Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Existence and concentration of solutions for a class of quasilinear problems

by

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Brasília

2021

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por

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Tese apresentada ao Departamento de Matemática da Universidade de Brasília como parte dos requisitos necessários para obtenção do grau de

DOUTOR EM MATEMÁTICA

18 de fevereiro de 2021

Comissão Examinadora:

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*O autor foi bolsista CAPES e CNPq durante a elaboração deste trabalho.

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Documento assinado eletronicamente por **Ricardo Ruviaro, Chefe do Departamento de Matemática do Instituto de Ciências Exatas.**, em 19/02/2021, às 15:49, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

Documento assinado eletronicamente por **Giovany de Jesus Malcher Figueiredo, Professor(a) de Magistério Superior do Departamento de Matemática do Instituto de Ciências Exatas**, em 19/02/2021, às 16:08, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

Documento assinado eletronicamente por **Gustavo Silvestre do Amaral Costa, Usuário Externo**, em 19/02/2021, às 17:47, conforme horário oficial de Brasília, com fundamento na Instrução da Reitoria 0003/2016 da Universidade de Brasília.

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CC837e

Costa, Gustavo Silvestre do Amaral
Existence and concentration of solutions for a class of quasilinear problems / Gustavo Silvestre do Amaral Costa; orientador Giovany de Jesus Malcher Figueiredo. -- Brasília, 2021.
103 p.

Tese (Doutorado - Doutorado em Matemática) -- Universidade de Brasília, 2021.

1. Existence and concentration of solutions. 2. Del Pino and Felmer's penalization method. 3. Exponential critical growth . 4. Subcritical growth. 5. Critical growth. I. Figueiredo, Giovany de Jesus Malcher, orient. II. Título.

Aos meus pais, à minha irmã

e à minha madrinha.

Às minhas avós (in memoriam).

"Aprender sem pensar é tempo perdido". (Confúcio)

Agradecimentos

Primeiramente, agradeço a Deus pela saúde e força para enfrentar as dificuldades da vida. Agradeço à minha família pelo apoio, que apesar das dificuldades sempre me deu condições para buscar meus sonhos.

Ao meu orientador, agradeço por toda a atenção, paciência, dedicação e motivação durante os quatro anos de doutorado, além de despertar em mim o gosto pela pesquisa.

Aos meus amigos e colegas de trabalho por estarem ao meu lado nos mais diversos momentos.

Agradeço também aos professores que comporam minha banca de defesa de tese.

Aos professores e servidores do Departamento de Matemática deixo os meus mais sinceros agradecimentos.

A CAPES, CNPq e FAPDF pelo apoio financeiro.

E a todos que contribuíram de forma direta ou indireta nessa conquista da minha vida, meus sinceros agradecimentos.

Resumo

Neste trabalho estudamos a existência e a concentração de soluções para uma classe de equações quaselineares. Mais precisamente, estudamos a seguinte classe de problemas

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right) + V(z)b\left(|u|^p\right)|u|^{p-2}u = f(u) \;\; \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \qquad (P_\epsilon)$$

onde $1 < p \leq q \leq N$ e $N \geq 2$. Estas soluções se concentram em torno do ponto de mínimo do potencial V quando $\epsilon \to 0$ e possuem decaimento exponencial. Consideramos a função f com três tipos diferentes de condições de crescimento: exponencial crítica, subcrítica e crítica. Aqui usamos métodos variacionais e a técnica de Del Pino e Felmer's [26] para superar a perda de compacidade .

Abstract

In this work we study the existence and concentration of the solutions for a class of quasilinear equations. More precisely, we study the following class of problems

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right) + V(z)b\left(|u|^p\right)|u|^{p-2}u = f(u) \;\; \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{P_ϵ}$$

where $1 < p \leq q \leq N$ and $N \geq 2$. These solutions concentrate around the minimum point of potential V as $\epsilon \to 0$ and have exponential decay at infinity. We consider the function f with three different types of growth: critical exponential, subcritical and critical. Here we use variational methods and Del Pino and Felmer's technique [26] in order to overcome the lack of compactness.

Contents

Introduction

In this work we are going study the existence and concentration of solutions for the following class of problems:

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right) + V(z)b\left(|u|^p\right)|u|^{p-2}u = f(u) \ \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{P_ϵ}$$

where $1 < p \leq q \leq N$ and $N \geq 2$. The hypotheses on the functions a and b are the following:

(a_1) the function a is of class C^1 and there exist constants $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^q \leq a(t^p)t^p \leq k_2 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(a_2) the mapping $t \mapsto A(t^p)$ is convex on $(0, \infty)$, where $A(t) = \displaystyle\int_0^t a(s)ds$;

(a_3) the mapping $t \mapsto \dfrac{a(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$;

(a_4) if $1 < p \leq q \leq 2 \leq N$ the mapping $t \mapsto a(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$ the mapping $t \mapsto a(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^q \leq b(t^p)t^p \leq k_4 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(b_2) the mapping $t \mapsto B(t^p)$ is convex on $(0, \infty)$, where $B(t) = \displaystyle\int_0^t b(s)ds$;

(b_3) the mapping $t \mapsto \dfrac{b(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$.

(b_4) if $1 < p \leq q \leq 2 \leq N$ the mapping $t \mapsto b(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$ the mapping $t \mapsto b(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

Using (a_3) and (b_3) we can prove that there exists a positive real constant $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma}a(t)t \leq A(t) \quad \text{and} \quad \frac{1}{\gamma}b(t)t \leq B(t), \quad \text{for all } t \geq 0. \tag{0.0.1}$$

The conditions on V are as follows:

(V_1) There is $V_0 > 0$ such that

$$0 < V_0 \leq V(z), \ \text{for all } z \in \mathbb{R}^N.$$

(V_2) There exists a bounded domain $\Omega \subset \mathbb{R}^N$, such that

$$0 < V_0 = \inf_{z\in\Omega} V(z) < \inf_{z\in\partial\Omega} V(z).$$

Such class of problems arises from applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction, as it can be seen for example in [37], [38] and [63]. For example, we can cite a particular case of (P_ϵ):

Problem 1: Let $a(t) = 1 + t^{\frac{q-p}{p}}$ and $b(t) = 1 + t^{\frac{q-p}{p}}$. In this case we are studying problem

$$-\Delta_p u - \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N.$$

The Problem 1 from a general reaction–diffusion system: $u_t = div(Du\nabla u) + g(x,u)$, where $Du := [|\nabla u|^{p-2} + |\nabla u|^{q-2}]$. In such applications, the function u describes a concentration, the term $div(Du\nabla u)$ corresponds to the diffusion with a diffusion coefficient Du and $g(\cdot,u)$ is the reaction and relates to source and loss processes. Usually, in chemical and biological applications, the reaction term $g(\cdot,u)$ is a polynomial of u with variable coefficients.

In order to illustrate the degree of generality of the kind of problems studied here, with adequate hypotheses on the functions a and b, in the following we present more some examples of problems which are also interesting from the mathematical point of view and have a wide range of applications in physics and related sciences.

Problem 2: Let $a(t) = t^{\frac{q-p}{p}}$ and $b(t) = t^{\frac{q-p}{p}}$. In this case we are studying problem

$$-\epsilon^q \Delta_q u + V(x)|u|^{q-2}u = f(u) \text{ in } \quad \mathbb{R}^N$$

and it is related to the main result showed in [9], [11], [12] in the case $p = 2$. In [5], [8], [45] the author have studied the case $1 < q \le N$.

Problem 3: Let $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1$. In this case we are studying problem

$$-\epsilon^p \ \text{div}(|\nabla u|^{p-2}\nabla u) - div\left(\frac{\epsilon^p|\nabla u|^{p-2}\nabla u}{(1+\epsilon^p|\nabla u|^p)^{\frac{p-2}{p}}}\right) + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N.$$

Problem 4: Let $a(t) = 1+t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1+t^{\frac{q-p}{p}}$. In this case we are studying problem

$$-\epsilon^p\Delta_p u - \epsilon^q\Delta_q u - \text{div}\left(\frac{\epsilon^p|\nabla u|^{p-2}\nabla u}{(1+\epsilon^p|\nabla u|^p)^{\frac{p-2}{p}}}\right) + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \text{ in } \mathbb{R}^N.$$

The interest in the study the class of $p\&q$ equations has increased because of the generality of the involved differential operators, see for example [48], [49], [56], [57]. Indeed [48] and [49] characterize the continuous spectrum of double-phase equations. On the other hand [56] and [57] deal, respectively, with the classes of Laplacian-like operators on Riemannian manifolds, and the existence of blow-up phenomena for A-Laplacian operators.

The concentration of solutions is motivated by the great interest in quantum mechanics, which, for instance, is the study of the nonlinear Schrodinger equation

$$i\epsilon\frac{\partial\Psi}{\partial t} = -\epsilon^2\Delta\Psi + (V(x)+E)\Psi - f(\Psi) \text{ for all } x \in \mathbb{R}^N, \qquad (NLS)$$

where $\epsilon > 0$. Knowledge of the solutions for the elliptic equation

$$-\epsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N \tag{$NLS1$}$$

has a great importance in the study of standing-wave solutions of (NLS). The behavior of the solutions, of the above equation, when $\epsilon \to 0$ has great physical interest since it describes the transition from quantum to classical mechanics, being called *semiclassical states*.

In [51], by a mountain pass argument, Rabinowitz proves the existence of positive solutions of $(NLS1)$, for $\epsilon > 0$ small, whenever

$$V_\infty = \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = \gamma > 0. \tag{R}$$

Later Wang [61] showed that these solutions concentrate at global minimum points of $V(x)$ as ϵ tends to 0. Wang also noted that the concentration of any family of solutions with energy uniformly bounded can only occur in a critical point of V.

In [26], del Pino and Felmer proved the existence of solutions, which are concentrated around local minimum of V by introducing a penalization method. More precisely, they assume that there is an open and bounded set Ω compactly contained in $\mathbb{R}^N$ such that

$$0 < \gamma = \inf_{x\in\mathbb{R}^N} V(x) \le V_0 = \inf_{x\in\Omega} V(x) < \min_{x\in\partial\Omega} V(x).$$

After this excellent paper [26], many authors have used the penalization method with different differential operators. There are more than four hundred quotes, which makes it almost impossible to cite all. However, this method has been little applied to show the existence of nodal solutions that concentrate at minimum points of potential V. For example, [9] and [12] with Laplacian operator, [8], [11] with Laplacian operator and nonlinearity of exponential type, [4], [31], [32] and [45] with p-Laplacian operator, [27] with quasilinear operator $-\Delta u - \Delta(u^2)u$, [36] with Laplacian operator and the nodal solutions concentrating on lower dimensional spheres, [52] with Laplacian operator and V with critical frequency.

As can be seen in [6], [34] and [30], $p\&q$ problems are generalizations of (R). However, as can seen below, we show that the arguments found in [26], [51] and [61] cannot be used directly. But before that, we are going to report some results on $p\&q$ problems type. There are interesting papers on such class of problems. We start with some problems in a bounded domain. For example, in [34] the author shows existence and multiplicity of solutions for a critical $p\&q$ problem considering nonlinearity of concave and convex type. The critical case with discontinuous nonlinearities was studied in [35]. In [15] and [24], the existence of solutions using non-variational methods is shown, such as sub-supersolutions and the principle of comparison.

Now we comment some results in $\mathbb{R}^N$. Existence results was studied in [23] and [30]. In [5] the authors studied concentration results in Orlicz-Sobolev spaces with subcritical nonlinearity and the potential satisfying the local condition introduced by Del pino and Felmer [26]. In [6] was showed the existence and concentration results with subcritical nonlinearity and the potential satisfying the global condition introduced by Rabinowitz [51](see also [61]).

This work is divided into three chapters and three appendices. In Chapter 1 we study existence and concentration of nodal solutions of (P_ϵ) with exponential critical growth. For this $q = N$ and the nonlinearity f is assumed to be a $C^1(\mathbb{R})$ odd function with critical exponential growth at $+\infty$, that is, f behaves $\exp(\alpha_0|t|^{N\backslash N-1})$, for some $\alpha_0 > 0$.

More precisely, we assume the following growth conditions in the origin and at infinity for the function $f : \mathbb{R} \to \mathbb{R}$ of C^1 class:

(f_1)

$$\lim_{|s|\to 0} \frac{f'(s)}{|s|^{N-2}} = 0.$$

(f_2) There exists $\alpha_0 > 0$ such that the function f satisfies

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = 0 \quad \text{for } \alpha > \alpha_0$$

and

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = \infty \quad \text{for } \alpha < \alpha_0,$$

where $S_{N-2}(\alpha, t) = \displaystyle\sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |t|^{Nk/(N-1)}$.

(f_3) There exists $\theta > \gamma p$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for } \ s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$ and $\gamma > 0$ was given in (0.0.1);

(f_4) $s \mapsto \dfrac{f(s)}{s^{N-1}}$ is nondecreasing in $s > 0$.

(f_5) There exist $r > N$ and $\tau > 1$ such that

$$sgn(t)f(t) \geq \tau |t|^{r-1},$$

for all $t \neq 0$.

In the last years, the research to find positive or nontrivial solutions for critical exponential elliptic problem has been made for many authors. For example [1], [2], [3], [8], [21], [25], [34], [41], [42], [46], [47], [53] and references therein.

However, the research to find nodal solutions for critical exponential elliptic problem has been made for few authors. In [10] the authors establish the existence and multiplicity of multi-bump nodal solutions for the class of problems involving the Laplacian operator. In [50] was studied existence of infinitely many sign-changing solutions for elliptic problems with critical exponential growth in bounded domain. In [11] and [12] the authors showed existence and concentration of solution using the penalization method.

Our arguments were strongly influenced by [10], [11], [12] and [50]. Below we list what we believe that are the main contributions of our chapter.

(i) As well-known, in order to overcome the difficult provoked by the exponential critical growth, it is sufficient to have some control on the norm of the minimizing sequence. We obtain such control using a solution of a problem in a bounded domain, as can be seen in Lemma 1.2.3.

(ii) Since the operator considered in this paper is not linear and nonhomegenous, some results that can be found in the papers above mentioned cannot be repeated here. For example, Lemma 1.2.2, Lemma 1.2.3, Lemma 1.2.4 , Lemma 1.2.5.

(iii) In this work we consider a large class of quasilinear operators that includes all operators considered in the papers above mentioned.

The main result of the Chapter 1 is the following:

Theorem 1. *Suppose that a, b, f and V satisfy $(a_1)-(a_4)$, $(b_1)-(b_4)$, $(f_1)-(f_5)$ and $(V_1)-(V_2)$ respectively. Then, there are $\epsilon_0>0$ and $\tau^*>1$ such that (P_ϵ) has a nodal solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N)\cap W^{1,N}(\mathbb{R}^N)$, for every $\epsilon\in(0,\epsilon_0)$ and for every $\tau>\tau^*$. Moreover, if P_ϵ^1 is the maximum point of w_ϵ and P_ϵ^2 is the minimum point of w_ϵ, then for $i=1,2$, we obtain*

$$\lim_{\epsilon\to 0} V(P_\epsilon^i)=V_0.$$

Moreover, there are positive constants C and α such that

$$|w_\epsilon(z)|\le C\left[\exp\left(-\alpha\left|\frac{z-P_\epsilon^1}{\epsilon}\right|\right)+\exp\left(-\alpha\left|\frac{z-P_\epsilon^2}{\epsilon}\right|\right)\right],$$

for all $\epsilon\in(0,\epsilon_0)$ and for all $z\in\mathbb{R}^N$.

In Chapter 2 we prove existence and concentration results for a family of nodal solutions for a (P_ϵ) with subcritical growth. We also show that each nodal solution changes sign exactly once in $\mathbb{R}^N$ and has a exponential decay at infinity. Here we use variational methods and Del Pino and Felmer's technique [26] in order to overcome the lack of compactness.

We would like to quote more articles that are directly related to the arguments that are used in this work. In [58] the author considers a strongly resonant Neumann problem driven by a general nonhomogeneous differential operator. In [59] the author proves the existence of at least two nontrivial solutions of a semilinear Robin problem, whose reaction makes difficult the direct application of variational methods on the energy functional of the problem. Then, the author passes to a suitable subspace where such techniques are applicable, using the Lyapunov-Schmidt reduction method. Finally, in [60] the authors use a classical variational approach based on the critical points theory to prove the existence of at least one nontrivial weak solution of a double phase Dirichlet problem. Here the differential operator of the problem is the sum of two r-Laplacian-type operators with variable exponents.

For this chapter the nonlinearity f is assumed to be a $C^1(\mathbb{R})$ odd function satisfying

(f_1)
$$\lim_{|s|\to 0}\frac{f'(s)}{|s|^{q-2}}=0.$$

(f_2) There exists $q<r<q^*=\frac{qN}{N-q}$ such that
$$\lim_{|s|\to\infty}\frac{f(s)}{|s|^{r-1}}=0.$$

(f_3) There exists $\theta\in(\gamma p,q^*)$ such that
$$0<\theta F(s)\le f(s)s,\quad \text{for}\quad s\neq 0,$$
where $F(s)=\displaystyle\int_0^s f(t)dt$ and $\gamma>0$ was given in (0.0.1);

(f_4) $s\mapsto \dfrac{f(s)}{s^{q-1}}$ is nondecreasing in $s>0$.

Our arguments were strongly influenced by [4], [11], [12], [26], [27], [36], [52]. Below we list what we believe that are the main contributions of our chapter.

(i) In this work we consider a large class of quasilinear operator that includes all operators considered in the papers above mentioned.

(ii) Since the operator considered in this paper is not linear and nonhomegenous, some results that can be found in the papers above mentioned cannot be repeated here. For example, Lemma 2.2.2, Lemma 2.3.3, Lemma 2.5.1 and Lemma 2.5.4.

The main result of the chapter is the following:

Theorem 2. *Suppose that a, b, f and V satisfy $(a_1)-(a_4)$, $(b_1)-(b_4)$, $(f_1)-(f_4)$ and $(V_1)-(V_2)$ respectively. Then there is $\epsilon_0>0$, such that (P_ϵ) has a nodal solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, for every $\epsilon\in(0,\epsilon_0)$. Moreover, if P^1_ϵ is the maximum point of w_ϵ and P^2_ϵ is the minimum point of w_ϵ, then for $i=1,2$, we obtain*

$$\lim_{\epsilon\to 0} V(P^i_\epsilon)=V_0.$$

Moreover, there are positive constants C and α such that

$$|w_\epsilon(z)|\leq C\left[\exp\left(-\alpha\left|\frac{z-P^1_\epsilon}{\epsilon}\right|\right)+\exp\left(-\alpha\left|\frac{z-P^2_\epsilon}{\epsilon}\right|\right)\right],$$

for all $\epsilon\in(0,\epsilon_0)$ and for all $z\in\mathbb{R}^N$.

In Chapter 3 we prove existence and concentration results for a family of positive solutions for a (P_ϵ) with critical growth. We use Mountain Pass Theorem and Del Pino & Felmer's arguments [26] associated to Lions's Concentration and Compactness Principle [39] in order to overcome the lack of compactness. For this chapter the nonlinearity f is assumed to be a $C^1(\mathbb{R})$ function satisfying

(f_1)

$$\lim_{|s|\to 0}\frac{f(s)}{|s|^{q-1}}=0.$$

(f_2) There exists $q<r<q^*=\frac{qN}{N-q}$ such that

$$\lim_{|s|\to \infty}\frac{f(s)}{|s|^{r-1}}=0.$$

(f_3) There exists $\theta\in(\gamma p,q^*)$ such that

$$0<\theta F(s)\leq f(s)s, \quad \text{for} \quad s>0,$$

where $F(s)=\displaystyle\int_0^s f(t)dt$ and $\gamma>0$ was given in (0.0.1);

(f_4) $s\mapsto \dfrac{f(s)}{s^{q-1}}$ is nondecreasing for $s>0$.

(f_5) There exist $\tau\in(q,q*)$ and $\lambda>1$

$$f(s)\geq \lambda s^{\tau-1}\ \ \forall\ s>0.$$

In this chapter is strongly influenced by the articles above. Below we list what we believe that are the main contributions of our paper.

(*i*) Unlike [6], [23] and [30], we show existence and concentration result considering the local potential introduced by Del Pino and Felmer [26].

(*ii*) Unlike [5], we are considering the critical nonlinearity.

(*iii*) Since the operator is not homogeneous, some estimates are different and more delicate than some estimates that can be found in [26] and [51] . For example, see Lemma 3.2.4, Proposition 3.3.1, Lemma 3.3.7 and all the Lemmas of Section 3.5.

(*iv*) In order to overcome the lack of compactness provoked by the critical growth, it is very common to use the Talenti's function (see [55]) to have some control on the minimax level, as can be seen in [20, Lemma 1.1]. The lack of homogenity of the $p\&q$ operator does not allow to use this argument. We overcome this difficulty using the solution of a problem in a bounded domain, as can be seen in Lemma 3.2.5.

The main result in the Chapter 3 is the following:

Theorem 3. *Suppose that a, b, f and V satisfy $(a_1) - (a_3)$, $(b_1) - (b_3)$, $(f_1) - (f_5)$ and $(V_1) - (V_2)$ respectively. Then there are $\epsilon_0 > 0$ and $\lambda^* > 1$ such that (P_ϵ) has a positive solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, for every $\epsilon \in (0, \epsilon_0)$ and for every $\lambda > \lambda^*$. In addition, if P_ϵ is the maximum point of w_ϵ, then*

$$\lim_{\epsilon \to 0} V(P_\epsilon) = V_0.$$

Moreover, there are positive constants C and α such that

$$|w_\epsilon(z)| \leq C \exp\left(-\alpha \left|\frac{z - P_\epsilon}{\epsilon}\right|\right),$$

for all $\epsilon \in (0, \epsilon_0)$ and for all $z \in \mathbb{R}^N$.

In both chapters we will use the technique used in [26], by del Pino and Felmer, and that autonomous problem has a ground-state solution, in other words solutions related to minimax level. Moreover, by arguments found in [6], which are related to the Moser iteration method [43], we can prove the exponential decay to solutions find here. In chapter 1 and 3 we need to prove the existence of ground-state solution of an auxiliary problem in bounded domain, these results are in Appendix A. The proof the technical results can be find in Appendix B. In the Appendix C is reserved for classic results that we will not prove.

In order not to go back to the Introduction and to make the chapters independent, we will state again, in each chapter, the main results, as well as the hypotheses on the functions a, b, V and f.

It is worth mentioning that all chapters are in articles that have been submitted for publication in specialized journals. The Chapter 2 was recently published in the journal Communications on Pure and Applied Analysis (see `http://www.aimsciences.org/article/doi/10.3934/cpaa.2020227`)

Notation

In this work we use the following notation:

c_i and C_i with $i = 0,\ 1,\ 2, ...$	(possibly different) positive constants;
$\rightharpoonup$	weak convergence;
$\to$	strong convergence;
$supp\ f$	support of the function f;
$B_R(z)$	open ball of radius R centered at z;
$a.e.$	almost everywhere;
$\|A\|$	measure of the set A;
$A_1 \subset\subset A_2$	A_1 strongly included in A_2, i.e., $\overline{A_1}$ is compact and $A_1 \subset A_2$;
$\nabla u = \left(\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_N}\right)$	gradient of the function u;
$\Delta u = \sum\limits_{i=1}^{N} \dfrac{\partial u}{\partial x_i} = div(\nabla u)$	Laplacian of u;
$\Delta_p u = div\left(\|\nabla u\|^{p-2} \nabla u\right)$	p-Laplacian of u.

Chapter 1

Existence and concentration of nodal solutions for a critical exponential $p\&N$ equation

In this chapter we show existence and concentration of nodal solutions of the quasilinear problem

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right)+V(z)b\left(|u|^p\right)|u|^{p-2}u=f(u) \ \text{ in } \mathbb{R}^N, \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,N}(\mathbb{R}^N), \\ u^+\neq 0 \text{ and } u^-\neq 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{P_ϵ}$$

where $\epsilon>0$, $1<p<N$, $N\geq 2$, $u^+(x):=\max\{u(x),0\}$ and $u^-(x):=\min\{u(x),0\}$. Notice that, in this case we have

$$u=u^++u^- \text{ and } |u|=u^+-u^-.$$

We show that such solutions changing of sign exactly once.

We say that a function $u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,N}(\mathbb{R}^N)$ is nodal solution of (P_ϵ) if $u^\pm\neq 0$ in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} a(\epsilon^p|\nabla u|^p)\epsilon^p|\nabla u|^{p-2}\nabla u\nabla v\ dz+\int_{\mathbb{R}^N} V(z)b(|u|^p)|u|^{p-2}uv\ dz=\int_{\mathbb{R}^N} f(u)v\ dz,$$

for all $v\in W^{1,p}(\mathbb{R}^N)\cap W^{1,N}(\mathbb{R}^N)$. The hypotheses on the functions a,b,f and V are the following:

(a_1) the function a is of class C^1 and there exist constants $k_1,k_2\geq 0$ such that

$$k_1t^p+t^N\leq a(t^p)t^p\leq k_2t^p+t^N, \quad \text{for all} \quad t>0;$$

(a_2) the mapping $t\mapsto A(t^p)$ is convex on $(0,\infty)$, where $A(t)=\displaystyle\int_0^t a(s)ds$;

(a_3) the mapping $t\mapsto \dfrac{a(t^p)}{t^{N-p}}$ is nonincreasing for $t>0$;

(a_4) if $1<p<2\leq N$ the mapping $t\mapsto a(t)$ is nondecreasing for $t>0$. If $2\leq p<N$ the mapping $t\mapsto a(t^p)t^{p-2}$ is nondecreasing for $t>0$.

As a direct consequence of (a_3) we obtain that the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(N-p)}{p} a(t) \text{ for all } t > 0. \tag{1.0.1}$$

Now if we define the function $h(t) = a(t)t - \frac{N}{p}A(t)$, using (1.0.1) we can prove that the function h is decreasing. Then, there exists a positive real constant $\gamma \geq \frac{N}{p}$ such that

$$\frac{1}{\gamma} a(t)t \leq A(t), \quad \text{for all } t \geq 0. \tag{1.0.2}$$

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^N \leq b(t^p)t^p \leq k_4 t^p + t^N, \quad \text{for all} \quad t > 0;$$

(b_2) the mapping $t \mapsto B(t^p)$ is convex on $(0, \infty)$, where $B(t) = \displaystyle\int_0^t b(s)ds$;

(b_3) the mapping $t \mapsto \dfrac{b(t^p)}{t^{N-p}}$ is nonincreasing for $t > 0$.

(b_4) if $1 < p < 2 \leq N$ the mapping $t \mapsto b(t)$ is nondecreasing for $t > 0$. If $2 \leq p < N$ the mapping $t \mapsto b(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

Using the hypothesis (b_3) and arguing as (1.0.1) and (1.0.2), we also can prove that there exists $\gamma \geq \frac{N}{p}$ such that

$$\frac{1}{\gamma} b(t)t \leq B(t), \quad \text{for all } t \geq 0. \tag{1.0.3}$$

The nonlinearity f is assumed to be a $C^1(\mathbb{R})$ odd function with critical exponential growth at $+\infty$, that is, f behaves as $\exp(\alpha_0 |t|^{N \setminus N-1})$, for some $\alpha_0 > 0$. More precisely, we assume the following growth conditions in the origin and at infinity for the function $f : \mathbb{R} \to \mathbb{R}$ of class C^1:

(f_1)

$$\lim_{|s| \to 0} \frac{f'(s)}{|s|^{N-2}} = 0.$$

(f_2) There exists $\alpha_0 > 0$ such that the function f satisfies

$$\lim_{t \to \infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = 0 \text{ for } \alpha > \alpha_0$$

and

$$\lim_{t \to \infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = \infty \text{ for } \alpha < \alpha_0,$$

where $S_{N-2}(\alpha, t) = \displaystyle\sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |t|^{Nk/(N-1)}$.

(f_3) There exists $\theta > \gamma p$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for} \quad s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$ and $\gamma > 0$ was given in (1.0.2);

(f_4) $s \mapsto \dfrac{f(s)}{s^{N-1}}$ is nondecreasing in $s > 0$.

(f_5) There exist $r > N$ and $\tau > 1$ such that

$$sgn(t)f(t) \geq \tau |t|^{r-1},$$

for all $t \neq 0$.

Before we give the main result, we need to put some hypotheses on the potential $V \in C(\mathbb{R}^N)$.

(V_1) There is $V_0 > 0$ such that

$$0 < V_0 \leq V(z), \text{ for all } z \in \mathbb{R}^N.$$

(V_2) There exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$0 < V_0 = \inf_{z \in \Omega} V(z) < \inf_{z \in \partial\Omega} V(z).$$

The main result is the following:

Theorem 1. Suppose that a, b, f and V satisfy $(a_1) - (a_4)$, $(b_1) - (b_4)$, $(f_1) - (f_5)$ and $(V_1) - (V_2)$ respectively. Then, there are $\epsilon_0 > 0$ and $\tau^* > 1$ such that (P_ϵ) has a nodal solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, for every $\epsilon \in (0, \epsilon_0)$ and for every $\tau > \tau^*$. Moreover, if P_ϵ^1 is the maximum point of w_ϵ and P_ϵ^2 is the minimum point of w_ϵ, then for $i = 1, 2$, we obtain

$$\lim_{\epsilon \to 0} V(P_\epsilon^i) = V_0.$$

Moreover, there are positive constants C and α such that

$$|w_\epsilon(z)| \leq C\left[\exp\left(-\alpha\left|\frac{z - P_\epsilon^1}{\epsilon}\right|\right) + \exp\left(-\alpha\left|\frac{z - P_\epsilon^2}{\epsilon}\right|\right)\right],$$

for all $\epsilon \in (0, \epsilon_0)$ and for all $z \in \mathbb{R}^N$.

The plan of the chapter is the following: In Section 1.1, we study an auxiliary problem obtained by Del Pino and Felmer's technique. In Section 1.2 and Section 1.3, we show existence and concentration of nodal solutions of the auxiliary problem. The proof of the main result is in Section 1.4. In Section 1.5, we show that the nodal solutions have exponential decay. In a appendix we study a problem in bounded domain.

1.1 Variational framework and an auxiliary problem

To prove Theorem 1, we will work with the problem below, which is equivalent to (P_ϵ) by change variable $z = \epsilon x$, which is given by

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + V(\epsilon x) b\left(|u|^p\right)|u|^{p-2}u = f(u) \ \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N), \end{cases} \qquad (\widetilde{P}_\epsilon)$$

where $\epsilon > 0$, $N \geq 2$ and $1 < p < N$.

In this work, we use the following version of the Trudinger-Moser inequality in the whole Euclidean space $\mathbb{R}^N$, which is due to do Ó [44].

Proposition 1.1.1. *If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int\limits_{\mathbb{R}^N} \left[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\right] dx < \infty.$$

Moreover, if $\|\nabla u\|_{L^N}^N \leq 1$, $\|u\|_{L^N} \leq K < \infty$ and $\alpha < \alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}$, then there exists a constant $C = C(N, K, \alpha)$, which depends only on N, K and α, such that

$$\int\limits_{\mathbb{R}^N} \left[\exp(\alpha|u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)\right] dx \leq C,$$

where ω_{N-1} is the $(N-1)$-dimensional measure of $(N-1)$ sphere.

To obtain solutions of $(\widetilde{P}_\epsilon)$, consider the following subspace of $W^{1,p}(\mathbb{R}^N) \bigcap W^{1,N}(\mathbb{R}^N)$ given by

$$W_\epsilon := \left\{ v \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x) b(|v|^p)|v|^p dx < +\infty \right\},$$

which is a Banach space when endowed with the norm

$$\|u\| = \|u\|_{1,p} + \|u\|_{1,N},$$

where

$$\|u\|_{1,m} = \left(\int_{\mathbb{R}^N} |\nabla u|^m dx + \int_{\mathbb{R}^N} V(\epsilon x)|u|^m dx \right)^{\frac{1}{m}}, \text{ for } m \geq 1.$$

We first notice that, by (f_1) and (f_2): given $\xi > 0$, $q \geq 0$ and $\alpha \geq 1$ there exists C_ξ, $\tilde{C}_\xi > 0$ such that

$$f(s)s \leq \xi|s|^N + C_\xi |s|^q \left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right] \quad \text{for all } s \in \mathbb{R}, \tag{1.1.1}$$

and

$$F(s) \leq \frac{\xi}{N}|s|^N + \tilde{C}_\xi |s|^q \left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right] \quad \text{for all } s \in \mathbb{R}, \tag{1.1.2}$$

for more details see Appendix B. Since the approach is variational, consider the energy functional associated $J_\epsilon : W_\epsilon \to \mathbb{R}$ given by

$$J_\epsilon(v) = \frac{1}{p} \int\limits_{\mathbb{R}^N} A\left(|\nabla v|^p\right) dx + \frac{1}{p} \int\limits_{\mathbb{R}^N} V(\epsilon x) B\left(|v|^p\right) dx - \int\limits_{\mathbb{R}^N} F(v) dx.$$

By standard arguments, one can prove that $J_\epsilon \in C^1(W_\epsilon, \mathbb{R})$. Let θ be the number given in (f_3), and let $\eta, \beta > 0$ be constants satisfying $\beta > \max\left\{ \dfrac{\theta p \gamma}{\theta - \gamma p}, N-1 \right\}$ and $\dfrac{f(\eta)}{|\eta|^{N-2}\eta} = \dfrac{V_0}{\beta}$, where V_0 appears in (V_1). Then, using the above numbers, we define the function of C^1 class given by

$$\widetilde{f}(s) = \begin{cases} f(s) \quad if \quad |s| \leq \frac{\eta}{2}, \\ \\ \dfrac{V_0}{\beta}|s|^{N-2}s \quad if \quad s > \eta, \\ \\ \dfrac{V_0}{\beta}|s|^{N-2}s, \quad if \quad s < -\eta. \end{cases}$$

Here we are defining the function $\widetilde{f}$ in $(-\eta,-\frac{\eta}{2})$ and $(\frac{\eta}{2},\eta)$ such that $\widetilde{f}$ is of class C^1. Note that by (f_1) and given $\xi>0$, we get

$$\widetilde{f}'(s) \leq \begin{cases} \xi|s|^{N-2} < (N-1)\frac{V_0}{\beta}|s|^{N-2} \quad if \quad |s|\leq \frac{\eta}{2}, \\ \\ (N-1)\dfrac{V_0}{\beta}|s|^{N-2} \quad if \quad s>\eta, \\ \\ (N-1)\dfrac{V_0}{\beta}|s|^{N-2}, \quad if \quad s<-\eta. \end{cases} \tag{1.1.3}$$

We now define the function

$$g(z,s) := \chi_\Omega(z)f(s) + (1-\chi_\Omega(z))\widetilde{f}(s),$$

and the auxiliary problem

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + V(\epsilon x)b\left(|u|^p\right)|u|^{p-2}u = g(\epsilon x,u) \;\; \text{in } \mathbb{R}^N, \\ u\in W_\epsilon, \end{cases} \tag{$P_{\epsilon_{aux}}$}$$

where χ_Ω is the characteristic function of the set Ω. It is easy to check that $(f_1)-(f_4)$ imply that g is a Carathéodory function and for $x\in\mathbb{R}^N$, the function $s\to g(\epsilon x,s)$ is of class C^1 and satisfies the following conditions, uniformly for $x\in\mathbb{R}^N$:

$$\lim_{|s|\to 0}\frac{g(\epsilon x,s)}{|s|^{N-1}} = 0, \tag{g_1}$$

$$g(\epsilon x,s)\leq f(s), \forall\ s>0 \text{ and } x\in\mathbb{R}^N, \tag{g_2}$$

$$0<\theta G(\epsilon x,s)\leq g(\epsilon x,s)s, \;\; \forall \epsilon x\in\Omega \;\text{ and }\; \forall s\neq 0, \tag{$(g_3)_i$}$$

and

$$0<NG(\epsilon x,s)\leq g(\epsilon x,s)s\leq \frac{1}{\beta}V(\epsilon x)|s|^N, \;\; \forall \epsilon x\notin\Omega \;\text{ and }\; \forall s\neq 0, \tag{$(g_3)_{ii}$}$$

where $G(\epsilon x,s)=\displaystyle\int_0^s g(\epsilon x,t)dt$.

For each $x\in\mathbb{R}^N$, the function

$$s\to\frac{g(\epsilon x,s)}{s^{N-1}} \;\text{ is nondecreasing for }\; s>0. \tag{g_4}$$

Remark 1. *Note that, for $z=\epsilon x$, if u_ϵ is a nodal solution of $(P_{\epsilon_{aux}})$ with $|u_\epsilon(z)|\leq\frac{\eta}{2}$ for every $\epsilon x\in\mathbb{R}^N\setminus\Omega$, then $u_\epsilon(x)$ is also a nodal solution of (P_ϵ).*

1.2 Existence of ground state nodal for the auxiliary problem

In this section we adapt some arguments found in Alves & Figueiredo [7], Alves & Soares [12] and Bartsch, Weth & Willem [18] to establish the existence of ground state nodal solution for problem $(P_{\epsilon_{aux}})$.

Hereafter, let us denote by I_ϵ the functional

$$I_\epsilon(v)=\frac{1}{p}\int_{\mathbb{R}^N}A\left(|\nabla v|^p\right)dx+\frac{1}{p}\int_{\mathbb{R}^N}V(\epsilon x)B\left(|v|^p\right)dx-\int_{\mathbb{R}^N}G(\epsilon x,v)dx,$$

which is well defined for $v \in W_\epsilon$ and by $\mathcal{N}_\epsilon$ the Nehari manifold associated given by

$$\mathcal{N}_\epsilon = \Big\{ u \in W_\epsilon : u \neq 0 \quad \text{and} \quad I'_\epsilon(u)u = 0 \Big\}.$$

Since g is C^1, the functional I_ϵ is of C^1 class. As we are looking for nodal solutions, we also define the following set

$$\mathcal{N}_\epsilon^{\pm} = \Big\{ u \in W_\epsilon : u^{\pm} \neq 0 \quad \text{and} \quad I'_\epsilon(u^{\pm})u^{\pm} = 0 \Big\},$$

where

$$u^{+}(z) = \max\{u(z), 0\} \text{ and } u^{-}(z) = \min\{u(z), 0\}.$$

The main result in this section is:

Theorem 1.2.1. *Let a satisfying $(a_1)-(a_4)$, b satisfying $(b_1)-(b_4)$, V satisfying $(V_1)-(V_2)$ and f satisfying $(f_1)-(f_5)$. Then, there is $\tau^* > 1$ such that $(P_{\epsilon_{aux}})$ has a nodal solution $u_\epsilon \in W_\epsilon$, for every $\tau > \tau^*$. Moreover, if $\frac{P_\epsilon^1}{\epsilon}$ is the maximum point of u_ϵ and $\frac{P_\epsilon^2}{\epsilon}$ is the minimum point of u_ϵ, then for $i = 1, 2$, we obtain*

$$\lim_{\epsilon \to 0} V(P_\epsilon^i) = V_0.$$

We begin with some information on the functional I_ϵ in $\mathcal{N}_\epsilon$ and in $\mathcal{N}_\epsilon^{\pm}$.

Lemma 1.2.2. *The functional I_ϵ satisfies the following conditions:*

(i) There is $C > 0$ such that

$$I_\epsilon(u) \geq C \left[\ \|u\|_{1,p}^p + \|u\|_{1,N}^N \ \right], \ \forall \ u \in \mathcal{N}_\epsilon \text{ and } \forall \ \epsilon > 0.$$

(ii) There exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}_\epsilon$ and $\|w^{\pm}\| \geq \rho$ for all $w \in \mathcal{N}_\epsilon^{\pm}$.

Proof. Since $u \in \mathcal{N}_\epsilon$ and (1.0.2), (1.0.3), (g_3) hold we have that

$$\begin{aligned} I_\epsilon(u) &= I_\epsilon(u) - \frac{1}{\theta}\left\langle I'_\epsilon(u), u \right\rangle \geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p \, dx \\ &+ \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} V(\epsilon x) b(|u|^p)|u|^p \, dx + \frac{1}{\theta} \int\limits_{\mathbb{R}^N \setminus \Omega_\epsilon} [g(\epsilon x, u)u - \theta G(\epsilon x, u)] \, dx, \end{aligned}$$

which implies, from (a_1), (b_1) and $(g_3)_{ii}$, that

$$\begin{aligned} I_\epsilon(u) &\geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} [k_1|\nabla u|^p + |\nabla u|^N] \, dx \\ &+ \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} V(\epsilon x)[k_3|u|^p + |u|^N]] \, dx - \frac{1}{\beta} \int\limits_{\mathbb{R}^N} [|\nabla u|^N + V(\epsilon x)|u|^N] \, dx. \end{aligned}$$

Then the item (i) holds because $\beta > \dfrac{\theta p \gamma}{\theta - \gamma p}$.

In order to prove (ii). Suppose by contradiction that there is a sequence (u_n) in $\mathcal{N}_\epsilon$ such that $u_n \to 0$ in W_ϵ. Then, from (a_1), (b_1) and (1.1.1), there exists $C_1 > 0$ such that

$$\begin{aligned} C_1[\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N] &\leq \xi \int\limits_{\mathbb{R}^N} |u_n|^N \, dx + C_\xi \int\limits_{\mathbb{R}^N} |u_n|^q \left[\exp\left(\alpha |u_n|^{N/N-1}\right) - S_{N-2}(\alpha, u_n)\right] \, dx \\ &\leq \xi \|u_n\|_{1,p}^N + C_\xi \int\limits_{\mathbb{R}^N} |u_n|^q \left[\exp\left(\alpha |u_n|^{N/N-1}\right) - S_{N-2}(\alpha, u_n)\right] dx. \end{aligned}$$

By Holder's Inequality with $s',\ s>1$, Sobolev embeddings and Proposition 1.1.1, there are $C_2, C_3>0$ such that

$$
\begin{aligned}
& C_2[\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N] \\
& \leq \|u_n\|^q_{L^{qs'}(\mathbb{R}^N)} \left(\int\limits_{\mathbb{R}^N} \left[\exp\left(s\alpha \|u_n\|^{N/N-1} \left(\frac{|u_n|}{\|u_n\|}\right)^{N/N-1}\right) - S_{N-2}\left(s\alpha, u_n\right) \right] dx \right)^{1/s} \\
& \leq C_3 \|u_n\|^q.
\end{aligned}
$$

Then, there exists $C_4>0$ such that $C_4 \leq \|u_n\|^{q-N}$. But the last inequality is impossible for $q>N$. Therefore, since $\mathcal{N}_\epsilon^{\pm} \subset \mathcal{N}_\epsilon$, the second item is proved. $\square$

From Lemma 1.2.2 is well defined the real number

$$
d_\epsilon = \inf_{\mathcal{N}_\epsilon^{\pm}} I_\epsilon. \tag{1.2.1}
$$

Moreover, from [17, Lemma 4.2, Lemma 4.3], for $u \in W_\epsilon$ with $u^{\pm} \neq 0$, there exist unique $t, s>0$ such that $tu^+ + su^- \in \mathcal{N}_\epsilon^{\pm}$. At this point, we can finally prove the existence of $u \in \mathcal{N}_\epsilon^{\pm}$ in which the infimum of I_ϵ is attained on $\mathcal{N}_\epsilon^{\pm}$.

Now we consider the problem

$$
\left\{
\begin{array}{l}
-k_2\Delta_p u - \Delta_N u + V_\infty (k_4|u|^{p-2}u + |u|^{N-2}u) = |u|^r \ \text{ in } \Omega, \\
u \in W_0^{1,N}(\Omega),
\end{array}
\right. \tag{P_r}
$$

where r is the constant which appears in the hypothesis (f_5) and V_∞ is a positive constant. We have associated to problem (P_r) the functional $I_r: W_0^{1,N}(\Omega) \to \mathbb{R}$, given by

$$
I_r(u) = \frac{1}{p}\int\limits_\Omega [k_2|\nabla u|^p + V_\infty k_4 |u|^p]\, dx + \frac{1}{N}\int\limits_\Omega \left[|\nabla u|^N + V_\infty |u|^N\right] dx - \frac{1}{r}\int\limits_\Omega |u|^r dx
$$

and the Nehari manifold

$$
\mathcal{N}_r = \{u \in W_0^{1,N}(\Omega) : u \neq 0 \text{ and } I_r'(u)u = 0\}
$$

and the set

$$
\mathcal{N}_r^{\pm} = \{u \in W_0^{1,N}(\Omega) :\ u^{\pm} \neq 0 \text{ and } I_r'(u^{\pm})u^{\pm} = 0\}.
$$

From Appendix A, there exists $w_r \in \mathcal{N}_r^{\pm}$ such that

$$
I_r(w_r) = c_r \quad \text{and} \quad I_r'(w_r^{\pm}) = 0,
$$

where

$$
c_r = \inf_{\mathcal{N}_r^{\pm}} I_r. \tag{1.2.2}
$$

Moreover,

$$
c_r \geq \left(\frac{r-N}{rN}\right) \int\limits_\Omega |w_r|^r dx, \tag{1.2.3}
$$

Lemma 1.2.3. *The value* $d_\epsilon := \inf\limits_{\mathcal{N}_\epsilon^{\pm}} I_\epsilon$ *satisfies*

$$
d_\epsilon \leq \left[\frac{r-p}{p\tau^{p/(r-p)}}\right] \frac{c_r N}{(r-N)}.
$$

Proof. Note that by the hypotheses (a_1), (b_1) and (f_5), we have

$$\int_{\mathbb{R}^N} a(|\nabla w_r^{\pm}|^p)|\nabla w_r^{\pm}|^p dx + \int_{\mathbb{R}^N} V(\epsilon x)b(|w_r^{\pm}|^p)|w_r^{\pm}|^p dx \le \int_{\Omega} \left[k_2|\nabla w_r^{\pm}|^p + V_\infty k_4 |w_r^{\pm}|^p\right] dx$$
$$+\int_{\Omega} \left[|\nabla w_r^{\pm}|^N + V_\infty |w_r^{\pm}|^N\right] dx = \int_{\Omega} |w_r^{\pm}|^r dx \le \int_{\Omega} f(w_r^{\pm})w_r^{\pm} dx \le \int_{\mathbb{R}^N} g(\epsilon x, w_r^{\pm})w_r^{\pm} dx,$$

where $V_\infty := \max_{x\in\overline{\Omega}} V(x)$. This inequality implies that $I'_\epsilon(w_r^{\pm})w_r^{\pm} \le 0$, then there exist $t, s \in (0,1)$ such that $tw_r^+ + sw_r^- \in \mathcal{N}_\epsilon^{\pm}$. Using (a_1), (b_1), (g_3) and (f_5), we obtain

$$d_\epsilon \le I_\epsilon(tw_r^+ + sw_r^-) \le \frac{t^p}{p}\int_{\Omega} \left[k_2|\nabla w_r^+|^p + V_\infty k_4|w_r^+|^p\right] dx$$
$$+\frac{s^p}{p}\int_{\Omega} \left[k_2|\nabla w_r^-|^p + V_\infty k_4|w_r^-|^p\right] dx + \frac{t^N}{N}\int_{\Omega} \left[|\nabla w_r^+|^N + V_\infty|w_r^+|^N\right] dx$$
$$+\frac{s^N}{N}\int_{\Omega} \left[|\nabla w_r^-|^N + V_\infty|w_r^-|^N\right] dx - \frac{\tau}{r}t^r\int_{\Omega} |w_r^+|^r dx - \frac{\tau}{r}s^r\int_{\Omega} |w_r^-|^r dx.$$

Since $t, s \in (0,1)$, $p < N$ and $I'_r(w_r^{\pm})w_r^{\pm} = 0$, we get

$$\begin{aligned}
d_\epsilon &\le I_\epsilon(tw_r^+ + sw_r^-) \le \frac{t^p}{p}\int_{\Omega} \left[k_2|\nabla w_r^+|^p + V_\infty k_4|w_r^+|^p\right] dx \\
&\quad +\frac{s^p}{p}\int_{\Omega} \left[k_2|\nabla w_r^-|^p + V_\infty k_4|w_r^-|^p\right] dx + \frac{t^p}{p}\int_{\Omega} \left[|\nabla w_r^+|^N + V_\infty|w_r^+|^N\right] dx \\
&\quad +\frac{s^p}{p}\int_{\Omega} \left[|\nabla w_r^-|^N + V_\infty|w_r^-|^N\right] dx - \frac{\tau}{r}t^r\int_{\Omega} |w_r^+|^r dx - \frac{\tau}{r}s^r\int_{\Omega} |w_r^-|^r dx \\
&\le \left[\frac{t^p}{p} - \tau\frac{t^r}{r}\right]\int_{\Omega} |w_r^+|^r dx + \left[\frac{s^p}{p} - \tau\frac{s^r}{r}\right]\int_{\Omega} |w_r^-|^r dx \\
&\le \max_{s\ge 0}\left[\frac{s^p}{p} - \tau\frac{s^r}{r}\right]\int_{\Omega} |w_r|^r dx.
\end{aligned}$$

Using (1.2.3) and by some straight forward algebric manipulations, we have

$$d_\epsilon \le \max_{s\ge 0}\left[\frac{s^p}{p} - \tau\frac{s^r}{r}\right]\frac{c_r N r}{(r-N)} = \left[\frac{r-p}{p\tau^{p/(r-p)}}\right]\frac{c_r N}{(r-N)}$$

and the result follows. □

Since τ is the parameter which appears in the hypothesis (f_5) we have that following result.

Lemma 1.2.4. *Let $(u_n) \subset \mathcal{N}_\epsilon^{\pm}$ be a minimizing sequence for d_ϵ, then there exists $\tau^* > 1$ such that*

$$\limsup_{n\to\infty} \|u_n\|^{N/N-1} \le \frac{\alpha_N}{4\alpha_0} \quad \text{if } \tau > \tau^*.$$

Proof. First notice that by (a_1), (b_1), (1.0.2), (1.0.3), $(g_3)_i$ and $(g_3)_{ii}$, we obtain

$$\begin{aligned}
d_\epsilon &= I_\epsilon(u_n) - \frac{1}{\theta}I'_\epsilon(u_n)u_n + o_n(1) \\
&\ge \left(\frac{1}{p\gamma} - \frac{1}{\theta} - \frac{1}{\beta}\right)\min\{1, k_1, k_3\}\left[\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N\right] + o_n(1).
\end{aligned}$$

Using the estimate on the value d_ϵ obtained in Lemma 1.2.3, we get

$$\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N \leq \left[\frac{r-p}{pr\tau^{p/(r-p)}}\right] \frac{c_r N r}{(r-N)} \left[\left(\frac{1}{p\gamma} - \frac{1}{\theta} - \frac{1}{\beta}\right) \min\{1, k_1, k_3\}\right]^{-1} + o_n(1).$$

Setting

$$\tau* := \max\left\{1, \left[\frac{(r-p)}{(r-N)} \frac{c_r N}{p}\right]^{r-p/p} \left[\left(\frac{1}{p\gamma} - \frac{1}{\theta} - \frac{1}{\beta}\right) \min\{1, k_1, k_3\} \min\left\{1, \left[\frac{\alpha_N}{2^{\frac{3N-2}{N-1}} \alpha_0}\right]^{N-1}\right\}\right]^{p-r/p}\right\}.$$

Therefore, if $\tau > \tau^*$, we can conclude that

$$\frac{1}{2^N}\|u_n\|^N \leq \|u_n\|_{1,p}^N + \|u_n\|_{1,N}^N \leq \|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N \leq \min\left\{1, \left[\frac{\alpha_N}{2^{\frac{3N-2}{N-1}} \alpha_0}\right]^{N-1}\right\} + o_n(1).$$

The last inequality implies

$$\frac{1}{2^N}\|u_n\|^N \leq \left[\frac{\alpha_N}{2^{\frac{3N-2}{N-1}} \alpha_0}\right]^{N-1} + o_n(1),$$

since $N \geq 2$ this completes the proof. □

Lemma 1.2.5. *If $(u_n) \subset \mathcal{N}_\epsilon^\pm$ is a minimizing sequence for d_ϵ, then there exists $s' > 1$ such that*

$$\liminf_{n\to\infty} \int_{\Omega_\epsilon} |u_n^\pm|^{qs'} dx > 0.$$

Proof. We know from (a_1), (b_1) and (1.1.1) that there exists a constant $C_1 > 0$ such that

$$C_1[\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N] \leq C_\xi \int_{\Omega_\epsilon} |u_n|^q \left[\exp\left(\alpha |u_n|^{N/N-1}\right) - S_{N-2}(\alpha, u_n)\right] dx.$$

Using Lemma 1.2.4, Proposition 1.1.1, $\alpha = 3\alpha_0$ and choosing $s > 1$ close to 1 we obtain a positive constant $C_2 > 0$ such that

$$\left(\int_{\mathbb{R}^N} \left[\exp\left(s\alpha \|u_n\|^{N/N-1} \left(\frac{|u_n|}{\|u_n\|}\right)^{N/N-1}\right) - S_{N-2}\left(s\alpha\|u_n\|^{N/N-1}, \frac{u_n}{\|u_n\|}\right)\right] dx\right)^{1/s} \leq C_2.$$

Therefore, by Sobolev embeddings, Holder's Inequality with $s', s > 1$ for s close to 1 and (1.1.1), we deduce that

$$C_1[\|u_n\|_{1,p}^p + \|u_n\|_{1,N}^N] \leq C_2 \left(\int_{\Omega_\epsilon} |u_n|^{qs'} dx\right)^{\frac{1}{s'}} + o_n(1).$$

Consequently, using Lemma 1.2.2 the result follows. □

Existence of nodal solution for the auxiliary problem

We are going to show that the infimum of I_ϵ on $\mathcal{N}_\epsilon^{\pm}$ is attained by some $u_\epsilon \in \mathcal{N}_\epsilon$, considering the cases $2 \leq p < N$ and $1 < p < 2 \leq N$.

Lemma 1.2.6. *If $2 \leq p < N$, then the functional I_ϵ is sequentially weakly lower semicontinous in W_ϵ. Moreover, the level d_ϵ is attained for some u_ϵ which is a nodal solution for problem $(P_{\epsilon_{aux}})$*

Proof. Firstly we prove that the functional I_ϵ is sequentially weakly lower semicontinous in W_ϵ. For this let us consider $(u_n) \subset W_\epsilon$ such that $u_n \rightharpoonup u$ in W_ϵ and $\Omega_\epsilon := \epsilon^{-1}\Omega$. From (a_2) and (b_2) it follows that

$$\int_{\Omega_\epsilon} A(|\nabla u|^p)\, dx \leq \liminf_{n\to+\infty} \int_{\Omega_\epsilon} A(|\nabla u_n|^p)\, dx, \tag{1.2.4}$$

$$\int_{\Omega_\epsilon} V(\epsilon x)B(|u|^p)\, dx \leq \liminf_{n\to+\infty} \int_{\Omega_\epsilon} V(\epsilon x)B(|u_n|^p)\, dx. \tag{1.2.5}$$

Moreover, by Sobolev embeddings, we get

$$\int_{\Omega_\epsilon} F(u)\, dx = \lim_{n\to+\infty} \int_{\Omega_\epsilon} F(u_n)\, dx. \tag{1.2.6}$$

Now we are going to prove that

$$I_{\epsilon,\mathbb{R}^N\backslash\Omega_\epsilon}(v) := \frac{1}{p}\int_{\mathbb{R}^N\backslash\Omega_\epsilon} (A(|\nabla v|^p) + V(\epsilon x)B(|v|^p)v^p)\, dx - \int_{\mathbb{R}^N\backslash\Omega_\epsilon} \tilde{F}(v)$$

is a strictly convex functional in $W_\epsilon(\mathbb{R}^N\backslash\Omega_\epsilon)$, where $\widetilde{F}(s) = \int\limits_0^s \widetilde{f}(t)dt$.

Observe that $I_\epsilon''(v)(w,w)$ is well-defined for $v, w \in W_\epsilon(\mathbb{R}^N)$, for $2 \leq p < N$. Then, for $v, w \in W_\epsilon(\mathbb{R}^N\backslash\Omega_\epsilon)$, $w \neq 0$, we have

$$\begin{aligned}
I_{\epsilon,\mathbb{R}^N\backslash\Omega_\epsilon}''(v)(w,w) &= p\int_{\mathbb{R}^N\backslash\Omega_\epsilon} a'(|\nabla v|^p)|\nabla v|^{2p-4}(\nabla v\nabla w)^2 dx \\
&+ (p-2)\int_{\mathbb{R}^N\backslash\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-4}(\nabla v\nabla w)^2 dx \\
&+ \int_{\mathbb{R}^N\backslash\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-2}|\nabla w|^2 dx \\
&+ p\int_{\mathbb{R}^N\backslash\Omega_\epsilon} V(\epsilon x)b'(|v|^p)|v|^{2p-4}(vw)^2 dx \\
&+ (p-2)\int_{\mathbb{R}^N\backslash\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-4}(vw)^2 dx \\
&+ \int_{\mathbb{R}^N\backslash\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2 dx - \int_{\mathbb{R}^N\backslash\Omega_\epsilon} \tilde{f}'(v)w^2 dx.
\end{aligned}$$

Using (2.1.1), (a_4) and (b_4), we deduce that

$$\begin{aligned}
I''_{\epsilon,\mathbb{R}^N\backslash\Omega_\epsilon}(v)(w,w) &\geq \int_{\mathbb{R}^N\backslash\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-2}|\nabla w|^2 dx + \int_{\mathbb{R}^N\backslash\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2 dx \\
&- \frac{N-1}{\beta}\int_{\mathbb{R}^N\backslash\Omega_\epsilon} V_0|v|^{N-2}w^2 dx.
\end{aligned}$$

Since we also have $\beta > N-1$, we finally get to $I''_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}(v)(w,w) > 0$. By convex analysis it follows that $I_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}$ is weakly lower semicontinuous.

From Lemma 1.2.2, there exists a bounded minimizing sequence (u_n) in $\mathcal{N}_\epsilon^\pm$ for d_ϵ and I_ϵ is coercive on $\mathcal{N}_\epsilon^\pm$. Hence, there exist $v, u_1, u_2 \in W_\epsilon$ such that

$$u_n \rightharpoonup v, \quad u_n^+ \rightharpoonup u_1, \quad u_n^- \rightharpoonup u_2 \quad \text{in } W_\epsilon.$$

Since the transformations $v \to v^+$ and $v \to v^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (see Lemma 2.3 in [22] with suitable adaptations), we have that $v^+ = u_1 \geq 0$ and $v^- = u_2 \leq 0$. By the Lemma 1.2.5, we conclude that $v^\pm \neq 0$, and therefore $v = v^+ + v^-$ is sign-changing, this implies that there exist $t, s > 0$ such that $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. we have the $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. Moreover, there exists a unique pair (t_v, s_v) of positive constants such that

$$I_\epsilon(t_v v^+ + s_v v^-) = \max_{t,s>0} I_\epsilon(tv^+ + sv^-).$$

Since I_ϵ is sequentially weakly lower semicontinous in W_ϵ and (u_n) in $\mathcal{N}_\epsilon^\pm$, we have

$$\begin{aligned} d_\epsilon &\leq I_\epsilon(u_\epsilon) = I_\epsilon(tv^+ + sv^-) \leq \liminf_{n\to+\infty} I_\epsilon(tu_n^+ + su_n^-) \\ &\leq \limsup_{n\to+\infty} I_\epsilon(tu_n^+ + su_n^-) \leq \lim_{n\to+\infty} I_\epsilon(u_n^+ + u_n^-) = \lim_{n\to+\infty} I_\epsilon(u_n) = d_\epsilon. \end{aligned}$$

□

Lemma 1.2.7. *For $1 < p < 2 \leq N$, the level d_ϵ is attained for some $u_\epsilon \in \mathcal{N}_\epsilon^\pm$. Moreover, u_ϵ is a nodal solution for problem $(P_{\epsilon_{aux}})$.*

Proof. From Lemma 1.2.2, there exists a bounded minimizing sequence (u_n) in $\mathcal{N}_\epsilon^\pm$ for d_ϵ and I_ϵ is coercive on $\mathcal{N}_\epsilon^\pm$. Hence, there exist $v, u_1, u_2 \in W_\epsilon$ such that

$$u_n \rightharpoonup v, \quad u_n^+ \rightharpoonup u_1, \quad u_n^- \rightharpoonup u_2 \quad \text{in } W_\epsilon.$$

Since the transformations $v \to v^+$ and $v \to v^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (see Lemma 2.3 in [22] with suitable adaptations), we have that $v^+ = u_1 \geq 0$ and $v^- = u_2 \leq 0$. By the Lemma 1.2.5, we conclude that $v^\pm \neq 0$, and therefore $v = v^+ + v^-$ is sign-changing, this implies that there exist $t, s > 0$ such that $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. we have the $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$.

On the order hand, using Sobolev embedding, we have

$$\lim_{n\to+\infty} \int_{\Omega_\epsilon} f(u_n^\pm)u_n^\pm dx = \int_{\Omega_\epsilon} f(v^\pm)v^\pm dx.$$

Then, using Fatou's Lemma and $(g_3)_{ii}$ we obtain that

$$\int_{\mathbb{R}^N} \left[a(|\nabla v^\pm|^p)|\nabla v^\pm|^p + V(\epsilon x)b(|v^\pm|^p)|v^\pm|^p\right] dx \leq \int_{\mathbb{R}^N} g(\epsilon x, v^\pm)v^\pm dx,$$

that is, $I'_\epsilon(v^\pm)v^\pm \leq 0$. Thus, $t,\ s \in (0,1]$.

Now, let us observe that assumptions (a_3), (b_3) and (g_4) imply the following monotonicity conditions:

$$t \longmapsto \frac{1}{p}A(t) - \frac{1}{N}a(t)t \text{ is increasing for } t \in (0,+\infty),$$

$$t \longmapsto \frac{1}{p}B(t) - \frac{1}{N}b(t)t \text{ is increasing for } t \in (0,+\infty),$$

$$t \longmapsto \frac{1}{N}g(\epsilon x,t)t - G(\epsilon x,t) \text{ is increasing for } t \in (0,+\infty),$$

Hence,

$$
\begin{aligned}
I_\epsilon(tv^+) &\leq \int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A(|\nabla(tv^+)|^p) - \frac{1}{N}a(|\nabla(tv^+)|^p)|\nabla(tv^+)|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} V(\epsilon x)\left(\frac{1}{p}B(|tv^+|^p) - \frac{1}{N}b(|tv^+|^p)|tv^+|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} \left(\frac{1}{N}g(\epsilon x, tv^+)tv^+ - G(\epsilon x, tv^+)\right) dx \\
&\leq \liminf_{n\to+\infty} \left[\int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A(|\nabla u_n^+|^p) - \frac{1}{N}a(|\nabla u_n^+|^p)|\nabla u_n^+|^p\right) dx\right. \\
&\quad + \int\limits_{\mathbb{R}^N} V(\epsilon x)\left(\frac{1}{p}B(|u_n^+|^p) - \frac{1}{N}b(|u_n^+|^p)|u_n^+|^p\right) dx \\
&\quad \left. + \int\limits_{\mathbb{R}^N} \left(\frac{1}{N}g(\epsilon x, u_n^+)u_n^+ - G(\epsilon x, u_n^+)\right) dx\right] = \liminf_{n\to+\infty} I_\epsilon(u_n^+).
\end{aligned}
$$

Using the same arguments as above one can immediately prove that $I_\epsilon(sv^-) \leq I_\epsilon(v^-)$. Then, using that g is and odd function and $u_\epsilon \in \mathcal{N}_\epsilon^\pm$, it follows that

$$
d_\epsilon \leq I_\epsilon(u_\epsilon) = I_\epsilon(tv^+) + I_\epsilon(sv^-) \leq \liminf_{n\to+\infty} I_\epsilon(u_n) = d_\epsilon.
$$

□

Remark 2. *Note that Lemma 1.2.7 is true for all $1 < p < N$, however the arguments used in Lemma 1.2.6 are new for nonhomogeneous operators.*

Proof of Theorem 1.2.1

Proof. The existence follows by Lemma 1.2.6 and Lemma 1.2.7. The proof that $I'_\epsilon(u_\epsilon) = 0$ and that u_ϵ has exactly two nodal domains or equivalently it changes sign exactly once can be seen in [17, pages 1230-1232]. □

1.3 Concentration results

In order to prove the concentration result, we consider the limit problem

$$
\begin{cases}
-div\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) + V_0 b(|u|^p)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N \\
u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)
\end{cases}
\tag{P_L}
$$

which the functional associated I_{V_0} is given by

$$
I_{V_0}(u) = \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)]dx - \int_{\mathbb{R}^N} F(u)dx,
$$

and by the corresponding Nehari manifold is given by

$$\mathcal{N}_{V_0} = \{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)\backslash\{0\}; I'_{V_0}(u)u = 0\}.$$

We also define

$$c_{V_0} = \inf_{\mathcal{N}_{V_0}} I_{V_0}.$$

We define the Palais-Smale compactness condition. We say that a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ is a Palais-Smale sequence at level c_{V_0} for the functional I_{V_0} if

$$I_{V_0}(u_n) \to c_{V_0}$$

and

$$\|I'_{V_0}(u_n)\| \to 0 \text{ in } (W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N))'.$$

If every Palais-Smale sequence for I_{V_0} has a converging subsequence in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, then one says that I_{V_0} satisfies the Palais-Smale condition ($(PS)_{c_{V_0}}$ for short).

The next result shows that problem (PL) has a solution that reaches c_{V_0}.

Lemma 1.3.1. *(A Compactness Lemma) Let $(u_n) \subset \mathcal{N}_{V_0}$ be a sequence satisfying $I_{V_0}(u_n) \to c_{V_0}$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that (v_n) has a convergent subsequence in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, where $v_n(x) := u_n(x + \tilde{y}_n)$. In particular, there exists a minimizer for c_{V_0}.*

Proof. Applying Ekeland's Variational Principle (see Theorem 8.5 in [62]), we may suppose that (u_n) is a $(PS)_{c_{V_0}}$ for I_{V_0}. From Lemma [6, Lemma 2.3] we can assume that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and $I'_{V_0}(u) = 0$.

If $u \neq 0$, then u is a ground state solution of the limit problem (P_{V_0}), that is, $I_{V_0}(u) = c_{V_0}$. In fact, using arguments found in [6, Lemma 2.3], we have that

$$\nabla u_n(x) \to \nabla u(x) \quad a.e \text{ in } \mathbb{R}^N \quad \text{and} \quad I'_{V_0}(u) = 0. \tag{1.3.1}$$

Then, by (1.0.2), (1.0.3) and the Fatou's Lemma,

$$\begin{aligned}
0 &\leq \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)]\, dx - \frac{1}{\theta}\int_{\mathbb{R}^N} [a(|\nabla u|^p)|\nabla u|^p + V_0 B(|u|^p)|u|^p]\, dx \\
&\leq \liminf_{n\to+\infty} \left\{ \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u_n|^p) + V_0 B(|u_n|^p)]\, dx \right. \\
&\qquad \left. - \frac{1}{\theta}\int_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p + V_0 B(|u_n|^p)|u_n|^p]\, dx \right\}
\end{aligned}$$

Hence, if $u \in \mathcal{N}_{V_0}$,

$$c_{V_0} \leq I_{V_0}(u) - \frac{1}{\theta} I'_{V_0}(u)u \leq \liminf_{n\to+\infty} \left[I_{V_0}(u_n) - \frac{1}{\theta} I'_{V_0}(u_n)u_n \right] = \lim_{n\to+\infty} I_{V_0}(u_n) = c_{V_0}.$$

By (1.3.1), (a_1), (b_1) and Lebesgue's theorem we conclude that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. Consequently, $I_{V_0}(u) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $u \equiv 0$, then in this case we cannot have $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ because $c_{V_0} > 0$. Hence, using [6, Proposition 2.1], there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that

$$v_n \rightharpoonup v \quad in \quad W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N),$$

where $v_n(x) := u_n(x + \tilde{y}_n)$. Therefore, v_n is also a $(PS)_{c_{V_0}}$ sequence for I_{V_0} and $v \not\equiv 0$. It follows from the above arguments that, up to a subsequence, (v_n) converges strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and the proof is complete. □

Proposition 1.3.2. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to c_{V_0}$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that (v_n) has a convergent subsequence in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, where $v_n(x) := u_n(x + \tilde{y}_n)$. Moreover, up to a subsequence, $y_n \to y \in \Omega$, where $y_n := \epsilon_n \tilde{y}_n$.*

Proof. Since $c_{V_0} > 0$, from Lemma [8, Proposition 5], there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants R and $\widetilde{\beta}$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} | u_n |^N \geq \widetilde{\beta} > 0,$$

and then, up to a subsequence, $v_n \rightharpoonup v \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. Let $t_n > 0$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$ then, since $v_n \in \mathcal{N}_{\epsilon_n}$ we obtain

$$c_{V_0} \leq I_{V_0}(\tilde{v}_n) \leq I_{\epsilon_n}(\tilde{v}_n) \leq I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u_n) = c_{V_0} + o_n(1), \tag{1.3.2}$$

which implies that

$$I_{V_0}(\tilde{v}_n) \to c_{V_0}.$$

From (1.3.2) and since (v_n) is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, we obtain that (t_n) is bounded. As a consequence, the sequence $(\tilde{v}_n)$ is also bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ which implies, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. We can assume that $t_n \to t_0 > 0$, and this limit implies that $\tilde{v} \not\equiv 0$. From Lemma 1.3.1, $\tilde{v_n} \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, and so $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$.

To conclude the proof of this proposition, we consider $y_n := \epsilon_n \tilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to y$ for $y \in \Omega$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denote by (y_n), verifying $|y_n| \to \infty$. From (a_1), (b_1) and (V_1) we have

$$\int\limits_{\mathbb{R}^N} \left[k_1|\nabla v_n|^p + |\nabla v_n|^N\right] dx + V_0 \int\limits_{\mathbb{R}^N} \left[k_3|v_n|^p + |v_n|^N\right] dx \leq \int\limits_{\mathbb{R}^N} g(\epsilon_n x + y_n, v_n) v_n dx.$$

Fix $R > 0$ such that $B_R(0) \supset \Omega$ and let $\mathcal{X}_{B_R(0)}$ be the characteristic function of $B_R(0)$. Since $\mathcal{X}_{B_R(0)}(\epsilon x + y_n) = o_n(1)$ for all $x \in B_R(0)$ and $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, then

$$\int\limits_{\mathbb{R}^N} \mathcal{X}_{B_R(0)}(\epsilon x + y_n) g(\epsilon x + y_n, v_n) v_n dx = o_n(1).$$

By definition of $\widetilde{f}$ we obtain that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} \left[k_1|\nabla v_n|^p + |\nabla v_n|^N\right] dx + V_0 \int\limits_{\mathbb{R}^N} \left[k_3|v_n|^p + |v_n|^N\right] dx \quad &\leq \int\limits_{\mathbb{R}^N \setminus B_R(0)} \widetilde{f}(v_n) v_n dx + o_n(1) \\
&\leq \frac{V_0}{\beta} \int\limits_{\mathbb{R}^N} |v_n|^N dx + o_n(1).
\end{aligned}$$

It follows that $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, obtain this way a contradiction because $c_{V_0} > 0$. Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to \overline{y} \in \mathbb{R}^N.$$

Arguing as above, if $\overline{y} \notin \overline{\Omega}$ we will obtain again $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, and then $\overline{y} \in \overline{\Omega}$. Now if $V(\overline{y}) = V_0$, we have $\overline{y} \notin \partial\Omega$ and consequently $\overline{y} \in \Omega$. Suppose by contradiction that $V(\overline{y}) > V_0$, then

$$c_{V_0} = I_{V_0}(\widetilde{v}) < \frac{1}{p} \int\limits_{\mathbb{R}^N} A(|\nabla \widetilde{v}|^p)dx + \frac{1}{p} \int\limits_{\mathbb{R}^N} V(\overline{y})B(|\widetilde{v}|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v})dx.$$

Using the fact that $\widetilde{v}_n \to \widetilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, from Fatou's Lemma we obtain that

$$c_{V_0} < \liminf_{n\to\infty} \left[\frac{1}{p} \int\limits_{\mathbb{R}^N} A(|\nabla \widetilde{v}_n|^p)dx + \frac{1}{p} \int\limits_{\mathbb{R}^N} V(\epsilon_n x + y_n)B(|\widetilde{v}_n|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v}_n)dx \right].$$

Therefore, since $(u_n) \in \mathcal{N}_{\epsilon_n}$,

$$c_{V_0} < \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_{V_0},$$

obtaining a contradiction. □

Lemma 1.3.3. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}^{\pm}_{\epsilon_n}$ be a nodal solution of problem $(P_{\epsilon_{aux}})$ such that $I_{\epsilon_n}(u^{\pm}_n) \to c_{V_0}$. Then $(v_{i,n})$ converges uniformly on compacts of $\mathbb{R}^N$, where $v_{1,n}(x) := u^+_n(x + \tilde{y}_{1,n})$ and $v_{2,n}(x) := u^-_n(x + \tilde{y}_{2,n})$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|v_{i,n}\|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi \quad \textit{for all} \quad n \geq n_0 \quad \textit{and} \quad i = 1, 2,$$

where $(\tilde{y}_{1,n})$ and $(\tilde{y}_{2,n})$ were given in Proposition 1.3.2.

Proof. Adapting some arguments explored in [6, Lemma 5.5], we have that the sequences $(v_{1,n})$ and $(v_{2,n})$ are bounded in $L^\infty(\mathbb{R}^N)$ and there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|v_{i,n}\|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} < \xi, \quad \text{for all} \quad n \geq n_0 \quad \text{and} \quad i = 1, 2.$$

Then, for any bounded domain $\Omega' \subset \mathbb{R}^N$, from (g_1) and (g_2) and continuity of V there exists $C > 0$ such that

$$|V(\epsilon_n x)|u_n|^{p-1} - g(\epsilon_n x, u_n)| \leq C, \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence,

$$|V(\epsilon_n x)|u_n|^{p-1} - g(\epsilon_n x, u_n)| \leq C + |\nabla u_n|^p, \quad \text{for all} \quad n \in \mathbb{N}.$$

Considering $\Psi(x) = C$, we get that $\Psi \in L^t(\Omega')$ with $t > \frac{p}{p-1}N$. From [28, Theorem 1], we have

$$|\nabla u_n| \in L^\infty_{loc}(\mathbb{R}^N)$$

Therefore, for all compact $K \subset \Omega'$ there exists a constant $C_0 > 0$, dependent only on C, N, p and $\text{dist}(K, \partial\Omega')$ such that

$$\|\nabla u_n\|_{\infty,K} \leq C_0.$$

Then,

$$|u_n|_{C^{0,\nu}_{loc}(\mathbb{R}^N)} \leq C, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad 0 < \nu < 1.$$

From Schauder's embedding, (u_n) has a subsequence convergent in $C^{0,\nu}_{loc}(\mathbb{R}^N)$. □

Lemma 1.3.4. *Given $\epsilon > 0$, the nodal solution u_ϵ of problem $(P_{\epsilon_{aux}})$ satisfies*

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

As a consequence

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon^+) = c_{V_0} \quad \textit{and} \quad \lim_{\epsilon \to 0} I_\epsilon(u_\epsilon^-) = c_{V_0}.$$

Proof. Consider $z_0 \in \Omega$ such that $V(z_0) = V_0$. Let us now consider $R > 0$ and set $Q_1,\ Q_2 \in \partial B_R(z_0)$ such that $|Q_1 - Q_2| = 2R$. If necessary, take R small enough such that $B(Q_i, R/4) \subset \Omega$. Taking $\psi_i : \mathbb{R}^N \to \mathbb{R}$ such that $\psi_i \equiv 1$ in $B(Q_i, R/4)$ and $\psi_i \equiv 0$ in $\mathbb{R}^N \backslash B(Q_i, R/2)$.

For $i = 1, 2$, let $w_i \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ be a ground-state positive solution (see Lemma 1.3.1) of the problem

$$-div(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(Q_i)b(|u|^p)(|u|^{p-2}u) = f(u) \quad \text{in } \mathbb{R}^N$$

which satisfies

$$C_{V(Q_i)} = I_{V(Q_i)}(w_i) = \inf_{v \in W_0 \backslash 0} \sup_{t \geq 0} I_{V(Q_i)}(tv),$$

where

$$I_{V(Q_i)}(v) = \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla v|^p) + V(Q_i)B(|v|^p)]\, dx - \int_{\mathbb{R}^N} G_\epsilon(\epsilon x, v) dx.$$

Then, we consider the function $w_{\epsilon, Q_i} : \mathbb{R}^N \to \mathbb{R}$ be given by

$$w_{\epsilon, Q_i}(x) := \psi_i(\epsilon x) w_i \left(x - \frac{Q_i}{\epsilon} \right) \in W_\epsilon$$

and $t_{\epsilon,i} > 0$, such that $t_{\epsilon,i} w_{\epsilon, Q_i} \in \mathcal{N}_\epsilon$. By the construction, we have

$$\overline{w}_\epsilon := t_{\epsilon,1} w_{\epsilon, Q_1} - t_{\epsilon,2} w_{\epsilon, Q_2} \in \mathcal{N}_\epsilon^\pm.$$

By $supp(w_{\epsilon,Q_1}) \cap supp(w_{\epsilon,Q_2}) = \emptyset$, once $B(Q_1, R) \cap B(Q_2, R) = \emptyset$, and w_i, for $i = 1, 2$, are positive solutions then

$$supp(\overline{w}_\epsilon^+) \cap supp(\overline{w}_\epsilon^-) = \emptyset, \quad \overline{w}_\epsilon^+ = t_{\epsilon,1} w_{\epsilon, Q_1} \text{ and } \overline{w}_\epsilon^- = -t_{\epsilon, Q_2} w_{\epsilon, Q_2} \ .$$

Then

$$I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-) \ \text{ and } \ I'_\epsilon(\overline{w}_\epsilon^\pm)\overline{w}_\epsilon^\pm = 0.$$

Hence

$$I_\epsilon(u_\epsilon) \leq I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-). \tag{1.3.3}$$

Therefore, with a direct computation we have

$$I_\epsilon(u_\epsilon) \leq I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-) = c_{V(Q_1)} + c_{V(Q_2)} + o_\epsilon(1).$$

Finally, taking $R \to 0$ in the last inequality and using the continuity of the minimax function (see [13], [51]) we get

$$\limsup_{\epsilon \to 0} I_\epsilon(u_\epsilon) \leq 2c_{V_0}.$$

Now let $t_\epsilon^{\pm} > 0$ be such that $t_\epsilon^{\pm} u_\epsilon^{\pm} \in \mathcal{N}_{V_0}$. Then,

$$2c_{V_0} \leq I_{V_0}(t_\epsilon^+ u_\epsilon^+) + I_{V_0}(t_\epsilon^- u_\epsilon^-) \leq I_\epsilon(t_\epsilon^+ u_\epsilon^+) + I_\epsilon(t_\epsilon^- u_\epsilon^-) \leq I_\epsilon(u_\epsilon^+) + I_\epsilon(u_\epsilon^-) = I_\epsilon(u_\epsilon).$$

Hence we have proved that

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

On the other hand, we know that $c_{V_0} \leq I_{V_0}(t_\epsilon^{\pm} u_\epsilon^{\pm}) \leq I_\epsilon(t_\epsilon^{\pm} u_\epsilon^{\pm}) \leq I_\epsilon(u_\epsilon^{\pm})$. Therefore,

$$c_{V_0} \leq \liminf_{\epsilon \to 0} I_\epsilon(u_\epsilon^{\pm}).$$

Assume by contradiction that at least one inequality is strict, then arguing as above we obtain

$$2c_{V_0} < \liminf_{\epsilon \to 0} \left(I_\epsilon(u_\epsilon^+) + I_\epsilon(u_\epsilon^-) \right) = I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

□

Lemma 1.3.5. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}_{\epsilon_n}^{\pm}$ be a solution of problem $(P_{\epsilon_{aux}})$. Then, there are $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that for $v_{1,n}(x) := u_n^+(x + \tilde{y}_{1,n})$ and $v_{2,n}(x) := u_n^-(x + \tilde{y}_{2,n})$, we have*

$$v_{1,n}(x) \geq \delta^*, \quad \text{for all} \quad x \in B_R(0) \ \text{ and } \ n \geq n_0$$

and

$$v_{2,n}(x) \leq -\delta^*, \quad \text{for all} \quad x \in B_R(0) \quad \text{and} \quad n \geq n_0,$$

where $R > 0$, $(\tilde{y}_{1,n})$ and $(\tilde{y}_{2,n})$ were given in Proposition 1.3.2.

Proof. Suppose by contradiction that $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N \setminus B_R(0))} \to 0$, for $i = 1$ or $i = 2$. Then by Lemma 1.3.3, we have $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N)} \to 0$. It follows from (f_1) that

$$|f(v_{i,n})| \leq \frac{V_0}{2} |v_{i,n}|^{N-1} \quad \text{for} \quad n \quad \text{sufficient large.} \tag{1.3.4}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} a(|\nabla v_{i,n}|^p)|\nabla v_{i,n}|^p dx \quad &+ \quad \int_{\mathbb{R}^N} V(\epsilon_n x + y_{i,n}) b(|v_{i,n}|^p)|v_{i,n}|^p dx \\ &= \quad \int_{\mathbb{R}^N} f(v_{i,n}) v_{i,n} dx + o_n(1) \\ &\leq \quad \frac{V_0}{2} \int_{\mathbb{R}^N} |v_{i,n}|^N dx + o_n(1), \end{aligned}$$

which implies from (a_1) and (b_1) that,

$$\|u_n^{\pm}\|_{W_{\epsilon_n}} \to 0,$$

which is a contradiction with Lemma 1.3.4. □

We are now ready to show the concentration result.

Lemma 1.3.6. *If $\frac{P_\epsilon^1}{\epsilon}$ is the maximum point of u_ϵ and $\frac{P_\epsilon^2}{\epsilon}$ is the minimum point of u_ϵ, then*

$$\lim_{\epsilon \to 0} V(P_\epsilon^i) = V_0 \quad \textit{for } i = 1, 2.$$

Proof. We first notice that using Lemma 1.3.5 there exist $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$v_{1,n}(q_n^1) := \max_{z\in\mathbb{R}^N} v_{1,n}(z) = u_n^+(q_n^1 + \tilde{y}_{1,n}) \geq u_n^+(x) \geq \delta^*, \quad \text{for all } n \geq n_0, \quad \text{for all } \ x \in B_R(0)$$

and

$$v_{2,n}(q_n^2) := \min_{z\in\mathbb{R}^N} v_{2,n}(z) = u_n^-(q_n^2 + \tilde{y}_{2,n}) \leq u_n^-(x) \leq -\delta^*, \quad \text{for all } n \geq n_0, \text{ for all } \ x \in B_R(0).$$

We claim that q_n^i, $i = 1, 2$ is bounded, otherwise using Lemma 1.3.3 and 1.3.5, there exists $R^* > 0$ such that $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N\setminus B_{R^*})} \leq \frac{\delta^*}{2}$, which implies that $|v_{i,n}(q_n^i)| \leq \frac{\delta^*}{2}$, where we obtain a contradiction. Then, $P^i_{\epsilon_n} = \epsilon_n q_n^i + y_{i,n}$ implies

$$\lim_{n\to+\infty} P^i_{\epsilon_n} = \lim_{n\to+\infty} y_{i,n} = \overline{y}_i \in \Omega.$$

Hence from continuity of V it follows that

$$\lim_{n\to+\infty} V(P^i_{\epsilon_n}) = V(\overline{y}_i) \geq V_0.$$

We claim that $V(\overline{y}_i) = V_0$. Indeed, suppose by contradiction that $V(\overline{y}_i) > V_0$. Using the same arguments of Proposition 1.3.2, we have that $\widetilde{v}_{i,n} \to \widetilde{v}_i$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and

$$c_{V_0} = \ I_{V_0}(\widetilde{v}_i) < \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla\widetilde{v}_i|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\overline{y}_i)B(|\widetilde{v}_i|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v}_i)dx.$$

Using that $\widetilde{v}_{i,n} \to \widetilde{v}_i$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and from Fatou's Lemma, we obtain

$$c_{V_0} < \liminf_{n\to\infty} \left[\frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla\widetilde{v}_{i,n}|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\epsilon_n x + y_{i,n})B(|\widetilde{v}_{i,n}|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v}_{i,n})dx\right],$$

and therefore

$$c_{V_0} < \liminf_{n\to\infty} I_{\epsilon_n}(t_{i,n}u_n^{\pm}) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n^{\pm}) = c_{V_0}.$$

This contradiction shows that $V(\overline{y}_i) = V_0$ for $i = 1, 2$. □

Lemma 1.3.7. *Let $\{\epsilon_n\}$ be a sequence of positive number such that $\epsilon_n \to 0$ as $n \to \infty$ and let $(x_n) \subset \overline{\Omega}_{\epsilon_n}$ be a sequence such that $u^+_{\epsilon_n}(x_n) \geq \Upsilon > 0$ or $u^-_{\epsilon_n}(x_n) \leq -\Upsilon < 0$ for each $n \in \mathbb{N}$ and for some Υ positive constant, where u_{ϵ_n} is a solution of $(P_{\epsilon_{aux}})$. Then,*

$$\lim_{n\to\infty} V(\overline{x}_n) = V_0$$

where $\overline{x}_n = \epsilon_n x_n$.

Proof. Up to a subsequence,

$$\overline{x}_n \to \overline{x} \in \overline{\Omega}.$$

From Lemma 1.3.4 we have that $u^+_{\epsilon_n} \in \mathcal{N}_{\epsilon_n}$,

$$I_{\epsilon_n}(u^+_{\epsilon_n}) \to c_{V_0},$$

and there exists a positive constants such that

$$\|u^+_{\epsilon_n}\| \leq C, \quad \forall\ n \in \mathbb{N} \quad \text{ and for some } \ C > 0.$$

Setting $v_n(z) := u^+_{\epsilon_n}(z + x_n)$, we have $\|v_n\| \le C$ and $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$. Recalling that

$$v_n(0) = u^+_{\epsilon_n}(x_n) \ge \Upsilon > 0,$$

we conclude that $v \not\equiv 0$.
Fix $t_n > 0$ verifying $\widetilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$, for each $n \in \mathbb{N}$. Hence

$$c_{V_0} \le I_{V_0}(\widetilde{v}_n) \le I_{\epsilon_n}(t_n v_n) \le I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u^+_n) = c_{V_0} + o_n(1).$$

Thus $I_{V_0}(\widetilde{v}_n) \to c_{V_0}$ with $\{\widetilde{v}_n\} \subset \mathcal{N}_{V_0}$. By Lemma 1.3.1, we have

$$\widetilde{v}_n \to \widetilde{v} \quad in \quad W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N) \;\text{ and }\; I_{V_0}(\widetilde{v}) = c_{V_0}. \tag{1.3.5}$$

Since $v \neq 0$, by Lemma 1.3.1 we have $y_n = 0$, for $n \in \mathbb{N}$. Moreover, recalling that V is continuous, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V(\overline{x}).$$

We claim that $V(\overline{x}) = V_0$. Indeed, suppose by contradiction that $V(\overline{x}) > V_0$, then

$$c_{V_0} = I_{V_0}(\widetilde{v}) < \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla \widetilde{v}|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\overline{x})B(|\widetilde{v}|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v})dx$$

and by (1.3.5) and Fatou's Lemma

$$\begin{aligned} c_{V_0} &< \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla \widetilde{v}_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + \overline{x}_n)B(|\widetilde{v}_n|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v}_n)dx\right] \\ &\le \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla t_n v_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + \overline{x}_n)B(|t_n v_n||^p)dx \right. \\ &\quad \left. - \int_{\mathbb{R}^N} G(\epsilon_n x + \overline{x}, t_n v_n)dx\right] \\ &= \liminf_{n\to\infty} I_{\epsilon_n}(t_n u^+_n) \le \liminf_{n\to\infty} I_{\epsilon_n}(u^+_n) = c_{V_0}, \end{aligned}$$

which leads an absurd. Consequently $\lim\limits_{n\to\infty} V(\overline{x}_n) = V_0$ and the lemma is proved. □

Lemma 1.3.8. *If m^+_ϵ is given by*

$$m^+_\epsilon := \sup\left\{\max_{\partial\Omega_\epsilon} u_\epsilon : u_\epsilon \in \mathcal{N}^\pm_\epsilon \quad \textit{is a solution of } (P_{\epsilon_{aux}})\right\}$$

and if m^-_ϵ is given by

$$m^-_\epsilon = \sup\left\{\min_{\partial\Omega_\epsilon} u_\epsilon : u_\epsilon \in \mathcal{N}^\pm_\epsilon \quad \textit{is a solution of } (P_{\epsilon_{aux}})\right\},$$

then there exists $\overline{\epsilon} > 0$ such that the sequences $(m^\pm_\epsilon)$ are bounded for all $\epsilon \in (0, \overline{\epsilon})$. Moreover, we have

$$\lim_{\epsilon\to 0} m^\pm_\epsilon = 0.$$

Proof. Suppose, by contradiction, $\lim\limits_{\epsilon\to 0} m^+_\epsilon = +\infty$ or $\lim\limits_{\epsilon\to 0} m^-_\epsilon = -\infty$, then there exist u_ϵ a solution of $(P_{\epsilon_{aux}})$ in $\mathcal{N}^\pm_\epsilon$ and $\Upsilon > 0$ such that

$$\max_{\partial\Omega_\epsilon} u^+_\epsilon \ge \Upsilon > 0$$

or

$$\max_{\partial\Omega_\epsilon} u_\epsilon^- \leq -\Upsilon < 0.$$

Thus there exists $\{\epsilon_n\} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ and there exists a sequence $\{x_n\} \subset \partial\Omega_{\epsilon_n}$ such that

$$u_{\epsilon_n}^+(x_n) \geq \Upsilon > 0 \quad \text{or} \quad u_{\epsilon_n}^-(x_n) \leq -\Upsilon < 0.$$

Thus, by Lemma 1.3.7, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V_0,$$

where $\overline{x}_n = \epsilon_n x_n$ and $\{\overline{x}_n\} \subset \partial\Omega$. Hence, up to a subsequence, we have $\overline{x}_n \to \overline{x}$ in $\partial\Omega$ and $V(\overline{x}) = V_0$, which does not make sense by (V_2). Hence, there exists $\overline{\epsilon} > 0$ such that $(m_\epsilon^\pm)$ is bounded, for all $\epsilon \in (0, \overline{\epsilon})$.

We have now to prove that $\lim_{\epsilon\to 0} m_\epsilon^\pm = 0$. Then, suppose by contradiction that there exists $\delta > 0$ and a sequence $\{\epsilon_n\} \subset \mathbb{R}^+$ satisfying

$$m_{\epsilon_n}^+ \geq \delta > 0$$

or

$$m_{\epsilon_n}^- \leq -\delta < 0.$$

Thus, there exists u_{ϵ_n} a solution of $(P_{\epsilon_{aux}})$ in $\mathcal{N}_{\epsilon_n}^\pm$ such that

$$m_{\epsilon_n}^+ - \frac{\delta}{2} < \max_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^+ \leq m_{\epsilon_n}^+$$

or

$$m_{\epsilon_n}^- \leq \min_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^- < m_{\epsilon_n}^- + \frac{\delta}{2}.$$

Hence,

$$\frac{\delta}{2} = \delta - \frac{\delta}{2} \leq m_{\epsilon_n}^+ - \frac{\delta}{2} < \max_{\partial\Omega_\epsilon} u_{\epsilon_n}^+,$$

$$\min_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^- < m_{\epsilon_n}^- + \frac{\delta}{2} < -\delta + \frac{\delta}{2} = -\frac{\delta}{2}$$

and then there exists a sequence $(x_n) \subset \partial\Omega_{\epsilon_n}$, such that

$$u_{\epsilon_n}^+(x_n) \geq \frac{\delta}{2}$$

or

$$u_{\epsilon_n}^-(x_n) \leq -\frac{\delta}{2}.$$

Repeating the above arguments, we will get an absurd. Thus, the proof is finished. □

Proof of Theorem 1.2.1

Proof. The proof is a consequence of Subsections 1.2 and 1.3. □

1.4 Proof of Theorem 1

Proof. Let u_ϵ be a solution of $(P_{\epsilon_{aux}})$. By Lemma 1.3.8, there exists $\overline{\epsilon} > 0$ such that $|m_\epsilon^{\pm}| < \frac{\eta}{2}$ for all $\epsilon \in (0, \overline{\epsilon})$, then $(u_\epsilon - \frac{\eta}{2})_+(x) \equiv 0$ for a neighborhood from $\partial\Omega_\epsilon$. Hence, $(u_\epsilon - \frac{\eta}{2})_+ \in W_0^{1,p}(\mathbb{R}^N \backslash \Omega_\epsilon) \cap W_0^{1,N}(\mathbb{R}^N \backslash \Omega_\epsilon)$ and the function $(u_\epsilon - \frac{\eta}{2})_+^* \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$, where

$$(u_\epsilon - \frac{\eta}{2})_+^*(x) = \begin{cases} 0 \text{ if } x \in \Omega_\epsilon, \\ \\ (u_\epsilon - \frac{\eta}{2})_+(x) \;\text{ if }\; x \in \mathbb{R}^N \backslash \Omega_\epsilon. \end{cases}$$

Using $(u_\epsilon - \frac{\eta}{2})_+^*$ as test function. Then, by (a_1), (b_1) and $(g_3)_{ii}$, we have

$$\begin{aligned} 0 \;\; \leq & \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} a(|\nabla u_\epsilon|^p)|\nabla (u_\epsilon - \frac{\eta}{2})_+^*|^p dx \\ & + \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} \left[V_0 b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{N-2} \right] ((u_\epsilon - \frac{\eta}{2})_+^*)^2 dx \\ & + \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} \left[V(\epsilon x) b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{N-2} \right] \frac{\eta}{2}(u_\epsilon - \frac{\eta}{2})_+^* dx = 0 \end{aligned}$$

The last equality implies

$$(u_\epsilon - \frac{\eta}{2})_+^* = 0, \;\; \text{a.e in } \; x \in \mathbb{R}^N \backslash \Omega_\epsilon.$$

Hence, $u_\epsilon \leq \frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$.

Since we can assume $m_\epsilon^- \leq -\frac{\eta}{2}$ for $\epsilon \in (0, \overline{\epsilon})$, working with the function $(u_\epsilon + \frac{\eta}{2})_-^*$, it is possible to prove that $u_\epsilon \geq -\frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$. This fact implies that $|u_\epsilon| \leq \frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$ and by Remark 1 the result follows. □

Finally, we are going to prove the exponential decay. First technical results.

1.5 Exponential decay

Lemma 1.5.1. *Consider $M, \alpha > 0$ and $\psi(x) := M \exp(-\alpha|x|)$. Then*

$$\begin{aligned} & i) - div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \\ & \qquad = \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1) \right) \right], \\ & ii) - div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \geq \left(\frac{(N-1)}{|x|} - \alpha(N-1) \right) a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}. \end{aligned}$$

Proof. Note that

$$\frac{\partial\psi}{\partial x_i}(x) = M\exp(-\alpha|x|)\frac{\partial}{\partial x_i}(-\alpha|x|) = M\exp(-\alpha|x|)(-\alpha)\frac{x_i}{|x|} = -\alpha\frac{x_i}{|x|}\psi(x),$$

which implies $|\nabla\psi| = \alpha\psi$. Then

$$-div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial x_i}\right]$$

$$= \alpha^{p-1}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[a(\alpha^p\psi^p)\,\psi^{p-1}\frac{x_i}{|x|}\right]$$

$$= \alpha^{p-1}\sum_{i=1}^{N}\left[a'(\alpha^p\psi^p)\frac{\partial}{\partial x_i}(\alpha^p\psi^p)\,\psi^{p-1}\frac{x_i}{|x|} + a(\alpha^p\psi^p)\frac{\partial}{\partial x_i}\left(\psi^{p-1}\frac{x_i}{|x|}\right)\right]$$

$$= \alpha^{p-1}\sum_{i=1}^{N}\left[a'(\alpha^p\psi^p)\,\alpha^p p\psi^{2p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|} + a(\alpha^p\psi^p)\left(\frac{|x|^2-x_i^2}{|x|^3}\psi^{p-1} + (p-1)\psi^{p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|}\right)\right]$$

$$= \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)\right],$$

this proves the first item.

To show item $ii)$ we are going to use (1.2) and item $i)$. Hence we have

$$-a'(\alpha^p\psi^p)\alpha^p\psi^p \geq -\frac{(N-p)}{p}a(\alpha^p\psi^p),$$

and

$$-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} \geq -\alpha\psi^{p-1}(N-p)a(\alpha^p\psi^p).$$

Consequently, by item $i)$,

$$-div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi)$$

$$\geq \alpha^{p-1}\left[-\alpha(N-p)a(\alpha^p\psi^p)\psi^{p-1} + \left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)a(\alpha^p\psi^p)\psi^{p-1}\right]$$

$$= \left(\frac{(N-1)}{|x|} - \alpha(N-1)\right)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}.$$

□

Corollary 1.5.2. *Since $V(x) \geq V_0$ in $\mathbb{R}^N$, then for $\alpha > 0$ small enough we have*

$$-div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{N-1} \geq 0 \text{ in } \mathbb{R}^N.$$

Proof. Using (a_1) and Lemma 1.5.1 we obtain that

$$\begin{aligned}
-div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) &\geq -\alpha(N-1)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}\\
&\geq -\alpha(N-1)\left(k_2\alpha^{p-1}\psi^{p-1} + \alpha^{N-1}\psi^{N-1}\right)\\
&= -\alpha(N-1)k_2\alpha^{p-1}\psi^{p-1} - \alpha(N-1)\alpha^{N-1}\psi^{N-1}
\end{aligned}$$

Moreover, since $V_0 > 0$ and $\alpha > 0$ is small enough we conclude that

$$k_3V_0 - \alpha(N-1)k_2\alpha^{p-1} \geq 0,$$

and

$$\frac{V_0}{4} - \alpha(N-1)\alpha^{N-1} \geq 0.$$

Consequently

$$-div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{N-1} \geq 0 \text{ in } \mathbb{R}^N.$$

□

Let us now relate the nodal solution u_ϵ to the exponential function ψ for small ϵ.

Lemma 1.5.3. *Let u_ϵ be the solution found in Theorem 1.2.1 and $v_{1,\epsilon}(x) := u_\epsilon^+(x + \tilde{y}_{1,\epsilon})$ and $v_{2,\epsilon}(x) := u_\epsilon^-(x + \tilde{y}_{2,\epsilon})$ given in Proposition 1.3.2. Setting $\varphi_{i,\epsilon} := \max\{|v_{i,\epsilon}| - \psi, 0\}$ for $i = 1, 2$, then for $\epsilon > 0$ small enough, we have*

$$\int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\,dx + k_3 V_0\int_{\mathbb{R}^N}|v_{i,\epsilon}|^{p-1}\varphi_{i,\epsilon}\,dx + \frac{V_0}{4}\int_{\mathbb{R}^N}|v_{i,\epsilon}|^{N-1}\varphi_{i,\epsilon}\,dx \le 0.$$

Proof. From Lemma 1.3.3, Lemma 1.3.4 and hypothesis (f_1), there exist $\rho_0 > 0$ such that $\epsilon > 0$ small enough,

$$\frac{f(|v_{i,\epsilon}|)}{|v_{i,\epsilon}|^{N-1}} \le \frac{3}{4}V_0, \quad \text{for all } \ |x| \ge \rho_0.$$

Since $\psi(x) := M\exp(-\alpha|x|)$ for $x \in \mathbb{R}^N$, we can find $\widetilde{M} > 0$ such that if $M \ge \widetilde{M}$, then $\varphi_{i,\epsilon} := \max\{|v_{i,\epsilon}| - \psi, 0\} \equiv 0$ in $B_{\rho_0}(0)$ and $\varphi_{i,\epsilon} \in W^{1,p}(|x| \ge \rho_0) \cap W^{1,N}(|x| \ge \rho_0)$. Therefore, the above inequality and (b_1),

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\,dx + V_0\int_{\mathbb{R}^N}\left[k_3|v_{i,\epsilon}|^{p-1}\varphi_{i,\epsilon} + |v_{i,\epsilon}|^{N-1}\varphi_{i,\epsilon}\right]dx \\
&\le \int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\,dx + \int_{\mathbb{R}^N} V(\epsilon x + y_{i,\epsilon})b(|v_{i,\epsilon}|^p)|v_{i,\epsilon}|^{p-2}v_{i,\epsilon}\varphi_{i,\epsilon}\,dx \\
&\le \int_{\mathbb{R}^N} f(|v_{i,\epsilon}|)\varphi_{i,\epsilon}\,dx \le \frac{3V_0}{4}\int_{\mathbb{R}^N}|v_{i,\epsilon}|^{N-1}\varphi_{i,\epsilon}dx
\end{aligned}$$

and the lemma is proved. □

Finally we are going to show the exponential decay for the functions u_ϵ.

Proposition 1.5.4. *There are $\epsilon_0 > 0$ and $C > 0$ such that*

$$|u_\epsilon(z)| \le C\left[\exp\left(-\alpha\left|\frac{z - P_\epsilon^1}{\epsilon}\right|\right) + \exp\left(-\alpha\left|\frac{z - P_\epsilon^2}{\epsilon}\right|\right)\right], \quad \textit{for all } z \in \mathbb{R}^N.$$

Proof. From [30, Lemma 2.4], we have that

$$\left\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x - y\right\rangle \ge 0, \ \forall\ x, y \in \mathbb{R}^N.$$

Consider $v_{1,\epsilon}(x) := u_\epsilon^+(x + \tilde{y}_{1,\epsilon})$, $v_{2,\epsilon}(x) := u_\epsilon^-(x + \tilde{y}_{2,\epsilon})$ and the set

$$\Lambda^i := \{x \in \mathbb{R}^N : |x| \ge \rho_0 \ \text{ and } \ |v_{i,\epsilon}| - \psi \ge 0\},$$

where ψ is the function is given by Lemma 1.5.1, $(\widetilde{y}_{1,n})$ and $(\widetilde{y}_{2,n})$ are given by

Proposition 1.3.2. Then, using Corollary 1.5.2 and Proposition 1.5.3, we obtain

$$
\begin{aligned}
0 &\geq \int\limits_{\mathbb{R}^N} \left\langle a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon} - a(|\nabla \psi|^p)|\nabla \psi|^{p-2}\nabla \psi, \nabla \varphi_{i,\epsilon}\right\rangle dx \\
&+V_0 k_3 \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right)\varphi_{i,\epsilon}\, dx + \frac{V_0}{4}\int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{N-1} - |\psi|^{N-1}\right)\varphi_{i,\epsilon}\, dx \\
&\geq V_0 k_3 \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right)\varphi^{\pm} dx + \frac{V_0}{4}\int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{N-1} - |\psi|^{N-1}\right)\varphi_{i,\epsilon} dx \\
&= V_0 k_3 \int\limits_{\Lambda^i} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right)(|v_{i,\epsilon}| - \psi) dx \\
&+\frac{V_0}{4}\int\limits_{\Lambda^i} \left(|v_{i,\epsilon}|^{N-1} - |\psi|^{N-1}\right)(|v_{i,\epsilon}| - \psi) dx \geq 0.
\end{aligned}
$$

Then $|\Lambda^i| = 0$, for $i = 1, 2$ and consequently

$$
|v_{1,\epsilon}(x)| + |v_{2,\epsilon}(x)| \leq 2M \exp(-\alpha|x|), \ \forall \ |x| \geq \rho_0.
$$

Considering $x = z - \tilde{y}_{i,\epsilon}$ and using Lemma 1.3.6 there exists a constant $C > 0$ satisfying

$$
\begin{aligned}
&|u_\epsilon^{\pm}(z)| \leq 2M \exp\left(-\alpha\left|\frac{z - y_{i,\epsilon}}{\epsilon}\right|\right) = 2M \exp\left(-\alpha\left|\frac{z - P_\epsilon^i + \epsilon q_\epsilon^i}{\epsilon}\right|\right) \\
&\leq 2M \exp\left(-\alpha\left|\frac{z - P_\epsilon^i}{\epsilon}\right|\right)\exp\left(-\alpha\left|q_\epsilon^i\right|\right) \leq C \exp\left(-\alpha\left|\frac{z - P_\epsilon^i}{\epsilon}\right|\right)
\end{aligned}
\tag{1.5.1}
$$

for all $|z - \tilde{y}_{i,\epsilon}| \geq \rho_0$ and for $\epsilon > 0$ small enough.

Now we are going to show the inequality (1.5.1) holds, for all $z \in \mathbb{R}^N$. Since $(y_{i,\epsilon})$ converges, it follows that

$$
|z| \geq \rho_0 - |\tilde{y}_{i,\epsilon}| = \rho_0 - \frac{|y_{i,\epsilon}|}{\epsilon} > \rho_0 - \frac{1 + |y_{i,\epsilon}|}{\epsilon} \to -\infty \ \text{ as } \ \epsilon \to 0.
$$

Then, there exists $\epsilon_0 > 0$ such that

$$
|u_\epsilon(z)| \leq C\left[\exp\left(-\alpha\left|\frac{z - P_\epsilon^1}{\epsilon}\right|\right) + \exp\left(-\alpha\left|\frac{z - P_\epsilon^2}{\epsilon}\right|\right)\right], \ \ \forall \ \ z \in \mathbb{R}^N \ \text{ and } \ \forall \ \ \epsilon \in (0, \epsilon_0).
$$

□

Chapter 2

Existence and concentration of nodal solutions for a subcritical $p\&q$ equation

In this chapter we prove existence and concentration results for a family of nodal solutions for a general quasilinear equation with subcritical growth. More precisely, we study the existence and concentration of nodal solutions to the following quasilinear equation

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right) + V(z)b\left(|u|^p\right)|u|^{p-2}u = f(u) \ \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{P_ϵ}$$

where $\epsilon > 0$, $1 < p < q < N$, $N \geq 2$ and $u^+ \neq 0$, $u^- \neq 0$ in $\mathbb{R}^N$ and

$$u^+(x) := \max\{u(x), 0\} \ \text{ and } \ u^-(x) := \min\{u(x), 0\}.$$

We show that such solutions changing of sign exactly once. We say that a function $u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ is nodal solution of (P_ϵ) if $u^\pm \neq 0$ in $\mathbb{R}^N$ and

$$\int\limits_{\mathbb{R}^N} a(\epsilon^p|\nabla u|^p)\epsilon^p|\nabla u|^{p-2}\nabla u\nabla v\, dz + \int\limits_{\mathbb{R}^N} V(z)b(|u|^p)|u|^{p-2}uv\, dz = \int\limits_{\mathbb{R}^N} f(u)v\, dz,$$

for all $v \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. The hypotheses on the functions a, b, f and V are the following:

(a_1) the function a is of class C^1 and there exist constants $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^q \leq a(t^p)t^p \leq k_2 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(a_2) the mapping $t \mapsto A(t^p)$ is convex on $(0, \infty)$, where $A(t) = \displaystyle\int_0^t a(s)ds$;

(a_3) the mapping $t \mapsto \dfrac{a(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$;

(a_4) if $1 < p < 2 \leq N$ the mapping $t \mapsto a(t)$ is nondecreasing for $t > 0$. If $2 \leq p < N$ the mapping $t \mapsto a(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

As a direct consequence of (a_3) we obtain that the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \text{ for all } t > 0. \tag{2.0.1}$$

Now if we define the function $h(t) = a(t)t - \frac{q}{p}A(t)$, using (2.0.1) we can prove that the function h is nonincreasing. Then, there exists a positive real constant $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma} a(t)t \leq A(t), \quad \text{for all } t \geq 0. \tag{2.0.2}$$

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^q \leq b(t^p)t^p \leq k_4 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(b_2) the mapping $t \mapsto B(t^p)$ is convex on $(0, \infty)$, where $B(t) = \displaystyle\int_0^t b(s)ds$;

(b_3) the mapping $t \mapsto \dfrac{b(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$.

(b_4) if $1 < p < 2 \leq N$ the mapping $t \mapsto b(t)$ is nondecreasing for $t > 0$. If $2 \leq p < N$ the mapping $t \mapsto b(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

Using the hypothesis (b_3) and arguing as (2.0.1) and (2.0.2), we also can prove that there exists $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma} b(t)t \leq B(t), \quad \text{for all } t \geq 0. \tag{2.0.3}$$

The nonlinearity f is assumed to be a $C^1(\mathbb{R})$ odd function satisfying

(f_1)

$$\lim_{|s| \to 0} \frac{f'(s)}{|s|^{q-2}} = 0.$$

(f_2) There exists $q < r < q^* = \frac{qN}{N-q}$ such that

$$\lim_{|s| \to \infty} \frac{f(s)}{|s|^{r-1}} = 0.$$

(f_3) There exists $\theta \in (\gamma p, q^*)$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for} \quad s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$ and $\gamma > 0$ was given in (2.0.2);

(f_4) $s \mapsto \dfrac{f(s)}{s^{q-1}}$ is nondecreasing in $s > 0$.

The condition on potential V are:

(V_1) There is $V_0 > 0$, such that

$$0 < V_0 \leq V(z), \text{ for all } z \in \mathbb{R}^N.$$

(V_2) There exists a bounded domain $\Omega \subset \mathbb{R}^N$, such that

$$0 < V_0 = \inf_{z\in\Omega} V(z) < \inf_{z\in\partial\Omega} V(z).$$

The main result is the following:

Theorem 2. Suppose that a, b, f and V satisfy $(a_1)-(a_4)$, $(b_1)-(b_4)$, $(f_1)-(f_4)$ and $(V_1)-(V_2)$ respectively. Then there is $\epsilon_0 > 0$, such that (P_ϵ) has a nodal solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, for every $\epsilon \in (0,\epsilon_0)$. Moreover, if P_ϵ^1 is the maximum point of w_ϵ and P_ϵ^2 is the minimum point of w_ϵ, then for $i = 1, 2$, we obtain

$$\lim_{\epsilon\to 0} V(P_\epsilon^i) = V_0.$$

Moreover, there are positive constants C and α, such that

$$|w_\epsilon(z)| \leq C\left[\exp\left(-\alpha\left|\frac{z-P_\epsilon^1}{\epsilon}\right|\right)+\exp\left(-\alpha\left|\frac{z-P_\epsilon^2}{\epsilon}\right|\right)\right], \quad \forall \;\; z\in\mathbb{R}^N.$$

for all $\epsilon \in (0,\epsilon_0)$.

To prove Theorem 2, we will work with the problem below, which is equivalent to (P_ϵ) by the change of variable $z = \epsilon x$, which is given by

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+V(\epsilon x)b\left(|u|^p\right)|u|^{p-2}u = f(u) \;\; \text{in } \mathbb{R}^N, \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \end{cases} \qquad (\widetilde{P}_\epsilon)$$

where $\epsilon > 0$, $1 < p < q < N$ and $N \geq 2$.

The plan of the paper is the following: In the section 2.1, we define an auxiliary problem. In section 2.2, we prove some results to auxiliary problem and we show existence of nodal solution for this auxiliary problem. The concentration of nodal solution of auxiliary problem is showed in section 2.3. The existence of one nodal solution of the original problem is showed in section 2.4. The exponential decay of the nodal solution of the original problem is proved in section 2.5.

2.1 Variational framework and an auxiliary problem

In order to obtain solutions of $(\widetilde{P}_\epsilon)$, consider the following subspace of $W^{1,p}(\mathbb{R}^N)\bigcap W^{1,q}(\mathbb{R}^N)$,

$$W_\epsilon := \left\{v\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)b(|v|^p)|v|^p dx < +\infty\right\},$$

which is a Banach space when endowed with the norm

$$\|u\| = \|u\|_{1,p} + \|u\|_{1,q},$$

where

$$\|u\|_{1,m} = \left(\int_{\mathbb{R}^N} |\nabla u|^m dx + \int_{\mathbb{R}^N} V(\epsilon x)|u|^m dx\right)^{\frac{1}{m}}, \text{ for } m \geq 1.$$

Since the approach is variational, consider the associated energy functional $J_\epsilon : W_\epsilon \to \mathbb{R}$ given by

$$J_\epsilon(v) = \frac{1}{p}\int\limits_{\mathbb{R}^N} A\left(|\nabla v|^p\right)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\epsilon x)B\left(|v|^p\right)dx - \int\limits_{\mathbb{R}^N} F(v)dx.$$

By standard arguments, one can prove that $J_\epsilon \in C^1(W_\epsilon, \mathbb{R})$. Let θ be the number given in (f_3), $\eta, \beta > 0$ be constants satisfying $\beta > \max\left\{\dfrac{\theta p\gamma}{\theta - \gamma p}, q-1\right\}$ and $\dfrac{f(\eta)}{\eta^{q-1}} = \dfrac{V_0}{\beta}$, where V_0 appears in (V_1). Using the above numbers, we define the function

$$\widetilde{f}(s) = \begin{cases} f(s) \quad if \quad |s| \leq \frac{\eta}{2}, \\ \\ \dfrac{V_0}{\beta} s^{q-1} \quad if \quad s > \eta, \\ \\ \dfrac{V_0}{\beta}|s|^{q-2}s, \quad if \quad s < -\eta. \\ \end{cases}$$

Here we are defining the function $\widetilde{f}$ in $(-\eta, -\frac{\eta}{2})$ $(\frac{\eta}{2}, \eta)$ such that $\widetilde{f}$ is C^1 class. Note that by (f_1), given $\xi > 0$, we get

$$\widetilde{f}'(s) \leq \begin{cases} \xi|s|^{q-2} < (q-1)\frac{V_0}{\beta}|s|^{q-2} \quad if \quad |s| \leq \frac{\eta}{2}, \\ \\ (q-1)\dfrac{V_0}{\beta}|s|^{q-2} \quad if \quad s > \eta, \\ \\ (q-1)\dfrac{V_0}{\beta}|s|^{q-2}, \quad if \quad s < -\eta. \\ \end{cases} \tag{2.1.1}$$

Now we define

$$g(z,s) = \chi_\Omega(z) f(s) + (1 - \chi_\Omega(z))\widetilde{f}(s),$$

and the auxiliary problem

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + V(\epsilon x) b\left(|u|^p\right)|u|^{p-2}u = g(\epsilon x, u) \ \text{ in } \mathbb{R}^N, \\ u \in W_\epsilon, \end{cases} \tag{$P_{\epsilon_{aux}}$}$$

where χ_Ω is the characteristic function of the set Ω. It is easy to check that $(f_1) - (f_4)$ imply that g is a Carathéodory function and for $x \in \mathbb{R}^N$, the function $s \to g(\epsilon x, s)$ is of class C^1 and satisfies the following conditions, uniformly for $x \in \mathbb{R}^N$:

$$\lim_{|s|\to 0} \frac{g(\epsilon x, s)}{|s|^{q-1}} = 0 \tag{g_1}$$

$$\lim_{|s|\to \infty} \frac{g(\epsilon x, s)}{|s|^{r-1}} = 0 \tag{g_2}$$

$$0 < \theta G(\epsilon x, s) \leq g(\epsilon x, s)s, \ \forall \epsilon x \in \Omega \ \text{ and } \forall s \neq 0 \tag{$(g_3)_i$}$$

and

$$0 < qG(\epsilon x, s) \leq g(\epsilon x, s)s \leq \frac{1}{\beta}V(\epsilon x)|s|^q, \ \forall \epsilon x \notin \Omega \ \text{ and } \forall s \neq 0, \tag{$(g_3)_{ii}$}$$

where $G(\epsilon x, s) = \displaystyle\int_0^s g(\epsilon x, t)dt$.

The function

$$s \to \frac{g(\epsilon x, s)}{|s|^{q-1}} \ \text{ is nondecreasing for each } \ x \in \mathbb{R}^N \ \text{ and for all } \ s \neq 0. \tag{g_4}$$

Remark 3. *Note that, for $z = \epsilon x$, if u_ϵ is a nodal solution of $(P_{\epsilon_{aux}})$ with $|u_\epsilon(z)| \leq \frac{\eta}{2}$ for every $\epsilon x \in \mathbb{R}^N \setminus \Omega$, then $u_\epsilon(x)$ is also a nodal solution of (P_ϵ).*

2.2 Existence of ground state nodal for the auxiliary problem

In this section we adapt some arguments found in Alves & Figueiredo [7] and Alves & Soares [12], Bartsch, Weth & Willem [18] to establish the existence of ground state nodal solution for problem $(P_{\epsilon_{aux}})$.

Hereafter, let us denote by I_ϵ the functional

$$I_\epsilon(v) = \frac{1}{p}\int\limits_{\mathbb{R}^N} A\left(|\nabla v|^p\right) dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\epsilon x)B\left(|v|^p\right) dx - \int\limits_{\mathbb{R}^N} G(\epsilon x, u)dx$$

and by $\mathcal{N}_\epsilon$ the Nehari manifold associated given by

$$\mathcal{N}_\epsilon = \Big\{u \in W_\epsilon : u \neq 0 \quad \text{and} \quad I'_\epsilon(u)u = 0\Big\}.$$

Since g is C^1, the functional I_ϵ is C^1 class. Since we are looking for nodal solutions, we also define the following set

$$\mathcal{N}_\epsilon^{\pm} = \Big\{u \in W_\epsilon : u^{\pm} \neq 0 \quad \text{and} \quad I'_\epsilon(u^{\pm})u^{\pm} = 0\Big\},$$

where

$$u^+(z) = \max\{u(z), 0\} \text{ and } u^-(z) = \min\{u(z), 0\}.$$

The main result in this section is:

Theorem 2.2.1. *Let a satisfying $(a_1) - (a_4)$, b satisfying $(b_1) - (b_4)$ and V such that $(V_1) - (V_2)$ hold. Then there is $\epsilon_0 > 0$, such that $(P_{\epsilon_{aux}})$ has nodal solution $u_\epsilon \in W_\epsilon$, for every $\epsilon \in (0, \epsilon_0)$. Moreover, if $\frac{P_\epsilon^1}{\epsilon}$ is the maximum point of u_ϵ and $\frac{P_\epsilon^2}{\epsilon}$ is the minimum point of u_ϵ, then for $i = 1, 2$, we obtain*

$$\lim_{\epsilon \to 0} V(P_\epsilon^i) = V_0.$$

We begin with some information on the functional I_ϵ in $\mathcal{N}_\epsilon$ and in $\mathcal{N}_\epsilon^{\pm}$.

Lemma 2.2.2. *(i) There is $C > 0$, such that*

$$I_\epsilon(u) \geq C\left[\ \|u\|_{1,p}^p + \|u\|_{1,q}^q\ \right],\ \forall\ u \in \mathcal{N}_\epsilon \text{ and } \forall\ \epsilon > 0.$$

(ii) There exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}_\epsilon$ and $\|w^{\pm}\| \geq \rho$ for all $w \in \mathcal{N}_\epsilon^{\pm}$.

(iii) There is $\rho_1 > 0$, such that,

$$0 < \rho_1 \leq \int\limits_{\Omega_\epsilon} (u^{\pm})^r dx,$$

for all $u \in \mathcal{N}_\epsilon^{\pm}$ and for all $\epsilon > 0$, where $\Omega_\epsilon := \epsilon^{-1}\Omega$.

Proof. Since $u \in \mathcal{N}_\epsilon$ and (2.0.2), (2.0.3), (g_3) holds, we have that

$$\begin{aligned} I_\epsilon(u) &= I_\epsilon(u) - \frac{1}{\theta}\left\langle I'_\epsilon(u), u\right\rangle \geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p\, dx \\ &+ \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N} V(\epsilon x)b(|u|^p)|u|^p\, dx + \frac{1}{\theta}\int\limits_{\mathbb{R}^N\setminus\Omega_\epsilon} \left[g(\epsilon x, u)u - \theta G(\epsilon x, u)\right]\, dx. \end{aligned}$$

Since $g(\epsilon x, s)s \geq 0$, from (a_1), (b_1) and $(g_3)_{ii}$, we obtain

$$\begin{aligned} I_\epsilon(u) &\geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} [k_1|\nabla u|^p + |\nabla u|^q] \, dx \\ &+ \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} V(\epsilon x)[k_3|u|^p + |u|^q]] \, dx - \frac{1}{\beta} \int\limits_{\mathbb{R}^N} [|\nabla u|^q + V(\epsilon x)|u|^q] \, dx. \end{aligned}$$

Now the result follows because $\beta > \dfrac{\theta p\gamma}{\theta - \gamma p}$.

In order to prove (ii), suppose, by contradiction, that there is a sequence (u_n) in $\mathcal{N}_\epsilon$ such that $u_n \to 0$ in W_ϵ. Then, from (a_1), (b_1), (g_1) and (g_2), given $\delta > 0$, there exist $C > 0$ and $C_\delta > 0$ such that

$$\begin{aligned} C[\|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q] &\leq \int\limits_{\mathbb{R}^N} a(|\nabla u_n|^p)|\nabla u_n|^p dx + \int\limits_{\mathbb{R}^N} V(\epsilon x) b(|u_n|^p)|u_n|^p dx \\ &\leq \delta \int\limits_{\mathbb{R}^N} |u_n|^q dx + C_\delta \int\limits_{\mathbb{R}^N} |u_n|^r dx. \end{aligned}$$

Using Sobolev embeddings we get

$$C\|u\|^q \leq \|u\|^r.$$

But the last inequality is impossible because $q < r$. Moreover, since $\mathcal{N}_\epsilon^\pm \subset \mathcal{N}_\epsilon$, the second item is over.

In order to prove (iii), from (a_1), (b_1), (g_1), (g_2) and Sobolev embeddings, for all $\delta > 0$ given, there are $C, C_\delta > 0$ such that

$$\begin{aligned} C[\|u^\pm\|_{1,p}^p + \|u^\pm\|_{1,q}^q] &\leq \int\limits_{\mathbb{R}^N} a(|\nabla u^\pm|^p)|\nabla u^\pm|^p dx + \int\limits_{\mathbb{R}^N} V(\epsilon x) b(|u^\pm|^p)|u^\pm|^p dx \\ &\leq \delta \int\limits_{\Omega_\epsilon} |u^\pm|^q dx + C_\delta \int\limits_{\Omega_\epsilon} |u^\pm|^r dx + \frac{1}{\beta} \int\limits_{\mathbb{R}^N \setminus \Omega_\epsilon} V(\epsilon x)|u^\pm|^q dx. \end{aligned}$$

Now the result follows by item (ii) and from arbitrariness of δ and because $\beta > 1$. $\square$

From Lemma 2.2.2 we have well defined the real number

$$d_\epsilon = \inf_{\mathcal{N}_\epsilon^\pm} I_\epsilon. \tag{2.2.1}$$

Moreover, from [17, Lemma 4.2, Lemma 4.3] , for $u \in W_\epsilon$ with the $u^\pm \neq 0$, there exist and are unique $t, s > 0$ such that $tu^+ + su^- \in \mathcal{N}_\epsilon^\pm$. At this point, we can finally prove the existence of $u \in \mathcal{N}_\epsilon^\pm$ in which the infimum of I_ϵ is attained on $\mathcal{N}_\epsilon^\pm$.

Existence of nodal solution for the auxiliary problem

We are going to show that the infimum of I_ϵ on $\mathcal{N}_\epsilon^\pm$ is attained by some $u_\epsilon \in \mathcal{N}_\epsilon$, considering the cases $2 \leq p < q < N$ and $1 < p < q < 2 \leq N$.

Lemma 2.2.3. *If $2 \leq p < q < N$, then the functional I_ϵ is sequentially weakly lower semicontinous in W_ϵ. Moreover, the level d_ϵ is attained for some u_ϵ which is a nodal solution for problem $(P_{\epsilon_{aux}})$*

Proof. Firstly we prove that the functional I_ϵ is sequentially weakly lower semicontinous in W_ϵ. For this let us consider $(u_n) \subset W_\epsilon$ such that $u_n \rightharpoonup u$ in W_ϵ and $\Omega_\epsilon := \epsilon^{-1}\Omega$. From (a_2) and (b_2) it follows that

$$\int_{\Omega_\epsilon} A(|\nabla u|^p)\,dx \le \liminf_{n\to+\infty} \int_{\Omega_\epsilon} A(|\nabla u_n|^p)\,dx, \tag{2.2.2}$$

$$\int_{\Omega_\epsilon} V(\epsilon x)B(|u|^p)\,dx \le \liminf_{n\to+\infty} \int_{\Omega_\epsilon} V(\epsilon x)B(|u_n|^p)\,dx. \tag{2.2.3}$$

Moreover, by Sobolev embeddings, we get

$$\int_{\Omega_\epsilon} F(u)\,dx = \lim_{n\to+\infty} \int_{\Omega_\epsilon} F(u_n)\,dx. \tag{2.2.4}$$

Now we are going to prove that

$$I_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}(v) := \frac{1}{p}\int_{\mathbb{R}^N\setminus\Omega_\epsilon} (A(|\nabla v|^p) + V(\epsilon x)B(|v|^p)v^p)\,dx - \int_{\mathbb{R}^N\setminus\Omega_\epsilon} \tilde{F}(v)$$

is a strictly convex functional in $W_\epsilon(\mathbb{R}^N\setminus\Omega_\epsilon)$, where $\widetilde{F}(s) = \int\limits_0^s \widetilde{f}(t)dt$.

Observe that $I_\epsilon{''}(v)(w,w)$ is well-defined for $v, w \in W_\epsilon(\mathbb{R}^N)$, for $2 \le p < q < N$. Then, for $v, w \in W_\epsilon(\mathbb{R}^N\setminus\Omega_\epsilon)$, $w \neq 0$, we have

$$\begin{aligned}
I_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}{''}(v)(w,w) &= p\int_{\mathbb{R}^N\setminus\Omega_\epsilon} a'(|\nabla v|^p)|\nabla v|^{2p-4}(\nabla v\nabla w)^2dx\\
&+ (p-2)\int_{\mathbb{R}^N\setminus\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-4}(\nabla v\nabla w)^2dx\\
&+ \int_{\mathbb{R}^N\setminus\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-2}|\nabla w|^2dx\\
&+ p\int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b'(|v|^p)|v|^{2p-4}(vw)^2dx\\
&+ (p-2)\int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-4}(vw)^2dx\\
&+ \int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2dx - \int_{\mathbb{R}^N\setminus\Omega_\epsilon} \tilde{f}'(v)w^2dx.
\end{aligned}$$

Using (2.1.1), (a_4) and (b_4), we deduce that

$$\begin{aligned}
I''_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}(v)(w,w) &\ge \int_{\mathbb{R}^N\setminus\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-2}|\nabla w|^2dx + \int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2dx\\
&- \frac{N-1}{\beta}\int_{\mathbb{R}^N\setminus\Omega_\epsilon} V_0|v|^{N-2}w^2dx.
\end{aligned}$$

Therefore from (b_1), we have

$$\begin{aligned}
I''_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}(v)(w,w) &\ge \int_{\mathbb{R}^N\setminus\Omega_\epsilon} a(|\nabla v|^p)|\nabla v|^{p-2}|\nabla w|^2dx + \int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2dx\\
&- \frac{N-1}{\beta}\int_{\mathbb{R}^N\setminus\Omega_\epsilon} V(\epsilon x)b(|v|^p)|v|^{p-2}|w|^2dx.
\end{aligned}$$

Since we also have $\beta > N-1$, we finally get to $I''_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}(v)(w,w) > 0$. By convex analysis it follows that $I_{\epsilon,\mathbb{R}^N\setminus\Omega_\epsilon}$ is weakly lower semicontinuous.

From Lemma 2.2.2, there exists a bounded minimizing sequence (u_n) in $\mathcal{N}_\epsilon^\pm$ for d_ϵ and I_ϵ is coercive on $\mathcal{N}_\epsilon^\pm$. Hence, there exist $v, u_1, u_2 \in W_\epsilon$ such that

$$u_n \rightharpoonup v, \quad u_n^+ \rightharpoonup u_1, \quad u_n^- \rightharpoonup u_2 \quad \text{in } W_\epsilon.$$

Since the transformations $v \to v^+$ and $v \to v^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (see Lemma 2.3 in [22] with suitable adaptations), we have that $v^+ = u_1 \geq 0$ and $v^- = u_2 \leq 0$. By item (iii) of Lemma 2.2.2, we conclude that $v^\pm \neq 0$, and therefore $v = v^+ + v^-$ is sign-changing, this implies that there exist $t, s > 0$ such that $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. we have the $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. Moreover, there exists a unique pair (t_v, s_v) of positive constants such that

$$I_\epsilon(t_v v^+ + s_v v^-) = \max_{t,s>0} I_\epsilon(tv^+ + sv^-).$$

Since I_ϵ is sequentially weakly lower semicontinous in W_ϵ and (u_n) in $\mathcal{N}_\epsilon^\pm$, we have

$$\begin{aligned} d_\epsilon &\leq I_\epsilon(u_\epsilon) = I_\epsilon(tv^+ + sv^-) \leq \liminf_{n\to+\infty} I_\epsilon(tu_n^+ + su_n^-) \\ &\leq \limsup_{n\to+\infty} I_\epsilon(tu_n^+ + su_n^-) \leq \lim_{n\to+\infty} I_\epsilon(u_n^+ + u_n^-) = \lim_{n\to+\infty} I_\epsilon(u_n) = d_\epsilon. \end{aligned}$$

□

Lemma 2.2.4. *For $1 < p < q < 2 \leq N$, the level d_ϵ is attained for some $u_\epsilon \in \mathcal{N}_\epsilon^\pm$. Moreover, u_ϵ is a nodal solution for problem $(P_{\epsilon_{aux}})$.*

Proof. From Lemma 2.2.2, there exists a bounded minimizing sequence (u_n) in $\mathcal{N}_\epsilon^\pm$ for d_ϵ and I_ϵ is coercive on $\mathcal{N}_\epsilon^\pm$. Hence, there exist $v, u_1, u_2 \in W_\epsilon$ such that

$$u_n \rightharpoonup v, \quad u_n^+ \rightharpoonup u_1, \quad u_n^- \rightharpoonup u_2 \quad \text{in } W_\epsilon.$$

Since the transformations $v \to v^+$ and $v \to v^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (see Lemma 2.3 in [22] with suitable adaptations), we have that $v^+ = u_1 \geq 0$ and $v^- = u_2 \leq 0$. By item (iii) of Lemma 1.2.2, we conclude that $v^\pm \neq 0$, and therefore $v = v^+ + v^-$ is sign-changing, this implies that there exist $t, s > 0$ such that $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$. we have the $u_\epsilon = tv^+ + sv^- \in \mathcal{N}_\epsilon^\pm$.

On the order hand, using Sobolev embedding, we have

$$\lim_{n\to+\infty} \int_{\Omega_\epsilon} f(u_n^\pm) u_n^\pm dx = \int_{\Omega_\epsilon} f(v^\pm) v^\pm dx.$$

Then, using Fatou's Lemma and $(g_3)_{ii}$ we obtain that

$$\int_{\mathbb{R}^N} \left[a(|\nabla v^\pm|^p)|\nabla v^\pm|^p + V(\epsilon x) b(|v^\pm|^p)|v^\pm|^p \right] dx \leq \int_{\mathbb{R}^N} g(\epsilon x, v^\pm) v^\pm dx,$$

that is, $I'_\epsilon(v^\pm)v^\pm \leq 0$. Thus, $t,\ s \in (0,1]$.

Now, let us observe that assumptions (a_3), (b_3) and (f_4) imply the following monotonicity conditions:

$$t \longmapsto \frac{1}{p}A(t) - \frac{1}{q}a(t)t \text{ is increasing for } t \in (0,+\infty),$$

$$t \longmapsto \frac{1}{p}B(t) - \frac{1}{q}b(t)t \text{ is increasing for } t \in (0,+\infty),$$

$$t \longmapsto \frac{1}{q}g(\epsilon x, t)t - G(\epsilon x, t) \text{ is increasing for } t \in (0,+\infty),$$

Hence,

$$
\begin{aligned}
I_\epsilon(tv^+) \quad &\leq \quad \int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A(|\nabla(tv^+)|^p) - \frac{1}{q}a(|\nabla(tv^+)|^p)|\nabla(tv^+)|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} V(\epsilon x)\left(\frac{1}{p}B(|tv^+|^p) - \frac{1}{q}b(|tv^+|^p)|tv^+|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} \left(\frac{1}{q}g(\epsilon x, tv^+)tv^+ - G(\epsilon x, tv^+)\right) dx \\
&\leq \liminf_{n\to+\infty}\Bigg[\int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A(|\nabla u_n^+|^p) - \frac{1}{q}a(|\nabla u_n^+|^p)|\nabla u_n^+|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} V(\epsilon x)\left(\frac{1}{p}B(|u_n^+|^p) - \frac{1}{q}b(|u_n^+|^p)|u_n^+|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} \left(\frac{1}{q}g(\epsilon x, u_n^+)u_n^+ - G(\epsilon x, u_n^+)\right) dx\Bigg] = \liminf_{n\to+\infty} I_\epsilon(u_n^+).
\end{aligned}
$$

Using the same arguments as above one can immediately prove that $I_\epsilon(sv^-) \leq I_\epsilon(v^-)$. Then, using that g is and odd function and $u_\epsilon \in \mathcal{N}_\epsilon^\pm$, it follows that

$$
d_\epsilon \leq I_\epsilon(u_\epsilon) = I_\epsilon(tv^+) + I_\epsilon(sv^-) \leq \liminf_{n\to+\infty} I_\epsilon(u_n) = d_\epsilon.
$$

□

Remark 4. *Note that Lemma 2.2.4 is true for all $1 < p < N$, however the arguments used in in Lemma 2.2.3 is new for nonhomogeneous operators.*

Proof of Theorem 2.2.1

Proof. The existence follows by Lemma 2.2.3 and Lemma 2.2.4. The proof that $I'_\epsilon(u_\epsilon) = 0$ and that u_ϵ has exactly two nodal domains or equivalently it changes sign exactly once can be seen in [17, pages 1230-1232]. □

2.3 Concentration results

In order to prove the concentration result, we consider the limit problem

$$
\begin{cases} -div\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) + V_0 b(|u|^p)|u|^{p-2}u = f(u) & \text{in } \mathbb{R}^N \\ W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \end{cases} \tag{P_L}
$$

whose associated functional is given by

$$
I_{V_0}(u) \quad = \quad \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)]dx - \int_{\mathbb{R}^N} F(u)dx,
$$

by the corresponding Nehari manifold is given by

$$\mathcal{N}_{V_0} = \{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)\backslash\{0\} : I'_{V_0}(u)u = 0\}.$$

We also define

$$c_{V_0} = \inf_{\mathcal{N}_{V_0}} I_{V_0}.$$

We define the Palais-Smale compactness condition. We say that a sequence $(u_n) \subset W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ is a Palais-Smale sequence at level c_{V_0} for the functional I_{V_0} if

$$I_{V_0}(u_n) \to c_{V_0}$$

and

$$\|I'_{V_0}(u_n)\| \to 0 \text{ in } (W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N))'.$$

If every Palais-Smale sequence of I_{V_0} has a strong convergent subsequence, then one says that I_{V_0} satisfies the Palais-Smale condition ($(PS)_{c_{V_0}}$ for short).

The next result shows that problem (P_L) has a solution that reaches c_{V_0}.

Lemma 2.3.1. *(A Compactness Lemma) Let $(u_n) \subset \mathcal{N}_{V_0}$ be a sequence satisfying $I_{V_0}(u_n) \to c_{V_0}$. Then, there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $v_n(x) = u_n(x + \tilde{y}_n)$ converges strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. In particular, there exists a minimizer for c_{V_0}.*

Proof. Applying Ekeland's Variational Principle (see Theorem 8.5 in [62]), we may suppose that (u_n) is a $(PS)_{c_{V_0}}$ for I_{V_0}. From Lemma [6, Lemma 2.3], going to a subsequence if necessary, we have that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and $I'_{V_0}(u) = 0$.

If $u \neq 0$, then u is a ground state solution of the limit problem (P_{V_0}), that is, $I_{V_0}(u) = c_{V_0}$. In fact, using arguments found in [6, Lemma 2.3], we have that

$$\nabla u_n(x) \to \nabla u(x) \quad a.e \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad I'_{V_0}(u) = 0. \tag{2.3.1}$$

Then, by (2.0.2), (2.0.3) and the Fatou's Lemma,

$$\begin{aligned}
0 &\leq \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)]\, dx - \frac{1}{\theta}\int_{\mathbb{R}^N} [a(|\nabla u|^p)|\nabla u|^p + V_0 B(|u|^p)|u|^p]\, dx \\
&\leq \liminf_{n\to+\infty} \Bigg\{ \frac{1}{p}\int_{\mathbb{R}^N} [A(|\nabla u_n|^p) + V_0 B(|u_n|^p)]\, dx \\
&\qquad - \frac{1}{\theta}\int_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p + V_0 B(|u_n|^p)|u_n|^p]\, dx \Bigg\}
\end{aligned}$$

Hence, if $u \in \mathcal{N}_{V_0}$,

$$c_{V_0} \leq I_{V_0}(u) - \frac{1}{\theta} I'_{V_0}(u)u \leq \liminf_{n\to+\infty} \left[I_{V_0}(u_n) - \frac{1}{\theta} I'_{V_0}(u_n)u_n \right] = \lim_{n\to+\infty} I_{V_0}(u_n) = c_{V_0}.$$

By (2.3.1), (a_1), (b_1) and Lebesgue's theorem we conclude that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Consequently, $I_{V_0}(u) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $u \equiv 0$, then in this case we cannot have $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ because $c_{V_0} > 0$. Hence, using [6, Proposition 2.1], there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that

$$v_n \rightharpoonup v \ \ in \ \ W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),$$

where $v_n = u_n(x + \tilde{y}_n)$. Therefore, v_n is also a $(PS)_{c_{V_0}}$ sequence of $I_{c_{V_0}}$ and $v \not\equiv 0$. It follows form above arguments that, up to a subsequence, (v_n) converges strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and the proof of the lemma is over. □

Proposition 2.3.2. *Let $\epsilon_n \to 0$ and $u_n \in \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to c_{V_0}$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to \overline{y} \in \Omega$, where $y_n = \epsilon_n \tilde{y}_n$.*

Proof. Since $c_{V_0} > 0$, from Lemma [6, Proposition 2.1], there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants R and β such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} | u_n |^q \geq \widetilde{\beta}, \quad \text{for some} \ \ \widetilde{\beta} > 0.$$

Thus, if $v_n(x) = u_n(x + \tilde{y}_n)$, up to a subsequence, $v_n \rightharpoonup v \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Let $t_n > 0$ be such that

$$\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}. \tag{2.3.2}$$

Then, since $v_n \in \mathcal{N}_{\epsilon_n}$, we obtain

$$c_{V_0} \leq I_{V_0}(\tilde{v}_n) \leq I_{\epsilon_n}(\tilde{v}_n) \leq I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u_n) = c_{V_0} + o_n(1), \tag{2.3.3}$$

which implies

$$I_{V_0}(\tilde{v}_n) \to c_{V_0} \ \text{ and } \ (\tilde{v}_n) \subset \mathcal{N}_{V_0}.$$

Since (v_n) is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, from (2.3.3), we get that (t_n) is bounded. As a consequence, the sequence $(\tilde{v}_n)$ also is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, thus for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and we can assume that $t_n \to t_0 > 0$, and this limit implies that $\tilde{v} \not\equiv 0$. From Lemma 2.3.1, $\tilde{v}_n \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, and so, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

To conclude the proof of the proposition, we consider $y_n = \epsilon_n \tilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to \overline{y}$ for $\overline{y} \in \Omega$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denote by (y_n), verifying $|y_n| \to \infty$. Note that from (a_1) and (b_1) we have

$$\int_{\mathbb{R}^N} [k_1 |\nabla v_n|^p + |\nabla v_n|^q] \, dx + V_0 \int_{\mathbb{R}^N} [k_3 |v_n|^p + |v_n|^q] \, dx \leq \int_{\mathbb{R}^N} g(\epsilon_n x + y_n, v_n) v_n dx.$$

Fix $R > 0$ such that $B_R(0) \supset \Omega$ and let $\mathcal{X}_{B_R(0)}$ be the characteristic function of $B_R(0)$. Since $\mathcal{X}_{B_R(0)}(\epsilon x + y_n) = o_n(1)$ for all $x \in B_R(0)$ and $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \mathcal{X}_{B_R(0)}(\epsilon x + y_n) g(\epsilon x + y_n, v_n) v_n dx = o_n(1).$$

By definition of $\widetilde{f}$ we obtain that

$$\int_{\mathbb{R}^N} [k_1|\nabla v_n|^p + |\nabla v_n|^q]\,dx + V_0 \int_{\mathbb{R}^N} [k_3|v_n|^p + |v_n|^q]\,dx \leq \int_{\mathbb{R}^N\setminus B_R(0)} \widetilde{f}(v_n)v_n dx + o_n(1)$$
$$\leq \frac{V_0}{\beta}\int_{\mathbb{R}^N} |v_n|^q dx + o_n(1).$$

It follows that $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, obtain this way a contradiction. Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to \overline{y} \in \mathbb{R}^N.$$

Arguing as above, if $\overline{y} \notin \overline{\Omega}$, we will obtain again $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, thus $\overline{y} \in \overline{\Omega}$. If $V(\overline{y}) = V_0$, we have $\overline{y} \notin \partial\Omega$ and consequently $\overline{y} \in \Omega$. Supposing by contradiction that $V(\overline{y}) > V_0$, we have

$$c_{V_0} = I_{V_0}(\widetilde{v}) < \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\overline{y})B(|\widetilde{v}|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v}).$$

Using again the fact that $\widetilde{v}_n \to \widetilde{v}$ in $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, from Fatou's Lemma

$$c_{V_0} < \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n z + y_n)B(|\widetilde{v}_n|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v}_n)\right]$$

that is, since $(u_n) \in \mathcal{N}_{\epsilon_n}$,

$$c_{V_0} < \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_{V_0},$$

obtaining a contradiction. □

Lemma 2.3.3. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}^{\pm}_{\epsilon_n}$ be a nodal solution of problem $(P_{\epsilon_{aux}})$ such that $I_{\epsilon_n}(u_n^{\pm}) \to c_{V_0}$. Then $(v_{i,n})$ converges uniformly on compacts of $\mathbb{R}^N$, where $v_{1,n}(x) := u_n^+(x+\tilde{y}_{1,n})$ and $v_{2,n}(x) := u_n^-(x+\tilde{y}_{2,n})$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|v_{i,n}\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} < \xi \quad \textit{for all} \quad n \geq n_0 \quad \textit{and} \quad i = 1, 2,$$

where $(\tilde{y}_{1,n})$ and $(\tilde{y}_{2,n})$ were given in Proposition 2.3.2.

Proof. Adapting some arguments explored in [6, Lemma 5.5], we have that the sequences $(v_{1,n})$ and $(v_{2,n})$ are bounded in $L^\infty(\mathbb{R}^N)$ and there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|v_{i,n}\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} < \xi, \quad \text{for all} \quad n \geq n_0 \quad \text{and} \quad i = 1, 2.$$

Then, for any bounded domain $\Omega' \subset \mathbb{R}^N$, from (g_1) and (g_2) and continuity of V there exists $C > 0$ such that

$$|V(\epsilon_n x)|u_n|^{p-1} - g(\epsilon_n x, u_n)| \leq C, \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence,

$$|V(\epsilon_n x)|u_n|^{p-1} - g(\epsilon_n x, u_n)| \leq C + |\nabla u_n|^p, \quad \text{for all} \quad n \in \mathbb{N}.$$

Considering $\Psi(x) = C$, we get that $\Psi \in L^t(\Omega')$ with $t > \frac{p}{p-1}N$. From [28, Theorem 1], we have

$$|\nabla u_n| \in L^{\infty}_{loc}(\mathbb{R}^N)$$

Therefore, for all compact $K \subset \Omega'$ there exists a constant $C_0 > 0$, dependent only on C, N, p and $\text{dist}(K, \partial\Omega')$ such that

$$\|\nabla u_n\|_{\infty,K} \le C_0.$$

Then,

$$|u_n|_{C^{0,\nu}_{loc}(\mathbb{R}^N)} \le C, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad 0 < \nu < 1.$$

From Schauder's embedding, (u_n) has a subsequence convergent in $C^{0,\nu}_{loc}(\mathbb{R}^N)$. □

Lemma 2.3.4. *Given $\epsilon > 0$, the nodal solution u_ϵ of problem $(P_{\epsilon_{aux}})$ satisfies*

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

As a consequence

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon^+) = c_{V_0} \quad \text{and} \quad \lim_{\epsilon \to 0} I_\epsilon(u_\epsilon^-) = c_{V_0}.$$

Proof. Consider $z_0 \in \Omega$ such that $V(z_0) = V_0$. Now let us consider $R > 0$ and set $Q_1,\ Q_2 \in \partial B_R(z_0)$ such that $|Q_1 - Q_2| = 2R$. If necessary, take R small enough such that $B(Q_i, R/4) \subset \Omega$. Taking $\psi_i : \mathbb{R}^N \to \mathbb{R}$ such that $\psi_i = 1$ in $B(Q_i, R/4)$ and $\psi_i = 0$ in $\mathbb{R}^N \backslash B(Q_i, R/2)$.

For $i = 1, 2$, let $w_i \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be a ground-state positive solution (see Lemma 2.3.1) of problem

$$-div(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(Q_i)b(|u|^p)(|u|^{p-2}u) = f(u) \quad \text{in } \mathbb{R}^N$$

which satisfies

$$C_{V(Q_i)} = I_{V(Q_i)}(w_i) = \inf_{v \in W_0 \backslash \{0\}} \sup_{t \ge 0} I_{V(Q_i)}(tv),$$

where

$$I_{V(Q_i)}(v) = \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla v|^p) + V(Q_i)B(|v|^p)]\, dx - \int_{\mathbb{R}^N} F(v)dx.$$

Consider the function $w_{\epsilon,Q_i} : \mathbb{R}^N \to \mathbb{R}$ be given by

$$w_{\epsilon,Q_i}(x) = \psi_i(\epsilon x) w_i\left(x - \frac{Q_i}{\epsilon}\right) \in W_\epsilon$$

and $t_{\epsilon,i} > 0$, such that $t_{\epsilon,i} w_{\epsilon,Q_i} \in \mathcal{N}_\epsilon$. By the construction, we have

$$\overline{w}_\epsilon := t_{\epsilon,1} w_{\epsilon,Q_1} - t_{\epsilon,2} w_{\epsilon,Q_2} \in \mathcal{N}_\epsilon^{\pm}.$$

By $supp(w_{\epsilon,Q_1}) \cap supp(w_{\epsilon,Q_2}) = \emptyset$, once $B(Q_1, R) \cap B(Q_2, R) = \emptyset$, and w_i, for $i = 1, 2$, are positives then

$$supp(\overline{w}_\epsilon^+) \cap supp(\overline{w}_\epsilon^-) = \emptyset, \quad \overline{w}_\epsilon^+ = t_{\epsilon,1} w_{\epsilon,Q_1} \text{ and } \overline{w}_\epsilon^- = -t_{\epsilon,Q_2} w_{\epsilon,Q_2}\ .$$

Then

$$I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-) \quad \text{and}$$
$$I'_\epsilon(\overline{w}_\epsilon^{\pm})\overline{w}_\epsilon^{\pm} = 0.$$

Hence

$$I_\epsilon(u_\epsilon) \le I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-). \tag{2.3.4}$$

Now with a direct computation we have

$$I_\epsilon(u_\epsilon) \le I_\epsilon(\overline{w}_\epsilon) = I_\epsilon(\overline{w}_\epsilon^+) + I_\epsilon(\overline{w}_\epsilon^-) = c_{V(Q_1)} + c_{V(Q_2)} + o_\epsilon(1).$$

Finally, taking $R \to 0$ in the last inequality and using the continuity of the minimax function (see [13], [51]) we get

$$\limsup_{\epsilon\to 0} I_\epsilon(u_\epsilon) \le 2c_{V_0}.$$

Now let $t_\epsilon^\pm > 0$ be such that $t_\epsilon^\pm u_\epsilon^\pm \in \mathcal{N}_{V_0}$. Then,

$$2c_{V_0} \le I_{V_0}(t_\epsilon^+ u_\epsilon^+) + I_{V_0}(t_\epsilon^- u_\epsilon^-) \le I_\epsilon(t_\epsilon^+ u_\epsilon^+) + I_\epsilon(t_\epsilon^- u_\epsilon^-) \le I_\epsilon(u_\epsilon^+) + I_\epsilon(u_\epsilon^-) = I_\epsilon(u_\epsilon).$$

Hence we have proved that

$$\lim_{\epsilon\to 0} I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

On the other hand, we know that $c_{V_0} \le I_{V_0}(t_\epsilon^\pm u_\epsilon^\pm) \le I_\epsilon(t_\epsilon^\pm u_\epsilon^\pm) \le I_\epsilon(u_\epsilon^\pm)$. Therefore,

$$c_{V_0} \le \liminf_{\epsilon\to 0} I_\epsilon(u_\epsilon^\pm).$$

Assume by contradiction that at least one inequality is strict, then arguing as above we obtain

$$2c_{V_0} < \liminf_{\epsilon\to 0} \left(I_\epsilon(u_\epsilon^+) + I_\epsilon(u_\epsilon^-)\right) = I_\epsilon(u_\epsilon) = 2c_{V_0}.$$

□

Lemma 2.3.5. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}_{\epsilon_n}^\pm$ be a solution of problem $(P_{\epsilon_{aux}})$. Then, there are $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that for $v_{1,n}(x) := u_n^+(x + \tilde{y}_{1,n})$ and $v_{2,n}(x) := u_n^-(x + \tilde{y}_{2,n})$, we have*

$$v_{1,n}(x) \ge \delta^*, \quad \textit{for all} \quad x \in B_R(0) \ \textit{and} \ \ n \ge n_0$$

and

$$v_{2,n}(x) \le -\delta^*, \quad \textit{for all} \quad x \in B_R(0) \quad \textit{and} \quad n \ge n_0,$$

where $R > 0$, $(\tilde{y}_{1,n})$ and $(\tilde{y}_{2,n})$ were given in Proposition 2.3.2.

Proof. Suppose by contradiction that $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} \to 0$, for $i = 1$ or $i = 2$. Then by Lemma 2.3.3, we have $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N)} \to 0$. It follows from (f_1) that

$$|f(v_{i,n})| \le \frac{V_0}{2}|v_{i,n}|^{q-1} \quad \text{for} \quad n \quad \text{sufficient large.} \tag{2.3.5}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} a(|\nabla v_{i,n}|^p)|\nabla v_{i,n}|^p dx \quad &+ \quad \int_{\mathbb{R}^N} V(\epsilon_n x + y_{i,n}) b(|v_{i,n}|^p)|v_{i,n}|^p dx \\ &= \quad \int_{\mathbb{R}^N} f(v_{i,n}) v_{i,n} dx + o_n(1) \\ &\le \quad \frac{V_0}{2} \int_{\mathbb{R}^N} |v_{i,n}|^q dx + o_n(1), \end{aligned}$$

which implies from (a_1) and (b_1) that,

$$\|u_n^\pm\|_{W_{\epsilon_n}} \to 0,$$

which is a contradiction with Lemma 2.3.4. □

Lemma 2.3.6. *For $i = 1, 2$, we have*

$$\lim_{\epsilon \to 0} V(P^i_\epsilon) = V_0.$$

Proof. We first notice that using Lemma 2.3.5 there exist $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$v_{1,n}(q^1_n) := \max_{z \in \mathbb{R}^N} v_{1,n}(z) = u^+_n(q^1_n + \tilde{y}_{1,n}) \geq u^+_n(x) \geq \delta^*, \quad \text{for all } n \geq n_0, \quad \text{for all } \ x \in B_R(0)$$

and

$$v_{2,n}(q^2_n) := \min_{z \in \mathbb{R}^N} v_{2,n}(z) = u^-_n(q^2_n + \tilde{y}_{2,n}) \leq u^-_n(x) \leq -\delta^*, \quad \text{for all } n \geq n_0, \text{ for all } \ x \in B_R(0).$$

We claim that q^i_n, $i = 1, 2$ is bounded, otherwise using Lemma 2.3.3 and 2.3.5, there exists $R^* > 0$ such that $\|v_{i,n}\|_{L^\infty(\mathbb{R}^N \setminus B_{R^*})} \leq \frac{\delta^*}{2}$, which implies that $|v_{i,n}(q^i_n)| \leq \frac{\delta^*}{2}$, where we obtain a contradiction. Then, $P^i_{\epsilon_n} = \epsilon_n q^i_n + y_{i,n}$ implies

$$\lim_{n \to +\infty} P^i_{\epsilon_n} = \lim_{n \to +\infty} y_{i,n} = \overline{y}_i \in \Omega.$$

Hence from continuity of V it follows that

$$\lim_{n \to +\infty} V(P^i_{\epsilon_n}) = V(\overline{y}_i) \geq V_0.$$

We claim that $V(\overline{y}_i) = V_0$. Indeed, suppose by contradiction that $V(\overline{y}_i) > V_0$. Using the same arguments of Proposition 2.3.2, we have that $\widetilde{v}_{i,n} \to \widetilde{v}_i$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and

$$c_{V_0} = I_{V_0}(\widetilde{v}_i) < \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \widetilde{v}_i|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\overline{y}_i) B(|\widetilde{v}_i|^p) dx - \int_{\mathbb{R}^N} F(\widetilde{v}_i) dx.$$

Using that $\widetilde{v}_{i,n} \to \widetilde{v}_i$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and from Fatou's Lemma, we obtain

$$c_{V_0} < \liminf_{n \to \infty} \left[\frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \widetilde{v}_{i,n}|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n x + y_{i,n}) B(|\widetilde{v}_{i,n}|^p) dx - \int_{\mathbb{R}^N} F(\widetilde{v}_{i,n}) dx \right],$$

and therefore

$$c_{V_0} < \liminf_{n \to \infty} I_{\epsilon_n}(t_{i,n} u^\pm_n) \leq \liminf_{n \to \infty} I_{\epsilon_n}(u^\pm_n) = c_{V_0}.$$

This contradiction shows that $V(\overline{y}_i) = V_0$ for $i = 1, 2$. □

Lemma 2.3.7. *Let $\{\epsilon_n\}$ be a sequence of positive number such that $\epsilon_n \to 0$ as $n \to \infty$ and let $(x_n) \subset \overline{\Omega}_{\epsilon_n}$ be a sequence such that $u^+_{\epsilon_n}(x_n) \geq \Upsilon > 0$ or $u^-_{\epsilon_n}(x_n) \leq -\Upsilon < 0$ for each $n \in \mathbb{N}$ and for some Υ positive constant, where u_{ϵ_n} is a solution of $(P_{\epsilon_{aux}})$. Then,*

$$\lim_{n \to \infty} V(\overline{x}_n) = V_0$$

where $\overline{x}_n = \epsilon_n x_n$.

Proof. Up to a subsequence,

$$\overline{x}_n \to \overline{x} \in \overline{\Omega}.$$

From Lemma 2.3.4 we have that $u^+_{\epsilon_n} \in \mathcal{N}_{\epsilon_n}$,

$$I_{\epsilon_n}(u^+_{\epsilon_n}) \to c_{V_0},$$

and there exists a positive constants such that

$$\|u_{\epsilon_n}^+\| \leq C, \quad \forall\, n \in \mathbb{N} \quad \text{and for some} \;\; C > 0.$$

Setting $v_n(z) := u_{\epsilon_n}^+(z + x_n)$, we have $\|v_n\| \leq C$ and $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Recalling that

$$v_n(0) = u_{\epsilon_n}^+(x_n) \geq \Upsilon > 0,$$

we conclude that $v \not\equiv 0$.
Fix $t_n > 0$ verifying $\widetilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$, for each $n \in \mathbb{N}$. Hence

$$c_{V_0} \leq I_{V_0}(\widetilde{v}_n) \leq I_{\epsilon_n}(t_n v_n) \leq I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u_n^+) = c_{V_0} + o_n(1).$$

Thus $I_{V_0}(\widetilde{v}_n) \to c_{V_0}$ with $\{\widetilde{v}_n\} \subset \mathcal{N}_{V_0}$. By Lemma 2.3.1, we have

$$\widetilde{v}_n \to \widetilde{v} \quad in \quad W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N) \;\; \text{and} \;\; I_{V_0}(\widetilde{v}) = c_{V_0}. \tag{2.3.6}$$

Since $v \neq 0$, by Lemma 2.3.1 we have $y_n = 0$, for $n \in \mathbb{N}$. Moreover, recalling that V is continuous, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V(\overline{x}).$$

We claim that $V(\overline{x}) = V_0$. Indeed, suppose by contradiction that $V(\overline{x}) > V_0$, then

$$c_{V_0} = I_{V_0}(\widetilde{v}) < \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\overline{x})B(|\widetilde{v}|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v})dx$$

and by (2.3.6) and Fatou's Lemma

$$\begin{aligned}
c_{V_0} &< \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + \overline{x}_n)B(|\widetilde{v}_n|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v}_n)dx\right] \\
&\leq \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla t_n v_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + \overline{x}_n)B(|t_n v_n||^p)dx\right. \\
&\quad \left. - \int_{\mathbb{R}^N} G(\epsilon_n x + \overline{x}, t_n v_n)dx\right] \\
&= \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n^+) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n^+) = c_{V_0},
\end{aligned}$$

which leads an absurd. Consequently $\lim_{n\to\infty} V(\overline{x}_n) = V_0$ and the lemma is proved. □

Lemma 2.3.8. *If m_ϵ^+ is given by*

$$m_\epsilon^+ := \sup\left\{\max_{\partial\Omega_\epsilon} u_\epsilon : u_\epsilon \in \mathcal{N}_\epsilon^\pm \;\; \textit{is a solution of } (P_{\epsilon_{aux}})\right\}$$

and if m_ϵ^- is given by

$$m_\epsilon^- = \sup\left\{\min_{\partial\Omega_\epsilon} u_\epsilon : u_\epsilon \in \mathcal{N}_\epsilon^\pm \;\; \textit{is a solution of } (P_{\epsilon_{aux}})\right\},$$

then there exists $\overline{\epsilon} > 0$ such that the sequences $(m_\epsilon^\pm)$ are bounded for all $\epsilon \in (0, \overline{\epsilon})$. Moreover, we have

$$\lim_{\epsilon\to 0} m_\epsilon^\pm = 0.$$

Proof. Suppose, by contradiction, $\lim_{\epsilon\to 0} m_\epsilon^+ = +\infty$ or $\lim_{\epsilon\to 0} m_\epsilon^- = -\infty$, then there exist u_ϵ a solution of $(P_{\epsilon_{aux}})$ in $\mathcal{N}_\epsilon^\pm$ and $\Upsilon > 0$ such that

$$\max_{\partial\Omega_\epsilon} u_\epsilon^+ \geq \Upsilon > 0 \quad \text{or} \quad \max_{\partial\Omega_\epsilon} u_\epsilon^- \leq -\Upsilon < 0.$$

Thus there exists $\{\epsilon_n\} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ and there exists a sequence $\{x_n\} \subset \partial\Omega_{\epsilon_n}$ such that

$$u_{\epsilon_n}^+(x_n) \geq \Upsilon > 0 \quad \text{or} \quad u_{\epsilon_n}^-(x_n) \leq -\Upsilon < 0.$$

Thus, by Lemma 2.3.7, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V_0,$$

where $\overline{x}_n = \epsilon_n x_n$ and $\{\overline{x}_n\} \subset \partial\Omega$. Hence, up to a subsequence, we have $\overline{x}_n \to \overline{x}$ in $\partial\Omega$ and $V(\overline{x}) = V_0$, which does not make sense by (V_2). Hence, there exists $\overline{\epsilon} > 0$ such that $(m_\epsilon^\pm)$ is bounded, for all $\epsilon \in (0, \overline{\epsilon})$.

We have now to prove that $\lim_{\epsilon\to 0} m_\epsilon^\pm = 0$. Then, suppose by contradiction that there exists $\delta > 0$ and a sequence $\{\epsilon_n\} \subset \mathbb{R}^+$ satisfying

$$m_{\epsilon_n}^+ \geq \delta > 0$$

or

$$m_{\epsilon_n}^- \leq -\delta < 0.$$

Thus, there exists u_{ϵ_n} a solution of $(P_{\epsilon_{aux}})$ in $\mathcal{N}_{\epsilon_n}^\pm$ such that

$$m_{\epsilon_n}^+ - \frac{\delta}{2} < \max_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^+ \leq m_{\epsilon_n}^+$$

or

$$m_{\epsilon_n}^- \leq \min_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^- < m_{\epsilon_n}^- + \frac{\delta}{2}.$$

Hence,

$$\frac{\delta}{2} = \delta - \frac{\delta}{2} \leq m_{\epsilon_n}^+ - \frac{\delta}{2} < \max_{\partial\Omega_\epsilon} u_{\epsilon_n}^+,$$

$$\min_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n}^- < m_{\epsilon_n}^- + \frac{\delta}{2} < -\delta + \frac{\delta}{2} = -\frac{\delta}{2}$$

and then there exists a sequence $(x_n) \subset \partial\Omega_{\epsilon_n}$, such that

$$u_{\epsilon_n}^+(x_n) \geq \frac{\delta}{2}$$

or

$$u_{\epsilon_n}^-(x_n) \leq -\frac{\delta}{2}.$$

Repeating the above arguments, we will get an absurd. Thus, the proof is finished. $\square$

2.4 Proof of Theorem 2

Proof. Let u_ϵ be a solution of $(P_{\epsilon_{aux}})$. By Lemma 2.3.8, there exists $\bar{\epsilon} > 0$ such that $|m_\epsilon^\pm| < \frac{\eta}{2}$, for all $\epsilon \in (0, \bar{\epsilon})$, then $(u_\epsilon - \frac{\eta}{2})_+(x) = 0$ for a neghborhood from $\partial\Omega_\epsilon$. Hence, $(u_\epsilon - \frac{\eta}{2})_+ \in W_0^{1,p}(\mathbb{R}^N \backslash \Omega_\epsilon) \cap W_0^{1,q}(\mathbb{R}^N \backslash \Omega_\epsilon)$ and the function $(u_\epsilon - \frac{\eta}{2})_+^* \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, where

$$(u_\epsilon - \frac{\eta}{2})_+^*(x) = \begin{cases} 0 \text{ if } x \in \Omega_\epsilon, \\ \\ (u_\epsilon - \frac{\eta}{2})_+(x) \;\text{ if }\; x \in \mathbb{R}^N \backslash \Omega_\epsilon. \end{cases}$$

Using $(u_\epsilon - \frac{\eta}{2})_+^*$ as test function. Then, by (a_1), (b_1) and $(g_3)_{ii}$, we have

$$\begin{aligned} 0 \;\; \leq & \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} a(|\nabla u_\epsilon|^p)|\nabla (u_\epsilon - \frac{\eta}{2})_+^*|^p dx \\ & + \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} \left[V_0 b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{q-2}\right]((u_\epsilon - \frac{\eta}{2})_+^*)^2 dx \\ & + \int\limits_{\mathbb{R}^N \backslash \Omega_\epsilon} \left[V(\epsilon x) b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{q-2}\right]\frac{\eta}{2}(u_\epsilon - \frac{\eta}{2})_+^* dx \leq 0 \end{aligned}$$

The last equality implies

$$(u_\epsilon - \frac{\eta}{2})_+^* = 0, \;\; \text{a.e in } \; x \in \mathbb{R}^N \backslash \Omega_\epsilon.$$

Hence, $u_\epsilon \leq \frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$.

Since we can assume $m_\epsilon^- \leq -\frac{\eta}{2}$ for $\epsilon \in (0, \bar{\epsilon})$, working with the function $(u_\epsilon + \frac{\eta}{2})_-^*$, it is possible to prove that $u_\epsilon \geq -\frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$. This fact implies that $|u_\epsilon| \leq \frac{\eta}{2}$ for $z \in \mathbb{R}^N \backslash \Omega_\epsilon$ and by Remark 3 the result follows. $\square$

2.5 Exponential decay

Lemma 2.5.1. *Consider $M, \alpha > 0$ and $\psi(x) = M\exp(-\alpha|x|)$. Then*

$$i) - div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi)$$
$$= \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)\right],$$

$$ii) - div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) \geq \left(\frac{(N-1)}{|x|} - \alpha(q-1)\right) a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}.$$

Proof. Note that

$$\frac{\partial\psi}{\partial x_i}(x) = M\exp(-\alpha|x|)\frac{\partial}{\partial x_i}(-\alpha|x|) = M\exp(-\alpha|x|)(-\alpha)\frac{x_i}{|x|} = -\alpha\frac{x_i}{|x|}\psi(x),$$

which implies $|\nabla\psi| = \alpha\psi$. Now we show the item $i)$.

$$-div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial x_i}\right]$$

$$\begin{aligned}
&= \alpha^{p-1}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[a\,(\alpha^p\psi^p)\,\psi^{p-1}\frac{x_i}{|x|}\right]\\
&= \alpha^{p-1}\sum_{i=1}^{N}\left[a'\,(\alpha^p\psi^p)\,\frac{\partial}{\partial x_i}\,(\alpha^p\psi^p)\,\psi^{p-1}\frac{x_i}{|x|} + a(\alpha^p\psi^p)\frac{\partial}{\partial x_i}\left(\psi^{p-1}\frac{x_i}{|x|}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \alpha^{p-1}\sum_{i=1}^{N}\left[a'\,(\alpha^p\psi^p)\,\alpha^p p\psi^{2p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|}\right]\\
&\quad+\alpha^{p-1}\sum_{i=1}^{N}\left[a(\alpha^p\psi^p)\left(\frac{|x|^2-x_i^2}{|x|^3}\psi^{p-1} + (p-1)\psi^{p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|}\right)\right]\\
&= \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)\right].
\end{aligned}$$

To show $ii)$ we will use (1.2) and item $i)$. By (1.2) we have

$$-a'(\alpha^p\psi^p)\alpha^p\psi^p \geq -\frac{(q-p)}{p}a(\alpha^p\psi^p),$$

where we get

$$-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} \geq -\alpha\psi^{p-1}(q-p)a(\alpha^p\psi^p).$$

Consequently, by $i)$,

$$-div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) \geq \left(\frac{(N-1)}{|x|} - \alpha(q-1)\right)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}.$$

□

Corollary 2.5.2. *Since $V(x) \geq V_0$ in $\mathbb{R}^N$ then, for small $\alpha > 0$,*

$$-div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{q-1} \geq 0 \ \textit{in}\ \mathbb{R}^N.$$

Proof. Using (a_1) and Lemma 2.5.1 we obtain that

$$\begin{aligned}
-div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) &\geq -\alpha(q-1)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}\\
&\geq -\alpha(q-1)\left(k_2\alpha^{p-1}\psi^{p-1} + \alpha^{q-1}\psi^{q-1}\right)\\
&= -\alpha(q-1)k_2\alpha^{p-1}\psi^{p-1} - \alpha(q-1)\alpha^{q-1}\psi^{q-1}
\end{aligned}$$

Moreover, since $V_0 > 0$ and $\alpha > 0$ is small we can conclude that

$$k_3V_0 - \alpha(q-1)k_2\alpha^{p-1} \geq 0,$$

and

$$\frac{V_0}{4} - \alpha(q-1)\alpha^{q-1} \geq 0.$$

Consequently

$$-div(a\,(|\nabla\psi|^p)\,|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{q-1} \geq 0 \text{ in } \mathbb{R}^N.$$

□

Now let us relate the nodal solution u_ϵ to the exponential function ψ for small ϵ.

Proposition 2.5.3. *Let u_ϵ be the solution found in Theorem 2.2.1 and $v_{1,\epsilon}(x) := u_\epsilon^+(x + \tilde{y}_{1,\epsilon})$ and $v_{2,\epsilon}(x) := u_\epsilon^-(x + \tilde{y}_{2,\epsilon})$ given in Proposition 2.3.2. Setting $\varphi_{i,\epsilon} := \max\{|v_{i,\epsilon}| - \psi, 0\}$ for $i = 1, 2$, then for $\epsilon > 0$ small enough, we have*

$$\int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\, dx + k_3 V_0 \int_{\mathbb{R}^N} |v_{i,\epsilon}|^{p-1}\varphi_{i,\epsilon}\, dx + \frac{V_0}{4}\int_{\mathbb{R}^N} |v_{i,\epsilon}|^{q-1}\varphi_{i,\epsilon}\, dx \leq 0.$$

Proof. From Lemma 2.3.3, Lemma 2.3.4 and hypothesis (f_1), there exist $\rho_0 > 0$ such that $\epsilon > 0$ small enough,

$$\frac{f(|v_{i,\epsilon}|)}{|v_{i,\epsilon}|^{q-1}} \leq \frac{3}{4}V_0, \quad \text{for all } \ |x| \geq \rho_0.$$

Since $\psi(x) := M\exp(-\alpha|x|)$ for $x \in \mathbb{R}^N$, we can find $\widetilde{M} > 0$ such that if $M \geq \widetilde{M}$, then $\varphi_{i,\epsilon} := \max\{|v_{i,\epsilon}| - \psi, 0\} \equiv 0$ in $B_{\rho_0}(0)$ and $\varphi_{i,\epsilon} \in W^{1,p}(|x| \geq \rho_0) \cap W^{1,q}(|x| \geq \rho_0)$. Therefore, the above inequality and (b_1),

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\, dx + V_0\int_{\mathbb{R}^N} \left[k_3|v_{i,\epsilon}|^{p-1}\varphi_{i,\epsilon} + |v_{i,\epsilon}|^{q-1}\varphi_{i,\epsilon}\right] dx \\
&\leq \int_{\mathbb{R}^N} a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon}\nabla\varphi_{i,\epsilon}\, dx + \int_{\mathbb{R}^N} V(\epsilon x + y_{i,\epsilon})b(|v_{i,\epsilon}|^p)|v_{i,\epsilon}|^{p-2}v_{i,\epsilon}\varphi_{i,\epsilon}\, dx \\
&\leq \int_{\mathbb{R}^N} f(|v_{i,\epsilon}|)\varphi_{i,\epsilon}\, dx \leq \frac{3V_0}{4}\int_{\mathbb{R}^N} |v_{i,\epsilon}|^{q-1}\varphi_{i,\epsilon} dx
\end{aligned}$$

and the lemma is proved. □

Finally we will show the exponential decay for functions u_ϵ.

Proposition 2.5.4. *There are $\epsilon_0 > 0$ and $C > 0$ such that*

$$|u_\epsilon(z)| \leq C\left[\exp\left(-\alpha\left|\frac{z - P_\epsilon^1}{\epsilon}\right|\right) + \exp\left(-\alpha\left|\frac{z - P_\epsilon^2}{\epsilon}\right|\right)\right],$$

for all $z \in \mathbb{R}^N$.

Proof. From [30, Lemma 2.4], we have that

$$\left\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x - y\right\rangle \geq 0, \ \forall\ x, y \in \mathbb{R}^N.$$

Consider $v_{1,\epsilon}(x) := u_\epsilon^+(x + \tilde{y}_{1,\epsilon})$, $v_{2,\epsilon}(x) := u_\epsilon^-(x + \tilde{y}_{2,\epsilon})$ and the set

$$\Lambda^i := \{x \in \mathbb{R}^N : |x| \geq \rho_0 \ \text{ and } \ |v_{i,\epsilon}| - \psi \geq 0\},$$

where ψ is the function is given by Lemma 2.5.1, $(\widetilde{y}_{1,n})$ and $(\widetilde{y}_{2,n})$ are given by

Proposition 2.3.2. Then, using Corollary 2.5.2 and Proposition 2.5.3, we obtain

$$
\begin{aligned}
0 \geq & \int\limits_{\mathbb{R}^N} \left\langle a(|\nabla v_{i,\epsilon}|^p)|\nabla v_{i,\epsilon}|^{p-2}\nabla v_{i,\epsilon} - a(|\nabla \psi|^p)|\nabla \psi|^{p-2}\nabla \psi, \nabla \varphi_{i,\epsilon} \right\rangle dx \\
& + V_0 k_3 \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right) \varphi_{i,\epsilon}\, dx + \frac{V_0}{4} \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{q-1} - |\psi|^{q-1}\right) \varphi_{i,\epsilon}\, dx \\
& \geq V_0 k_3 \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right) \varphi^{\pm} dx + \frac{V_0}{4} \int\limits_{\mathbb{R}^N} \left(|v_{i,\epsilon}|^{q-1} - |\psi|^{q-1}\right) \varphi_{i,\epsilon} dx \\
& = V_0 k_3 \int\limits_{\Lambda^i} \left(|v_{i,\epsilon}|^{p-1} - |\psi|^{p-1}\right) (|v_{i,\epsilon}| - \psi) dx \\
& + \frac{V_0}{4} \int\limits_{\Lambda^i} \left(|v_{i,\epsilon}|^{q-1} - |\psi|^{q-1}\right) (|v_{i,\epsilon}| - \psi) dx \geq 0.
\end{aligned}
$$

Then $|\Lambda^i| = 0$, for $i = 1, 2$ and consequently

$$|v_{1,\epsilon}(x)| + |v_{2,\epsilon}(x)| \leq 2M \exp(-\alpha|x|), \ \forall \ |x| \geq \rho_0.$$

Considering $x = z - \tilde{y}_{i,\epsilon}$ and using Lemma 2.3.6 there exists a constant $C > 0$ satisfying

$$
\begin{aligned}
|u_\epsilon^{\pm}(z)| &\leq 2M \exp\left(-\alpha \left|\frac{z - y_{i,\epsilon}}{\epsilon}\right|\right) = 2M \exp\left(-\alpha \left|\frac{z - P_\epsilon^i + \epsilon q_\epsilon^i}{\epsilon}\right|\right) \\
&\leq 2M \exp\left(-\alpha \left|\frac{z - P_\epsilon^i}{\epsilon}\right|\right) \exp\left(-\alpha \left|q_\epsilon^i\right|\right) \leq C \exp\left(-\alpha \left|\frac{z - P_\epsilon^i}{\epsilon}\right|\right)
\end{aligned}
\tag{2.5.1}
$$

for all $|z - \tilde{y}_{i,\epsilon}| \geq \rho_0$ and for $\epsilon > 0$ small enough.

Now we are going to show the inequality (3.5.1) holds, for all $z \in \mathbb{R}^N$. Since $(y_{i,\epsilon})$ converges, it follows that

$$|z| \geq \rho_0 - |\tilde{y}_{i,\epsilon}| = \rho_0 - \frac{|y_{i,\epsilon}|}{\epsilon} > \rho_0 - \frac{1 + |y_{i,\epsilon}|}{\epsilon} \to -\infty \quad \text{as} \quad \epsilon \to 0.$$

Then, there exists $\epsilon_0 > 0$ such that

$$|u_\epsilon(z)| \leq C \left[\exp\left(-\alpha \left|\frac{z - P_\epsilon^1}{\epsilon}\right|\right) + \exp\left(-\alpha \left|\frac{z - P_\epsilon^2}{\epsilon}\right|\right)\right], \quad \forall \ \ z \in \mathbb{R}^N \ \text{ and } \ \forall \ \ \epsilon \in (0, \epsilon_0).$$

$\square$

Chapter 3

Existence and concentration of positive solutions for a critical $p\&q$ equation

In this chapter we are concerned with a class of problems, named $p\&q$ problems type. More precisely, we show existence and concentration results of positive solutions for the critical problem given by

$$\begin{cases} -div\left(a\left(\epsilon^p|\nabla u|^p\right)\epsilon^p|\nabla u|^{p-2}\nabla u\right)+V(z)b\left(|u|^p\right)|u|^{p-2}u=f(u)+|u|^{q^*-2}u \ \text{ in } \mathbb{R}^N, \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \end{cases} \tag{P_ϵ}$$

where $\epsilon>0$, $1<p\leq q<N$ and $N\geq 2$. We say that a function $u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ is positive solution of (P_ϵ) if $u>0$ in $\mathbb{R}^N$ and

$$\int\limits_{\mathbb{R}^N}\epsilon^p a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla v\ dx+\int\limits_{\mathbb{R}^N}V(z)b(|u|^p)|u|^{p-2}uv\ dx=\int\limits_{\mathbb{R}^N}[f(u)v+u^{q^*-1}v]dx,$$

for all $v\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$. The hypotheses on the functions a,b,f and V are the following:

(a_1) the function a is of class C^1 and there exist constants $k_1,k_2\geq 0$ such that

$$k_1t^p+t^q\leq a(t^p)t^p\leq k_2t^p+t^q,\quad \text{for all}\quad t>0;$$

(a_2) the mapping $t\mapsto \dfrac{a(t^p)}{t^{q-p}}$ is nonincreasing for $t>0$;

(a_3) if $1<p<2\leq N$ the mapping $t\mapsto a(t)$ is nondecreasing for $t>0$. If $2\leq p<N$ the mapping $t\mapsto a(t^p)t^{p-2}$ is nondecreasing for $t>0$.

As a direct consequence of (a_2) we obtain that the map a and its derivative a' satisfy

$$a'(t)t\leq \frac{(q-p)}{p}a(t) \text{ for all } t>0. \tag{3.0.1}$$

Now if we define the function $h(t)=a(t)t-\frac{q}{p}A(t)$, using (3.0.1) we can prove that the function h is decreasing. Then, there exists a positive real constant $\gamma\geq\frac{q}{p}$ such that

$$\frac{1}{\gamma}a(t)t\leq A(t),\quad \text{for all } t\geq 0. \tag{3.0.2}$$

The hypotheses on the function b are the following:

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^q \leq b(t^p)t^p \leq k_4 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(b_2) the mapping $t \mapsto \dfrac{b(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$.

(b_3) if $1 < p < 2 \leq N$ the mapping $t \mapsto b(t)$ is nondecreasing for $t > 0$. If $2 \leq p < N$ the mapping $t \mapsto b(t^p)t^{p-2}$ is nondecreasing for $t > 0$.

Using the hypothesis (b_2) and arguing as (3.0.1) and (3.0.2), we also can prove that there exists $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma} b(t)t \leq B(t), \quad \text{for all } t \geq 0. \tag{3.0.3}$$

The nonlinearity f is assumed to be a C^1 function with the following hypotheses:

(f_1)

$$\lim_{|s| \to 0} \frac{f(s)}{|s|^{q-1}} = 0.$$

(f_2) There exists $q < r < q^* = \frac{qN}{N-q}$ such that

$$\lim_{|s| \to \infty} \frac{f(s)}{|s|^{r-1}} = 0.$$

(f_3) There exists $\theta \in (\gamma p, q^*)$ such that

$$0 < \theta F(s) \leq f(s)s \quad \text{for} \quad s > 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$ and $\gamma > 0$ was given in (3.0.2);

(f_4) $s \mapsto \dfrac{f(s)}{s^{q-1}}$ is nondecreasing for $s > 0$.

(f_5) There exist $\tau \in (q, q*)$ and $\lambda > 1$

$$f(s) \geq \lambda s^{\tau - 1} \quad \forall\ s > 0.$$

Before we give the main result, we need to put some hypotheses on the potential $V \in C(\mathbb{R}^N)$.

(V_1) There is $V_0 > 0$ such that

$$0 < V_0 \leq V(z), \text{ for all } z \in \mathbb{R}^N.$$

(V_2) There exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$0 < V_0 = \inf_{z \in \Omega} V(z) < \inf_{z \in \partial\Omega} V(z).$$

The main result is the following:

Theorem 3. Suppose that a, b, f and V satisfy $(a_1)-(a_3)$, $(b_1)-(b_3)$, $(f_1)-(f_5)$ and $(V_1)-(V_2)$ respectively. Then there are $\epsilon_0>0$ and $\lambda^*>1$ such that (P_ϵ) has a positive solution $w_\epsilon \in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$, for every $\epsilon\in(0,\epsilon_0)$ and for every $\lambda>\lambda^*$. In addition, if P_ϵ is the maximum point of w_ϵ, then

$$\lim_{\epsilon\to 0} V(P_\epsilon)=V_0.$$

Moreover, there are positive constants C and α such that

$$|w_\epsilon(z)|\le C\exp\left(-\alpha\left|\frac{z-P_\epsilon}{\epsilon}\right|\right),$$

for all $\epsilon\in(0,\epsilon_0)$ and for all $z\in\mathbb{R}^N$.

This chapter is organized as follows. In Section 3.1 we define an auxiliary problem using the penalization argument introduced by Del Pino and Felmer [26]. The existence of solution for the auxiliary problem was showed in Section 3.2. In order to show the concentration result, in Section 3.2 we studied the autonomous problem. The concentration result was showed in Section 3.3. In Section 3.4 we showed that the solutions of the auxiliary problem are solutions of the original problem. In Section 3.5 we showed the exponential decay of these solutions. To conclude the paper, we showed in an appendix the existence of a problem in a bounded domain that was important to overcome the lack of compactness.

3.1 Variational framework and an auxiliary problem

To prove Theorem 3, we will work with the problem below, which is equivalent to (P_ϵ) by change variable $z=\epsilon x$, which is given by

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+V(\epsilon x)b\left(|u|^p\right)|u|^{p-2}u=f(u)+|u|^{q^*-2}u \ \text{ in } \mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N),\end{cases} \qquad (\widetilde{P}_\epsilon)$$

where $\epsilon>0$, $N\ge 2$ and $1<p\le q<N$.

In order to obtain solutions of $(\widetilde{P}_\epsilon)$, consider the following subspace of $W^{1,p}(\mathbb{R}^N)\bigcap W^{1,q}(\mathbb{R}^N)$ given by

$$W_\epsilon:=\left\{v\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(\epsilon x)b(|v|^p)|v|^p dx<+\infty\right\},$$

which is a Banach space when endowed with the norm

$$\|u\|=\|u\|_{1,p}+\|u\|_{1,q},$$

where

$$\|u\|_{1,m}=\left(\int_{\mathbb{R}^N}|\nabla u|^m dx+\int_{\mathbb{R}^N}V(\epsilon x)|u|^m dx\right)^{\frac{1}{m}},\ \text{ for } m\ge 1.$$

Since the approach is variational, consider the energy functional associated $J_\epsilon: W_\epsilon\to\mathbb{R}$ given by

$$J_\epsilon(v)=\frac{1}{p}\int_{\mathbb{R}^N}A\left(|\nabla v|^p\right)dx+\frac{1}{p}\int_{\mathbb{R}^N}V(\epsilon x)B\left(|v|^p\right)dx-\int_{\mathbb{R}^N}F(v)dx-\frac{1}{q^*}\int_{\mathbb{R}^N}v_+^{q^*}dx,$$

where $u_+ = max\{u, 0\}$. By standard arguments, one can prove that $J_\epsilon \in C^1(W_\epsilon, \mathbb{R})$. As we are interested in nonnegative solutions we can assume that $f(s) = 0$ for $s \leq 0$.

Let β be a positive number satisfying $\beta > \max\left\{\dfrac{p\gamma\theta}{q(\theta - p\gamma)}, \dfrac{V_0 p\gamma}{q}, 1\right\}$, where θ was given in (f_3) and V_0 appeared in (V_1). From (f_4), there exists $\eta > 0$ such that $\dfrac{f(\eta) + \eta^{q^*-1}}{\eta^{q-1}} = \dfrac{V_0}{\beta}$. Then, using the above numbers, we define the function of C^1 class given by

$$\widetilde{f}(s) = \begin{cases} 0 & if \quad s \leq 0, \\ \\ f(s) + s^{q^*-1} & if \quad 0 < s \leq \frac{\eta}{2}, \\ \\ \dfrac{V_0}{\beta}|s|^{q-2}s & if \quad s > \eta. \end{cases}$$

We now define the function

$$g(z, s) := \chi_\Omega(z)[f(s) + (s^+)^{q^*-1}] + (1 - \chi_\Omega(z))\widetilde{f}(s),$$

and the auxiliary problem

$$\begin{cases} -div\left(\epsilon a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + V(\epsilon x)b\left(|u|^p\right)|u|^{p-2}u = g(\epsilon x, u) \ \text{ in } \mathbb{R}^N, \\ u \in W_\epsilon, \end{cases} \tag{$P_{\epsilon_{aux}}$}$$

where χ_Ω is the characteristic function of the set Ω. It is easy to check that $(f_1) - (f_4)$ imply that g is a Carathéodory function and for $x \in \mathbb{R}^N$, the function $s \to g(\epsilon x, s)$ is of class C^1 and satisfies the following conditions, uniformly for $x \in \mathbb{R}^N$:

$$\lim_{|s| \to 0} \frac{g(\epsilon x, s)}{|s|^{q-1}} = 0 \tag{g_1}$$

$$g(\epsilon x, s) \leq f(s) + s^{q^*-1}, \ \forall \ s > 0 \text{ and } x \in \mathbb{R}^N \tag{g_2}$$

$$0 < \theta G(\epsilon x, s) \leq g(\epsilon x, s)s, \ \ \forall \epsilon x \in \Omega \ \text{ and } \forall s > 0 \tag{$(g_3)_i$}$$

and

$$0 < qG(\epsilon x, s) \leq g(\epsilon x, s)s \leq \frac{1}{\beta}V(\epsilon x)|s|^q, \ \ \forall \epsilon x \notin \Omega \ \text{ and } \forall s > 0, \tag{$(g_3)_{ii}$}$$

where $G(\epsilon x, s) = \displaystyle\int_0^s g(\epsilon x, t)dt$.

The function

$$s \to \frac{g(\epsilon x, s)}{|s|^{q-1}} \ \text{ is nondecreasing.} \tag{g_4}$$

Remark 5. *Note that, for $z = \epsilon x$, if u_ϵ is a positive solution of $(P_{\epsilon_{aux}})$ with $|u_\epsilon(z)| \leq \frac{\eta}{2}$ for every $\epsilon x \in \mathbb{R}^N \setminus \Omega$, then $u_\epsilon(x)$ is also a positive solution of (P_ϵ).*

3.2 Existence of ground state for the auxiliary problem

Hereafter, let us denote by $I_\epsilon : W_\epsilon \to \mathbb{R}$ the functional given by

$$I_\epsilon(v) = \frac{1}{p}\int_{\mathbb{R}^N} A\left(|\nabla v|^p\right)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)B\left(|v|^p\right)dx - \int_{\mathbb{R}^N} G(\epsilon x, v)dx.$$

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of I_ε, that is,

$$\mathcal{N}_\varepsilon := \{u \in W_\varepsilon \setminus \{0\} : \langle I'_\varepsilon(u), u\rangle = 0\}$$

and define the number b_ε by setting

$$b_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u). \tag{3.2.1}$$

Using (f_1), (f_2) and (g_2) we have: for every $\xi > 0$ there exists C_ξ such that

$$|g(\varepsilon x, s)| \le \xi|s|^{q-1} + C_\xi|s|^{r-1} + |s|^{q^*-1} \ \text{ for all } x \in \mathbb{R}^N, \ s \in \mathbb{R}. \tag{3.2.2}$$

Then, by definition of g and (3.2.2), there is $r_\varepsilon > 0$ such that

$$\|u\| \ge r_\varepsilon > 0 \ \text{ for all } u \in \mathcal{N}_\varepsilon. \tag{3.2.3}$$

The main result in this section is:

Theorem 3.2.1. *Let a satisfying $(a_1)-(a_3)$, b satisfying $(b_1)-(b_3)$, f satisfying $(f_1)-(f_5)$ and V such that $(V_1)-(V_2)$ hold. Then, there is $\lambda^* > 1$ such that $(P_{\epsilon_{aux}})$ has positive solution $u_\epsilon \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, for every $\lambda > \lambda^*$. Moreover, if $\frac{P_\epsilon}{\epsilon}$ is the maximum point of u_ϵ then*

$$\lim_{\epsilon\to 0} V(P_\epsilon) = V_0.$$

In order to use the Mountain Pass Theorem [14], we define the Palais-Smale compactness condition. We say that a sequence $(u_n) \subset W_\epsilon$ is a Palais-Smale sequence at level c for the functional I_ϵ if

$$I_\epsilon(u_n) \to c \quad \text{and} \quad \|I'_\epsilon(u_n)\| \to 0 \text{ in } (W_\epsilon)',$$

where

$$c := \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I_\epsilon(\eta(t)) > 0 \quad \text{and} \quad \Gamma := \{\eta \in C([0,1], X) : \eta(0) = 0, \ I_\epsilon(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I_ϵ has a strong convergent subsequence, then one says that I_ϵ satisfies the Palais-Smale condition ((PS) for short).

Lemma 3.2.2. *The functional I_ϵ satisfies the following conditions*

(i) There are $\alpha, \rho > 0$ such that

$$I_\epsilon(u) \ge \alpha, \quad \text{if} \ \ \|u\| = \rho$$

.

(ii) For any $u \in C_0^\infty(\Omega_\epsilon, [0,\infty))$, we have

$$\lim_{t\to\infty} I_\epsilon(tu) = -\infty.$$

Proof. Using (a_1), (b_1) and (3.2.2) we obtain

$$I_\epsilon(u) \ge \frac{\min\{k_1, k_3\}}{p}\|u\|_{1,p}^p + \frac{1}{q}\|u\|_{1,q}^q - \frac{\xi}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{C_\xi}{r}\int_{\mathbb{R}^N} |u|^r dx - \frac{1}{q^*}\int_{\mathbb{R}^N} |u|^{q^*} dx.$$

By Sobolev embeddings, choosing $\xi > 0$ appropriate and taking $\|u\| < 1$ there are positive constants $C_1,\ C_2,\ C_3$, such that

$$I_\epsilon(u) \geq C_1 \left[\|u\|_{1,p}^p + \|u\|_{1,q}^q\right] - C_2\|u\|^r - C_3\|u\|^{q^*} \geq C_4\|u\|^q - C_2\|u\|^r - C_3\|u\|^{q^*}.$$

Then the item (i) follows.

Now we show that the item (ii) holds. Consider a positive function $w \in C_0^\infty(\Omega_\epsilon)$, $t > 0$ and using (a_1), (b_1), (f_3) and Sobolev embedding, we have

$$I_\epsilon(tw) \leq \frac{t^p}{p}\max\{k_2, k_4\}\|w\|_{1,p}^p + \frac{t^q}{q}\|w\|_{1,q}^q - \frac{t^{q^*}}{q^*}\int\limits_{\Omega_\epsilon} |w|^{q^*}dx$$

this proves the second item.

□

Hence, there exists a Palais-Smale sequence $(u_n) \subset W_\epsilon$ at level c_ϵ. Using (a_2), (b_2) and (f_4), it is possible to prove that

$$c_\epsilon = b_\epsilon = \inf_{u\in W_\epsilon\setminus\{0\}} \sup_{t\geq 0} I_\epsilon(tu),$$

where b_ϵ was defined in (3.2.1).

In order to prove the Palais-Smale condition, we need to prove the next lemma.

Lemma 3.2.3. *Let (u_n) be a $(PS)_d$ sequence for I_ϵ, then the sequence (u_n) is bounded W_ϵ. Moreover, for each $\xi > 0$ there exists $R = R(\xi) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} [a(|\nabla u_n|^p)|\nabla u_n|^p + V(\epsilon x)b(|u_n|^p)|u_n|^p]dx < \xi.$$

Proof. Since (u_n) is a $(PS)_d$ sequence for functional I_d, then using (3.0.1), (3.0.3), (g_i) and (g_{ii}) we have that

$$\begin{aligned}
o_n(1) + d + o_n(1)\|u_n\| &= I_\epsilon(u_n) - \frac{1}{\theta}I'_\epsilon(u_n)u_n \\
&\geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N} \left[a(|\nabla u_n|^p)|\nabla u_n|^p + [1+\mu V(x)]\, b(|u_n|^p)|u_n|^p\right]\, dx \\
&\quad -\frac{1}{\beta}\int\limits_{\mathbb{R}^N} \left[|\nabla u|^q + V(\epsilon x)|u|^q\right]\, dx \\
&\geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\left(\min\{k_1,k_3\}\|u_n\|_{1,p}^p + \left(1 - \frac{1}{\beta}\right)\|u_n\|_{1,q}^q\right).
\end{aligned}$$

Then, arguing as the [6, Lemma 2.3] , we can concluded that (u_n) is bounded in W_ϵ.

Let $\eta_R \in C^\infty(\mathbb{R}^N)$ such that $\eta_R(x) = 0$ if $x \in B_{R/2}(0)$ and $\eta_R(x) = 1$ if $x \notin B_R(0)$, with $0 \leq \eta_R(x) \leq 1$ and $|\nabla\eta_R| \leq \dfrac{C}{R}$, where C is a constant independent of R. Since the sequence $(\eta_R u_n)$ is bounded in W_ϵ, and fixing $R > 0$ such that $\Omega_\epsilon \subset B_{R/2}(0)$ we obtain, by definition of the functional I_ϵ,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R(0)} \left[a(|\nabla u_n|^p)|\nabla u_n|^p + V(\epsilon x)b(|u_n|^p)|u_n|^p\right]dx = I_\epsilon(u_n)u_n\eta_R + \int_{\mathbb{R}^N} g(\epsilon x, u_n)u_n\eta_R dx \\
&\qquad - \int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\nabla\eta_R dx + o_n(1).
\end{aligned}$$

Using $(g_3)_{ii}$ we estimate

$$\left(1-\frac{1}{\beta}\right)\int_{\mathbb{R}^N\setminus B_R(0)}\left[a(|\nabla u_n|^p)|\nabla u_n|^p+V(\epsilon x)b(|u_n|^p)|u_n|^p\right]dx$$
$$\leq \int_{\mathbb{R}^N}|u_n|a(|\nabla u_n|^p)|\nabla u_n|^{p-1}|\nabla\eta_R|dx+o_n(1).$$

As (u_n) is bounded in W_ϵ and $|\nabla\eta_R|\leq\dfrac{C}{R}$. Passing to the limit in the last estimate, we get

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_R}[a(|\nabla u_n|^p)|\nabla u_n|^p+V(\epsilon x)b(|u_n|^p)|u_n|^p]dx<\xi.$$

for some R sufficiently large and for some fixed $\xi>0$. □

In the next result we show that the functional I_ϵ satisfies the Palais-Smale condition for some levels. For this work we are denoting by S the best Sobolev constant for the embedding of $D^{1,q}(\mathbb{R}^N)$ into $L^{q^*}(\mathbb{R}^N)$, that is, the largest positive constant S such that

$$S\left(\int_{\mathbb{R}^N}|u|^{q^*}dx\right)^{\frac{q}{q^*}}\leq\int_{\mathbb{R}^N}|\nabla u|^q dx \quad \text{for every } u\in W^{1,q}(\mathbb{R}^N). \tag{3.2.4}$$

Lemma 3.2.4. *The functional I_ϵ satisfies the Palais-Smale condition at any level*

$$d<\left(\frac{1}{\theta}-\frac{1}{q^*}\right)S^{N/q}.$$

Proof. Let $(u_n)\subset W_\epsilon$ be a Palais-Smale sequence at level $d<\left(\dfrac{1}{\theta}-\dfrac{1}{q^*}\right)S^{N/q}$ for the functional I_ϵ. Arguing as Lemma [6, Lemma 2.3] we have that (u_n) is bounded in W_ϵ. Then by Sobolev embeddings we deduce, up to a subsequence, that

$$\begin{cases} u_n \rightharpoonup u \ \text{ weakly in } \ W_\epsilon, \\ \nabla u_n(x)\to\nabla u(x) \ \ a.e \ \text{ in } \ \mathbb{R}^N, \\ u_n\to u \ \text{ strongly in } \ L^s_{loc}(\mathbb{R}^N) \ \text{ for any } \ p\leq s<q^*, \\ u_n(x)\to u(x) \ \text{ for a.e } \ x\in\mathbb{R}^N. \end{cases} \tag{3.2.5}$$

Using the same kind of ideias contained [6, Lemma 2.3], we may conclude that u is a critical point of I_ϵ. From Lemma 3.2.3 and for each $\xi>0$ given there exists $R>0$ such that

$$\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N\setminus B_R(0)}[a(|\nabla u_n|^p)|\nabla u_n|^p+V(\epsilon x)b(|u_n|^p)|u_n|^p]dx<\xi.$$

This inequality, (a_1), (b_1), (f_1), (f_2), (g_2) and the Sobolev embeddings imply, for n large enough, there exists a positive constant C_1 such that

$$\left|\int\limits_{\mathbb{R}^N\setminus B_R(0)}g(\epsilon x,u_n)u_n dx\right|\leq C_1\left(\xi+\xi^{r/q}+\xi^{q^*/q}\right). \tag{3.2.6}$$

On the other hand, taking R large enough, we suppose that

$$\left|\int\limits_{\mathbb{R}^N\setminus B_R(0)}g(\epsilon x,u)udx\right|<\xi. \tag{3.2.7}$$

Therefore, by (3.2.6) and (3.2.7),

$$\int\limits_{\mathbb{R}^N\backslash B_R(0)} g(\epsilon x, u_n)u_n dx = \int\limits_{\mathbb{R}^N\backslash B_R(0)} g(\epsilon x, u)u \; dx + o_n(1). \tag{3.2.8}$$

We claim that

$$\int\limits_{B_R(0)\cap(\mathbb{R}^N\backslash\Omega_\epsilon)} g(\epsilon x, u_n)u_n dx = \int\limits_{B_R(0)\cap(\mathbb{R}^N\backslash\Omega_\epsilon)} g(\epsilon x, u)u \; dx + o_n(1). \tag{3.2.9}$$

Indeed, we have, in view of the definition of g,

$$g(\epsilon x, u_n)u_n \le f(u_n)u_n + \left(\frac{\eta}{2}\right)^{q^*} + \frac{V_0}{\beta}|u_n|^q \;\text{ for any } x \in \mathbb{R}^N\backslash\Omega_\epsilon.$$

Since the set $B_R(0)\cap(\mathbb{R}^N\backslash\Omega_\epsilon)$ is bounded we can use the above estimate, (f_1), (f_2), (3.2.5) and Lebesgue's Theorem to conclude that the convergence (3.2.9) holds.

Finally, we now prove the following convergence

$$\int\limits_{\Omega_\epsilon} |u_n|^{q^*} dx = \int\limits_{\Omega_\epsilon} |u|^{q^*} \; dx + o_n(1). \tag{3.2.10}$$

Since (u_n) is bounded in W_ϵ and using the Lions's Concentration Compactness Principle [39], we may suppose that

$$|\nabla u_n|^q \rightharpoonup \mu \qquad \text{and} \qquad |u_n|^{q^*} \rightharpoonup \nu.$$

Then we obtain an at most countable index set Γ, sequences $(x_i) \subset \mathbb{R}^N$ and (μ_i), $(\nu_i) \subset (0,\infty)$, such that

$$\mu \ge |\nabla u|^q + \sum_{i\in\Gamma} \mu_i \delta x_i, \quad \nu = |u|^{q^*} + \sum_{i\in\Gamma} \nu_i \delta x_i \;\text{ and }\; S\nu_i^{q/q^*} \le \mu_i, \tag{3.2.11}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$. Thus it is sufficient to show that $\{x_i\}_{i\in\Gamma}\cap\Omega_\epsilon = \emptyset$. Then, we suppose by contradiction that $x_i \in \Omega_\epsilon$ for some $i \in \Gamma$. Consider $R > 0$ and the function $\psi_R := \psi(x_i - x)$, where $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is such that $\psi \equiv 1$ in $B_R(x_i)$, $\psi \equiv 0$ in $\mathbb{R}^N\backslash B_{2R}(x_i)$, $|\nabla\psi|_\infty \le 2$, where $R > 0$ will be chosen in such way that the support of ψ is contained in Ω_ϵ. Then, as $(\psi_R u_n)$ is bounded and $I'_\epsilon(u_n)\psi_R u_n = o_n(1)$,

$$\int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi_R \; dx + \int_{\mathbb{R}^N} \psi_R a(|\nabla u_n|^p)|\nabla u_n|^p \; dx$$

$$+ \int_{\mathbb{R}^N} \psi_R V(\epsilon x) b(|u_n|^p)|u_n|^p \; dx = \int_{\mathbb{R}^N} f(x, u_n)\psi_R u_n \; dx + \int_{\mathbb{R}^N} \psi_R |u_n|^{q^*} \; dx + o_n(1).$$

Note that, using (a_1), (b_1) and that the function f has subcritical growth, we have

$$\lim_{R\to 0}\left[\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n^p \cdot \nabla\psi_R \; dx\right] = 0,$$

$$\lim_{R\to 0}\left[\lim_{n\to\infty} \int_{\mathbb{R}^N} V(\epsilon x) b(|u_n|^p)|u_n|^p \psi_R \; dx\right] = 0,$$

and

$$\lim_{R\to 0}\left[\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_n)\psi_R u_n dx\right] = 0.$$

Therefore, by (a_1) again,

$$\int\limits_{\mathbb{R}^N} \psi_R |\nabla u_n|^q dx \leq \int\limits_{\mathbb{R}^N} |u_n|^{q^*} \psi_R dx + o_n(1).$$

Since ψ_R has compact support and letting $n \to \infty$ in the above expression, we see that

$$\int_{\mathbb{R}^N} \psi_R \mathrm{d}\mu \leq \int_{\mathbb{R}^N} \psi_R \mathrm{d}\nu,$$

which implies

$$\mu_i \leq \nu_i.$$

From this inequality and (3.2.11) one easily sees that $S^{N/q} \leq \nu_i$. As $\beta > \dfrac{p\gamma\theta}{q(\theta - p\gamma)}$ and $S^{N/q} \leq \nu_i$ we have, by previous arguments,

$$\begin{aligned} c = I_\epsilon(u_n) - \frac{1}{\theta} I'_\epsilon(u_n)u_n + o_n(1) \quad &\geq \quad \left(\frac{\theta - p\gamma}{p\gamma\theta} - \frac{1}{q\beta}\right) \|u_n\|^q_{1,q} + \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int\limits_{\Omega_\epsilon} |u_n|^{q^*} dx + o_n(1) \\ &\geq \quad \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int\limits_{\Omega_\epsilon} \psi_R |u_n|^{q^*} dx + o_n(1). \end{aligned}$$

Hence, taking the limit and using (3.2.11), we get

$$c \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \sum_{i \in \Gamma} \psi_R(x_i)\nu_i = \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \nu_i \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) S^{N/q}$$

which does not make sense. Thus we obtain the convergence (3.2.10).

Therefore

$$\int\limits_{\mathbb{R}^N} g(\epsilon x, u_n)u_n \, dx = \int\limits_{\mathbb{R}^N} g(\epsilon x, u)u \, dx + o_n(1). \tag{3.2.12}$$

Finally, we prove that, up to a subsequence, $u_n \to u$ in W_ϵ. Since $I'_\epsilon(u_n)u_n = o_n(1)$, $I'_\epsilon(u) = 0$, (3.2.12) and Fatou's Lemma we have

$$\begin{aligned} 0 \quad \leq \quad & \int\limits_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p - a(|\nabla u|^p)|\nabla u|^p] \, dx + \int\limits_{\mathbb{R}^N} V(\epsilon x) \, [b(|u_n|^p)|u_n|^p - b(|u|^p)|u|^p] \, dx \\ & + \int\limits_{\mathbb{R}^N} [g(\epsilon x, u)u - g(\epsilon x, u_n)u_n] \, dx = o_n(1). \end{aligned}$$

Then, using (a_1) and (b_1), we obtain $\|u_n - u\| = o_n(1)$, that is, the sequence (u_n) converges strongly to u. $\square$

Let us now consider the following problem

$$\begin{cases} -k_2\Delta_p u - \Delta_q u + V_\infty(k_4|u|^{p-2}u + |u|^{q-2}u) = |u|^\tau \ \text{ in } \Omega, \\ u \in W_0^{1,q}(\Omega), \end{cases} \tag{P_∞}$$

where τ is the constant which appears in the hypothesis (f_5) and V_∞ is a positive constant. We have associated to problem (P_∞) the functional

$$I_\infty(u) = \frac{1}{p} \int\limits_\Omega [k_2|\nabla u|^p + V_\infty k_4|u|^p] \, dx + \frac{1}{q} \int\limits_\Omega [|\nabla u|^q + V_\infty|u|^q] \, dx - \frac{1}{\tau} \int\limits_\Omega |u|^\tau dx$$

and the associated Nehari manifold

$$\mathcal{N}_\infty = \{u \in W_0^{1,q}(\Omega) : \; u \neq 0 \text{ and } I'_\infty(u)u = 0\}.$$

From Appendix A there exists $w_\tau \in W_0^{1,q}(\Omega)$ such that

$$I_\infty(w_\tau) = c_\infty, \; I'_\infty(w_\tau) = 0$$

and

$$c_\infty \geq \left(\frac{\tau - q}{\tau q}\right) \int\limits_{\mathbb{R}^N} |w_\tau|^\tau dx. \tag{3.2.13}$$

Since λ is the parameter which appears in the hypothesis (f_5) we have the following result.

Lemma 3.2.5. *There exists $\lambda^* > 1$, such that if $\lambda > \lambda^*$, then $c_\epsilon < \left(\frac{1}{\theta} - \frac{1}{q^*}\right) S^{N/q}$.*

Proof. First of all, by the hypotheses (a_1), (b_1) and (f_5), we obtain

$$\begin{aligned}&\int\limits_{\mathbb{R}^N} a(|\nabla w_\tau|^p)|\nabla w_\tau|^p dx + \int\limits_{\mathbb{R}^N} V(\epsilon x) b(|w_\tau|^p)|w_\tau|^p dx \leq \int\limits_\Omega [k_2|\nabla w_\tau|^p + V_\infty k_4 |w_\tau|^p]\, dx \\ &+ \int\limits_\Omega [|\nabla w_\tau|^q + V_\infty |w_\tau|^q]\, dx = \int\limits_\Omega |w_\tau|^\tau dx \leq \int\limits_\Omega f(w_\tau) w_\tau dx \leq \int\limits_{\mathbb{R}^N} g(\epsilon x, w_\tau) w_\tau dx,\end{aligned}$$

where $V_\infty := \max\limits_{x \in \overline{\Omega}} V(x)$. This inequality implies that $I'_\epsilon(w_\tau^\pm) w_\tau^\pm \leq 0$, and then there exists $t \in (0,1)$ such that $t w_\tau \in \mathcal{N}_\epsilon$.

Using (a_1), (b_1) and (f_5), we obtain

$$\begin{aligned} c_\epsilon &\leq I_\epsilon(t w_\tau) \\ &\leq \frac{t^p}{p} \int\limits_\Omega [k_2|\nabla w_\tau|^p + V_\infty k_4|w_\tau|^p]\, dx + \frac{t^q}{q} \int\limits_\Omega [|\nabla w_\tau|^q + V_\infty |w_\tau|^q]\, dx - \frac{\lambda}{\tau} t^\tau \int\limits_\Omega |w_\tau|^\tau dx. \end{aligned}$$

Since $t \in (0,1)$, $p \leq q$ and $I'_\infty(w_\tau) w_\tau = 0$, we get

$$\begin{aligned} c_\epsilon &\leq I_\epsilon(t w_\tau) \\ &\leq \frac{t^p}{p} \int\limits_\Omega [k_2|\nabla w_\tau|^p + V_\infty k_4|w_\tau|^p]\, dx + \frac{t^p}{p} \int\limits_\Omega [|\nabla w_\tau|^q + V_\infty |w_\tau|^q]\, dx - \frac{\lambda}{\tau} t^\tau \int\limits_\Omega |w_\tau|^\tau dx \\ &= \left[\frac{t^p}{p} - \lambda \frac{t^\tau}{\tau}\right] \int\limits_\Omega |w_\tau|^\tau dx \leq \max_{s \geq 0} \left[\frac{s^p}{p} - \lambda \frac{s^\tau}{\tau}\right] \int\limits_\Omega |w_\tau|^\tau dx. \end{aligned}$$

Using (3.2.13), we have

$$c_\epsilon \leq \max_{s \geq 0} \left[\frac{s^p}{p} - \lambda \frac{s^\tau}{\tau}\right] \frac{c_\infty q \tau}{(\tau - q)} \leq \left[\frac{\tau - p}{p \lambda^{p/(\tau - p)}}\right] \frac{c_\infty q}{(\tau - q)}$$

By some straight forward algebric manipulations, we get

$$c_\epsilon \leq \left[\frac{\tau - p}{p \lambda^{p/(\tau - p)}}\right] \frac{c_\infty q}{(\tau - q)}.$$

Then, if we choose $\lambda > \lambda^* := \max \left\{ 1, \left[\frac{(\tau - p)}{(\tau - q)} \frac{q}{p} \frac{\theta q^*}{(q^* - \theta)} \frac{c_\infty}{S^{N/q}}\right]^{(\tau - p)/p} \right\}$ in the hypothesis (f_5), the proof is complete. $\square$

Proof of the Theorem 3.2.1

Proof. The proof is a consequence of Lemma 3.2.2, Lemma 3.2.4 and Lemma 3.2.5. $\square$

The Autonomous Problem

In order to prove the concentration result, we consider the following problem

$$\begin{cases} -div\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) + V_0 b(|u|^p)|u|^{p-2}u = f(u) + |u|^{q^*-1} & \text{in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \end{cases} \tag{P_0}$$

which the functional associated I_0 is given by

$$I_0(u) = \frac{1}{p}\int_{\mathbb{R}^N}[A(|\nabla u|^p) + V_0 B(|u|^p)]dx - \int_{\mathbb{R}^N} F(u)dx - \frac{1}{q^*}\int_{\mathbb{R}^N}|u|^{q^*}dx,$$

and the corresponding Nehari manifold is given by

$$\mathcal{N}_0 = \{u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)\backslash\{0\}; I_0'(u)u = 0\}.$$

We also define

$$c_0 = \inf_{\mathcal{N}_0} I_0.$$

Using the same arguments of prove of Lemma 3.2.5, we conclude that

$$c_0 < \left(\frac{1}{\theta} - \frac{1}{q*}\right)S^{N/q}. \tag{3.2.14}$$

The next result allows to show that problem (P_0) has a solution that reaches c_0.

Lemma 3.2.6. *Let $(u_n) \subset \mathcal{N}_0$ be a sequence such that $I_0(u_n) \to c_0$. Then there are a sequence $(y_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that*

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^q dx \geq \eta. \tag{3.2.15}$$

Proof. Suppose that (3.2.15) is not satisfied. Since (u_n) is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we have, by in [40, Lemma 2.1],

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^s dx = 0 \quad \text{for all} \quad s \in (q, q^*).$$

Hence, from $(f_1) - (f_3)$,

$$\int_{\mathbb{R}^N} f(u_n)u_n dx = o_n(1).$$

Since we also have (g_3) and that $I'_{\epsilon_n}(u_n)u_n = o_n(1)$, we get

$$\int_{\mathbb{R}^N}|u_n|^{q^*}dx = \int_{\mathbb{R}^N}[a(|\nabla u_n|^p)|\nabla u_n|^p dx + V_0 b(|u_n|^p)|u_n|^p]\,dx + o_n(1) := l$$

We claim that $l > 0$. Indeed, if the claim is not true then, by (a_1) and (b_1), we have $c_0 = 0$ which is a contradiction. Therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{q^*} dx = l > 0. \tag{3.2.16}$$

By definition of the constant S, we have

$$S \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^q dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_n|^{q^*} dx\right)^{q/q^*}} \leq l^{q/N}. \tag{3.2.17}$$

Thus, using (3.0.2), (3.0.3) and (f_3), we deduce that

$$c_0 + o_n(1) = I_0(u_n) - \frac{1}{\theta} I_0(u_n)u_n \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) \int_{\mathbb{R}^N} |u_n|^{q^*} dx + o_n(1).$$

Using (3.2.16), (3.2.17) and that $c_0 > 0$, we obtain $c_0 \geq \left(\frac{1}{\theta} - \frac{1}{q^*}\right) S^{N/q}$ which is a contradiction with (3.2.14). $\square$

Now we are ready to show that the problem (P_0) has a solution that reaches c_0.

Lemma 3.2.7. *(A Compactness Lemma) Let $(u_n) \subset \mathcal{N}_0$ be a sequence satisfying $I_0(u_n) \to c_0$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $v_n(x) = u_n(x + \widetilde{y}_n)$ converges strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.*
In particular, there exists a minimizer for c_0.

Proof. Applying Ekeland's Variational Principle (see Theorem 8.5 in [62]), we may suppose that (u_n) is a $(PS)_{c_0}$ for I_0. Since (u_n) is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we can assume, up to subsequences, that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

Using arguments found in [6, Lemma 2.3], we have that

$$\nabla u_n(x) \to \nabla u(x) \;\; a.e \;\text{ in }\; \mathbb{R}^N \quad \text{and} \quad I_0'(u) = 0. \tag{3.2.18}$$

Then, by (3.0.2), (3.0.3) and the Fatou's Lemma,

$$\begin{aligned} 0 &\leq \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla u|^p) + V_0 B(|u|^p)]\, dx - \frac{1}{\theta} \int_{\mathbb{R}^N} [a(|\nabla u|^p)|\nabla u|^p + V_0 B(|u|^p)|u|^p]\, dx \\ &\leq \liminf_{n\to+\infty} \Bigg\{ \frac{1}{p} \int_{\mathbb{R}^N} [A(|\nabla u_n|^p) + V_0 B(|u_n|^p)]\, dx \\ &\qquad - \frac{1}{\theta} \int_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p + V_0 B(|u_n|^p)|u_n|^p]\, dx \Bigg\} \end{aligned}$$

Hence, if $u \in \mathcal{N}_0$,

$$c_0 \leq I_0(u) - \frac{1}{\theta} I_0'(u)u \leq \liminf_{n\to+\infty} \left[I_0(u_n) - \frac{1}{\theta} I_0'(u_n)u_n\right] = \lim_{n\to+\infty} I_0(u_n) = c_0.$$

By (3.2.18), (a_1), (b_1) and Lebesgue's theorem we conclude that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Consequently, $I_0(u) = c_0$ and the sequence $(\widetilde{y}_n)$ is the sequence null.

If $u \equiv 0$, then in that case we cannot have $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ because $c_{V_0} > 0$. Hence, using Lemma 3.2.6, there exists a sequence $\{\widetilde{y}_n\} \subset \mathbb{R}^N$ such that

$$v_n \rightharpoonup v \ \ in \ \ W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),$$

where $v_n := u_n(x + \widetilde{y}_n)$. Therefore, (v_n) is also a $(PS)_{c_0}$ sequence for I_0 and $v \not\equiv 0$. It follows from the above arguments that, up to a subsequence, (v_n) converges strongly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and the proof is complete. □

3.3 Concentration results

In this section we prove some technical results in order to show the concentration result.

Proposition 3.3.1. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to c_0$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) := u_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in \Omega$, where $y_n = \epsilon_n \tilde{y}_n$.*

Proof. Since V satisfies (V_1) and $c_0 > 0$, we repeat the same arguments in Lemma 3.2.6 to conclude that there exist positive constants R and $\widetilde{\beta}$ and a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} | u_n |^q \geq \widetilde{\beta} > 0.$$

Since the sequence (u_n) is bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we immediately obtain, up to a subsequence, $v_n \rightharpoonup v \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, where $v_n(x) := u_n(x + \tilde{y}_n)$. Let $t_n > 0$ be such that

$$\tilde{v}_n = t_n v_n \in \mathcal{N}_0. \tag{3.3.1}$$

Then, since $u_n \in \mathcal{N}_{\epsilon_n}$, we have

$$c_0 \leq I_0(\tilde{v}_n) \leq I_{\epsilon_n}(\tilde{v}_n) \leq I_{\epsilon_n}(v_n) = I_{\epsilon_n}(u_n) = c_0 + o_n(1), \tag{3.3.2}$$

which implies that $I_0(\tilde{v}_n) \to c_0$, as $n \to +\infty$.

From boundedness of (v_n) and (3.3.2), we obtain that (t_n) is bounded. As a consequence, the sequence $(\tilde{v}_n)$ is also bounded in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ which implies, up to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

Note that we can assume that $t_n \to t_0 > 0$. Then, this limit implies that $\tilde{v} \not\equiv 0$. From Lemma 3.2.7, we conclude that $\tilde{v_n} \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and this implies that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

To conclude the proof of this proposition, we consider $y_n := \epsilon_n \tilde{y}_n$. Our goal is to show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n \to y$ for $y \in \Omega$. First of all, we claim that (y_n) is bounded. Indeed, suppose that there exists a subsequence, still denote by (y_n), verifying $|y_n| \to \infty$. From (a_1), (b_1) and (V_1) we have

$$\int_{\mathbb{R}^N} [k_1 |\nabla v_n|^p + |\nabla v_n|^q] \, dx + V_0 \int_{\mathbb{R}^N} [k_3 |v_n|^p + |v_n|^q] \, dx \leq \int_{\mathbb{R}^N} g(\epsilon_n x + y_n, v_n) v_n dx.$$

Fix $R > 0$ such that $B_R(0) \supset \Omega$ and let $\mathcal{X}_{B_R(0)}$ be the characteristic function of $B_R(0)$. Since $\mathcal{X}_{B_R(0)}(\epsilon x + y_n) = o_n(1)$ for all $x \in B_R(0)$ and $v_n \to v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \mathcal{X}_{B_R(0)}(\epsilon x + y_n) g(\epsilon x + y_n, v_n) v_n dx = o_n(1).$$

By definition of $\widetilde{f}$ we obtain that

$$\begin{aligned}\int\limits_{\mathbb{R}^N} [k_1|\nabla v_n|^p + |\nabla v_n|^q]\, dx + V_0 \int\limits_{\mathbb{R}^N} [k_3|v_n|^p + |v_n|^q]\, dx \ &\leq \int\limits_{\mathbb{R}^N\setminus B_R(0)} \widetilde{f}(v_n)v_n dx + o_n(1)\\ &\leq \frac{V_0}{\beta}\int\limits_{\mathbb{R}^N} |v_n|^q dx + o_n(1).\end{aligned}$$

It follows that $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, obtain this way a contradiction because $c_0 > 0$.

Hence (y_n) is bounded and, up to a subsequence,

$$y_n \to \overline{y} \in \mathbb{R}^N.$$

Arguing as above, if $\overline{y} \notin \overline{\Omega}$ we will obtain again $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, and then $\overline{y} \in \overline{\Omega}$. Now if $V(\overline{y}) = V_0$, we have $\overline{y} \notin \partial\Omega$ and consequently $\overline{y} \in \Omega$. Suppose by contradiction that $V(\overline{y}) > V_0$. Then, we have

$$c_0 = I_0(\widetilde{v}) < \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla\widetilde{v}|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\overline{y})B(|\widetilde{v}|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v})dx - \int\limits_{\mathbb{R}^N} |\widetilde{v}|^{q^*} dx.$$

Using the fact that $\widetilde{v}_n \to \widetilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, from Fatou's Lemma we obtain

$$c_0 < \liminf_{n\to\infty}\left[\frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla\widetilde{v}_n|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(\epsilon_n z + y_n)B(|\widetilde{v}_n|^p)dx - \int\limits_{\mathbb{R}^N} F(\widetilde{v}_n)dx - \int\limits_{\mathbb{R}^N} |\widetilde{v}_n|^{q^*} dx.\right]$$

Since $u_n \in \mathcal{N}_{\epsilon_n}$, this implies that

$$c_0 < \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_0,$$

obtaining a contradiction. □

Lemma 3.3.2. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ a solution of problem $(P_{\epsilon_{aux}})$. Then (v_n) converges uniformly on compacts of $\mathbb{R}^N$, where $v_n(x) := u_n(x + \tilde{y}_n)$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} < \xi \quad \textit{for all} \quad n \geq n_0,$$

where $(\tilde{y}_n)$ is the sequence of Proposition 3.3.1.

Proof. Note that v_n is a solution of problem

$$\begin{cases} -div\left(a\left(|\nabla v_n|^p\right)|\nabla v_n|^{p-2}\nabla v_n\right) + V(\epsilon x + y_n)b\left(|v_n|^p\right)|v_n|^{p-2}v_n = g(\epsilon x + y_n, v_n) \ \text{ in } \mathbb{R}^N,\\ v_n \in W_\epsilon,\end{cases}$$

where $y_n = \epsilon_n\tilde{y}_n$. Adapting some arguments explored in [6, Lemma 5.5], we have that the sequence (v_n) is bounded in $L^\infty(\mathbb{R}^N)$ and there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|v_n\|_{L^\infty(\mathbb{R}^N\setminus B_R(0)} < \xi, \quad \text{for all} \quad n \geq n_0.$$

Then, for any bounded domain $\Omega' \subset \mathbb{R}^N$, from $(g_1) - (g_2)$ and continuity of V there exists $C > 0$ such that

$$|V(\epsilon x + y_n)v_n^{p-1} - g(\epsilon x + y_n, v_n)| \leq C, \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence,

$$|V(\epsilon x + y_n)v_n^{p-1} - g(\epsilon x + y_n, v_n)| \leq C + |\nabla v_n|^p, \quad \text{for all} \quad n \in \mathbb{N}.$$

Considering $\Psi(x) = C$, we get that $\Psi \in L^t(\Omega')$ with $t > \frac{p}{p-1}N$. From [28, Theorem 1], we have

$$\|\nabla v_n\| \in L^{\infty}_{loc}(\mathbb{R}^N).$$

Therefore, for all compact $K \subset \Omega'$ there exists a constant $C_0 > 0$, dependent only on C, N, p and $\mathrm{dist}(K, \partial\Omega')$, such that

$$|\nabla v_n|_{\infty,K} \leq C_0.$$

Then,

$$|v_n|_{C^{0,\nu}_{loc}(\mathbb{R}^N)} \leq C, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad 0 < \nu < 1.$$

From Schauder's embedding, (v_n) has a subsequence convergent in $C^{0,\nu}_{loc}(\mathbb{R}^N)$. □

Lemma 3.3.3. *Given $\epsilon > 0$, the solution u_ϵ of problem $(P_{\epsilon_{aux}})$ satisfies*

$$\lim_{\epsilon \to 0} I_\epsilon(u_\epsilon) = c_{V_0}.$$

Proof. Consider $z_0 \in \Omega$ such that $V(z_0) = V_0$. Let us now consider $R > 0$ and set $Q \in \partial B_R(z_0)$. If necessary, take R small enough such that $B(Q, R/4) \subset \Omega$. Taking $\psi : \mathbb{R}^N \to \mathbb{R}$ such that $\psi \equiv 1$ in $B(Q, R/4)$ and $\psi \equiv 0$ in $\mathbb{R}^N \backslash B(Q, R/2)$.

Let $w_0 \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ be a ground-state positive solution of the problem (P_0) which satisfies $c_0 = I_0(w_0)$ (see Lemma 3.2.7). Then, we consider the function $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ be given by

$$w_\epsilon(x) := \psi_i(\epsilon x) w_0 \left(x - \frac{z_0}{\epsilon} \right) \in W_\epsilon$$

and $t_\epsilon > 0$, such that $t_\epsilon w_\epsilon \in \mathcal{N}_\epsilon$. Then, with a direct computation, we have

$$I_\epsilon(u_\epsilon) \leq I_\epsilon(t_\epsilon w_\epsilon) = c_0 + o_\epsilon(1).$$

Finally, taking $R \to 0$ in the last inequality and using the continuity of the minimax function (see [13], [51]) we get

$$\limsup_{\epsilon \to 0} I_\epsilon(u_\epsilon) \leq c_0.$$

Let $t_{\epsilon,0} > 0$ be such that $t_{\epsilon,0} u_\epsilon \in \mathcal{N}_0$. Then,

$$c_0 \leq I_0(t_{\epsilon,0} u_\epsilon) \leq I_\epsilon(t_{\epsilon,0} u_\epsilon) \leq I_\epsilon(u_\epsilon)$$

and the proof is complete. □

Lemma 3.3.4. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be a solution of problem $(P_{\epsilon_{aux}})$. Then, there are $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that, for $v_n(x) = u_n(x + \tilde{y}_n)$, we have*

$$v_n(x) \geq \delta^*, \quad \textit{for all} \quad x \in B_R(0) \quad \textit{and} \quad n \geq n_0,$$

where $R > 0$ and $(\tilde{y}_n)$ were given in Lemma 3.3.2.

Proof. Suppose, by contradiction, that $\|u_n\|_{L^\infty(|x|<R)} = \|u_n\|_{L^\infty(|x-\tilde{y}_n|<R)} \to 0$. By Lemma 3.3.2, we have $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$. It follows from (f_1) that

$$|f(v_n) + v_n^{q^*-1}| \leq \frac{V_0}{2}|v_n|^{q-1} \quad \text{for} \quad n \quad \text{sufficient large.} \tag{3.3.3}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^N} a(|\nabla v_n|^p)|\nabla v_n|^p dx \quad &+ \quad \int_{\mathbb{R}^N} V(\epsilon_n x + y_n) b(|v_n|^p)|v_n|^p dx \\ &= \quad \int_{\mathbb{R}^N} f(v_n) v_n dx + o_n(1) \\ &\leq \quad \frac{V_0}{2} \int_{\mathbb{R}^N} |v_n|^q dx + o_n(1),\end{aligned}$$

which implies from (a_1) and (b_1) that,

$$\|u_n^{\pm}\|_{W_{\epsilon_n}} \to 0,$$

which is a contradiction with Lemma 3.3.3. □

We are now ready to show the concentration of the ground state solution.

Lemma 3.3.5. *If $\frac{P_\epsilon}{\epsilon}$ is the maximum point of u_ϵ, then*

$$\lim_{\epsilon \to 0} V(P_\epsilon) = V_0.$$

Proof. We first notice that using Lemma 3.3.4 there exist $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$v_n(q_n) := \max_{z \in \mathbb{R}^N} v_n(z) = u_n(q_n + \tilde{y}_n) \geq u_n(x) \geq \delta^*, \quad \text{for all } n \geq n_0, \quad \text{for all} \quad x \in B_R(0).$$

We claim that (q_n) is bounded, otherwise using Lemma 3.3.2 and 3.3.4, there exists $R^* > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N \setminus B_{R^*})} \leq \frac{\delta^*}{2}$, which implies that $|v_n(q_n)| \leq \frac{\delta^*}{2}$, where we obtain a contradiction.

Then, $P_{\epsilon_n} = \epsilon_n q_n + y_n$ which implies

$$\lim_{n \to +\infty} P_{\epsilon_n} = \lim_{n \to +\infty} y_n = \overline{y} \in \Omega.$$

Hence from continuity of V it follows that

$$\lim_{n \to +\infty} V(P_{\epsilon_n}) = V(\overline{y}) \geq V_0.$$

We claim that $V(\overline{y}) = V_0$. Indeed, suppose by contradiction that $V(\overline{y}) > V_0$. Then, we have

$$c_0 = I_0(\tilde{v}) < \frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\overline{y}) B(|\tilde{v}|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}) - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}|^{q^*} dx.$$

Using that $\tilde{v}_n \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we obtain, from Fatou's Lemma,

$$c_0 < \liminf_{n \to \infty} \left[\frac{1}{p} \int_{\mathbb{R}^N} A(|\nabla \tilde{v}_n|^p) dx + \frac{1}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + y_n) B(|\tilde{v}_n|^p) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n) - \frac{1}{q^*} \int_{\mathbb{R}^N} |\tilde{v}_n|^{q^*} dx \right],$$

and therefore

$$c_0 < \liminf_{n \to \infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n \to \infty} I_{\epsilon_n}(u_n) = c_0.$$

This contradiction shows that $V(\overline{y}) = V_0$. □

Lemma 3.3.6. *Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\epsilon_n \to 0$ as $n \to \infty$ and let $(x_n) \subset \overline{\Omega}_{\epsilon_n}$ be a sequence such that $u_{\epsilon_n}(x_n) \geq \Upsilon > 0$ for some constant Υ, where for each $n \in \mathbb{N}$, u_{ϵ_n} is a solution of $(P_{\epsilon_{aux}})$. Then,*

$$\lim_{n\to\infty} V(\overline{x}_n) = V_0$$

where $\overline{x}_n = \epsilon_n x_n$.

Proof. Up to a subsequence,

$$\overline{x}_n \to \overline{x} \in \overline{\Omega}.$$

From Lemma 3.3.3 we have that

$$I_{\epsilon_n}(u_{\epsilon_n}) \to c_0,$$

and there exists a positive constant C such that

$$\|u_{\epsilon_n}\| \leq C, \quad \forall\ n \in \mathbb{N} \quad , \text{ for some } \ C > 0.$$

Setting $v_n(z) := u_{\epsilon_n}(z + x_n)$, we have $\|v_n\| \leq C$ and $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. Recalling that

$$v_n(0) = u_{\epsilon_n}(x_n) \geq \Upsilon > 0,$$

we conclude that $v \not\equiv 0$.
Fix $t_n > 0$ verifying $\widetilde{v}_n = t_n v_n \in \mathcal{N}_0$, for each $n \in \mathbb{N}$. Hence,

$$c_0 \leq I_0(\widetilde{v}_n) \leq I_{\epsilon_n}(t_n v_n) \leq I_\epsilon(u_n) = c_0 + o_n(1).$$

Thus, $I_0(\widetilde{v}_n) \to c_0$, with $\{\widetilde{v}_n\} \subset \mathcal{N}_0$. By Lemma 3.2.7, we have

$$\widetilde{v}_n \to \widetilde{v} \quad in \quad W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \ \text{ and } \ I_0(\widetilde{v}) = c_0. \tag{3.3.4}$$

Moreover, recalling that V is continuous, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V(\overline{x}).$$

We claim that $V(\overline{x}) = V_0$. Indeed, Suppose by contradiction that $V(\overline{x}) > V_0$, then

$$c_0 = I_0(\widetilde{v}) < \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\overline{x})B(|\widetilde{v}|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v})dx - \frac{1}{q^*}\int_{\mathbb{R}^N} |\widetilde{v}|^{q^*}dx.$$

Thus, by (3.3.4) and Fatou's Lemma, we have

$$\begin{aligned} c_0 &< \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla\widetilde{v}_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n z + \overline{x}_n)B(|\widetilde{v}_n|^p)dx - \int_{\mathbb{R}^N} F(\widetilde{v}_n)dx - \frac{1}{q^*}\int_{\mathbb{R}^N}|\widetilde{v}|^{q^*}dx\right] \\ &\leq \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla t_n v_n|^p)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n z + \overline{x}_n)B(|t_n v_n||^p)dx - \int_{\mathbb{R}^N} G(\epsilon_n z + \overline{x}, t_n v_n)dx\right] \\ &= \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_0, \end{aligned}$$

which leads a absurd. Consequently $\lim_{n\to\infty} V(\overline{x}_n) = V_0$. □

Lemma 3.3.7. *If m_ϵ is given by $m_\epsilon = \sup\{\max\limits_{\partial\Omega_\epsilon} u_\epsilon : \text{is a solution of } (P_{\epsilon_{aux}})\ \}$, then there exists $\overline{\epsilon} > 0$ such that the sequence (m_ϵ) is bounded for all $\epsilon \in (0, \overline{\epsilon})$. Moreover, we have $\lim\limits_{\epsilon \to 0} m_\epsilon = 0$.*

Proof. Suppose, by contradiction, $\lim\limits_{\epsilon\to 0} m_\epsilon = +\infty$, then there exist u_ϵ a solution of $(P_{\epsilon_{aux}})$ in $\mathcal{N}_\epsilon$ and $\Upsilon > 0$ such that

$$\max_{\partial\Omega_\epsilon} u_\epsilon \geq \Upsilon > 0$$

Thus there exists $\{\epsilon_n\} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ and there exists a sequence $\{x_n\} \subset \partial\Omega_{\epsilon_n}$ such that

$$u_{\epsilon_n}(x_n) \geq \Upsilon > 0.$$

Thus, by Lemma 3.3.6, we have

$$\lim_{n\to\infty} V(\overline{x}_n) = V_0,$$

where $\overline{x}_n = \epsilon_n x_n$ and $\{\overline{x}_n\} \subset \partial\Omega$. Hence, up to a subsequence, we have $\overline{x}_n \to \overline{x}$ in $\partial\Omega$ and $V(\overline{x}) = V_0$, which does not make sense by (V_2). Hence, there exists $\overline{\epsilon} > 0$ such that (m_ϵ) is bounded, for all $\epsilon \in (0, \overline{\epsilon})$.

Suppose by contradiction that there exists $\delta > 0$ and a sequence $\{\epsilon_n\} \subset \mathbb{R}^+$ satisfying

$$m_{\epsilon_n} \geq \delta > 0$$

Thus, there exists u_{ϵ_n} a solution of $(P_{\epsilon_{aux}})$ such that

$$m_{\epsilon_n} - \frac{\delta}{2} < \max_{\partial\Omega_{\epsilon_n}} u_{\epsilon_n} \leq m_{\epsilon_n}.$$

Hence,

$$\frac{\delta}{2} = \delta - \frac{\delta}{2} \leq m_{\epsilon_n} - \frac{\delta}{2} < \max_{\partial\Omega_\epsilon} u_{\epsilon_n},$$

and then there exists a sequence $(x_n) \subset \partial\Omega_{\epsilon_n}$, such that

$$u_{\epsilon_n}(x_n) \geq \frac{\delta}{2}.$$

Repeating the above arguments, we will get an absurd. Thus, the proof is finished. $\square$

3.4 Proof of Theorem 3

Proof. Let u_ϵ be a solution of $(P_{\epsilon_{aux}})$. By Lemma 3.3.7, there exists $\overline{\epsilon} > 0$ such that $m_\epsilon < \frac{\eta}{2}$ for all $\epsilon \in (0, \overline{\epsilon})$, then $(u_\epsilon - \frac{\eta}{2})_+(x) \equiv 0$ for a neighborhood from $\partial\Omega_\epsilon$. Hence, $(u_\epsilon - \frac{\eta}{2})_+ \in W_0^{1,p}(\mathbb{R}^N \backslash \Omega_\epsilon) \cap W_0^{1,q}(\mathbb{R}^N \backslash \Omega_\epsilon)$ and the function $(u_\epsilon - \frac{\eta}{2})_+^* \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, where

$$(u_\epsilon - \frac{\eta}{2})_+^*(x) := \begin{cases} 0 \text{ if } x \in \Omega_\epsilon, \\ (u_\epsilon - \frac{\eta}{2})_+(x) \text{ if } x \in \mathbb{R}^N \backslash \Omega_\epsilon. \end{cases}$$

Using $(u_\epsilon - \frac{\eta}{2})_+^*$ as test function. Then, by (a_1), (b_1) and $(g_3)_{ii}$, we have

$$
\begin{aligned}
0 \quad \leq & \int\limits_{\mathbb{R}^N\backslash\Omega_\epsilon} a(|\nabla u_\epsilon|^p)|\nabla(u_\epsilon - \frac{\eta}{2})_+^*|^p dx \\
& + \int\limits_{\mathbb{R}^N\backslash\Omega_\epsilon} \left[V_0 b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{q-2}\right]((u_\epsilon - \frac{\eta}{2})_+^*)^2 dx \\
& + \int\limits_{\mathbb{R}^N\backslash\Omega_\epsilon} \left[V(\epsilon x) b(|u_\epsilon|^p)|u_\epsilon|^{p-2} - \frac{V_0}{\beta}|u_\epsilon|^{q-2}\right]\frac{\eta}{2}(u_\epsilon - \frac{\eta}{2})_+^* dx = 0
\end{aligned}
$$

The last equality implies

$$(u_\epsilon - \frac{\eta}{2})_+^* = 0, \ \text{ a.e in } \ x \in \mathbb{R}^N\backslash\Omega_\epsilon.$$

This implies that $|u_\epsilon| \leq \frac{\eta}{2}$ for $z \in \mathbb{R}^N\backslash\Omega_\epsilon$, and by Remark 5 the result follows. □

3.5 Exponential decay

Finally, we are going to prove the exponential decay. First technical results

Lemma 3.5.1. *Consider $M, \alpha > 0$ and $\psi(x) := M\exp(-\alpha|x|)$. Then*

$$
\begin{aligned}
& i) - div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \\
& \qquad = \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)\right], \\
& ii) - div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) \geq \left(\frac{(N-1)}{|x|} - \alpha(q-1)\right)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}.
\end{aligned}
$$

Proof. Note that

$$\frac{\partial\psi}{\partial x_i}(x) = M\exp(-\alpha|x|)\frac{\partial}{\partial x_i}(-\alpha|x|) = M\exp(-\alpha|x|)(-\alpha)\frac{x_i}{|x|} = -\alpha\frac{x_i}{|x|}\psi(x),$$

which implies $|\nabla\psi| = \alpha\psi$. Then

$$
\begin{aligned}
& -div(a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\nabla\psi) = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left[a(|\nabla\psi|^p)|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial x_i}\right] \\
& = \alpha^{p-1}\sum_{i=1}^N \frac{\partial}{\partial x_i}\left[a(\alpha^p\psi^p)\psi^{p-1}\frac{x_i}{|x|}\right] \\
& = \alpha^{p-1}\sum_{i=1}^N \left[a'(\alpha^p\psi^p)\frac{\partial}{\partial x_i}(\alpha^p\psi^p)\psi^{p-1}\frac{x_i}{|x|} + a(\alpha^p\psi^p)\frac{\partial}{\partial x_i}\left(\psi^{p-1}\frac{x_i}{|x|}\right)\right] \\
& = \alpha^{p-1}\sum_{i=1}^N \left[a'(\alpha^p\psi^p)\alpha^p p\psi^{2p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|} + a(\alpha^p\psi^p)\left(\frac{|x|^2 - x_i^2}{|x|^3}\psi^{p-1} + (p-1)\psi^{p-2}\frac{\partial\psi}{\partial x_i}\frac{x_i}{|x|}\right)\right] \\
& = \alpha^{p-1}\left[-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} + a(\alpha^p\psi^p)\psi^{p-1}\left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)\right],
\end{aligned}
$$

this prove the first item.

To prove the item $ii)$ we are going to use (1.2) and the item $i)$. Hence we have

$$-a'(\alpha^p\psi^p)\alpha^p\psi^p \geq -\frac{(q-p)}{p}a(\alpha^p\psi^p),$$

and consequently

$$-p\alpha^{p+1}a'(\alpha^p\psi^p)\psi^{2p-1} \geq -\alpha\psi^{p-1}(q-p)a(\alpha^p\psi^p).$$

Therefore, by the item $i)$,

$$\begin{aligned}
&-div(a\left(|\nabla\psi|^p\right)|\nabla\psi|^{p-2}\nabla\psi) \\
&\geq \alpha^{p-1}\left[-\alpha(q-p)a(\alpha^p\psi^p)\psi^{p-1} + \left(\frac{(N-1)}{|x|} - \alpha(p-1)\right)a(\alpha^p\psi^p)\psi^{p-1}\right] \\
&= \left(\frac{(N-1)}{|x|} - \alpha(q-1)\right)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1}.
\end{aligned}$$

□

Corollary 3.5.2. *Since $V(x) \geq V_0$ in $\mathbb{R}^N$, then for $\alpha > 0$ small enough we have*

$$-div(a\left(|\nabla\psi|^p\right)|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{q-1} \geq 0 \text{ in } \mathbb{R}^N.$$

Proof. Using (a_1) and Lemma 3.5.1 we obtain that

$$\begin{aligned}
-div(a\left(|\nabla\psi|^p\right)|\nabla\psi|^{p-2}\nabla\psi) &\geq -\alpha(q-1)a(\alpha^p\psi^p)\alpha^{p-1}\psi^{p-1} \\
&\geq -\alpha(q-1)\left(k_2\alpha^{p-1}\psi^{p-1} + \alpha^{q-1}\psi^{q-1}\right) \\
&= -\alpha(q-1)k_2\alpha^{p-1}\psi^{p-1} - \alpha(q-1)\alpha^{q-1}\psi^{q-1}
\end{aligned}$$

Moreover, since $V_0 > 0$ and $\alpha > 0$ is small enough, we concluded that

$$k_3V_0 - \alpha(q-1)k_2\alpha^{p-1} \geq 0$$

and

$$\frac{V_0}{4} - \alpha(q-1)\alpha^{q-1} \geq 0.$$

Consequently

$$-div(a\left(|\nabla\psi|^p\right)|\nabla\psi|^{p-2}\nabla\psi) + k_3V_0\psi^{p-1} + \frac{V_0}{4}\psi^{q-1} \geq 0 \text{ in } \mathbb{R}^N.$$

□

Let us now relate the positive solution v_ϵ to the exponential function ψ for small ϵ.

Lemma 3.5.3. *Let u_ϵ be the solution found in Theorem 3.2.1 and $v_\epsilon(x) := u_\epsilon(x+\tilde{y}_\epsilon)$ given in Proposition 3.3.1. For For $\varphi_\epsilon = \max\{v_\epsilon - \psi, 0\}$ and $\epsilon > 0$ sufficient small, we have*

$$\int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon\nabla\varphi_\epsilon\, dx + k_3V_0\int_{\mathbb{R}^N}|v_\epsilon|^{p-1}\varphi_\epsilon\, dx + \frac{V_0}{4}\int_{\mathbb{R}^N}|v_\epsilon|^{q-1}\varphi_\epsilon\, dx \leq 0.$$

Proof. From Lemma 3.3.2, Lemma 3.3.3 and hypothesis (f_1), there exist $\rho_0 > 0$ such that $\epsilon > 0$ small enough,

$$\frac{f(v_\epsilon) + v_\epsilon^{q^*-1}}{|v_\epsilon|^{q-1}} \leq \frac{3}{4} V_0, \quad \text{for all } \ |x| \geq \rho_0.$$

Since $\psi(x) := M \exp(-\alpha|x|)$ for $x \in \mathbb{R}^N$, we can find $\widetilde{M} > 0$ such that if $M \geq \widetilde{M}$, then $\varphi_\epsilon := \max\{|v_{i,\epsilon}| - \psi, 0\} \equiv 0$ in $B_{\rho_0}(0)$ and $\varphi_\epsilon \in W^{1,p}(|x| \geq \rho_0) \cap W^{1,q}(|x| \geq \rho_0)$. Therefore, the above inequality and (b_1),

$$\int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \nabla \varphi_\epsilon \, dx + V_0 \int_{\mathbb{R}^N} \left[k_3 |v_\epsilon|^{p-1}\varphi_\epsilon + |v_\epsilon|^{q-1}\varphi_\epsilon\right] dx$$

$$\leq \int_{\mathbb{R}^N} a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon \nabla \varphi_\epsilon \, dx + \int_{\mathbb{R}^N} V(\epsilon x + y_\epsilon) b(|v_\epsilon|^p)|v_\epsilon|^{p-2} v_\epsilon \varphi_\epsilon \, dx$$

$$\leq \int_{\mathbb{R}^N} f(v_\epsilon)\varphi_\epsilon \, dx \leq \frac{3V_0}{4} \int_{\mathbb{R}^N} |v_\epsilon|^{q-1}\varphi_\epsilon dx$$

and the lemma is proved. □

Finally we are going to show the exponential decay for the functions u_ϵ.

Proposition 3.5.4. *There are $\epsilon_0 > 0$ and $C > 0$ such that*

$$|u_\epsilon(z)| \leq C \exp\left(-\alpha \left|\frac{z - P_\epsilon}{\epsilon}\right|\right), \quad \textit{for all} \ \ z \in \mathbb{R}^N.$$

Proof. From [**?**, Lemma 2.4], we have that

$$\left\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y, x - y \right\rangle \geq 0, \ \forall \ x, y \in \mathbb{R}^N.$$

Consider $v_\epsilon(x) := u_\epsilon(x + \tilde{y}_\epsilon)$ the set

$$\Lambda := \{x \in \mathbb{R}^N : |x| \geq \rho_0 \ \text{ and } \ |v_\epsilon| - \psi \geq 0\},$$

where ψ is the function is given by Lemma 3.5.1, $(\widetilde{y}_n)$ is given by Proposition 3.3.1. Then, using Corollary 3.5.2 and Proposition 3.5.3, we obtain

$$0 \geq \int_{\mathbb{R}^N} \left\langle a(|\nabla v_\epsilon|^p)|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon - a(|\nabla \psi|^p)|\nabla \psi|^{p-2}\nabla \psi, \nabla \tilde{\varphi}\right\rangle dx$$

$$+V_0 k_3 \int_{\mathbb{R}^N} \left(|v_\epsilon|^{p-1} - |\psi|^{p-1}\right)\tilde{\varphi}\, dx + \frac{V_0}{4}\int_{\mathbb{R}^N} \left(|v_\epsilon|^{q-1} - |\psi|^{q-1}\right)\tilde{\varphi}\, dx$$

$$\geq V_0 k_3 \int_{\mathbb{R}^N} \left(|v_\epsilon|^{p-1} - |\psi|^{p-1}\right)\tilde{\varphi} dx + \frac{V_0}{4}\int_{\mathbb{R}^N} \left(|v_\epsilon|^{q-1} - |\psi|^{q-1}\right)\tilde{\varphi} dx$$

$$= V_0 k_3 \int_{\Lambda} \left(|v_\epsilon|^{p-1} - |\psi|^{p-1}\right)(v_\epsilon - \psi) dx$$

$$+\frac{V_0}{4}\int_{\Lambda} \left(|v_\epsilon|^{q-1} - |\psi|^{q-1}\right)(v_\epsilon - \psi) dx \geq 0.$$

Then $|\Lambda| = 0$ and consequently

$$v_\epsilon(x) \leq M \exp(-\alpha|x|), \ \forall \ |x| \geq \rho_0.$$

Considering $x = z - \tilde{y}_\epsilon$ and using Lemma 3.3.5 there exists a constant $C > 0$ satisfying

$$\begin{aligned} |u_\epsilon(z)| &\leq M \exp\left(-\alpha \left|\frac{z - y_\epsilon}{\epsilon}\right|\right) = M \exp\left(-\alpha \left|\frac{z - P_\epsilon + \epsilon q_\epsilon}{\epsilon}\right|\right) \\ &\leq M \exp\left(-\alpha \left|\frac{z - P_\epsilon}{\epsilon}\right|\right) \exp\left(-\alpha \left|q_\epsilon\right|\right) \leq C \exp\left(-\alpha \left|\frac{z - P_\epsilon}{\epsilon}\right|\right), \end{aligned} \tag{3.5.1}$$

for all $|z - \tilde{y}_\epsilon| \geq \rho_0$ and for $\epsilon > 0$ small enough.

Now we are going to show the inequality (3.5.1) holds, for all $z \in \mathbb{R}^N$. Since (y_ϵ) converges, it follows that

$$|z| \geq \rho_0 - |\tilde{y}_\epsilon| = \rho_0 - \frac{|y_\epsilon|}{\epsilon} > \rho_0 - \frac{1 + |y_\epsilon|}{\epsilon} \to -\infty \ \text{ as } \ \epsilon \to 0.$$

Then, there exists $\epsilon_0 > 0$ such that

$$|u_\epsilon(z)| \leq C \exp\left(-\alpha \left|\frac{z - P_\epsilon}{\epsilon}\right|\right), \ \ \forall \ \ z \in \mathbb{R}^N \ \text{ and } \ \forall \ \ \epsilon \in (0, \epsilon_0).$$

□

Chapter 4

Appendix A

In this appendix we show the existence of a nodal solution for auxiliary problem (P_r) and of a positive solution for auxiliary problems (P_∞). The auxiliary problems (P_r) and (P_∞) are used in the chapters 1 and 3, respectively.

Problem (P_r)

In this appendix we show the existence of nodal solution for the problem

$$\begin{cases} -k_2\Delta_p u - \Delta_N u + V_\infty(k_4|u|^{p-2}u + |u|^{N-2}u) = |u|^{r-2}u \ \text{ in } \Omega, \\ u \in W_0^{1,N}(\Omega), \end{cases} \tag{P_r}$$

where r is the constant that appears in the hypothesis (f_5) and V_∞ is a positive constant. We have associated to the problem (P_r) the functional

$$I_r(u) = \frac{1}{p}\int_\Omega [k_2|\nabla u|^p + V_\infty k_4|u|^p]\,dx + \frac{1}{N}\int_\Omega \left[|\nabla u|^N + V_\infty|u|^N\right]dx - \frac{1}{r}\int_\Omega |u|^r dx$$

and the set

$$\mathcal{N}_r^\pm = \{u \in W_0^{1,N}(\Omega)|\ u^\pm \neq 0 \text{ and } I_r'(u)u^\pm = 0\}$$

Then, we can prove that there exists $w_r \in \mathcal{N}_r^\pm$ such that

$$I_r(w_r) = c_r := \inf_{\mathcal{N}_r^\pm} I_r \quad \text{and} \quad I_r'(w_r) = 0. \tag{4.0.1}$$

Lemma 4.0.1. *For each $u \in W_0^{1,N}(\Omega)$ such that $u^\pm \neq 0$, there exists a unique pair $(t,s) \in (0,+\infty)\times(0,+\infty)$, such that $tu^+ + su^- \in \mathcal{N}_r^\pm$..*

Proof. Note that if $u^\pm \in W_0^{1,N}(\Omega)\backslash\{0\}$ and $\gamma > 0$, we have

$$\begin{aligned} \frac{I_r(\gamma u^\pm)}{\gamma^r} &= \frac{\gamma^{p-r}}{p}\int_\Omega \left[k_2|\nabla u^\pm|^p + V_\infty k_4|u^\pm|^p\right]dx \\ &+ \frac{\gamma^{N-r}}{N}\int_\Omega \left[|\nabla u^\pm|^N + V_\infty|u^\pm|^N\right]dx - \frac{1}{r}\int_\Omega |u^\pm|^r dx. \end{aligned}$$

Then,

$$\lim_{\gamma\to 0}\frac{I_r(\gamma u)}{\gamma^r} = +\infty \ \text{ and } \lim_{\gamma\to+\infty}\frac{I_r(\gamma u)}{\gamma^r} = -\frac{1}{r}\int_\Omega |u|^r dx < 0.$$

Consequently, there exists $t,\ s \in (0,+\infty)$ such that

$$I_r(tu^+) := \sup_{\gamma\geq 0} I_r(\gamma u^+) \quad\text{and}\quad I_r(su^-) := \sup_{\gamma\geq 0} I_r(\gamma u^-).$$

This implies that

$$I_r'(tu^+ + su^-)(tu^+ + su^-) = I_r'(tu^+)tu^+ + I_r'(su^-)su^- = 0$$

In order to show the unicity of t and s, consider $f(s) = s^r$ and note that $\dfrac{f(t)}{t^N}$ is nondecreasing in $t > 0$, see [17]. □

Lemma 4.0.2. *The following properties hold:*

(i) There exists $\rho_r > 0$ such that $\left[\int_\Omega |\nabla u^\pm|^N dx\right]^{1/N} \geq \rho_r$*, for all $u \in \mathcal{N}_r^\pm$;*

(ii) There exists a constant $C_r > 0$ such that $I_r(u) \geq C_r \int_\Omega |\nabla u|^N dx$, for all $u \in \mathcal{N}_r^\pm$.

Proof. Using that $I_r'(u)u^\pm = 0$ and by Sobolev embeddings there exists $C > 0$ such that

$$\begin{aligned}\int_\Omega |\nabla u^\pm|^N dx &\leq \int_\Omega \left[k_2|\nabla u^\pm|^p + V_\infty k_4|u^\pm|^p\right] dx + \int_\Omega \left[|\nabla u^\pm|^N + V_\infty|u^\pm|^N\right] dx \\ &= \int_\Omega |u^\pm|^r dx \leq C\left[\int_\Omega |\nabla u^\pm|^N dx\right]^{r/N}.\end{aligned}$$

Since $r > N$, the item (i) follows.

To verify the second assertion observe that

$$\begin{aligned}I_r(u) = I_r(u) - \frac{1}{r}I_r'(u)u &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega \left[k_2|\nabla u|^p + V_\infty k_4|u|^p\right] dx \\ &+ \left(\frac{1}{N} - \frac{1}{r}\right)\int_\Omega \left[|\nabla u|^N + V_\infty|u|^N\right] dx \geq \left(\frac{1}{N} - \frac{1}{r}\right)\int_\Omega |\nabla u|^N dx.\end{aligned}$$

□

Proposition 4.0.3. *There exists $w_r \in W_0^{1,N}(\Omega)$ such that w_r is a solution of (P_r) and $I_r(w_r) = \inf_{\mathcal{N}_r^\pm} I_r$.*

Proof. Let (u_n) be a minimizing sequence for I_r in $\mathcal{N}_r^\pm$, i.e, a sequence $\{u_n\} \subset \mathcal{N}_r^\pm$ such that $I_r(u_n) = c_r + o_n(1)$. Note that, by Lemma 4.0.2, (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$. Then there exists $u_r \in W_0^{1,N}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_r$ in $W_0^{1,N}(\Omega)$. Arguing as in Lemma 2.3 in [22], it is possible to show that $v \mapsto v^\pm$ is a continuous function of $W_0^{1,N}(\Omega)$ into itself, from which it follows that $u_n^\pm \rightharpoonup u_r^\pm$ in $W_0^{1,N}(\Omega)$. Moreover, by Sobolev embeddings,

$$\begin{cases} u_n^\pm \to u_r^\pm \ \text{ strongly in } \ L^s(\Omega) \ \text{ for any } \ 1 \leq s < +\infty, \\ u_n^\pm(x) \to u_r^\pm(x) \ \text{ for a.e } \ x \in \Omega. \end{cases} \tag{4.0.2}$$

First we are going to show that $u_r^\pm \neq 0$. In fact, if $u_r^\pm \equiv 0$ then, using the Lemma 4.0.2 and that $I_r'(u_n)u_n^\pm = 0$ for all $n \in \mathbb{N}$, we have the following contradiction

$$\rho_r^N \leq \int_\Omega |\nabla u_n^\pm|^N dx \leq \int_\Omega |u_n^\pm|^r dx = o_n(1).$$

It follows from Fatou's Lemma that

$$\int_\Omega |\nabla u_r^\pm|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n^\pm|^p dx \quad \text{and} \quad \int_\Omega |\nabla u_r^\pm|^N dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n^\pm|^N dx.$$

Therefore, using compact embedding and Lemma 4.0.1 we get

$$\begin{aligned} c_r &\leq I_r(tu_r^+ + su_r^-) \leq \liminf_{n\to\infty} [\, I_r(tu_n^+) + I_r(su_n^-)\,] \leq \liminf_{n\to\infty} [\, I_r(u_n^+) + I_r(u_n^-)\,] \\ &= \liminf_{n\to\infty} I_r(u_n) + o_n(1) = c_r. \end{aligned}$$

Considering $w_r = tu_r^+ + su_r^-$, we obtain $I_r(w_r) = c_r$ and using a Deformation Lemma [35, Proof of Theorem 1.1], we conclude that $I_r'(w_r) = 0$. $\square$

Problem (P_∞)

Finally, we show existence of positive solution for the problem

$$\begin{cases} -k_2\Delta_p u - \Delta_q u + V_\infty k_4 |u|^{p-2}u + V_\infty |u|^{q-2}u = |u|^{\tau-2}u & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{P_∞}$$

where Ω is a bounded domain in $\mathbb{R}^N$, k_2, k_4, V_∞ are positive constants and τ is the constant which appears in the hypothesis (f_5). We have associated to problem (P_∞) the functional

$$I_\infty(u) = \frac{1}{p}\int_\Omega [k_2|\nabla u|^p + V_\infty k_4 |u|^p]\, dx + \frac{1}{q}\int_\Omega [|\nabla u|^q + V_\infty |u|^q]\, dx - \frac{1}{\tau}\int_\Omega |u|^\tau dx$$

and the Nehari manifold

$$\mathcal{N}_\infty = \{u \in W_0^{1,q}(\Omega) : \; u \neq 0 \text{ and } I_\infty'(u)u = 0\}$$

Lemma 4.0.4. *For all $u \in W_0^{1,q}(\Omega)\backslash\{0\}$ there exists a unique $t_u \in (0, +\infty)$, such that $tu \in \mathcal{N}_\infty$.*

Proof. Note that if $u \in W_0^{1,q}(\Omega)\backslash\{0\}$ and $t > 0$, we have

$$I_\infty(tu) = t^\tau \left[\frac{t^{p-\tau}}{p}\int_\Omega [k_2|\nabla u|^p + V_\infty k_4|u|^p]\, dx + \frac{t^{q-\tau}}{q}\int_\Omega [|\nabla u|^q + V_\infty |u|^q]\, dx - \frac{1}{\tau}\int_\Omega |u|^\tau dx \right].$$

Then,

$$\lim_{t\to 0} \frac{I_\infty(tu)}{t^\tau} = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \frac{I_\infty(tu)}{t^\tau} = -\frac{1}{\tau}\int_\Omega |u|^\tau dx < 0.$$

Consequently, there exists $t_u \in (0, +\infty)$ such that $I_\infty(t_u u) = \sup_{t\geq 0} I_\infty(tu)$ and $t_u u \in \mathcal{N}_\infty$.

In order to show the unicity of t_u, consider $f(t) = t^\tau$ and note that $\frac{f(t)}{t^q}$ is increasing, see [17]. $\square$

Lemma 4.0.5. *The following properties hold:*

(i) There exists $\rho_\tau > 0$ such that $\left[\int\limits_\Omega |\nabla u|^q dx\right]^{1/q} \geq \rho_\tau$, *for all $u \in \mathcal{N}_\infty$;*

(ii) There exists a constant $C_\tau > 0$ such that $I_\infty(u) \geq C_\tau \int\limits_\Omega |\nabla u|^q dx$, for all $u \in \mathcal{N}_\infty$.

Proof. By Sobolev's embeddings, there exists $C > 0$ such that

$$\int\limits_\Omega |\nabla u|^q dx \leq \int\limits_\Omega \left[k_2 |\nabla u|^p + V_\infty k_4 |u|^p\right] dx + \int\limits_\Omega \left[|\nabla u|^q + V_\infty |u|^q\right] dx = \int\limits_\Omega |u|^\tau dx$$
$$\leq C \left[\int\limits_\Omega |\nabla u|^q dx\right]^{\tau/q}.$$

Since $\tau > q$, the item (i) follows.

To verify the second assertion observe that

$$I_\infty(u) = I_\infty(u) - \frac{1}{\tau} I'_\infty(u)u \geq \left(\frac{1}{p} - \frac{1}{\tau}\right) \int\limits_\Omega \left[k_2 |\nabla u|^p + V_\infty k_4 |u|^p\right] dx$$
$$+ \left(\frac{1}{q} - \frac{1}{\tau}\right) \int\limits_\Omega \left[|\nabla u|^q + V_\infty |u|^q\right] dx \geq \left(\frac{1}{q} - \frac{1}{\tau}\right) \int\limits_\Omega |\nabla u|^q dx.$$

□

Proposition 4.0.6. *There exists $u_\tau \in W_0^{1,q}(\Omega)$ such that u_τ is a solution of (P_∞) and $I_\infty(u_\tau) = \inf\limits_{\mathcal{N}_\infty} I_\infty$.*

Proof. Let (u_n) be a minimizing sequence for I_∞ in $\mathcal{N}_\infty$. By Lemma 4.0.5, we conclude that (u_n) is bounded in $W_0^{1,q}(\Omega)$. Then there exists $u_\tau \in W_0^{1,q}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_\tau$ in $W_0^{1,q}(\Omega)$ and

$$\begin{cases} u_n \to u \text{ strongly in } L^s(\Omega) \text{ for any } 1 \leq s < q^*, \\ u_n(x) \to u(x) \text{ for a.e } x \in \Omega. \end{cases} \tag{4.0.3}$$

Since $\tau \in (q, q^*)$ we have, by Lemma 4.0.5 again, that $u \neq 0$. Hence,

$$c_\infty \leq I_\infty(t_u u) \leq \liminf_{n\to\infty} I_\infty(t_u u_n) \leq \liminf_{n\to\infty} I_\infty(u_n) + o_n(1) = c_\infty.$$

Considering $u_\tau = t_u u$ we have $I_\infty(u_\tau) = c_\tau$ and using Implicit Theorem and arguing as in [16, Lemma 2.5.17] we conclude that $I'_\infty(u_\tau) = 0$. □

Chapter 5

Appendix B

Lemma 5.0.1. *From (f_1) and (f_2) we obtain that: given $\xi > 0$, $q \geq 0$ and $\alpha \geq 1$ there exists $C_\xi > 0$ such that*

$$f(s)s \leq \xi|s|^N + C_\xi|s|^q\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right], \quad \textit{for all } s \in \mathbb{R}$$

and

$$F(s) \leq \frac{\xi}{N}|s|^N + \widetilde{C}_\xi|s|^q\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right] \quad \textit{for all } s \in \mathbb{R}.$$

Proof. Note first the since (f_1) holds, then for every $\xi > 0$ there exists $\delta > 0$ such that

$$|f(s)| \leq \xi|s|^{N-1}, \quad \forall \ \ |\xi| \leq \delta. \tag{5.0.1}$$

On the other hand, from (f_2), we obtain for every $\xi > 0$ and some $\alpha \geq 1$ there exists $R > 0$ such that

$$f(s) \leq \xi\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right], \quad \forall \ \ |s| \geq R.$$

Choosing $R > \max\{1, \delta\}$ and $q \geq 0$ we obtain

$$f(s) \leq \xi\frac{|s|^{q-1}}{R^{q-1}}\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right], \quad \forall \ \ |s| \geq R.$$

From above inequality and the continuity of the function on $[\delta, R]$ there exists $C_1 > 0$ such that

$$f(s) \leq \xi\frac{|s|^{q-1}}{R^{q-1}}\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right] + C_1, \quad \forall \ \ |s| \geq \delta..$$

Since $\max_{t\geq\delta}\left[1 - \frac{C_1}{\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha,s)}\right] > 0$ we can concluded that there exists $C_\xi > 0$ such that

$$f(s) \leq C_\xi|s|^{q-1}\left[\exp(\alpha|s|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, s)\right], \quad \forall \ \ |s| \geq \delta. \tag{5.0.2}$$

Then, by (5.0.1) and (5.0.2), the result follows. □

In the proof of the next lemma, we adapted some arguments found in [6].

Lemma 5.0.2. *Let (ϵ_n) be a sequence such that $\epsilon_n \to 0$ and for each $n \in \mathbb{N}$, let $(u_n) \subset \mathcal{N}^{\pm}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$ be a solution of problem $(P_{\epsilon_{aux}})$. Then (v_n) converges uniformly on compacts of $\mathbb{R}^N$, where $v_{i,n}(x) := u^{\pm}_n(x + \tilde{y}_{i,n})$ for $i = 1, 2$. Moreover, given $\xi > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|v_n|_{L^\infty(\mathbb{R}^N\setminus B_R(0))} < \xi, \quad \textit{for all} \ \ n \geq n_0,$$

where $(\tilde{y}_{i,n})$ are given in Proposition 1.3.2 (or 2.3.2 or 3.3.1).

Proof. Observe that in the chapter 1, when $q = N$ and f has exponential growth we have, by Lemma 1.2.4,

$$\int_{\mathbb{R}^N} f(u_n^{\pm})u_n^{\pm}dx \leq \xi \int_{\mathbb{R}^N} |u_n^{\pm}|^N dx + C_\xi \int_{\mathbb{R}^N} |u_n^{\pm}|^q \left[\exp(\alpha|u_n^{\pm}|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u_n^{\pm})\right] dx.$$

Applying Hölder's inequality for s to closed 1 and Proposition 1.1.1 $\alpha = 3\alpha_0$, we have

$$\int_{\mathbb{R}^N} f(u_n^{\pm})u_n^{\pm}dx \leq \xi \int_{\mathbb{R}^N} |u_n^{\pm}|^N dx$$

$$+\left(\int_{\mathbb{R}^N} |u_n|^{qs'}dx\right)^{1/s'}\left(\int_{\mathbb{R}^N} [\exp(s\alpha\|u_n\|^{N/N-1}\left(\frac{|u_n|}{\|u_n\|}\right)^{N/N-1})-S_{N-2}(s\alpha\|u_n\|^{N/N-1}, \frac{u_n}{\|u_n\|})]dx\right)^{1/s}$$

$$\leq \xi \int_{\mathbb{R}^N} |u_n^{\pm}|^N dx + \bar{C}_\xi \left(\int_{\mathbb{R}^N} |u_n|^{qs'}dx\right)^{1/s'}$$

On the other hand, in the chapters 2 and 3 we have, from the growth conditions of function f,

$$f(t) \leq \xi t^{q-1} + C_\xi t^{q^*-1}, \forall\ t \geq 0. \tag{5.0.3}$$

For simplicity we will work with (5.0.3) and we will consider $(\tilde{y}_{i,n}) = (y_n)$.

Let us now $R_0 > 0$ and consider $\eta \in C^\infty(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$, $|\nabla\eta| \leq 4/R_0$ and

$$\eta(x) = \begin{cases} 0 \text{ if } |x| \leq R_0/2, \\ 1 \text{ if } |x| \geq R_0. \end{cases} \tag{5.0.4}$$

Defining $\eta_n(x) := \eta(x - \widetilde{y}_n)$, then $0 \leq \eta_n \leq 1$ and $|\nabla_n\eta| \leq 4/R_0$. For each $n \in \mathbb{N}$ and for $L > 0$, let

$$u_{L,n}^{\pm}(x) = \begin{cases} u_n^{\pm}(x), & u_n^{\pm}(x) \leq L, \\ L, & u_n^{\pm}(x) > L. \end{cases} \tag{5.0.5}$$

and

$$z_{L,n}^{\pm} := \eta_n^q (u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}$$

with $\gamma > 1$ to be determined later. Taking $z_{L,n}^{\pm}$ as a test function, we obtain $I'_{\epsilon_n}(u)z_{L,n}^{\pm} = 0$ and then

$$q\int_{\mathbb{R}^N} (u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}\eta_n^{q-1}a(|\nabla u_n^{\pm}|^p)|\nabla u_n^{\pm}|^{p-2}\nabla u_n^{\pm}\nabla\eta_n dx + \int_{\mathbb{R}^N} (u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q a(|\nabla u_n^{\pm}|^p)|\nabla u_n^{\pm}|^p dx$$

$$+q(\gamma-1)\int_{\mathbb{R}^N} (u_{L,n}^{\pm})^{q(\gamma-1)-1}u_n^{\pm}\eta_n^q a(|\nabla u_n^{\pm}|^p)|\nabla u_n^{\pm}|^{p-2}\nabla u_n^{\pm}\nabla u_{L,n}^{\pm}dx$$

$$\int_{\mathbb{R}^N} V(\epsilon_n x)b(|u_n^{\pm}|^p)|u_n^{\pm}|^p\eta_n^q(u_{L,n}^{\pm})^{q(\beta-1)}dx = \int_{\mathbb{R}^N} g(\epsilon_n x, u_n^{\pm})\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}dx.$$

Using (a_1), (b_1) and (5.0.3) we obtain

$$\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[k_1|\nabla u_n^{\pm}|^p+|\nabla u_n^{\pm}|^q]dx+q(\gamma-1)\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[k_1|\nabla u_{L,n}^{\pm}|^p+|\nabla u_{L,n}^{\pm}|^q]dx$$

$$+V_0\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}[k_3|u_n^{\pm}|^p+|u_n^{\pm}|^q]dx\le\xi\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^qdx+C_\xi\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^{q^*}dx$$

$$+\frac{V_0}{\beta}\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^qdx-q\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}\eta_n^{q-1}a(|\nabla u_n^{\pm}|^p)|\nabla u_n^{\pm}|^{p-2}\nabla u_n^{\pm}\nabla\eta_ndx$$

Then for a $\xi>0$ sufficiently small and using (a_1) we have the following the inequality

$$\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[k_1|\nabla u_n^{\pm}|^p+|\nabla u_n^{\pm}|^q]dx+q(\gamma-1)\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[k_1|\nabla u_{L,n}^{\pm}|^p+|\nabla u_{L,n}^{\pm}|^q]dx$$

$$+V_0k_3\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^pdx+\frac{V_0}{2}\left(\frac{\beta-1}{\beta}\right)\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^qdx$$

$$\le C_\xi\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^{q^*}dx+q\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}\eta_n^{q-1}[k_2|\nabla u_n^{\pm}|^{p-1}|\nabla\eta_n|+|\nabla u_n^{\pm}|^{q-1}|\nabla\eta_n|]dx.$$

Moreover, using Young's Inequality we obtain for each $\overline{\xi}>0$,

$$q\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}u_n^{\pm}\eta_n^{q-1}[k_2|\nabla u_n^{\pm}|^{p-1}|\nabla\eta_n|+|\nabla u_n^{\pm}|^{q-1}|\nabla\eta_n|]dx$$

$$\le qk_2\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^{q-p}(|\nabla u_n^{\pm}|\eta_n)^{p-1}(u_n^{\pm}|\nabla\eta_n|)dx+q\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}(|\nabla u_n^{\pm}|\eta_n)^{q-1}(u_n^{\pm}|\nabla\eta_n|)dx$$

$$\le qk_2\overline{\xi}\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q|\nabla u_n^{\pm}|^pdx+qk_2C_{\overline{\xi}}\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^{q-p}|u_n^{\pm}|^p|\nabla\eta_n|^pdx$$

$$+q\overline{\xi}\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q|\nabla u_n^{\pm}|^qdx+qC_{\overline{\xi}}\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^q|\nabla\eta_n|^qdx$$

Choosing $\overline{\xi}>0$ sufficiently small we obtain $C_1>0$ such that

$$\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[|\nabla u_n^{\pm}|^p+|\nabla u_n^{\pm}|^q]dx+\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^q[|\nabla u_{L,n}^{\pm}|^p+|\nabla u_{L,n}^{\pm}|^q]dx$$

$$+\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^pdx+\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^qdx\le C_1\int_{\mathbb{R}^N}\eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^{q^*}dx$$

$$+C_1\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}\eta_n^{q-p}|u_n^{\pm}|^p|\nabla\eta_n|^pdx+C_1\int_{\mathbb{R}^N}(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^q|\nabla\eta_n|^qdx \tag{5.0.6}$$

Note that, again by Young's Inequality and by

$$(u_n^\pm)^p (u_{L,n}^\pm)^{q(\gamma-1)} \le (u_n^\pm)^p (u_{L,n}^\pm)^{p(\gamma-1)} + (u_n^\pm)^q (u_{L,n}^\pm)^{q(\gamma-1)},$$

we obtain for each $\overline{\xi}_1 > 0$ such that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^{q-p} |u_n^\pm|^p |\nabla \eta_n|^p dx \le \overline{\xi}_1 \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q |u_n^\pm|^p dx \\
&+C_{\overline{\xi}_1} \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^p |\nabla \eta_n|^q dx \le \overline{\xi}_1 \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q |u_n^\pm|^p dx \\
&+C_{\overline{\xi}_1} \int\limits_{\mathbb{R}^N} [(u_{L,n}^\pm)^{p(\gamma-1)} |u_n^\pm|^p + (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^q] |\nabla \eta_n|^q dx
\end{aligned} \tag{5.0.7}$$

Then using (5.0.6), (5.0.7) and choosing $\overline{\xi}_1 > 0$ sufficiently small, there exists $C_2 > 0$ such that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q [|\nabla u_n^\pm|^p + |\nabla u_n^\pm|^q] dx + \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q [|\nabla u_{L,n}^\pm|^p + |\nabla u_{L,n}^\pm|^q] dx \\
&+ \int\limits_{\mathbb{R}^N} \eta_n^q (u_{L,n}^\pm)^{q(\gamma-1)} [|u_n^\pm|^p + |u_n^\pm|^q] dx \le C_2 \int\limits_{\mathbb{R}^N} [(u_{L,n}^\pm)^{p(\gamma-1)} |u_n^\pm|^p + (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^q] |\nabla \eta_n|^q dx \\
&+C_2 \int\limits_{\mathbb{R}^N} \eta_n^q (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^{q^*} dx
\end{aligned} \tag{5.0.8}$$

We now consider the function $\widehat{u}_{L,n} := \eta_n u_n^\pm (u_{L,n}^\pm)^{\gamma-1}$. Then there exists $C_3 > 0$ such that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} |\nabla \widehat{u}_{L,n}^\pm|^q dx \le 4^q \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} (u_n^\pm)^q |\nabla \eta_n|^q dx + 4^q \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q |\nabla u_n^\pm|^q dx \\
&+4^q (\gamma-1)^q \int\limits_{\mathbb{R}^N} (u_{L,n}^\pm)^{q(\gamma-1)} \eta_n^q |\nabla u_{L,n}^\pm|^q dx \\
&\le C_3 \gamma^q \int\limits_{\mathbb{R}^N} [(u_{L,n}^\pm)^{p(\gamma-1)} |u_n^\pm|^p + (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^q] |\nabla \eta_n|^q dx \\
&+C_3 \gamma^q \int\limits_{\mathbb{R}^N} \eta_n^q (u_{L,n}^\pm)^{q(\gamma-1)} |u_n^\pm|^{q^*} dx
\end{aligned}$$

Hence there exists $C_4 > 0$ such that for all $\gamma > 1$ we have

$$\|\widehat{u}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} \leq S \int\limits_{\mathbb{R}^N} |\nabla \widehat{u}^{\pm}_{L,n}|^q dx$$

$$\leq C_4\gamma^q \int\limits_{\mathbb{R}^N} [(u^{\pm}_{L,n})^{p(\gamma-1)}|u^{\pm}_n|^p + (u^{\pm}_{L,n})^{q(\gamma-1)}|u^{\pm}_n|^q]|\nabla \eta_n|^q dx \tag{5.0.9}$$

$$+C_4\gamma^q \int\limits_{\mathbb{R}^N} \eta^q_n (u^{\pm}_{L,n})^{q(\gamma-1)}|u^{\pm}_n|^{q^*} dx$$

where S is the best Sobolev constant of the embedding $W^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$.

Now we can prove that exists $n_0 \in \mathbb{R}^N$ and $R > R_0 > 0$ such that

$$u_n \in L^{\frac{q^{*2}}{q}}(|x - \widetilde{y}_n| \geq R), \quad \forall\, n \geq n_0.$$

In fact, considering $\gamma = \dfrac{q^{*2}}{q}$ in (5.0.9), using Hölder's Inequality and that $u^{\pm}_{L,n} \leq u^{\pm}_n$ we obtain a constant $C_5 > 0$ such that

$$\|\widehat{u}^{\pm}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} \leq C_4\gamma^q \int\limits_{\mathbb{R}^N} [|u^{\pm}_{L,n}|^{\frac{p}{q}(q^*-q)}|u^{\pm}_n|^p + |u^{\pm}_{L,n}|^{(q^*-q)}|u^{\pm}_n|^q]|\nabla \eta_n|^q dx$$

$$+C_4\gamma^q \int\limits_{\mathbb{R}^N} |\widehat{u}_{L,n}|^q |u^{\pm}_n|^{q^*-q} dx \leq C_4\gamma^q \int\limits_{\mathbb{R}^N} |u^{\pm}_n|^{\frac{p}{q}(q^*-q)}|u^{\pm}_n|^p dx + C_4\gamma^q \int\limits_{\mathbb{R}^N} |u^{\pm}_n|^{q^*} dx$$

$$+C_4\gamma^q \int\limits_{\mathbb{R}^N} |\widehat{u}_{L,n}|^q |u^{\pm}_n|^{q^*-q} dx \leq C_4\gamma^q \left[\int\limits_{\mathbb{R}^N} |u^{\pm}_n|^{\frac{p}{q}q^*} dx\right]^{\frac{q^*-q}{q^*}} \left[\int\limits_{\mathbb{R}^N} |u^{\pm}_n|^{\frac{p}{q}q^*} dx\right]^{\frac{q}{q^*}}$$

$$+C_4\gamma^q \int\limits_{\mathbb{R}^N} |u^{\pm}_n|^{q^*} dx + C_4\gamma^q \|\widehat{u}^{\pm}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} \|u^{\pm}_n\|^{\frac{q^*-q}{q^*}}_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}.$$

Note that $p < \dfrac{p}{q}q^* < q^*$. Then from interpolation inequality, we have

$$\begin{aligned}\|\widehat{u}^{\pm}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} &\leq C_4\gamma^q \|u^{\pm}_n\|^{\theta}_{L^p(\mathbb{R}^N)} \|u^{\pm}_n\|^{1-\theta}_{L^{q^*}(\mathbb{R}^N)} + C_4\gamma^q \|u^{\pm}_n\|^{q^*}_{L^{q^*}(\mathbb{R}^N)} \\ &+C_4\gamma^q \|\widehat{u}^{\pm}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} \|u^{\pm}_n\|^{\frac{q^*-q}{q^*}}_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))},\end{aligned} \tag{5.0.10}$$

where $\theta = \dfrac{q-p}{q^*-p} < 1$. Moreover, since $v_n \to v$ in W_ϵ then, for every $\xi > 0$, there exist $n_0 \in \mathbb{R}^N$ and $R > R_0 > 0$ such that

$$\|u^{\pm}_n\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))} < \xi, \quad \forall n \geq n_0. \tag{5.0.11}$$

Therefore, using (5.0.11) in (5.0.10) we obtain

$$\frac{1}{2}\|\widehat{u}^{\pm}_{L,n}\|^q_{L^{q^*}(\mathbb{R}^N)} \leq C_4\gamma^q \|u^{\pm}_n\|^{\theta}_{L^p(\mathbb{R}^N)} \|u^{\pm}_n\|^{1-\theta}_{L^{q^*}(\mathbb{R}^N)} + C_4\gamma^q \|u^{\pm}_n\|^{q^*}_{L^{q^*}(\mathbb{R}^N)} < \infty. \tag{5.0.12}$$

Using Fatou's Lemma in the variable L, we finally obtain that

$$\frac{1}{2}\left(\int\limits_{|x-\widetilde{y}_n|\geq R} |u_n^{\pm}|^{\frac{q^{*2}}{q}}dx\right)^{q/q^*} \leq \liminf_{L\to\infty}\frac{1}{2}\|\widehat{u}_{L,n}^{\pm}\|_{L^{q^*}(\mathbb{R}^N)}^q \leq \infty. \tag{5.0.13}$$

Now we are going to consider $\eta \in C^\infty(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$, $|\nabla\eta| \leq 4/R_0$ and

$$\eta(x) = \begin{cases} 0 \text{ if } |x| \leq R_0, \\ 1 \text{ if } |x| \geq 2R_0. \end{cases} \tag{5.0.14}$$

Defining $\eta_n(x) := \eta(x-\widetilde{y}_n)$, then $0 \leq \eta_n \leq 1$ and $|\nabla\eta_n| \leq 4/R_0$. Using the same arguments in (5.0.9) and that $\widehat{u}_{L,n}^{\pm} \leq u_n^{\pm}$ we have, for $R > 2R_0$, that

$$\begin{aligned}
\|\widehat{u}_{L,n}^{\pm}\|_{L^{q^*}(\mathbb{R}^N)}^q &\leq C_4\gamma^q \int\limits_{|x-\widetilde{y}_n|\geq R} [(u_{L,n}^{\pm})^{p(\gamma-1)}|u_n^{\pm}|^p + (u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^q]|\nabla\eta_n|^q dx \\
&+C_4\gamma^q \int\limits_{|x-\widetilde{y}_n|\geq R} \eta_n^q(u_{L,n}^{\pm})^{q(\gamma-1)}|u_n^{\pm}|^{q^*}dx \leq C_4\gamma^q \int\limits_{|x-\widetilde{y}_n|\geq R} [|u_n^{\pm}|^{p\gamma} + |u_n^{\pm}|^{q\gamma}]|\nabla\eta_n|^q dx \\
&+C_4\gamma^q \int\limits_{|x-\widetilde{y}_n|\geq R} |u_n^{\pm}|^{\gamma q}|u_n^{\pm}|^{q^*-q}dx.
\end{aligned} \tag{5.0.15}$$

Choosing $\gamma = \gamma_0 := q^*\dfrac{t-1}{qt}$ with $t = \dfrac{(q^*)^2}{q(q^*-q)} = \dfrac{q^*}{q}\dfrac{q^*}{q^*-q} > 1$ and using Hölder's inequality, we obtain

$$\begin{aligned}
\|\widehat{u}_{L,n}^{\pm}\|_{L^{q^*}(\mathbb{R}^N)}^q &\leq C_5\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{q^*}dx\right]^{\frac{p(t-1)}{qt}} + C_5\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{q^*}dx\right]^{\frac{t-1}{t}} \\
&+C_4\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{\frac{q\gamma_0 t}{t-1}}dx\right]^{\frac{t-1}{t}}\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{t(q^*-q)}dx\right]^{\frac{1}{t}} \leq C_5\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{q^*}dx\right]^{\frac{\gamma_0 p}{q^*}} \\
&+C_5\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{q^*}dx\right]^{\frac{\gamma_0 q}{q^*}} + C_5\gamma_0^q\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{q^*}dx\right]^{\frac{\gamma_0 q}{q^*}}\left[\int\limits_{|x-\widetilde{y}_n|\geq R}|u_n^{\pm}|^{\frac{(q^*)^2}{q}}dx\right]^{\frac{1}{t}}.
\end{aligned}$$

Arguing as in (5.0.11) and using (5.0.13) and the interpolation inequality as in (5.0.10) we have that

$$\begin{aligned}
\|\widehat{u}_{L,n}^{\pm}\|_{L^{q^*}(\mathbb{R}^N)}^q &\leq C_6\gamma_0^q\left[\|u_n^{\pm}\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}^{p\gamma_0} + 2\|u_n^{\pm}\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}^{q\gamma_0}\right] \\
&\leq 3C_6\gamma_0^q\|u_n^{\pm}\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}^{q\gamma_0} \\
&= C_7\gamma_0^q\|u_n^{\pm}\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}^{q\gamma_0}.
\end{aligned}$$

Observe that

$$\|u^{\pm}_{L,n}\|^{q\gamma_0}_{L^{q^*\gamma_0}(\mathbb{R}^N/B_R(\widetilde{y}_n))} = \left[\int\limits_{|x-\widetilde{y}_n|\geq R} |u^{\pm}_{L,n}|^{q^*\gamma_0}dx\right]^{\frac{q}{q^*}} \leq \|\widehat{u}^{\pm}_{L,n}\|^{q}_{L^{q^*}(\mathbb{R}^N)}$$

$$\leq C_7\gamma_0^q\|u^{\pm}_n\|^{q\gamma_0}_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}.$$

Consequently, applying Fatou's lemma in the variable L,

$$\|u^{\pm}_n\|_{L^{q^*\gamma_0}(\mathbb{R}^N/B_R(\widetilde{y}_n))} \leq C_7^{\frac{1}{q\gamma_0}}\gamma_0^{\frac{1}{\gamma_0}}\|u^{\pm}_n\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}.$$

Repeating the arguments from (5.0.15) for $\gamma = \gamma_0^2,\ ...,\gamma_0^m$, with $m \in \mathbb{N}$, we deduce that

$$\|u^{\pm}_n\|_{L^{q^*\gamma_0^m}(\mathbb{R}^N/B_R(\widetilde{y}_n))} \leq C_7^{\sum\limits_{i=0}^{m}\frac{1}{q\gamma_0^{i-1}}}\gamma_0^{\sum\limits_{i=0}^{m}\frac{i-1}{\gamma_0^{i-1}}}\|u^{\pm}_n\|_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}.$$

which implies, once that $\sum\limits_{i=0}^{m}\frac{1}{q\gamma_0^{i-1}} < \infty$ and $\sum\limits_{i=0}^{m}\frac{i-1}{\gamma_0^{i-1}} < \infty$, that

$$\|u^{\pm}_n\|_{L^{\infty}(\mathbb{R}^N/B_R(\widetilde{y}_n))} \leq C_7\left[\|u^{\pm}_n\|^{p}_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))} + \|u^{\pm}_n\|^{q}_{L^{q^*}(\mathbb{R}^N/B_R(\widetilde{y}_n))}\right]^{\frac{1}{q}}.$$

Considering the change of variable $z = x - \widetilde{y}_n$ and that $v_n \to v$ in W_ϵ, then there are $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|v^{\pm}_n\|_{L^{\infty}(\mathbb{R}^N/B_R(0))} < \frac{\xi}{2}, \qquad \forall\ \ n \geq n_0.$$

Thus the proof is complete. □

Chapter 6

Appendix C

Theorem 6.0.1 (dominated convergence theorem, Lebesgue)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) $f_n(x) \to f(x)$ a.e on Ω,

(b) there is a function $g \in L^1$ such that for all n, $|f_n(x)| \leq g(x)$ a.e on Ω.

Then $f \in L^1$ and $\|f_n - f\| \to 0$.

Proof. See [19]. □

Theorem 6.0.2 (Fatou's lemma)**.** *Let (f_n) be a sequence of functions in L^1 that satisfy*

(a) for all n, $f_n \geq 0$ a.e,

(b) $\sup\limits_n \int f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf\limits_{n\to\infty} f_n(x) \leq +\infty$. Then $f \in L^1$ and

$$\int f dx \leq \liminf_{n\to\infty} \int f_n dx.$$

Proof. See [19]. □

Theorem 6.0.3 (Hölder's inequality)**.** *Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| dx \leq \left(\int |f|^p dx\right)^{1/p} \left(\int |g|^{p'} dx\right)^{1/p'}$$

Proof. See [19]. □

Theorem 6.0.4. *Let (f_n) be a sequence in L_p and let $f \in L_p$ be such that $\|f_n - f\|_p \to 0$. Then, there exist a subsequence (f_{n_k}) and a function $h \in L_p$ such that*

(a) $f_{n_k}(x) \to f(x)$ a.e on Ω,

(b) $|f_{n_k}(x)| \leq h(x)\ \forall\ k \in \mathbb{N}$, a.e on Ω.

Proof. See [19]. □

Theorem 6.0.5 (Sobolev embedding theorem)**.** ***1. Case*** $\Omega = \mathbb{R}^N$*:*

$$\text{If } \ 1 \le p < N \quad \text{then } \ W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N), \quad \forall s \in [p, \ p^*],$$

$$\text{If } \ p = N \quad \text{then } \ W^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N), \quad \forall s \in [p, \ +\infty),$$

$$\text{If } \ p > N \quad \text{then } \ W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N).$$

2. Ω ***is an with bounded open set of class*** C^1***:***

$$\text{If } \ 1 \le p < N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall s \in [1, \ p^*],$$

$$\text{If } \ p = N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall s \in [1, \ +\infty),$$

$$\text{If } \ p > N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow L^\infty(\mathbb{R}^N).$$

Proof. See [19]. □

Theorem 6.0.6 (Rellich–Kondrachov)**.** *. Suppose that* Ω *is bounded and of class* C^1*. Then we have the following compact injections:*

$$\text{If } \ 1 \le p < N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall s \in [1, \ p^*),$$

$$\text{If } \ p = N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow L^s(\Omega), \quad \forall s \in [1, \ +\infty),$$

$$\text{If } \ p > N \quad \text{then } \ W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}).$$

Proof. See [19]. □

Definition 6.0.1 (Palais-Smale sequence)**.** *We say that a sequence* $(u_n) \subset V$ *is a Palais-Smale sequence at level c for (* $(PS)_c$ *for short) the functional* I *if*

$$I(u_n) \to c$$

and

$$\|I'(u_n)\| \to 0 \ \text{in} \ (V)'.$$

Definition 6.0.2 (Palais-Smale condition)**.** *If every Palais-Smale sequence of* I *has a strong convergent subsequence, then one says that* I *satisfies the Palais-Smale condition ((PS) for short).*

Theorem 6.0.7 (Mountain pass theorem)**.** *Suppose that* V *is a Banach space and a functional* $I \in C^1(V)$ *that satisfies the condition* $(PS)_c$*. Assume that*

1) $I(0) = 0$;

2) $\exists \ \rho > 0, \ \alpha > 0 : \ \|u\| = \rho \Rightarrow I(u) \ge \alpha$;

3) $\exists \ e \in V : \ \|u\| \ge \rho$ *and* $I(e) < 0$.

where

$$c = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], X) : \eta(0) = 0, \ I(\eta(1)) < 0\}.$$

Then c *is a critical value.*

Proof. See [14] or [54]. □

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