

The normal closure of a set X is the smallest normal subgroup containing X and it is possible to see that $\overline{X} = \langle g^{-1}Xg \mid g \in G \rangle$.

Let X be a set and $F = F(X)$ be the free group on X. Suppose that R is a subset of F, $N = \overline{R}$ is the normal closure of R in F and G is the factor group F/N. So, we write $G = \langle X|R \rangle$ and this is called a free presentation, or simply a presentation of G. The elements of X are called generators and those of R defining relators. A group G is called finitely presented if it has a presentation with both X and R finite sets.

Lemma 1.3.6. *[29, Lemma 1] Let F, G, H be groups and $v : F \to G$, $\alpha : F \to H$ homomorphisms such that $\mathrm{Im}(v) = G$ and $\ker(v) \subseteq \ker(\alpha)$. Then there is a homomorphism $\alpha' : G \to H$ such that $v\alpha' = \alpha$.*

$$\begin{tikzcd}
\ker(\alpha) \arrow[dr] & & G \arrow[dd, dashed, "\alpha'"] \\
& F \arrow[ur, "v"] \arrow[dr, "\alpha"'] & \\
\ker(v) \arrow[uu, hook] \arrow[ur] & & H
\end{tikzcd}$$

The last lemma is very important for the proof of the next two propositions, which describe factor groups and homomorphisms using the definition of presentation of a group.

Proposition 1.3.7. *[29, Proposition 2] (von Dyck) If $G = \langle X|R \rangle$ and $H = \langle X|S \rangle$, where $R \subseteq S \subseteq F(X)$, then there is an epimorphism $\phi : G \to H$ fixing every $x \in X$ and such that $\ker \phi = \overline{S \backslash R}$. Conversely, every factor group of $G = \langle X|R \rangle$ has a presentation $\langle X|S \rangle$ with $R \subseteq S$.*

Proposition 1.3.8. *[29, Proposition 3] (Substitution Test) Suppose we are given a presentation $G = \langle X|R \rangle$, a group H, and a mapping $\theta : X \to H$. Then θ extends to a homomorphism $\theta' : G \to H$ if and only if, for all $x \in X$ and all $r \in R$, the result of substituting $\theta(x)$ for x in r yields the identity of H.*

A very useful result in the context of presentations is given in the next proposition, which shows us how the presentation of the direct product of two groups is.

Proposition 1.3.9. *[29, Proposition 4] If G, H are groups presented by $\langle X|R \rangle$, $\langle Y|S \rangle$ respectively, then their direct product $G \times H$ has the presentation*

$$G \times H = \langle X, Y \mid R, S, [X, Y] \rangle,$$

where $[X, Y]$ *denotes the set of commutators* $\{x^{-1}y^{-1}xy \mid x \in X, y \in Y\}$.

The concept of group presentation can be used to define other products, such as the free product for two groups.

Definition 1.3.10. *Let* $G_1 = \langle X|R\rangle$ *and* $G_2 = \langle Y|S\rangle$ *be groups. Their free product is given by the presentation* $G_1 * G_2 = \langle X, Y|R, S\rangle$.

In general the free product is defined using a universal property similarly done for free groups. Therefore $G_1 * G_2$ does not depend of the presentations of G_1 and G_2. Moreover, like as free groups, subgroups of free products is also a free product and there exists a bound for their rank (The Kuroš Subgroup Theorem [42, 6.3.1]). More details about this topic can be found in [42].

1.4 Polycyclic groups

Definition 1.4.1. *A group* G *is said to be polycyclic if it has a descending chain of subgroups* $G = G_1 \geq G_2 \geq \ldots \geq G_{n+1} = 1$ *in which each* G_{i+1} *is a normal subgroup of* G_i, *and the quotient group* G_i/G_{i+1} *is cyclic. Such a chain of subgroups is called a polycyclic series.*

It is clear that polycylic groups are soluble. Moreover, this class is closed with respect to forming subgroups, images and extensions.

Consider the following polycyclic series $G = G_1 \geq G_2 \geq \ldots \geq G_{n+1} = 1$, for the polycyclic group G. Since the quotient group G_i/G_{i+1} is cyclic, there exist elements $x_i \in G$ with $\langle x_i G_{i+1}\rangle = G_i/G_{i+1}$ for every index i. The sequence of elements $X = [x_1, \ldots, x_n]$ satisfying this condition is called a *polycyclic sequence* for G. Moreover, the group G is generated by the elements in the sequence X and the polycyclic series is uniquely determined by X.

Definition 1.4.2. *Let* X *be a polycyclic sequence for* G. *The sequence* $R(X) = (r_1, \ldots, r_n)$ *defined by* $r_i = |G_i : G_{i+1}| \in \mathbb{N} \cup \{\infty\}$ *is called the sequence of relative orders for* X. *The set* $\{i \in \{1, \ldots, n\} \mid r_i \text{ is finite}\}$ *is denoted by* $I(X)$.

The sequence $R(X)$ and the set $I(X)$ depend on the underlying polycyclic series $G_1, \ldots, G_n$ of X only. The number https://pt.overleaf.com/project/5dbc3621a1c83c0001ce53abof infinite entries in the sequence of relative orders is an invariant for G, which is called the *Hirsch length* of G. Moreover, the sequence $R(X)$ exhibit some basic information about the group G. For example, the group G is finite if and only if every entry in $R(X)$ is finite or, equivalently, if and only if $I(X) = \{1, \ldots, n\}$. If G is finite, then $|G| = r_1 \cdot \ldots \cdot r_n$, the product of the entries in $R(X)$.

Lemma 1.4.3. *[28, Lemma 8.3] Let X be a polycyclic sequence for G with the relative orders $R(X) = (r_1, \ldots, r_n)$. Then for every $g \in G$ there exists a unique sequence $(e_1, \ldots, e_n)$, with $e_i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $0 \leq e_i < r_i$ if $i \in I(X)$, such that $g = x_1^{e_1} \cdots x_n^{e_n}$.*

The expression $g = x_1^{e_1} \cdots x_n^{e_n}$ given in the last lemma is called the *normal form* of the element $g \in G$ with respect to X. The sequence $(e_1, \ldots, e_n)$ is the *exponent vector* of g with respect to X, denoted by $exp_X(g) = (e_1, \ldots, e_n)$.

The exponent vectors of the elements in a polycyclic group can be used to describe relations for G in its generators X. These relations are the first and fundamental step towards a polycyclic presentation for G. In this way, we have the following lemma, which describes these relations.

Lemma 1.4.4. *[28, Lemma 8.6] Let $X = [x_1, \ldots, x_n]$ be a polycyclic sequence for G with relative orders $R(X) = (r_1, \ldots, r_n)$.*

(i) Let $i \in I(X)$. Then the normal form of a power $x_i^{r_i}$ is of the form $x_i^{r_i} = x_{i+1}^{a_{i,i+1}} \cdots x_n^{a_{i,n}}$.

(ii) Let $1 \leq j < i \leq n$. Then the normal form of a conjugate $x_j^{-1} x_i x_j$ is of the form $x_j^{-1} x_i x_j = x_{j+1}^{b_{i,j,j+1}} \cdots x_n^{b_{i,j,n}}$.

(iii) Let $1 \leq j < i \leq n$. Then the normal form of a conjugate $x_j x_i x_j^{-1}$ is of the form $x_j x_i x_j^{-1} = x_{j+1}^{c_{i,j,j+1}} \cdots x_n^{c_{i,j,n}}$.

Now, we have the definition of a polycyclic presentation of a polycyclic group.

Definition 1.4.5. *A presentation $\langle x_1, \ldots, x_n \mid R \rangle$ is called a polycyclic presentation if there exists a sequence $S = (s_1, \ldots, s_n)$ with $s_i \in \mathbb{N} \cup \{\infty\}$ and integers $a_{i,k}$, $b_{i,j,k}$, $c_{i,j,k}$ such that R consists of the following relations:*

$$
\begin{aligned}
x_i^{s_i} &= R_{i,i} \text{ with } R_{i,i} := x_{i+1}^{a_{i,i+1}} \cdots x_n^{a_{i,n}} \text{ for} 1 \leq i \leq n \text{ with } s_i < \infty, \\
x_j^{-i} x_i x_j &= R_{i,j} \text{ with } R_{i,j} := x_{j+1}^{b_{i,j,j+1}} \cdots x_n^{b_{i,j,n}} \text{ for} 1 \leq j < i \leq n, \\
x_j x_i x_j^{-1} &= R_{j,i} \text{ with } R_{j,i} := x_{j+1}^{c_{i,j,j+1}} \cdots x_n^{c_{i,j,n}} \text{ for} 1 \leq j < i \leq n.
\end{aligned}
$$

These relations are called polycyclic relations. The first type are the power relations and the second and third types are the conjugate relations. And S is called the sequence of power exponents of the presentation.

Conjugate relations of the form $x_j^{-1}x_ix_j = x_i$ or $x_jx_ix_j^{-1} = x_i$ are called *trivial polycyclic relations.* They are often omitted from a polycyclic presentation to simplify the notation. This means that polycyclic presentations have to be distinguished from arbitrary finite presentations. For this purpose we denote them with $Pc\langle x_1, \ldots, x_n \mid R\rangle$. A group that is defined as and represented by a polycyclic presentation is known as a *PC-group.*

Every polycyclic group G has a polycyclic sequence X and every such sequence induces a complete set of polycyclic relations as outlined in the last lemma. The power exponents S of the presentation is equal the relative orders $R(X)$ in this case. On the other hand, every polycyclic presentation defines a polycyclic group. These properties are summarized in the next two results.

Theorem 1.4.6. *[28, Theorem 8.8] Every polycyclic sequence determines a (unique) polycyclic presentation. Thus every polycyclic group can be defined by a polycyclic presentation.*

Theorem 1.4.7. *[28, Theorem 8.9] Let $Pc\langle x_1, \ldots, x_n \mid R\rangle$ be a polycyclic presentation and let G be the group defined by this presentation. Then G is polycyclic and $X = [x_1, \ldots, x_n]$ is a polycyclic sequence for G. Its relative orders $R(X) = (r_1, \ldots, r_n)$ satisfies $r_i \leq s_i$ for $1 \leq i \leq n$.*

Definition 1.4.8. *A polycyclic presentation $Pc\langle X|R\rangle$ with power exponents S is called consistent (or confluent) if $R(X) = S$.*

Theorem 1.4.9. *[28, Theorem 8.11] Every polycyclic sequence determines a consistent polycyclic presentation. Thus every polycyclic group can be defined by a consistent polycyclic presentation.*

Many algorithms have been derived and implemented for computation of non-abelian tensor squares in the context of polycyclic groups (see for instance [45, 20, 19, 11, 17]).

1.5 The Schur Multiplier

In the beginning of the twentieth century I. Schur [49] introduced an invariant of a group G, which became to be known as the Schur multiplier, $M(G)$, during his studies of projective representations of groups. This group was defined in the following way.

Definition 1.5.1. *Let G be a finite group. The Schur Multiplier of G, denoted by $M(G)$, is by definition the second cohomology group of G with complex coefficients with trivial G-action. That is, $M(G) = H^2(G, \mathbb{C}^*)$.*

In 1907, Schur proved the following formula for the Schur Multiplier, using free presentations. It is possible to prove that this formula is independent of their presentation. This is known as the Hopf's formula.

Theorem 1.5.2. *[31, 2.4.6(i)] Let G be a finite group and let $G = F/R$ where F is a free group of rank n. Then*

$$M(G) \cong \frac{F' \cap R}{[F,R]}.$$

Let G be a group and let K_i, for $i = 1,2,3$, be the free abelian group on i-tuples $(x_1,\ldots,x_i)$, where the x_j run through all nonidentity elements of the G. For convenience, put $(x_1,\ldots,x_i) = 1$ whenever some $x_j = 1$. Define a chain of homomorphisms $K_3 \xrightarrow{\delta_3} K_2 \xrightarrow{\delta_2} K_1$ by the rules

$$\begin{aligned}\delta_3(x,y,z) &= (y,z)(x,yz)(xy,z)^{-1}(x,y)^{-1}\\ \delta_2(x,y) &= (x)(y)(xy)^{-1}, \quad \text{for all} \quad x,y,z \in G.\end{aligned}$$

Observe that

$$\delta_2(\delta_3(x,y,z)) = (y)(z)(yz)^{-1}(x)(yz)(xyz)^{-1}(xy)^{-1}(z)^{-1}(xyz)(x)^{-1}(y)^{-1}(xy) = 1$$

for all $x,y,z \in G$, it follows that $Im\delta_3 \subseteq \ker\delta_2$. We now define the second homology group $H_2(G,\mathbb{Z})$ by

$$H_2(G,\mathbb{Z}) = \frac{\ker\delta_2}{Im\delta_3}.$$

Theorem 1.5.3. *[31, 2.7.3] Let G be an arbitrary group and let $G = F/R$ be a free presentation of G. Then*

$$\frac{F' \cap R}{[F,R]} \cong H_2(G,\mathbb{Z}).$$

In particular, if G is finite, then $M(G) \cong H_2(G,\mathbb{Z})$.

Let q be a non-negative integer. Considering the trivial G-module $\mathbb{Z}_q$ as coefficients we can define the second homology group of the group G, denoted by $H_2(G,\mathbb{Z}_q)$. Joining the Hopf's formula for to the second integer homology group $H_2(G,\mathbb{Z})$ (see [35] for more details) with the short exact sequency of coefficients:

$$\mathbb{Z} \overset{\times q}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_q$$

give us the following formula

$$H_2(G,\mathbb{Z}_q) \cong \frac{R \cap F'F^q}{R^q[R,F]}$$

where $R \rightarrowtail F \twoheadrightarrow G$ is a free presentation of G.

Definition 1.5.4. *The q-multiplier of the group G, $M^q(G)$, is defined by the quotient group*

$$M^q(G) = \frac{R}{R^q[R,F]}.$$

1.6 Finite p-groups

Let p be a prime number. A group is a p-group if the orders of all of its elements are powers of p. In the finite context this means that the order of G is p^n for some $n \in \mathbb{N}$, which is denoted by $|G| = p^n$. The theory about finite p-groups is very long and contains a lot of results. In this section we will present the main about this theory used in our text. We give some generic properties and later we specifies the families of p-groups studied, as well as their main characteristics. Throughout this section G denotes a non-trivial finite p-group.

It is obvious that subgroups and quotient groups of a finite p-group are again finite p-groups. Note that in a group of order p every non-trivial element generates a subgroup of order that divides p and hence equals p, so this is a cyclic group. Moreover, every group of order p^2 is abelian.

The first property of a finite p-group is about its nilpotency and a bound of its nilpotency class.

Lemma 1.6.1. *[33, Lemma 1.2.2] Let $|G| = p^n$ where $n > 1$. Then G is nilpotent of class at most $n-1$.*

For finite p-groups we can define two special characteristics subgroups based on the order and on the p-th powers of its elements. These subgroups allow us to define more series which are very meaningful in the study of the power structure of a p-group.

Definition 1.6.2. *Let $i > 0$ be an integer. Then*

$$G^{\{p^i\}} = \{x^{p^i} \mid x \in G\} \quad \text{and} \quad G^{p^i} = \langle G^{\{p^i\}} \rangle,$$

$$\Omega_{\{i\}}(G) = \{x \in G \mid x^{p^i} = 1\} \quad \text{and} \quad \Omega_i(G) = \langle \Omega_{\{i\}}(G) \rangle.$$

$\Pi_i(G)$ for $i \geq 0$ is defined inductively by:

$$\Pi_0(G) = G \quad \text{and} \quad \Pi_i(G) = (\Pi_{i-1}(G))^p \text{ for } i > 0.$$

When the p-group G is abelian we have the equalities $G^{\{p^i\}} = G^{p^i}$ and $\Omega_{\{i\}}(G) = \Omega_i(G)$. Moreover, $|G : \Omega_i(G)| = G^{p^i}$. In literature it is common to say that a finite p-group is *power abelian* if it satisfies these three equalities for all i. Other families of finite p-groups have the same behaviour, like regular and powerful p-groups for example.

The exponent of the group is, by definition, the least common multiple of the orders of its elements. It is denoted by $\exp(G)$. For p-groups this means that the exponent is simply the maximum order of its elements. When G is a finite p-group we have $\exp(G) = p^e$ for some $e \in \mathbb{N}$.

Lemma 1.6.3. *[33, Lemma 1.2.6 - 1.2.7] Let G have exponent p^e for some integer $e > 0$.*

(i) $G = \Omega_e(G) \geq \Omega_{e-1}(G) \geq \cdots \geq \Omega_1(G) > \Omega_0(G) = \langle 1 \rangle$.

(ii) $G = G^{p^0} > G^{p^1} > \cdots > G^{p^{e-1}} > G^{p^e} = \langle 1 \rangle$.

(iii) $G^{p^i} \leq \Omega_{e-i}(G)$ *for* $1 \leq i \leq e$.

(iv) $G = \Pi_0(G) > \Pi_1(G) > \cdots > \Pi_e(G) \geq \langle 1 \rangle$.

(v) $G^{p^{i+j}} \leq (G^{p^i})^{p^j}$ *and* $\Omega_i(\Omega_j(G)) = \Omega_k(G)$ *where* $k = min\{i, j\}$.

(vi) $(G^{p^j})^{p^{i-j}} \leq \Pi_i(G)$ *for all* $j \leq i$.

Definition 1.6.4. *The Frattini subgroup of a group G, denoted by $\Phi(G)$, is the intersection of all maximal subgroups of G. If the group G does not have any maximal subgroup, then we define $\Phi(G) = G$.*

It is possible to prove that the Frattini subgroup of any group G coincide with the set of all nongenerators of the group. An element g is called a nongenerator of G if $G = \langle g, X \rangle$ always implies that $G = \langle X \rangle$ when X is a subset of G. The particular case of finite p-groups we have a very useful characterization for the Frattini subgroup. Moreover, the quotient $G/\Phi(G)$ can be seen as a vector space.

Recall the rank of a finite group G is defined to be $rk(G) = \sup\{d(H) \mid H \leq G\}$, where $d(H)$ is the minimal cardinality of a generating set of H.

Proposition 1.6.5. *[33, Proposition 1.2.4]*

Let G be a non-trivial finite p-group. Then

(i) The Frattini subgroup is $\Phi(G) = G'G^p$. The rank of the elementary abelian group $G/\Phi(G)$ is denoted by $d(G)$.

(ii) **(Burnside's Basis Theorem)** *Every generating set for G contains a generating subset with exactly $d(G)$ elements. Also $\{x_1, \ldots, x_{d(G)}\}$ generates G if and only if $\{x_1\Phi(G), \ldots, x_{d(G)}\Phi(G)\}$ is a basis for the vector space $G/\Phi(G)$.*

In abelian groups and in some families of finite p-groups, the product of n-th powers is an n-th power, that is, for all elements x, y and integer n there exist z such that $x^n y^n = z^n$. In the abelian case $z = xy$. This is not true in an arbitrary group. The Philip Hall's Compilation Formula or Hall-Petrescu's Formula give us another relation that is valid in an arbitrary group.

In an arbitrary group G, for all elements x, y and $n \in \mathbb{N}$, $(xy)^n$ and $x^n y^n$ are equals modulo an element of G', that is we can write $x^n y^n = (xy)^n c$ for some $c \in G'$. By Philip Hall's Compilation Formula, the element c is a product of commutators, as in the next result.

Theorem 1.6.6. *[18, Appendix A] Let x, y be elements of a group G and n a positive integer. Then*

$$x^n y^n = (xy)^n c_2^{\binom{n}{2}} \ldots c_i^{\binom{n}{i}} \ldots c_{n-1}^n c_n$$

where $c_i \in \gamma_i(G)$ for each $i \in \{1, \ldots, n\}$.

Each c_i can be construct as a product of commutators of length at least i. Moreover, the only elements that appears in this product are x and y. So, we can consider each $c_i = c_i(x, y) \in \gamma_i(\langle x, y\rangle)$ for each i.

If we consider $n = p^k$, for a positive integer k, then we have a particular case of this formula very interesting to apply in the theory of finite p-groups. In this case, the binomial coefficient $\binom{p^k}{i}$ is divisible by p^{k-j} for $p^j \leq i \leq p^{j+1}$. Therefore we have the following reformulation.

Theorem 1.6.7. *[22, Theorem 2.1]*(Hall's collection formula) *Let G be a group and let x, y be two elements of G. Let $H = \langle x, y\rangle$ and $L = \langle x, [x, y]\rangle$. Then, for all $k \geq 0$,*

$$(xy)^{p^k} \equiv x^{p^k} y^{p^k} \pmod{\gamma_2(H)^{p^k} \gamma_p(H)^{p^{k-1}} \gamma_{p^2}(H)^{p^{k-2}} \ldots \gamma_{p^k}(H)} \tag{1.1}$$

and

$$[x, y]^{p^k} \equiv [x^{p^k}, y] \pmod{\gamma_2(L)^{p^k} \gamma_p(L)^{p^{k-1}} \gamma_{p^2}(L)^{p^{k-2}} \ldots \gamma_{p^k}(L)}. \tag{1.2}$$

Considering a finite number of elements of G, we have an immediate application of the equation (1.1), as the next corollary.

Corollary 1.6.8. *[24, Corollary 2.6] Let G be a p-group and $x_1, \ldots, x_r$ elements of G. Then for any $k \geq 0$ we have*

$$(x_1 \ldots x_r)^{p^k} \equiv x_1^{p^k} \ldots x_r^{p^k} \pmod{\gamma_2(L)^{p^k} \gamma_p(L)^{p^{k-1}} \gamma_{p^2}(L)^{p^{k-2}} \gamma_{p^3}(L)^{p^{k-3}} \cdots \gamma_{p^k}(L)},$$

where $L = \langle x_1, \ldots, x_r \rangle$.

The next result represents an important tool in the demonstrations of our results and it is used to prove an application of Hall's compilation formula for normal subgroups of the finite p-group. This application show us a congruence between two subgroups.

Lemma 1.6.9. *[22, Lemma 2.2] Let G be a finite p-group, p a prime number, and let M and N be normal subgroups of G. If $N \leq M[N,G]N^p$ then $N \leq M$.*

Theorem 1.6.10. *[22, Theorem 2.4] Let G be a finite p-group and let M and N be normal subgroups of G. Then*

$$[N^{p^k}, M] \equiv [N, M]^{p^k} \pmod{[M,_p N]^{p^{k-1}} [M,_{p^2} N]^{p^{k-2}} \cdots [M,_{p^k} N]},$$

for $k \in \mathbb{N}$.

The last result of this section show us a bound for the exponent of the subgroup $\Omega_i(G)$ which depends on the exponent of a p-group G under some conditions. This is a particular case of the theorem proved by Fernández-Alcober et al. in [22].

Lemma 1.6.11 ([22])**.** *Let G be a finite p-group and $k \geq 1$. Assume that $\gamma_{k(p-1)}(G) \leq \gamma_r(G)^{p^s}$ for some r and s such that $k(p-1) < r + s(p-1)$. Then the exponent $\exp(\Omega_i(G))$ is at most p^{i+k-1} for all i.*

1.6.1 Powerful p-groups

The first family of finite p-groups we consider is that of powerful p-groups. The basic theory about these groups was introduced by Avinoam Mann, and developed by him and by Alexander Lubotzky, in a work published in 1987 ([34]). The abelian p-groups are contained in this class and we can obtained some properties and generalizations of results as for abelian groups.

For a powerful p-group the minimum number of generators of its subgroups is bounded by the number of generators of the whole group, which is valid for abelian groups, too. Besides that, for all finite p-group of there exists a powerful characteristic subgroup whose index is bounded by a function depending on the rank of the group. This property gave importance to this family in the recent advances in the theory of finite p-groups. These results can be found in the work of Lubotzky and Mann ([34]) or in books like [18] (Theorems 2.9 and 2.13). In this section we will focus on the power structure of powerful p-groups, and their subgroup lattice.

Definition 1.6.12. *A finite p-group G is powerful if either p is odd and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$. A subgroup N of a finite p-group G is powerfully embedded in G if either p is odd and $[N, G] \leq N^p$, or $p = 2$ and $[N, G] \leq N^4$.*

Since the quotient G/G^2 is abelian and G/G' is the largest abelian quotient, it follows that $G' \leq G^2$ in any 2-group. For this reason, there exists a difference between the definition of powerful when p is odd and when p is even. For abbreviation, we write N *p.e.* G instead of N is powerfully embedded in G. Observe that if N *p.e.* G, then N is normal in G and N is powerful. Moreover, in the odd case, G is powerful if and only if $\Phi(G) = G^p$.

One of the first questions about a new family of groups is if its property are inherited by their subgroups. For powerful p-groups it is not true. The dihedral group of size 8 is not a powerful p-group, but it is possible to construct a direct product which has a copy of this group as subgroup. In the same way we can do this for the odd case. For more details about these examples we refer to Section 6.1 of [33]. We summarize in the next lemma some properties involving some subgroups satisfying this condition embedded.

Lemma 1.6.13. *[33, Lemma 6.1.2]*

(i) G is powerful if and only if G p.e. G.

(ii) If N p.e. G, then N is powerful and G centralizes N/N^p.

(iii) If $K \trianglelefteq G$ and N p.e. G, then NK/K p.e. G/K. In particular, quotients of powerful p-groups are powerful, too.

(iv) If N p.e. G and $x \in G$, then $\langle x \rangle N$ is powerful.

(v) If M p.e. G and N p.e. G, then MN p.e. G.

One interesting fact using this definition of the embedding property is that we can obtain a powerful subgroup of a p-group even without the whole group being. So the next result shows that if a subgroup N, of any finite p-group, is powerfully embedded, then $[N, G]$ and N^p are also powerfully embedded.

Lemma 1.6.14. *[33, Lemma 6.1.5] If N p.e. G, then $[N, G]$ p.e. G and N^p p.e. G.*

With the hypothesis that G is powerful we can conclude that there exists some important subgroups of G that have the embedded property and, with this, these subgroups are also powerful, as the next two results show us.

Corollary 1.6.15. *[33, Corollary 6.1.6] If G is powerful, then the subgroups $\Pi_i(G)$ and $\gamma_i(G)$ are powerfully embedded in G for all $i \geq 1$.*

Theorem 1.6.16. *[33, Theorem 6.1.7] If G is powerful, then $G^{p^i} = \Pi_i(G)$ and G^{p^i} p.e. G for all $i \geq 0$.*

As commented at the beginning of this section about our interesting in the power structure of this family, the next results demonstrate that the subgroup of the p-th powers of G can be seen as the set of the p-th powers. Furthermore, in this class of p-groups we have the equality $G^{p^2} = (G^p)^p$, while in the general case it holds only the inclusion $(G^{p^2} \leq (G^p)^p)$. The equality can be extended for others p-th powers in a powerful p-groups.

Corollary 1.6.17. *[33, Corollary 6.1.8] Let G be powerful.*

(i) $(G^{p^i})^{p^j} = G^{p^{i+j}}$ *for all i and j.*

(ii) For p odd, $[G^{p^i}, G] \leq G^{p^{i+1}}$ and $\gamma_{i+1}(G) \leq G^{p^i}$ for all $i \geq 0$, and for $p = 2$, $[G^{2^{i-1}}, G] \leq G^{2^{i+1}}$ and $\gamma_{i+1}(G) \leq G^{4^i}$ for all $i \geq 0$.

(iii) $G^{p^{i+1}} = \Phi(G^{p^i})$ *for all $i \geq 0$.*

Theorem 1.6.18. *[33, Theorem 6.1.10] If G is powerful, then $G^{p^i} = G^{\{p^i\}}$ for all $i \geq 0$.*

The last two results of this section are not commonly found in the books containing the theory about powerful p-groups. However, they already appear in the work of Lubotzky and Mann, in 1987. The first one deals with the normal closure of a subset of a powerful p-group, while the second one is about a bound for the nilpotency class and the order of a powerful p-group.

Proposition 1.6.19. *[34, Proposition 1.10] Let $N \leq G$ be a subgroup powerfully embedded in G. If N is the normal closure of some subset of G, then N is actually generated by this subset.*

Proposition 1.6.20. *[34, Proposition 2.5] Let G be powerful, with $d(G) = d$ and $\exp(G) = p^e$. Then $cl(G) \leq e$, $|G| \leq p^{de}$ and $|M(G)| \leq p^{\frac{d(d-1)e}{2}}$.*

1.6.2 Potent p-groups

In 1969, Arganbright introduced another interesting family of p-groups, see [2], which have the property that $\gamma_n(G) \leq \gamma_m(G)^p$ for $1 < n/m < p$ where n and m are integers. In particular, he proved that if G is a finite p-group satisfying $\gamma_{p-1}(G) \leq G^p$, then the product of p-th powers of elements of G is the p-th power of one element of G.

Considering the case $n = p-1$ and $m = 1$ in Arganbright's work, González-Sánchez and Jaikin-Zapirain defined a new family of p-groups named potent (see [24]). They studied the main properties and the structure of normal subgroups of this new class of p-groups.

Definition 1.6.21. *Let G be a finite p-group. We say that G is potent if $G' \leq G^4$ for $p = 2$ or $\gamma_{p-1}(G) \leq G^p$ for $p > 2$.*

Observe that the definition of potent p-groups coincide with the of one powerful for primes 2 and 3. Moreover, in general a powerful p-group, is also potent, while the converse is not true in general. The basic properties of potent p-groups are collected in the next theorem.

Theorem 1.6.22. *[24, Theorem 2.1] Let G be a potent p-group. Then the following properties hold:*

(i) If $p = 2$, then $\gamma_{k+1}(G) \leq \gamma_k(G)^4$, and if $p > 2$, then $\gamma_{p-1+k}(G) \leq \gamma_{k+1}(G)^p$, for all $k \geq 1$.

(ii) $\gamma_k(G)$ is potent, for all $k \in \mathbb{N}$.

(iii) $\langle x, [G, G]\rangle$ is potent, for all $x \in G$.

(iv) G^p is precisely the set of p-th powers of G.

(v) If N is a normal subgroup of G, then G/N is potent.

Similarly as defined for powerful p-groups, we can define the property of being potently embedded for a normal subgroup. Observe again that in this definition the group G, where N is contained, is not necessarily potent. Moreover, if N have this property, then N is also potent.

Definition 1.6.23. *Let G be a p-group and N a normal subgroup of G. We say that N is potently embedded in G if $[N, G] \leq N^4$ for $p = 2$ and $[N, {}_{p-2}G] \leq N^p$ for p odd.*

In the sequence we have some subgroups that inherit this property when in the hypothesis there is potently embedded subgroups.

Theorem 1.6.24. *[24, Theorem 3.1] Let N and M be potently embedded subgroups of G. Then $[N^p, M] = [N, M]^p$.*

Theorem 1.6.25. *[24, Theorem 3.2] Let G be a potent p-group and N, M potently embedded subgroups of G. Then we have that*

(i) NM *is potently embedded.*

(ii) $[N, G]$ *is potently embedded.*

(iii) N^p *is potently embedded.*

The main results concerning potent p-groups have been collected in the next theorem. Observe that for the odd case if G is potent, then G is power abelian, sharing this property with abelian and powerful p-groups.

Theorem 1.6.26. *[24, Theorem 1.1] Let G be a finite potent p-group.*

(i) If $p = 2$, then

(a) The exponent of $\Omega_i(G)$ is at most 2^{i+1}, for all $i \in \mathbb{N}$.

(b) The nilpotency class of $\Omega_i(G)$ is at most $\lfloor (i+3)/2 \rfloor$.

(c) If $N \trianglelefteq G$ and $N \leq G^2$, then N is power abelian.

(d) If $N \trianglelefteq G$ and $N \leq G^4$, then N is powerful.

(ii) If $p > 2$, then

(a) The exponent of $\Omega_i(G)$ is at most p^i, for all $i \in \mathbb{N}$.

(b) The nilpotency class of $\Omega_i(G)$ is at most $(p-2)i + 1$.

(c) If $N \trianglelefteq G$, then N is power abelian.

(d) If $N \trianglelefteq G$ and $N \leq G^p$, then N is powerful.

Here, the notations $\lceil r \rceil$ and $\lfloor r \rfloor$ mean the smallest integer greater than or equal to r and the greatest integer less than or equal to r, respectively.

1.6.3 p-Groups of maximal class and coclass r

The last family of finite p-groups studied is the class of p-groups of maximal class and coclass r. Groups of maximal class are groups of coclass 1.

Firstly we can describe some properties and results for the theory of p-groups of maximal class, introduced for Blackburn in his basic material about these groups published in 1958, [9]. We based this section on [33].

Definition 1.6.27. *A group of order p^n with $n > 3$ is of maximal class if it has nilpotency class $n-1$.*

In this definition we can consider $n \geq 2$, but the p-groups of maximal class of order p^2 and p^3 are all well-known. The groups of order p^2 are always abelian, so they have class 1. There are two isomorphism classes of non-abelian groups of order p^3, which correspond to dihedral and quaternion for $p = 2$ and to the groups $M_{p^3} = \langle a, b \mid a^{p^2} = b^p = 1, a^b = a^{1+p} \rangle$ and $E_{p^3} = \langle a, b, c \mid a^p = b^p = c^p = 1, a^c = ab, [a, b] = [b, c] = 1 \rangle$ for odd p. These groups has maximal class and, so, it suffices to consider $n > 3$.

Throughout this subsection G denotes a p-group of maximal class of order p^n. The lower and the upper central series of G are

$$\begin{aligned} &G = \gamma_1(G) > \gamma_2(G) > \ldots \gamma_n(G) = \langle 1 \rangle \quad \text{and} \\ &G = \zeta_{n-1}(G) > \zeta_{n-2}(G) > \ldots > \zeta_0(G) = \langle 1 \rangle. \end{aligned}$$

They coincide and the sections of these series are all of order p, except for the first one which has order p^2 and is not cyclic. The quotients $G/\gamma_i(G)$ is also maximal class for $4 \leq i \leq n-1$.

A very interesting property of this class is about the normal subgroups and the number of maximal subgroups.

Proposition 1.6.28. *[33, Proposition 3.1.2] Let G be a p-group of maximal class. If $N \trianglelefteq G$ and $|G : N| = p^r$ for $r \geq 2$, then $N = \gamma_r(G)$. Also G has precisely $p+1$ maximal subgroups.*

Definition 1.6.29. *Let G be a finite p-group. For each i with $2 \leq i \leq n-2$ the 2-step centralizer K_i in G is defined to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$.*

Proposition 1.6.30. *[33, Proposition 3.1.4] Let G be a p-group of maximal class. Then the 2-step centralizers $K_2, \ldots, K_{n-2}$ are maximal subgroups of G.*

In this family it is common to fix the following notation: if G is a p-group of maximal class of order p^n, and if K_2 is the 2-th 2-step centralizer in G, then we define G_i by:

$$G_0 = G, \quad G_1 = K_2, \quad \text{and} \quad G_i = \gamma_i(G) \quad \text{for} \quad 2 \leq i \leq n.$$

With this notation we can describe the relation between p-th powers and commutators in a p-group of maximal class. Before, we need to define another family of finite p-groups, namely that of regular p-groups.

Definition 1.6.31. *A finite p-group G is called regular if $x^p y^p \equiv (xy)^p \pmod{H^p}$ for every $x, y \in G$, where $H = H(x,y) = \langle x, y\rangle'$.*

Corollary 1.6.32. *[33, Corollary 3.3.6] Assume that $n > p+1$.*

(i) If $1 \leq i \leq n-p+1$ then $\Omega_1(G_i) = G_{n-p+1}$ and $G_i^p = G_{i+p-1}$.

(ii) If M is a maximal subgroup of G different from G_1, then M is of maximal class and is not regular.

Remember that $K_2 = G_1$ is defined to be 2-step centralizer in G of $\gamma_2(G)/\gamma_4(G) = G_2/G_4$, that is $G_1 = C_G(G_2/G_4)$. In other words G_1 consists of the elements $x \in G$ such that $[x, G_2] \leq G_4$. The subgroup G_1 is very important as already highlighted in many works about this theory. The next lemma shows the relationship of this subgroup with the family of the potent p-groups.

Lemma 1.6.33. *Let G be a p-group of maximal class. Then G has a potent maximal subgroup or a maximal subgroup of class at most $p-1$.*

Proof. For $p = 2$ all p-groups of maximal class are known and these are known to be either dihedral, semidihedral or quaternion groups. So, for $p = 2$, G has cyclic maximal subgroup. Thus we can assume that p is odd.

On the one hand, if $|G| \leq p^{p+1}$, then its maximal subgroups have order p^p and then nilpotency class at most $p-1$.

On the other hand, assume that $|G| \geq p^{p+2}$. In this case $[G_1, G_1] = [G_1, \gamma_2(G)]$, since $|G_1 : \gamma_2(G)| = p$. Thus, as $[G_1, G_1] \leq \gamma_4(G)$, it follows that

$$\gamma_{p-1}(G_1) = [[G_1, G_1], {}_{p-3}\, G_1] \leq [\gamma_4(G), {}_{p-3}\, G_1] \leq \gamma_p(G) = G_1^p,$$

and G_1 is a potent maximal subgroup □

Now we generalize the definition of p-groups of maximal class, that is, p-groups of coclass r. Blackburn's work ([9]) has been the inspiration behind the coclass project.

Definition 1.6.34. *A group of order p^n and nilpotency class $n-r$ is said to have coclass r.*

Lemma 1.6.35. *[33, Lemma 4.1.13]*

(i) If $N \trianglelefteq G$, then the coclass of G/N is at most the coclass of G.

(ii) If $N \trianglelefteq G$ and the coclass of G is equal to the coclass of G/N, then $N = \gamma_i(G)$ for some $i \geq 2$.

Let p be a prime and r a positive integer, we define the integer $m(p,r)$ by $m(p,r) = (p-1)p^{r-1}$ for p odd and $m(2,r) = 2^{r+2}$. It is well-known that if G is a p-group of coclass r, then $\gamma_{m(p,r)}(G)$ is powerful, see for instance [33, Theorem 6.3.1 and 6.3.2]. Moreover, we have the following two results.

Theorem 1.6.36. *[33, Theorem 6.3.8] Let G be a finite 2-group of coclass r and nilpotency class c and let $m = m(2,r)$ and $s = d(\gamma_m(G))$. If $c \geq 2^{r+3}$, then the following hold:*

(i) $\gamma_i(G)^2 = \gamma_{i+s}(G)$ *for all* $i \geq m$.

(ii) $s = 2^d$ *with* $0 \leq d \leq r+1$.

Theorem 1.6.37. *[33, Theorem 6.3.9] Let G be a finite p-group of coclass r and nilpotency class c for p odd, and let $m = m(p,r)$ and $s = d(\gamma_m(G))$. If $c \geq 2p^r$, then the following hold:*

(i) $\gamma_i(G)^p = \gamma_{i+s}(G)$ *for all* $i \geq m$.

(ii) $s = (p-1)p^d$ *with* $0 \leq d \leq r-1$.

Chapter

2

The group $\nu^q(G)$

In this chapter we define the non-abelian tensor product $G \otimes H$ of two distinct groups acting "compatibly" on each other and some related constructions. Moreover, we present some generalizations of these constructions.

2.1 The non-abelian tensor square of a group and the group $\nu(G)$

An action of the group G on a group H is defined by a homomorphism $\theta : G \to Aut(H), g \mapsto \theta_g$ for all $g \in G$, where $Aut(H)$ is the group of all automorphisms of H. We write $(h)\theta_g = h^g$ which represents a right action of G on H. If θ is the trivial homomorphism, then we say that G acts trivially on H.

Let G and H be groups each of which acts upon the other (on the right) and upon themselves by conjugation such that

$$g^{(h^{g_1})} = \left(\left(g^{g_1^{-1}}\right)^h\right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left(\left(h^{h_1^{-1}}\right)^g\right)^{h_1},$$

for all $g, g_1 \in G$ and $h, h_1 \in H$. Such actions are said to be *compatible* and we say that G and H act compatibly on each other.

Considering this situation, the *non-abelian tensor product* $G \otimes H$ of G and H, as defined by Brown and Loday in [14], is the group generated by the symbols $g \otimes h$, where $g \in G$ and $h \in H$, subject to the defining relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}),$$

where $g, g_1 \in G$ and $h, h_1 \in H$. This product generalizes the usual tensor product $G/G' \otimes_{\mathbb{Z}} H/H'$ of the abelianized groups. When G acts trivially on H and H acts trivially on G, these products coincide (see [14, Proposition 1.4]). If $G = H$ and the actions are by conjugation, then we have the *non-abelian tensor square* $G \otimes G$.

Let G and G^φ be groups, isomorphic via $\varphi : G \to G^\varphi$, $g \mapsto g^\varphi$, for all $g \in G$. Remember that for arbitrary elements $x, y \in G$ we write x^y to mean the conjugate $y^{-1}xy$ of x by y; the commutator of x and y is then $[x, y] = x^{-1}y^{-1}xy$ and our commutators are left normed: $[x, y, z] = [[x, y], z]$. In 1991, Rocco [44] introduced the group $\nu(G)$ (see also Ellis and Leonard [20]), defined as

$$\nu(G) = \left\langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k^\varphi}, \forall g, h, k \in G \right\rangle. \tag{2.1}$$

The following basic properties are consequences of the defining relations of $\nu(G)$ and the commutator rules. More information on the non-abelian tensor square and the construction of $\nu(G)$ can be found in the works of Kappe ([30]) and Nakaoka and Rocco ([41]).

Lemma 2.1.1. *[44, Section 2] The following relations hold in $\nu(G)$ for all $g, h, x, y \in G$.*

(i) $[g, h^\varphi]^{[x, y^\varphi]} = [g, h^\varphi]^{[x, y]}$.

(ii) $[g, h^\varphi, x^\varphi] = [g, h, x^\varphi] = [g, h^\varphi, x] = [g^\varphi, h, x^\varphi] = [g^\varphi, h^\varphi, x] = [g^\varphi, h, x]$.

(iii) $[g, g^\varphi]$ *is central in* $\nu(G)$, $\forall g \in G$.

(iv) $[g, h^\varphi][h, g^\varphi]$ *is central in* $\nu(G)$, $\forall g, h \in G$.

(v) $[g, g^\varphi] = 1$, $\forall g \in G'$.

(vi) $[[g, h^\varphi], [x, y^\varphi]] = [[g, h], [x, y]^\varphi]$.

The first interesting result about this group is that it inherits some properties of the initial group, such as nilpotency and solubility.

Proposition 2.1.2. *[44, Proposition 2.4] Let G be a finite π-group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\nu(G)$ is also a finite π-group, finite nilpotent or solvable of finite degree.*

Let N be a normal subgroup of a finite group G. We set $\overline{G}$ for the quotient group G/N and the canonical epimorphism $\pi : G \to \overline{G}$ gives rise to an epimorphism $\widetilde{\pi} : \nu(G) \to \nu(\overline{G})$ such that $g \mapsto \overline{g}$, $g^\varphi \mapsto \overline{g^\varphi}$, where $\overline{G^\varphi} = G^\varphi/N^\varphi$ is identified with $\overline{G}^\varphi$.

Lemma 2.1.3. *[44, Proposition 2.5 and Remark 3] With the above notation we have* *(i)* $[N, G^{\varphi}] \trianglelefteq \nu(G)$, $[G, N^{\varphi}] \trianglelefteq \nu(G)$.

(ii) $\ker(\widetilde{\pi}) = [N, G^{\varphi}][G, N^{\varphi}] \cdot \langle N, N^{\varphi}\rangle = ([N, G^{\varphi}][G, N^{\varphi}] \cdot N) \cdot N^{\varphi}$.

(iii) There is an exact sequence

$$1 \to [N, G^{\varphi}][G, N^{\varphi}] \to [G, G^{\varphi}] \to [G/N, (G/N)^{\varphi}] \to 1.$$

We will denote by $\mathcal{Q}$ the subgroup $[N, G^{\varphi}][G, N^{\varphi}] \cdot \langle N, N^{\varphi}\rangle$ in $\nu(G)$, where the dot means internal semidirect product.

The main motivation to introduce $\nu(G)$ is that its subgroup $\Upsilon(G) = [G, G^{\varphi}]$ is isomorphic to the non-abelian tensor square $G \otimes G$, as defined by Brown and Loday in their seminal paper [14].

Proposition 2.1.4. *[44, Proposition 2.6] The map $\tau : G \otimes G \to \Upsilon(G)$ defined on the generators by $\tau(g \otimes h) = [g, h^{\varphi}]$ extends to an isomorphism from $G \otimes G$ to $\Upsilon(G)$.*

Since $\Upsilon(G)$ is a normal subgroup of $\nu(G)$, by defining relations, we have the following equality $\nu(G) = (\Upsilon(G) \cdot G) \cdot G^{\varphi}$. This product gives an elegant description for the lower central series and the derived series of $\nu(G)$.

Theorem 2.1.5. *[44, Theorems 3.1 and 3.3]*

(i) $\gamma_i(\nu(G)) = \gamma_i(G)\gamma_i(G^{\varphi})[\gamma_{i-1}(G), G^{\varphi}]$, *for* $i \geq 2$.

(ii) $\nu(G)^{(i)} = G^{(i)}(G^{\varphi})^{(i)}[G^{(i-1)}, (G^{\varphi})^{(i-1)}]$, *for* $i \geq 2$.

Corollary 2.1.6. *[44, Corollary 3.2 and 3.4]*

(i) Let G be a nilpotent group of class c. Then $\nu(G)$ is a nilpotent group of class at most $c + 1$.

(ii) Let G be a solvable group of derived length l. Then $\nu(G)$ is solvable of derived length at most $l + 1$.

During this work, we establish some bounds for the exponents of the $\nu(G)$ and some of its sections.In [45], Rocco considered two epimorphisms

$$\begin{array}{rccc} \rho : & \nu(G) & \to & G, \\ & g & \mapsto & g \\ & g^{\varphi} & \mapsto & g \end{array} \quad \text{and} \quad \begin{array}{rccc} \rho' = \rho|_{\Upsilon(G)} : & \Upsilon(G) & \to & G', \\ & [g, h^{\varphi}] & \mapsto & [g, h] \end{array} . \qquad (2.2)$$

We define $\Theta(G)$ to be the kernel of ρ and $\mu(G)$ to the kernel of ρ'. It then follows that $\mu(G) = \Theta(G) \cap [G, G^{\varphi}]$. Using the epimorphism ρ we have the following exact sequences

$$1 \to [G, G^{\varphi}] \to \nu(G) \to G \times G \to 1$$

and

$$1 \to \Theta(G) \to \nu(G) \to G \to 1.$$

Considering these exact sequences we can deduce that $\nu(G)/\mu(G)$ is isomorphic to a subgroup of $G \times G \times G$ and so, $\exp(\nu(G))$ divides $\exp(G) \cdot \exp(\mu(G))$.

The group $\nu(G)$ has a section isomorphic to the Schur Multiplier of the group G. Remember that for the finite case the Schur Multiplier coincides with the second homology group. The next proposition show us this relationship.

Proposition 2.1.7. *[45, Proposition 2.8] The section $\mu(G)/\Delta(G)$ of $\nu(G)$ is isomorphic to the second homology group $H_2(G, \mathbb{Z}) = M(G)$, where $\Delta(G) = \langle [g, g^{\varphi}] \mid g \in G \rangle$.*

The subgroup $\Delta(G)$, which by Lemma 2.2.1 is central in $\nu(G)$, is such that the quotient $\Upsilon(G)/\Delta(G)$ is isomorphic to the *exterior square* $G \wedge G$ (see [35] or [13] for more details). The isomorphism of the last proposition can be written as the following exact sequence

$$1 \to \Delta(G) \to \mu(G) \to M(G) \to 1.$$

Using this exact sequence, it follows that $\exp(\mu(G))$ divides $\exp(M(G)) \cdot \exp(\Delta(G))$. Moreover, as $[g^j, g^{\varphi}] = [g, g^{\varphi}]^j$ for any $g \in G$, we have $\exp(\Delta(G))$ divides $\exp(G)$. Consequently,

$$\exp(\nu(G)) \text{ divides } \exp(G)^2 \cdot \exp(M(G)).$$

Assume that 2 does not divide $|G^{ab}|$, where $G^{ab} = G/G'$. According to [10, Corollary 1.4], we deduce that $\mu(G) \cong M(G) \times \Delta(G)$ and so,

$$\exp(\nu(G)) \text{ divides } \exp(G) \cdot \max\{\exp(G), \exp(M(G))\}.$$

Recently, in 2017([6]), Bastos, Nakaoka e Rocco establishes a sufficient condition for a finitely generated non-abelian tensor product to be finite and using this they examined certain finiteness conditions for $\nu(G)$. But, in 1994, Rocco ([45]) gave an alternative proof for the finiteness of the group $\nu(G)$ when G is a finite group. This proof was based on the relation between $\nu(G)$ and the Sidki's group, $\chi(G)$. The group $\chi(G)$ was introduced by Sidki in [50] and it is defined by

$$\chi(G) = \langle G \cup G^{\varphi} \mid [g, g^{\varphi}] = 1, \forall g \in G \rangle, \tag{2.3}$$

where again G^{φ} is an isomorphic copy of the group G.

The weak commutativity group, $\chi(G)$, as this group is known maps onto G by $g \mapsto g$, $g^{\varphi} \mapsto g$ with kernel $L(G) = \langle g^{-1}g^{\varphi} \mid g \in G \rangle$, and it maps onto $G \times G$ by $g \mapsto (g, 1), g^{\varphi} \mapsto (1, g)$ with kernel $D(G) = [G, G^{\varphi}]$. It is an important fact that $L(G)$ and $D(G)$ commute. Moreover, we can define $T(G)$ to be the subgroup of $G \times G \times G$ generated by $\{(g, g, 1), (1, g, g) \mid g \in G\}$. Then $\chi(G)$ maps onto $T(G)$ by $g \mapsto (g, g, 1)$, $g^{\varphi} \mapsto (1, g, g)$, with kernel $W(G) = L(G) \cap D(G)$, an abelian group. In particular, the quotient $\chi(G)/W(G)$ is isomorphic to a subgroup of $G \times G \times G$. A further normal subgroup of $\chi(G)$ is $R(G) = [G, L(G), G^{\varphi}]$, where the quotient $W(G)/R(G)$ is isomorphic to the Schur Multiplier $M(G)$. The relation between the constructions $\nu(G)$ and $\chi(G)$ is given by the next theorem.

Theorem 2.1.8. *[45, Remark 4] For all finite group G,*

$$\frac{\chi(G)}{R(G)} \cong \frac{\nu(G)}{\Delta(G)}.$$

As in Proposition 2.1.2, the Sidki's group satisfies the same result.

Theorem 2.1.9. *[50, Theorem C(i)] Let $\mathcal{P}$ be one of the following group theoretic properties: finite π-group (π a set of primes), finite nilpotent, solvable of finite degree, perfect. Then*

$$G \textit{ is a } \mathcal{P} - \textit{group} \Rightarrow \chi(G) \textit{ is a } \mathcal{P} - \textit{group}.$$

In [43], Rocco describes some bounds for the order and the nilpotency class of $\chi(G)$, when G is a finite p-group, p odd. In particular, he obtained some commutator relation for the subgroups of $\chi(G)$.

Lemma 2.1.10. *[43, Lemma 3.2.1] Let p be an odd prime and G a finite p-group.*

(i) $[D(G), G] = [D(G), \chi(G)]$.

(ii) $[D(G), G] \leq [\gamma_2(G), G^{\varphi}]$.

Later, Gupta, Rocco and Sidki ([27]) showed an interesting description of some commutator subgroups in $\chi(G)$

$$[G^{\varepsilon_1}, \ldots, G^{\varepsilon_n}] = [G^{\varphi},_{n-1} G]\gamma_n(G)^{\chi(G)}\gamma_n(G^{\varphi})^{\chi(G)},$$

where $\varepsilon_i \in \{1, \varphi\}$, $i \in \{1, \ldots, n\}$, and $H^{\chi(G)}$ denotes the normal closure of H in $\chi(G)$ [27, Lemma 2.2]. The next result is an immediate consequence of the above results.

Lemma 2.1.11. *Let p be an odd prime. Let G be a finite p-group with nilpotency class at most 2. Then $\gamma_3(\chi(G)) = [\gamma_2(G), G^\varphi]$.*

2.2 An extension to the group $\nu^q(G)$

A modular version of the operator ν was considered in [15], where for any non-negative integer q the authors introduced and studied a group $\nu^q(G)$, which in turn is an extension of the so called non-abelian q-tensor square of G, denoted by $G \otimes^q G$, first defined by Conduché and Rodriguez-Fernandez in [16] (see also [21], [12]). Following [21], we start this section by defining this group.

Let E be a group and q a non-negative integer. Consider G and H normal subgroups of E, where the actions involved are by conjugations. The *non-abelian q-tensor product*, $G \otimes^q H$, of G and H, is defined as the group generated by the symbols $g \otimes h$ and $\widehat{k}$, where $g \in G$, $h \in H$ and $k \in G \cap H$, subject to the defining relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h) \tag{2.4}$$

$$g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}) \tag{2.5}$$

$$(g \otimes h)^{\widehat{k}} = g^{k^q} \otimes h^{k^q} \tag{2.6}$$

$$\widehat{kk_1} = \widehat{k} \prod_{i=1}^{q-1} (k \otimes (k_1^{-i})^{k^{q-1-i}}) \widehat{k} \tag{2.7}$$

$$[\widehat{k}, \widehat{k_1}] = k^q \otimes k_1^q \tag{2.8}$$

$$\widehat{[g,h]} = (g \otimes h)^q \tag{2.9}$$

for all $g, g_1 \in G$, $h, h_1 \in H$ and $\widehat{k}, \widehat{k_1} \in G \cap H$.

If $G = H$ we have the definition of the non-abelian q-tensor square of the group G, $G \otimes^q G$. For $q = 0$ we define the 0-tensor product $G \otimes^0 H$ as the group generated by the symbols $g \in G$ and $h \in H$ considering only relations (2.4) and (2.5). This means that for $q = 0$, the 0-tensor product is the non-abelian tensor product $G \otimes H$.

Now, we are in a position to define a modular version of the operator ν. In order to briefly describe the group $\nu^q(G)$, again consider G and G^φ be groups, isomorphic via $\varphi : G \to G^\varphi$, $g \mapsto g^\varphi$, for all $g \in G$ and let $\mathcal{K} = \{\widehat{k} \mid k \in G\}$ be a set of symbols, one for each element of G. For $q = 0$ we set $\mathcal{K}$ equal to the empty set. Write $F(\mathcal{K})$ for the free group on $\mathcal{K}$ and consider the free product $\nu(G) * F(\mathcal{K})$. Denote by J the normal closure in $\nu(G) * F(\mathcal{K})$ of the following elements:

$$g^{-1}\widehat{k}g(\widehat{k^g})^{-1} \tag{2.10}$$

$$(g^\varphi)^{-1}\widehat{k}(g^\varphi)(\widehat{k^g})^{-1} \tag{2.11}$$

$$(\widehat{k})^{-1}[g,h^\varphi]\widehat{k}[g^{k^q},(h^{k^q})^\varphi]^{-1} \tag{2.12}$$

$$(\widehat{k})^{-1}\widehat{kk_1}(\widehat{k_1})^{-1}\prod_{i=1}^{q-1}[k,(k_1^{-i})^\varphi]^{-k^{q-1-i}} \tag{2.13}$$

$$[\widehat{k},\widehat{k_1}][k^q,(k_1^q)^\varphi]^{-1} \tag{2.14}$$

$$\widehat{[g,h]}[g,h^\varphi]^{-q} \tag{2.15}$$

for all $\widehat{k},\widehat{k_1} \in \mathcal{K}$ and $g,h \in G$.

According to [15], the group $\nu^q(G)$ is then defined to be the quotient group

$$\nu^q(G) := (\nu(G) * F(\mathcal{K}))/J. \tag{2.16}$$

Notice that for $q = 0$ the sets of relations (2.10) to (2.15) are empty; in this case we have $\nu^0(G) = \nu(G) * F(\mathcal{K}))/J \cong \nu(G)$.

Let $R_1,\ldots,R_6$ be the sets of relations corresponding to (2.10), ..., (2.15), respectively, and let R be their union, $R = \cup_{i=1}^6 R_i$. Moreover, consider $S = S_1 \cup S_2$ where

$$S_1 = \{[g^x,(h^x)^\varphi] = [g,h^\varphi]^x \mid g,h,x \in G\}$$
$$S_2 = \{[g^x,(h^x)^\varphi] = [g,h^\varphi]^{x^\varphi} \mid g,h,x \in G\}.$$

Therefore the group $\nu^q(G)$ has the following presentation $\nu^q(G) = \langle G, G^\varphi, \mathcal{K} \mid S, R\rangle$.

We denote by K the subgroup of $\nu^q(G)$ generated by the images of $\mathcal{K}$ in $\nu^q(G)$. By relation (2.12) we see that K normalizes the subgroup $[G,G^\varphi]$ in $\nu^q(G)$, and is normalized by G and G^φ; therefore, $\Upsilon^q(G) := [G,G^\varphi]K$ is a normal subgroup of $\nu^q(G)$. Then we have

$$\nu^q(G) = (\Upsilon^q(G)\cdot G)\cdot G^\varphi, \tag{2.17}$$

where the dots indicate (internal) semidirect products.

For the sake of completeness, we reproduce in sequel some results of [15] concerning $\nu^q(G)$, which extend similar results about $\nu(G)$, found in [44]. For example, in the next lemma, they summarized some basic consequences of the defining relations of $\nu^q(G)$. Some of their properties are the same property of $\nu(G)$.

Lemma 2.2.1. *[15, Lemma 2.4] Suppose that $q \geq 0$. The following relations hold in $\nu^q(G)$, for all $g, h, x, y \in G$.*

(i) $[g, h^\varphi]^{[x,y^\varphi]} = [g, h^\varphi]^{[x,y]}$.

(ii) $[g, h^\varphi, x^\varphi] = [g, h, x^\varphi] = [g, h^\varphi, x] = [g^\varphi, h, x^\varphi] = [g^\varphi, h^\varphi, x] = [g^\varphi, h, x]$.

(iii) If $h \in G'$ (or if $g \in G'$), then $[g, h^\varphi][h, g^\varphi] = 1$.

(iv) $[\widehat{x}, [g, h^\varphi]] = [\widehat{x}, [g, h]]$.

(v) $(\widehat{x})^g = \widehat{x}[x^q, g^\varphi]$.

(vi) If $[g, h] = 1$, then $[g, h^\varphi]$ and $[h, g^\varphi]$ are central elements of $\nu^q(G)$, of the same finite order dividing q. If, in addition, g, h are torsion elements of orders $o(g), o(h)$, respectively, then the order of $[g, h^\varphi]$ divides $gcd(q, o(g), o(h))$.

(vii) $[g, g^\varphi]$ is central in $\nu^q(G)$, for all $g \in G$.

(viii) $[g, h^\varphi][h, g^\varphi]$ is central in $\nu^q(G)$.

(ix) $[g, g^\varphi] = 1$, for all $g \in G'$.

(x) If $[x, g] = 1 = [x, h]$, then $[g, h, x^\varphi] = 1 = [[g, h]^\varphi, x]$.

Furthermore, as it was for $\nu(G)$ in the particular case where $q = 0$, an elegant description of the lower central series and the derived series was given in [15]. This description was very useful in our work.

Proposition 2.2.2. *[15, Proposition 2.7] Let G be any group and $q \geq 0$. Then, for all $i \geq 2$,*

(i) $\gamma_i(\nu^q(G)) = ([\gamma_{i-1}(G), G^\varphi] \cdot \gamma_i(G)) \cdot \gamma_i(G^\varphi)$.

(ii) $\nu^q(G)^{(i)} = ([G^{(i-1)}, (G^\varphi)^{(i-1)}] \cdot G^{(i)}) \cdot (G^\varphi)^{(i)}$.

A natural question that arises when we build a new structure based on an already known structure is the following: is a certain property in the base structure inherited by the new structure? In other words, what properties does the group $\nu^q(G)$ inherit from the group G? The next theorem gives us some answers to this question.

Recall that a group G is said to be polycyclic if it has a cyclic series, by which we mean a series with cyclic factors.

Theorem 2.2.3. *[15, Theorem 2.8] Let G be a group and q a non-negative integer.*

(i) If G is a finite π-group (π a set of primes), then $\nu^q(G)$ is a finite π-group.

(ii) If G is polycyclic, then $\nu^q(G)$ is polycyclic.

(iii) If G is nilpotent of class c, then $\nu^q(G)$ is nilpotent of class at most $c+1$.

(iv) If G is solvable of derived length l, then $\nu^q(G)$ is solvable of derived length at most $l+1$.

The relationship between the two constructions, $\nu^q(G)$ and $G \otimes^q G$, which generalizes Rocco's result in [44, Proposition 2.6], is given by the next proposition. Thus, the new construction has proved to be a very good extension for $\nu(G)$, obviously with the necessary adaptations.

Proposition 2.2.4. *[15, Proposition 2.9] $\Upsilon^q(G)$ is isomorphic to the non-abelian q-tensor square $G \otimes^q G$, for any $q \geq 0$.*

Using the description of the group $\nu^q(G)$ given by the equation (2.17), it is possible to see that there exists homomorphisms

$$\begin{array}{rccc} \rho: & \nu^q(G) & \to & G, \\ & g & \mapsto & g \\ & g^{\varphi} & \mapsto & g \\ & \widehat{k} & \mapsto & k^q \end{array} \quad \text{and} \quad \begin{array}{rccc} \rho' = \rho|_{\Upsilon^q(G)}: & \Upsilon^q(G) & \to & G, \\ & [g, h^{\varphi}] & \mapsto & [g,h] \\ & \widehat{k} & \mapsto & k^q \end{array}. \tag{2.18}$$

Let $\theta^q(G) := ker(\rho)$, $\mu^q(G) := ker(\rho') = \Upsilon^q(G) \cap \theta^q(G)$ and let $\Delta^q(G) := \langle [g, g^{\varphi}] \mid g \in G \rangle \leq \mu^q(G)$. Thus we have

$$\frac{\Upsilon^q(G)}{\mu^q(G)} \cong Im(\rho') = G'G^q,$$

where G^q is the subgroup of G generated by all q-th powers g^q for all $g \in G$. Moreover, in [15] the authors showed a conection between this construction and the second homology group of G with coefficients in the trivial G-module $\mathbb{Z}_q$.

Theorem 2.2.5. *[15, Theorem 2.12]) Let $H_2(G, \mathbb{Z}_q)$ be the second homology group of G with coefficients in the trivial G-module $\mathbb{Z}_q$. Then*

$$\frac{\mu^q(G)}{\Delta^q(G)} \cong \mathrm{H}_2(G, \mathbb{Z}_q).$$

The last quotient of this construction we will highlight is

$$\frac{\Upsilon^q(G)}{\Delta^q(G)} \cong G \wedge^q G,$$

the q-exterior square of G, for all $q \geq 0$, which is defined by the quotient of $G \otimes^q G$ by its normal subgroup generated by the elements $k \otimes k$ for all $k \in G$.

Using the previous considerations about the group $\nu^q(G)$ we can construct the following diagram, according to [15].

$$\begin{tikzcd}
 & \nu^q(G) \arrow[d, no head] \arrow[ddl, dashed, no head, bend right, "G \times G"'] & \\
 & \Upsilon^q(G)\theta^q(G) \arrow[dl, no head] \arrow[dr, no head] & \\
\Upsilon^q(G) \arrow[dr, no head, "G'G^q"] \arrow[ddr, dashed, no head, bend right, "G \wedge^q G"'] & & \theta^q(G) \arrow[dl, no head] \\
 & \mu^q(G) \arrow[d, no head, "H_2(G,\mathbb{Z}_q)"] & \\
 & \Delta^q(G) \arrow[d, no head] & \\
 & \{1\} &
\end{tikzcd}$$

In addition, the structure of this construction for quotient groups is showed. For this, let N be a normal subgroup of a finite group G. We set $\overline{G}$ for the quotient group G/N and the canonical epimorphism $\pi : G \to \overline{G}$ extends to an epimorphism $\overline{\pi} : \nu^q(G) \to \nu^q(\overline{G})$ such that $g \mapsto \overline{g}$, $h^\varphi \mapsto \overline{h^\varphi}$ and $\widehat{k} \mapsto \overline{k} = \widehat{\overline{k}}$, for all $g \in G$, $h^\varphi \in G^\varphi$ and $\widehat{k} \in \mathcal{K}$ where $\overline{G^\varphi} = G^\varphi/N^\varphi$ is identified with $\overline{G}^\varphi$ and $\overline{\mathcal{K}} = \overline{\pi}(\mathcal{K}) = \{\overline{k} \mid k \in G\} = \{\widehat{\overline{k}} \mid \overline{k} \in \overline{G}\}$ is a set in one-to-one correspondence with $\overline{G}$. Using this notation it is possible to generalize the results obtained by Rocco for the case $q = 0$ in [44, Proposition 2.5 and Remark 3].

Lemma 2.2.6. *[15, Lemma 2.14] With the above notation, let $\mathcal{N} = \{\widehat{x} \mid x \in N\}$. Then:*

(i) $[N, G^\varphi]$ *and* $[G, N^\varphi]$ *are normal subgroups of* $\nu^q(G)$.

(ii) $\ker(\overline{\pi}) = \langle N, N^\varphi, \mathcal{N}\rangle [N, G^\varphi][G, N^\varphi]$.

(iii) $\ker(\overline{\pi}) \cap \Upsilon^q(G) = [N, G^\varphi][G, N^\varphi]\langle\mathcal{N}\rangle$.

We will denote $\ker(\overline{\pi})$ by $\mathcal{R}$. Using the above notation there is an exact sequence

$$1 \to [N, G^\varphi][G, N^\varphi]\langle\mathcal{N}\rangle \to [G, G^\varphi]K \to \left[\frac{G}{N}, \left(\frac{G}{N}\right)^\varphi\right] \left\langle \widehat{\frac{G}{N}} \right\rangle \to 1, \qquad (2.19)$$

where we can use the identification $\Upsilon^q(\frac{G}{N}) = \left[\frac{G}{N}, \left(\frac{G}{N}\right)^{\varphi}\right] \left\langle \widehat{\frac{G}{N}} \right\rangle$.

To finish this section, we will summarize the descriptions of the group $\nu^q(G)$ when G is a cyclic group (finite and infinite), obtained in the Section 3 of [15].

Theorem 2.2.7. *[15, Theorem 3.1] Let C_n (resp. C_∞) be the cyclic group of order n (resp. ∞), q a non-negative integer and $d = gcd(n, q)$. Then*

$$C_\infty \otimes^q C_\infty \cong C_\infty \times C_q,$$
$$C_n \otimes^q C_n \cong \begin{cases} C_n \times C_d, & if\ d \text{ is odd}, \\ C_n \times C_d, & if\ d \text{ is even and either } 4|n \text{ or } 4|q, \\ C_{2n} \times C_{d/2}, & \text{otherwise}. \end{cases}$$

To simplify some our computations we can write the group $\nu^q(G)$ considering G as a cyclic group of order n and the infinity cyclic. For this we will use the last theorem and the useful description of the group $\nu^q(G)$ as the semi-direct product $(\Upsilon^q(G) \cdot G) \cdot G^{\varphi}$

Lemma 2.2.8. *Let C_n (resp. C_∞) be the cyclic group of order n (resp. ∞), q a non-negative integer and $d = gcd(n, q)$. Then*

$$\nu^q(C_\infty) \cong ((C_\infty \times C_q) \cdot C_\infty) \cdot C_\infty,$$
$$\nu^q(C_n) \cong \begin{cases} ((C_n \times C_d) \cdot C_n) \cdot C_n, & if\ d \text{ is odd}, \\ ((C_n \times C_d) \cdot C_n) \cdot C_n, & if\ d \text{ is even and either } 4|n \text{ or } 4|q, \\ ((C_{2n} \times C_{d/2}) \cdot C_n) \cdot C_n, & \text{otherwise}. \end{cases}$$

Chapter

3

Some investigations for the groups $\nu(G)$ and $\chi(G)$

In this chapter we present some results obtained in joint works with Bastos, de Melo, Monetta and Nunes ([5] and [4]) concerning the groups $\nu(G)$ and $\chi(G)$, by considering a finite p-group G satisfying some additional property such as being powerful or potent, among others.

In the first section we discuss and we prove results related to the Sidki's *weak commutativity group* $\chi(G)$. In the second section we present results in the context of Rocco's group $\nu(G)$ obtained in [5] and [4]. The results presented in the last section will be proved in subsequent chapters, as particular cases of a most general situation.

3.1 The group $\chi(G)$

In [38], Moravec proved that if G is a powerful p-group, then the non-abelian tensor square $[G, G^{\varphi}]$ and the derived subgroup $\nu(G)'$ are powerfully embedded in $\nu(G)$. Moreover, the exponent $\exp(\nu(G)')$ divides $\exp(G)$. The following result is an extension of Moravec's results [38] to the context of the weak commutativity group.

Theorem 3.1.1. *[5, Theorem C] Let p be an odd prime and let G be a powerful p-group with* $\exp(G) = p^e$.

(i) If $k \geqslant 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $D(G) = [G, G^{\varphi}]$ are powerfully embedded in $\chi(G)$.

(ii) If $p = 3$, then $\exp(\chi(G))$ *divides* $3 \cdot \exp(G)$. *If $p \geq 5$, then* $\exp(\chi(G)) = exp(G)$.

Proof. (i) First we will prove that $D(G)$ is powerfully embedded in $\chi(G)$, that is, $[D(G),\chi(G)] \leq D(G)^p$. By Lemma 2.1.10,

$$[D(G),\chi(G)] = [D(G),G] \leq [\gamma_2(G),G^{\varphi}] \leq [G^p,G^{\varphi}].$$

So, it remains to prove that $[G^p,G^{\varphi}] \leq D(G)^p$. Let $x,y \in G$. By Theorem 1.6.7

$$[x^p,y^{\varphi}] \equiv [x,y^{\varphi}]^p \pmod{\gamma_2(L)^p\gamma_p(L)},$$

where $L = \langle x,[x,y^{\varphi}]\rangle$. Note that $\gamma_2(L) \leq [D(G),G] \leq D(G)$. Therefore $\gamma_2(L)^p \leq D(G)^p$. On the other hand, $\gamma_p(L) \leq [D(G),\,_{p-1}\,\chi(G)] \leq [D(G),\,_2\,\chi(G)]$. Consequently, $[x^p,y^{\varphi}] \in D(G)^p[D(G),\,_2\,\chi(G)]$ for any $x,y \in G$. Therefore $[D(G),\chi(G)] \leq D(G)^p[D(G),\,_2\,\chi(G)]$ and by Lemma 1.6.9 we obtain that $[D(G),\chi(G)] \leq D(G)^p$ as required.

It remains to prove that $\gamma_k(\chi(G))$ is powerfully embedded in $\chi(G)$. First, assume that $k=2$. We need to prove that

$$\gamma_3(\chi(G)) \leq \gamma_2(\chi(G))^p.$$

Note that $M = \gamma_2(\chi(G))^p\gamma_4(\chi(G))$ is a normal subgroup of $\chi(G)$. By Lemma 1.6.9, without loss of generality we can assume that $M=1$. In particular, we have that $[G^{\varphi},G^{\varphi}]^p = [G,G]^p = [G,G^{\varphi}]^p = 1$. Since G is powerful, it follows that G and G^{φ} are nilpotent groups of class at most 2. By Lemma 2.1.11, $\gamma_3(\chi(G)) = [\gamma_2(G),G^{\varphi}] \leq [G^p,G^{\varphi}]$. As in the previous case we have that $[G^p,G^{\varphi}] \leq D(G)^p \leq \gamma_2(\chi(G))^p$. Since $M=1$, follows that $\gamma_2(\chi(G))^p = 1$. By Lemma 1.6.9 this means that $\gamma_3(\chi(G)) \leq \gamma_2(\chi(G))^p$, as wished.

Now assume that $k>2$. We will prove by induction on k. Suppose by induction hypothesis that $[\gamma_k(\chi(G)),\chi(G)] \leq \gamma_k(\chi(G))^p$. Using Theorem 1.6.10 it follows that

$$\begin{aligned}
[\gamma_{k+1}(\chi(G)),\chi(G)] &\leq [\gamma_k(\chi(G))^p,\chi(G)]\\
&\leq [\gamma_k(\chi(G)),\chi(G)]^p[\chi(G),\,_p\,\gamma_k(\chi(G))]\\
&\leq \gamma_{k+1}(\chi(G))^p[\gamma_{k+1}(\chi(G)),\chi(G),\,_{p-2}\,\chi(G)]\\
&\leq \gamma_{k+1}(\chi(G))^p[\gamma_{k+1}(\chi(G)),\chi(G),\chi(G)].
\end{aligned}$$

Therefore, by Lemma 1.6.9 we obtain

$$[\gamma_{k+1}(\chi(G)),\ \chi(G)] \leq \gamma_{k+1}(\chi(G))^p.$$

(ii) First suppose $p=3$. Note that, by the first item,

$$\gamma_{2(3-1)}(\chi(G)) = [\gamma_3(\chi(G)), \chi(G)] \leq \gamma_3(\chi(G))^3.$$

Applying Lemma 1.6.11 to $k=2, r=p=3$ and $s=1$, we have that $\exp(\Omega_e(\chi(G)))$ divides p^{e+1}. Consequently, $\exp(\Omega_e(\chi(G))) = \exp(\chi(G))$ divides p^{e+1} since $\chi(G)$ is generated by G and G^{φ}.

Now, assume that $p \geqslant 5$. It is clear that $\gamma_{p-1}(\chi(G)) \leq [\gamma_2(\chi(G)), \chi(G)]$. Now, again by last item, we deduce that

$$\gamma_{p-1}(\chi(G)) \leq [\gamma_2(\chi(G)), \chi(G)] \leq \chi(G)^p.$$

Then, we can apply Lemma 1.6.11 to $k=s=r=1$ obtaining that $\exp(\Omega_e(\chi(G))) = \exp(\chi(G))$ divides p^e. Moreover, $\chi(G)$ is a potent p-group. In particular (see [24] for details), $\chi(G)$ is a power abelian p-group whenever p is an odd prime. As $\chi(G)$ is generated by G and G^{φ} we have $\exp(\chi(G)) = \exp(G)$, which completes the proof. □

The above result is no longer valid if we drop the assumption that p is an odd prime (see Example 3.1.2, below). Moreover, we do not know if the last theorem can be extended to potent p-groups, $p \geqslant 5$. What is still lacking is a convenient description of all terms of the lower central series $\gamma_n(\chi(G))$, see [27, Section 2] and [43] for more details.

Example 3.1.2. *[5, Example 4.3] Let $G = \langle a, b, c\rangle$ be the elementary abelian 2-group of rank 3. Then $\chi(G)$ is a nilpotent group of class 3 and order 1024. The following is a consistent polycyclic presentation of $\chi(G)$:*

$\chi(G) = \langle x_1, \cdots, x_{10} \mid x_1^2=1,\ x_2^2=1,\ x_3^2=1,\ x_4^2=1,\ x_5^2=1,\ x_6^2=1,\ x_7^2=1,\ x_8^2=1,\ x_9^2=1,\ x_{10}^2=1,\ x_4^{x_2}=x_4x_7,\ x_4^{x_3}=x_4x_8,\ x_5^{x_1}=x_5x_7,\ x_5^{x_3}=x_5x_9,\ x_6^{x_1}=x_6x_8,\ x_6^{x_2}=x_6x_9,\ x_7^{x_3}=x_7x_{10},\ x_7^{x_6}=x_7x_{10},\ x_8^{x_2}=x_8x_{10},\ x_8^{x_5}=x_8x_{10},\ x_9^{x_1}=x_9x_{10},\ x_9^{x_4}=x_9x_{10}\rangle$.

In terms of the original generators we have $x_1=a, x_2=b, x_3=c, x_4=a^{\varphi}, x_5=b^{\varphi}, x_6=c^{\varphi}, x_7=[a,b^{\varphi}], x_8=[a,c^{\varphi}], x_9=[b,c^{\varphi}], x_{10}=[a,b^{\varphi},c]$. Therefore

$$D(G) = \langle [a,b^{\varphi}], [a,c^{\varphi}], [b,c^{\varphi}], [a,b^{\varphi},c]\rangle \cong \mathbb{Z}_2^4$$

and

$$[D(G), \chi(G)] = [D(G), G] = \langle [a,b^{\varphi},c]\rangle \cong \mathbb{Z}_2.$$

Thus $D(G)^4 = 1$ while $[D(G),\chi(G)] \neq 1$. This shows that $D(G)$ is not powerfully embedded in $\chi(G)$ in the even case.

Theorem 3.1.3. *Let p be an odd prime and G a powerful p-group. Then the i-th term of the derived series $\chi(G)^{(i)}$ is powerfully embedded in $\chi(G)$, for $i \geq 1$.*

Proof. We will prove it by induction on i. Since $\gamma_2(\chi(G)) = \chi(G)^{(1)}$, the base step of our induction was proved in the last result. Now, suppose by induction that

$$[\chi(G)^{(i)}, \chi(G)] \leq (\chi(G)^{(i)})^p \text{ if } p > 2.$$

The particular case of the Three Subgroups Lemma and Theorem 1.6.10 give us

$$\begin{aligned}
[\chi(G)^{(i+1)}, \chi(G)] &= [\chi(G)^{(i)}, \chi(G)^{(i)}, \chi(G)] \leq [\chi(G), \chi(G)^{(i)}, \chi(G)^{(i)}] \\
&\leq [(\chi(G)^{(i)})^p, \chi(G)^{(i)}] \\
&\leq [\chi(G)^{(i)}, \chi(G)^{(i)}]^p[\chi(G)^{(i)}, {}_p\, \chi(G)^{(i)}] \\
&\leq (\chi(G)^{(i+1)})^p[\chi(G)^{(i)}, \chi(G)^{(i)}, {}_{p-1}\, \chi(G)] \\
&\leq (\chi(G)^{(i+1)})^p[\chi(G)^{(i+1)}, \chi(G), \chi(G)].
\end{aligned}$$

By Lemma 1.6.9 we then get $[\chi(G)^{(i+1)}, \chi(G)] \leq (\chi(G)^{(i+1)})^p$, as wished. □

3.2 The group $\nu(G)$

In this section we present the results about the group $\nu(G)$ for some type of finite p-group. In each case we add some type of power commutator relation on the group G.

In [38], Moravec proved that if G is a powerful p-group, then the non-abelian tensor square $[G, G^\varphi]$ and the derived subgroup $\nu(G)'$ are powerfully embedded in $\nu(G)$. Moreover, the exponent $\exp(\nu(G)')$ divides $\exp(G)$. Our main goal is to extend it for potent's family.

For this we need to prove the next proposition.

Proposition 3.2.1. *[5, Proposition 3.3] Let $p \geq 3$ be a prime and $n \in \mathbb{N}$ such that $1 < n < p$. Suppose that G is a finite p-group such that $\gamma_n(G) \leq G^p$. Then*

(i) $\gamma_{n+1}(\nu(G)) \leq \gamma_2(G)^p \gamma_2(G^\varphi)^p [G, G^\varphi]^p$.

(ii) $[[G, G^\varphi], {}_{n-1}\, \nu(G)] \leq [G, G^\varphi]^p$.

Now, we are in a position to state the theorems for the potent's family.

Theorem 3.2.2. *[5, Theorem A] Let p be a prime and let G be a potent finite p-group.*

(i) The non-abelian tensor square $[G, G^\varphi]$ is potently embedded in $\nu(G)$.

(ii) If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\nu(G))$ is potently embedded in $\nu(G)$.

Theorem 3.2.3. *[5, Theorem B] Let p be a prime and let G be a p-group with $\exp(G) = p^e$.*

(i) If G is potent, then $\exp(\nu(G))$ divides p^{e+1}.

(ii) If $\gamma_{p-2}(G) \leq G^p$, then $\nu(G)$ is a potent p-group. In particular, $\exp(\nu(G)) = p^e$.

The following results are immediate consequences of the last theorem.

Corollary 3.2.4. *[5, Corollary 1.1] Let $p \geq 5$ be a prime and let G be a powerful finite p-group. Then $\nu(G)$ is a potent p-group. In particular, $\exp(\nu(G)) = \exp(G)$.*

Corollary 3.2.5. *[5, Corollary 1.2] Let p be a prime and G a potent finite p-group. Then $\exp(M(G))$ and $\exp(\mu(G))$ divide $p \cdot \exp(G)$.*

In [22] the authors defined the term *power-commutator condition*. These kind of conditions were thought as being inclusion relations of certain power-commutator subgroups (that is, a subgroup formed by taking commutators and powers in any order). For instance, inclusions satisfied for powerful or potent finite p-group, $\gamma_2(G) \leq G^4$, $\gamma_2(G) \leq G^p$ or $\gamma_{p-1}(G) \leq G^p$. In the family of p-groups of coclass r, where $r \geq 1$ (in particular, for the family of p-groups of maximal class where $r = 1$), there exists a similar condition, which is given by the equality $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$, where m and s are positive integers such that $m \geq s$. In some sense the next result can be viewed as an extension of the results found by Moravec [38] and by Bastos et al. [5], now considering this equality as a hypothesis.

Theorem 3.2.6. *[4, Theorem 1.1] Let p be a prime and G a p-group. Let m and s be positive integers such that $m \geq s$ and suppose that $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$. Then*

(i) $\gamma_{i+s+1}(\nu(G)) = \gamma_{i+1}(\nu(G))^p$ for $i > m$.

(ii) If p is odd, then $\exp(\gamma_{m+1}(\nu(G)))$ divides $\exp(\gamma_m(G))$.

(iii) If $p = 2$ and $\gamma_m(G)$ is powerful, then $\exp(\gamma_{m+1}(\nu(G)))$ divides $\exp(\gamma_m(G))$.

Throughout the sequel N denotes a normal subgroup of a finite p-group G. For the sake of brevity, we write $\mathcal{Q} = \ker(\widetilde{\pi}) = \langle N, N^{\varphi}\rangle[N, G^{\varphi}][G, N^{\varphi}]$.

Proposition 3.2.7. *[4, Proposition 4.1] With the above notation, we have that* *(i)* $\gamma_s(\mathcal{Q}) = \gamma_s(N)\gamma_s(N^\varphi)[\gamma_{s-1}(N), N^\varphi][N, \gamma_{s-1}(N^\varphi)]$ *for* $s \geq 2$.

(ii) If $p \geq 3$, $n \in \mathbb{N}$ *such that* $1 < n < p$ *and* $\gamma_n(N) \leq N^p$, *then* $\gamma_{n+1}(\mathcal{Q}) \leq \gamma_2(N)^p\gamma_2(N^\varphi)^p[N, N^\varphi]^p$.

(iii) If $p = 2$ *and* N *is powerful, then* $\gamma_3(\mathcal{Q}) \leq \gamma_2(N)^4\gamma_2(N^\varphi)^4[N, N^\varphi]^4$.

Corollary 3.2.8. *[4, Corollary 4.2] If* N *is potent and* $s \geq 2$, *then the* s*-th term of the lower central series* $\gamma_s(\mathcal{Q})$ *is potently embedded in* $\mathcal{Q}$.

Since $\mathcal{Q} = \ker(\widetilde{\pi})$, the last two results were used to get a bound for the exponent of $\mathcal{Q}$ for, subsequently, to limit the exponent of $\nu(G)$.

Remember that we denote $\overline{p} = p$ if $p \geq 3$ and $\overline{p} = 4$ if $p = 2$.

Corollary 3.2.9. *[4, Corollary 4.3] If* N *is potent or* $\gamma_p(N) = 1$, *then* $\exp(\mathcal{Q})$ *divides* $\overline{p} \cdot \exp(N)$.

In order to obtain bounds for the exponent of $\nu(G)$ in terms of some specific normal subgroups of G, we can use the last corollary and Lemma 2.1.3.

Theorem 3.2.10. *[4, Theorem 1.2] Let* p *be a prime and* N *a normal subgroup of a* p*-group* G.

(i) If N *is potent or* $\gamma_p(N) = 1$, *then* $\exp(\nu(G))$ *divides* $\overline{p} \cdot \exp(\nu(G/N)) \cdot \exp(N)$.

(ii) If $\gamma_{p-2}(N) \leq N^p$, *then* $\exp(\nu(G))$ *divides* $\exp(\nu(G/N)) \cdot \exp(N)$.

In [13], the authors showed that if G is a 2-group of maximal class, then $\exp([G, G^\varphi])$ divides $\exp(G)$ (see [13, Propositions 13–15]). Therefore, $\exp(\nu(G))$ divides $\exp(G)^2$. Besides that, in [39], Moravec proved that if G is a p-group of maximal class, then $\exp(M(G))$ divides $\exp(G)$. Since all p-group G of maximal class has a potent maximal subgroup or a maximal subgroup of class at most $p-1$, we can use the last theorem to prove a similar result.

Corollary 3.2.11. *[4, Corollary 1.3] Let* p *be a prime and* G *a* p*-group of maximal class. Then* $\exp(\nu(G))$ *divides* $\overline{p}^2 \cdot \exp(G)$.

In the literature, the exponent of several sections of the group $\nu(G)$, like $G \otimes G$, $\mu(G)$ and $M(G)$, has been investigated (see [47] and the references therein). In [21], Ellis proved that if G is a p-group of class $c \geq 2$, then $\exp([G, G^\varphi])$ divides $\exp(G)^{c-1}$. In [36], Moravec showed that if G is a p-group of class $c \geq 2$, then $\exp(M(G))$ divides $\exp(G)^{2\lfloor \log_2(c) \rfloor}$. Later, in [48], Sambonet proved that if G is a p-group of class $c \geq 2$, then $\exp(M(G))$

divides $\exp(G)^{\lfloor \log_{p-1}(c) \rfloor + 1}$ if $p > 2$ and $\exp(M(G))$ divides $2^{\lfloor \log_2(c) \rfloor} \cdot \exp(G)^{\lfloor \log_2(c) \rfloor + 1}$ if $p = 2$. In [1], Antony et al. demonstrated that $\exp(M(G))$ divides $\exp(G)^{1+\lceil \log_{p-1}(\frac{c+1}{p+1}) \rceil}$ if $p \leq c$, improving all the previous bounds. In this way we get the next theorem for the $\exp([G, G^{\varphi}])$ which improves the bound obtained in [1].

Theorem 3.2.12. *[4, Theorem 1.2] Let p be a prime and G a p-group of nilpotency class c. Let $n = \lceil \log_p(c+1) \rceil$. Then $\exp([G, G^{\varphi}])$ divides $\exp{(G)}^n$.*

In the context of the p-groups of coclass r, Moravec [39] proved that if G is a p-group of coclass r, then $\exp(M(G))$ and $\exp(G \wedge G)$ divide $\exp(G)^{r+1+2\lfloor \log_2(m-1) \rfloor}$, where $m = (p-1)p^{r-1}$ if $p \geq 3$ or $m = 2^{r+2}$ if $p = 2$. Here, we have the next theorem and its corollary.

Theorem 3.2.13. *[4, Theorem 1.5] Let p be a prime and G a p-group of coclass r.*

(i) If p is odd, then $\exp([G, G^{\varphi}])$ divides $(\exp(G))^r \cdot \exp(\gamma_m(G))$, where $m = (p-1)p^{r-1}$.

(ii) If $p = 2$, then $\exp([G, G^{\varphi}])$ divides $(\exp(G))^{r+3} \cdot \exp(\gamma_m(G))$, where $m = 2^{r+2}$.

Corollary 3.2.14. *[4, Corollary 1.6] Let p be a prime and G a p-group of coclass r.*

(i) If $p \geq 3$, then $\exp(M(G))$ and $\exp(\mu(G))$ divides $\exp(G)^{r+1}$.

(ii) If $p = 2$, then $\exp(M(G))$ and $\exp(\mu(G))$ divides $\exp(G)^{r+4}$.

It is worth to mention that for every prime p, the bounds obtained in the last corollary improve those obtained in [39] when the coclass r is at least 2 (cf. [39, Corollary 4.8]). Furthermore, in the context of Sambonet theorem [48, Theorem 3.3], the improvement occurs for $e \leq r$ if $p > 2$ and $e \leq r + 2$ if $p = 2$, where $\exp(G) = p^e$.

Chapter 4

Technical results

In this chapter we prove some technical results involving the terms of the lower central series and some variations of these terms using the construction $\nu^q(G)$, $q \geq 0$, and its subgroups $\Upsilon^q(G)$ and $\mathcal{R}$. Since the definition of powerful and potent p-groups involve the terms of its lower central series, these technical results are useful in the theorems of the next chapter. Some results of this chapter can be found in [5, 4] in the version when $q = 0$.

Lemma 4.0.1. *Let G be a group and q be a non-negative integer. Then, for all $k \geq 1$, $[[G, G^{\varphi}],\ _k\ \nu^q(G)] = [\gamma_{k+1}(G),\ G^{\varphi}]$.*

Proof. The proof is by induction on k. For $k = 1$, consider the generator $[g, h^{\varphi}] \in [G, G^{\varphi}]$. Since $\nu^q(G) = \langle G \cup G^{\varphi} \cup \mathcal{K} \rangle$ it suffices to prove that the three elements $[g, h^{\varphi}, g_1], [g, h^{\varphi}, g_1^{\varphi}]$ and $[g, h^{\varphi}, \widehat{k}]$ are in $[\gamma_2(G), G^{\varphi}]$, for all $g, h, g_1, k \in G$. We see that

$$[g, h^{\varphi}, g_1] = [g, h^{\varphi}, g_1^{\varphi}] = [g, h, g_1^{\varphi}] \in [\gamma_2(G), G^{\varphi}]$$

$$\begin{aligned} [g, h^{\varphi}, \widehat{k}] &= [g, h^{\varphi}]^{-1}[g, h^{\varphi}]^{\widehat{k}} = [g, h^{\varphi}]^{-1}[g^{k^q}, (h^{k^q})^{\varphi}] \\ &= [g, h^{\varphi}]^{-1}[g, h^{\varphi}]^{k^q} = [g, h^{\varphi}, k^q] = [g, h, (k^q)^{\varphi}] \in [\gamma_2(G), G^{\varphi}]. \end{aligned}$$

But if we take a generic element $\alpha = \prod_{i=1}^{n}[g_i, h_i^{\varphi}] \in [G, G^{\varphi}]$, using the normality and the relation about commutator we have that

$$[\alpha, \beta] = \Big[\prod_{i=1}^{n}[g_i, h_i^{\varphi}], \beta\Big] = \Big[\prod_{i=1}^{n-1}[g_i, h_i^{\varphi}], \beta\Big]^{[g_n, h_n^{\varphi}]} \cdot [[g_n, h_n^{\varphi}], \beta] \in [\gamma_2(G), G^{\varphi}],$$

for all $\beta \in \nu^q(G)$. This means that $[G, G^{\varphi}, \nu^q(G)] \leq [\gamma_2(G), G^{\varphi}]$. Since the equality $[G, G^{\varphi}, G^{\varphi}] = [G, G, G^{\varphi}]$ holds the other inclusion is clear. Therefore $[G, G^{\varphi}, \nu^q(G)] = [\gamma_2(G), G^{\varphi}]$.

Now suppose that $[G, G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}]$. By induction hypothesis we have that

$$[G, G^{\varphi}, {}_{k+1}\ \nu^q(G)] := [G, G^{\varphi}, {}_k\ \nu^q(G), \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}, \nu^q(G)].$$

For this consider $[\delta, h^{\varphi}] \in [\gamma_{k+1}(G), G^{\varphi}]$, $\delta \in \gamma_{k+1}(G)$ and $h \in G$, and let $g \in G$, $\widehat{k} \in \mathcal{K}$. So

$$\begin{aligned}
&[\delta, h^{\varphi}, g] = [\delta, h^{\varphi}, g^{\varphi}] = [\delta, h, g^{\varphi}] \in [\gamma_{k+2}(G), G^{\varphi}];\\
&[\delta, h^{\varphi}, \widehat{k}] \;=\; [\delta, h^{\varphi}]^{-1}[\delta, h^{\varphi}]^{\widehat{k}} = [\delta, h^{\varphi}]^{-1}[\delta^{k^q}, (h^{k^q})^{\varphi}] =\\
&\qquad\quad = [\delta, h^{\varphi}]^{-1}[\delta, h^{\varphi}]^{k^q} = [\delta, h^{\varphi}, k^q] = [\delta, h, (k^q)^{\varphi}] \in [\gamma_{k+2}(G), G^{\varphi}].
\end{aligned}$$

If we consider the elements $\delta_n = \prod_{i=1}^{n}[\alpha_i, h_i^{\varphi}] \in [\gamma_{k+1}(G), G^{\varphi}]$ and $\beta \in \nu^q(G)$, we have that

$$[\delta_n, \beta] = \Big[\prod_{i=1}^{n-1}[\alpha_i, h_i^{\varphi}], \beta\Big]^{[\alpha_n, h_n^{\varphi}]}[\alpha_n, h_n^{\varphi}, \beta] \in [\gamma_{k+2}(G), G^{\varphi}]$$

by normality of $[\gamma_{k+2}(G), G^{\varphi}]$. So each generator of $[\gamma_{k+1}(G), G^{\varphi}, \nu^q(G)]$ is contained in $[\gamma_{k+2}(G), G^{\varphi}]$, this means that $[\gamma_{k+1}(G), G^{\varphi}, \nu^q(G)] \leq [\gamma_{k+2}(G), G^{\varphi}]$. Since the other inclusion is trivial, it follows that the equality. Therefore, $[[G, G^{\varphi}], {}_k\ \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}]$, as wished. $\square$

Lemma 4.0.2. *Let G be a group and q be a non-negative integer. For i, j, k positive integers we have*

(i) $[\gamma_i(G), G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_{i+k}(G), G^{\varphi}]$.

(ii) $[\gamma_i(G), \gamma_j(G^{\varphi}), {}_k\ \nu^q(G)] \leq [\gamma_{i+j+k-1}(G), G^{\varphi}]$.

Proof. (i) The proof is by induction on i and k. By the last lemma for $i = k = 1$, we have $[G, G^{\varphi}, \nu^q(G)] = [G, G^{\varphi}, G] = [\gamma_2(G), G^{\varphi}]$.

In the next step of our induction we need to prove that the equality is valid for $i = 1$ and its valid for all k. However, this demonstrated in the last lemma.

Therefore, remains see that the equality is hold for all i and for all k. Suppose by induction that $[\gamma_i(G), G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_{i+k}(G), G^{\varphi}]$. We will show that $[\gamma_{i+1}(G), G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_{i+k+1}(G), G^{\varphi}]$. Indeed

$$[\gamma_{i+1}(G), G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_i(G), G, G^{\varphi}, {}_k\ \nu^q(G)] = [\gamma_i(G), G^{\varphi}, G, {}_k\ \nu^q(G)].$$

Using the similar argument as in the end of the last lemma we have that

$$[\gamma_i(G), G^{\varphi}, G, {}_k\ \nu^q(G)] = [\gamma_i(G), G^{\varphi}, \nu^q(G), {}_k\ \nu^q(G)] = [\gamma_i(G), G^{\varphi}, {}_k\ \nu^q(G), \nu^q(G)]$$

and by the induction hypothesis it follows that

$$[\gamma_i(G), G^{\varphi},\ _k\ \nu^q(G), \nu^q(G)] = [\gamma_{i+k}(G), G^{\varphi}, \nu^q(G)].$$

Now, we can use a similar argument used in the end of the proof of the last lemma and properties of the group $\nu^q(G)$ in a commutator for to obtain that

$$[\gamma_{i+k}(G), G^{\varphi}, \nu^q(G)] = [\gamma_{i+k}(G), G^{\varphi}, G] = [\gamma_{i+k}(G), G, G^{\varphi}] = [\gamma_{i+k+1}(G), G^{\varphi}].$$

(ii) The proof of this item is by induction too, but our induction is on the sum $i+j+k$. The base step happens when $i+j+k=3$, that is $i=j=k=1$, and this case is a special case of the first item when $i=k=1$.

Now, suppose by induction that the inequality is valid for the sum $i+j+k$ and we prove that for $i+j+k+1$. Observe that $i+j+k+1 = i+j+(k+1) = (i+1)+j+k = i+(j+1)+k$, so

$$\begin{aligned}
[\gamma_i(G), \gamma_j(G^{\varphi}),\ _{k+1}\ \nu^q(G)] &= [\gamma_i(G), \gamma_j(G^{\varphi}),\ _k\ \nu^q(G),\ \nu^q(G)] \\
&\leq [\gamma_{i+j+k-1}(G), G^{\varphi}, \nu^q(G)] = [\gamma_{i+j+k-1}(G), G^{\varphi}, G] \\
&= [\gamma_{i+j+k}(G), G^{\varphi}].
\end{aligned}$$

For the other cases we will separate in two cases $k=1$ and $k>1$. If $k=1$ we can use a similar argument used in Lemma 4.0.1 and it follows that

$$\begin{aligned}
[\gamma_{i+1}(G), \gamma_j(G^{\varphi}), \nu^q(G)] &= [\gamma_{i+1}(G), \gamma_j(G^{\varphi}), G] \\
&= [\gamma_{i+1}(G), \gamma_j(G), G^{\varphi}] \leq [\gamma_{i+j+1}(G), G^{\varphi}] \\
[\gamma_i(G), \gamma_{j+1}(G^{\varphi}), \nu^q(G)] &= [\gamma_i(G), \gamma_{j+1}(G^{\varphi}), G^{\varphi}] \\
&= [\gamma_i(G), \gamma_{j+1}(G), G^{\varphi} \leq [\gamma_{i+j+1}(G), G^{\varphi}].
\end{aligned}$$

On the other hand, if $k>1$ we have that

$$\begin{aligned}
[\gamma_{i+1}(G), \gamma_j(G^{\varphi}),\ _k\ \nu^q(G)] &= [\gamma_{i+1}(G), \gamma_j(G^{\varphi}),\ _{k-1}\ \nu^q(G), \nu^q(G)] \\
&\leq [\gamma_{i+j+k-1}(G), G^{\varphi}, \nu^q(G)] = [\gamma_{i+j+k-1}(G), G^{\varphi}, G] \\
&= [\gamma_{i+j+k-1}(G), G, G^{\varphi}] = [\gamma_{i+j+k}(G), G^{\varphi}] \\
[\gamma_i(G), \gamma_{j+1}(G^{\varphi}),\ _k\ \nu^q(G)] &= [\gamma_i(G), \gamma_{j+1}(G^{\varphi}),\ _{k-1}\ \nu^q(G), \nu^q(G)] \\
&\leq [\gamma_{i+j+k-1}(G), G^{\varphi}, \nu^q(G)] = [\gamma_{i+j+k-1}(G), G^{\varphi}, G] \\
&= [\gamma_{i+j+k-1}(G), G, G^{\varphi}] = [\gamma_{i+j+k}(G), G^{\varphi}]
\end{aligned}$$

Therefore the proof is complete.

□

In the next two lemmas we need to add some hypothesis. First we suppose that the exponent of the group G divides the non-negative integer q. After we add the hypothesis that $\gamma_n(G) \leq G^p$, where n is natural number such that $1 < n < p$.

Lemma 4.0.3. *Let G be a group and q be a non-negative integer. If $exp(G)$ divides q, then $[\Upsilon^q(G), {}_k\, \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}]$, for all $k \geq 1$.*

Proof. The proof is by induction on k. For the base step of the induction to simplify the notation we write T for the commutator $[G, G^{\varphi}]$, so that $\Upsilon^q(G) = [G, G^{\varphi}]K = TK$ and hence

$$[\Upsilon^q(G), \nu^q(G)] = [T, \nu^q(G)][K, \nu^q(G)],$$

by normality of T in $\nu^q(G)$. Now, under the assumption that $\exp(G)$ divides q, we get that K is abelian (by defining relation (2.14)), while G and G^{φ} centralize K (by Lemma 2.2.1(v)). Consequently, $[K, \nu^q(G)] = 1$, since $\nu^q(G) = (TK)GG^{\varphi}$ (by (2.17)). In addition, $[T, G^{\varphi}] = [T, G] = [G, G, G^{\varphi}]$, by Lemma 2.2.1 (ii). Therefore,

$$[T, \nu^q(G)] = [G, G^{\varphi}, (TK)GG^{\varphi}] = [G, G^{\varphi}, G][G, G^{\varphi}, G^{\varphi}] = [G, G, G^{\varphi}].$$

This proves the base step of our induction.

Assume the formula holds for k; we will prove it for $k+1$. By hypothesis of induction $[\Upsilon^q(G), {}_{k+1}\, \nu^q(G)] := [\Upsilon^q(G), {}_k\, \nu^q(G), \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}, \nu^q(G)]$.

By the first item of the last lemma we have that

$$[\gamma_{k+1}(G), G^{\varphi}, \nu^q(G)] = [\gamma_{k+1}(G), G^{\varphi}, G] = [\gamma_{k+1}(G), G, G^{\varphi}].$$

Therefore the proof is complete. □

Considering that $exp(G)$ divides q we can conclude from the last two lemmas that $[[G, G^{\varphi}], {}_k\, \nu^q(G)] = [\Upsilon^q(G), {}_k\, \nu^q(G)] = [\gamma_k(G), G^{\varphi}]$, for all $k \geq 1$.

Proposition 4.0.4. *Let $p \geq 3$ be a prime, q a non-negative integer and $n \in \mathbb{N}$ such that $1 < n < p$. Suppose that G is a finite p-group such that $\gamma_n(G) \leq G^p$. Then $\gamma_{n+1}(\nu^q(G)) \leq \gamma_2(G)^p \gamma_2(G^{\varphi})^p [G, G^{\varphi}]^p$.*

Proof. By Proposition 2.2.2 (i), we have $\gamma_{n+1}(\nu^q(G)) = \gamma_{n+1}(G)\gamma_{n+1}(G^{\varphi})[\gamma_n(G), G^{\varphi}]$. As $\gamma_n(G) \leq G^p$ we obtain that

$$\gamma_{n+1}(\nu^q(G)) \leq [G^p, G][(G^{\varphi})^p, G^{\varphi}][G^p, G^{\varphi}].$$

First we prove that $[G^p, G] \leq \gamma_2(G)^p$. Since $n < p$ and by Theorem 1.6.10 we have

$$\begin{aligned}[G^p, G] &\leq [G, G]^p[G,_p G] = [G, G]^p[\gamma_{p-1}(G), G, G] \\ &\leq [G, G]^p[\gamma_n(G), G, G] \leq [G, G]^p[G^p, G, G].\end{aligned}$$

Applying Lemma 1.6.9 to $N = [G^p, G]$ and $M = [G, G]^p$, we deduce that $[G^p, G] \leq [G, G]^p$. Clearly, in the same way we have $[(G^\varphi)^p, G^\varphi] \leq \gamma_2(G^\varphi)^p$.

Now, it remains to prove that $[G^p, G^\varphi] \leq [G, G^\varphi]^p$. For this let $x,\ y \in G$. By Theorem 1.6.7,

$$[x^p, y^\varphi] \equiv [x, y^\varphi]^p \pmod{\gamma_2(L)^p\gamma_p(L)},$$

where $L = \langle x, [x, y^\varphi]\rangle$. Note that $\gamma_2(L)^p \leq [G, G^\varphi, G]^p = [G, G, G^\varphi]^p \leq [G, G^\varphi]^p$. By Lemma 4.0.1 and by hypothesis on n, it follows that

$$\begin{aligned}\gamma_p(L) &\leq [G, G^\varphi, G,_{p-2}\nu^q(G)] \leq [G, G^\varphi,_{p-2}\nu^q(G), \nu^q(G)] \\ &= [\gamma_{p-1}(G), G^\varphi, \nu^q(G)] \leq [\gamma_n(G), G^\varphi, \nu^q(G)] \leq [G^p, G^\varphi, \nu^q(G)].\end{aligned}$$

Considering all the elements $x, y \in G$, we deduce that

$$[G^p, G^\varphi] \leq [G, G^\varphi]^p[G^p, G^\varphi, \nu^q(G)].$$

Note that $[G^p, G^\varphi]$ and $[G, G^\varphi]^p$ are normal subgroups of $\nu^q(G)$. Applying Lemma 1.6.9 to $N = [G^p, G^\varphi]$ and $M = [G, G^\varphi]^p$, we obtain $[G^p, G^\varphi] \leq [G, G^\varphi]^p$, which completes the proof. □

If G is a p-group and M, N are subgroups of G, then always holds that $N^pM^p \leq (NM)^p$. Using this fact and the description of the lower central series of the group $\nu^q(G)$ we can conclude of the last proposition that $\gamma_{n+1}(\nu^q(G)) \leq \gamma_2(\nu^q(G))^p$.

The next result of this chapter is a kind of generalization of the last proposition, because in this case we will consider that the finite p-group satisfies another type of power commutator condition.

Proposition 4.0.5. *Let G be a p-group, with p a prime. Let m and s be positive integers such that $m \geq s$ and suppose that $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$. Consider $q \geq 0$. Then*

(i) $\gamma_{i+s+1}(\nu^q(G)) \leq \gamma_{i+1}(\nu^q(G))^p$ *if $p > 2$ and $i \geq m$ or if $i > m$.*

(ii) $\gamma_{i+1}(\nu^q(G))^p \leq \gamma_{i+s+1}(\nu^q(G))$ *for all prime p and $i \geq m$.*

Proof. (i) We will prove that $\gamma_{i+s+1}(\nu^q(G)) \leq \gamma_{i+1}(\nu^q(G))^p$, when $p > 2$ or $i > m$. From Proposition 2.2.2 and by hypothesis we have

$$\begin{aligned}\gamma_{i+s+1}(\nu^q(G)) &= \gamma_{i+s+1}(G)\ \gamma_{i+s+1}(G^\varphi)\ [\gamma_{i+s}(G), G^\varphi] \\ &= \gamma_{i+1}(G)^p\ \gamma_{i+1}(G^\varphi)^p\ [\gamma_i(G)^p, G^\varphi].\end{aligned}$$

By definition $\gamma_{i+1}(\nu^q(G))^p = \left(\gamma_{i+1}(G)\gamma_{i+1}(G^\varphi)[\gamma_i(G), G^\varphi]\right)^p$. So both $\gamma_{i+1}(G)^p$ and $\gamma_{i+1}(G^\varphi)^p$ are contained in $\gamma_{i+1}(\nu^q(G))^p$. Therefore, it suffices to prove that $[\gamma_i(G)^p, G^\varphi] \leq \gamma_{i+1}(\nu^q(G))^p$. For this let $x \in \gamma_i(G)$ and $y^\varphi \in G^\varphi$, by Hall's collection formula (1.2) we have

$$[x^p, y^\varphi] \equiv [x, y^\varphi]^p \pmod{\gamma_2(L)^p\ \gamma_p(L)},$$

where $L = \langle x, [x, y^\varphi]\rangle$. Observe that $\gamma_2(L) \leq [\gamma_i(G), G^\varphi, \gamma_i(G)]$. Therefore

$$\gamma_2(L)^p \leq [\gamma_i(G), G^\varphi, \gamma_i(G)]^p \leq \gamma_{2i+1}(\nu^q(G))^p \leq \gamma_{i+1}(\nu^q(G))^p.$$

For the subgroup $\gamma_p(L)$, we will need to separate into two cases. For every prime p, $i > m$ implies that $i \geq s+1$ (remember that $m \geq s$), then

$$\begin{aligned}\gamma_p(L) \leq \gamma_2(L) \leq [\gamma_i(G), G^\varphi, \gamma_i(G)] \leq \gamma_{2i+1}(\nu^q(G)) &\leq \gamma_{i+s+2}(\nu^q(G)) \\ &= \gamma_{i+s+2}(G)\gamma_{i+s+2}(G^\varphi)[\gamma_{i+s+1}(G), G^\varphi] \\ &= \gamma_{i+2}(G)^p\gamma_{i+2}(G^\varphi)^p[\gamma_i(G)^p, G, G^\varphi] \\ &\leq \gamma_{i+1}(\nu^q(G))^p[\gamma_i(G)^p, G, G^\varphi] \leq \gamma_{i+1}(\nu^q(G))^p[\gamma_i(G)^p, G^\varphi, \nu^q(G)].\end{aligned}$$

On the other hand, $p \geq 3$ implies $2i + p - 3 \geq i + s$ and we have

$$\begin{aligned}\gamma_p(L) &\leq [\gamma_i(G), G^\varphi, \gamma_i(G),\ {}_{p-2}\ \nu^q(G)] = [\gamma_{i+1}(G), \gamma_i(G^\varphi),\ {}_{p-3}\ \nu^q(G), \nu^q(G)] \\ &\leq [\gamma_{2i+p-3}(G), G^\varphi, \nu^q(G)] \leq [\gamma_{i+s}(G), G^\varphi, \nu^q(G)] = [\gamma_i(G)^p, G^\varphi, \nu^q(G)].\end{aligned}$$

Therefore, in any case it follows that each element $[x^p, y^\varphi] \in [\gamma_i(G)^p, G^\varphi]$ is contained in $\gamma_{i+1}(\nu^q(G))^p[\gamma_i(G)^p, G^\varphi, \nu^q(G)]$, which yields

$$[\gamma_i(G)^p, G^\varphi] \leq \gamma_{i+1}(\nu^q(G))^p[\gamma_i(G)^p, G^\varphi, \nu^q(G)].$$

Applying Lemma 1.6.9 with $N = [\gamma_i(G)^p, G^\varphi]$ and $M = \gamma_{i+1}(\nu^q(G))^p$, we can conclude that $[\gamma_i(G)^p, G^\varphi] \leq \gamma_{i+1}(\nu^q(G))^p$.

(ii) In order to prove that $\gamma_{i+1}(\nu^q(G))^p \leq \gamma_{i+s+1}(\nu^q(G))$, for all prime p and $i \geq m$, consider the subgroup $W = \gamma_{i+1}(G)^p\ \gamma_{i+1}(G^\varphi)^p\ [\gamma_i(G), G^\varphi]^p$. Firstly, we show that

$$W \equiv \gamma_{i+1}(\nu(G))^p \pmod{\gamma_{i+s+1}(\nu^q(G))}.$$

By definition, $W \leq \gamma_{i+1}(\nu^q(G))^p \leq \gamma_{i+1}(\nu^q(G))^p\gamma_{i+s+1}(\nu^q(G))$, so it remains to prove that $\gamma_{i+1}(\nu^q(G))^p \leq W\gamma_{i+s+1}(\nu^q(G))$. Take $\alpha \in \gamma_{i+1}(G)$, $\beta \in \gamma_{i+1}(G^\varphi)$ e $\delta \in [\gamma_i(G), G^\varphi]$ then, applying Corollary 1.6.8 we have

$$(\alpha\beta\delta)^p \equiv \alpha^p\beta^p\delta^p \pmod{\gamma_2(J)^p\gamma_p(J)},$$

where $J = \langle \alpha, \beta, \delta \rangle$. It is immediate that $\alpha^p\beta^p\delta^p \in W$. Moreover, all the generators of $\gamma_2(J)$ belong to $\gamma_{i+s+1}(\nu^q(G))$. Indeed,

$$\begin{aligned}
&[\alpha, \beta] \in [\gamma_{i+1}(G), \gamma_{i+1}(G^\varphi)] \leq [\gamma_{i+1}(\nu^q(G)), \gamma_{i+1}(\nu^q(G))] \leq \gamma_{i+s+1}(\nu^q(G));\\
&[\delta, \alpha] \in [\gamma_i(G), G^\varphi, \gamma_{i+1}(G)] \leq [\gamma_{i+1}(\nu^q(G)), \gamma_{i+1}(\nu^q(G)] \leq \gamma_{i+s+1}(\nu^q(G));\\
&[\delta, \beta] \in [\gamma_i(G), G^\varphi, \gamma_{i+1}(G^\varphi)] \leq [\gamma_{i+1}(\nu^q(G)), \gamma_{i+1}(\nu^q(G)] \leq \gamma_{i+s+1}(\nu^q(G)).
\end{aligned}$$

Therefore, $\gamma_2(J)^p \leq \gamma_2(J) \leq \gamma_{i+s+1}(\nu^q(G))$ and $\gamma_p(J) \leq \gamma_2(J) \leq \gamma_{i+s+1}(\nu^q(G))$. It follows that $(\alpha\beta\delta)^p \in W\gamma_{i+s+1}(\nu^q(G))$, and so

$$\gamma_{i+1}(\nu^q(G))^p \leq W\gamma_{i+s+1}(\nu^q(G)).$$

To conclude, we demonstrate that $W \leq \gamma_{i+s+1}(\nu^q(G))$, that is, $[\gamma_i(G), G^\varphi]^p \leq \gamma_{i+s+1}(\nu^q(G))$. Let $\alpha = \alpha_1^p \ldots \alpha_n^p \in [\gamma_i(G), G^\varphi]^p$. Since each $\alpha_j \in [\gamma_i(G), G^\varphi]$, we can write $\alpha_j = [x_{j1}, y_{j1}^\varphi] \ldots [x_{jl}, y_{jl}^\varphi]$, with $x_{jk} \in \gamma_i(G)$ and $y_{jk}^\varphi \in G^\varphi$, for all $k \in \{1, \ldots, l\}$, where l depends on j. Applying Corollary 1.6.8

$$([x_{j1}, y_{j1}^\varphi] \ldots [x_{jl}, y_{jl}^\varphi])^p \equiv [x_{j1}, y_{j1}^\varphi]^p \ldots [x_{jl}, y_{jl}^\varphi]^p \pmod{\gamma_2(S)^p\gamma_p(S)},$$

where $S = \langle [x_{j1}, y_{j1}^\varphi], \ldots, [x_{jl}, y_{jl}^\varphi] \rangle$. Observe that

$$\begin{aligned}
&\gamma_p(S) \leq \gamma_2(S) \leq [\gamma_i(G), G^\varphi, [\gamma_i(G), G^\varphi]] \leq \gamma_{2i+1}(\nu^q(G)) \leq \gamma_{i+s+1}(\nu^q(G));\\
&\gamma_2(S)^p \leq \gamma_{i+s+1}(\nu^q(G))^p \leq \gamma_{i+s+1}(\nu^q(G)).
\end{aligned}$$

Furthermore each element $[x_{jk}, y_{jk}^\varphi]^p$ is contained in $\gamma_{i+s+1}(\nu(G))$. Indeed, again by

Hall's collection formula (1.2) we have

$$[x_{jk}, y_{jk}^{\varphi}]^p \equiv [x_{jk}^p, y_{jk}^{\varphi}] \pmod{\gamma_2(H)^p\gamma_p(H)},$$

where $H = \langle x_{jk}, [x_{jk}, y_{jk}^{\varphi}]\rangle$. Here we see that

$$\begin{aligned}
&[x_{jk}^p, y_{jk}^{\varphi}] \in [\gamma_i(G)^p, G^{\varphi}] \le \gamma_{i+s+1}(\nu^q(G));\\
&\gamma_p(H) \le \gamma_2(H) \le [\gamma_i(G), G^{\varphi}, \gamma_i(G)] \le \gamma_{2i+1}(\nu^q(G)) \le \gamma_{i+s+1}(\nu^q(G));\\
&\gamma_2(H)^p \le \gamma_{i+s+1}(\nu^q(G))^p \le \gamma_{i+s+1}(\nu^q(G)).
\end{aligned}$$

This means that each $\alpha_j^p \in \gamma_{i+s+1}(\nu^q(G))$, so $\alpha \in \gamma_{i+s+1}(\nu^q(G))$.

Therefore $[\gamma_i(G), G^{\varphi}]^p \le \gamma_{i+s+1}(\nu^q(G))$, which completes the proof.

□

Recall that we wrote $\mathcal{R} = \langle N, N^{\varphi}, \mathcal{N}\rangle[N, G^{\varphi}][G, N^{\varphi}]$ for $\ker(\overline{\pi})$, where N is a normal subgroup of G. We will finish this chapter by giving an useful description of the lower central series of this kernel.

Proposition 4.0.6. *Let N be a normal subgroup of the group G and let $\mathcal{R}$ as defined above. For $s \ge 2$, $\gamma_s(\mathcal{R}) = \gamma_s(N)\gamma_s(N^{\varphi})[\gamma_{s-1}(N), N^{\varphi}]$.*

Proof. It is clear that $\gamma_s(N)\gamma_s(N^{\varphi})[\gamma_{s-1}(N), N^{\varphi}] \le \gamma_s(\mathcal{R})$, for every $s \ge 2$. So, we need to prove only the other inclusion. For this, we will argue by induction on s.

Let $X = \{n_1, n_2^{\varphi}, \widehat{n_3}, [n_4, g_1^{\varphi}], [g_2, n_5^{\varphi}] \mid n_i \in N, g_j \in G\}$ be a set of generators of $\mathcal{R}$. Assume that $s = 2$. Since $\gamma_2(N)\gamma_2(N^{\varphi})[N, N^{\varphi}]$ is a normal subgroup of $\nu^q(G)$, it is sufficient to show that each commutator of weight 2 in the generators belongs to it. Let $n, n', n_1, n_2 \in N$ and $g_1, g_2 \in G$. Then

$$[n_1, n_2] \in \gamma_2(N); \qquad [n_1^{\varphi}, n_2^{\varphi}] \in \gamma_2(N^{\varphi}); \qquad [\widehat{n_1}, \widehat{n_2}] = [n_1^q, (n_2^q)^{\varphi}] \in [N, N^{\varphi}]$$

$$[n_1, n_2^{\varphi}] \in [N, N^{\varphi}]; \qquad [n_1, \widehat{n_2}] = [n_1^{\varphi}, \widehat{n_2}] = [n_1^{\varphi}, n_2^q] \in [N, N^{\varphi}];$$

$$[n_1, g_1^{\varphi}, n_2] = [n_1, g_1^{\varphi}, n_2^{\varphi}] = [n_1, g_1, n_2^{\varphi}] = [n, n_2^{\varphi}] \in [N, N^{\varphi}];$$

$$\begin{aligned}
[n_1, g_1^{\varphi}, \widehat{n_2}] &= [n_1, g_1^{\varphi}]^{-1}[n_1, g_1^{\varphi}]^{\widehat{n_2}} = [n_1, g_1^{\varphi}]^{-1}[n_1^{n_2^q}, (g_1^{n_2^q})^{\varphi}] = [n_1, g_1^{\varphi}]^{-1}[n_1, g_1^{\varphi}]^{n_2^q}\\
&= [n_1, g_1^{\varphi}, n_2^q] = [n_1, g_1, (n_2^q)^{\varphi}] = [n, (n_2^q)^{\varphi}] \in [N, N^{\varphi}];
\end{aligned}$$

$$[g_2, n_2^{\varphi}, n_1] = [g_2, n_2^{\varphi}, n_1^{\varphi}] = [g_2, n_2, n_1^{\varphi}] = [n', n_2^{\varphi}] \in [N, N^{\varphi}];$$

$$
\begin{aligned}
[g_2, n_2^\varphi, \widehat{n_1}] &= [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{\widehat{n_1}} = [g_2, n_2^\varphi]^{-1}[g_2^{n_1^q}, (n_2^{n_1^q})^\varphi] = [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{n_1^q} \\
&= [g_2, n_2^\varphi, n_1^q] = [g_2, n_2, (n_2^q)^\varphi] = [n', (n_2^q)^\varphi] \in [N, N^\varphi];
\end{aligned}
$$

$$
\begin{aligned}
[[g_2, n_2^\varphi], [n_1, g_1^\varphi]] &= [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{[n_1, g_1^\varphi]} = [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{[n_1, g_1]} \\
&= [[g_2, n_2^\varphi], [n_1, g_1]] = [[g_2, n_2^\varphi], n] = [g_2, n_2, n^\varphi] = [n', n^\varphi] \in [N, N^\varphi];
\end{aligned}
$$

$$
\begin{aligned}
[[g_2, n_2^\varphi], [g_1, n_1^\varphi]] &= [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{[g_1, n_1^\varphi]} = [g_2, n_2^\varphi]^{-1}[g_2, n_2^\varphi]^{[g_1, n_1]} = [[g_2, n_2^\varphi], [g_1, n_1]] \\
&= [[g_2, n_2^\varphi], n^{-1}] = [g_2, n_2, (n^{-1})^\varphi] = [n', (n^{-1})^\varphi] \in [N, N^\varphi];
\end{aligned}
$$

$$
\begin{aligned}
[[n_2, g_2^\varphi], [n_1, g_1^\varphi]] &= [n_2, g_2^\varphi]^{-1}[n_2, g_2^\varphi]^{[n_1, g_1^\varphi]} = [n_2, g_2^\varphi]^{-1}[n_2, g_2^\varphi]^{[n_1, g_1]} = [[n_2, g_2^\varphi], [n_1, g_1]] \\
&= [n_2, g_2^\varphi, n^{-1}] = [n_2, g_2, (n^{-1})^\varphi] = [(n')^{-1}, (n^{-1})^\varphi] \in [N, N^\varphi].
\end{aligned}
$$

Then for $s = 2$ the inclusion holds. Now assume $s \geq 2$ and suppose by induction hypothesis that $\gamma_s(\mathcal{R}) = \gamma_s(N)\gamma_s(N^\varphi)[\gamma_{s-1}(N), N^\varphi]$. In particular $Y = \{x_1,\ x_2^\varphi,\ [y, n^\varphi] \mid x_i \in \gamma_s(N),\ y \in \gamma_{s-1}(N),\ n \in N\}$ is a set of generators for $\gamma_s(\mathcal{R})$. Since $\gamma_{s+1}(\mathcal{R}) = [\gamma_s(\mathcal{R}), \mathcal{R}]$, we need to show that

$$[\alpha, \beta] \in \gamma_{s+1}(N)\gamma_{s+1}(N^\varphi)[\gamma_s(N), N^\varphi]$$

for every $\alpha \in X$ and $\beta \in Y$, similarly with the base step. Let $x \in \gamma_s(N)$, $y \in \gamma_{s-1}(N)$, $m, n, n', n_1, n_2 \in N$ and $g_1, g_2 \in G$. Then

$$[x, m] \in \gamma_{s+1}(N); \qquad [x, m^\varphi] \in [\gamma_s(N), N^\varphi];$$

$$[x^\varphi, m] \in [\gamma_s(N^\varphi), N]; \qquad [x^\varphi, m^\varphi] \in \gamma_{s+1}(N^\varphi);$$

$$[x^\varphi, \widehat{m}] = [x, \widehat{m}] = [x^\varphi, m^q] \in [\gamma_s(N^\varphi), N];$$

$$[n_1, g_1^\varphi, x] = [n_1, g_1^\varphi, x^\varphi] = [n_1, g_1, x^\varphi] = [n, x^\varphi] \in [N, \gamma_s(N^\varphi)];$$

$$[g_2, n_2^\varphi, x] = [g_2, n_2^\varphi, x^\varphi] = [g_2, n_2, x^\varphi] = [n', x^\varphi] \in [N, \gamma_s(N^\varphi)];$$

$$[[y, m^\varphi], n_1] = [[y, m^\varphi], n_1^\varphi] = [y, m, n_1^\varphi] \in [\gamma_s(N), N^\varphi];$$

$$
\begin{aligned}
[[y, m^\varphi], \widehat{n_1}] &= [y, m^\varphi]^{-1}[y, m^\varphi]^{\widehat{n_1}} = [y, m^\varphi]^{-1}[y^{n_1^q}, (m^{n_1^q})^\varphi] \\
&= [y, m^\varphi]^{-1}[y, m^\varphi]^{n_1^q} = [[y, m^\varphi], n_1^q] = [y, m, (n_1^q)^\varphi] \in [\gamma_s(N), N^\varphi];
\end{aligned}
$$

$$[[y, m^\varphi], [n_1, g_1^\varphi]] = [y, m^\varphi, [n_1, g_1]] = [y, m^\varphi, n] = [y, m, n^\varphi] \in [\gamma_s(N), N^\varphi];$$

$$[[y, m^\varphi], [g_2, n_2^\varphi]] = [y, m^\varphi, [g_2, n_2]] = [y, m^\varphi], n'] = [y, m, n'^\varphi] \in [\gamma_s(N), N^\varphi].$$

This suffices to conclude that $\gamma_{s+1}(\mathcal{R}) \leq \gamma_{s+1}(N)\gamma_{s+1}(N^\varphi)[\gamma_s(N), N^\varphi]$, which establishes the formula. $\square$

Chapter 5

Generalizations for $q \geq 0$

In this chapter our focus is on extending the results obtained in [38] and [5] for the modular version of the operator ν, that is, we obtain similar results, which involve exponents, number of generators and some series, for the group $\nu^q(G)$ and some of its sections, where q is a non-negative integer and G is a finite p-group, powerful, potent or it has another power commutator condition.

The first question that arises about such an extension is whether a given property of the group G is inherited by $\nu^q(G)$ and for which integers q this occurs.

Taking into account that abelian groups are both powerful and potent, we start with an example that shows that if $G = C_p$, the cyclic group of order p, then $\nu^q(G)$ does not inherit these properties when p divides q. This also generalizes the same example given by Moravec in [38] for the case $q = 0$. We make use of the description $\nu^q(C_p)$ given by Bueno and Rocco in [15].

Example 5.0.1. *In this example we describe $\nu^q(C_p)$ for arbitrary integers $q \geq 0$, where C_p is the cyclic group of order p (see [15, Section 3.1] for details). Here we consider the cases $p \mid q$ and $p \nmid q$.*

- **Case** $p \mid q$. *This case can still be divided into three subcases: $p = 2$, $p = 3$ and $p \geq 5$.*

  If $p \geq 3$, we know that $\nu^q(C_p) \cong (C_p \times C_p \times C_p) \rtimes C_p (\cong \mathcal{H}_1)$, where

$$\begin{aligned} \mathcal{H}_1 \ = \ & \langle u, v, x, y \mid u^p = v^p = x^p = y^p = 1, [u, v] = y, \\ & [u, x] = [u, y] = [v, x] = [v, y] = [x, y] = 1 \rangle. \end{aligned} \tag{5.1}$$

  Thus the exponent of $\nu^q(C_p)$ is p, but $\gamma_2(\nu^q(C_p)) \cong \gamma_2(\mathcal{H}_1) = \langle y \rangle \cong C_p$. Therefore, *if $p = 3$ then $\nu^q(C_p)$ is not powerful nor potent, since the definition is the same. But, if $p \geq 5$ then $\nu^q(C_p)$ is not powerful however it is potent, since $\gamma_{p-1}(\nu^q(G)) \leq \gamma_3(\nu^q(G)) = 1$.*

If $p = 2$, then we need to consider two subcases: 4 divides q and $q = 2r$, with r odd. If 4 divides q, then $\nu^q(C_2) \cong (C_2 \times C_2 \times C_2) \rtimes C_2$ ($\cong \mathcal{H}$), where

$$\begin{aligned} \mathcal{H} &= \langle u, v, x, y \mid u^2 = v^2 = x^2 = y^2 = 1, [u, v] = y, \\ &\quad [x, u] = [x, v] = [y, u] = [y, v] = [x, y] = 1 \rangle. \end{aligned} \tag{5.2}$$

This group has exponent 2 and $\gamma_2(\mathcal{H}) \cong C_2 \cong \gamma_2(\nu^q(C_2))$. Therefore, again, $\nu^q(C_2)$ is not powerful nor potent. If $q = 2r$, with r odd, then we have $\nu^q(C_2) \cong (C_2 \times C_4) \rtimes C_2$, which is isomorphic to the group

$$\begin{aligned} \mathcal{H}_2 &= \langle u, v, x, y \mid u^2 = v^2 = x^4 = 1, [u, v] = y, x^2 = y, \\ &\quad [x, u] = [x, v] = [y, u] = [y, v] = [x, y] = 1 \rangle. \end{aligned} \tag{5.3}$$

Then $(\nu^q(C_2))^4 = 1$ and $\gamma_2(\nu^q(C_2)) \cong \langle y \rangle \cong C_2$. Also in this case $\nu^q(C_2)$ is not powerful nor potent.

In particular, if $q = 0$, we have $\Upsilon(C_p) \cong C_p \otimes C_p = C_p$ and hence $\nu^q(C_p) = \nu(C_p) \cong (C_p \times C_p) \rtimes C_p$, which is a nonabelian group of order p^3 and exponent p. So, $\nu(C_p)$ is not a powerful p-group (cf. [38]), but it is a potent p-group.

- **Case** $p \nmid q$. *Here we have $\nu^q(C_p) \cong ((\Upsilon^q(C_p)) \times C_p) \times C_p^{\varphi} \cong (C_p \times C_p) \times C_p$. This implies that $\gamma_2(\nu^q(C_p)) = 1 = (\nu^q(C_p))^p$ and $\gamma_2(\nu^q(C_2)) = 1 = (\nu^q(C_2))^4$, for all p. Consequently, $\nu^q(C_p)$ is a powerful (so potent) p-group in this last case.*

This examples show that $\nu^q(G)$ is not necessarily powerful or potent, depending on the prime p and on the non-negative integer q.

5.1 Lower central series of $\nu^q(G)$ and the q-tensor square

We gave an example where G is a powerful and potent p-group such that $\nu^q(G)$ is not necessarily powerful or potent. But, the interesting fact is that on considering a powerful or potent p-group G we can prove that there exist important normal subgroups of $\nu^q(G)$ which are powerful or potent, as we can see by the results proved in this section.

These results extend to the modular version of the operator ν results found by Moravec (powerful's family [38]) and by Bastos et al. (potent's family [5]).

Theorem 5.1.1. *Let G be a powerful finite p-group. Then $[\nu^q(G), \nu^q(G)]$ is powerfully embedded in $\nu^q(G)$.*

Proof. **Case p odd.** We need to show that $[[\nu^q(G), \nu^q(G)], \nu^q(G)] \leq [\nu^q(G), \nu^q(G)]^p$. By hypothesis $\gamma_2(G) \leq G^p$ and $p \geq 3$, we can apply the Proposition 4.0.4 considering $n = 2$. So $\gamma_3(\nu^q(G)) \leq \gamma_2(G)^p \gamma_2(G^\varphi)^p [G, G^\varphi]^p$. And this implies that $\gamma_3(\nu^q(G)) \leq (\gamma_2(G)\gamma_2(G^\varphi)[G, G^\varphi])^p = \gamma_2(\nu^q(G))^p$.

Case $p = 2$. Here we want to show that $\gamma_3(\nu^q(G)) \leq [\nu^q(G), \nu^q(G)]^4$. By the Proposition 2.2.2 we have that $\gamma_3(\nu^q(G)) = \gamma_3(G)\gamma_3(G^\varphi)[\gamma_2(G), G^\varphi]$. Since G is powerful, it follows that $\gamma_3(G) = [G, G, G] \leq [G^4, G] = \gamma_2(G)^4$. Similarly, $\gamma_3(G^\varphi) \leq \gamma_2(G^\varphi)^4$.

Moreover, we have that $[\gamma_2(G), G^\varphi] \leq [G^4, G^\varphi]$. Now we need to prove that $[G^4, G^\varphi] \leq [G, G^\varphi]^4$. To this end, let $g, h \in G$; the Hall's collection formula (1.2) asserts that

$$[g^4, h^\varphi] \equiv \ [g, h^\varphi]^4 \pmod{\gamma_2(L)^4 \gamma_2(L)^2 \gamma_4(L)},$$

where $L = \langle g, [g, h^\varphi] \rangle$. Observe that $\gamma_2(L) \leq [G, G^\varphi, G] = [G, G, G^\varphi]$, this implies that

$$\begin{aligned}
&\gamma_2(L)^4 \leq [G, G, G^\varphi]^4 \leq [G, G^\varphi]^4 \\
&\gamma_2(L)^2 \leq [G, G, G^\varphi]^2 \leq [G^4, G^\varphi]^2 \\
&\gamma_4(L) \leq [G, G, G^\varphi, L, L] \leq [G, G, G^\varphi, \nu^q(G), \nu^q(G)] \leq [G^4, G^\varphi, \nu^q(G)].
\end{aligned}$$

Therefore $[g^4, h^\varphi] \in [G, G^\varphi]^4 [G^4, G^\varphi, \nu^q(G)][G^4, G^\varphi]^2$, for all $g, h \in G$, which implies that

$$[G^4, G^\varphi] \leq [G, G^\varphi]^4 [G^4, G^\varphi, \nu^q(G)][G^4, G^\varphi]^2.$$

By normality of the subgroups $[G^4, G^\varphi]$ and $[G, G^\varphi]^4$ in $\nu^q(G)$, we can apply Lemma 1.6.9, obtaining $[G^4, G^\varphi] \leq [G, G^\varphi]^4$.

Therefore,

$$\gamma_3(\nu^q(G)) \leq \gamma_2(G)^4 \gamma_2(G^\varphi)^4 [G^4, G^\varphi] \leq \gamma_2(G)^4 \gamma_2(G^\varphi)^4 [G, G^\varphi]^4 \leq \gamma_2(\nu^q(G))^4$$

and the proof is complete. $\square$

Note that $\gamma_2(\nu^q(G))$, which is powerfully embedded in $\nu^q(G)$, is powerful, since

$$[\gamma_2(\nu^q(G)), \gamma_2(\nu^q(G))] \leq [\gamma_2(\nu^q(G)), \nu^q(G)] \leq \gamma_2(\nu^q(G))^{\overline{p}},$$

where $\overline{p} = p$ if p is odd, or $\overline{p} = 4$ if $p = 2$.

Theorem 5.1.2. *Let G be a powerful finite p-group. Then*

(i) $\gamma_i(\nu^q(G))$ is powerfully embedded in $\nu^q(G)$, for $i \geq 2$.

(ii) $\nu^q(G)^{(i)}$ is powerfully embedded in $\nu^q(G)$, for $i \geq 1$.

Proof. We will prove both parts by induction on i; Theorem 5.1.1 corresponds to the base step of our induction.

(i) Firstly consider $p > 2$. Suppose by induction that $[\gamma_i(\nu^q(G)), \nu^q(G)] \leq \gamma_i(\nu^q(G))^p$. By Theorem 1.6.10 we then get

$$\begin{aligned}
[\gamma_{i+1}(\nu^q(G)), \nu^q(G)] &\leq [\gamma_i(\nu^q(G))^p, \nu^q(G)] \\
&\leq [\gamma_i(\nu^q(G)), \nu^q(G)]^p[\nu^q(G), {}_p\ \gamma_i(\nu^q(G))] \\
&\leq \gamma_{i+1}(\nu^q(G)^p[\gamma_{i+1}(\nu^q(G)), {}_{p-1}\ \nu^q(G)] \\
&\leq \gamma_{i+1}(\nu^q(G))^p[\gamma_{i+1}(\nu^q(G)), \nu^q(G), \nu^q(G)].
\end{aligned}$$

Applying Lemma 1.6.9 we obtain $[\gamma_{i+1}(\nu^q(G)), \nu^q(G)] \leq (\gamma_{i+1}(\nu^q(G)))^p$, for $p > 2$.

Now let $p = 2$ and suppose that $[\gamma_i(\nu^q(G)), \nu^q(G)] \leq \gamma_i(\nu^q(G))^4$. Again, by Theorem 1.6.10 we have

$$\begin{aligned}
[\gamma_{i+1}(\nu^q(G)), \nu^q(G)] &\leq [\gamma_i(\nu^q(G))^4, \nu^q(G)] \\
&\leq [\gamma_i(\nu^q(G)), \nu^q(G)]^4[\nu^q(G),_2\ \gamma_i(\nu^q(G))]^2[\nu^q(G),_4\ \gamma_i(\nu^q(G))] \\
&\leq \gamma_{i+1}(\nu^q(G))^4[\gamma_{i+1}(\nu^q(G)), \nu^q(G)]^2[\gamma_{i+1}(\nu^q(G)), {}_3\ \nu^q(G)] \\
&\leq \gamma_{i+1}(\nu^q(G))^4[\gamma_{i+1}(\nu^q(G)), \nu^q(G)]^2[\gamma_{i+1}(\nu^q(G)), \nu^q(G), \nu^q(G)].
\end{aligned}$$

By applying Lemma 1.6.9 we obtain $[\gamma_{i+1}(\nu^q(G)), \nu^q(G)] \leq \gamma_{i+1}(\nu^q(G))^4$ if $p = 2$.

(ii) Suppose by induction that:

$$[\nu^q(G)^{(i)}, \nu^q(G)] \leq (\nu^q(G)^{(i)})^p \text{ if } p > 2.$$

The Three Subgroups Lemma and Theorem 1.6.10 give us

$$\begin{aligned}
[\nu^q(G)^{(i+1)}, \nu^q(G)] &= [\nu^q(G)^{(i)}, \nu^q(G)^{(i)}, \nu^q(G)] \leq [\nu^q(G), \nu^q(G)^{(i)}, \nu^q(G)^{(i)}] \\
&\leq [(\nu^q(G)^{(i)})^p, \nu^q(G)^{(i)}] \\
&\leq [\nu^q(G)^{(i)}, \nu^q(G)^{(i)}]^p[\nu^q(G)^{(i)}, {}_p\ \nu^q(G)^{(i)}] \\
&\leq (\nu^q(G)^{(i+1)})^p[\nu^q(G)^{(i)}, \nu^q(G)^{(i)}, {}_{p-1}\ \nu^q(G)] \\
&\leq (\nu^q(G)^{(i+1)})^p[\nu^q(G)^{(i+1)}, \nu^q(G), \nu^q(G)].
\end{aligned}$$

By Lemma 1.6.9 we then get $[\nu^q(G)^{(i+1)}, \nu^q(G)] \leq \nu^q(G)^{(i+1)})^p$, for $p > 2$.

The proof is complete. $\square$

Theorem 5.1.3. *Let G be a powerful finite p-group. Assume that* $\exp(G)$ *divides q. Then $\Upsilon^q(G)$ is powerfully embedded in $\nu^q(G)$.*

Proof. Firstly, let p be odd. To show that $[\Upsilon^q(G), \nu^q(G)] \leq \Upsilon^q(G)^p$ we can make use of Lemma 1.6.9, with $N = [\Upsilon^q(G), \nu^q(G)]$ and $M = \Upsilon^q(G)^p$. To this end it suffices to show that $N \leq W$, where $W := \Upsilon^q(G)^p[\Upsilon^q(G), \nu^q(G), \nu^q(G)]$. Using the base step of the Lemma 4.0.3 and the fact that G is powerful we get that $N = [G', G^\varphi] \leq [G^p, G^\varphi]$. So, let us prove that $[G^p, G^\varphi] \leq W$. For arbitrary elements $x, y \in G$, by Theorem 1.6.7 we have

$$[x^p, y^\varphi] \equiv [x, y^\varphi]^p \pmod{\gamma_2(L)^p \gamma_p(L)},$$

where $L = \langle x, [x, y^\varphi] \rangle$. Now we note that $\gamma_2(L)^p \leq [G, G^\varphi, G]^p = [G, G, G^\varphi]^p \leq [G, G^\varphi]^p \leq \Upsilon^q(G)^p \leq W$. Furthermore, since in this case $p - 1 \geq 2$, we have

$$\gamma_p(L) \leq [G, G^\varphi, G, {}_{p-2}\, \nu^q(G)] \leq [\Upsilon^q(G), {}_2\, \nu^q(G)] \leq W.$$

As $[x, y^\varphi]^p \in [G, G^\varphi]^p \leq W$, it follows that $[x^p, y^\varphi] \in W$. Consequently, $[G^p, G^\varphi] \leq W$. This proves the Theorem, for p odd.

If $p = 2$ we use similar arguments as above, but now we set $W_1 := \Upsilon^q(G)^4[\Upsilon^q(G), \nu^q(G), \nu^q(G)][\Upsilon^q(G), \nu^q(G)]^2$ and prove that $[G^4, G^\varphi] \leq W_1$. Using Theorem 1.6.7 again we can write, for all $x, y \in G$,

$$[x^4, y^\varphi] \equiv [x, y^\varphi]^4 \pmod{\gamma_2(L)^4 \gamma_2(L)^2 \gamma_4(L)}$$

where $L = \langle x, [x, y^\varphi] \rangle$. We observe that

$$\begin{aligned}
&\gamma_2(L)^4 \leq [G, G, G^\varphi]^4 \leq [G, G^\varphi]^4 \leq W_1 \\
&\gamma_2(L)^2 \leq [G, G^\varphi, G]^2 \leq [\Upsilon^q(G), \nu^q(G)]^2 \leq W_1, \\
&\gamma_4(L) \leq [G, G^\varphi, G, {}_2\, \nu^q(G)] \leq [\Upsilon^q(G), {}_3\, \nu^q(G)] \leq [\Upsilon^q(G), {}_2\, \nu^q(G)] \leq W_1.
\end{aligned}$$

In addition, as $[x, y^\varphi]^4 \in [G, G^\varphi]^4 \leq W_1$, it follows that $[x^4, y^\varphi] \in W_1$. Consequently, $[G^4, G^\varphi] \leq W_1$. This completes the proof. $\square$

The last theorem states that $\Upsilon^q(G)$ is powerfully embedded in $\nu^q(G)$, when $\exp(G)$ divides q. By Lemma 1.6.13 follows that $\frac{\Upsilon^q(G)}{\Delta^q(G)}$ is powerfully embedded in $\frac{\nu^q(G)}{\Delta^q(G)}$. This means that this quotient is powerful. Since $\frac{\Upsilon^q(G)}{\Delta^q(G)}$ is isomorphic to $G \wedge^q G$, the q-exterior square of the group G, we have that $G \wedge^q G$ is powerful too.

Changing from the family of powerful to that of potent p-groups we obtain similar results, the differences appear in the terms of the derived series. In the potent's family we need to prove directly that the terms of the derived series are potent, instead of proving that these terms are potently embedded, as we did in the powerful's family.

Theorem 5.1.4. *Let p be a prime, q be a positive integer and G be a potent finite p-group.*

(i) If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\nu^q(G))$ is potently embedded in $\nu^q(G)$.

(ii) Moreover, if $exp(G)$ divides q, then the q-tensor square $\Upsilon^q(G)$ is potently embedded in $\nu^q(G)$.

Proof. Recall that the definitions of powerful and potent p-groups coincide for $p = 2$ and 3. Using the Theorem 5.1.2 for the first item and the Theorem 5.1.3 for the the second item we will conclude the wished. Now, it remains to consider potent p-groups with $p \geq 5$.

(i) We will prove it by induction on k. If $k = 2$ we apply Proposition 4.0.4 for $n = p-1$ to obtain

$$\gamma_p(\nu^q(G)) = [\gamma_2(\nu^q(G)),\ _{p-2}\ \nu^q(G)] \leq \gamma_2(\nu^q(G))^p.$$

Suppose by induction hypothesis that $[\gamma_k(\nu^q(G)),\ _{p-2}\ \nu^q(G)] \leq \gamma_k(\nu^q(G))^p$. By Theorem 1.6.10,

$$\begin{aligned}
[\gamma_{k+1}(\nu^q(G)),\ _{p-2}\ \nu^q(G)] = [\gamma_k(\nu^q(G)),\ _{p-2}\ \nu^q(G), \nu^q(G)] &\leq [\gamma_k(\nu^q(G))^p, \nu^q(G)] \\
&\leq [\gamma_k(\nu^q(G)), \nu^q(G)]^p[\nu^q(G),\ _p\ \gamma_k(\nu^q(G))] \\
&\leq \gamma_{k+1}(\nu^q(G))^p[\gamma_{k+1}(\nu^q(G)),\ _{p-2}\ \nu^q(G), \nu^q(G)].
\end{aligned}$$

Therefore, by Lemma 1.6.9 with $N = [\gamma_{k+1}(\nu^q(G)),\ _{p-2}\ \nu^q(G)]$ and $M = \gamma_{k+1}(\nu^q(G))^p$ we have

$$[\gamma_{k+1}(\nu^q(G)),\ _{p-2}\ \nu^q(G)] \leq \gamma_{k+1}(\nu^q(G))^p,$$

which is the desired conclusion for this item.

(ii) Suppose that $exp(G)$ divides q. We need to prove that

$$[\Upsilon^q(G),\ _{p-2}\ \nu^q(G)] \leq \Upsilon^q(G)^p.$$

By Lemma 4.0.3, for $k = p-2$, we have that $[\Upsilon^q(G),\ _{p-2}\ \nu^q(G)] = [\gamma_{p-1}(G), G^\varphi] \leq [G^p, G^\varphi]$. Since $[G, G^\varphi]^p \leq \Upsilon^q(G)^p$, it is suffices to prove that $[G^p, G^\varphi] \leq [G, G^\varphi]^p$. But this follows by similar arguments those used in the Proposition 4.0.4. This completes the proof.

$\square$

Notice that if $p \nmid q$ then G is q-perfect (that is, $G = G'G^q$) and thus G is a homomorphic image of $G \otimes^q G$ by the homomorphism ρ' (Cf. (2.18)); consequently, $\exp(G \otimes^q G) = \exp(G)$. More precisely, in this case we also have $\mathrm{H}_2(G, \mathbb{Z}_q) = 1 = \Delta^q(G)$, giving $\ker(\rho') = 1$ and thus $G \otimes^q G \cong G \cong G \wedge^q G$. In particular, obviously $G \otimes^q G$ is powerful or potent if G is.

Theorem 5.1.5. *Let p be a prime, q be a positive integer and G a potent finite p-group. Then the i-th term of the derived series $\nu^q(G)^{(i)}$ is a potent p-group, for $i \geq 1$.*

Proof. We just need to consider the odd case. We will prove it by induction on i. In the Theorem 5.1.4 we proved that $\gamma_2(\nu^q(G)) = \nu^q(G)^{(1)}$ is potently embedded in $\nu^q(G)$, which implies that $\nu^q(G)^{(1)}$ is potent.

Assume that the result holds for degree i, that is, $\gamma_{p-1}(\nu^q(G)^{(i)}) \leq (\nu^q(G)^{(i)})^p$. The induction hypothesis and Theorem 1.6.10 give us

$$\begin{array}{rcl}
\gamma_{p-1}(\nu^q(G)^{(i+1)}) & = & [[\nu^q(G)^{(i)}, \nu^q(G)^{(i)}],\ _{p-2}[\nu^q(G)^{(i)}, \nu^q(G)^{(i)}]] \\
& \leq & [\nu^q(G)^{(i)}, \nu^q(G)^{(i)},\ _{p-2}\nu^q(G)^{(i)}] = [\gamma_{p-1}(\nu^q(G)^{(i)}),\ \nu^q(G)^{(i)}] \\
& \leq & [(\nu^q(G)^{(i)})^p\ , \nu^q(G)^{(i)}] \leq [\nu^q(G)^{(i)}, \nu^q(G)^{(i)}]^p[\nu^q(G)^{(i)},\ _p\nu^q(G)^{(i)}] \\
& \leq & (\nu^q(G)^{(i+1)})^p[\gamma_{p-1}(\nu^q(G)^{(i)}), \nu^q(G)^{(i)},\ \nu^q(G)] \\
& \leq & (\nu^q(G)^{(i+1)})^p[(\nu^q(G)^{(i)})^p, \nu^q(G)^{(i)}, \nu^q(G)].
\end{array}$$

By Lemma 1.6.9 we then get $[(\nu^q(G)^{(i)})^p, \nu^q(G)^{(i)}] \leq (\nu^q(G)^{(i+1)})^p$, which implies that $\gamma_{p-1}(\nu^q(G)^{(i+1)}) \leq (\nu^q(G)^{(i+1)})^p$, for $p > 2$. $\square$

Now, instead of considering a finite powerful or potent p-group G, we will suppose that G is any finite p-group which contains a potent normal subgroup N. In this situation we will show that the subgroup $\mathcal{R} = \langle N, N^\varphi, \mathcal{N}\rangle[N, G^\varphi][G, N^\varphi]$ is potent, by proving that $\mathcal{R}$ is potently embedded in itself, using the next proposition.

Proposition 5.1.6. *Let N be a normal subgroup of a finite p-group G and let $\mathcal{R} = \langle N, N^\varphi, \mathcal{N}\rangle[N, G^\varphi][G, N^\varphi]$.*

(i) If $p \geq 3$ and N is potent, then $\gamma_p(\mathcal{R}) \leq \gamma_2(\mathcal{R})^p$;

(ii) If $p = 2$ and N is powerful, then $\gamma_3(\mathcal{R}) \leq \gamma_2(\mathcal{R})^4$.

Proof. (i) Since N is a potent p-group, by Theorem 1.6.22 (i) and $p \geq 3$, we have $\gamma_p(N) = \gamma_{p-1+1}(N) \leq \gamma_2(N)^p$ so

$$\gamma_p(\mathcal{R}) = \gamma_p(N)\gamma_p(N^\varphi)[\gamma_{p-1}(N), N^\varphi] \leq \gamma_2(N)^p\gamma_2(N^\varphi)^p[N^p, N^\varphi].$$

We will prove that $[N^p, N^\varphi] \leq [N, N^\varphi]^p$. Let $n, m \in N$, by Hall's collection formula (1.2),

$$[n^p, m^\varphi] \equiv [n, m^\varphi]^p \pmod{\gamma_2(L)^p \gamma_p(L)},$$

where $L = \langle n, [n, m^\varphi] \rangle$. Note that $\gamma_2(L)^p \leq [N, N^\varphi, N]^p = [N, N, N^\varphi]^p \leq [N, N^\varphi]^p$ and

$$\begin{aligned} \gamma_p(L) &\leq [N, N^\varphi, N, {}_{p-2}N] = [\gamma_{p-1}(N), N, N^\varphi] \\ &\leq [N^p, N, N^\varphi] = [N^p, N^\varphi, N^\varphi] \leq [N^p, N^\varphi, \nu^q(G)]. \end{aligned}$$

Thus, it follows that

$$[N^p, N^\varphi] \leq [N, N^\varphi]^p [N^p, N^\varphi, \nu^q(G)].$$

Note that $[N^p, N^\varphi]$ and $[N, N^\varphi]^p$ are normal subgroups of $\nu^q(G)$. So, applying Lemma 1.6.9 considering these normal subgroups we obtain $[N^p, N^\varphi] \leq [N, N^\varphi]^p$. Therefore

$$\gamma_p(N)\gamma_p(N^\varphi)[N^p, N^\varphi] \leq \gamma_2(N)^p \gamma_2(N^\varphi)^p [N, N^\varphi]^p,$$

which completes this item.

(ii) Now, consider $p = 2$. Since N is a powerful p-group, so is potent too, by Theorem 1.6.22 (i) we have that $\gamma_3(N) \leq \gamma_2(N)^4$ thus

$$\gamma_3(\mathcal{R}) = \gamma_3(N)\gamma_3(N^\varphi)[\gamma_2(N), N^\varphi] \leq \gamma_2(N)^4 \gamma_2(N^\varphi)^4 [N^4, N^\varphi].$$

Again we need to prove that $[N^4, N^\varphi] \leq [N, N^\varphi]^4$. So, for $n, m \in N$, the Hall's collection formula (1.2) asserts that

$$[n^4, m^\varphi] \equiv [n, m^\varphi]^4 \pmod{\gamma_2(L)^4 \gamma_2(L)^2 \gamma_4(L)},$$

where $L = \langle n, [n, m^\varphi] \rangle$. Observe that $\gamma_2(L) \leq [N, N^\varphi, N] = [N, N, N^\varphi]$; this implies that

$$\gamma_2(L)^4 \leq [N, N, N^\varphi]^4 \leq [N, N^\varphi]^4$$

$$\gamma_2(L)^2 \leq [N, N, N^\varphi]^2 \leq [N^4, N^\varphi]^2$$

$$\gamma_4(L) \leq [N, N, N^\varphi, L, L] \leq [N, N, N^\varphi, \nu^q(G), \nu^q(G)] \leq [N^4, N^\varphi, \nu^q(G)].$$

Therefore $[n^4, m^\varphi] \in [N, N^\varphi]^4 [N^4, N^\varphi, \nu^q(G)][N^4, N^\varphi]^2$, for all $n, m \in N$, which implies that

$$[N^4, N^\varphi] \leq [N, N^\varphi]^4 [N^4, N^\varphi, \nu^q(G)][N^4, N^\varphi]^2.$$

By normality of the subgroups $[N^4, N^\varphi]$ and $[N, N^\varphi]^4$ in $\nu^q(G)$, we can apply Lemma 1.6.9 to obtain $[N^4, N^\varphi] \leq [N, N^\varphi]^4$.

Therefore

$$\gamma_3(\mathcal{R}) \leq \gamma_3(N)\gamma_3(N^\varphi)[N^4, N^\varphi] \leq \gamma_2(N)^4 \gamma_2(N^\varphi)^4 [N, N^\varphi]^4$$

and the proof is complete.

□

Corollary 5.1.7. *Let N be a potent normal subgroup of the group G and consider $s \geq 2$. Then $\gamma_s(\mathcal{R})$ is potently embedded in $\mathcal{R}$. In particular, $\gamma_s(\mathcal{R})$ is potent.*

Proof. We will prove by induction on s. The base step was done in the last proposition. So, we need only prove the induction step, for this suppose that

$$[\gamma_s(\mathcal{R}),\ _{p-2}\ \mathcal{R}] \leq \gamma_s(\mathcal{R})^p,\ p \geq 3$$
$$[\gamma_s(\mathcal{R}), \mathcal{R}] \leq \gamma_s(\mathcal{R})^4,\ p = 2.$$

Let $p \geq 3$. By induction hypothesis and Theorem 1.6.10 we get

$$\begin{aligned}
[\gamma_{s+1}(\mathcal{R}),\ _{p-2}\ \mathcal{R}] &= [\gamma_s(\mathcal{R}),\ _{p-2}\ \mathcal{R}, \mathcal{R}] \leq [\gamma_s(\mathcal{R})^p, \mathcal{R}] \\
&\leq [\gamma_s(\mathcal{R}), \mathcal{R}]^p [\mathcal{R},\ _p\ \gamma_s(\mathcal{R})] \\
&\leq \gamma_{s+1}(\mathcal{R})^p [\gamma_{s+1}(\mathcal{R}),\ _{p-2}\ \mathcal{R}, \mathcal{R}]
\end{aligned}$$

Applying Lemma 1.6.9 we obtain $[\gamma_{s+1}(\mathcal{R}),\ _{p-2}\ \mathcal{R}] \leq \gamma_{s+1}(\mathcal{R})^p$, for $p \geq 3$.

Now let $p = 2$. In a similar way as above we have

$$\begin{aligned}
[\gamma_{s+1}(\mathcal{R}), \mathcal{R}] &= [\gamma_s(\mathcal{R}), \mathcal{R}, \mathcal{R}] \leq [\gamma_s(\mathcal{R})^4, \mathcal{R}] \\
&\leq [\gamma_s(\mathcal{R}), \mathcal{R}]^4 [\mathcal{R},\ _2\ \gamma_s(\mathcal{R})]^2 [\mathcal{R},\ _4\ \gamma_s(\mathcal{R})] \\
&\leq \gamma_{s+1}(\mathcal{R})^4 [\gamma_{s+1}(\mathcal{R}), \mathcal{R}]^2 [\gamma_{s+1}(\mathcal{R}), \mathcal{R}, \mathcal{R}]
\end{aligned}$$

By applying Lemma 1.6.9 we obtain $[\gamma_{s+1}(\mathcal{R}), \mathcal{R}] \leq \gamma_{s+1}(\mathcal{R})^4$, if $p = 2$. □

5.2 Bounding exponents and numbers of generators

In this last section of this chapter we investigate what happens to the number of generators and to the exponent of the group $\nu^q(G)$ and some of its sections comparing with those of the group G. For the powerful's family we get similar results due to Moravec in [38], but for the potent's family results about the number of generators have not been reached.

Theorem 5.2.1. *Let G be a powerful finite p-group. Assume that* $\exp(G)$ *divides q and let $d = d(G)$. Then $d(G \otimes^q G) \leq d(d+1)$.*

Proof. Suppose that $G = \langle g_1, \dots, g_d \rangle$. By definition the subgroup $\Upsilon^q(G) = [G, G^\varphi]K$ is the normal closure in $\nu^q(G)$ of the set

$$X = \{[g_i, g_j^\varphi], \widehat{g_k} \mid 1 \leq i, j, k \leq d\}.$$

Since $\Upsilon^q(G)$ is powerfully embedded in $\nu^q(G)$, by Proposition 1.6.19 it follows that this subgroup is actually generated by X. Thus, having in mind the isomorphism $\Upsilon^q(G) \cong G \otimes^q G$, an upper bound for $d(G \otimes^q G)$ is given by $|X| = d^2 + d = d(d+1)$, as desired. $\square$

In [46] the authors proved that if G is a nilpotent group of class two with $d(G) = d$, then $d(G \otimes^q G) \leq \frac{1}{3}d(d^2 + 3d + 2)$, for all $q \geq 0$, and this bound is attained, for instance, for the free nilpotent group of class two and rank d. On the other hand, by considering, for example, the group $G = \langle x, y \mid x^9 = 1 = y^3, x^y = x^4 \rangle$, which is a powerful 3-group of nilpotency class 2 then, with the help of the GAP system [23], we found for $q = 9$ that $d(G \otimes^9 G) = 6$. So this group G satisfies the better bound given in the last theorem. In [3], the author was some investigates about the number of generators of a nilpotent group of class two considering the case when $q = 0$, which is one of the first works about this.

Moreover, recalling that $\nu^q(G) = (\Upsilon^q(G) \cdot G) \cdot G^\varphi$, using the last theorem we can give an upper bound for the number of generators of $\nu^q(G)$, when G is a powerful, d-generated p-group with $\exp(G)$ dividing q: $d(\nu^q(G)) \leq d(d+1) + 2d = d(d+3)$.

Corollary 5.2.2. *Let G be a powerful finite p-group with $d(G) = d$. Then, for all $q \geq 0$, $d([\nu^q(G), \nu^q(G)]) \leq d(2d-1)$.*

Proof. Let G be generated by $\{g_1, \dots, g_d\}$. By Proposition 2.2.2 we see that $\gamma_2(\nu^q(G))$ is the normal closure in $\nu^q(G)$ of the set

$$Y = \{[g_i, g_j], [g_i^\varphi, g_j^\varphi], [g_k, g_l^\varphi] \mid 1 \leq i < j \leq d,\ 1 \leq k, l \leq d\}.$$

As $[\nu^q(G), \nu^q(G)]$ is powerfully embedded in $\nu^q(G)$, again by Proposition 1.6.19, it follows that $[\nu^q(G), \nu^q(G)]$ is generated by Y. Thus we get $\mathrm{d}([\nu^q(G), \nu^q(G)]) \leq |Y| = \binom{d}{2} + \binom{d}{2} + d^2 = d(2d-1)$. $\square$

In order to establish some bounds for exponents first we can mention that, for the case $q = 0$, Moravec [37] showed that if G is a group of nilpotency class ≤ 3 with finite exponent, then $\exp(G \otimes G)$ divides $\exp(G)$. Independent of the nilpotency class, this result is also true if we consider a powerful finite p-group, as also proved again by Moravec in [38]. Based on this result we want a limitation for the exponent of $\Upsilon^q(G)$, for all $q \geq 0$. Since $\Upsilon^q(G) = [G, G^{\varphi}]K$ and $[G, G^{\varphi}] \leq \gamma_2(\nu^q(G))$, we use a limitation of the exponent of the subgroups $\gamma_2(\nu^q(G))$ and K to obtain our desired upper bounds.

Theorem 5.2.3. *Let G be a powerful p-group. Then*

(i) $\exp([\nu^q(G), \nu^q(G)])$ *divides* $\exp(G)$.

(ii) $\exp(K)$ *divides* $\exp(G)$ *if p is odd.*

(iii) $\exp(K)$ *divides* $\exp(G)$ *if $p = 2$ and either q is odd or 4 divides q.*

(iv) $\exp(K)$ *divides* $2\exp(G)$ *if $p = 2$ and 4 does not divide q.*

Proof. (i)

Our proof starts with the observation that it is possible, using Theorem 5.1.1 and a double induction on $j \in \mathbb{N}$ and $k \geq 2$, to show that:

$$\begin{aligned} \gamma_k(\nu^q(G))^{p^j} &\leq \gamma_2(\nu^q(G))^{p^{j+k-2}} \text{ if } p \geq 3; \\ \gamma_k(\nu^q(G))^{2^j} &\leq \gamma_2(\nu^q(G))^{2^{j+2(k-2)}} \text{ if } p = 2. \end{aligned} \tag{5.4}$$

Consider $\exp(G) = p^e$ and suppose that $\exp(\gamma_2(\nu^q(G))$ does not divide $\exp(G) = p^e$. So, without loss of generality we can assume that $\exp(\gamma_2(\nu^q(G))) = p^{e+1}$. This means that $\gamma_2(\nu^q(G))^{p^{e+1}} = 1$ and that there exists an element $[a, b]^{p^e} \neq 1$ with $a, b \in \nu^q(G)$.

Since $\nu^q(G) = \langle G \cup G^{\varphi} \cup \mathcal{K} \rangle$, by its defining relations we have the following possibilities for the element $[a, b]$:

$$\begin{array}{lll} [a,b] = [g_1, g_2]; & [a,b] = [h_1^{\varphi}, g_1]; & [a,b] = [\widehat{k}_1, g_1] = [k_1^q, g_1^{\varphi}]; \\ [a,b] = [g_1, h_1^{\varphi}]; & [a,b] = [h_1^{\varphi}, h_2^{\varphi}]; & [a,b] = [\widehat{k}_1, h_1^{\varphi}] = [k_1^q, h_1^{\varphi}]; \\ [a,b] = [g_1, \widehat{k}_1]; & [a,b] = [h_1^{\varphi}, \widehat{k}_1]; & [a,b] = [\widehat{k}_1, \widehat{k}_2] = [k_1^q, (k_2^q)^{\varphi}], \end{array}$$

where $g_1, g_2 \in G$, $h_1^{\varphi}, h_2^{\varphi} \in G^{\varphi}$ and $\widehat{k}_1, \widehat{k}_2 \in \mathcal{K}$. Then we can assume that a is a element in G or G^{φ}.

Considering $p > 2$ and using Hall's collection formula (1.2) we can expand the expression $[a^{p^e}, b]$ in the following way

$$[a,b]^{p^e} \equiv [a^{p^e}, b] \left(\mathrm{mod}\ \gamma_2(L)^{p^e} \prod_{i=1}^{e} \gamma_{p^i}(L)^{p^{e-i}} \right) \tag{5.5}$$

where $L = \langle a, [a,b] \rangle$. Observe that $\gamma_2(L) \leq \gamma_3(\nu^q(G))$. As $e - i + p^i - 1 > e + 1$, for $i \in \{1, \ldots e\}$, using the first observation of this proof we can write

$$\begin{aligned}
&\gamma_2(L)^{p^e} \leq \gamma_3(\nu^q(G))^{p^e} \leq \gamma_2(\nu^q(G))^{p^{e+1}} = 1; \\
&\gamma_{p^i}(L))^{p^{e-i}} \leq \gamma_{p^i+1}(\nu^q(G))^{p^{e-i}} \leq \gamma_2(\nu^q(G))^{p^{e-i+p^i+1-2}} \leq \gamma_2(\nu^q(G))^{p^{e+1}} = 1.
\end{aligned}$$

Since $a \in G \cup G^\varphi$, it follows that $a^{p^e} = 1$. So the congruence (5.5) can be rewritten as $[a,b]^{p^e} = [a^{p^e}, b] = 1$, a contradiction.

Now if $p = 2$, Hall's collection formula (1.2) for $[a^{p^e}, b]$ becames

$$[a,b]^{2^e} \equiv [a^{2^e}, b] \left(\mathrm{mod}\ \gamma_2(L)^{2^e} \gamma_2(L)^{2^{e-1}} \prod_{i=2}^{e} \gamma_{2^i}(L)^{2^{e-i}} \right) \tag{5.6}$$

where $L = \langle a, [a,b] \rangle$. Observe that $\gamma_2(L) \leq \gamma_3(\nu^q(G))$. As $e - i + 2^{i+1} - 2 > e + 1$, for $i \in \{2, \ldots e\}$; using the initial observation again we have

$$\begin{aligned}
&\gamma_2(L)^{2^e} \leq \gamma_2(L)^{2^{e-1}} \leq \gamma_3(\nu^q(G))^{2^{e-1}} \leq \gamma_2(\nu^q(G))^{2^{e-1+2(3-2)}} = \gamma_2(\nu^q(G))^{2^{e+1}} = 1; \\
&\gamma_{2^i}(L))^{2^{e-i}} \leq \gamma_{2^i+1}(\nu^q(G))^{2^{e-i}} \leq \gamma_2(\nu^q(G))^{2^{e-i+2^{i+1}-2}} \leq \gamma_2(\nu^q(G))^{2^{e+1}} = 1.
\end{aligned}$$

Since $a \in G \cup G^\varphi$, it follows that $a^{2^e} = 1$. So the congruence (5.6) can be rewritten as $[a,b]^{2^e} = [a^{2^e}, b] = 1$, again a contradiction. Therefore, $\exp(\gamma_2(\nu^q(G))$ divides $\exp(G)$.

(ii), (iii) and (iv)

Suppose that G has exponent p^e and let $\alpha = \widehat{k_1}^{\epsilon_1} \cdot \ldots \cdot \widehat{k_n}^{\epsilon_n}$ be an arbitrary element in K, where $\epsilon_j = \pm 1$, $j = 1, \ldots, n$. By Hall's collection formula (1.1) we can write, for any $m \geq 0$,

$$(\widehat{k_1}^{\epsilon_1} \cdots \widehat{k_n}^{\epsilon_n})^{p^m} \equiv \widehat{k_1}^{\epsilon_1 p^m} \cdots \widehat{k_n}^{\epsilon_n p^m} \left(\mathrm{mod}\ \gamma_2(H)^{p^m} \prod_{i=1}^{m} \gamma_{p^i}(H)^{p^{m-i}} \right), \tag{5.7}$$

where $H = \langle \widehat{k_1}, \ldots, \widehat{k_n} \rangle$.

Set $m = e$ if $p \geq 3$, or $m = e + 1$ if $p = 2$. By relation (5.4) we have the following inclusions of subgroups:

$$\begin{aligned}
\gamma_2(H)^{p^{e+1}} &\leq \gamma_2(H)^{p^e} \leq \gamma_2(\nu^q(G))^{p^e}, \text{ for all } p;\\
\gamma_{p^i}(H)^{p^{e-i}} &\leq \gamma_{p^i}(\nu^q(G))^{p^{e-i}} \leq \gamma_2(\nu^q(G))^{p^{e-i+p^i-2}}, \text{ for } p \geq 3, \text{ and}\\
\gamma_{2^i}(H)^{2^{e+1-i}} &\leq \gamma_{2^i}(\nu^q(G))^{2^{e+1-i}} \leq \gamma_2(\nu^q(G))^{2^{e+1-i+2(2^i-2)}}, \text{ for } p = 2.
\end{aligned}$$

But, by part (i) the right hand side of each of the above inclusions is the trivial group, since, for $1 \leq i \leq m$, $-i+p^i-2 \geq 0$ if $p \geq 3$, and $1-i+2^{i+1}-4 \geq 0$ if $p = 2$. Therefore, in any case, equation (5.7) can be rewritten as $(\widehat{k_1}^{\epsilon_1} \cdots \widehat{k_n}^{\epsilon_n})^{p^m} = \widehat{k_1}^{\epsilon_1 p^m} \cdots \widehat{k_n}^{\epsilon_n p^m}$.

We claim that $(\widehat{k})^{p^m} = 1$ for any generator $\widehat{k}$ of K. Induction using relations (2.13) and Lemma 2.2.1 gives, for all integers l,

$$\widehat{k^{-l}} = \widehat{k}^{-l}[k, k^{\varphi}]^{-\frac{(l+1)l}{2}\cdot\frac{q(q-1)}{2}}.$$

Since $[k, k^{\varphi}]$ is central in $\nu^q(G)$ we have the followings cases:

- If $p \geq 3$, it follows that

$$\begin{aligned}
1 = \widehat{k^{p^e}} &= \widehat{k}^{p^e}[k, k^{\varphi}]^{-\frac{(p^e-1)p^e}{2}\cdot\frac{q(q-1)}{2}}\\
&= \widehat{k}^{p^e}[k, k^{\varphi}]^{p^e\frac{-p^e+1}{2}\cdot\frac{q(q-1)}{2}}\\
&= \widehat{k}^{p^e}[k^{p^e}, k^{\varphi}]^{\frac{-p^e+1}{2}\cdot\frac{q(q-1)}{2}}\\
&= \widehat{k}^{p^e}.
\end{aligned}$$

- If $p = 2$ and 4 divides q, then $q = 4s$, with $s \in \mathbb{N}$, we have

$$\begin{aligned}
1 = \widehat{k^{2^e}} &= \widehat{k}^{2^e}[k, k^{\varphi}]^{-\frac{(2^e-1)2^e}{2}\cdot\frac{4s(4s-1)}{2}}\\
&= \widehat{k}^{2^e}[k, k^{\varphi}]^{-(2^e-1)2^e\cdot s(4s-1)}\\
&= \widehat{k}^{2^e}[k^{2^e}, k^{\varphi}]^{(-2^e+1)\cdot s(4s-1)}\\
&= \widehat{k}^{2^e}.
\end{aligned}$$

- If $p = 2$ and q is odd, then

$$\begin{aligned}
1 = \widehat{k^{2^e}} &= \widehat{k}^{2^e}[k, k^{\varphi}]^{-\frac{(2^e-1)2^e}{2}\cdot\frac{q(q-1)}{2}}\\
&= \widehat{k}^{2^e}[k, k^{\varphi}]^{-\frac{(2^e-1)2^e}{2}\cdot q\frac{(q-1)}{2}}\\
&= \widehat{k}^{2^e}[k^q, k^{\varphi}]^{\frac{(-2^e+1)2^e}{2}\cdot\frac{q-1}{2}}\\
&= \widehat{k}^{2^e}.
\end{aligned}$$

- If $p = 2$ and 4 does not divide q, then we see that

$$\begin{aligned} 1 = \widehat{k^{2^{e+1}}} &= \widehat{k}^{2^{e+1}}[k,k^{\varphi}]^{-\frac{(2^{e+1}-1)2^{e+1}}{2}\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^{e+1}}[k,k^{\varphi}]^{(-2^{e+1}+1)2^{e}\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^{e+1}}[k^{2^e},k^{\varphi}]^{(-2^{e+1}+1)\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^{e+1}}. \end{aligned}$$

This completes the proof. □

It follows easily from Theorem 5.1.1 that $\gamma_2(\nu^q(G))$ is powerful. Using the limitation for the number of generators, Corollary 5.2.2, and the limitation for the exponent, Theorem 5.2.3(i), of $\gamma_2(\nu^q(G))$, we can apply Proposition 1.6.20 to obtain a limitation for their order. So, $|\gamma_2(\nu^q(G))| \leq p^{ed(2d-1)}$.

It should be noted that an important structural difference between the non-abelian tensor square $G \otimes G$ (case $q = 0$) and the more general q-tensor square $G \otimes^q G$, $q \geq 0$, is the presence in $G \otimes^q G$ of the subgroup $K \trianglelefteq \nu^q(G)$, generated by $\mathcal{K} = \{\widehat{k} \mid k \in G\}$, which is trivial in the case $q = 0$. In fact, while the non-abelian tensor square $G \otimes G \cong [G, G^{\varphi}]$ is embedded in the derived group $\nu(G)'$, in general $G \otimes^q G$ is far from being isomorphic to the subgroup $[G, G^{\varphi}] \leq \nu^q(G)$ if $q > 0$. The subgroup K plays an important role in the structure of $\Upsilon^q(G)$ and so it has an influence on the exponent of $G \otimes^q G$ ($\cong \Upsilon^q(G)$).

Corollary 5.2.4. *Let G be a powerful p-group. Then*

(i) $\exp(\Upsilon^q(G))$ *divides* $\exp(G)$ *if p is odd.*

(ii) $\exp(\Upsilon^q(G))$ *divides* $\exp(G)$ *if $p = 2$ and either q is odd or 4 divides q.*

(ii) $\exp(\Upsilon^q(G))$ *divides* $2\exp(G)$ *if $p = 2$ and 4 does not divide q.*

Proof. Let α be an arbitrary element of $\Upsilon^q(G) = [G, G^{\varphi}]K$. Thus we can write $\alpha = xy$, where $x \in [G, G^{\varphi}]$ and $y \in K$. By Theorem 5.2.3 we know that $x^{\exp(G)} = 1$, while $y^{p^m} = 1$ where, as above, $p^m = p^e = \exp(G)$ if $p \geq 3$ or if $p = 2$ and either q is odd or 4 divides q, and $p^m = 2^{e+1} = 2\exp(G)$ if $p = 2$ and 4 does not divide q. Using Hall's collection formula (1.1) once again we get

$$(xy)^{p^m} \equiv x^{p^m} y^{p^m} \Big(\text{mod } \gamma_2(H)^{p^m} \prod_{i=1}^{m} \gamma_{p^i}(H)^{p^{m-i}}\Big),$$

where $H = \langle x, y \rangle$. A similar argument as in the proof of Theorem 5.2.3 shows that $\gamma_2(H)^{p^m} \prod_{i=1}^{m} \gamma_{p^i}(H)^{p^{m-i}} = 1$. Therefore, $\alpha^{p^m} = (xy)^{p^m} = x^{p^m} y^{p^m} = 1$. This proves our corollary. □

We see that the upper bound given in Corollary 5.2.4 (ii) is attained by the cyclic group C_2 with $q = 2$, since $C_2 \otimes^2 C_2 \cong C_4$ (Cf. [15, Theorem 3.1]); also, for the Klein four group V_4 we have, according to [15, Remark 2.17]), $V_4 \otimes^2 V_4 \cong C_4 \times C_4 \times C_2$.

Using the Proposition 1.6.20 and the Corollary 5.2.4 we can establish a bound for the order of $\Upsilon^q(G)$ considering the exponent and the number of generators of the group G.

Corollary 5.2.5. *Let G be a powerful p-group, with $d(G) = d$ and $\exp(G) = p^e$. If $\exp(G)$ divides q then*

(i) $|\Upsilon^q(G)| \leq p^{ed(d+1)}$ *and* $|G \wedge^q G| \leq p^{\frac{ed(d+1)}{2}}$ *if p is odd.*

(ii) $|\Upsilon^q(G)| \leq p^{ed(d+1)}$ *and* $|G \wedge^q G| \leq p^{\frac{ed(d+1)}{2}}$ *if $p = 2$ and either q is odd or 4 divides q.*

(iii) $|\Upsilon^q(G)| \leq 2^{(e+1)d(d+1)}$ *and* $|G \wedge^q G| \leq 2^{\frac{(e+1)d(d+1)}{2}}$ *if $p = 2$ and 4 does not divide q.*

Proof. Suppose that $G = \langle g_1, \ldots, g_d \rangle$. The order $|\Upsilon^q(G)|$ is a simple application of the last corollary and the Theorem 5.2.1, in the Proposition1.6.20.

Since $G \wedge^q G$ is powerful too, firstly we need to determinate their number of generators and their exponent, for this remember the isomorphism $G \wedge^q G \cong \frac{\Upsilon^q(G)}{\Delta^q(G)}$. As $[g^j, g^{\varphi}] = [g, g^{\varphi}]^j$ for any $g \in G$, we have $\exp(\Delta^q(G))$ divides $\exp(G)$. By Corollary 5.2.4, the exponent of $\Upsilon^q(G)$ is limited by a function of G, therefore the exponent of the quotient, $\frac{\Upsilon^q(G)}{\Delta^q(G)}$, will have the limitation of the exponent of $\Upsilon^q(G)$.

Remains to calculate the number of generators of this quotient. Since $\Upsilon^q(G)$ is generated by the set $X = \{[g_i, g_j^{\varphi}], \widehat{g_k} \mid 1 \leq i, j, k \leq d\}$ and $[g_i, g_i^{\varphi}] \in \Delta^q(G)$, it follows that the number of generators of the quotient is given $\binom{d}{2} + d = \frac{d(d+1)}{2}$. Now, applying the Proposition 1.6.20 we obtain the corollary. □

Now we are interested in finding a limitation for the exponent of the whole group $\nu^q(G)$ or for some of its subgroups, considering the exponent of the group G, where G is a potent finite p-group. For this, firstly we will show under what conditions for i and q we have the equality $\Omega_i(\nu^q(G)) = \nu^q(G)$, where $i \geq 1$ and $q \geq 0$.

Lemma 5.2.6. *Let G be a p-group and suppose that $\exp(G) = p^e$. Then*

(i) $\nu^q(G)$ *is generated by elements of order less than or equal to p^e if $p \geq 3$.*

(ii) $\nu^q(G)$ *is generated by elements of order less than or equal to p^e if $p = 2$ and either q is odd or 4 divides q.*

(iii) $\nu^q(G)$ *is generated by elements of order at most 2^{e+1} if $p = 2$ and 4 does not divide q.*

Proof. Firstly, notice that $\nu^q(G)$ is generated by G, G^φ and $\mathcal{K}$. Since $\exp(G) = p^e = \exp(G^\varphi)$, the only point remaining concerns the behaviour of the exponent of $\mathcal{K}$. Let $\widehat{k} \in \mathcal{K}$ e remember that $[k, k^\varphi]$ is central in $\nu^q(G)$, so we have the followings cases:

- If $p \geq 3$, then

$$\begin{aligned} 1 = \widehat{k^{p^e}} &= \widehat{k}^{p^e}[k, k^\varphi]^{\frac{-(p^e-1)p^e}{2}\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{p^e}[k, k^\varphi]^{\frac{-p^e+1}{2}p^e\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{p^e}. \end{aligned}$$

- If $p = 2$ and 4 divides q, then $q = 4s$, with $s \in \mathbb{N}$, and hence

$$\begin{aligned} 1 = \widehat{k^{2^e}} &= \widehat{k}^{2^e}[k, k^\varphi]^{\frac{-(2^e-1)2^e}{2}\cdot\frac{4s(4s-1)}{2}} \\ &= \widehat{k}^{2^e}[k, k^\varphi]^{(-2^e+1)2^e\cdot s(4s-1)} \\ &= \widehat{k}^{2^e}. \end{aligned}$$

- If $p = 2$ and q is odd, then we see that

$$\begin{aligned} 1 = \widehat{k^{2^e}} &= \widehat{k}^{2^e}[k, k^\varphi]^{\frac{-(2^e-1)2^e}{2}\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^e}[k, k^\varphi]^{\frac{(-2^e+1)2^e}{2}\cdot q\frac{q-1}{2}} \\ &= \widehat{k}^{2^e}. \end{aligned}$$

- If $p = 2$ and 4 does not divide q, it follows

$$\begin{aligned} 1 = \widehat{k^{2^{e+1}}} &= \widehat{k}^{2^{e+1}}[k, k^\varphi]^{\frac{-(2^{e+1}-1)2^{e+1}}{2}\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^{e+1}}[k, k^\varphi]^{(-2^{e+1}+1)2^e\cdot\frac{q(q-1)}{2}} \\ &= \widehat{k}^{2^{e+1}}. \end{aligned}$$

Therefore, we conclude that $\nu^q(G) = \Omega_e(\nu^q(G))$, if $p \geq 3$, $\nu^q(G) = \Omega_e(\nu^q(G))$, if $p = 2$ and either q is odd or 4 divides q, and $\nu^q(G) = \Omega_{e+1}(\nu^q(G))$, if $p = 2$ and 4 does not divide q. $\square$

Corollary 5.2.7. *Let G be a potent p-group and suppose that $\exp(G) = p^e$. Then*

(i) $\exp(\nu^q(G))$ *divides* p^{e+1} *if* $p \geq 3$.

(ii) $\exp(\nu^q(G))$ *divides* 2^{e+2} *if* $p = 2$ *and either* q *is odd or* 4 *divides* q.

(ii) $\exp(\nu^q(G))$ *divides* 2^{e+3} *if* $p = 2$ *and* 4 *does not divide* q.

Proof. (i) Assume that $p \geq 3$. According to Theorem 5.1.4, we have that

$$\gamma_{2(p-1)}(\nu^q(G)) = [\gamma_p(\nu^q(G)),\ _{p-2}\ \nu^q(G)] \leq \gamma_p(\nu^q(G))^p.$$

Applying Lemma 1.6.11 to $k = 2, r = p$ and $s = 1$, we obtain that $\exp(\Omega_e(\nu^q(G)))$ divides p^{e+1}. But, by the last lemma $\Omega_e(\nu^q(G)) = \nu^q(G)$. Therefore, $\exp(\nu^q(G))$ divides p^{e+1}.

(ii) Now, consider $p = 2$ and either q is odd or 4 dividing q. By Theorem 5.1.4,

$$\gamma_3(\nu^q(G)) = [\gamma_2(\nu^q(G)), \nu^q(G)] \leq \gamma_2(\nu^q(G))^4.$$

Again by Lemma 1.6.11 to $k = 3, r = 2$ and $s = 2$, we have $\exp(\Omega_e(\nu^q(G)))$ divides 2^{e+2}. Since $\Omega_e(\nu^q(G)) = \nu^q(G)$, when 4 divides q, follows that $\exp(\nu^q(G))$ divides 2^{e+2}.

(iii) If $p = 2$ and 4 does not divide q, by Theorem 5.1.4, we have that $\gamma_3(\nu^q(G)) \leq \gamma_2(\nu^q(G))^4$. Again by Lemma 1.6.11 to $k = 3, r = 2$ and $s = 2$, it follows that $\exp(\Omega_{e+1}(\nu^q(G)))$ divides 2^{e+3}. Since $\Omega_{e+1}(\nu^q(G)) = \nu^q(G)$, when 4 does not divide q follows that $\exp(\nu^q(G))$ divides 2^{e+3}.

□

In the last corollary the same conclusion are obtained when we substitute the assumption that G is a powerful p-group with the assumption that it is a potent p-group, since if G is powerful then G is potent too.

In order to obtain another bound for the exponent of the group $\nu^q(G)$ in terms of the exponent of the group G we will give a bound for the kernel of the epimorphism $\overline{\pi}$, where the normal subgroup used is a potent p-group or has nilpotency class $p - 1$.

Lemma 5.2.8. *Let N be a normal subgroup of the p-group G and suppose that* $\exp(N) = p^e$. *Consider* $\mathcal{R} = \langle N, N^{\varphi}, \mathcal{N}\rangle[N, G^{\varphi}][N^{\varphi}, G]$. *Then*

(i) $\mathcal{R}$ *is generated by elements of order at most* p^e *if* $p \geq 3$.

(ii) $\mathcal{R}$ *is generated by elements of order at most* p^e *if* $p = 2$ *and either* q *is odd or* 4 *divides* q.

(ii) $\mathcal{R}$ *is generated by elements of order at most* 2^{e+1} *if* $p = 2$ *and* 4 *does not divide* q.

Proof. Firstly, notice that K is generated by N^{G^φ}, $(N^\varphi)^G$ and $\mathcal{N}$. Since $\exp(N) = p^e = \exp(N^\varphi)$, the only point remaining concerns the behaviour of the exponent of $\mathcal{N}$. But we can proceed analogously to the proof of Lemma 5.2.6.

Therefore, we can conclude that $\mathcal{R} = \Omega_e(\mathcal{R})$, if $p \geq 3$,or if $p = 2$ and either q is odd or 4 divides q, and $\mathcal{R} = \Omega_{e+1}(\mathcal{R})$, if $p = 2$ and 4 does not divide q. □

Corollary 5.2.9. *Let N and $\mathcal{R}$ be groups as in the last lemma. Assume that* $\exp(N) = p^e$, *then*

(i) Suppose that $\gamma_p(N) = 1$ or that N is a potent normal subgroup of G. If p is odd. Then $\exp(\mathcal{R})$ *divides* p^{e+1}.

(ii) Suppose that $\gamma_2(N) = 1$ or that N is a potent normal subgroup of G. If $p = 2$ and either q is odd or 4 divides q. Then $\exp(\mathcal{R})$ *divides* 2^{e+2}.

(iii) Suppose that $\gamma_2(N) = 1$ or that N is a potent normal subgroup of G. If $p = 2$ and 4 does not divide q. Then $\exp(\mathcal{R})$ *divides* 2^{e+3}

Proof. (i) Assume that $p \geq 3$. If $\gamma_p(N) = 1$, by Proposition 4.0.6, we have $\gamma_{p+1}(\mathcal{R}) = 1$. Since $2p - 2 \geq p + 1$, we see that $\gamma_{2(p-1)}(\mathcal{R}) \leq \gamma_{p+1}(\mathcal{R}) = 1$. On the other hand, if N is a normal subgroup potent of G, by Corollary 5.1.7 we conclude that $\gamma_{2(p-1)}(\mathcal{R}) = [\gamma_p(\mathcal{R}),_{p-2} \mathcal{R}] \leq \gamma_p(\mathcal{R})^p$.

So, in any case, we can apply the Lemma 1.6.11 to $k = 2, r = p$ and $s = 1$, for to obtain that $\exp(\Omega_e(\mathcal{R}))$ divides p^{e+1}. But, by Lemma 5.2.8 (i), $\Omega_e(\mathcal{R}) = \mathcal{R}$. Therefore $\exp(\Omega_e(\mathcal{R})) = \exp(\mathcal{R})$ divides p^{e+1}.

(ii) Consider $p = 2$ and suppose that q is odd or that 4 divides q. If $\gamma_2(N) = 1$, by Proposition 4.0.6, we have $\gamma_3(\mathcal{R}) = 1$. But, if N is a normal subgroup potent of G, by Corollary 5.1.7 we have $\gamma_{3(2-1)}(\mathcal{R}) = \gamma_3(\mathcal{R}) \leq \gamma_2(\mathcal{R})^4$.

So, in any case, we can apply the Lemma 1.6.11 to $k = 3, r = 2$ and $s = 2$, for to obtain that $\exp(\Omega_e(\mathcal{R}))$ divides 2^{e+2}. But, by Lemma 5.2.8 (ii), in this case, it follows that $\Omega_e(\mathcal{R}) = \mathcal{R}$. Therefore $\exp(\Omega_e(\mathcal{R})) = \exp(\mathcal{R})$ divides 2^{e+2}.

(iii) Now, assume that $p = 2$ and 4 does not divide q. Again, if $\gamma_2(N) = 1$, then $\gamma_3(\mathcal{R}) = 1$, by Proposition 4.0.6. If N is a normal subgroup potent of G we have $\gamma_3(\mathcal{R}) \leq \gamma_2(\mathcal{R})^4$, by Corollary 5.1.7.

Applying the Lemma 1.6.11 to $k = 3, r = 2$ and $s = 2$, we obtain that $\exp(\Omega_{e+1}(\mathcal{R}))$ divides 2^{e+3}. In this case $\Omega_{e+1}(\mathcal{R}) = \mathcal{R}$, it follows that $\exp(\mathcal{R})$ divides 2^{e+3}.

□

Theorem 5.2.10. *Let p be a prime and N be a normal subgroup of the p-group G. Suppose that $\gamma_p(N) = 1$ or that N is a potent subgroup of G. Then*

(i) $\exp(\nu^q(G))$ *divides* $p \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ *if* $p \geq 3$.

(ii) $\exp(\nu^q(G))$ *divides* $4 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ *if* $p = 2$ *and either* q *is odd or* 4 *divides* q.

(iii) $\exp(\nu^q(G))$ *divides* $8 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ *if* $p = 2$ *and* 4 *dos not divide* q.

Proof. We prove both items at the same time. Remember that, by Lemma 2.2.6, the kernel of the canonical epimorphism $\overline{\pi} : \nu^q(G) \to \nu^q(G/N)$ is defined by the subgroup $\mathcal{R} = \langle N, N^{\varphi}, \mathcal{N}\rangle[N, G^{\varphi}][N^{\varphi}, G]$. Moreover, $\exp(\nu^q(G))$ divides $\exp(\nu^q(G/N)) \cdot \exp(\mathcal{R})$. In the Corollary 5.2.9, we deduce that $\exp(\mathcal{R})$ divides $p \cdot \exp(N)$, if $p \geq 3$, or $\exp(\mathcal{R})$ divides $4 \cdot \exp(N)$ if $p = 2$ and either 4 divides q or q is odd, and $\exp(\mathcal{R})$ divides $8 \cdot \exp(N)$ if $p = 2$ and 4 does not divide q, which completes the proof. □

Since any p-group of maximal class has a potent maximal subgroup or a maximal subgroup of class at most $p - 1$, we can use Corollary 5.2.9 to obtain a simple limitation for the exponent of $\nu^q(G)$ in terms of the exponent of the group G.

Corollary 5.2.11. *Let G be a p-group of maximal class.*

(i) $\exp(\nu^q(G))$ *divides* $p^2 \cdot \exp(G)$ *if* $p \geq 3$.

(ii) $\exp(\nu^q(G))$ *divides* $4 \cdot 2 \cdot \exp(G)$ *if* $p = 2$ *and either* q *is odd or* 4 *divides* q.

(iii) $\exp(\nu^q(G))$ *divides* $2 \cdot 4^2 \cdot \exp(G)$ *if* $p = 2$ *and* 4 *does not divide* q.

Proof. Remember that if G is a p-group of maximal class, then G has a potent maximal subgroup or a maximal subgroup of class at most $p-1$. This means that we can consider N be a maximal subgroup of G such that N is potent or $\gamma_p(N) = 1$. Therefore we have the same hypothesis under N of the last theorem. In this way we obtain that $\exp(\nu^q(G))$ divides $p \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ if $p \geq 3$, or divides $4 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ if $p = 2$ and either q is odd or 4 divides q, or divides $8 \cdot \exp(\nu^q(G/N)) \cdot \exp(N)$ if $p = 2$ and 4 does not divide q.

Since N is a maximal subgroup contained in a p-group of maximal class, it follows that $G/N \equiv C_p$, for all prime p. So, the proof is completed by showing a limitation of the exponent of the $\nu^q(C_p)$. In Example 5.0.1 we saw that $\exp(\nu^q(C_p)) = p$, if $p \geq 3$, or if $p = 2$ and either q is odd or 4 divides q, and $\exp(\nu^q(C_2)) = 4$, if $p = 2$ and 4 does not divide q, which conclude our argument. □

In the next theorem, which gives us a relationship between the exponent of one term of the lower central series of $\nu^q(G)$ and the exponent of one term of the lower central series of the group G.

Theorem 5.2.12. *Let G be a p-group, with p a prime. Let m and s be positive integers such that $m \geq s$ and suppose that $\gamma_{i+s}(G) = \gamma_i(G)^p$ for every $i \geq m$. Consider $q \geq 0$. Then*

(i) $\exp(\gamma_{m+1}(\nu^q(G))) = \exp(\gamma_m(G))$ *if p is odd.*

(ii) $\exp(\gamma_{m+1}(\nu^q(G))) = \exp(\gamma_m(G))$ *if $p = 2$ and $\gamma_m(G)$ is powerful.*

Proof. The proofs of both items are similar and we will prove them at the same time. The only difference is the hypothesis that $\gamma_m(G)$ is powerful in the second item, but we can observe that if p is odd, then $\gamma_m(G)$ is a powerful p-group, too. Indeed, by hypothesis we have

$$[\gamma_m(G), \gamma_m(G)] \leq \gamma_{2m}(G) \leq \gamma_{m+s}(G) = \gamma_m(G)^p.$$

The fact that the subgroup $\gamma_m(G)$ is powerful implies that we can apply Theorem 1.6.16, that is, it is valid that $\Pi_j(\gamma_m(G))) = \gamma_m(G)^{p^j}$ for all $j \geq 1$ and for every p. Let p^t be the exponent of $\gamma_m(G)$. Thus

$$\gamma_{m+ts}(G) = \gamma_m(G)^{p^t} = 1.$$

Therefore, from Proposition 2.2.2 we obtain that $\gamma_{m+ts+1}(\nu^q(G)) = 1$. By Proposition 4.0.5 we have that $\gamma_{i+1}(\nu^q(G))^p \leq \gamma_{i+s+1}(\nu^q(G))$ for $i \geq m$ and for all prime. Therefore,

$$\gamma_{m+1}(\nu^q(G))^{p^t} \leq \Pi_t(\gamma_{m+1}(\nu^q(G))) \leq \gamma_{m+ts+1}(\nu^q(G)) = 1.$$

This means that $\exp(\gamma_{m+1}(\nu^q(G)))$ divides $\exp(\gamma_m(G))$.

For the other inequality, suppose by absurd that $\exp(\gamma_m(G)) = p^t$ is greater than $\exp(\gamma_{m+1}(\nu^q(G))) = p^k$, this implies that $k \leq t-1$.

Since $\gamma_m(G)$ is powerful and their exponent is p^t, it follows that $\gamma_{m+(t-1)s}(G) = \gamma_m(G)^{p^{t-1}} \neq 1$. This implies that $\gamma_{m+(t-1)s+1}(\nu^q(G)) \neq 1$. As $\gamma_{m+1}(\nu^q(G))$ is powerful and $m + ks + 1 \leq m + (t-1)s + 1$, we have that

$$\gamma_{m+(t-1)s+1}(\nu^q(G)) \leq \gamma_{m+ks+1}(\nu^q(G)) = \gamma_{m+1}(\nu^q(G))^{p^k} = 1.$$

But, this is an absurd with the above fact that $\gamma_{m+(t-1)s+1}(\nu^q(G)) \neq 1$. Therefore $\exp(\gamma_{m+1}(G)) \leq \exp(\gamma_m(\nu^q(G)))$ and the proof is complete. □

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