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Processos de Poisson e o Custo Mínimo Esperado de Transporte com Sensores

por

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De tudo ficam três coisas: A certeza de que estamos começando, a certeza de que é preciso continuar e a certeza de que podemos ser interrompidos antes de terminar. Fazer da interrupção um camino novo, fazer da queda um passo de dar, do medo uma escola, do sonho uma ponte, da procura um encontro, e, assim, terá valido a pena existir!

(Fernando Sabino)

Para minha amada família

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Resumo

Neste trabalho, primeiro obtivemos uma fórmula fechada para a distância esperada $E\big[|X_{k+r} - Y_k|\big]$ entre eventos de dois processos de Poisson independentes com tempos de chegada $X_1, X_2, \ldots$ e $Y_1, Y_2, \ldots$ e, respectivas, taxas de chegada λ_1 e λ_2. Em seguida, foi encontrado um intervalo para a soma $C_{opt} = \sum_{i=1}^{n} E\big[|X_k - Y_k|\big]$. Para o caso particular em que as taxas de chegada dos dois processos λ_1 e λ_2 são iguais a $\lambda > 0$, a fórmula analítica fechada para o custo mínimo esperado de transporte

$$C_{opt}(\lambda, n) = \frac{2n}{3\lambda}\binom{n+\frac{1}{2}}{n},$$

foi determinada por Kranakis (2014).

Como segundo resultado, com o uso da função H de Fox, encontramos o a-ésimo momento absoluto da diferença entre eventos de dois processos de Poisson independentes com tempos de chegadas $X_1, X_2, \ldots$ e $Y_1, Y_2, \ldots$ e, respectivas taxas λ_1 e λ_2,

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!(-1)^a}{\lambda_2^a}\sum_{j=0}^{a}\frac{(k+r)^{(j)}}{j!}\frac{k^{(a-j)}}{(a-j)!}\left(\frac{-\lambda_2}{\lambda_1}\right)^j - 2I_2 \times 1_{[\text{mod}2]}(a),$$

em que

$$I_2 = \frac{(-1)^a(\lambda_1/\lambda_2)^{k+r}\,\Gamma(a+1)\Gamma(a+r+2k)}{\lambda_2^a\,\Gamma(k)\Gamma(1+k+r+a)} \times {}_2F_1\big(a+2k+r; k+r; 1+k+r+a; -\frac{\lambda_1}{\lambda_2}\big),$$

$1_{[\mathrm{mod}2]}(a) = \begin{cases} 1, & a \text{ ímpar} \\ 0, & a \text{ par} \end{cases}$ e $\,_2F_1$ é a função hipergeométrica de Gauss.

Uma potencial aplicação de $C_{opt}(\lambda_1, \lambda_2, n)$ é para o cálculo do custo mínimo de transporte do movimento de sensores alocados conforme os processos $\{X_i, Y_j\}$.

Abstract

In this work, we first obtained a closed formula for the expected distance $E\big[|X_{k+r}-Y_k|\big]$ between events of two independent Poisson processes with arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ and respective arrival rates λ_1 and λ_2. Then, a interval was found for the sum $C_{opt} = \sum\limits_{i=1}^{n} E\big[|X_k - Y_k|\big]$. For the particular case, in which the arrival rates of the two processes λ_1 and λ_2 are equal to $\lambda > 0$, the closed analytical formula for the expected minimum cost of transportation

$$C_{opt}(\lambda, n) = \frac{2n}{3\lambda}\binom{n+\frac{1}{2}}{n},$$

was determined by Kranakis (2014).

As a second result, using Fox's H function, we find the absolute a -th absolute moment of difference between events of two independent Poisson processes with arrival times $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ and, respective rates λ_1 and λ_2,

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!(-1)^a}{\lambda_2^a}\sum_{j=0}^{a}\frac{(k+r)^{(j)}}{j!}\frac{k^{(a-j)}}{(a-j)!}\left(\frac{-\lambda_2}{\lambda_1}\right)^j - 2I_2 \times 1_{[\mathrm{mod}2]}(a),$$

where

$$I_2 = \frac{(-1)^a(\lambda_1/\lambda_2)^{k+r}\,\Gamma(a+1)\Gamma(a+r+2k)}{\lambda_2^a\,\Gamma(k)\Gamma(1+k+r+a)} \times {}_2F_1\big(a+2k+r; k+r; 1+k+r+a; -\frac{\lambda_1}{\lambda_2}\big),$$

$$1_{[\mathrm{mod}2]}(a) = \begin{cases} 1, & a \text{ odd} \\ 0, & a \text{ even} \end{cases}$$ and ${}_2F_1$ is the hypergeometric function.

A potential application of $C_{opt}(\lambda_1, \lambda_2, n)$ is for calculating the minimum cost of transporting the movement of sensors allocated according to the processes $\{X_i, Y_j\}$.

Contents

Chapter 1

Introdução

Sensores móveis são utilizados no monitoramento e comunicação de dados para diversos fins, como pesquisa oceanográfica (Pérez et al., 2011), análise de ar tropical (Tudose et al., 2011), robótica (Teng et al., 2007), monitoramento e segurança (Ma et al., 2020), entre outros.

Um dos principais tópicos de pesquisa nesta área é a determinação de uma alocação ótima dos sensores de forma a gerar uma boa cobertura a um custo mínimo.

Por meio da tecnologia de sensor móvel, uma boa cobertura pode ser obtida colocando-se os sensores nas posições desejadas. No entanto, os sensores móveis são geralmente equipados com uma bateria e o gasto de energia é muito maior durante o movimento do sensor do que durante sua função de detecção. Portanto, é importante minimizar os movimentos do sensor para aumentar sua vida útil e manter a confiabilidade da rede a que pertence.

Existem duas abordagens para estudar o custo mínimo esperado de transporte: a soma ou o máximo dos movimentos dos sensores desde suas posições iniciais até o destino. Com relação à soma, Ajtai, Komlós, and Tusnády (1984) considerou 2n sensores, n azul $X_1, X_2, \cdots X_n$ e n vermelho $Y_1, Y_2, \cdots Y_n$, distribuídos de forma independente e uniforme em um quadrado unitário e provou que o custo mínimo esperado de transporte, denotado por T_n e definido por $T_n := \min_{\pi} \sum_{i=1}^{n} d(X_{\pi(i)}, Y_i)$ pertence ao intervalo assintótico $\Theta\big(\sqrt{n \ \log \ n}\big)$. Kranakis (2014), ao assumir que os sensores se movem aleatoriamente em uma reta de acordo com dois processos de Poisson independentes e identicamente distribuídos com taxas de chegada λ e respectivos tempos de chegada $X_1, X_2, \cdots X_n$ e $Y_1, Y_2, \cdots Y_n$, determinou um intervalo assintótico para o custo mínimo esperado de transporte $C_T := \sum_{k=1}^{n} E\big[|X_k - Y_k|\big]$. Kapelko (2015) generalizou o resultado do Kranakis (2014). Ele considerou as mesmas hipóteses de Kranakis (2014) e determinou uma expressão assintótica para o custo mínimo esperado, potência $a > 0$, $C_T^a = \sum_{k=1}^{n} M_{pop}^a$, com $M_{pop}^a = E\big[|X_k - Y_k|^a\big]$.

Recentemente, Kapelko (2017), ao considerar dois processos aleatórios gerais idênticos e independentes, determinou expressões assintóticas para o custo mínimo de transporte esperado na potência $b > 0$, C_T^b.

Kapelko (2018) generalizou o resultado de Kranakis (2014), obtendo uma fórmula analítica fechada para o a-ésimo momento da distância absoluta dos tempos de chegada, M_{pop}^a, de dois processos de Poisson i.i.d com taxas λ.

Kapelko (2020) investigou sobre a energia para deslocamento de sensores aleatórios para conectividade e interferência. Para isso, ele determinou M_{pop} entre eventos de processos d-dimensionais, independentes e idênticos com taxas de chegada $\lambda > 0$.

Um problema de custo de transporte mais geral do que os abordados nos artigos citados acima ocorre quando assumimos que os sensores se movem de acordo com dois processos estocásticos, não necessariamente com a mesma distribuição. Nesse trabalho, estudamos este problema mais geral.

Ao considerar, em uma rede de sensores, pares $\{X_i, Y_j\}$, onde $X_1, X_2, \cdots$ são azuis e, $Y_1, Y_2, \cdots$ são vermelhos, incialmente colocados de acordo com dois processos de Poisson com taxas $\lambda_1 > 0$ e $\lambda_2 > 0$, respectivamente, estudamos o custo mínimo de transporte esperado, por duas abordagens.

A primeira abordagem, apresentada no Capítulo 2, consiste em determinar um intervalo para C_T. Dessa forma, nossos resultados generalizam os de Kranakis (2014).

O principal resultado com a segunda abordagem, apresentada no Capítulo 3, é uma fórmula fechada para o a-ésimo momento M^a_{pop}.

No Capítulo 2, iniciamos introduzindo as definições integrais das funções gama, gama incompleta, beta e beta incompleta e identidades envolvendo-as. Em seguida, propomos no Teorema 2.2.1 a seguinte fórmula fechada, em termos da função beta incompleta, para a distância esperada entre eventos de dois processos de Poisson independentes com taxas λ_1 e λ_2 quaisquer:

$$E\big[|X_{k+r} - Y_k|\big] = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + 2k(k+r)\binom{2k+r}{k}\left[\frac{B_p(k+r,k+1)}{\lambda_2} - \frac{B_p(k+r+1,k)}{\lambda_1}\right],$$

válida para inteiros $r \geq 0$ e $k \geq 1$, em que $B_p(r,t)$ representa a função beta incompleta com parâmetros $r, t \in \mathbb{N}$ e $p \in (0,1)$.

Para a demonstração do Teorema 2.2.1, foram necessárias as propriedades de esperança condicional, as funções gama e beta definidas previamente e suas relações com binômios de Newton. Por fim, aplicamos a seguinte identidade combinatória que relaciona a soma polinomial ponderada por coeficientes binomiais com a função beta incompleta (ver DiDonato and Jarnagin, 1966):

$$\sum_{s=0}^{L}\binom{n+s}{s}p^s = \frac{1-(L+1)\binom{L+n+1}{n}B_p(L+1,n+1)}{(1-p)^{n+1}}.$$

No Corolário 2.2.2, sob as hipóteses de igualdade das taxas e de correspondência entre os tempos de chegadas dos dois processos de Poisson independentes, mostramos a validade da seguinte fórmula:

$$E\big[|X_k - Y_k|\big] = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}.$$

No Corolário 2.2.3, ao assumir que os processos de Poisson independentes sejam idênticos, obtivemos a mesma fórmula analítica fechada em termos de polinômio de Pochhammer para distância esperada entre os tempos de chegadas X_{k+r} e Y_k apresentada como resultado principal por Kranakis (2014) :

$$E\left[|X_{k+r}-Y_k|\right]=\frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}\left(1+\sum_{s=0}^{r-1}\frac{r-s}{(2k+s)\,2^s}\frac{(2k+1)^{(s)}}{(k+1)^{(s)}}\right),$$

A prova deste Corolário foi baseada essencialmente na identidade combinatória que envolve a função beta incompleta regularizada e coeficiente binomial com a aplicação do Princípio da Indução Matemática.

Sob as mesmas hipóteses do Teorema 2.2.1, obtivemos no Teorema 2.3.1 um intervalo para o Custo Mínimo Esperado de Transporte. Demonstramos os limites inferior e superior deste intervalo através de relações de recorrência das funções beta incompletas regularizadas.

No final do Capítulo 2, discutimos a convergência do Custo Amostral. Através de técnicas de inferência estatística, determinamos um intervalo de confiança para o Custo Mínimo de Transporte, após encontrar uma quantidade pivotal baseada no Custo Amostral.

No capítulo 3, tratamos sobre o a- ésimo momento absoluto da diferença entre eventos de dois processos de Poisson. Baseamos nas funções especiais gama, gama incompleta, H-Fox e hipergeométrica de Gauss.

A função H de Fox é bem útil para resolver problemas advindos do cálculo fracionário e solucionar integrais que possuem como integrando as funções gamas. Segundo Mathai, Saxena, and Haubold (2010), a função H de Fox tem grande aplicabilidade em problemas da física, matemática, engenharia e estatística. A sua importância reside no fato de quase todas as funções especiais que ocorrem em matemática e estatística são casos particulares dessa função. A função hipergeométrica de Gauss, por exemplo, é um caso particular da função H de Fox.

Para encontrar o a-ésimo momento, M^a_{pop}, utilizamos as funções especiais H de Fox, hipergeométrica de Gauss, gama, gama incompleta superior e inferior. Essas funções possibilitaram a obtenção de uma fórmula fechada para o a-ésimo momento absoluto da diferença entre os tempos de chegada de dois processos de Poisson independentes.

Apresentamos o Teorema 3.4.1 que generaliza o resultado principal de Kapelko (2018).

Ao considerar dois processos de Poisson independentes com tempos de chegada $X_1, X_2, \ldots$ e $Y_1, Y_2, \ldots$ e, respectivas taxas λ_1 e λ_2, mostramos a validade da seguinte fórmula para o a-ésimo momento absoluto:

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{(k+r)^{(j)}}{j!} \frac{k^{(a-j)}}{(a-j)!} \left(\frac{-\lambda_2}{\lambda_1}\right)^j - 2I_2 \times 1_{[\mathrm{mod}2]}(a),$$

em que

$$I_2 = \frac{(-1)^a (\lambda_1/\lambda_2)^{k+r}\ \Gamma(a+1)\Gamma(a+r+2k)}{\lambda_2^a\ \Gamma(k)\Gamma(1+k+r+a)} \times {}_2F_1\big(a+2k+r; k+r; 1+k+r+a; -\frac{\lambda_1}{\lambda_2}\big),$$

$$1_{[\mathrm{mod}2]}(a) = \begin{cases} 1, & a \ \text{ímpar} \\ 0, & a \ \text{par} \end{cases} \quad \text{e} \quad {}_2F_1 \text{ é a função hipergeométrica.}$$

Para provar o Teorema 3.4.1, demonstramos, primeiramente, os Lemas 3.4.2 e 3.4.3.

No Lema 3.4.2, mostramos a seguinte identidade, válida para todo a inteiro:

$$\int_0^\infty \int_0^\infty (t-y)^a\ f_2(y)\ f_1(t)\ dt\ dy = \frac{a!(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{i^{(j)}}{j!} \frac{k^{(a-j)!}}{(a-j)!} \left(-\frac{\lambda_2}{\lambda_1}\right)^j,$$

onde $f_1(\cdot)$ e $f_2(\cdot)$ são, respectivamente, as densidades das distribuições $Gama(i, \lambda_1)$ e $Gama(k, \lambda_2)$.

No Lema 3.16 mostramos a validade da identidade:

$$I_2 = \frac{\lambda_1^i \lambda_2^k \Gamma(a+1)(-1)^a}{\Gamma(i)\Gamma(k)} \frac{\Gamma(a+i+k)\Gamma(i)}{\Gamma(1+i+a)} \times_2 F_1(a+i+k, i; 1+i+a; -\frac{\lambda_1}{\lambda_2}),$$

onde $I_2 = \int_0^\infty \int_0^y (t-y)^a f_1(t) f_2(y) dt dy$.

Para sua demonstração, aplicamos as transformadas de Euler e de Laplace da função H de Fox, reduzindo a uma função hipergeométrica.

Para ordem a ímpar, apresentamos o Corolário 3.4.4. Sob as hipóteses do teorema e igualdade das taxas de chegadas, obtivemos o resultado apresentado no teorema 3 de Kapelko (2018):

$$E\big[|X_k - Y_k|^a\big] = \frac{a!}{\lambda^a}\,\frac{\Gamma\big(\frac{a}{2}+k\big)}{\Gamma(k)\,\Gamma\big(\frac{a}{2}+1\big)}.$$

Provamos o resultado reescrevendo os polinômios de Pochhammer em função de números binomiais. Aplicamos em seguida, a identidade combinatória e finalizamos colocando a expressão em termos da função gama.

Já para ordem a par, mostramos no Corolário 3.4.5 que nosso resultado coincide com o teorema 9 de Kapelko (2018). Provamos por meio da identidade (Teorema de kummer):

$${}_2F_1(a+2k, k\ ;\ 1+k+a\ ;\ -1) = \frac{\Gamma(1+a+2k-k)\,\Gamma\big(1+\frac{1}{2}(a+2k)\big)}{\Gamma(1+a+2k)\,\Gamma\big(1+\frac{1}{2}(a+2k)-k\big)}.$$

Por fim, tanto no Capítulo 2 como no Capítulo 3, inserimos também tabelas e ilustrações gráficas de nossos resultados geradas a partir de simulações dos processos através do software computacional R Core Team (2020). No apêndice A, encontra-se o código da programação.

Chapter 2

Expected Distance and Interval for Transport Cost

2.1 Introduction

Mobile sensors are used in data monitoring and communication for various purposes, such as oceanographic research (Pérez et al., 2011), tropical air analysis (Tudose et al., 2011), robotics (Teng et al., 2007), and security monitoring (Ma et al., 2020), among others.

One of the main research topics in this area is the determination of an optimal allocation of the sensors in order to generate good coverage at a minimum cost.

Through mobile sensor technology, good coverage can be achieved by placing the sensor in the desired positions. However, mobile sensors are generally equipped with a battery and the energy expenditure is much greater during the movement of the sensor than during its detection function. Therefore, it is important to minimize the movements of the sensor to increase its useful life and maintain the reliability of the network to which it belongs.

There are two approaches to studying the minimum expected cost of transport: the sum or maximum of the movements of the sensors from their initial positions to the destination. With respect to the sum, Ajtai, Komlós, and Tusnády (1984) considered 2n sensors, n blue

$X_1, X_2, \cdots X_n$ and n red $Y_1, Y_2, \cdots Y_n$, distributed independently and uniformly in a unit square, and proved that the expected minimum cost of transportation, denoted by T_n and defined by $T_n := \min_{\pi} \sum_{i=1}^{n} d(X_{\pi(i)}, Y_i)$, belongs to the asymptotic interval $\Theta\big(\sqrt{n \ \log \ n}\big)$. Kranakis (2014), when assuming that the sensors move randomly on a line according to two independent and identically distributed Poisson processes with arrival rate λ and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2 \cdots$ determined an interval for the expected minimum cost of transport, defined by $C_T = \sum_{k=1}^{n} E\big[|X_k - Y_k|\big]$. Kapelko (2015) generalized the result of Kranakis (2014). He considered the same hypotheses as Kranakis (2014) and determined an asymptotic expression for the expected minimum cost at power $a > 0$, $C_T^a = \sum_{k=1}^{n} E\big[|X_k - Y_k|^a\big]$. Recently, Kapelko (2017), when considering two identical and independent general random processes, determined asymptotic expressions for the expected minimum transport cost at power $b > 0$, C_T^b.

A more general transportation cost problem than that addressed in the articles cited above occurs when it is assumed that the sensors move according to two independent stochastic processes, not necessarily with the same distribution. In this paper, we study this more general problem. Our results generalize Kranakis (2014).

We obtain an exact interval for the transport cost, $C_{opt} = C_T$, by considering a network of two sensors $\{X_i, Y_j\}$, where $X_1, X_2, \cdots$ are blue and $Y_1, Y_2, \cdots$ are red, that initially randomly allocated according to a Poisson processes with arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Note that λ_1 can be different from λ_2, so the sensors $\{X_i\}$ and $\{Y_i\}$ follow a different law. In addition to obtaining an interval for the expected transport cost, here we carry out a study of statistical inference and verify that the sample transport cost is a consistent estimator of the theoretical transport cost found.

Kranakis (2014), Kapelko (2015) and Kapelko (2017) based their results on combinatorial theory, but for the proof of our results we also use results of the following special functions: gamma function, upper and lower incomplete gamma functions, beta function, and incomplete beta function. These functions are defined, respectively, by:

$$\Gamma(a) := \int\limits_0^\infty t^{a-1}e^{-t}dt \ , \tag{2.1}$$

$$\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}dt, \tag{2.2}$$

$$\gamma(a,x) := \int_0^x t^{a-1}e^{-t}dt \ , \tag{2.3}$$

$$B(a,b) := \int\limits_0^1 t^{a-1}(1-t)^{b-1}dt \tag{2.4}$$

and

$$B_x(a,b) := \int\limits_0^x t^{a-1}(1-t)^{b-1}dt. \tag{2.5}$$

The following identities (see Gradshteyn and Ryzhik, 2014) are also used:

$$\Gamma(a) = \gamma(a,x) + \Gamma(a,x), \tag{2.6}$$

$$\Gamma(n+1,x) = n! \ e^{-x}\sum_{r=0}^{n}\frac{x^r}{r!}, \tag{2.7}$$

$$\gamma(n+1,x) = n!\left(1 - e^{-x}\sum_{r=0}^{n}\frac{x^r}{r!}\right), \tag{2.8}$$

and

$$\frac{\Gamma(x+h)}{\Gamma(x)} = x^{(h)} : \quad \text{Pochhammer polynomial}$$
$$= x(x+1)(x+2)\cdots(x+h-1), \text{if } h \geq 1. \tag{2.9}$$

The rest of the chapter is organized as follows: Section 2.2 describes our main results; Section 2.4, presents the statistical inference results about transport cost and illustrations of the results generated from Monte Carlo simulation experiments; and Section 2.5 concludes.

2.2 Main Results

2.2.1 Expected Distance

In this subsection we present Theorem 2.2.1 in which we determine a closed analytic expression for $E\big[|X_k - Y_k|\big]$. Let X_i and Y_k be random variables that represent the i-th and k-th arrival times of two independent Poisson processes with rates λ_1 and λ_2. Then, X_i and Y_k have gamma distribution. With the notation

$$X_i \sim Gama(i, \lambda_1) \quad \text{e} \quad Y_k \sim Gama(k, \lambda_2),$$

the random variables X_i and Y_k have probability density functions (pdf‘s)

$$f_{X_i}(x) := f_1(x) = \frac{\lambda_1^i}{\Gamma(i)} x^{i-1} e^{-\lambda_1 x}, \qquad x > 0 \tag{2.10}$$

and

$$f_{Y_k}(y) := f_2(y) = \frac{\lambda_2^k}{\Gamma(k)} y^{k-1} e^{-\lambda_2 y}, \qquad y > 0, \tag{2.11}$$

respectively. The shape parameters are i and k and the scale parameters are $\lambda_1 > 0$ e $\lambda_2 > 0$. The particular cases of our results are in the Corollaries 2.2.2 and 2.2.3. These results correspond to the main results of Kranakis, 2014.

Theorem 2.2.1. *Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively; $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$. Then*

$$E\big[|X_{k+r} - Y_k|\big] = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + 2k(k+r)\binom{2k+r}{k}\left[\frac{B_p(k+r,k+1)}{\lambda_2} - \frac{B_p(k+r+1,k)}{\lambda_1}\right], \tag{2.12}$$

for non-negative integers $r \geq 0$ and $k \geq 1$.

Proof. By using the conditional expectation property, we have:

$$\begin{aligned} E\big[|X_i - Y_k|\big] &= E\big[E\big(|X_i - Y_k| \bigm| Y_k\big)\big] \\ &= \int_0^\infty E\big[|X_i - y|\big]\ f_2(y)\ dy. \end{aligned} \tag{2.13}$$

To find the the expected value of (2.13), consider:

$$E\big[|X_i - y|\big] = I_1 + I_2, \quad \text{with} \tag{2.14}$$

$I_1 = \int_0^y -(x-y) f_1(x)\ dx$ and $I_2 = \int_y^\infty (x-y) f_1(x)\ dx.$

By combining (2.10), I_1 and I_2, we deduce that:

$$I_1 = \frac{\lambda_1^i}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_1 x}\ dx - \frac{i}{\lambda_1} + \frac{y\lambda_1^i}{\Gamma(i)} \int_0^y x^{i-1} e^{-\lambda_1 x}\ dx. \tag{2.15}$$

and

$$I_2 = \frac{\lambda_1^i}{\Gamma(i)} \int_y^\infty x^i e^{-\lambda_1 x}\, dx - \frac{y\lambda_1^i}{\Gamma(i)} \int_y^\infty x^{i-1} e^{-\lambda_1 x}\, dx. \tag{2.16}$$

Now (2.15) and (2.16) are replaced in equation (2.14). The expected value result in terms of the incomplete gamma functions is:

$$E\big[|X_i - y|\big] = \frac{2}{\lambda_1 \Gamma(i)} \Gamma(i+1, \lambda_1 y) - \frac{i}{\lambda_1} + \frac{y}{\Gamma(i)} \Big(\gamma(i, \lambda_1 y) - \Gamma(i, \lambda_1 y)\Big). \tag{2.17}$$

And by substituting (2.17) in (2.13) we obtain an expression composed of the following three new integrals:

$$E\big[|X_i - Y_k|\big] = -\frac{i}{\lambda_1} + \frac{2}{\lambda_1 \Gamma(i)} J_1 + \frac{1}{\Gamma(i)} J_2 - \frac{1}{\Gamma(i)} J_3, \tag{2.18}$$

where

$$J_1 := \int_0^\infty \Gamma(i+1, \lambda_1 y) f_2(y)\, dy,$$

$$J_2 := \int_0^\infty y\, \gamma(i, \lambda_1 y) f_2(y)\, dy$$

and

$$J_3 := \int_0^\infty y\, \Gamma(i, \lambda_1 y) f_2(y)\, dy.$$

These integrals are calculated using the series representation of the incomplete gamma functions and the density (2.11). After algebraic manipulations, we deduce that:

$$J_1 = \frac{\Gamma(i+1)}{\Gamma(k)} q^k \sum_{s=0}^{i} \left[p^s \frac{\Gamma(s+k)}{s!} \right], \tag{2.19}$$

$$J_2 = \frac{\Gamma(i)k}{\lambda_2} - \frac{\Gamma(i)}{\lambda_2\Gamma(k)} q^{k+1} \sum_{s=0}^{i-1} \left[p^s \frac{\Gamma(s+k+1)}{s!} \right] \tag{2.20}$$

and

$$J_3 = \frac{\Gamma(i)}{\lambda_2\Gamma(k)} q^{k+1} \sum_{s=0}^{i-1} \left[p^s \frac{\Gamma(s+k+1)}{s!} \right]. \tag{2.21}$$

with $p = \lambda_1/(\lambda_1 + \lambda_2)$ and $q = 1 - p$.

By combining integrals (2.19), (2.20) and (2.21) in (2.18), we get:

$$\begin{aligned} E\big[|X_i - Y_k|\big] &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\ q^k}{\lambda_1\Gamma(k)} \sum_{s=0}^{i} p^s \frac{\Gamma(s+k)}{s!} - \frac{2q^{k+1}}{\lambda_2\ \Gamma(k)} \sum_{s=0}^{i-1} p^s \frac{\Gamma(s+k+1)}{s!} \\ &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\ q^k}{\lambda_1} \sum_{s=0}^{i} \binom{s+k-1}{s} p^s - \frac{2k\ q^{k+1}}{\lambda_2} \sum_{s=0}^{i-1} \binom{s+k}{s} p^s, \end{aligned} \tag{2.22}$$

where $p = \lambda_1/(\lambda_1 + \lambda_2)$ and $q = 1 - p$.

Finally, we update equation (2.22), to obtain:

$$\begin{aligned} E\big[|X_i - Y_k|\big] &= \frac{k}{\lambda_2} - \frac{i}{\lambda_1} + \frac{2i\ q^k}{\lambda_1} \left[\frac{1 - (i+1)\binom{i+k}{k-1} B_p(i+1,k)}{q^k} \right] \\ &\quad - \frac{2k\ q^{k+1}}{\lambda_2} \left[\frac{1 - i\binom{i+k}{k} B_p(i,k+1)}{q^{k+1}} \right] \\ &= \frac{i}{\lambda_1} - \frac{k}{\lambda_2} + 2ik\binom{i+k}{k} \left(\frac{B_p(i,k+1)}{\lambda_2} - \frac{B_p(i+1,k)}{\lambda_1} \right), \end{aligned} \tag{2.23}$$

by using the identity (see DiDonato and Jarnagin, 1966)

$$\sum_{s=0}^{L} \binom{n+s}{s} p^s = \frac{1-(L+1)\binom{L+n+1}{n} B_p(L+1, n+1)}{(1-p)^{n+1}}. \tag{2.24}$$

Replacing i by $k+r$ in (2.23) finishes the proof. $\square$

An expression equivalent to (2.12), in terms of the regularized incomplete beta function, is provided below. To obtain this result, just replace the expressions

$$B(k+r, k+1) = \frac{1}{(k+r)\binom{2k+r}{k}} \tag{2.25}$$

and

$$B(k+r+1, k) = \frac{1}{k\binom{2k+r}{k}} \tag{2.26}$$

in the regularized incomplete beta function

$$I_p(a,b) := \frac{B_p(a,b)}{B(a,b)}.$$

Then

$$E\big[|X_{k+r} - Y_k|\big] = \frac{k+r}{\lambda_1} - \frac{k}{\lambda_2} + \frac{2k\ I_p(k+r, k+1)}{\lambda_2} - \frac{2(k+r)\ I_p(k+r+1, k)}{\lambda_1} \tag{2.27}$$

is equivalent to (2.12).

Corollary 2.2.2. *Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. If $r = 0$, $k \in \mathbb{Z}_{\geq 1}$ and*

$\lambda_1 = \lambda_2 = \lambda > 0$, *then*

$$E\big[|X_k - Y_k|\big] = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}. \tag{2.28}$$

Proof. From Theorem 2.2.1, for $r = 0$ and $p = \lambda_1/(\lambda_1 + \lambda_2) = 1/2$, we have

$$E\big[|X_{k+r} - Y_k|\big] = \frac{2k^2}{\lambda}\binom{2k}{k}\big[B_{\frac{1}{2}}(k, k+1) - B_{\frac{1}{2}}(k+1, k)\big]. \tag{2.29}$$

The identity

$$B_x(a; n+1-a) = B(a; n+1-a)\sum_{j=a}^{n}\binom{n}{j}x^j(1-x)^{n-j}, \tag{2.30}$$

(see DiDonato and Jarnagin, 1966), allows rewriting the difference in equation (2.29) as

$$\begin{aligned} B_{\frac{1}{2}}(k, k+1) - B_{\frac{1}{2}}(k+1, k) &= \frac{B(k, k+1)}{2^{2k}}\binom{2k}{k} \\ &= \frac{1}{k2^{2k}}, \end{aligned} \tag{2.31}$$

by using the identity

$$B(k, k+1) = \frac{1}{2k}\binom{2k-1}{k-1}^{-1} = \frac{1}{k}\binom{2k}{k}^{-1}. \tag{2.32}$$

The result (2.28) is obtained replacing (2.31) by (2.29). $\square$

Corollary 2.2.3. *Consider two independent Poisson processes with arrival times* $X_1, X_2, \cdots$ *and* $Y_1, Y_2, \cdots$ *and arrival rates* $\lambda_1 > 0$ *and* $\lambda_2 > 0$*, respectively. If* $r > 0$, $k \in \mathbb{Z}_{\geq 1}$ *and* $\lambda_1 = \lambda_2 = \lambda > 0$*, then*

$$E\big[|X_{k+r} - Y_k|\big] = \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k}\left(1 + \sum_{s=0}^{r-1}\frac{r-s}{(2k+s)\,2^s}\frac{(2k+1)^{(s)}}{(k+1)^{(s)}}\right), \tag{2.33}$$

Proof. For $\lambda_1 = \lambda_2 = \lambda$ and $r > 0$, from Theorem 2.2.1, we have:

$$E\big[|X_{k+r} - Y_k|\big] = \frac{r}{\lambda} + \frac{2k(k+r)}{\lambda}\binom{2k+r}{k}\big[B_{\frac{1}{2}}(k+r, k+1) - B_{\frac{1}{2}}(k+r+1, k)\big]. \quad (2.34)$$

Equation (2.34) is updated by rewriting $B_{\frac{1}{2}}(k+r, k+1)$ and $B_{\frac{1}{2}}(k+r+1, k)$ with identity (2.30) as

$$\begin{aligned} E\big[|X_{k+r} - Y_k|\big] &= \frac{-r}{\lambda} + \frac{2(k+r)}{\lambda 2^k}\sum_{s=0}^{k+r}\binom{s+k-1}{s}\frac{1}{2^s} - \frac{2k}{\lambda 2^{k+1}}\sum_{s=0}^{k+r-1}\binom{s+k}{s}\frac{1}{2^s} \\ &= -\frac{r}{\lambda} + \frac{2(k+r)}{2^k\lambda\Gamma(k)} \times \sum_{j=0}^{1} H_j, \end{aligned} \quad (2.35)$$

where

$$\begin{aligned} H_0 &= \sum_{s=0}^{k+r}\frac{(k-1)!}{2^s}\binom{s+k-1}{s} \\ &\overset{(s-k:=t)}{=} \Gamma(k)\left[2^{k-1} + 2^{-k}\sum_{t=0}^{r}\binom{t+2k-1}{k-1}2^{-t}\right] \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} H_1 &= \sum_{s=0}^{k+r-1}\Big(-\frac{1}{k+r}\Big)\frac{k!}{2^{s+1}}\binom{s+k}{s} \\ &= -\frac{k!}{2(k+r)}\left[2^k - \binom{2k}{k}2^{-k} + 2^{-k}\sum_{t=0}^{r-1}\frac{2k+t}{k}\binom{2k+t-1}{k-1}2^{-t}\right]. \end{aligned} \quad (2.37)$$

By replacing H_0 and H_1 in (2.35), we get:

$$\begin{aligned} E\big[|X_{k+r} - Y_k|\big] &= \frac{k2^{-2k}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t} \\ &+ \frac{2^{-2k}}{\lambda}\left[2^{-r}(2k+r)C_{2k+r-1}^{k-1} + \sum_{t=0}^{r} t\binom{2k+t-1}{k-1}2^{-t}\right]. \end{aligned} \quad (2.38)$$

Now, the identity

$$\sum_{t=0}^{r} t\binom{2k+t-1}{k-1}2^{-t} = 2k\binom{2k-1}{k-1} - 2^{-r}(2k+r)\binom{2k+r-1}{k-1},$$

valid for $k \geq 1$, is applied in (2.38), so:

$$\begin{aligned} E\big[|X_{k+r} - Y_k|\big] &= \frac{k2^{-2k}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t} + \frac{k2^{-2k+1}}{\lambda}\binom{2k-1}{k-1} \\ &= \frac{k2^{-2k+1}}{\lambda}\binom{2k}{k} + \frac{2^{-2k+1}}{\lambda}\sum_{t=0}^{r-1}(r-t)\binom{2k+t-1}{k-1}2^{-t}. \end{aligned} \tag{2.39}$$

Finally, by replacing the binomial identity

$$\binom{2k+s-1}{k-1} = k\binom{2k}{k}\frac{1}{2k+s}\frac{(2k+1)^{(s)}}{(k+1)^{(s)}} \tag{2.40}$$

in (2.39), the result (2.33) is obtained. □

2.3 Minimum expected transport cost

In this section, we present an interval for the expected transport cost of a pair $\{X_i, Y_k\}$ of sensors placed randomly in the interval $[0,\infty)$. The position of the i-th sensors (blue) and the k-th (red) are determined by the arrival times X_i and Y_k, according to two Poisson process with arrival rates λ_1 and λ_2, respectively. This expected transport cost corresponds to:

$$C_{opt}(\lambda_1,\lambda_2,n) = \sum_{k=1}^{n} E\big[|X_k - Y_k|\big].$$

Theorem 2.3.1. *Consider two independent Poisson processes with arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ and arrival rates $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively, such that $\lambda_1 \geq \lambda_2$. Then*

$$C_{opt}(\lambda_1,\lambda_2,n) \in \big[\ l_n\ ,\ s_n\ \big], \tag{2.41}$$

where

$$l_n = \frac{n(n+1)}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2} \times S(n, \lambda_1, \lambda_2), \tag{2.42}$$

$$s_n = \frac{n(n+1)}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \times S(n, \lambda_1, \lambda_2) \tag{2.43}$$

and

$$S(n, \lambda_1, \lambda_2) = \sum_{k=1}^{n} \frac{(pq)^k}{B(k+1,k)}. \tag{2.44}$$

Proof. For $r = 0$, from (2.27) we have:

$$E\big[|X_k - Y_k|\big] = \frac{k}{\lambda_1} - \frac{k}{\lambda_2} + 2k\left[\frac{I_p(k,k+1)}{\lambda_2} - \frac{I_p(k+1,k)}{\lambda_1}\right]. \tag{2.45}$$

By applying identity

$$I_p(a,b) = I_p(a-1,b+1) - \frac{p^{a-1}q^b}{bB(a,b)},$$

in (2.45), for $a = k+1$, $b = k$ and $\lambda_1 \geq \lambda_2$, results in:

$$E\big[|X_k - Y_k|\big] \geq k\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + 2k\left[\frac{I_p(k,k+1)}{\lambda_2} - \frac{I_p(k+1,k)}{\lambda_2}\right] \tag{2.46}$$

$$= k\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2k}{\lambda_2}\frac{(pq)^k}{kB(k+1,k)} \tag{2.47}$$

$$= k\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2}\frac{(pq)^k}{B(k+1,k)}. \tag{2.48}$$

So, the lower bound of the sum is:

$$\sum_{k=1}^{n} E\big[|X_k - Y_k|\big] \geq \frac{n(n+1)}{2}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{2}{\lambda_2}\sum_{k=1}^{n}\frac{(pq)^k}{B(k+1,k)}, \tag{2.49}$$

To obtain the upper limit of the sum, we replace the identities

$$I_p(a,b) = I_p(a,b+1) - \frac{p^a q^b}{bB(a,b)}$$

and

$$I_p(a,b) = I_p(a+1,b) + \frac{p^a q^b}{aB(a,b)}$$

in (2.45).Then, for $a = b = k$, we get:

$$\begin{aligned} \frac{I_p(k,k+1)}{\lambda_2} - \frac{I_p(k+1,k)}{\lambda_1} &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) I_p(k,k) \\ &+ \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \frac{(pq)^k}{kB(k,k)}. \end{aligned} \tag{2.50}$$

The upper limit of the sum is obtained from (2.45), (2.50) and the fact that $I_x(k,k) \leq 1, \ \ \forall\, k \in \mathbb{Z}^+$. That is:

$$E\big[|X_k - Y_k|\big] \leq k\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + 2\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \frac{(pq)^k}{B(k,k)} \tag{2.51}$$

and

$$\sum_{k=1}^{n} E\big[|X_k - Y_k|\big] \leq \frac{n(n+1)}{2}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) \sum_{k=1}^{n} \frac{(pq)^k}{B(k+1,k)}. \tag{2.52}$$

Finally, the proof of Proposition finishes by combining inequalities (2.49) and (2.52). □

Corollary 2.3.2. *Consider two independent Poisson processes with arrival times* $X_1, X_2, \cdots$ *and* $Y_1, Y_2, \cdots$ *and arrival rates* $\lambda_1 = \lambda_2 = \lambda$. *Then*

$$C_{opt}(\lambda, \lambda, n) = \frac{2n}{3\lambda}\binom{n + \frac{1}{2}}{n}. \tag{2.53}$$

Proof. From (2.41), in Theorem 2.3.1, we have

$$\frac{2}{\lambda} \times S(n, \lambda) \ \leq \ \sum_{k=1}^{n} E\big[|X_k - Y_k|\big] \ \leq \ \frac{2}{\lambda} \times S(n, \lambda) \ .$$

That is

$$\sum_{k=1}^{n} E\big[|X_k - Y_k|\big] = \frac{2}{\lambda} \times S(n, \lambda). \tag{2.54}$$

The proof of (2.53) follows directly from(2.54) and (2.26), because $p = q = 1/2$ and $S(n, \lambda) = \sum_{k=1}^{n} k2^{-2k}\binom{2k}{k}$. □

Equation (2.53) is one of the main results of Kranakis (2014).

2.3.1 Graphic illustrations of C_{opt}

In order to illustrate our results regarding C_{opt}, here we show some graphs of the interval of C_{opt}. These graphs were generated by considering some fixed values of the parameters λ_1 and λ_2 through the Project R for Statistical Computing R Core Team, 2020.

2.4 Statistical Inference of C_{opt}

2.4.1 Estimation

Let $X_i = (X_{i1}, X_{i2}, \cdots, X_{im})$ and $Y_i = (Y_{i1}, Y_{i2}, \cdots, Y_{im})$; $i \in \{1, 2, \cdots, n\}$ be random samples from the Poisson processes with arrival rates λ_1 and λ_2 and respective arrival times

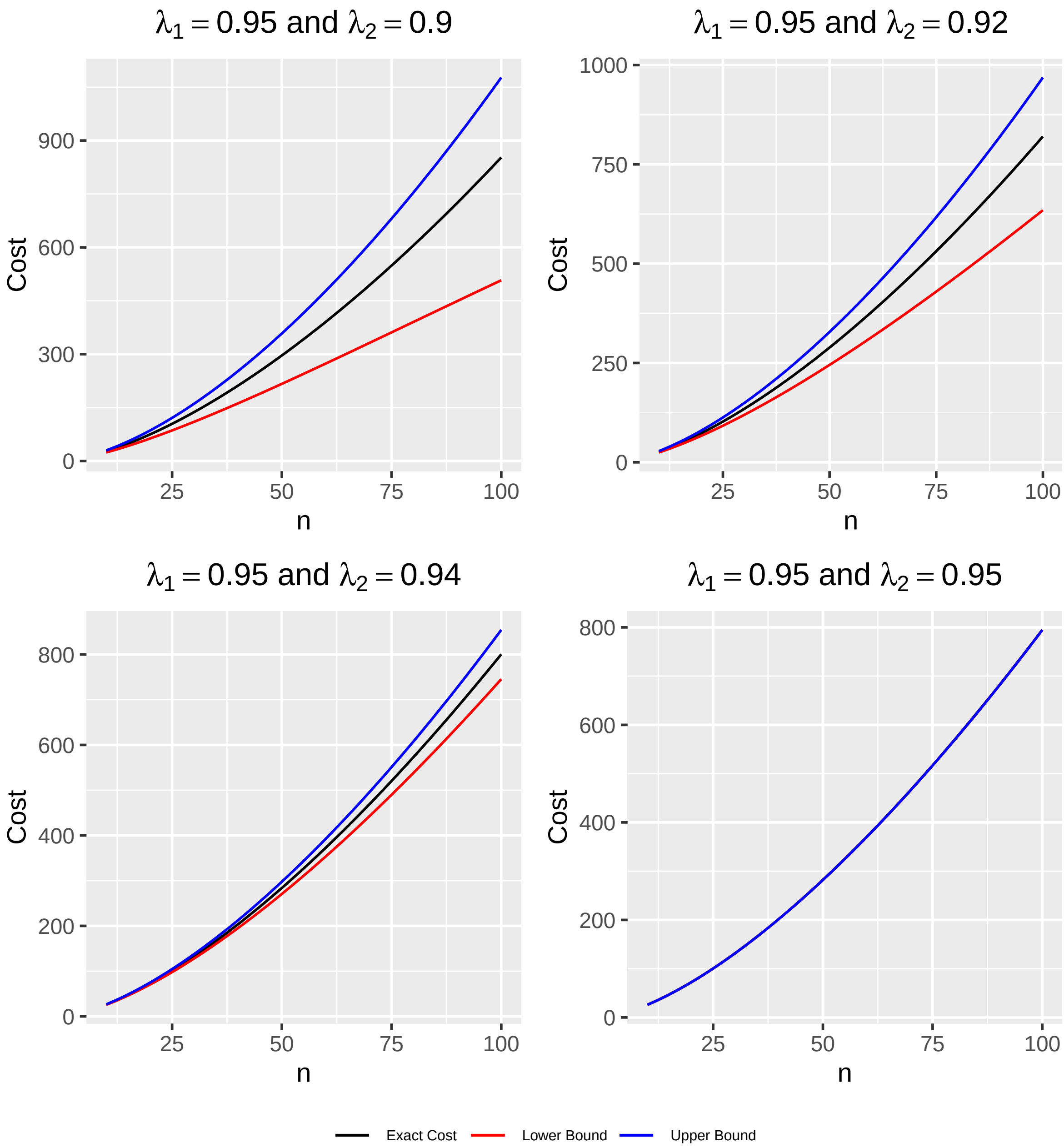


Figure 2.1: Graphs related to the intervals of the minimum expected cost of transport as in (2.41).

$X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$. Then:

$$X_{ij} \sim Gamma(i, \lambda_1) \text{ and } Y_{ij} \sim Gamma(i, \lambda_2)$$

for all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$.

Consider the sample minimum cost

$$\hat{C}_{opt}(n, m) = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} \left| X_{ij} - Y_{ij} \right|. \tag{2.55}$$

Here, we prove that (2.55) is a good estimator of

$$C_{opt}(\lambda_1, \lambda_2, n)$$

obtained in (2.27). In addition, we prove the asymptotic normality of (2.55) and then define a confidence interval of $C_{opt}(\lambda_1, \lambda_2, n)$.

Since $\left\{ \sum_{i=1}^{n} \left| X_{ij} - Y_{ij} \right| \right\}_{j \geq 1}$ is an infinite sequence of independent and identically distributed (i.i.d.) terms with expected value

$$E\left[\hat{C}_{opt}(n, m)\right] = C_{opt}(\lambda_1, \lambda_2, n), \tag{2.56}$$

by the strong law of large numbers, (see Billingsley, 1995) we have that $\hat{C}_{opt}(n, m)$ converges almost surely to the expected value $C_{opt}(\lambda_1, \lambda_2, n)$, that is:

$$\hat{C}_{opt}(n, m) \xrightarrow{a.s.} C_{opt}(\lambda_1, \lambda_2, n), \quad m \to \infty \tag{2.57}$$

or

$$P\left(\lim_{m \to \infty} \hat{C}_{opt}(n, m) = C_{opt}(\lambda_1, \lambda_2, n) \right) = 1.$$

Therefore, from (2.56) and (2.57), $\hat{C}_{opt}(n,m)$ is an **unbiased estimator** of $C_{opt}(\lambda_1,\lambda_2,n)$. On the other hand, since the variance

$$Var(\hat{C}_{opt}) = \frac{n(n+1)}{2m}\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right) < \infty \tag{2.58}$$

then as m approaches infinity, the sample minimum cost $\hat{C}_{opt}(n,m)$ converges, in distribution, to $N\left(E(\hat{C}_{opt}), Var(\hat{C}_{opt})\right)$. That is:

$$\hat{C}_{opt}(n,m) \overset{d}{\sim} N\left(C_{opt}, \frac{n(n+1)}{2m}\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right)\right), \quad m \to \infty. \tag{2.59}$$

From (2.57) and (2.59), we define the confidence interval for $C_{opt}(\lambda_1,\lambda_2,n)$, with confidence coefficient of $1-\alpha$, by:

$$I_{100(1-\alpha)\%}\big(C_{opt}\big) = \left[\ \hat{C}_{opt} - z_{\alpha/2}\sqrt{Var(\hat{C}_{opt})}\ \ ,\ \ \hat{C}_{opt} + z_{\alpha/2}\sqrt{Var(\hat{C}_{opt})}\ \right]. \tag{2.60}$$

2.4.2 Numerical illustrations

The performance of statistic (2.55) was tested by Monte Carlo simulation with 10 combinations of λ_1 and λ_2, as defined in Table 1. We use the algorithm implemented in the computational software R Core Team, 2020, version R 4.0.1 (June, 2020).

Algoritmo 1: Monte Carlo Simulation for Sample Minimum Cost

Input: Rates: λ_1 and λ_2
Number of Replications: m
Sizes of Sample (vector): n
Output: Sample Minimum Cost ($\hat{C}_{opt}(n, m)$)

Function generate.Sample.Cost
  $\hat{C}_{opt} = [\ ]$
  for $j \leftarrow 1$ *to length(n)* **do**
    for $i \leftarrow 1$ *to* n_j **do**
      P_1: Generate Random Sample (size=m) of Gamma(i, λ_1);

      P_2: Generate Random Sample (size=m) of Gamma(i, λ_2).
    end

    Dif.Abs:= Determine the absolute values of $(P_2 - P_1)$;

    Mean.Dif := Calculate the means of *Dif.Abs*;

    Sum.Mean:= Add the values of *Mean.Dif*.
  end
  $\hat{C}_{opt}$:= *Sum.Mean*.
  return $\hat{C}_{opt}$
end

Table 2.1 report the results of the mean estimates of $\hat{C}_{opt}(n,m)$, the values of $C_{opt}(\lambda_1,\lambda_2,n)$ as (2.41), the bias and the mean square error (MSE) of $\hat{C}_{opt}(n,m)$. These results, in Tables 2.1-2.5, confirm the convergence of (2.57). In addition, Figures 2.2a and 2.2b illustrate the good performance of the estimator.

Table 2.1: $C_{opt}(\lambda_1,\lambda_2,n)$, mean estimates, MSE and bias of $\hat{C}_{opt}(n,m)$ with $n = 50$ and $m = 500$.

λ_1	λ_2	$C_{opt}(\lambda_1,\lambda_2,n)$	$\hat{C}_{opt}(n,m)$	MSE	bias
0.55	0.92	951.65	950.02	14.10	-1.63
0.87	0.42	1578.21	1581.17	26.58	2.96
0.59	0.93	816.06	818.44	15.98	2.39
0.70	0.66	404.71	408.61	26.34	3.91
0.86	0.70	451.87	451.46	8.82	-0.41
0.97	0.47	1405.62	1406.26	14.66	0.64
0.43	0.93	1600.54	1598.93	19.33	-1.61
0.87	0.56	841.31	839.06	16.59	-2.26
0.74	0.79	361.95	359.99	12.58	-1.96
0.53	0.90	1007.33	1005.11	17.19	-2.23

Table 2.2: $C_{opt}(\lambda_1,\lambda_2,n)$, mean estimates, MSE and bias of $\hat{C}_{opt}(n,m)$ with $n = 50$ and $m = 1000$.

λ_1	λ_2	$C_{opt}(\lambda_1,\lambda_2,n)$	$\hat{C}_{opt}(n,m)$	MSE	bias
0.55	0.92	951.65	954.80	15.63	3.15
0.87	0.42	1578.21	1579.04	9.60	0.83
0.59	0.93	816.06	819.84	19.49	3.79
0.70	0.66	404.71	408.16	17.46	3.45
0.86	0.70	451.87	449.20	11.45	-2.67
0.97	0.47	1405.62	1406.27	7.56	0.66
0.43	0.93	1600.54	1596.28	26.56	-4.26
0.87	0.56	841.31	837.15	23.09	-4.16
0.74	0.79	361.95	360.87	5.54	-1.08
0.53	0.90	1007.33	1007.90	6.44	0.57

In Table 2.6 we show the results of the confidence interval of $C_{opt}(\lambda_1,\lambda_2,n)$ with 95%, obtaided from (2.60). The results are satisfactory. The graphs of these results are shown in Figure 2.3.

Table 2.3: $C_{opt}(\lambda_1, \lambda_2, n)$, mean estimates, MSE and bias of $\hat{C}_{opt}(n, m)$ with $n = 50$ and $m = 1e4$.

λ_1	λ_2	$C_{opt}(\lambda_1, \lambda_2, n)$	$\hat{C}_{opt}(n, m)$	MSE	bias
0.55	0.92	951.65	951.03	0.95	-0.62
0.87	0.42	1578.21	1577.51	1.38	-0.70
0.59	0.93	816.06	814.87	1.92	-1.19
0.70	0.66	404.71	404.87	0.58	0.16
0.86	0.70	451.87	451.19	0.90	-0.69
0.97	0.47	1405.62	1406.41	1.34	0.79
0.43	0.93	1600.54	1601.29	1.40	0.75
0.87	0.56	841.31	841.84	0.85	0.52
0.74	0.79	361.95	361.81	0.46	-0.13
0.53	0.90	1007.33	1007.02	0.71	-0.31

Table 2.4: $C_{opt}(\lambda_1, \lambda_2, n)$, mean estimates, MSE and bias of $\hat{C}_{opt}(n, m)$ with $n = 50$ and $m = 1e5$.

λ_1	λ_2	$C_{opt}(\lambda_1, \lambda_2, n)$	$\hat{C}_{opt}(n, m)$	MSE	bias
0.55	0.92	951.65	951.47	0.09	-0.18
0.87	0.42	1578.21	1578.34	0.11	0.14
0.59	0.93	816.06	816.00	0.05	-0.06
0.70	0.66	404.71	404.65	0.06	-0.06
0.86	0.70	451.87	451.57	0.14	-0.30
0.97	0.47	1405.62	1405.99	0.21	0.37
0.43	0.93	1600.54	1601.10	0.40	0.56
0.87	0.56	841.31	841.78	0.27	0.46
0.74	0.79	361.95	361.93	0.04	-0.02
0.53	0.90	1007.33	1007.48	0.08	0.15

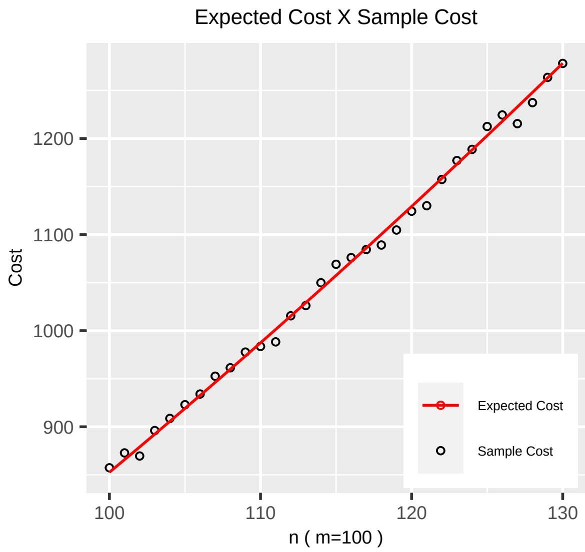
Graphically, the result (2.59) is illustrated in Figure 2.4.

Table 2.5: $C_{opt}(\lambda_1, \lambda_2, n)$, mean estimates, MSE and bias of $\hat{C}_{opt}(n, m)$ with $n = 50$ and $m = 1e6$.

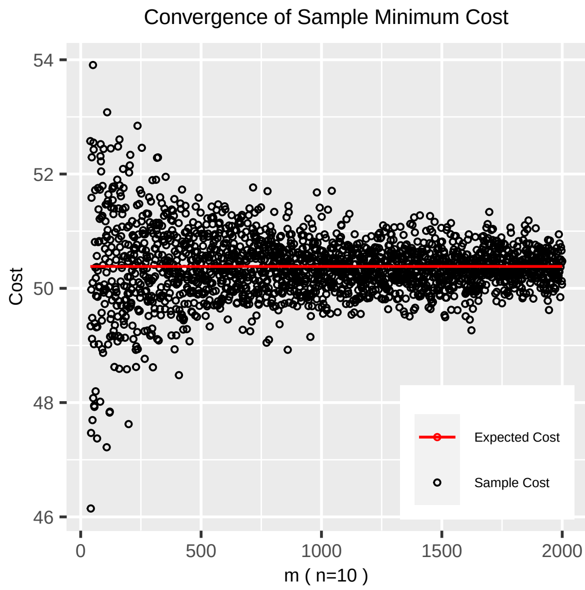
λ_1	λ_2	$C_{opt}(\lambda_1, \lambda_2, n)$	$\hat{C}_{opt}(n, m)$	MSE	bias
0.55	0.92	951.65	951.62	0.01	-0.02
0.87	0.42	1578.21	1578.16	0.01	-0.05
0.59	0.93	816.06	816.10	0.01	0.04
0.70	0.66	404.71	404.71	0.01	0.00
0.86	0.70	451.87	451.85	0.00	-0.02
0.97	0.47	1405.62	1405.64	0.01	0.02
0.43	0.93	1600.54	1600.55	0.01	0.01
0.87	0.56	841.31	841.14	0.04	-0.17
0.74	0.79	361.95	361.91	0.01	-0.04
0.53	0.90	1007.33	1007.34	0.01	0.01

Table 2.6: Confidence interval for $C_{opt}(\lambda_1, \lambda_2, n)$ with 95% of confidence.

λ_1	λ_2	$\hat{C}_{opt}(n, m)$	$C_{opt}(\lambda_1, \lambda_2, n)$	$I_{95\%}(C_{opt})$
0.26	0.25	3065.70	3065.53	[3062.16, 3077.46]
0.55	0.28	8801.69	8802.98	[8797.76, 8808.89]
0.91	0.30	11188.47	11188.42	[11184.60, 11194.31]
0.22	0.04	93329.40	93341.57	[93297.73, 93363.09]
0.46	0.90	5389.82	5391.14	[5384.39, 5391.20]
0.78	0.99	1479.48	1478.87	[1477.50, 1482.05]
0.61	0.60	1243.93	1244.34	[1241.60, 1248.07]
0.20	0.56	15906.65	15905.23	[15895.58, 15910.23]
0.20	0.11	20375.64	20375.64	[20366.78, 20395.68]
0.41	0.07	57903.33	57913.29	[57896.82, 57936.20]



(a) Graph of $\hat{C}_{opt}(n, m)$ versus $C_{opt}(\lambda_1, \lambda_2, n)$, for $\lambda_1 = 0.95$ and $\lambda_2 = 0.90$.



(b) Graph of $\hat{C}_{opt}(n, m)$ varying m (black) and $C_{opt}(\lambda_1, \lambda_2, n)$ (red), for $\lambda_1 = 0.55$ and $\lambda_2 = 0.95$.

Figure 2.2: Convergence of sample cost

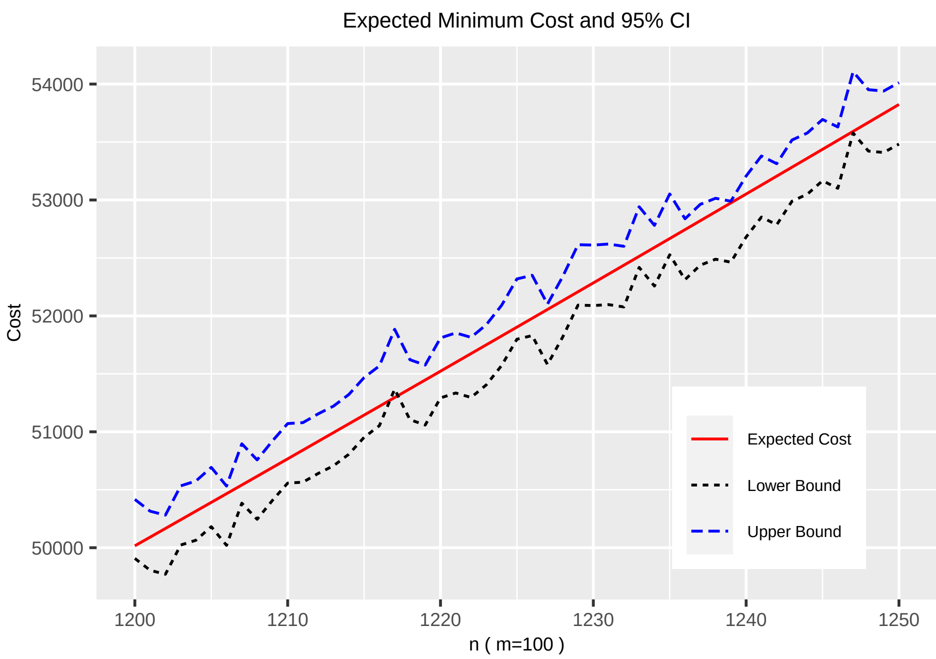


Figure 2.3: Illustration of the confidence interval for $C_{opt}(\lambda_1, \lambda_2, n)$, with 95% of confidence, for $\lambda_1 = 0.95$ and $\lambda_2 = 0.90$.

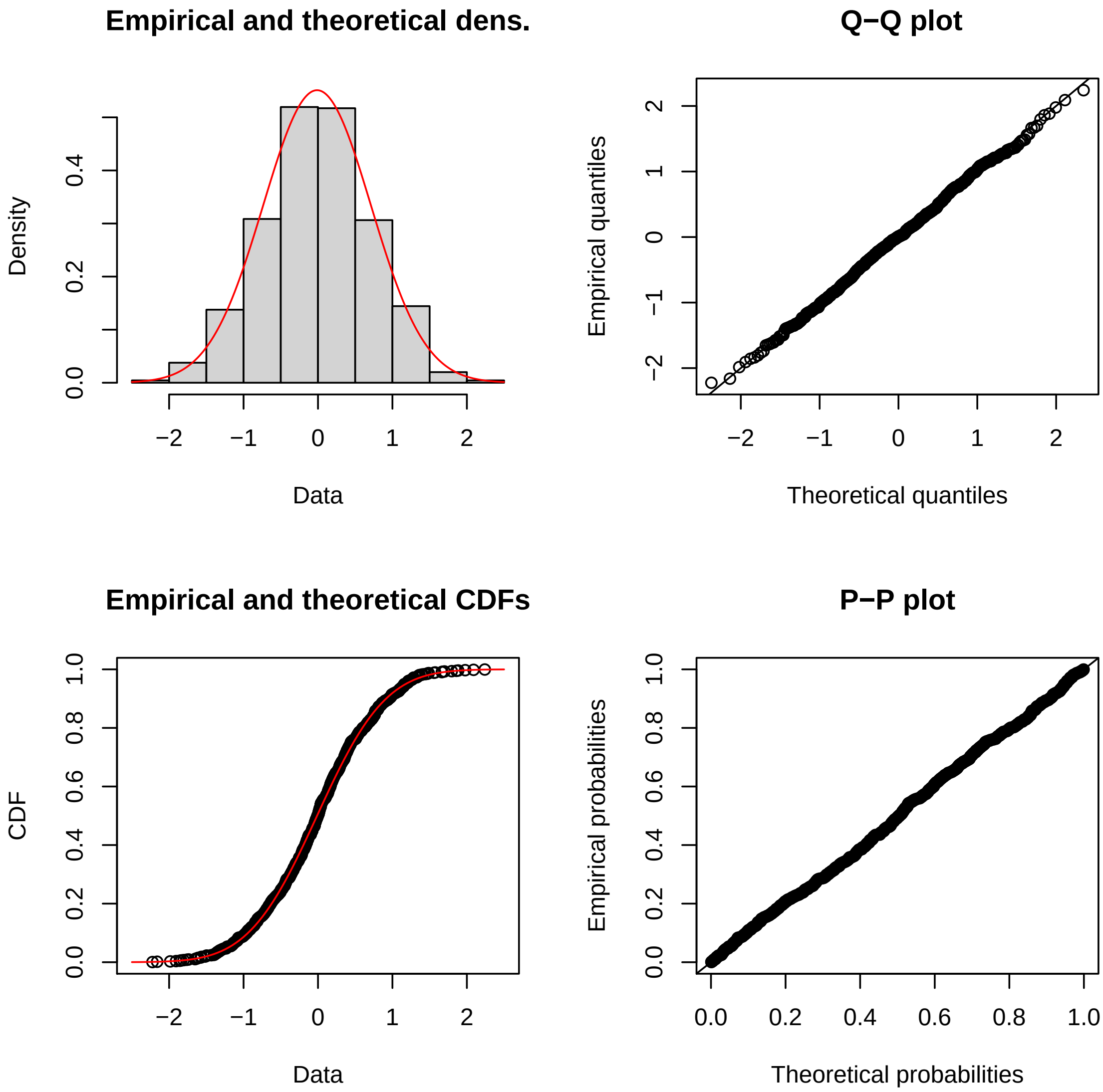


Figure 2.4: Illustration of the asymptotic normality of the estimator $\hat{C}_{opt}(n, m)$ as (2.59).

2.5 Conclusion

In this article, we derive an interval for the sum of the expected absolute difference between two Poisson processes that can have different rates. Our results generalize those of Kranakis (2014), and to apply our results we calculate the minimum transport cost of a random two-color combination when two sensors are initially placed according to two Poisson processes with different or equal laws. We perform a complete statistical inference study, prove asymptotic normality of the cost estimator, and perform a simulation study to show the consistency of the cost estimator.

Chapter 3

Generalized moments in Poisson processes

3.1 Introduction

The problem of determining the cost of moving a sensor, in a network of sensors, measured as the sum or maximum of movements of sensors from their initial positions to target destinations, has been studied by several authors, such as Kranakis (2014). Works related to this subject, considering that the sensors are placed in the network according to a stochastic process, are Ajtai, Komlós, and Tusnády (1984), Kranakis et al. (2013), Kranakis (2014), Kapelko (2018) and Kapelko (2020) and Moltchanov (2012).

Results already available regarding the distance between points of two spatial point processes, the homogeneous Poisson process on $\mathbb{R}^2$ and the process with points uniformly distributed on $\mathbb{R}^2$, were unified by Moltchanov (2012). In the same work, some applications to sensors and mobile wireless networks were also discussed. Some of the papers in which the initial objective is to determine the distance between events of two stochastic processes are Ajtai, Komlós, and Tusnády (1984) and Kranakis et al. (2013). They considered two stochastic processes, in which the arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ are independent and uniformly distributed. Kranakis (2014) and Kapelko (2018) derived the expected distance between two independent homogeneous Poisson processes with the same arrival rate and with the respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. Recently, Kapelko (2020) investigated about the energy for displacement of random sensors for connectivity and interference. For this, he determined the distance between identical and independent events of d-dimensional Poisson processes with arrival rate $\lambda > 0$.

Further regarding Poisson processes, Kranakis (2014) derived an analytical formula for the absolute distance between arrival times of two independent and identically distributed Poisson processes with arrival rate λ. Kapelko (2018) generalized this result by obtaining a closed analytical formula for the a-th moment of the absolute distance from the arrival times of these same stochastic processes.

In this work, we provide a closed analytical formula for the a-th moment, $E[|X_{k+r} - Y_k|]^a$, by considering two independent Poisson processes, but with any arrival rates, λ_1 and λ_2 and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. We also show that the results of Kapelko (2018) are particular cases of our formulation. In addition, we present numerical illustrations of our result.

The remainder of this paper is structured as follows. In the rest of this section, we present preliminary results of the hypergeometric function and Fox's H-function. In Section 2, we present the main results and, in Section 3, we show numerical results and graphic illustrations .

3.2 Generalized hypergeometric function

In this subsection, we present some basic concepts of the Hypergeometric function and introduce the Fox H-function, which will be useful in the next section.

The generalized hypergeometric function is one of a class of special functions that are very useful in calculating probability. In this article, we use it to show the relationship between our result and Kapelko's results.

According to Srivastava (2019), the generalized hypergeometric function can be defined, in terms of Pochhammer polynomials,

$$x^{(q)} := \begin{cases} \dfrac{\Gamma(x+q)}{\Gamma(x)} = x(x+1)\cdots(x+q-1), & q \geq 1 \\ 1, & q = 0\ , \end{cases} \tag{3.1}$$

by:

$${}_pF_q(a_1, \cdots, a_p\ ;\ b_1, \cdots, b_q\ ;\ z) = \sum_{n=0}^{\infty} \frac{a_1^{(n)} \cdots a_p^{(n)}}{b_1^{(n)} \cdots b_q^{(n)}} \frac{z^n}{n!}\ , \tag{3.2}$$

where no parameter b_j of the denominator can be zero or a negative integer. Also, If any parameter a_j of the numerator the equation (3.2) is zero or a negative integer, the series ends.

$$\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \ \ \alpha > 0$$

is the gamma function.

When $p = 2$ and $q = 1$, a function ${}_pF_q$ is called a hypergeometric function. For the study of ${}_2F_1$, see Oliveira, 2012.

For particular values of z, we have:

1. Case $z = 1$ (Gauss' theorem): Let α, β, ϕ be complex numbers, such that $Re(\phi - \beta - \alpha) > 0$. So,

$${}_2F_1(\alpha, \beta;\ \phi;\ 1) = \frac{\Gamma(\phi)\ \Gamma(\phi - \alpha - \beta)}{\Gamma(\phi - \alpha)\ \Gamma(\phi - \beta)}. \tag{3.3}$$

2. Case $z = -1$ (Kummer's theorem): Let α and β be complex numbers, so,

$$ {}_2F_1(\alpha, \beta; 1+\alpha-\beta; -1) = \frac{\Gamma(1+\alpha-\beta)\,\Gamma\big(1+\frac{\alpha}{2}\big)}{\Gamma(1+\alpha)\,\Gamma\big(1+\frac{\alpha}{2}-\beta\big)}. \tag{3.4} $$

The computation of the hypergeometric function is available in various programs, including Wolfram Mathematica, Mathematica, Maple and R-Project.

3.3 Fox's H-function

The H function, as introduced by Fox (1961), is a complex contour integral that contains gamma functions in its members (see Mathai, Saxena, and Haubold, 2010), and is defined by:

$$ \begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{cccccc} (a_1, A_1), & \ldots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \ldots, & (a_p, A_p) \\ (b_1, B_1), & \ldots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \ldots, & (b_q, B_q) \end{array} \right] \\ &= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)} z^{-s} ds, \end{aligned} \tag{3.5} $$

where all A_j and B_j are positive real numbers and all a_j and b_j may be complex numbers. The contour L runs from $c - i\infty$ to $c + i\infty$ such that the poles of $\Gamma(b_j + B_j s),\ j = 1, \ldots, m$ lie to the left of L and the poles of $\Gamma(1 - a_j - A_j s),\ j = 1, \ldots, n$ lie to the right of L.

The H function contains a large number of analytical functions as special cases. Here, we present two special cases of the H function that are useful in proving our main result (see the book by Mathai, Saxena, and Haubold (2010), pages 23 and 24).

$$ H_{0\,1}^{1\,0}\left[z \,\middle|\, (\alpha, \beta) \right] = \beta^{-1} z^{\frac{\alpha}{\beta}} e^{-z^{1/\beta}}, \tag{3.6} $$

and

$$H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix} (1-\alpha,1) & (1-\beta,1) \\ (0,1) & (1-\xi,1) \end{matrix}\right] = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\xi)}\,{}_2F_1(\alpha,\beta;\xi;-z), \tag{3.7}$$

where ${}_2F_1$ is the hypergeometric function. From (3.7), we can verify that the generalized hypergeometric function is a particular case of Fox's H-function.

The H function allows solving different problems arising from probability, physics and engineering, due to its properties and the fact that it has several transforms, such as the Mellin transform, Laplace transform, Laplace inverse transform and Euler transform.

In this work, we use the Laplace transform of $x^{\rho-1}H_{p,q}^{m,n}(\alpha x)$ and the Euler transform of $H_{p,q}^{m,n}(\beta x)$ given, respectively, by:

$$\int_0^\infty e^{-sx}x^{\rho-1}H_{p,q}^{m,n}\left[\alpha x \,\middle|\, \begin{matrix} (a_p,A_p) \\ (b_q,B_q) \end{matrix}\right] dx = s^{-\rho}H_{p,q}^{m,n}\left[\alpha s^{-\sigma} \,\middle|\, \begin{matrix} (1-\rho,\sigma) & (a_p,A_p) \\ & (b_q,B_q) \end{matrix}\right] \tag{3.8}$$

and

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1}H_{p,q}^{m,n}(\beta t^r)dt = \Gamma(\sigma)x^{\rho+\sigma-1}H_{p+1,q+1}^{m,n+1}\left[\beta x^r \,\middle|\, \begin{matrix} (1-\rho,r) & (a_p,A_p) \\ (b_q,B_q) & (1-\rho-\sigma,r) \end{matrix}\right], \tag{3.9}$$

valid for $\rho \in \mathbb{C}$, $\alpha > 0,\ \sigma > 0$, and $k > 0$. See identity (2.19), pag. 50 and identity (2.51), pag. 58 of (Mathai, Saxena, and Haubold, 2010).

3.4 Main Results

Consider two independent homogeneous Poisson processes with rates λ_1 and λ_2 characterized by the i-th and the k-th arrival times, denoted by X_i and Y_k. Hence, X_i and Y_k are random variables with a gamma distribution, denoted by:

$$X_i \sim Gamma(i, \lambda_1) \quad \text{and} \quad Y_k \sim Gamma(k, \lambda_2) \ .$$

So, the probability density functions of these random variables are

$$f_{X_i}(x) := f_1(x) = \frac{\lambda_1^i}{\Gamma(i)} x^{i-1} e^{-\lambda_1 x} \quad , \qquad x > 0 \tag{3.10}$$

and

$$f_{Y_k}(x) := f_2(y) = \frac{\lambda_2^k}{\Gamma(k)} x^{k-1} e^{-\lambda_2 y} \quad , \qquad y > 0 \ , \tag{3.11}$$

respectively.

Theorem 3.4.1. *Consider two independent Poisson processes with arrival rates λ_1 and λ_2 and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. Let $k \geq 1$, $r \geq 0$, $a \geq 1$ be integers and $\lambda_1, \lambda_2 > 0$. Then:*

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{(k+r)^{(j)}}{j!} \frac{k^{(a-j)}}{(a-j)!} \left(\frac{-\lambda_2}{\lambda_1}\right)^j - 2I_2 \times 1_{[\text{mod2}]}(a), \tag{3.12}$$

where

$$I_2 = \frac{(-1)^a (\lambda_1/\lambda_2)^{k+r}\ \Gamma(a+1)\Gamma(a+r+2k)}{\lambda_2^a\ \Gamma(k)\Gamma(1+k+r+a)} \times {}_2F_1\big(a+2k+r; k+r; 1+k+r+a; -\frac{\lambda_1}{\lambda_2}\big),$$

$$1_{[\mathrm{mod}2]}(a) = \begin{cases} 1, & a \ \ odd \\ 0, & a \ \ even \end{cases} \quad \textit{and} \quad {}_2F_1 \textit{ is the hipergeometric function.}$$

Lemma 3.4.2. *Consider the hypotheses of Theorem 3.4.1, a integer and*

$$I_1 := \int_0^\infty f_2(y) \int_0^\infty (t-y)^a \ f_1(t) \ dt \ dy.$$

Then, for the $a \geq 1$ integer, the following identity is valid:

$$I_1 = \frac{a!(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{i^{(j)}}{j!} \frac{k^{(a-j)!}}{(a-j)!} \left(-\frac{\lambda_2}{\lambda_1} \right)^j. \tag{3.13}$$

Proof. The k-th moment of a random variable $T \sim Gamma(m, \lambda)$ is given by:

$$\begin{aligned} E(T^k) &= \frac{1}{\lambda^k} \frac{\Gamma(m+k)}{\Gamma(m)} \\ &= \frac{m^{(k)}}{\lambda^k}, \end{aligned} \tag{3.14}$$

using the relation between gamma function and Pochhammer polynomial (3.1). To calculate I_1 we use (3.14) and Newton's binomial

$$\begin{aligned} I_1 &= \int_0^\infty f_2(y) \sum_{j=0}^{a} \binom{a}{j} (-1)^{a-j} \ y^{a-j} \ \frac{i^{(j)}}{\lambda_1^j} \ dy \\ &= \sum_{j=0}^{a} \binom{a}{j} (-1)^{a-j} \ \frac{i^{(j)}}{\lambda_1^j} \int_0^\infty y^{a-j} \ f_2(y) \ dy \\ &= \frac{a! \ (-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{i^{(j)}}{j!} \ \frac{k^{(a-j)}}{(a-j)!} \left(-\frac{\lambda_2}{\lambda_1} \right)^j. \end{aligned} \tag{3.15}$$

□

In the proof of the following lemma we use the theory given in the previous section, about Fox's H-function.

Lemma 3.4.3. *Consider the hypotheses of Theorem 3.4.1,* $a \geq 1$ *integer and*

$$I_2 = \int_0^\infty \int_0^y (t-y)^a f_1(t) f_2(y) dt dy.$$

Then, the following identity is valid:

$$I_2 = \frac{\lambda_1^i \lambda_2^k \Gamma(a+1)(-1)^a}{\Gamma(i)\Gamma(k)} \frac{\Gamma(a+i+k)\Gamma(i)}{\Gamma(1+i+a)} \times_2 F_1(a+i+k, i; 1+i+a; -\frac{\lambda_1}{\lambda_2}). \tag{3.16}$$

Proof. By replacing (3.10) and (3.11) in I_2, we get:

$$\begin{aligned} I_2 &= \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)} \int_0^\infty y^{k-1} e^{-\lambda_2 y} \left(\int_0^y t^{i-1} (t-y)^a e^{-\lambda_1 t} dt \right) dy \\ &= \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)} \int_0^\infty y^{k-1} e^{-\lambda_2 y} \left(I_{21} \right) dy. \end{aligned} \tag{3.17}$$

Now, using representation (3.6) for the exponential function of I_{21}, we have:

$$I_{21} = (-1)^a \int_0^y t^{i-1} (y-t)^{(a+1)-1} H_{0\,1}^{1\,0} \left[\lambda_1 t \,\middle|\, (0,1) \right] dt. \tag{3.18}$$

Since the integral in (3.18) is an Euler transform of the H function, we use the identity (3.9) and obtain

$$I_{21} = (-1)^a y^{i+a} \Gamma(a+1) H_{1\,2}^{1\,1} \left[\lambda_1 y \,\middle|\, \begin{matrix} (1-i,1) & \\ (0,1) & (-i-a,1) \end{matrix} \right]. \tag{3.19}$$

We then update (3.17) by replacing (3.19) in (3.17)

$$I_2 = \frac{\lambda_1^i \lambda_2^k \Gamma(a+1)(-1)^a}{\Gamma(i)\Gamma(k)} \int_0^\infty y^{i+a+k-1} e^{-\lambda_2 y} \left(H_{1\,2}^{1\,1} \left[\lambda_1 y \,\middle|\, \begin{matrix} (1-i,1) & \\ (0,1) & (-i-a,1) \end{matrix} \right] \right) dy. \tag{3.20}$$

Finally, since the integral of (3.20) is a Laplace transform of $y^{a+i+k}G_{1\,2}^{1\,1}(\lambda_1 y)$, we use identity (3.8) and obtain:

$$I_2 = \frac{\lambda_1^i \lambda_2^k \Gamma(a+1)(-1)^a}{\Gamma(i)\Gamma(k)} \times$$
$$\times\, \lambda_2^{-(a+i+k)} H_{2,2}^{1,2}\left[\lambda_1\lambda_2^{-1} \,\middle|\, \begin{matrix} (1-a-i-k,1) & (1-i,1) \\ (0,1) & (-i-a,1) \end{matrix}\right]. \tag{3.21}$$

The result (3.16) is obtained by using the identity (3.7) in (3.21). □

Proof. **(Theorem 3.4.1)** By the law of total expectation, we have:

$$\begin{aligned} E\big[|X_i - Y_k|^a\big] &= E\Big[E\big[|X_i - Y_k|^a\big]\Big|Y_k\Big] \\ &= \int_0^\infty E\big[|X_i - y|^a\big]\ f_2(y)\ dy. \end{aligned} \tag{3.22}$$

The expectation (3.22) is determined for both the odd and even cases, as follows:

When a is even, by definition, we have that

$$\begin{aligned} E\big[|X_i - y|^a\big] &= \int_0^\infty (t-y)^a\ f_1(t)\ dt \\ &= \int_y^\infty (t-y)^a\ f_1(t)\ dt + \int_0^y (y-t)^a\ f_1(t)\ dt \end{aligned} \tag{3.23}$$

Replacing expression (3.23) in (3.22), results in:

$$E\big[|X_i - Y_k|^a\big] = I_1. \tag{3.24}$$

Now, we substitute (3.13) in (3.24), with $i = k + r$. Then:

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!\,(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{(k+r)^{(j)}}{j!}\,\frac{k^{(a-j)}}{(a-j)!}\,\Big(-\frac{\lambda_2}{\lambda_1}\Big)^j. \tag{3.25}$$

When a is odd, by definition, we have that:

$$\begin{aligned} E\big[|X_i - y|^a\big] &= \int_y^\infty (t-y)^a\, f_1(t)\, dt + \int_0^y (y-t)^a\, f_1(t)\, dt \\ &= \int_0^\infty (t-y)^a\, f_1(t)\, dt - 2\int_0^y (t-y)^a\, f_1(t)\, dt. \end{aligned} \tag{3.26}$$

Finally we rewrite (3.22) with (3.26) and obtain:

$$E\big[|X_i - Y_k|^a\big] = I_1 - 2I_2. \tag{3.27}$$

The result follows when applying Lemmas 3.4.2 and 3.4.3, equations (3.13) and (3.16) with $i = k + r$. □

In the next corollaries we show that the identity derived by Kapelko (2018) about the a - th absolute moment of the difference between the arrival times of two identical and independent Poisson processes with rate λ is a particular result of our identity (3.12). Our Corollary 3.4.4 corresponds to Theorem 3 of Kapelko (2018) and our Corollary 3.4.5 corresponds to Theorem 9 of Kapelko (2018).

Corollary 3.4.4. *Consider two identical and independent Poisson processes with arrival rate $\lambda > 0$ and respective arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$.*

If $k \geq 1$, $a \geq 1$ are integers with a even, then:

$$E\big[|X_k - Y_k|^a\big] = \frac{a!}{\lambda^a}\,\frac{\Gamma\big(\frac{a}{2}+k\big)}{\Gamma(k)\,\Gamma\big(\frac{a}{2}+1\big)}. \tag{3.28}$$

Proof. For $\lambda_1 = \lambda_2 = \lambda$, from Theorem 3.4.1 follows that

$$
\begin{aligned}
E\big[|X_k - Y_k|^a\big] &= \frac{a!\,(-1)^a}{\lambda_2^a}\sum_{j=0}^{a}\frac{(k)^{(j)}}{j!}\,\frac{k^{(a-j)}}{(a-j)!}\,\big(-1)^j \\
&= \frac{a!}{\lambda^a}\sum_{j=0}^{a}(-1)^j\binom{k-1+j}{j}\binom{k-1+a-j}{a-j} \\
&= \frac{a!}{\lambda^a}\binom{k-1+a/2}{a/2}\frac{1+(-1)^a}{2}. \qquad (3.29)
\end{aligned}
$$

For the last equality (see eq. 3.36; page 40 of Gould, 1972). Equation (3.28) follows from (3.29) when we rewrite the combinations in terms of the gamma function.

□

Corollary 3.4.5. *Consider two identical and independent Poisson processes with arrival rate* $\lambda > 0$ *and respective arrival times* $X_1, X_2, \cdots$ *and* $Y_1, Y_2, \cdots$. *If* $k \geq 1$, $a \geq 1$ *are integers with* a *odd, then:*

$$
E\big[|X_k - Y_k|^a\big] = \frac{a!}{\lambda^a}\,\frac{\Gamma\big(\frac{a}{2}+k\big)}{\Gamma(k)\,\Gamma\big(\frac{a}{2}+1\big)} \qquad (3.30)
$$

Proof. Since a is an odd number, from Theorem 3.4.1 we obtain:

$$
\begin{aligned}
E\big[|X_k - Y_k|^a\big] &= -2I_2 \\
&= 2\frac{\Gamma(a+1)}{\lambda^a\,\Gamma(k)}\,\frac{\Gamma(a+2k)}{\Gamma(1+k+a)} \times {}_2F_1\big(a+2k; k; 1+k+a; -1\big), \qquad (3.31)
\end{aligned}
$$

because,

$$
\frac{a!\,(-1)^a}{\lambda_2^a}\sum_{j=0}^{a}\frac{(k)^{(j)}}{j!}\,\frac{k^{(a-j)}}{(a-j)!}\,(-1)^j = \frac{(-1)^a}{\lambda^a}\sum_{j=0}^{a}\binom{a}{j}(-1)^j k^{(j)}k^{(a-j)} = 0. \qquad (3.32)
$$

For identity (3.4), we have that:

$$ {}_2F_1(a+2k,k\ ;\ 1+k+a\ ;\ -1) = \frac{\Gamma(1+a+2k-k)\,\Gamma\big(1+\frac{1}{2}(a+2k)\big)}{\Gamma(1+a+2k)\,\Gamma\big(1+\frac{1}{2}(a+2k)-k\big)}. \tag{3.33} $$

The result (3.30) follows when replacing (3.33) in (3.31) and performing some algebraic manipulations. □

Theorem 3.4.6. *Consider two identical and independent Poisson processes with arrival rate $\lambda_1 > 0$ and $\lambda_2 > 0$ and respectives arrival times $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$. If $k \geq 1$, $a \geq 1$ are integers with a odd and $\lambda_2 > \lambda_1$, then:*

$$ C^a_{opt} = \frac{2\,a!\,(-1)^{a+1}\,\lambda_1\,e^{\lambda_1\lambda_2/(\lambda_1+\lambda_2)}}{\lambda_2^a\,(\lambda_1+\lambda_2)\,\Gamma(n)}\,\Gamma\left(n\,,\ \frac{\lambda_1\,\lambda_2}{\lambda_1+\lambda_2}\right). \tag{3.34} $$

Lemma 3.4.7. *Let $f_1(\cdot)$ and $f_2(\cdot)$ be the densities of the distributions $Gamma(i,\lambda_1)$ and $Gamma(k,\lambda_2)$, respectively, with $a \geq 1$ integer, $\lambda_1 > 0$ and $\lambda_2 > 0$. Then, for $y > 0$:*

$$ \int_0^\infty\int_0^y (t-y)^a\,f_1(t)f_2(y)\,dt\,dy = \frac{\lambda_1^i\lambda_2^{k-a-i-1}a!(-1)^a}{\Gamma(k)}{}_2F_1(i,a+i+1;a+i+1;-\lambda_1/\lambda_2). \tag{3.35} $$

In particular, when $\lambda_2 > \lambda_1$, you have:

$$ I_2 = \frac{(-1)^a\,a!\,\lambda_1^i\lambda_2^{k-a-i-1}}{\Gamma(k)}\left(1+\frac{\lambda_1}{\lambda_2}\right)^{-i}. \tag{3.36} $$

Proof.

$$I_2 := \int_0^\infty \int_0^y (t-y)^a\, f_1(t) f_2(y)\, dt\, dy \tag{3.37}$$

$$= \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)} \int_0^\infty \int_0^y (t-y)^a\, t^{i-1} e^{-\lambda_1 t} y^{k-1} e^{-\lambda_2 y}\, dt\, dy \tag{3.38}$$

$$\overset{(t=uy)}{=} \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)} \int_0^\infty y^{a+i} e^{-\lambda_2 y} \left\{ \int_0^1 (u-1)^a u^{i-1} e^{-\lambda_1 y u}\, du \right\} dy \tag{3.39}$$

$$= \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)} (-1)^a \int_0^\infty y^{a+i} e^{-\lambda_2 y} \underbrace{\left\{ \int_0^1 u^{i-1} (1-u)^a\, e^{-(\lambda_1 y) u}\, du \right\}}_{I_{21}} dy. \tag{3.40}$$

Let $I_{21} := \int_0^1 u^{i-1}(1-u)^a\, e^{-(\lambda_1 y)u}\, du$.

From the integral representation of the confluent hypergeometric function (see proof of result $R1$ in the Appendix A)

$${}_1F_1(d; c; x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 e^{xt}\, t^{d-1} (1-t)^{c-d-1}\, dt, \tag{3.41}$$

with $d = i,\ c = a + i + 1$ e $x = -\lambda_1 y$, it follows that

$$I_{21} = \frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)} {}_1F_1(i; a+i+1; -\lambda_1 y). \tag{3.42}$$

Replacing the Eq.(3.42) in (3.40), results in

$$I_2 = \frac{\lambda_1^i \lambda_2^k}{\Gamma(i)\Gamma(k)}(-1)^a \int_0^\infty y^{a+i} e^{-\lambda_2 y} \frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)} {}_1F_1(i; a+i+1; -\lambda_1 y)\, dy \tag{3.43}$$

$$= \frac{\lambda_1^i \lambda_2^k (-1)^a}{\Gamma(i)\Gamma(k)} \frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)} \underbrace{\int_0^\infty y^{a+i} e^{-\lambda_2 y} {}_1F_1(i; a+i+1; -\lambda_1 y)\, dy}_{I_{22}}. \tag{3.44}$$

Let $I_{22} := \int_0^\infty y^{a+i} e^{-\lambda_2 y} {}_1F_1(i; a+i+1; -\lambda_1 y)\, dy$.

From the following result (see proof of result $R2$ in the Appendix A)

$$\int_0^\infty t^{\mu-1} e^{-xt} {}_1F_1(d;\ c;\ qt)\, dt = \Gamma(\mu) x^{-\mu} {}_2F_1(d, \mu; c; q/x), \tag{3.45}$$

with $d = i,\ c = a+i+1,\ x = \lambda_2,\ \mu = a+i+1$ and $q = -\lambda_1$, follow that

$$I_{22} = \Gamma(a+i+1) \lambda_2^{-(a+i+1)} {}_2F_1(i, a+i+1; a+i+1; -\lambda_1/\lambda_2), \tag{3.46}$$

for $\lambda_1 > 0$ and $\lambda_2 > 0$.

Therefore

$$I_2 = \frac{\lambda_1^i \lambda_2^k (-1)^a}{\Gamma(i)\Gamma(k)} \frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)} \Gamma(a+i+1) \lambda_2^{-(a+i+1)} {}_2F_1(i, a+i+1; a+i+1; -\lambda_1/\lambda_2) \tag{3.47}$$

$$= \frac{\lambda_1^i \lambda_2^{k-a-i-1} a! (-1)^a}{\Gamma(k)} {}_2F_1(i, a+i+1; a+i+1; -\lambda_1/\lambda_2). \tag{3.48}$$

And, under the hypothesis that $\lambda_2 > \lambda_1 > 0$, we can find another value for the integral I_2.

Assuming $\lambda_2 > \lambda_1 > 0$, we have by the result below (see proof of result $R3$ in the Appendix

A)

$$\int_0^\infty t^{\mu-1}e^{-xt}{}_1F_1(d;\,\mu\,;qt)\,dt = \Gamma(\mu)x^{-\mu}\left(1-\frac{q}{x}\right)^{-d}, \tag{3.49}$$

with $x = \lambda_2,\ q = -\lambda_1,\ d = i,$ and $\mu = a+i+1$, that

$$\begin{aligned} I_{22} &= \int_0^\infty y^{a+i}e^{-\lambda_2 y}{}_1F_1(i; a+i+1; -\lambda_1 y)\,dy \quad (3.50)\\ &= \Gamma(a+i+1)\lambda_2^{-(a+i+1)}\left(1+\frac{\lambda_1}{\lambda_2}\right)^{-i}. \quad (3.51) \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &= \frac{\lambda_1^i\lambda_2^k}{\Gamma(i)\Gamma(k)}(-1)^a\int_0^\infty y^{a+i}e^{-\lambda_2 y}\frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)}{}_1F_1(i; a+i+1; -\lambda_1 y)\,dy \quad (3.52)\\ &= \frac{\lambda_1^i\lambda_2^k(-1)^a}{\Gamma(i)\Gamma(k)}\,\frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)}\underbrace{\int_0^\infty y^{a+i}e^{-\lambda_2 y}{}_1F_1(i; a+i+1; -\lambda_1 y)\,dy}_{I_{22}} \quad (3.53)\\ &= \frac{\lambda_1^i\lambda_2^k(-1)^a}{\Gamma(i)\Gamma(k)}\,\frac{\Gamma(i)\Gamma(a+1)}{\Gamma(a+i+1)}\,\Gamma(a+i+1)\lambda_2^{-(a+i+1)}\left(1+\frac{\lambda_1}{\lambda_2}\right)^{-i} \quad (3.54)\\ &= \frac{(-1)^a\,a!\,\lambda_1^i\lambda_2^{k-a-i-1}}{\Gamma(k)}\left(1+\frac{\lambda_1}{\lambda_2}\right)^{-i}. \quad (3.55) \end{aligned}$$

□

Proof. (**Theorem 3.4.6**) Considering a odd, we saw that $E\big[|X_k - Y_k|^a\big] = -2I_2\big|_{i=k}$. So, when

$\lambda_2 > \lambda_1 > 0$, it follows that

$$C^a_{opt} = \sum_{k=1}^{n} E\big[|X_k - Y_k|^a\big] \tag{3.56}$$

$$= \frac{2\,(-1)^{a+1}\,a!}{\lambda_2^{a+1}} \sum_{k=1}^{n} \left\{ \frac{\lambda_1^k}{\Gamma(k)} \left(1 + \frac{\lambda_1}{\lambda_2}\right)^{-k} \right\} \tag{3.57}$$

$$= \frac{2\,a!\,(-1)^{a+1}}{\lambda_2^{a+1}} \sum_{k=1}^{n} \left\{ \frac{1}{\Gamma(k)} \left(\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\right)^{k} \right\} \tag{3.58}$$

$$= \frac{2\,a!\,(-1)^{a+1}\,\lambda_1\,e^{\lambda_1\lambda_2/(\lambda_1+\lambda_2)}}{\lambda_2^a\,(\lambda_1+\lambda_2)\,\Gamma(n)}\,\Gamma\!\left(n\,,\ \frac{\lambda_1\,\lambda_2}{\lambda_1+\lambda_2}\right). \tag{3.59}$$

$\square$

3.5 Numerical and Graphical Results

For a random sample of size N of each process

$$X_k \sim Gamma(k, \lambda) \quad \text{e} \quad Y_k \sim Gamma(k, \lambda),$$

it is natural to consider that the statistic

$$M^a_k = \frac{1}{N} \sum_{j=0}^{N} \left|X_{k_j} - Y_{k_j}\right|^a \tag{3.60}$$

is a moments estimator of

$$E\big[|X_{k+r} - Y_k|^a\big] = \frac{a!(-1)^a}{\lambda_2^a} \sum_{j=0}^{a} \frac{(k+r)^{(j)}}{j!} \frac{k^{(a-j)}}{(a-j)!} \left(\frac{-\lambda_2}{\lambda_1}\right)^j - 2I_2 1_{[\text{mod}2]}(a)$$

$$= M^a_{pop}, \tag{3.61}$$

given in (3.12).

3.5.1 Bias of M_k^a

In this subsection, we present the results obtained, via Monte Carlo simulation, of the bias of M_k^a; $Bias(M_k^a) = \left(\frac{1}{m}\sum_{k=1}^{m} M_k^a\right) - M_{pop}^a$. The result of the bias is close to zero for different experiments.

We computationally implemented M_{pop}^a in R Core Team (2020), version R 4.0.1 (June, 2020) and simulated m samples of size N of the times X_k and Y_k for each value of $k \in \{1, \cdots, 12\}$.

For $\lambda_1 = \lambda_2 = 2$, we simulated $m = 12$ samples of size $N = 1 \times 10^6$, one for each $k \in \{1, \ldots 12\}$. $M_{pop}^a,\ a = 1, 2, 3, 4$, versus sample moments, $M_k^a,\ a = 1, 2, 3, 4$. Table 3.2 shows the bias of these sample estimates.

The general case is when $\lambda_1 \neq \lambda_2$. By considering $\lambda_1 = 10$ and $\lambda_2 = 20$ for processes $X_k \sim Gamma(k, \lambda_1)$ e $Y_k \sim Gamma(k, \lambda_2)$, we simulated $m = 12$ samples of size $N = 1 \times 10^7$, one for each $k \in \{1, \ldots 12\}$. The results of the first four population moments and sample moments are shown in Table 3.3 and the bias corresponding to sample estimates is given in Table 3.4.

From the results of Tables 3.2 and 3.4, we can conclude that the statistic (3.60) has a very little bias.

Table 3.1: Population moments versus sample moments for equal rates.

k	M_{pop}^1	M_k^1	M_{pop}^2	M_k^2	M_{pop}^3	M_k^3	M_{pop}^4	M_k^4
1	0.5000	0.5003	0.5000	0.5010	0.7499	0.7527	1.5000	1.5048
2	0.7499	0.7502	1.0000	1.0023	1.8750	1.8826	4.5000	4.5148
3	0.9374	0.9372	1.5000	1.4992	3.2812	3.2787	9.0000	8.9785
4	1.0937	1.0930	2.0000	1.9964	4.9218	4.9017	15.0000	14.8648
5	1.2304	1.2300	2.5000	2.4975	6.7675	6.7529	22.5000	22.4084
6	1.3535	1.3537	3.0000	3.0043	8.7978	8.8340	31.5000	31.7730
7	1.4663	1.4665	3.5000	3.5019	10.9973	11.0109	42.0000	42.1298
8	1.5710	1.5693	4.0000	3.9905	13.3538	13.3086	54.0000	53.7901
9	1.6692	1.6716	4.5000	4.5112	15.8577	15.9202	67.5000	67.9509
10	1.7619	1.7637	5.0000	5.0089	18.5006	18.5418	82.5000	82.7077
11	1.8500	1.8489	5.5000	5.4907	21.2757	21.2321	99.0000	98.8811
12	1.9341	1.9343	6.0000	5.9989	24.1770	24.1542	117.0000	116.7288

Table 3.2: Estimator bias for equal rates.

k	$Bias(M_k^1)$	$Bias(M_k^2)$	$Bias(M_k^3)$	$Bias(M_k^4)$
1	-0.0003	-0.0010	-0.0027	-0.0048
2	-0.0002	-0.0023	-0.0076	-0.0148
3	0.0002	0.0007	0.0025	0.0214
4	0.0006	0.0035	0.0200	0.1351
5	0.0004	0.0024	0.0146	0.0915
6	-0.0002	-0.0043	-0.0361	-0.2730
7	-0.0002	-0.0019	-0.0136	-0.1298
8	0.0017	0.0094	0.0452	0.2098
9	-0.0024	-0.0112	-0.0625	-0.4509
10	-0.0018	-0.0089	-0.0411	-0.2077
11	0.0011	0.0092	0.0436	0.1188
12	-0.0001	0.0010	0.0228	0.2711

Table 3.3: Population moments versus sample moments for different rates.

k	M_{pop}^1	M_k^1	M_{pop}^2	M_k^2	M_{pop}^3	M_k^3	M_{pop}^4	M_k^4
1	0.083	0.083	0.015	0.015	0.004	0.004	0.002	0.002
2	0.137	0.137	0.035	0.035	0.013	0.013	0.006	0.006
3	0.186	0.186	0.060	0.060	0.027	0.027	0.015	0.015
4	0.233	0.233	0.090	0.090	0.046	0.046	0.029	0.029
5	0.280	0.280	0.125	0.125	0.073	0.073	0.051	0.051
6	0.327	0.327	0.165	0.165	0.106	0.106	0.082	0.082
7	0.374	0.374	0.210	0.210	0.148	0.148	0.124	0.124
8	0.421	0.421	0.260	0.260	0.199	0.199	0.179	0.179
9	0.469	0.469	0.315	0.315	0.260	0.260	0.250	0.250
10	0.517	0.516	0.375	0.375	0.331	0.331	0.338	0.338
11	0.565	0.565	0.440	0.440	0.414	0.414	0.447	0.447
12	0.613	0.613	0.510	0.510	0.508	0.508	0.579	0.580

Table 3.4: Estimator bias for different rates.

k	$Bias(M_k^1)$	$Bias(M_k^2)$	$Bias(M_k^3)$	$Bias(M_k^4)$
1	-0.00004	-0.00002	0.00000	0.00000
2	-0.00002	-0.00001	-0.00001	0.00000
3	0.00002	0.00002	0.00002	0.00002
4	-0.00001	0.00000	0.00002	0.00005
5	-0.00003	-0.00001	-0.00001	0.00000
6	0.00010	0.00010	0.00010	0.00020
7	0.00001	0.00002	0.00002	0.00001
8	0.00010	0.00010	0.00010	0.00010
9	-0.00003	-0.00004	-0.00010	-0.00010
10	0.00010	0.00010	0.00010	0.00010
11	-0.00010	-0.00003	0.00004	0.00020
12	-0.00001	-0.00002	-0.00010	-0.00040

Algoritmo 2: Monte Carlo Simulation for Bias and Moments Cost

Input: Arrival Rates: $\lambda_1 > 0$ and $\lambda_2 > 0$.
Arrival Time Vectors k; Power Vector a and Lag r.
M_p function for calculating the population moment.

Output: Matrices with bias and moment values

function Moments.Bias
- $Bias$:= null matrix with $n = length(k)$ lines and $m = length(a)$ columns;
- Mom:= null matrix with $n = length(k)$ lines and $m = 2 * length(a)$ columns.
- **for** $i \leftarrow 1$ *to length(n)* **do**
  - Generate n samples of $X \sim Gamma(\alpha_i = k_i + r, \lambda_1)$;
  - Generate n samples of $Y \sim Gamma(k_i, \lambda_2)$.
  - **for** $j \leftarrow 1$ *to* m **do**
    - $Mom[i, 2j-1] \leftarrow$ Apply the M_p function at the point $(a_j, \alpha_i, k_i, r, \lambda_1, \lambda_2)$
    - $Mom[i, 2j] \leftarrow$ Calculate the average $|X - Y|_j^a$
    - $Bias[i, j] \leftarrow \mathrm{Mom[i, 2j-1]} - \mathrm{Mom[i, 2j]}$
  - **end**
- **end**
- $Bias$; Mom
- **return** Bias matrix and Moments matrix

end

3.5.2 Graphic illustrations

For each a fixed, $M^a_{pop} = M^a_{pop}(\lambda_1, \lambda_2)$ is a bivariate function. In Figures 3.1 and 3.2, we show its behavior. These graphs were obtained with the aid of packages *ggplot2: Elegant Graphics for Data Analysis*, *gridExtra: Miscellaneous Functions for "Grid" Graphics* and *plot3D: Plotting Multi-Dimensional Data* of software R Core Team (2020), version 4.0.1 (June, 2020)

If $\lambda_1 = \lambda_2 = \lambda$, then the function $M^a_{pop} = M^a_{pop}(\lambda)$ is univariate. Figure 3.1 corresponds to the graph of $M^a_{pop}(\lambda)$ constructed from generating a sequence of 1000 points where $\lambda \in [0.1,\ 5]$. From this figure, the function decreases faster when a increases.

Graphically, we also show that the function $M^a_{pop} = M^a_{pop}(\lambda_1, \lambda_2)$ is more complex for different rates than for equal rates, because the graph has three dimensions. In order to illustrate the behavior of this function, we fixed the time at $k = 4$, $r = 0$ and allowed $\lambda_1,\ \lambda_2$ to vary in $[2, 6]$. The graphs of $M^a_{pop}(\lambda_1, \lambda_2)$, generated with 500 points, are shown in Figure 3.2.

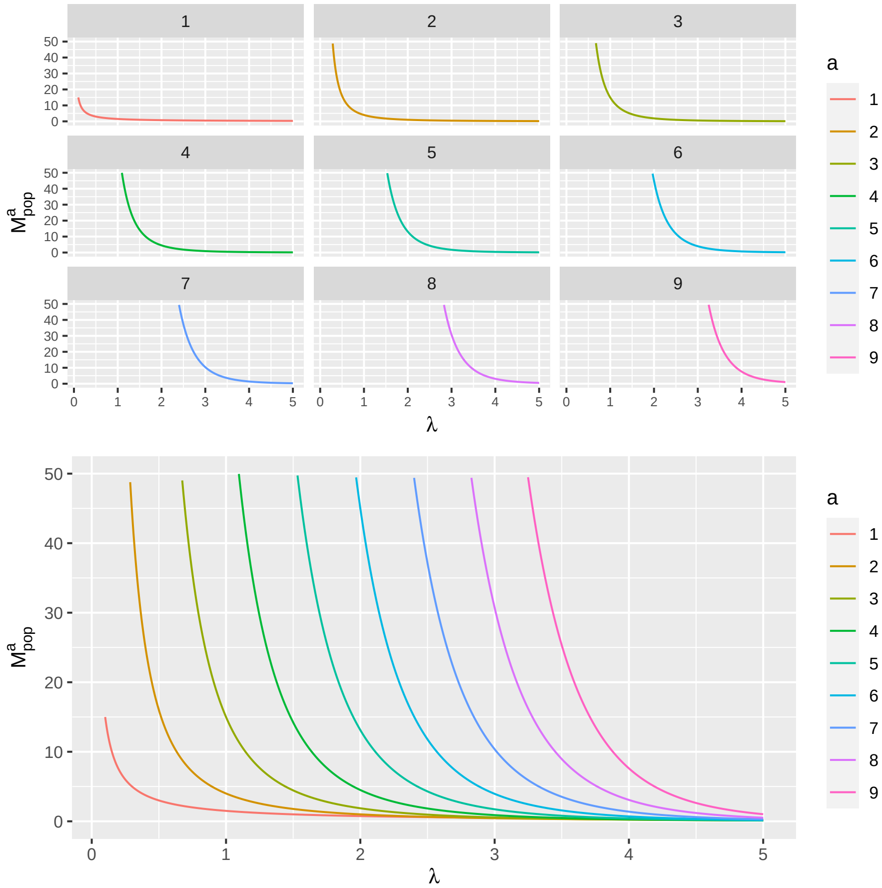


Figure 3.1: Graph of $M^a_{pop}(\lambda)$ for $\lambda \in [0.1,\ 5]$, $k = 4$ and $r = 0$.

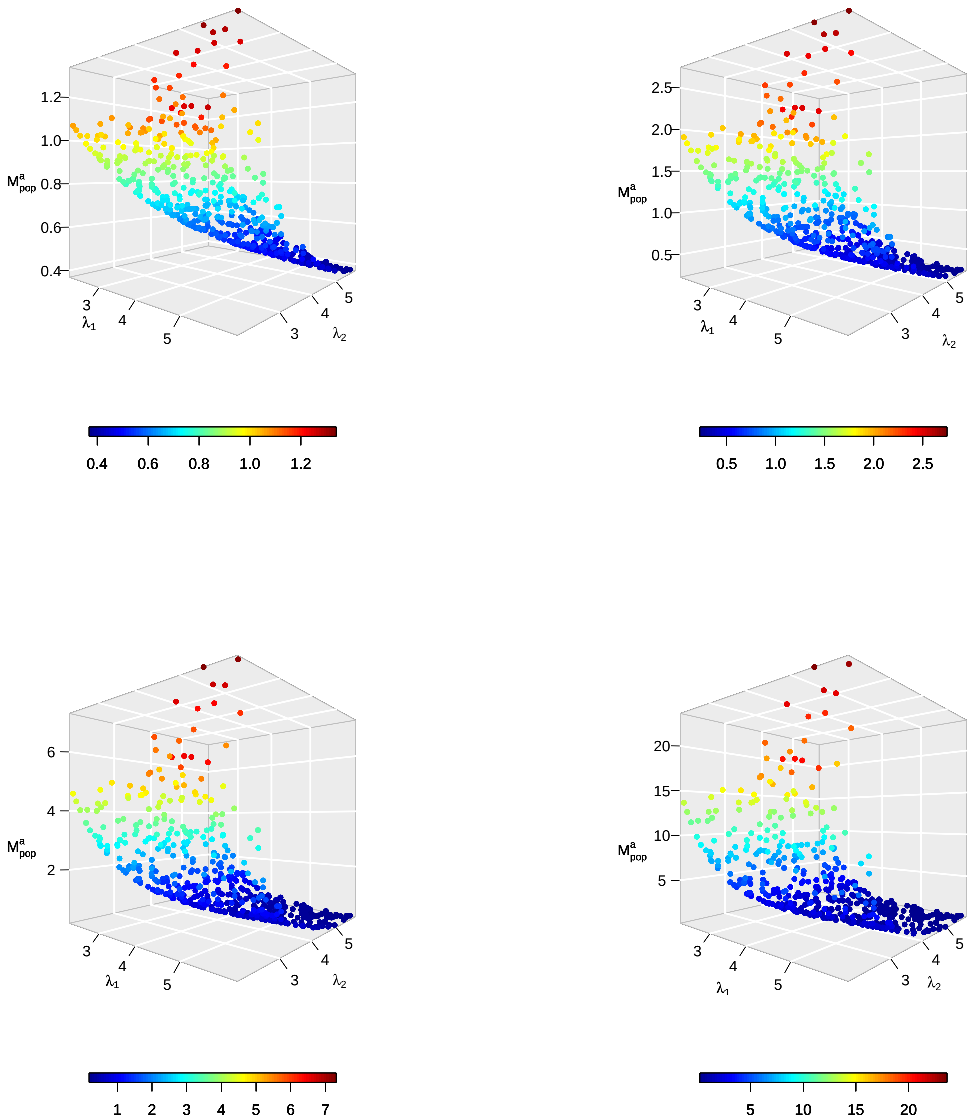


Figure 3.2: Graph of $M^a_{pop}(\lambda_1, \lambda_2)$ for $a = 1, 2, 3, 4$, $k = 4$, $r = 0$, and λ_1, λ_2 in $[2, 6]$.

Algoritmo 3: Generate 3D Graph for population moments

Input: $k \geq 1$ and $r \geq 0$ integers: $\lambda_1 > 0$ and $\lambda_2 > 0$.

λ_1 and λ_2, generate two random samples of $U(2,6)$ of size 500.

M_p function for calculating the population moment.

Output: 3D Graph of the $M^a_{pop}(\lambda_1, \lambda_2)$ function of two Poisson processes.

function 3DGraph

  $Data$:=null matrix with $n = length(\lambda_1)$ rows and $m = 4$ columns and a vector $a = (1, 2, 3, 4)$.

  for $i \leftarrow 1$ *to* n **do**

    for $j \leftarrow 1$ *to* 4 **do**

      $Data[i, j] \leftarrow$ Value of the M_p at the $(a_j, k, r, \lambda_{1_i}, \lambda_{2_i})$

    end

  end

  for $t \leftarrow 1$ *to* 4 **do**

    Apply the scatter3D function of the plot3D package

    in the arguments $(\lambda_1, \lambda_2, Data[*, t], \mathrm{phi} = 0, \mathrm{bty} = \text{"g"})$

  end

  return 3D Graph of the $M^a_{pop}(\lambda_1, \lambda_2)$

  Remark: $[*, t] :=$ row by row in column t of the data matrix

end

3.6 Conclusion

For independent and identically distributed Poisson processes with rate λ, as commented by Kapelko (2018), it is combinatorially challenging to derive the closed form formula for the $E|X_{k+r} - Y_k|^a$ when a is odd. However, when Fox's H-function is used, in this work, we show that the proof is direct. In the case that $\lambda_1 > 0$ and $\lambda_2 > 0$ are not necessarily the same, we present here an elegant proof for the expression of the absolute moment of the difference between the arrival times of the two independent Poisson processes. Our results are generalizations of Kapelko (2018). We also present numerical results and graphical illustrations of our results. One possible application is in calculation of the minimum transport cost in a network of two mobile sensors.

Appendix A

Resultados para o cálculo da integral I_2

A.1 Preliminares

Para uma nova solução da integral I_2, podem ser utilizados os Resultados $R1$, $R2$ e $R3$ abaixo, cujas demonstrações encontram-se no final deste apêndice.

- $R1$) **(Representação Integral da função Hipergeométrica Confluente)**

  A seguinte representação é válida

$$ {}_1F_1(d;c;x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int\limits_0^1 e^{xt}\, t^{d-1}(1-t)^{c-d-1}\, dt, \tag{A.1}$$

  onde $\mathcal{R}(c) > \mathcal{R}(d) > 0$.

- $R2$) **(Transformada de Laplace de uma função Hipergeométrica Confluente)**

  A seguinte identidade é válida

$$ \int\limits_0^\infty t^{\mu-1} e^{-xt}{}_1F_1(d;\; c\,;\; qt)\, dt = \Gamma(\mu) x^{-\mu}{}_2F_1(d,\mu;c;q/x). \tag{A.2}$$

  onde $\mathcal{R}(x) > 0$, $\mathcal{R}(x) > \mathcal{R}(q)$, $\mathcal{R}(\mu) > 0$ e ${}_2F_1$ é uma função hipergeométrica.

- $R3$) **(Caso Particular do Resultado $R2$, para parâmetros c e μ iguais)**

$$\int_0^\infty t^{\mu-1} e^{-xt} {}_1F_1(d;\, \mu\,; qt)\, dt = \Gamma(\mu) x^{-\mu} \left(1 - \frac{q}{x}\right)^{-d}, \tag{A.3}$$

onde $\mathcal{R}(x) > \mathcal{R}(q)$, $|x| > |q|$, $\mathcal{R}(\mu) > 0$ e ${}_1F_1$ é uma função hipergeométrica confluente.

Para o estudo das funções hipergeométrica ${}_2F_1$ e hipergeométrica confluente ${}_1F_1$, adotamos a referência Oliveira (2012). Nesta, a primeira função encontra-se no Capítulo 5 e, a segunda, no Capítulo 8.

A.2 Demonstração dos Resultados

A.2.1 Resultado $R1$

A seguinte representação é válida

$${}_1F_1(d; c; x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 e^{xt}\, t^{d-1} (1-t)^{c-d-1}\, dt, \tag{A.4}$$

onde $\mathcal{R}(c) > \mathcal{R}(d) > 0$.

Proof. Da representação integral associada à função Hipergeométrica:

$${}_2F_1(d, b; c; z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^\infty t^{b-1} (1-t)^{c-b-1} (1-zt)^{-d}\, dt, \tag{A.5}$$

com $\mathcal{R}(c) > \mathcal{R}(b) > 0$, decorre que, considerando a mudança de variável $z = x/b$,

$${}_2F_1(d, b; c; x/b) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \left(1 - \frac{xt}{b}\right)^{-d} dt. \tag{A.6}$$

Agora, tomando o limite para $b \to \infty$, em A.6, obtemos

$$ {}_1F_1(d;c;x) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^1 e^{xt}\, t^{d-1}(1-t)^{c-d-1}\, dt. \tag{A.7}$$

□

A.2.2 Resultado $R2$

Mostre que

$$ \int_0^\infty t^{\mu-1} e^{-xt} {}_1F_1(d;\, c\,;\, qt)\, dt = \Gamma(\mu) x^{-\mu} {}_2F_1(d\,,\, \mu\,;\, c\,;\, q/x). \tag{A.8}$$

onde $\mathcal{R}(x) > 0$, $\mathcal{R}(x) > \mathcal{R}(q)$, $\mathcal{R}(\mu) > 0$ e ${}_2F_1$ é uma função hipergeométrica.

Proof. Para a prova, utilizaremos as representações em séries de potência para as funções hipergeométrica confluente e hipergeométrica.

Com efeito, pela representação em séries de potência, decorre que

$$\int_0^\infty t^{\mu-1}e^{-xt}{}_1F_1(d;\,c\,;\,qt)\,dt = \int_0^\infty t^{\mu}e^{-xt}\left\{\sum_{n=0}^{\infty}\frac{d^{(n)}}{c^{(n)}}\frac{(qt)^n}{n!}\right\}dt \tag{A.9}$$

$$= \frac{\Gamma(c)}{\Gamma(d)}\int_0^\infty t^{\mu-1}e^{-xt}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(d+n)}{\Gamma(c+n)}\frac{(qt)^n}{n!}\right\}dt \tag{A.10}$$

$$= \frac{\Gamma(c)}{\Gamma(d)}\sum_{n=0}^{\infty}\left\{\frac{\Gamma(d+n)}{\Gamma(c+n)}\frac{q^n}{n!}\underbrace{\int_0^\infty t^{\mu+n-1}e^{-xt}\,dt}_{Nucleo\ Gama(\mu+n,x)}\right\} \tag{A.11}$$

$$= \frac{\Gamma(c)}{\Gamma(d)}\sum_{n=0}^{\infty}\left\{\frac{\Gamma(d+n)q^n}{\Gamma(c+n)}\,\frac{\Gamma(\mu+n)}{x^{\mu+n}}\right\} \tag{A.12}$$

$$= \Gamma(\mu)x^{-\mu}\underbrace{\frac{\Gamma(c)}{\Gamma(d)\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{\Gamma(d+n)\Gamma(\mu+n)}{\Gamma(c+n)}\frac{(q/x)^n}{n!}}_{{}_2F_1(d\,,\,\mu\,;\,c\,;\,q/x)} \tag{A.13}$$

$$= \Gamma(\mu)x^{-\mu}{}_2F_1(d\,,\,\mu\,;\,c\,;\,q/x), \tag{A.14}$$

em que a igualdade na Eq.(A.27) é justificada pela representação em séries de potência da função hipergeométrica confluente e, a passagem da Eq.(A.31) para a Eq.(A.32), pela representação em séries de potência da hipergeométrica. □

A.2.3 Resultado $R3$

Mostre que

$$\int_0^\infty t^{\mu-1}e^{-xt}{}_1F_1(d;\,\mu\,;qt)\,dt = \Gamma(\mu)x^{-\mu}\left(1-\frac{q}{x}\right)^{-d}, \tag{A.15}$$

onde $\mathcal{R}(x) > \mathcal{R}(q)$, $|x| > |q|$, $\mathcal{R}(\mu) > 0$ e ${}_1F_1$ é uma função hipergeométrica confluente.

Proof. De fato, da representação em séries para função hipergeométrica confluente, tem-se

$$\int_0^\infty t^{\mu-1} e^{-xt} {}_1F_1(d;\, \mu\,; qt)\, dt = \frac{\Gamma(\mu)}{\Gamma(d)} \int_0^\infty t^{\mu-1} e^{-xt} \left\{ \sum_{n=0}^\infty \frac{\Gamma(d+n)}{\Gamma(\mu+n)} \frac{(qt)^n}{n!} \right\} dt \tag{A.16}$$

$$= \frac{\Gamma(\mu)}{\Gamma(d)} \sum_{n=0}^\infty \left\{ \frac{\Gamma(d+n)}{\Gamma(\mu+n)} \frac{q^n}{n!} \int_0^\infty t^{n+\mu-1} e^{-xt}\, dt \right\} \tag{A.17}$$

$$= \frac{\Gamma(\mu)}{\Gamma(d)} \sum_{n=0}^\infty \left\{ \frac{\Gamma(d+n)}{\Gamma(\mu+n)} \frac{q^n}{n!} \frac{\Gamma(\mu+n)}{x^{\mu+n}} \right\} \tag{A.18}$$

$$= \Gamma(\mu) x^{-\mu} \sum_{n=0}^\infty \frac{\Gamma(d+n)}{\Gamma(d)} \frac{(q/x)^n}{n!} \tag{A.19}$$

$$= \Gamma(\mu) x^{-\mu} \sum_{n=0}^\infty \binom{n+d-1}{n} (q/x)^n \tag{A.20}$$

$$= \Gamma(\mu) x^{-\mu} \left(1 - \frac{q}{x}\right)^{-d}, \tag{A.21}$$

em que, na passagem da Eq.(A.38) para Eq.(A.39), usamos a identidade combinatória

$$(1-h)^{-d} = \sum_{n\geq 0} \binom{n+d-1}{d-1} h^n. \tag{A.22}$$

□

Appendix B

programming code in R

```
#==================================================================#
#=========== Custo  Minimo Esperado (Kranakis [2014]) ============#
#==================================================================#
## Funcao Distancia Esperada
E0<-function(k,l){(k*2^(-2*k+1)/l)*choose(2*k,k)}
## Grafico para o Custo Minimo Esperado
Graf_Int.Custo0<-function(n,l){
                L<-length(n)
                custo<-LI<-LS<-numeric(L)
                for (j in 1:L){
                     LI[j]<-(n[j]*n[j]^0.5)/(2*exp(1)*(2*pi)^0.5*l)
                     LS[j]<-(2*exp(1/24)*n[j]*n[j]^0.5)/(pi^0.5*l)
                     custo[j]<-sum(E0(1:n[j],l))
                     }
   plot(n,custo,type="l", # Plote do Custo
       main="Custo Minimo Esperado",
       ylab=expression(C(n,lambda)))
   lines(n,LI,col="red")  # Limite Inferior
```

```
   lines(n,LS,col="blue") # Limite Superior
}
## Plote dos Graficos
l<-c(0.35,0.55,0.75,0.95) ; x11() ; par(mfrow=c(2,2))
for (i in 1:4){
  Graf_Int.Custo0(v,l[i])
  legend("topleft", legend=bquote(lambda==.(l[[i]])) )
}
#================================================================#
#====================== CUSTO GENERALIZADO ========================#
#================================================================#
## Funcao Distancia Esperada
E<-function(k,l1,l2){
 x<-l1/(l1+l2)
 res<-k*(1/l1-1/l2)+2*k*((1/l2)*pbeta(x,k,k+1)-(1/l1)*pbeta(x,k+1,k))
 return(res)
}
## Grafico para o Custo Generalizado
Graf_Int.Custo<-function(n,l1,l2){
              L<-length(n) ; x<-l1/(l1+l2) ; y<-1-x
              custo<-LI<-LS<-numeric(L)
              for (i in 1:L) {
                   LI[i]<-(n[i]*(n[i]+1)/2)*(1/l1-1/l2)+(2/l2)*
                          sum((x*y)^(1:n[i])/beta(1:n[i]+1,1:n[i]))
                   LS[i]<-(n[i]*(n[i]+1)/2)*(1/l2-1/l1)+(1/l2+1/l1)*
                          sum(((x*y)^(1:n[i]))/beta(1:n[i]+1,1:n[i]))
                   custo[i]<-sum(E(1:n[i],l1,l2))
                   }
```

```
  plot(n,custo,type="l",main="Custo Minimo Esperado",
        ylab=expression(C(n,lambda[1],lambda[2])))
  lines(n,LI,col="red") ; lines(n,LS,col="blue")
}
## Plote dos Graficos
v<-seq(10,100,1)
l1<-rep(0.95,4)
l2<-c(0.90,0.92,0.94,0.95)
x11(); par(mfrow=c(2,2))
for (i in 1:4) {
     Graf_Int.Custo(v,l1[i],l2[i])
     legend("topleft", legend=bquote(lambda[1]==.(l1[[i]])~","~
           lambda[2]==.(l2[[i]])) )
}
#=================================================================#
#===========DISTANCIA ESPERADA E CUSTO TEORICO (EXATO)===========#
#=================================================================#
E<-function(k,l1,l2){
      x<-l1/(l1+l2)
     res<-k*(1/l1-1/l2)+2*k*((1/l2)*pbeta(x,k,k+1)-(1/l1)*pbeta(x,k
        +1,k))
     return(res)
}
Custo_teorico<-function(n,l1,l2){
  custo<-numeric(length(n))
  for (j in 1:length(n)){
        custo[j]<-sum(E(1:n[j],l1,l2))}
return(custo)
```

```
}
#============================================================#
#=============== GRAFICO CUSTO TEORICO ======================#
#============================================================#
G_Int<-function(n,l1,l2){
  L<-length(n)
  x<-l1/(l1+l2)
  y<-1-x
  custo<-LI<-LS<-numeric(L)
  for (i in 1:L) {
    LI[i]<-(n[i]*(n[i]+1)/2)*(1/l1-1/l2)+(2/l2)*
    sum((x*y)^(1:n[i])/beta(1:n[i]+1,1:n[i]))
   LS[i]<-(n[i]*(n[i]+1)/2)*(1/l2-1/l1)+(1/l2+1/l1)*
   sum(((x*y)^(1:n[i]))/beta(1:n[i]+1,1:n[i]))
   custo[i]<-sum(E(1:n[i],l1,l2))
 }
plot(n,custo,type="l",main="Expected Minimum Cost",
     ylab=expression(C[opt](n,lambda[1],lambda[2])),
     cex.main=0.75,cex=0.4,cex.lab=.7,cex.axis=0.7)
lines(n,LI,col="red") ; lines(n,LS,col="blue")
}
v<-seq(10,100,1)
l1<-rep(0.95,4)
l2<-c(0.90,0.92,0.94,0.95)
x11(width = 3.8, height=4.5)
par(mfrow=c(2,2))
for (i in 1:4) {
  G_Int(v,l1[i],l2[i])
```

```
 legend("topleft", legend=bquote(lambda[1]==.(l1[[i]]) ~ "," ~lambda
    [2]==.(l2[[i]])),cex=0.42)
 legend("bottomright",inset=0.02,legend=c("Upper Bound","Exact Cost",
    "Lower Bound"),col=c("blue","black","red"),lty=c(1,1),cex=0.33)
}
## GGPLOT2
Int<-function(n,l1,l2){
       L<-length(n) ; x<-l1/(l1+l2) ; y<-1-x
       custo<-LI<-LS<-numeric(L)
   for (i in 1:L) {
      LI[i]<-(n[i]*(n[i]+1)/2)*(1/l1-1/l2)+(2/l2)*
      sum((x*y)^(1:n[i])/beta(1:n[i]+1,1:n[i]))
      LS[i]<-(n[i]*(n[i]+1)/2)*(1/l2-1/l1)+(1/l2+1/l1)*
      sum(((x*y)^(1:n[i]))/beta(1:n[i]+1,1:n[i]))
      custo[i]<-sum(E(1:n[i],l1,l2))
}
df<-data.frame(Linf=LI, Custo=custo, Lsup=LS)
     return(df)
}
v<-seq(10,100,1)
l1<-rep(0.95,4)
l2<-c(0.90,0.92,0.94,0.95)
list.df<-list()
for (i in 1:4) {
  list.df[[i]]<-Int(v,l1[i],l2[i])
}
library(ggplot2)
graf<-list()
```

```
df.new<-list()
for (i in 1:4){
  l<-l1[i] ; u<-l2[i]
  df.new[[i]]<-data.frame(v,list.df[i])
}
p2<-ggplot(df.new[[2]], aes(x=v,y=Custo))+
  geom_line(aes(colour="Exact Cost"))+
  geom_line(aes(x=v, y=Linf, colour="Lower Bound"))+
  geom_line(aes(x=v, y=Lsup, colour="Upper Bound"))+
  scale_colour_manual('',values=c("Exact Cost"="black","Lower Bound"=
     "red","Upper Bound"="blue"))+
  labs(x="n",
       y="Cost",
       title=expression(paste(lambda[1]==0.95,"and ",lambda[2]==0.92)
          ))+
  theme(legend.background=element_rect(fill = alpha("white", 0)),
        legend.key=element_rect(fill = alpha("white", .5)))+
  theme(legend.position = c(0.81, 0.16),
        legend.text = element_text(size = 6),
        legend.background = element_rect(size=0.3),
        plot.title = element_text(size=8,hjust=0.5),
        axis.text = element_text(size = 7),
        axis.title = element_text(size= 7))
library(gridExtra)
n <- length(graf)
nCol <- floor(sqrt(n))
do.call("grid.arrange", c(graf, ncol=nCol))
#install.packages("ggpubr",dependencies=TRUE)
```

```
library(ggpubr)
gg<-ggarrange(p1, p2, p3, p4, ncol=2, nrow=2, common.legend = TRUE,
    legend="bottom")
geom_point(shape=1,size=0.8,
           aes(colour="Sample Cost",shape="Sample Cost",linetype="
              Sample Cost"))+
           stat_function(fun=function(x) C_Teo,
           aes(colour="Expected Cost",shape="Expected Cost",linetype=
              "Expected Cost"))+
  scale_colour_manual('',values=c("Sample Cost"="black","Expected
     Cost"="red"))+
  scale_shape_manual('',values=c("Sample Cost"=1,"Expected Cost"=NA))
     +
  scale_linetype_manual('',values=c("Sample Cost"=0,"Expected Cost"
     =1))+
  labs( x="m ( n=10 )", y="Cost", title = "Convergence of Sample
     Minimum Cost")+
  theme(legend.position = c(0.81, 0.16),
        legend.text = element_text(size = 5),
        legend.title = element_blank(),
        legend.background = element_rect(size=0.3),
        plot.title = element_text(size=8,hjust=0.5),
        axis.text = element_text(size = 7),
        axis.title = element_text(size= 7))
#==================================================================#
#================DISTRIBUICAO EMPIRICA DO CUSTO=====================#
#==================================================================#
## R Base
```

```
n<-100:120 ; l1<-c(2,4,6,8) ; l2<-c(2,3,5,7)
set.seed(12345) ; par(mfrow=c(2,2))
for (i in 1:4) {
  plot.ecdf(Custo_teorico(n,l1[i],l2[i]),main="Distribuicao Empirica do Custo Esperado")
  legend("topleft",legend=bquote(lambda[1]==.(l1[[i]]) ~ "," ~lambda[2]==.(l2[[i]])) )
}
Var_Custo.est<-function(n,m,l1,l2){(n*(n+1)/(2*m))*(1/l1^2+1/l2^2)}
set.seed(12345)
par(mfrow=c(1,1))
l1<-0.55 ; l2<-0.95 ; m<-40:2000
Custo.est<-function(n,m,l1,l2) {
  # l1: taxa do processo 1
  # l2: taxa do processo 2
  # n: quantidade de amostras de tempo; (n primeiras chegadas)
  # m: tamanho de cada amostra do tempo X_i
  Custo<-numeric(length(n))
  for (j in 1:length(n)){
     mat.X<-matrix(0,nrow=n[j],ncol=m)
     mat.Y<-matrix(0,nrow=n[j],ncol=m)
     for (i in 1:n[j]){
       mat.X[i,]<-rgamma(m,i,l1) # m tempos do 1o processo
       mat.Y[i,]<-rgamma(m,i,l2) # m tempos do 2o processo
    }
  Custo[j]<-sum(rowMeans(abs(mat.X-mat.Y)))
  }
return(Custo)
```

```
}
#================================================================#
#============== CONVERGENCIA DO CUSTO ===========================#
#================================================================#
C_est<-sapply(m,Custo.est,n=10,l1,l2)
C_Teo<-Custo_teorico(10,l1,l2)
### Grafico no R Base
x11()
plot(m,C_est,xlab="m (n=10)",ylab="Cost",cex.axis=0.6,cex.lab=.7,
     mgp=c(1.2,0.5,0), cex=0.6, tck=0.02,)
legend("bottomright",inset=0.02,legend=c("Expected Cost","Sample Cost
   "),
       col=c("red","black"),lty=c(1,1),cex=0.4)
abline(h=C_Teo,col="red")
title(main="Convergence of Sample Minimum Cost",cex.main=0.62)
### Grafico no ggplot2
library(ggplot2)
df2<-data.frame(m,C_est)
g2<-ggplot(data=df2, aes(x=m, y=C_est))+
     geom_point(shape=1,size=0.8,aes(colour="Sample Cost",shape="
        Sample Cost",linetype="Sample Cost"))+
                stat_function(fun=function(x) C_Teo,aes(colour="
                   Expected Cost",shape="Expected Cost",linetype="
                   Expected Cost"))+
                scale_colour_manual('',values=c("Sample Cost"="black"
                   ,"Expected Cost"="red"))+
                scale_shape_manual('',values=c("Sample Cost"=1,"
                   Expected Cost"=NA))+
```

```
                 scale_linetype_manual('',values=c("Sample Cost"=0,"
                    Expected Cost"=1))+
                 labs( x="m ( n=10 )", y="Cost", title = "Convergence
                    of Sample Minimum Cost")+
                 theme(legend.position = c(0.81, 0.16),
                          legend.text = element_text(size = 5),
                          legend.title = element_blank(),
                          legend.background = element_rect(size=0.3),
                          plot.title = element_text(size=8,hjust=0.5),
                          axis.text = element_text(size = 7),
                          axis.title = element_text(size= 7))
#===================================================================#
#=================CUSTO ESPERADO x CUSTO AMOSTRAL===================#
#===================================================================#
l1<-0.95 ; l2<-0.90
n<-100:130
CUSTO<-sapply(n,Custo_teorico,l1=l1,l2=l2)
set.seed(12345)
CUSTO.est<-sapply(n,Custo.est,m=100,l1=l1,l2=l2)
plot(n,CUSTO.est,type="l",ylab="Custo",xlab="n (m=100)")
lines(n,CUSTO,col="red")
title(main="Custo Esperado X Custo Amostral")
legend("topleft", expression(lambda[1]~"= 0.95, "~lambda[2]~" = 0.90"
   ))
legend("bottomright",inset=0.02,legend=c("Custo Esperado","Custo
   Amostral"),
       col=c("red","black"),lty=c(1,1),cex=0.8)
x11(width=3.8,height=2.9)
```

```
plot(n,CUSTO.est,pch=1,ylab="Cost",xlab="n  (m=100)",
     main = "Expected Cost X Sample Cost ",
     cex.lab= 0.7,
     cex.axis=0.5,
     cex.main=0.7,
     mgp=c(1.2,0.5,0), cex=0.6,
     )
lines(n,CUSTO, col="red")
df1<-data.frame(n,CUSTO,CUSTO.est)
dev.off()
g1<-ggplot(data=df1, aes(x=n, y= CUSTO.est))+
  geom_point(shape=1,size=1,aes(colour="Sample Cost",shape="Sample
     Cost",linetype="Sample Cost"))+
  geom_line(aes(x=n, y= CUSTO, shape="Expected Cost", colour="
     Expected Cost",linetype="Expected Cost"))+
  scale_colour_manual('',values=c("Sample Cost"= "black","Expected
     Cost"="red"))+
  scale_shape_manual('',values=c("Sample Cost"=1,"Expected Cost"= NA)
     )+
  scale_linetype_manual('',values=c("Sample Cost"=0,"Expected Cost"
     =1))+
  labs( x="n ( m=100 )", y="Cost", title = "Expected Cost X Sample
     Cost")+
  theme(legend.position = c(0.81, 0.16),
          legend.text = element_text(size = 5),
          legend.title = element_blank(),
          legend.background = element_rect(size=0.3),
          plot.title = element_text(size=8,hjust=0.5),
```

```
          axis.text = element_text(size = 7),
          axis.title = element_text(size= 7))
setwd("C:/Users/adolfoamds/Documents")
path<-getwd()
ggsave(filename="convergen.pdf", plot=g1, device="pdf",
       path=path, height=3.0, width=3.2, units="in",dpi=500)
#================================================================#
#===================== GRAFICO AJUSTE DA NORMAL ================#
#================================================================#
Q_Pivo<-function(n,m,l1,l2){
  res<-(Custo.est(n,m,l1,l2) - Custo_teorico(n,l1,l2))/sqrt(Var_Custo
     .est(n,m,l1,l2))
  return(res)
}
Qtd_Pivotal<-sapply(100:1000,Q_Pivo,m=100,l1=0.95,l2=0.90)
#install.packages("fitdistrplus")
library(fitdistrplus)
ajuste_normal<-fitdist(Qtd_Pivotal,"norm")
x11(width = 3.8, height=5)
plot(ajuste_normal,cex.main=0.75,cex=0.4,cex.lab=.7,cex.axis=0.7)
#install.packages("ggplot2") ; install.packages("gridExtra")
library(ggplot2)
## Intervalo de Confianca ao nivel de 95%
Intervalo<-function(n,m,l1,l2){
 int<-Custo.est(n,m,l1,l2)+c(-1,1)*qnorm(0.975)*sqrt(Var_Custo.est(n,
    m,l1,l2))
return( int )
}
```

```
set.seed(12345) ; n<-1200:1250
Int<-sapply(n,Intervalo,m=100,l1=0.95,l2=0.9)
LI<-Int[1,]
LS<-Int[2,]
CUSTO<-sapply(n,Custo_teorico,l1=0.95,l2=0.90)
df0 <- data.frame( n = n,
                   custo = CUSTO,
                   li   =   LI,
                   ls = LS
)
grafico0<-ggplot(df0, aes(n, custo)) +
  geom_point() +
  geom_line() +
  geom_errorbar(aes(ymin = li, ymax = ls)) +
  labs(x = "n ( m=100 )",
       y = "Cost",
       title = "Expected Minimum Cost and 95% CI") +
  theme( plot.title = element_text(size=11) )
path<-getwd()
ggsave(filename="gg-formatted.png", plot=grafico0, device="png",
       path=path, height=3.5, width=4, units="in", dpi=500)
N<-rep(n,3)
Custo<-c(LS,CUSTO,LI)
Legenda<-rep( c("Upper Bound", "Expected Cost","Lower Bound"),each=
    length(n))
df<-data.frame(n=N,Legenda,Custo)
grafico1<-ggplot(df, aes(x = n, y = Custo)) +
  geom_line(aes(color = Legenda, linetype = Legenda)) +
```

```
  scale_color_manual(values = c("red", "black","blue"))+
  labs( x="n  ( m=100 )", y="Cost",
        title = "Expected Minimum Cost and 95% CI")+
  theme(legend.position = c(0.80, 0.22),
        legend.text = element_text(size = 6),
        legend.title = element_blank(),
        legend.background = element_rect(size=0.3),
        plot.title = element_text(size=8,hjust=0.5),
        axis.text = element_text(size = 7),
        axis.title = element_text(size= 7))
path<-getwd()
ggsave(filename="Afvbgg.pdf", plot=grafico1, device="pdf",
       path=path, height=3, width=3.5, units="in",dpi=500)
gridExtra::grid.arrange(grafico0, grafico1, ncol=2)
INT <- function(n,m,l1,l2,cl) {
  # cl: nivel de confianca
  alpha <- 1-cl/100
  # CI para Custo Teorico (Esperado)
  dp_am <-sqrt(Var_Custo.est(n,m,l1,l2))     # desvio padrao amostral
  z_s <- qnorm(1 - alpha/2)                  # quantil da normal
  li <- Custo.est(n,m,l1,l2) - z_s*dp_am     # limite inferior
  ls <- Custo.est(n,m,l1,l2) + z_s*dp_am     # limite superior
  c(limite_Inferior=li, limite_superior=ls)
}
# Gerar N vezes os Intervalos para Custo Teorico
simulacao <- function(N,n,m,l1,l2,cl){
  set.seed(123)
  sim<-t(replicate(N, INT(n,m,l1,l2,cl)))
```

```
  return(list("Qtd de Intervalos que Contem o Custo"=
                N*mean(sim[,1] <= Custo_teorico(n,l1,l2) &
                sim[,2]>= Custo_teorico(n,l1,l2)),"Total de
                   Intervalos"=N))
}
#================= CHAPTER 3 ====================================#
#=============== a-th moment by Kapelko (2017) =================#
#===============================================================#
E<-function(a,k,l) {
  nnn(gamma(a+1)/l^a)*( gamma(a/2+k) / (gamma(k)*gamma(a/2+1)) ) }
#================================================================#
#=============Gauss hypergemetric function=======================#
#================================================================#
## used packages: "gsl"
#install.packages("gsl")
library(gsl)
Gauss2F1 <- function(a,b,c,z){
  if(z>=0 & z<1){
    hyperg_2F1(a,b,c,z)
  }else{
    hyperg_2F1(a,c-b,c,1-1/(1-z))/(1-z)^a
  }
}
#===================================================================#
#======================== Pochhammer polynomial ====================#
#===================================================================#
p.c <- function( x, q ){
  if ( q < 0 )
```

```
    stop( "q e negativo" )
  else if ( q == 0 )
    return ( 1 )
  else {
    res <- 1
    for ( i in 1:q ) {
      res <- res * ( x + i - 1 )
        }
      return ( res )
    }
  return ( NULL )
}
#==================================================================#
#==================generalized moment function=====================#
#==================================================================#
## Parameters:
# a := order of the moment
# k := arrival time order
# r := lag between times
# l1 := first process arrival rate
# l2 := second process arrival rate
Mg<-function(a,k,r,l1,l2){
 I_1<-NULL ; I_2<-NULL
 for (i in 1:length(l1)) {
  f<-function(j){(factorial(a)*(-1)^a/l2[i]^a)*(p.c(k+r,j)*p.c(k,a-j)
     /(factorial(j)*factorial(a-j)))*(-l2[i]/l1[i])^j}
    j<-0:a
  I_1[i]<-sum(sapply(j,f))
```

```
  I_2[i]<-((l1[i]/l2[i])^(k+r)*l2[i]^(-a)*gamma(a+1)*(-1)^(a)*gamma(a
     +2*k+r)/(gamma(k)*gamma(1+k+r+a)))*Gauss2F1(a+2*k+r,k+r,1+k+r+a
     ,-l1[i]/l2[i])
  }
  return(I_1-2*I_2*ifelse(a%%2==1,1,0))
}
#==========================================================================#
#======programming code for corollaries 1 and 2 (Kapelko (2017)====#
#==========================================================================#
## Parameters:
# a := order of the moment
# k := arrival time order
# l := rate of arrival
E1<-function(a,k,l){
  res<-NULL
 for (i in 1:length(l)){
   res[i]<-(gamma(a+1)/l[i]^a)*(gamma(a/2+k)/(gamma(k)*gamma(a/2+1)))
  }
 return(res)
}
#========================================================================#
#===================programming code for graphics=================#
#===============plot function for graphic 1=======================#
## used packages: "gridExtra" ; "ggplot2" ; "Plot3D"
#install.packages("gridExtra") ; install.packages("ggplot2")
library(ggplot2) ; require(gridExtra)
## Parameters:
# n:= n first moments
```

```
# k:= arrival time order
# l_in  :=  start rate
# l_fin :=  end rate
# comp  := number of rates between l_in and l_fim
# curve := 0 (detach); 1 (joins)
g2<-function(n, k, l_in, l_fim, comp, curve){
      v<-seq(l_in,l_fim,length.out=comp)
      a<-(1:n)
      arg3<-rep(v,length(a))
      arg1<-rep(a,each=length(v))
      arg2<-rep(k,length(a)*length(v))
      y<-mapply(E2,arg1,arg2,0,arg3,arg3) ; arg1<-as.factor(arg1)
      dd<-data.frame(x=arg3,y=y,a=arg1)
   if( curva==0) {
     ggplot(data=dd,aes( x=x, y=y, color=a)) +
      geom_line()+ facet_wrap(~ a)+ylim(0,50) +
      labs(x=expression(lambda),y=expression(M[pop]^a)) +
      theme(plot.title=element_text(size=7, face="bold",
            color="black",hjust=0.5),
            axis.text=element_text(size=7))
    } else {
    ggplot(data=dd,aes( x=x, y=y, color=a)) +
      geom_line()+ylim(0,50)+
      labs(x=expression(lambda),y=expression(M[pop]^a))+
      theme(plot.title=element_text(size=9,face="bold",
      color="black",hjust=0.5))}
  }
x11()
```

```
grid.arrange(g2(9,2,0.1,5,1000,0) , g2(9,2,0.1,5,1000,1),nrow=2)
#=============== plot function for graphic 2 =================#
### used packages: "plot3D"
install.packages("plot3D") ; library(plot3D)
l1<-sort(runif(100,2,5)) ;  l2<-l1
dd<-data.frame(x=l1,y=l2,e1=E2(1,4,0,l1,l2),e2=E2(2,4,0,l1,l2),
e3=E2(3,4,0,l1,l2),e4=E2(4,4,0,l1,l2) )
par(mfrow=c(2,2))
for (i in 3:6){
   x11()
     scatter3D(l1,l2,dd[,i],phi=0,type="h",bty="g",ticktype="detailed"
        ,pch=19,cex=0.5,main="",xlab="",ylab="",zlab= "", colkey =
        list(length = 0.5, width = 0.5, cex.clab = 0.5, side=1))
      text3D(-14,7,0.7, labels = expression(M[pop]^a), add = TRUE)
      text3D(3.5,1.2,0.4, labels = expression(lambda[1]), add = TRUE)
      text3D(5.7,3.0,0.4, labels = expression(lambda[2]), add = TRUE)
}
#==================plot function for graphic 3===============#
### used packages: "Plot3D"
library(plot3D)
l1<-runif(500,2,6) ;  l2<-runif(500,2,6)
dd<-data.frame(x=l1,y=l2,e1=E2(1,4,0,l1,l2),e2=E2(2,4,0,l1,l2),
               e3=E2(3,4,0,l1,l2),e4=E2(4,4,0,l1,l2) ) ; par(mfrow=c
                  (2,2))
for (i in 3:6){
  x11()
  scatter3D(l1,l2,dd[,i],phi=0, type="p",bty="g",ticktype="detailed",
            pch=19,cex=0.7,main="",xlab="",ylab="",zlab="",
```

```
            colkey = list(length = 0.5, width = 0.5, cex.clab =0.5,
               side=1))
  text3D(-14,7,0.7, labels = expression(M[pop]^a), add = TRUE)
  text3D(5.0,0.8,0.4, labels = expression(lambda[1]), add = TRUE)
  text3D(6.5,2.0,0.4, labels = expression(lambda[2]), add = TRUE)
}
#============= programming code for simulation =========#
## Parameters:
# n := sample size
# a := order of the moment
# k := arrival time order
# r := lag between times
# l1 := first process arrival rate
# l2 := second process arrival rate
simula <- function(n,a,k,r,l1,l2){
            Dif<-matrix(0,nrow=length(k),ncol=length(a))
            Mom<-matrix(0,nrow=length(k),ncol=2*length(a))
            colnames(Dif)<-rep("",length(a)) ; colnames(Mom)<-rep(""
               ,2*length(a))
              for (i in 1:length(k)){
                 X<-rgamma(n,k[i]+r[i],l1) ;  Y<-rgamma(n,k[i],l2)
                    for (j in 1:length(a)){
                        Mom[i,2*j-1]<-E2(a[j],k[i]+r[i],0,l1,l2)
                        Mom[i,2*j]<-mean((abs(X-Y))^a[j])
                        Dif[i,j]<- Mom[i,2*j-1] - Mom[i,2*j]
                        colnames(Dif)[j]<-paste("D",j)
                        colnames(Mom)[2*j-1]<-paste("Mp",j)
                        colnames(Mom)[2*j]<-paste("Ma",j)
```

```
            }
      }
  return(list(Moments=cbind(k,r,trunc(Mom,prec=4)),
                Moments_Difference=cbind(k,r,trunc(Dif,prec=4))))
}
## Simulation 1
simula(1000000,1:4,1:12,rep(0,12),2,2)
## Simulation 2
simula(10000000,1:4,1:12,rep(0,12),10,20)
```

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