Universidade de Brasília
Instituto de Ciências Exatas
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Graded Algebras with the Neutral Component Satisfying a Polynomial Identity of Degree 2

por

Antonio Marcos Duarte de França

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por

Antonio Marcos Duarte de França †

sob orientação da

Profa. Dra. Irina Sviridova

Tese de Doutorado apresentada ao Programa de Pós-Graduação em Matemática do Departamento de Matemática da Universidade de Brasília, PPGMat–UnB, como parte dos requisitos necessários para obtenção do título de Doutor em Matemática.

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"Tudo é possível. O impossível apenas demora mais".

Dan Brown

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Friedrich Nietzsche

Quem alcança seu ideal, vai além dele.

Friedrich Nietzsche

Resumo

Seja $\mathfrak{A}$ uma álgebra associativa sobre um corpo $\mathbb{F}$ graduada por um grupo G, e "e" a unidade de G. Nesse trabalho, nós estudamos e respodemos os seguintes questionamentos: o que podemos dizer sobre $\mathfrak{A}$ quando $\mathfrak{A}_e$ é: 1) um anel nil? 2) um anel nilpotente? 3) uma subálgebra central em $\mathfrak{A}$? Nesse sentindo, nós estudamos a classe de todos os anéis graduados cuja a componente neutra é nil, e a classe de todas as álgebras graduadas com a componente neutra central na álgebra. Dessa forma, nós provamos que, dado um anel associativo $\mathfrak{R}$ com uma S-graduação finita, onde S é um monóide à esquerda cancelativo, se $\mathfrak{R}_e$ é nil (resp. nil de índice limitado) e f-comutativo, então $\mathfrak{R}$ também é um anel nil (resp. de índice limitado). Entre outros resultados, usando o Teorema de Dubnov-Ivanov-Nagata-Higman, nós obtemos uma importante aplicação de nossos resultados: dada uma $\mathbb{F}$-álgebra $\mathfrak{R}$ com uma finita S-graduação, se $\mathsf{char}(\mathbb{F}) = 0$ e $\mathfrak{R}_e$ é nil de índice limitado, então $\mathfrak{R}$ é nilpotente. Além disso, nós exibimos uma considerável relação entre anéis graduados e o Problema de Köthe. Na sequência, nós estudamos a variedade definida pelo conjunto de polinômios G-graduados $\{[x^{(e)}, y^{(g)}] : g \in \mathsf{G}\}$, onde G é um grupo. Dessa forma, nós provamos que se G é um grupo finito e abeliano, e $\mathbb{F}$ é um corpo algebricamente fechado de característica zero, então nós descrevemos um portador para variedade de todas as álgebras G-graduadas com a componente neutra central. Finalmente, nós provamos que, em certas condições, se uma álgebra graduada $\mathfrak{A}_e$ satisfaz uma identidade polinomial f de grau 2, então $\mathfrak{A}$ é nilpotente ou $\mathfrak{A}$ tem a componente neutra comutativa.

Palavras-chave: álgebra associativa G-graduada, anel associativo S-graduado, anel nil, componente neutra central, problema de Köthe, teorema de Dubnov-Ivanov-Nagata-Higman, $\mathsf{G}PI$-álgebra, identidades graduadas.

Abstract

Let $\mathfrak{A}$ be an associative algebra over a field $\mathbb{F}$ graded by a group G, and e the unit of G. In this work, we study and we answer the following questions: what can we say about $\mathfrak{A}$ when $\mathfrak{A}_e$ is: 1) a nil ring? 2) a nilpotent ring? 3) a central subalgebra in $\mathfrak{A}$? In this sense, we study the class of all graded rings whose neutral component is nil, and the class of all graded algebras whose neutral component is central in the algebra. Namely, we prove that, given an associative ring $\mathfrak{R}$ with a finite S-grading, where S is a left cancellative monoid, if $\mathfrak{R}_e$ is nil (resp. nil of bounded index) and f-commutative, then $\mathfrak{R}$ is a nil ring (resp. of bounded index). Among other results, using Dubnov-Ivanov-Nagata-Higman Theorem we obtain an important application of our results: given an $\mathbb{F}$-algebra $\mathfrak{R}$ with a finite S-grading, if $\mathsf{char}(\mathbb{F}) = 0$ and $\mathfrak{R}_e$ is nil of bounded index, then $\mathfrak{R}$ is nilpotent. Besides that, we exhibit a considerable relation between graded rings and Köthe's Problem. Next, we study a graded variety defined by a set of G-graded polynomials $\{[x^{(e)}, y^{(g)}] : g \in \mathsf{G}\}$, where G is a group. Namely, we prove that if G is a finite abelian group, and $\mathbb{F}$ an algebraically closed field of characteristic zero, then we describe a carrier to the variety of all the G-graded algebras with the central neutral component. Finally, we prove that, in suitable conditions, if a graded algebra $\mathfrak{A}_e$ satisfies a polynomial identity f of degree 2, then either $\mathfrak{A}$ is nilpotent or $\mathfrak{A}$ has the commutative neutral component.

Keywords: S-graded associative ring, nil ring, central neutral component, Köthe's Problem, Dubnov-Ivanov-Nagata-Higman Theorem. G-graded associative algebra, $\mathsf{G}PI$-algebra, graded identities.

CONTENTS

INTRODUCTION

Let $\mathbb{F}$ be an arbitrary field, G an arbitrary group, and $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ the free G-graded associative algebra over $\mathbb{F}$ generated by a countable infinite set $X^{\mathsf{G}} = \bigcup_{g\in\mathsf{G}} X_g$ where $X_g = \{x_1^{(g)}, x_2^{(g)}, \dots\}$ for all $g \in \mathsf{G}$. The indeterminates of X_g are said to be homogeneous of degree g. Given a monomial $m = x_{i_1}^{(g_1)} x_{i_2}^{(g_2)} \cdots x_{i_s}^{(g_s)} \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$, the *homogeneous degree* of m, denoted by $\mathsf{deg}(m)$, is defined by $g_1 g_2 \cdots g_s$. Therefore, it is natural to write $\mathbb{F}\langle X^{\mathsf{G}}\rangle = \bigoplus_{g\in\mathsf{G}} \mathbb{F}_g$, where $\mathbb{F}_g$ is the subspace of the algebra $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ generated by all the monomials having homogeneous degree g. It is easy to check that $\mathbb{F}_g\mathbb{F}_h \subseteq \mathbb{F}_{gh}$ for all $g, h \in \mathsf{G}$. The above decomposition into direct sum makes $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ a G-grading algebra. Hence, $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ is the free G-graded associative algebra generated by the sets X_g, $g \in \mathsf{G}$.

Now, let $\mathfrak{A}$ be an algebra over $\mathbb{F}$ with a G-grading Γ, i.e., $\Gamma : \mathfrak{A} = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g$ with $\mathfrak{A}_g$ subspace of $\mathfrak{A}$ and $\mathfrak{A}_g\mathfrak{A}_h \subseteq \mathfrak{A}_{gh}$ for all $g, h \in \mathsf{G}$. We say that $\mathfrak{A}$ is an associative $\mathsf{G}PI$-algebra over $\mathbb{F}$ (or simply $\mathsf{G}PI$-algebra) if there exists a nonzero $f = f(x_1^{(g_1)}, x_2^{(g_2)}, \dots, x_n^{(g_n)}) \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ such that $f(a_1, a_2, \dots, a_n) = 0$ for all $a_1 \in \mathfrak{A}_{g_1}, a_2 \in \mathfrak{A}_{g_2}, \dots, a_n \in \mathfrak{A}_{g_n}$. In this case, we write $f \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}$ and we say that f is a G-graded polynomial identity of $\mathfrak{A}$. We denote by $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$ the set of all G-graded identities of $\mathfrak{A}$. In other words, $\mathsf{T}^{\mathsf{G}}(\mathfrak{A}) = \{f \in \mathbb{F}\langle X^{\mathsf{G}}\rangle : f \equiv_{\mathsf{G}} 0 \text{ in } \mathfrak{A}\}$. It is easy to check that $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$ is a G-graded ideal of $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ invariant under G-endomorphisms of $\mathbb{F}\langle X^{\mathsf{G}}\rangle$, called $\mathsf{G}T$-ideal of G-graded identities of $\mathfrak{A}$. Consider $\mathsf{Supp}(\Gamma) = \{g_1, \dots, g_d\}$ finite, where $\mathsf{Supp}(\Gamma) = \{g \in \mathsf{G} : \mathfrak{A}_g \neq 0\}$. For each $i = 1, 2, \dots,$ put $x_i = \sum_{j=1}^{d} x_i^{(g_j)}$. Let $\mathbb{F}\langle X\rangle$ be the free associative algebra generated by set $X = \{x_1, x_2, \dots\}$. Consider the set $\mathsf{T}(\mathfrak{A}) \subseteq \mathbb{F}\langle X\rangle$ of polynomial (ordinary) identities of $\mathfrak{A}$, i.e. $\mathsf{T}(\mathfrak{A}) = \{f \in \mathbb{F}\langle X\rangle : f \equiv 0 \text{ in } \mathfrak{A}\}$. We have that $\mathsf{T}(\mathfrak{A})$ is an ideal of $\mathbb{F}\langle X\rangle$ invariant by its endomorphisms, called T-ideal of identities of $\mathfrak{A}$. Note that $\mathsf{T}(\mathfrak{A}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$.

One of central problems in the study of graded algebras is to obtain non-graded (ordinary) properties from the analysis of gradings of a given algebra and vice versa. In this sense, given a graded algebra, we try to determine relationships between its graded identities and its non-graded identities. Let $\mathfrak{A} = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g$ be a G-graded algebra, G a finite group with neutral element e. In [5], Bergen and Cohen showed that if $\mathfrak{A}_e$ is a PI-algebra, then $\mathfrak{A}$ is also a PI-algebra. In that work, a bound for the degree of the polynomial identity satisfied by $\mathfrak{A}$ was not found. On the other hand, in [2], Bahturin, Giambruno and Riley proved the same result, but, in addition, they gave a bound for the minimal degree of the polynomial identity satisfied by $\mathfrak{A}$. Namely, the following result was shown:

Theorem 5.3, [2]: Let $\mathbb{F}$ be an arbitrary field and G be a finite group. Suppose that $\mathfrak{A}$ is a G-graded associative $\mathbb{F}$-algebra such that $\mathfrak{A}_e$ satisfies a polynomial identity of degree d. Then $\mathfrak{A}$ satisfies a polynomial identity of degree n, where n is any integer satisfying the inequality

$$\frac{|\mathsf{G}|^n(|\mathsf{G}|d-1)^{2n}}{(|\mathsf{G}|d-1)!} < n!.$$

In particular, if n is the least integer such that $\mathbf{e}|\mathsf{G}|(|\mathsf{G}|d-1)^2 \leqslant n$, then $\mathfrak{A}$ satisfies a polynomial identity of degree n, where $\mathbf{e}$ is the base of the natural logarithm.

Therefore, in this thesis work, we analyse specific cases of the statement in the previous theorem. We study and answer the following questions: *what can we say about a graded algebra* $\mathfrak{A}$ *when* $\mathfrak{A}_e$ *is:* **1)** *a nilpotent ring?* **2)** *a nil ring?* **3)** *a central algebra?* **4)** *a commutative algebra?* And so, we divided this work into 4 (four) chapters: **1)** *Graded Algebras, Graded Bimodules and Graded Identities*; **2)** *Second Cohomology Group*; **3)** *Graded Rings with Nil Neutral Component*; **4)** *Graded Algebras with Central Neutral Component*. In what follows, let us discuss each one of these chapters a little.

In the first chapter, we introduce notations and definitions which are necessary for a better presentation of other chapters. We define here all the algebraic structures that we use in this work. Furthermore, we exhibit various properties of these algebraic structures. The most important part of this chapter is the last section, it is the key to prove the main theorem in Chapter 4. We admit to be known the concepts of a *monoid*, *group*, *ring*, *field* and *vector space over a field.* In the whole text, all rings and algebras are assumed to be associative, G denotes a group, $\mathbb{F}$ and $\mathbb{K}$ denote fields. For more details about the basic structures that we use here, see [6, 8, 9, 23, 25, 26, 31, 40].

In the second chapter, we present an overview of the objects of the theory of cohomology of groups. Here we give all definitions necessary to understand the problems exposed, and some of the main results. Here, the most important result, being an algebraically closed field, is that the 2nd cohomology group of a finite group is finite. In the second section, we present more interesting results. The main result is the following:

Corollary 2.2.7: *If* $[\mathsf{G} : H] < \infty$*, H is central in* G *and* M *is an abelian group with a trivial* G*-action, then*

$$\mathsf{H}^2(H, \mathsf{M}) = \mathsf{res}_{\mathsf{H}}^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right) .$$

Our goal in the second chapter is to determine suitable conditions to ensure that the restriction homomorphism from $\mathsf{H}^2(\mathsf{G}, \mathsf{M})$ into $\mathsf{H}^2(H, \mathsf{M})$ is surjective, where H is a subgroup of a group G.

In the third chapter, we consider a left cancellative monoid S, i.e. $gh = gt$ implies that $h = t$ for any $g, h, t \in \mathsf{S}$, and an associative ring $\mathfrak{R}$ with a finite S-grading Γ. Our principal goal in this chapter is to present some results which are direct implications of the case "$\mathfrak{R}_e$ is nilpotent" or "$\mathfrak{R}_e$ is nil", where e is the neutral element of S. In this sense, we give some upper bounds for $\mathsf{nd}(\mathfrak{R})$, the nilpotency index of $\mathfrak{R}$. Here, we are interested in studying associative rings with an S-grading, whose neutral component is nil. Unless otherwise stated, $\mathfrak{R}$ is an associative ring, and S is a left cancellative monoid, with the neutral element e. Let $\mathfrak{R}$ be a nilpotent ring, that is, there exists an integer $n > 0$ such that $x_1 x_2 \cdots x_n = 0$ for any $x_1, x_2, \ldots, x_n \in \mathfrak{R}$ ($\mathfrak{R}^n = \{0\}$). We define the *nilpotency index* of $\mathfrak{R}$, denoted by $\mathsf{nd}(\mathfrak{R})$, as the smallest number $d \in \mathbb{N}$ such that $\mathfrak{R}^d = \{0\}$. Analogously, if $\mathfrak{R}$ is a nil ring of bounded index, i.e. there exists some integer $n > 0$ such that $y^n = 0$ for any $y \in \mathfrak{R}$, we define the *nil index* of $\mathfrak{R}$, denoted by $\mathsf{nd}_{nil}(\mathfrak{R})$, as the smallest number $r \in \mathbb{N}$ such that $a^r = 0$ for any $a \in \mathfrak{R}$. Consequently, any nilpotent ring also is a nil ring (of bounded index). Therefore, for any nilpotent ring $\mathfrak{R}$, $\mathsf{nd}_{nil}(\mathfrak{R}) \leqslant \mathsf{nd}(\mathfrak{R})$.

In [24], the authors proved that if a finite solvable group G acts by automorphisms on a ring $\mathfrak{R}$ without non-zero fixed points, i.e. $\mathfrak{R}^{\mathsf{G}} = \{0\}$, and without $|\mathsf{G}|$-torsion, then $\mathfrak{R}^{|\mathsf{G}|} = \{0\}$. They also proved that if G is a finite group acting on a ring $\mathfrak{R}$ without $|\mathsf{G}|$-torsion, and $\mathfrak{R}^{\mathsf{G}}$ is nilpotent, then $\mathfrak{R}$ is nilpotent. Other result proved by these authors is that if $\mathfrak{R}$ is a ring graded by a finite cyclic group, such that $\mathfrak{R}_e$ is central, then the commutator ideal of $\mathfrak{R}$ is nil. Already in [28], E.I. Khukhro presents the following result (Corollary 4.3.8 (p. 101)): *if a Lie ring admits a regular automorphism of prime order, then it is nilpotent.* And N. Yu. Makarenko, in [32], using techniques created by E.I. Khukhro, showed that given a G-graded associative algebra $\mathfrak{A}$, where G is a finite group of order n, if $\mathfrak{A}_e$ has a nilpotent two-sided ideal of finite codimension in $\mathfrak{A}_e$, then $\mathfrak{A}$ has a homogeneous nilpotent two-sided ideal of nilpotency index bounded by a function on n and of finite codimension. In our work, we generalize some of these results.

In this third chapter, we study the following:

Problem*: Does $\mathfrak{R}_e$ nil imply that $\mathfrak{R}$ is nil, where $\mathfrak{R}_e$ is the neutral component of Γ?

In this sense, we study the class of all the S-graded rings whose neutral component is nil. Among other results, we obtain a positive solution for **Problem*** in the class of all f-commutative rings, where an associative ring $\mathfrak{R}$ is said to be **f-commutative** if there exist a semigroup $\mathfrak{S}$ that acts on the left of $\mathfrak{R}$, and a mapping $\mathsf{f} : \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{S}$ such that $ab - \mathsf{f}(a, b)ba = 0$ for any $a, b \in \mathfrak{R}$. More precisely, we prove

Theorem 3.2.14: Any ring with a finite grading whose neutral component is nil and f-commutative is a nil ring.

Moreover, adding in the previous theorem the hypotheses "$\mathfrak{R}_e$ is finitely generated", we obtain that $\mathfrak{R}$ is a nilpotent ring. In general, the assumptions "finitely generated" and "f-commutative" are necessary to guarantee that $\mathfrak{R}$ is nilpotent, and so we present some counterexamples.

The importance of our results arises when we relate them to *Dubnov-Ivanov-Nagata-Higman Theorem* and *Köthe Problem*. Let us present these two problems.

In [33] and [22], Nagata and Higman (and in [12], Dubnov and Ivanov), respectively, proved that, under some suitable conditions, any associative nil algebra is also a nilpotent algebra. Firstly, Nagata proved the validity of the result over a field of characteristic zero, in [33]. Afterwards, Higman established the result in a more general case, in [22]. A similar result was previously also published in Russian ([12]).

In this way, we have some natural questions: how to characterize a G-graded algebra whose neutral component is nil/nilpotent? Does the nil neutral component implies that the algebra is nilpotent? If so, what are the possible limits for its nilpotency index? Thus, we prove the following theorem that is a generalization of Nagata-Higman Theorem:

Theorem 3.3.3: Let S be a left cancellative monoid , and $\mathfrak{A}$ an associative algebra over a field $\mathbb{F}$ with a finite S-grading, $\mathsf{char}(\mathbb{F}) = p$. Suppose that $\mathfrak{A}_e$ is a nil algebra of bounded index. If $p = 0$ or $p > \mathsf{nd}_{nil}(\mathfrak{A}_e)$, then $\mathfrak{A}$ is a nilpotent algebra.

Finally, we exhibit a considerable relation between graded rings and *Köthe's Problem.* This problem was proposed in 1930 by G. Köthe in [29] and still has not a general solution. Köthe's Problem asks, whether the sum of two right nil ideals of a ring is nil, or equivalently, if a ring $\mathfrak{R}$ has no nonzero nil ideals, then $\mathfrak{R}$ has no nonzero one-sided nil ideals. Various mathematicians have studied this problem since 1930, and we can cite some of the works: [15, 14, 43, 42]. The Köthe conjecture has several different formulations. Among others, we have the following equivalent statements:

Theorem: The following statements are equivalent:

i) If a ring has no nonzero nil ideals, then it has no nonzero one-sided nil ideals;

ii) The sum of two right nil ideals in any ring is nil;

iii) For any nil ring $\mathfrak{R}$, the ring of 2×2 matrices over $\mathfrak{R}$ is nil;

iv) For any nil ring $\mathfrak{R}$, the ring of $n \times n$ matrices over $\mathfrak{R}$ is nil.

This theorem can be found in [43]. In our work, we prove that Köthe's Problem has a positive solution in the class of f-commutative rings graded by a monoid. Moreover, we show that

Theorem 3.3.7: A positive answer to **Problem*** implies that Köthe's Problem has a positive solution.

Equivalently, a counterexample to Köthe's Problem would yield a counterexample to **Problem***.

Finally, in the fourth and last chapter, we study the G-graded algebras with the central neutral component, and we study the variety of G-graded algebras defined by G-graded polynomial identities $[x^{(e)}, y^{(g)}]$ for all $g \in \mathsf{G}$, where G is an abelian finite group, and the base field is algebraically closed fo characteristic zero. In other words, in this chapter, we exhibit results concerning to the variety $\mathfrak{V}^{\mathsf{G}}$ of all G-graded algebras whose neutral component is central, i.e. $\mathfrak{V}^{\mathsf{G}} := \mathsf{var}^{\mathsf{G}}\left(\{[x^{(e)}, y^{(g)}] : g \in \mathsf{G}\}\right)$. We present some properties of algebras which belong to the variety $\mathfrak{V}^{\mathsf{G}}$, and in suitable conditions, we give a description of $\mathfrak{V}^{\mathsf{G}}$, in the language of a carrier. Here, we assume that G is a finite abelian group, $\mathbb{F}$ is an algebraically closed field of characteristic zero, and all considered algebras are associative $\mathbb{F}$-algebras, unless otherwise stated.

Firstly, we present some results on rings graded by a cancellative monoid that have a small support, and whose neutral component is central. The main result here is that any graded ring with the central neutral component and the support of order at most 3 is Lie nilpotent, i.e. satisfies the polynomial identity $[x_1, \ldots, x_n] = 0$ for some $n \in \mathbb{N}$. We exhibit also some counterexamples for the case of rings with order of support greater than 3. In the second section of 4th Chapter, we introduce the variety of $\mathfrak{V}^{\mathsf{G}}$ of all G-graded PI-algebras with the central neutral component. Basically, we exhibit our objects of study. In the third section, we study the graded algebra of finite dimension whose neutral component is central in the algebra. We show some properties of these algebras, we apply some concepts of cohomology of groups, and combinatorial arguments. Our two main results in this section are the following:

Theorem 4.3.8: *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, G a finite abelian group, H a subgroup of G, $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and $\mathfrak{A} = \mathbb{F}^\sigma[H] \oplus \mathsf{J}$ a finite dimensional G-graded unitary $\mathbb{F}$-algebra. Suppose that $\mathsf{Supp}(\Gamma_{\mathfrak{A}}) = H$. If $\mathfrak{A}_e$ is central in $\mathfrak{A}$, then*

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \mathbb{F}^\sigma[H] \ .$$

Moreover, $\mathfrak{A}$ belongs to $\mathsf{var}^{\mathsf{G}}(\mathbb{F}^\gamma[\mathsf{G}])$ for some $\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ which extends σ.

Theorem 4.3.14: *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A}$ a finitely generated G-graded algebra. If $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$, there exists a finite dimensional G-graded algebra*

$$\mathsf{C}_{\mathsf{G},\mathfrak{A}} = \bigtimes_{H \trianglelefteq \mathsf{G}} \left(\bigtimes_{[\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)} \left(\mathbb{F}^\sigma[H] \oplus \mathsf{J}_{(H,[\sigma])}\right) \right) ,$$

where each $\mathsf{J}_{(H,[\sigma])}$ is a finite dimensional G-graded nilpotent algebra ($\mathsf{J}_{(H,[\sigma])}$ is the Jacobson radical of $\mathfrak{A}_{(H,[\sigma])} := \mathbb{F}^\sigma[H] \oplus \mathsf{J}_{(H,[\sigma])}$), satisfying

$$\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_\infty \mathsf{T}^{\mathsf{G}}(\mathfrak{A}) \ .$$

Moreover, if $\mathfrak{A}$ *is unitary, then* $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$.

In the 4th section, we study the variety $\mathfrak{V}^{\mathsf{G}}$ the all the G-graded algebras with the central neutral component. Here, our main results are the following.

Theorem 4.5.4: *Let* G *be a finite abelian group,* H *a subgroup of* $\mathsf{G} \times \mathbb{Z}_2$, $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, *and* $\mathbb{F}$ *an algebraically closed field of characteristic zero. Let* $\mathfrak{A} = \mathbb{F}^\sigma[H] \oplus \mathsf{J}$ *be a finite dimensional* $\mathsf{G} \times \mathbb{Z}_2$*-graded unitary algebra, with the semisimple part* $\mathfrak{B} = \mathbb{F}^\sigma[H]$, *and* J *is the Jacobson radical of* $\mathfrak{A}$. *Suppose that one of the following hypotheses is true:*

1) $\mathfrak{A} = \mathfrak{A}_{\pi(H)}$;

2) $H = \mathsf{G} \times \{0\}$;

3) $\pi(\mathsf{Supp}(\Gamma_{\mathsf{J}})) \subseteq \pi(H)$;

4) $H \leqslant \mathsf{G} \times \{0\}$ *and* $(\mathsf{Supp}(\Gamma_{\mathsf{J}})) \subseteq \pi(H)$;

If $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ *is central in* $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, *then* J *is generated as a* $\mathsf{G} \times \mathbb{Z}_2$*-graded* $\mathfrak{B}$*-bimodule by a nilpotent subalgebra* $\hat{\mathsf{N}}$ *of* J, *which is super-central in* $\mathfrak{A}$, *and*

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathbb{F}^\sigma[H]) \ .$$

In particular, in the cases 2) and 4) we have that

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathbb{F}^\sigma[H] \equiv_{\mathsf{G}PI} \mathbb{F}^{\tilde{\sigma}}[\pi(H)] \ ,$$

for some $\tilde{\sigma} \in \mathsf{Z}^2(\pi(H), \mathbb{F}^*)$.

Theorem 4.5.5: *Let* G *be a finite abelian group, and* $\mathbb{F}$ *an algebraically closed field of characteristic zero. There exists a finite dimensional* $\mathsf{G} \times \mathbb{Z}_2$*-graded unitary algebra*

$$\mathsf{C}_{\mathsf{G}} = \bigtimes_{\substack{H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2 \\ (e,1) \notin H}} \left(\bigtimes_{[\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)} \left(\mathbb{F}^\sigma[H] \oplus \mathsf{J}_{(H,[\sigma])} \right) \right) \ ,$$

such that $\mathsf{J}_{(H,[\sigma])}$ *is a finite dimensional* $\mathsf{G} \times \mathbb{Z}_2$*-graded nilpotent algebra (*$\mathsf{J}_{(H,[\sigma])}$ *is the Jacobson radical of* $\mathfrak{A}_{(H,[\sigma])} := \mathbb{F}^\sigma[H] \oplus \mathsf{J}_{(H,[\sigma])}$*), satisfying*

$$\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}})) \ .$$

In the last section of 4th chapter, we prove that if a graded algebra $\mathfrak{A}$ satisfies a polynomial identity f of degree 2, in suitable conditions, then either $\mathfrak{A}$ is nilpotent or $\mathfrak{A}$ has the commutative neutral component.

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CHAPTER 1

GRADED ALGEBRAS, GRADED BIMODULES AND GRADED IDENTITIES

The aim of this chapter is to introduce notations and definitions, that are necessary for a better presentation of the next chapters. We define here all algebraic structures that we use in this work. Furthermore, we exhibit various properties of these algebraic structures.

The most important part of this chapter is the last section, which is the key to prove the main theorem in Chapter 4.

In the whole text, all rings and algebras are assumed to be associative, G denotes a group, $\mathbb{F}$ and $\mathbb{K}$ denote fields. For more details about the basic structures that we use here, see [6, 8, 9, 23, 25, 26, 31].

1.1 Basic Definitions and Properties

In this section, we present some definitions and properties of the basic structures which are needed to understand the next chapters better. Let us comment briefly the definitions of semigroups, monoids, groups, rings and vector spaces. Afterwards, we define (bi)modules over algebras, gradings on algebras and modules. Finally, we present some properties of these important structures.

Let S be a non-empty set and "$*$" a map from $S \times S$ into S. We say that

P1) $(S, *)$ is **associative** if $(x * y) * z = x * (y * z)$ for any $x, y, z \in S$;

P2) $(S, *)$ is **commutative** if $y * x = x * y$ for any $x, y \in S$;

P3) $(S, *)$ has a **neutral element** (or **unity**) if there exists $1_S \in S$ such that $1_S * x = x * 1_S = x$ for any $x \in S$. In this case, $(S, *)$ is called **unitary**;

P4) $x \in (S, *)$ is **invertible** if $(S, *)$ is unitary and there exists $y \in S$, called **inverse** of x, such that $x * y = y * x = 1_S$.

A non-empty set $\mathfrak{S}$ with a binary map "$*$" is said to be a **semigroup** if $(\mathfrak{S}, *)$ satisfies *P1*.

A non-empty set S with a binary map "$*$" is said to be a **monoid** if $(\mathsf{S}, *)$ satisfies *P1* and *P3*. Observe that any unitary semigroup is a monoid.

Now, given a non-empty set G with a map $* : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, which is called a map **multiplication**. We say that $(\mathsf{G}, *)$ is a **group** if it satisfies *P1*, *P3* and *P4*, for any $x \in \mathsf{G}$. A group G is said to be **abelian** if satisfies *P2*. Note that any group is a monoid such that its elements are invertible.

Let S be a monoid. We say that S is **left cancellative** (resp. **right cancellative**) if $gh = gt$ (resp. $hg = tg$) implies $h = t$, for any $g, h, t \in \mathsf{S}$. We say that S is bf cancellative if S is both left cancellative and right cancellative. Observe that any group is a cancellative monoid.

Remark 1.1.1 *Let G be a group. Given an element $g \in \mathsf{G} - \{e\}$, where e is the unity of G, if there exists an number $m \in \mathbb{N}$ such that $g^m = e$, then we say that the* **order** *of g is the smallest number $n \in \mathbb{N}$ such that $g^n = e$, and in this case we denote $\mathsf{o}(g) = n$. If there is no $m \in \mathbb{N}$ such that $g^m = e$, then we say that g has an infinite order, and we denote $\mathsf{o}(g) = \infty$. Note that when G is finite, we have that all elements of G have a finite order, and $\mathsf{o}(g) \mid |\mathsf{G}|$., for any $g \in \mathsf{G}$.*

Take a non-empty set $\mathfrak{R}$ with two maps $+, \cdot : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$. We say that $(\mathfrak{R}, +, \cdot)$ is a **ring** if $(\mathfrak{R}, +)$ is an abelian group, and $x(y + z) = xy + xz$, and $(x + y)z = xz + yz$ hold, for any $x, y, z \in \mathfrak{R}$ (distributivity). A ring $(\mathfrak{R}, +, \cdot)$ is **associative** if $(\mathfrak{R}, \cdot)$ satisfies *P1*; **commutative** if $(\mathfrak{R}, \cdot)$ satisfies *P2*; and **unitary** if $(\mathfrak{R}, \cdot)$ satisfies *P3*.

A **field** $\mathbb{F}$ is a unitary commutative associative ring $(\mathbb{F}, +, \cdot)$ such that $(\mathbb{F}, \cdot)$ satisfies *P4* for any $x \in \mathbb{F} - \{0\}$, where 0 is the neutral element of $(\mathbb{F}, +)$.

Finally, a **vector space** over a field $\mathbb{F}$ (or simply, $\mathbb{F}$-vector space) is a non-empty set V together with two maps $+ : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$, and $\cdot : \mathbb{F} \times \mathsf{V} \to \mathsf{V}$ such that $(\mathsf{V}, +)$ is an abelian group, and $(\lambda + \gamma)x = \lambda x + \gamma x$, $\lambda(x + y) = \lambda x + \lambda y$, $(\lambda\gamma)x = \lambda(\gamma x)$, for any $x, y \in \mathsf{V}$ and $\lambda, \gamma \in \mathbb{F}$, and $1_{\mathbb{F}} x = x$ for any $x \in \mathsf{V}$.

A **subring** (resp. a **subspace**) is a subset which is also a ring (resp. a vector space) with the same operations.

An **ideal** I of a ring $\mathfrak{R}$ is a subring which is invariant with respect to multiplication by $\mathfrak{R}$, i.e. $I\mathfrak{R}, \mathfrak{R}I \subseteq I$. We can define a **left ideal** (resp. a **right ideal**) of a ring $\mathfrak{R}$ requiring that $\mathfrak{R}I \subseteq I$ (resp. $I\mathfrak{R} \subseteq I$).

Definition 1.1.2 *Let $\mathfrak{A}$ be a vector space over a field $\mathbb{F}$. We say that $\mathfrak{A}$ is an* $\mathbb{F}$**-algebra** *if there exists a map "$\cdot$" from $\mathfrak{A} \times \mathfrak{A}$ into $\mathfrak{A}$ that satisfies the following properties:*

i) $c \cdot (a + b) = c \cdot a + c \cdot b$;

ii) $(a + b) \cdot c = a \cdot c + b \cdot c$;

iii) $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$,

for any $a, b, c \in \mathfrak{A}$ and $\lambda \in \mathbb{F}$.

We say that $\mathfrak{A}$ is **associative** if $(\mathfrak{A}, \cdot)$ satisfies *P1*; **unitary** if $(\mathfrak{A}, \cdot)$ satisfies *P3*; **commutative** if $(\mathfrak{A}, \cdot)$ satisfies *P2*. A **subalgebra** of $\mathfrak{A}$ is a subset of $\mathfrak{A}$ which is also an algebra, and an *ideal* I of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ which is invariant with respect to the multiplication of $\mathfrak{A}$, that is, $I\mathfrak{A}, \mathfrak{A}I \subseteq I$. We define a **left ideal** and a **right ideal** of an algebra, being sufficient to require $\mathfrak{A}I \subseteq I$ and $I\mathfrak{A} \subseteq I$, respectively.

A nonzero ideal I of an algebra $\mathfrak{A}$ is called a **minimal ideal** if for any ideal J of $\mathfrak{A}$ which is contained in I, one has $J = \{0\}$ or $J = I$. Analogously, we define a **minimal left ideal** and **minimal right ideal** of an algebra.

Given an algebra $\mathfrak{A}$ without unit, it is always possible to obtain a unitary algebra derived from $\mathfrak{A}$. In fact, consider the algebra $\mathfrak{A}^{\#} = \mathfrak{A} \oplus 1_{\mathfrak{A}^{\#}}\mathbb{F}$ whose product is defined as following: for any $a, b \in \mathfrak{A}$ and $\lambda, \gamma \in \mathbb{F}$

$$(a + \lambda 1_{\mathfrak{A}^{\#}})(b + \gamma 1_{\mathfrak{A}^{\#}}) = (ab + \lambda b + \gamma a) + \lambda\gamma 1_{\mathfrak{A}^{\#}}. \tag{1.1}$$

The algebra $\mathfrak{A}^{\#}$ is called an **algebra derived from $\mathfrak{A}$ by adjoining the unit** $1_{\mathfrak{A}^{\#}}$.

Two important classes of algebras are the class of *nil algebras* and the class of *nilpotent algebras*. An algebra $\mathfrak{A}$ is said to be **nil** if for any $a \in \mathfrak{A}$ there is an integer $n = n(a) > 0$ such that $a^n = 0$. A nil algebra $\mathfrak{A}$ has a **bounded index** when there exists an integer $n_0 > 0$ such that $b^{n_0} = 0$ for any $b \in \mathfrak{A}$, and thus, $\mathfrak{A}$ is a **nil algebra of bounded index**.

We say that $\mathfrak{A}$ is a **nilpotent algebra** if there exists an integer $d > 0$ such that $a_1a_2\cdots a_d = 0$ for any $a_1, a_2, \ldots, a_d \in \mathfrak{A}$. Notice that any nilpotent algebra is also a nil algebra (of bounded index). The reciprocal is not true. The definitions of **nil rings** (of bounded index) and **nilpotent rings** are analogous.

From now on, all the rings and algebras are assumed to be associative.

Definition 1.1.3 *Let $\mathfrak{A}$ be an algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. We define the* **center of $\mathfrak{B}$ in $\mathfrak{A}$**, *denoted by $\mathcal{Z}_{\mathfrak{A}}(\mathfrak{B})$, as being the set*

$$\mathcal{Z}_{\mathfrak{A}}(\mathfrak{B}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{B}\}.$$

When $\mathfrak{B} = \mathfrak{A}$, we write $\mathcal{Z}_{\mathfrak{A}}(\mathfrak{B}) = \mathcal{Z}(\mathfrak{A})$, and $\mathcal{Z}(\mathfrak{A})$ is called **center of $\mathfrak{A}$**. *Notice that $\mathfrak{A}$ is* **commutative** *if $\mathcal{Z}(\mathfrak{A}) = \mathfrak{A}$, and we say that $\mathfrak{B}$ is* **central** *in $\mathfrak{A}$ if $\mathfrak{B} \subseteq \mathcal{Z}(\mathfrak{A})$.*

The **center of a ring** is defined analogously.

We define the center of a multiplicative group. Let G be a group with multiplicative notation and S a subset of G. We define the **center of S in G** as the set

$$\mathcal{Z}_{\mathsf{G}}(S) := \{g \in \mathsf{G} : gs = sg, \text{ for any } s \in S\}.$$

When $S = \mathsf{G}$, we write $\mathcal{Z}_{\mathsf{G}}(\mathsf{G}) = \mathcal{Z}(\mathsf{G})$, and $\mathcal{Z}(\mathsf{G})$ is called **center of G**.

Let us consider now a generalization of commutativity.

Definition 1.1.4 *Consider a semigroup $\mathfrak{S}$, and an associative ring $\mathfrak{R}$. A* **left action** *of $\mathfrak{S}$ on $\mathfrak{R}$ is a mapping $\cdot : \mathfrak{S} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ satisfying*

$$(\lambda\gamma)\cdot x = \lambda(\gamma \cdot x) \quad \textit{and} \quad \lambda \cdot (xy) = (\lambda \cdot x)y \ ,$$

for any $\lambda, \gamma \in \mathfrak{S}$ and $x, y \in \mathfrak{R}$, and it is called an **action by semigroup**. *Consider* *a map* $\mathsf{f} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{S}$. *If* $ab = \mathsf{f}(a,b)ba$ *for any* $a, b \in \mathfrak{R}$, *then we say that* $\mathfrak{R}$ *is an* **f-commutative** *ring.*

We will consider f-commutativity of a ring with more details in Chapter 3.

Let us recall the definition of a *linear transformation* between two $\mathbb{F}$-vector spaces. Given two vector spaces V and $\tilde{\mathsf{V}}$ over the same field $\mathbb{F}$, a map $\psi : \mathsf{V} \longrightarrow \tilde{\mathsf{V}}$ is said to be a **linear transformation** if $\psi(a + b) = \psi(a) + \psi(b)$ and $\psi(\lambda a) = \lambda\psi(a)$ for any $a, b \in \mathsf{V}$ and $\lambda \in \mathbb{F}$. The **kernel** and **image** of ψ are, respectively, defined by $\mathsf{ker}(\psi) = \{a \in \mathsf{V} : \psi(a) = 0\}$ and $\mathsf{im}(\psi) = \{\psi(a) \in \tilde{\mathsf{V}} : a \in \mathsf{V}\}$. We say that ψ is an **epimorphism** if it is a surjective map, i.e. $\mathsf{im}(\psi) = \tilde{\mathsf{V}}$; ψ is a **monomorphism** if it is an injective map, i.e. $\psi(a) = \psi(b)$ implies $a = b$ in V; and ψ is an **isomorphism** if it is an epimorphism and a monomorphism. Notice that $\mathsf{ker} = \{0\}$ iff ψ is injective. It is not difficult to see that $\mathsf{ker}(\psi)$ and $\mathsf{im}(\psi)$ are subspaces of V and $\tilde{\mathsf{V}}$, respectively. For more details, see [23].

Definition 1.1.5 *Let* $\mathfrak{A}$ *and* $\tilde{\mathfrak{A}}$ *be two* $\mathbb{F}$*-algebras and* $\phi : \mathfrak{A} \to \tilde{\mathfrak{A}}$ *a linear transformation. We say that* ϕ *is a* **homomorphism of algebras** *if* ϕ *satisfies* $\phi(ab) = \phi(a)\phi(b)$ *for any* $a, b \in \mathfrak{A}$.

The definitions of a **kernel**, **image**, **epimorphism of algebras**, **monomorphism of algebras** and **isomorphism of algebras** are inherited from linear transformations. It is not difficult to see that $\mathsf{ker}(\phi)$ and $\mathsf{im}(\phi)$ are subalgebras of $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$, respectively. In particular, $\mathsf{ker}(\phi)$ is an ideal of $\mathfrak{A}$. We write $\mathfrak{A} \cong \tilde{\mathfrak{A}}$ when there exists an isomorphism of algebras between $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$. Otherwise, we write $\mathfrak{A} \ncong \tilde{\mathfrak{A}}$.

Let us now define *gradings* on algebras and rings, *modules over an algebra* and *gradings on modules*. For more details, see [6, 34, 35].

We denote by the symbol "$\bigoplus$" the direct sum of additive subgroups on a ring or the direct sum of $\mathbb{F}$-subspaces on an $\mathbb{F}$-algebra, i.e. $\mathfrak{A} = \bigoplus_{i\in I} \mathfrak{A}_i$ means $\mathfrak{A}_i \cap (\sum_{j\in I, j\neq i} \mathfrak{A}_j) = \{0\}$, where $\mathfrak{A}_i \subset \mathfrak{A}$ are additive subgroups of a ring $\mathfrak{A}$ or $\mathbb{F}$-subspaces of an $\mathbb{F}$-algebra $\mathfrak{A}$.

Definition 1.1.6 *Let* $\mathfrak{R}$ *be a ring, and* S *a monoid. An* S*-grading on* $\mathfrak{R}$ *is a decomposition of* $\mathfrak{R}$ *as a direct sum of its additive subgroups* $\mathfrak{R}_g \subseteq \mathfrak{R}$, $g \in \mathsf{S}$,

$$\mathfrak{R} = \bigoplus_{g\in\mathsf{S}} \mathfrak{R}_g \ ,$$

such that $\mathfrak{R}_g\mathfrak{R}_h \subseteq \mathfrak{R}_{gh}$ for any $g, h \in \mathsf{S}$. We say that $\mathfrak{R}$ is an S-graded ring, or that $\mathfrak{R}$ is a ring graded by the monoid S.

The $\mathfrak{R}_s$'s are called **homogeneous components**. For each $s \in \mathsf{S}$, any element $r \in \mathfrak{R}_s$ is called a **homogeneous element** of degree s, and we write $\mathsf{deg}(r) = s$.

We can also define a grading by a quotient group in a natural way. In fact, let $\mathfrak{R}$ be a ring graded by a group S. Given a normal subgroup $\tilde{\mathsf{S}}$ of S, consider the quotient group $\mathsf{S}/\tilde{\mathsf{S}}$. Being $\mathfrak{R} = \bigoplus_{s\in\mathsf{S}} \mathfrak{R}_s$ an S-grading on $\mathfrak{R}$, we have that

$$\mathfrak{R} = \bigoplus_{\bar{s}\in\mathsf{S}/\tilde{\mathsf{S}}} \mathfrak{R}_{\bar{s}}$$

defines an $\mathsf{S}/\tilde{\mathsf{S}}$-grading on $\mathfrak{R}$, where $\mathfrak{R}_{\bar{s}} = \bigoplus_{r\in\tilde{S}} \mathfrak{R}_{sr}$.

Definition 1.1.7 *Let $\mathfrak{A}$ be an $\mathbb{F}$-algebra and G a group. A G-grading on $\mathfrak{A}$ is a decomposition of $\mathfrak{A}$ as the direct sum of subspaces $\mathfrak{A}_g \subset \mathfrak{A}$, $g \in \mathsf{G}$,*

$$\mathfrak{A} = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g \ ,$$

such that $\mathfrak{A}_g\mathfrak{A}_h \subseteq \mathfrak{A}_{gh}$ for any $g, h \in \mathsf{G}$. We say that $\mathfrak{A}$ is a G-graded algebra.

The $\mathfrak{A}_g$'s are called **homogeneous components**. For each $g \in \mathsf{G}$, any element $a \in \mathfrak{A}_g$ is called a **homogeneous element** of degree g, and we write $\mathsf{deg}(a) = g$.

Let $\mathfrak{A}$ be a G-graded algebra. Denote by $\Gamma_{\mathfrak{A}} : \mathfrak{A} = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g$ the G-grading on $\mathfrak{A}$ considered. The **support** of $\Gamma_{\mathfrak{A}}$, denoted by $\mathsf{Supp}(\Gamma_{\mathfrak{A}})$, is given by the set

$$\mathsf{Supp}(\Gamma_{\mathfrak{A}}) = \{g \in \mathsf{G} : \mathfrak{A}_g \neq \{0\}\} \ .$$

When no confusion can arise, we write only

Example 1.1.8 *Given a group G and a field $\mathbb{F}$, consider the* **group algebra** *$\mathbb{F}\mathsf{G}$, where the elements of $\mathbb{F}\mathsf{G}$ are the finite formal sums $\sum_{g\in\mathsf{G}} \lambda_g\eta_g$, where $\lambda_g \in \mathbb{F}$. We assume that the set $\{\eta_g : g \in \mathsf{G}\}$, where each element η_g corresponds to element $g \in \mathsf{G}$, is an $\mathbb{F}$-basis of $\mathbb{F}\mathsf{G}$, and $\eta_h\eta_g = \eta_{hg}$ for any $h, g \in \mathsf{G}$. The multiplication "$\cdot$" is linearly extended on the whole $\mathbb{F}\mathsf{G}$. A natural example of a G-graded $\mathbb{F}$-algebra (and also G-graded ring) is the group algebra $\mathbb{F}\mathsf{G}$, where $\mathfrak{A}_g = \mathsf{span}_{\mathbb{F}}\{\eta_g \in \mathfrak{A}\}$.*

Example 1.1.9 *Consider a group* G *and* $\mathfrak{A} = M_n(\mathbb{F})$*, the algebra of matrices of order* n*. Fixed an* n*-tuple* $\xi = (g_1, \ldots, g_n) \in \mathsf{G}^n$*, the* G*-grading on* $\mathfrak{A}$ *given by* $\mathfrak{A} = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g$*, where* $\mathfrak{A}_g = \mathsf{span}_{\mathbb{F}}\{E_{ij} \in \mathfrak{A} : g_i^{-1}g_j = g\}$*, is called the* **elementary G-grading** *defined by* ξ*. More generally, if* $\mathfrak{B} = M_n(\mathbb{F}\mathsf{G})$*, the algebra of matrices of order* n *over* $\mathbb{F}\mathsf{G}$*, and fixed an* n*-tuple* $\tilde{\xi} = (\tilde{g}_1, \ldots, \tilde{g}_n) \in \mathsf{G}^n$*, the* G*-grading on* $\mathfrak{B}$ *given by* $\mathfrak{B} = \bigoplus_{g\in\mathsf{G}} \mathfrak{B}_g$*, where* $\mathfrak{B}_g = \mathsf{span}_{\mathbb{F}}\{E_{ij}\eta_h \in \mathfrak{B} : \tilde{g}_i^{-1}h\tilde{g}_j = g\}$*, is called the* **canonical elementary G-grading** *defined by* $\tilde{\xi}$*.*

Let $\mathfrak{A}$ be a G-graded algebra. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is a **graded subalgebra** if $\mathfrak{B} = \bigoplus_{g\in\mathsf{G}}(\mathfrak{B} \cap \mathfrak{A}_g)$. This is means that, given $b \in \mathfrak{B}$ with $b = \sum_{g\in\mathsf{G}} b_g$ $(b_g \in \mathfrak{A}_g)$, we have that $b_g \in \mathfrak{B}$ for any $b \in \mathsf{G}$.

An ideal I of $\mathfrak{A}$ is said to be **graded** if $I = \oplus_{g\in\mathsf{G}}(I \cap \mathfrak{A}_g)$, i.e. if for any $x = \sum_{g\in\mathsf{G}} x_g \in I$, with $x_g \in \mathfrak{A}_g$, then $x_g \in I$ for any $g \in \mathsf{G}$.

It is clear that, if $\mathfrak{A}$ is a unitary G-graded algebra (resp. ring), then $1_{\mathfrak{A}} \in \mathfrak{A}_e$, where e is the neutral element of G. For more details, see Chapter 3 in [17].

Definition 1.1.10 *A* G*-graded algebra* $\mathfrak{A}$ *is said to be (left)* **G-simple** *(or* **simple graded***, or* **minimal graded***) if* $\mathfrak{A}^2 \neq \{0\}$ *and* $\mathfrak{A}$ *does not have proper* G*-graded (left) ideals, i.e. if* I *is a graded (left) ideal of* $\mathfrak{A}$*, then either* $I = \{0\}$ *or* $I = \mathfrak{A}$*. Moreover, assuming* $\mathfrak{A}$ *be unitary,* $\mathfrak{A}$ *is a* **G-division** *(or* **division G-graded***) algebra if all its nonzero homogeneous elements are inversible in* $\mathfrak{A}$*, i.e. for any* $a \in \bigcup_{g\in\mathsf{G}} \mathfrak{A}_g$*,* $a \neq 0$*, there exists* $a^{-1} \in \mathfrak{A}$ *such that* $aa^{-1} = a^{-1}a = 1$*. Note that* a^{-1} *is also homogeneous of degree* $\mathsf{deg}(a^{-1}) = (\mathsf{deg}(a))^{-1}$*.*

It is not difficult to show that any division algebra is a division G-graded algebra, for any G-grading on $\mathfrak{A}$, but there are division graded algebras that are not division algebras. For example, $\mathbb{F}\mathsf{G}$ is a division G-graded algebra, but it is not a division algebra for any field $\mathbb{F}$ and group G of order greater than or equal to 2. Moreover, any division graded algebra is also a simple graded algebra.

A graded ideal I of a G-graded algebra $\mathfrak{A}$ is called **minimal graded ideal** when $\{0\}$ and I are the only graded ideals of $\mathfrak{A}$ contained in I.

Notice that, given a subgroup H of G, the subset $\tilde{\mathfrak{A}} = \bigoplus_{g\in H} \mathfrak{A}_g$ of $\mathfrak{A}$ is a graded subalgebra of $\mathfrak{A}$. In general, $\mathsf{Supp}(\Gamma)$ is not a subgroup of G. Indeed, consider $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}_4$, and $\mathfrak{B} = M_3(\mathbb{F})$. Put $\xi = (g_1, g_2, g_3) \in \mathsf{G}^3$ with $g_1 = (0,0)$, $g_2 = (1,0), g_3 = (1,1)$. Notice that $\mathfrak{B}$ with the elementary G-grading defined by ξ has the support $\mathsf{Supp}(\Gamma) = \{g_1, g_2, g_3, g_3^{-1}, g_2g_3, g_3^{-1}g_2\}$, which obviously is not a subgroup of G.

Definition 1.1.11 *Let $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ be two* G*-graded* $\mathbb{F}$*-algebras and* $\phi : \mathfrak{A} \to \tilde{\mathfrak{A}}$ *a homomorphism of algebras. We say that* ϕ *is a* **graded homomorphism** *of* G*-graded algebras if* $\phi(\mathfrak{A}_g) \subseteq \tilde{\mathfrak{A}}_g$ *for any* $g \in \mathsf{G}$.

We say that a graded homomorphism of algebras ϕ is a **graded epimorphism** if it is surjective; ϕ is a **graded monomorphism** if it is injective; and ϕ is a **graded isomorphism** if it is bijective. We write $\mathfrak{A} \cong_{\mathsf{G}} \tilde{\mathfrak{A}}$ when there exists a graded isomorphism between two G-graded algebras $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$. Otherwise, we write $\mathfrak{A} \ncong_{\mathsf{G}} \tilde{\mathfrak{A}}$. It is not difficult to see that $\mathsf{ker}(\phi)$ and $\mathsf{im}(\phi)$ are graded subalgebras of $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$, respectively. Moreover, $\mathsf{ker}(\phi)$ is a graded ideal of $\mathfrak{A}$.

Let us now define bimodules over algebras and graded bimodules over graded algebras.

Definition 1.1.12 *Let* $\mathbb{F}$ *be a field,* $\mathfrak{A}$ *an* $\mathbb{F}$*-algebra (not necessarily unitary), and* M *an* $\mathbb{F}$*-vector space. We say that* M *is a* **left $\mathfrak{A}$-module** *if there exists a well defined map from* $\mathfrak{A} \times \mathsf{M}$ *into* M *that satisfies the following conditions:*

i) $a(m + m_1) = am + am_1$,

ii) $(a + a_1)m = am + a_1m$,

iii) $(aa_1)m = a(a_1m)$,

iv) $(\lambda a)m = \lambda(am) = a(\lambda m)$,

for any $a, a_1 \in \mathfrak{A}$, $\lambda \in \mathbb{F}$ *and* $m, m_1 \in \mathsf{M}$. *If* $\mathfrak{A}$ *is a unitary algebra, then we require that*

$$1_{\mathfrak{A}}m = m$$

for any $m \in \mathsf{M}$, *and hence, we say that* M *is a* **unitary** *left* $\mathfrak{A}$*-module.*

Analogously, we define a **right $\mathfrak{A}$-module**.

Definition 1.1.13 *Let* $\mathbb{F}$ *be a field,* $\mathfrak{A}$ *an* $\mathbb{F}$*-algebra (not necessarily unitary), and* M *an* $\mathbb{F}$*-vector space. We say that* M *is a* **right $\mathfrak{A}$-module** *if there exists a well defined map from* $\mathsf{M} \times \mathfrak{A}$ *into* M *that satisfies the following conditions:*

i) $(m + m_1)a = ma + m_1a$,

ii) $m(a + a_1) = ma + ma_1$,

iii) $m(aa_1) = (ma)a_1$,

iv) $m(\lambda a) = (\lambda m)a = \lambda(ma)$,

for any $a, a_1 \in \mathfrak{A}$, $\lambda \in \mathbb{F}$ *and* $m, m_1 \in \mathsf{M}$. *If* $\mathfrak{A}$ *is a unitary algebra, then we require that*

$$m1_{\mathfrak{A}} = m$$

for any $m \in \mathsf{M}$, *and hence, we say that* M *is a* **unitary** *right* $\mathfrak{A}$*-module.*

Given an left $\mathfrak{A}$-module M, we say that M is a left 0-module if $\mathfrak{A}\mathsf{M} = \{0\}$, i.e. $am = 0$ for any $a \in \mathfrak{A}$ and $m \in \mathsf{M}$.

It is easy to see that if $\mathfrak{A}$ is a commutative algebra, then the definitions of a left $\mathfrak{A}$-module and a right $\mathfrak{A}$-module are the same, and hence, we often say "an $\mathfrak{A}$-module". All the results for right $\mathfrak{A}$-modules and left $\mathfrak{A}$-modules are similar. For more details about (one-sided) modules over algebras, see [9].

Definition 1.1.14 *Let* $\mathbb{F}$ *be a field,* $\mathfrak{A}$ *and* $\tilde{\mathfrak{A}}$ *two* $\mathbb{F}$*-algebras (not necessarily unitary), and* M *an* $\mathbb{F}$*-vector space. We say that* M *is an* $(\mathfrak{A}, \tilde{\mathfrak{A}})$**-bimodule** *if it is a left* $\mathfrak{A}$*-module and a right* $\tilde{\mathfrak{A}}$*-module, and its two scalar multiplications satisfy the associative law:*

$$r(ms) = (rm)s$$

for any $r \in \mathfrak{A}$, $s \in \tilde{\mathfrak{A}}$ *and* $m \in \mathsf{M}$. *When* $\mathfrak{A} = \tilde{\mathfrak{A}}$, *we say that* M *is an* $\mathfrak{A}$**-bimodule**. *When* $\mathfrak{A}$ *and* $\tilde{\mathfrak{A}}$ *are unitary,* M *is a* **unitary** $\mathfrak{A}$**-bimodule** *iff* M *is a unitary left* $\mathfrak{A}$*-module and a unitary right* $\tilde{\mathfrak{A}}$*-module.*

Let $\mathfrak{A}$ be an algebra, and M a left (resp. right) $\mathfrak{A}$-module. A **submodule** N of M is a subspace of M which is $\mathfrak{A}$-invariant, i.e. N is also a left $\mathfrak{A}$-module. Given a subset S of M, we define the **submodule of** M **generated by** S, denoted by ${}_{\mathfrak{A}}S$ (resp. $S_{\mathfrak{A}}$), as being the set given by

$${}_{\mathfrak{A}}S = \left\{ \sum_{k=1}^{n} r_k m_k \in \mathsf{M} : n \in \mathbb{N}, r_i \in \mathfrak{A} \cup \mathbb{F}, m_i \in S \right\}$$

$$\left(\text{resp. } S_{\mathfrak{A}} = \left\{ \sum_{k=1}^{n} m_k s_k \in \mathsf{M} : n \in \mathbb{N}, s_i \in \mathfrak{A} \cup \mathbb{F}, m_i \in S \right\}\right) .$$

Observe that, necessarily, S is a subset of ${}_{\mathfrak{A}}S$ (resp. $S_{\mathfrak{A}}$). Let us consider also a submodule $\mathfrak{A}S$ (resp. $S\mathfrak{A}$) of M defined by

$$\mathfrak{A}S = \left\{ \sum_{i=1}^{n} r_i m_i \in \mathsf{M} : n \in \mathbb{N}, r_i \in \mathfrak{A}, m_i \in S \right\}$$

$$\left(\text{resp. } S\mathfrak{A} = \left\{ \sum_{j=1}^{\tilde{n}} \tilde{m}_j s_j \in \mathsf{M} : \tilde{n} \in \mathbb{N}, s_j \in \mathfrak{A}, \tilde{m}_j \in S \right\}\right).$$

If $S = \{m\}$, we denote $\mathfrak{A}S$ (resp. $S\mathfrak{A}$) by $\mathfrak{A}m$ (resp. $m\mathfrak{A}$) when no confusion can arise. Observe that not always S is a subset of $\mathfrak{A}S$ (resp. $S\mathfrak{A}$), and ${}_{\mathfrak{A}}S = \mathfrak{A}S + \mathsf{span}_{\mathbb{F}}\{m \in S\}$ (resp. $S_{\mathfrak{A}} = S\mathfrak{A} + \mathsf{span}_{\mathbb{F}}\{x \in S\}$). If $\mathfrak{A}$ is unitary, and M is a unitary left (resp. right) $\mathfrak{A}$-module, then ${}_{\mathfrak{A}}S = \mathfrak{A}S$ (resp. $S_{\mathfrak{A}} = S\mathfrak{A}$), and ${}_{\mathfrak{A}}S$ (resp. $S_{\mathfrak{A}}$) is a unitary left (resp. right) $\mathfrak{A}$-module.

Now, let $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ be two algebras, and M an $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule. A **subbimodule** N of M is a subspace of M which is also an $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule. Given a subset S of M, we define the **subbimodule of M generated by** S, denoted by ${}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}}$, as being the set given by

$$\begin{aligned} {}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}} &= \left\{ \sum_{k=1}^{n} r_k m_k s_k \in \mathsf{M} : n \in \mathbb{N}, r_i \in \mathfrak{A} \cup \mathbb{F}, s_i \in \tilde{\mathfrak{A}} \cup \mathbb{F}, m_i \in S \right\} \\ &= \left\{ \sum_{l=1}^{n_0} \lambda_l m_l + \sum_{i=1}^{n_1} r_i \breve{m}_i + \sum_{j=1}^{n_2} \tilde{m}_j s_j + \sum_{k=1}^{n_3} q_k \hat{m}_k p_k \in \mathsf{M} : \begin{array}{c} n_t \in \mathbb{N}_0, m_l, \breve{m}_i, \tilde{m}_j, \hat{m}_j \in S, \\ \lambda_l \in \mathbb{F}, r_i, q_k \in \mathfrak{A}, s_j, p_k \in \tilde{\mathfrak{A}} \end{array} \right\} , \end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let us consider also a submodule $\mathfrak{A}S\tilde{\mathfrak{A}}$ of M defined by

$$\mathfrak{A}S\tilde{\mathfrak{A}} = \left\{ \sum_{k=1}^{n} r_k m_k s_k \in \mathsf{M} : n \in \mathbb{N}, r_i \in \mathfrak{A}, s_i \in \tilde{\mathfrak{A}}, m_i \in S \right\}.$$

If $S = \{m\}$, we denote $\mathfrak{A}S\tilde{\mathfrak{A}}$ by $\mathfrak{A}m\tilde{\mathfrak{A}}$ when no confusion can arise. Observe that ${}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}} = \mathsf{span}_{\mathbb{F}}\{m \in S\} + \mathfrak{A}S + S\tilde{\mathfrak{A}} + \mathfrak{A}S\tilde{\mathfrak{A}}$, and hence, $\mathfrak{A}S\tilde{\mathfrak{A}} \subseteq {}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}}$ and $S \subseteq {}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}}$. When $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are unitary, and M is a unitary $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule, we have that ${}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}} = \mathfrak{A}S\tilde{\mathfrak{A}}$, and ${}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}}$ is a unitary $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule.

Observe that a subbimodule of a graded bimodule is graded iff it can be generated as a bimodule by homogeneous elements.

A left (resp. right) $\mathfrak{A}$-module M is called **irreducible** (or **simple**) if $\mathfrak{A}\mathsf{M} \neq \{0\}$ (resp. $\mathsf{M}\mathfrak{A} \neq \{0\}$), and $\{0\}$ and M are the only submodules of M. Therefore, for any irreducible left (resp. right) $\mathfrak{A}$-module M, we have $\mathsf{M} = \mathfrak{A}m$ (resp. $\mathsf{M} = m\mathfrak{A}$) for any nonzero $m \in \mathsf{M}$. Really, considering $N = \{m \in \mathsf{M} : \mathfrak{A}m = \{0\}\}$, we have that N is a submodule of M, and $N \neq \mathsf{M}$ because $\mathfrak{A}N = \{0\}$ and $\mathfrak{A}\mathsf{M} \neq \{0\}$, hence, $N = \{0\}$ (since M is irreducible), and consequently, for any $m \neq 0$ ($m \notin N$), we have $\mathfrak{A}m \neq \{0\}$ and $\mathfrak{A}m = \mathsf{M}$. Notice that, given a subalgebra I of $\mathfrak{A}$ such that $\mathfrak{A}I \neq \{0\}$ (resp. $I\mathfrak{A} \neq \{0\}$), I is a (minimal) left (resp. right) ideal of $\mathfrak{A}$ iff I is a (irreducible) left (resp. right) $\mathfrak{A}$-module.

Definition 1.1.15 *An* $(\mathfrak{A}, \tilde{\mathfrak{A}})$*-bimodule* M *is called* **irreducible** *(or* **simple***) if* $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$*, and* $\{0\}$ *and* M *are the only subbimodules of* M*. Particularly, the condition* $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$ *means that* $am\tilde{a} \neq 0$ *for some* $a \in \mathfrak{A}$*,* $\tilde{a} \in \tilde{\mathfrak{A}}$ *and* $m \in \mathsf{M}$*.*

For any irreducible $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule M, we have $\mathsf{M} = \mathfrak{A}m\tilde{\mathfrak{A}}$ for any nonzero $m \in \mathsf{M}$. Indeed, it is sufficient to see that $N = \{m \in \mathsf{M} : \mathfrak{A}m\tilde{\mathfrak{A}} = \{0\}\}$ is a subbimodule of M, and hence, we can conclude that $N = \{0\}$. When M is a unitary $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule (hence $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are unitary also), the condition $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$ is equivalent to conditions $\mathfrak{A}\mathsf{M} \neq \{0\}$ and $\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$.

Notice that, given a subalgebra I of $\mathfrak{A}$ such that $\mathfrak{A}I\mathfrak{A} \neq \{0\}$, I is a (minimal) (two-sided) ideal of $\mathfrak{A}$ iff I is an (irreducible) $\mathfrak{A}$-bimodule.

Let $\mathfrak{A}$ be an algebra, and M a left (resp. right) $\mathfrak{A}$-module. We say that M is **faithful** if $a\mathsf{M} = \{0\}$ (resp. $\mathsf{M}a = \{0\}$), where $a \in \mathfrak{A}$, implies $a = 0$. This means that the set $\mathsf{Ann}_{\mathfrak{A}}(\mathsf{M}) := \{a \in \mathfrak{A} : am = 0, \forall m \in \mathsf{M}\}$ (resp. $\mathsf{Ann}_{\mathfrak{A}}(\mathsf{M}) := \{a \in \mathfrak{A} : ma = 0, \forall m \in \mathsf{M}\}$) is null. It is easy to prove that $I = \mathsf{Ann}_{\mathfrak{A}}(\mathsf{M})$ is an (two-sided) ideal of $\mathfrak{A}$, for any left (resp. right) $\mathfrak{A}$-module M. From this, observe that if $\mathfrak{A}$ is a simple algebra, then any left (resp. right) $\mathfrak{A}$-module M is either a null left (resp. right) module or a faithful left (resp. right) $\mathfrak{A}$-module, and any irreducible left (resp. rigth) $\mathfrak{A}$-module is faithful, since $\mathfrak{A}\mathsf{M} \neq \{0\}$ (resp. $\mathsf{M}\mathfrak{A} \neq \{0\}$). Observe that M is also a left (resp. right) $\mathfrak{A}/I$-module, which is faithful, since $\mathsf{Ann}_{\mathfrak{A}/I}(\mathsf{M}) = \{0\}$.

Let N be a submodule of a left (resp. right) $\mathfrak{A}$-module M. The **left quotient $\mathfrak{A}$-module** (resp. **right quotient $\mathfrak{A}$-module**) M/N is defined as follow:

i) $\mathsf{M}/\mathsf{N} = \{\overline{m} = m + N : m \in \mathsf{M}\}$ is a quotient vector space;

ii) $a\overline{m} = \overline{am}$ (resp. $\overline{m}a = \overline{ma}$) for any $a \in \mathfrak{A}$ and $m \in \mathsf{M}$.

By the two above items, note that M/N is a left (resp. right) $\mathfrak{A}$-module naturally.

Analogously we define a *quotient bimodule*. Let N be a subbimodule of an $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule M. The **quotient** $(\mathfrak{A}, \tilde{\mathfrak{A}})$**-bimodule** M/N is defined as follow:

i) $\mathsf{M}/\mathsf{N} = \{\overline{m} = m + N : m \in \mathsf{M}\}$ is a quotient vector space;

ii) $a\overline{m} = \overline{am}$ for any $a \in \mathfrak{A}$ and $m \in \mathsf{M}$;

iii) $\overline{m}b = \overline{mb}$ for any $b \in \tilde{\mathfrak{A}}$ and $m \in \mathsf{M}$;

It is clear that $a\overline{m}b = \overline{am}b = \overline{amb} = \overline{amb}$ for any $a \in \mathfrak{A}$, $b \in \tilde{\mathfrak{A}}$ and $m \in \mathsf{M}$. By the three above items, note that M/N is an $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule naturally.

Definition 1.1.16 *Let $\mathfrak{A}$ be an algebra, M and $\tilde{\mathsf{M}}$ two left (resp. right) $\mathfrak{A}$-modules and $\phi : \mathsf{M} \longrightarrow \tilde{\mathsf{M}}$ a linear transformation. We say that ϕ is a* **homomorphism of left** *(resp.* **right***)* $\mathfrak{A}$**-modules** *if ϕ satisfies $\phi(am) = a\phi(m)$ (resp. $\phi(ma) = \phi(m)a$) for any $a \in \mathfrak{A}$ and $m \in \mathsf{M}$.*

Definition 1.1.17 *Let $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ be two algebras, M and $\tilde{\mathsf{M}}$ two $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodules and $\varphi : \mathsf{M} \longrightarrow \tilde{\mathsf{M}}$ a linear transformation. We say that φ is a* **homomorphism of** $(\mathfrak{A}, \tilde{\mathfrak{A}})$**-bimodules** *if φ satisfies $\varphi(am) = a\varphi(m)$ and $\varphi(mb) = \varphi(m)b$ for any $a \in \mathfrak{A}$, $b \in \tilde{\mathfrak{A}}$ and $m \in \mathsf{M}$.*

The definitions of **kernel**, **image**, **epimorphism**, **monomorphism** and **isomorphism** of left (resp. right) $\mathfrak{A}$-modules are inherited from linear transformations. It is not difficult to see that $\mathsf{ker}(\phi)$ and $\mathsf{im}(\phi)$ are submodules of M and $\tilde{\mathsf{M}}$, respectively. We write $\mathsf{M} \cong \tilde{\mathsf{M}}$ when there exists an isomorphism of left (resp. right) $\mathfrak{A}$-modules between two $\mathfrak{A}$-modules M and $\tilde{\mathsf{M}}$. Similar definitions we have for $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodules.

Now, let us define structures of G-gradings on modules.

Definition 1.1.18 *Let $\mathbb{F}$ be a field, G a group, $\mathfrak{A}$ an $\mathbb{F}$-algebra, and M a left (resp. right) $\mathfrak{A}$-module. Suppose that $\mathfrak{A}$ has a G-grading. A G-grading on M is a decomposition of M as a direct sum of $\mathbb{F}$-subspaces $\mathsf{M}_g \subseteq \mathsf{M}$, $g \in \mathsf{G}$,*

$$\mathsf{M} = \bigoplus_{g \in \mathsf{G}} \mathsf{M}_g \ ,$$

such that $\mathfrak{A}_g\mathsf{M}_h \subseteq \mathsf{M}_{gh}$ *(resp.* $\mathsf{M}_h\mathfrak{A}_g \subseteq \mathsf{M}_{hg}$*) for any* $g, h \in \mathsf{G}$*. We say that* M *is a* **G-graded left** *(resp.* **right***)* **$\mathfrak{A}$-module***.*

Analogously, given two G-graded algebras $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$, and an $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule M, we define a G-grading on M.

Definition 1.1.19 *Let* G *be a group,* $\mathfrak{A}$ *and* $\tilde{\mathfrak{A}}$ *two* G*-graded algebras and* M *an* $(\mathfrak{A}, \tilde{\mathfrak{A}})$*-bimodule. A* G*-grading on* M *is a decomposition of* M *in a direct sum of* $\mathbb{F}$*-subspaces* $\mathsf{M}_g \in \mathsf{M}$*,* $g \in \mathsf{G}$*, satisfying* $\mathfrak{A}_g\mathsf{M}_h \subseteq \mathsf{M}_{gh}$ *and* $\mathsf{M}_h\tilde{\mathfrak{A}}_t \subseteq \mathsf{M}_{ht}$*, for any* $g, h, t \in \mathsf{G}$*. In this case, we say that* M *is a* **G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule***.*

A special case of a bimodule graded by a group occurs when $\mathfrak{A} = \tilde{\mathfrak{A}}$, then we have a **$\mathsf{G}$-graded $\mathfrak{A}$-bimodule**. In the next section, we will detail the study of the G-graded $\mathfrak{A}$-bimodules.

Let M be a G-graded left (resp. right) $\mathfrak{A}$-module, where G is a group and $\mathfrak{A}$ is a G-graded algebra. A submodule N of M is called **graded submodule** if $N = \bigoplus_{g\in\mathsf{G}}(N \cap \mathsf{M}_g)$. This means that if $m = \sum_{g\in\mathsf{G}} m_g \in N$, with $m_g \in \mathsf{M}_g$, then $m_g \in N$ for any $g \in \mathsf{G}$. Similarly, a subbimodule N' of a G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule M', where $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are G-graded, is said to be a **graded subbimodule** if $N' = \bigoplus_{g\in\mathsf{G}}(N' \cap \mathsf{M}'_g)$.

A G-graded left (resp. right) $\mathfrak{A}$-module M is called **irreducible graded** if $\mathfrak{A}\mathsf{M} \neq \{0\}$ (resp. $\mathsf{M}\mathfrak{A} \neq \{0\}$, and M does not have proper graded submodules. This means that M is an irreducible graded left (resp. right) $\mathfrak{A}$-module iff $\mathfrak{A}\mathsf{M} \neq \{0\}$ (resp. $\mathsf{M}\mathfrak{A} \neq \{0\}$), and $\{0\}$ and M are the only graded submodules of M.

Remark 1.1.20 *Let* G *be a group,* $\mathfrak{A}$ *a* G*-graded algebra, and* I *a graded left ideal of* $\mathfrak{A}$*, i.e.* I *is a left ideal of* $\mathfrak{A}$ *such that* $I = \bigoplus_{g\in\mathsf{G}}(I \cap \mathfrak{A}_g)$*. We have that* I *is a* G*-graded left* $\mathfrak{A}$*-module naturally. So, when* $\mathfrak{A}I \neq \{0\}$*,* I *is a minimal* G*-graded left ideal of* $\mathfrak{A}$ *if and only if* I *is an irreducible* G*-graded left* $\mathfrak{A}$*-module (see Definition 1.1.10). For right ideals and two-sided ideals we can deduce analogue result.*

Definition 1.1.21 *Let* M *be a* G*-graded* $(\mathfrak{A}, \tilde{\mathfrak{A}})$*-bimodule. We say that* M *is* **irreducible graded** *if* $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$*, and* M *does not have proper graded subbimodules. This means that* M *is an irreducible* G*-graded* $(\mathfrak{A}, \tilde{\mathfrak{A}})$*-bimodule iff* $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$*, and* $\{0\}$ *and* M *are the only graded subbimodules of* M*. Particularly, the condition* $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}} \neq \{0\}$ *means that* $am\tilde{a} \neq 0$ *for some homogeneous elements* $a \in \mathfrak{A}$*,* $m \in \mathsf{M}$ *and* $\tilde{a} \in \mathsf{M}$

For any irreducible G-graded $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule M, we have $\mathsf{M}=\mathfrak{A}m\tilde{\mathfrak{A}}$ for any nonzero homogeneous element $m\in\mathsf{M}$. In fact, it is sufficient to see that $N=\{m\in\mathsf{M}:\mathfrak{A}m\tilde{\mathfrak{A}}=\{0\}\}$ is a graded subbimodule of M. Hence, we can conclude that $N=\{0\}$, because $\mathfrak{A}\mathsf{M}\tilde{\mathfrak{A}}\{0\}$, and $\mathfrak{A}N\tilde{\mathfrak{A}}\{0\}$, and so $N\neq\mathsf{M}$.

Notice that, given a subalgebra I of $\mathfrak{A}$ such that $\mathfrak{A}I\mathfrak{A}\neq\{0\}$, I is a (minimal) (two-sided) ideal of $\mathfrak{A}$ iff I is an (irreducible) $\mathfrak{A}$-bimodule.

Now, let M be a G-graded left $\mathfrak{A}$-module (resp. G-graded $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule). Given a graded submodule (resp. graded subbimodule) N of M, we have that left quotient $\mathfrak{A}$-module (resp. quotient $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule) M/N is a G-graded left $\mathfrak{A}$-module (resp. $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule) naturally. In fact, since $\mathsf{M}=\bigoplus_{g\in\mathsf{G}}\mathsf{M}_g$ and $N=\bigoplus_{g\in\mathsf{G}}(N\cap\mathsf{M}_g)$, we have the quotient space $\mathsf{M}_g/(N\cap\mathsf{M}_g)$ is well defined, for any $g\in\mathsf{G}$. It is easy to see that

$$\frac{\mathsf{M}}{N}=\bigoplus_{g\in\mathsf{G}}\frac{\mathsf{M}_g}{N\cap\mathsf{M}_g}\ ,$$

and thus, M/N is a G-graded left $\mathfrak{A}$-module (resp. G-graded $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule) called **graded quotient left $\mathfrak{A}$-module** (resp. **graded quotient $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule**). For quotient right $\mathfrak{A}$-modules, we obtain a similar result.

Definition 1.1.22 *Let $\mathfrak{A}$ be a G-graded algebra, M and $\tilde{\mathsf{M}}$ two G-graded left (resp. right) $\mathfrak{A}$-modules and $\psi:\mathsf{M}\to\tilde{\mathsf{M}}$ a homomorphism of left (resp. right) $\mathfrak{A}$-modules. We say that ψ is a* **homogeneous homomorphism** *of degree $h_0\in\mathsf{G}$ of G-graded left (resp. right) $\mathfrak{A}$-modules if $\psi(\mathsf{M}_g)\subseteq\tilde{\mathsf{M}}_{gh_0}$ (resp. $\psi(\mathsf{M}_g)\subseteq\tilde{\mathsf{M}}_{h_0g}$) for any $g\in\mathsf{G}$. A finite sum of homogeneous homomorphisms of left (resp. right) $\mathfrak{A}$-modules is called a* **graded homomorphism of left** *(resp.* **right***)* **$\mathfrak{A}$-modules**.

Similarly to above definition we define homogeneous homomorphisms of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules.

Definition 1.1.23 *Let G be a group, $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ two G-graded algebras, M and $\tilde{\mathsf{M}}$ two G-graded $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules, and $\varphi:\mathsf{M}\to\tilde{\mathsf{M}}$ a homomorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules. We say that φ is a* **homogeneous homomorphism** *of degree $h_0\in\mathsf{G}$ of G-graded $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules if $\varphi(\mathsf{M}_g)\subseteq\tilde{\mathsf{M}}_{gh_0}=\tilde{\mathsf{M}}_{h_0g}$ for any $g\in\mathsf{G}$. A finite sum of homogeneous homomorphisms of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules is called a* **graded homomorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules**.

Observe that the last definition is correct for any $h_0 \in \mathcal{Z}(\mathsf{G})$. Particularly, this is correct for any $h_0 \in \mathsf{G}$ if G is abelian.

Notice that any homogeneous homomorphism of G-graded left (resp. right) $\mathfrak{A}$-modules is also a graded homomorphism. Not always the kernel or image of a graded homomorphism are graded submodules, but if a homomorphism of graded left modules (resp. right modules, bimodules) is homogeneous of degree h, it is easy to see that its kernel and image are graded submodules. When two G-graded left (resp. right) $\mathfrak{A}$-modules M and $\tilde{\mathsf{M}}$ are homogeneously isomorphic, i.e. there exists a homogeneous isomorphism $\psi : \mathsf{M} \to \tilde{\mathsf{M}}$, we write $\mathsf{M} \cong_{\mathsf{G}} \tilde{\mathsf{M}}$. Similar definitions and notations are used also for G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodules.

Let M be a G-graded left $\mathfrak{A}$-module. Considering another G-graded algebra $\tilde{\mathfrak{A}}$, we have that M is a right $\tilde{\mathfrak{A}}$-module with the trivial product, i.e. $ma = 0$ for any $m \in \mathsf{M}$ and $a \in \tilde{\mathfrak{A}}$, and hence, M is a G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule naturally. Analogously, we can assume that a right $\tilde{\mathfrak{A}}$-module is also a $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule. Therefore, some results for bimodules are also valid for left and right modules. The next two theorems are also true for graded left and right modules, and the proofs of them are similar to the proofs for bimodules.

Theorem 1.1.24 *Let G be a group, $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ two G-graded algebras, and M a G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule.*

i) If M' is a G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule and $\psi : \mathsf{M} \longrightarrow \mathsf{M}'$ is a homogeneous homomorphism of degree h, then

$$\frac{\mathsf{M}}{\mathsf{ker}(\psi)} \cong_{\mathsf{G}} \mathsf{im}(\psi)$$

as G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodules;

ii) If N, W are G-graded subbimodules of M, then

$$\frac{N + W}{W} \cong_{\mathsf{G}} \frac{N}{N \cap W}$$

as G-graded $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodules;

iii) If N, W are G-graded subbimodules of M with $N \subseteq W$, then

$$\frac{\mathsf{M}}{W} \cong_{\mathsf{G}} \frac{\mathsf{M}/N}{W/N}$$

as G*-graded* $(\mathfrak{A},\tilde{\mathfrak{A}})$*-bimodules.*

Proof: *i)* Since ψ is a homogeneous homomorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules, we have that $\mathsf{ker}(\psi)$ is a graded subbimodule of M, and $\mathsf{im}(\psi)$ is a graded subbimodule of $\tilde{\mathsf{M}}$. Consider the quotient $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodule $\mathsf{M}/\mathsf{ker}(\psi)$, and the map

$$\begin{array}{rccc} \tilde{\psi}: & \dfrac{\mathsf{M}}{\mathsf{ker}(\psi)} & \longrightarrow & \mathsf{im}(\psi) \\ & \overline{m}=m+\mathsf{ker}(\psi) & \longmapsto & \tilde{\psi}(\overline{m})=\psi(m) \end{array} \ , \ m\in\mathsf{M} \ ,$$

which is well defined because if $\overline{m}=\overline{m'}$, then $m-m'\in\mathsf{ker}(\psi)$, and consequently,

$$\tilde{\psi}(\overline{m})=\psi(m)=\psi(m')=\tilde{\psi}(\overline{m'}) \ .$$

It is easy to see that ψ is a homogeneous isomorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules, since ψ is a homogeneous homomorphism.

ii) Consider the map

$$\begin{array}{rccc} \phi: & N & \longrightarrow & \dfrac{N+W}{W} \\ & m & \longmapsto & \phi(m)=m+W \end{array} \ , \ m\in\mathsf{M} \ .$$

Obviously ϕ is a homogeneous epimorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules. By item *i)*, we have that

$$\frac{N}{\mathsf{ker}(\phi)}\cong_{\mathsf{G}}\frac{N+W}{W} \ .$$

Notice that $N\cap W\subseteq\mathsf{ker}(\phi)$. Conversely, take $m\in\mathsf{ker}(\phi)$. Hence, $m\in N$ and $\overline{0}=\phi(m)=\overline{m}$, and hence, $m\in W$. Consequently, $\mathsf{ker}(\phi)=N\cap W$. The result follows.

iii) Consider the map

$$\begin{array}{rccc} \varphi: & \mathsf{M} & \longrightarrow & \dfrac{\mathsf{M}/N}{W/N} \\ & m & \longmapsto & \varphi(m)=(m+N)+W/N \end{array} \ , \ m\in\mathsf{M} \ .$$

Obviously φ is a homogeneous epimorphism of $(\mathfrak{A},\tilde{\mathfrak{A}})$-bimodules. By item *i)*, we have that

$$\frac{\mathsf{M}}{\mathsf{ker}(\varphi)}\cong_{\mathsf{G}}\frac{\mathsf{M}/N}{W/N} \ .$$

Note that $W\subseteq\mathsf{ker}(\varphi)$. Conversely, take $m\in\mathsf{ker}(\varphi)$. Hence, $\overline{0}=\varphi(m)=(m+N)+W/N$, and hence, $m + N \in W/N$. Consequently, we have that $m \in W$, and so $\mathsf{ker}(\phi) \subseteq W$. The result follows. ■

Theorem 1.1.25 *Let* G *be a group,* $\mathfrak{A}$ *and* $\tilde{\mathfrak{A}}$ *two* G*-graded algebras and* M *a* G*-graded* $(\mathfrak{A}, \tilde{\mathfrak{A}})$*-bimodule. Suppose* N *is a graded subbimodule of* M*. Then any graded subbimodule of the quotient bimodule* M/N *is of the form* $P/N = \{x + N : x \in P\}$*, where* P *is a graded subbimodule of* M *such that* $N \subset P \subset \mathsf{M}$*. The correspondence between graded subbimodules of* M/N *and graded subbimodules of* M *which contain* N *is a bijection.*

Proof: Analogous to the nongraded case (see Theorem 6.22, [40]). ■

1.2 Properties of Graded Algebras

In this section, we present main properties of graded algebras. We also exhibit some well-known results which help us to develop this work.

Here, $\mathbb{F}$ denotes a field and G denotes a group. By $\mathbb{F}^* = \mathbb{F}\backslash\{0\}$ we denote the multiplicative group of $\mathbb{F}$. By convention, we assume that G is a group with multiplicative notation.

1.2.1 Cocycles and Coboundaries

Definition 1.2.1 *The mapping* $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ *which satisfies*

$$\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z) \text{ for all } x, y, z \in \mathsf{G}$$

is called a **2-cocycle** *on* G *with values in* $\mathbb{F}^*$*. The set of all* 2*-cocycles from* G *into* $\mathbb{F}^*$ *is denoted by* $\mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$*.*

Example 1.2.2 *The application* σ *from* $\mathsf{G}\times\mathsf{G}$ *to* $\mathbb{F}^*$ *given by* $\sigma(x, y) = 1$*, for any* $x, y \in \mathsf{G}$*, is a* 2*-cocycle called the* **trivial 2-cocycle***.*

Example 1.2.3 *Given a group* G*, a field* $\mathbb{F}$ *and a map* $f : \mathsf{G} \to \mathbb{F}^*$*, the application* $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ *defined by*

$$\sigma(x, y) = \frac{f(xy)}{f(x)f(y)}, \quad x, y \in G,$$

is a 2*-cocycle, which is called* 2-**coboundary**. *The set of all* 2*-coboundaries from* G *into* $\mathbb{F}^*$ *is denoted by* $\mathsf{B}^2(\mathsf{G}, \mathbb{F}^*)$.

Given two 2-cocycles $\sigma, \rho : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$, we say that σ and ρ are **equivalent** if there exists a 2-coboundary $f : \mathsf{G} \longrightarrow \mathbb{F}^*$ such that

$$\frac{\sigma(x, y)}{\rho(x, y)} = \frac{f(xy)}{f(x)f(y)}$$

for any $x, y \in G$. We write $[\sigma] = [\rho]$ in this case. It is not difficult to show that this relation is an equivalence relation on the set of all 2-cocycles on G. In this case, we write $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*) := \{[\sigma] : \sigma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)\}$.

Definition 1.2.4 *Given a* 2*-cocycle* $\sigma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$, *we say that* σ *is* **symmetric** *when* $\sigma(x, y) = \sigma(y, x)$ *for any* $x, y \in \mathsf{G}$. *When* $\sigma(x, y) = -\sigma(y, x)$ *for any* $x, y \in \mathsf{G}$, σ *is called* **antisymmetric**.

It is immediate of the above definition that any 2-coboundary is a symmetric 2-cocycle.

Example 1.2.5 *Let* $G = (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. *The map* $\sigma : \mathsf{G} \times \mathsf{G} \to \mathbb{C}^*$ *given by the following table*

σ	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$
$(\bar{0}, \bar{0})$	1	1	1	1
$(\bar{0}, \bar{1})$	1	1	1	1
$(\bar{1}, \bar{0})$	1	-1	1	-1
$(\bar{1}, \bar{1})$	1	-1	1	-1

defines a 2*-cocycle, i.e.* σ *belongs to* $\mathsf{Z}^2(\mathsf{G}, \mathbb{C})$. *Notice that* σ *is neither symmetric nor antisymmetric.*

Proposition 1.2.6 *Let* G *be a group,* $\mathbb{F}$ *a field and* $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ *a* 2*-cocycle on* G. *If* $e \in \mathsf{G}$ *is the neutral element (unit of* G*), then for any* $x, y \in \mathsf{G}$ *we have*

$$\sigma(x, e) = \sigma(e, y).$$

In particular, $\sigma(x, e) = \sigma(e, x) = \sigma(e, e)$ *for any* $x \in \mathsf{G}$. *In addition,* $\sigma(x, x^{-1}) = \sigma(x^{-1}, x)$, *for any* $x \in \mathsf{G}$.

Proof: By Definition 1.2.1, doing $y = e$, we have

$$\sigma(x,e)\sigma(xe,z) = \sigma(x,ez)\sigma(e,z)$$
$$\sigma(x,e)\sigma(x,z) = \sigma(x,z)\sigma(e,z)$$
$$\sigma(x,e) = \sigma(e,z)$$

for any $x, z \in \mathsf{G}$. In particular, for $z = x$, and after for $z = e$, we have

$$\sigma(x,e) = \sigma(e,x)$$
$$\sigma(x,e) = \sigma(e,e)\ .$$

Now, again by Definition 1.2.1, doing $y = x^{-1}$ and $z = x$, we have

$$\sigma(x,x^{-1})\sigma(xx^{-1},x) = \sigma(x,x^{-1}x)\sigma(x^{-1},x)$$
$$\sigma(x,x^{-1})\sigma(e,x) = \sigma(x,e)\sigma(x^{-1},x)$$
$$\sigma(x,x^{-1}) = \sigma(x^{-1},x)$$

for any $x \in \mathsf{G}$, where here we use the equality $\sigma(e,w) = \sigma(w,e)$ for any $w \in \mathsf{G}$. ■

Consider any element $\lambda \in \mathbb{F}$. We write $\sqrt[n]{\lambda} \in \mathbb{F}$ if the polynomial $x^n - \lambda = 0$ has a solution in $\mathbb{F}$, i.e. if there exists an element $\gamma \in \mathbb{F}$ such that $\gamma^n = \lambda$. Hence, we write $\gamma = \sqrt[n]{\lambda}$ to denote that $\gamma^n = \lambda$.

Proposition 1.2.7 *Let G be a finite cyclic group of order n generated by g, $\mathbb{F}$ a field and $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ a 2-cocycle. If*

$$\sqrt[n]{\sigma(g,e)\sigma(g,g)\sigma(g,g^2)\cdots\sigma(g,g^{n-1})} \in \mathbb{F}\ ,$$

then σ is a 2-coboundary. In particular, if $\mathbb{F}$ is an algebraically closed field and G a finite cyclic group, then any 2-cocycle on G is a 2-coboundary.

Proof: Suppose $\mathsf{G} = \langle g \rangle$ with $|\mathsf{G}| = n$. Take σ a 2-cocycle on G, and assume that $\lambda = \sqrt[n]{\sigma(g,e)\sigma(g,g)\cdots\sigma(g,g^{n-1})} \in \mathbb{F}^*$. Consider the map $\phi : \mathsf{G} \longrightarrow \mathbb{F}^*$ defined by $\phi(e) = \dfrac{1}{\sigma(g,e)}$, $\phi(g) = \dfrac{1}{\lambda}$ and $\phi(g^i) = \dfrac{1}{\lambda^i}\sigma(g,g)\cdots\sigma(g,g^{i-1})$ for $2 \leqslant i \leqslant n-1$. It is not difficult to see that

$$\sigma(t,h) = \frac{\phi(th)}{\phi(t)\phi(h)}$$

for any $t, h \in \mathsf{G}$, since

$$\sigma(g^r, g^s) = \frac{\sigma(g, g^{r+s-1})\sigma(g, g^{r+s-2})\cdots\sigma(g, g^{s+2})\sigma(g, g^{s+1})\sigma(g, g^s)}{\sigma(g, g^{r-1})\sigma(g, g^{r-2})\cdots\sigma(g, g^2)\sigma(g, g)}$$

for any $r, s = 1, \ldots, n$. The result follows. ∎

In the previous proposition, the condition "$\mathbb{F}$ is an algebraically closed field" can be changed by "$\mathbb{F}$ contains $\sqrt[n]{\lambda}$ for any $\lambda \in \mathbb{F}$", then any 2-cocycle on G with values in $\mathbb{F}^*$ is a 2-coboundary.

Corollary 1.2.8 *Any 2-cocycle on a finite cyclic group is symmetric.*

Proof: Let G be a finite cyclic group, and $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ a 2-cocycle. Consider an algebraically closed extension $\mathbb{K}$ of $\mathbb{F}$, and the application $\tilde{\sigma} : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{K}^*$ defined by $\tilde{\sigma}(g, h) = \sigma(g, h)$ for any $g, h \in \mathsf{G}$. We have that $\tilde{\sigma}$ is a 2-cocycle, and since $\mathbb{K}$ is an algebraically closed field, by Proposition 1.2.7, it follows that there exists an application $f : \mathsf{G} \longrightarrow \mathbb{K}^*$ such that $\tilde{\sigma}(g, h) = \dfrac{f(gh)}{f(g)f(h)}$ for any $g, h \in \mathsf{G}$. Hence, we have

$$\sigma(g, h) = \tilde{\sigma}(g, h) = \frac{f(gh)}{f(g)f(h)} = \frac{f(hg)}{f(h)f(g)} = \tilde{\sigma}(h, g) = \sigma(h, g)$$

for any $g, h \in \mathsf{G}$. Therefore, σ is symmetric. ∎

It follows from the previous corollary that the restriction of σ to a cyclic subgroup is symmetric, i.e. given a 2-cocycle $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ and a cyclic subgroup H of G, we have that $\sigma_H : H \times H \longrightarrow \mathbb{F}^*$ defined by $\sigma_H(g, h) := \sigma(g, h)$ for any $g, h \in H$ is a symmetric 2-cocycle on H.

Remark 1.2.9 *Let $\mathsf{G} = H_1 \times H_2$ be a group, and $\mathbb{F}$ a field. Given a 2-cocycle σ_i on H_i, $i = 1, 2$, with values in $\mathbb{F}^*$, it is easy to see that the map $\sigma := \sigma_1\sigma_2 : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ defined by*

$$\sigma(x, y) = \sigma((x_1, x_2), (y_1, y_2)) = \sigma_1(x_1, y_1)\sigma_2(x_2, y_2) \ ,$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathsf{G}$, is a 2-cocycle on G.

1.2.2 Some results on G-graded algebras

Let $\mathbb{F}$ be a field, and G a group. Consider the group algebra $\mathbb{F}\mathsf{G}$ (see Example 1.1.8). Observe that $\mathbb{F}\mathsf{G}$ is an associative algebra with unity; if G is commutative (resp. finite), then $\mathbb{F}\mathsf{G}$ is a commutative (resp. finite dimensional) algebra. Also, $\mathbb{F}\mathsf{G}$ has a natural G-grading. Let us consider a more general situation.

Definition 1.2.10 *Let G be a group, $\mathbb{F}$ a field, and $\sigma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ a 2-cocycle on G. Consider the $\mathbb{F}$-vector space*

$$\mathbb{F}^\sigma[\mathsf{G}] = \left\{ \sum_{g \in \mathsf{G}} \alpha_g \eta_g : \alpha_g \in \mathbb{F}, g \in \mathsf{G} \right\} ,$$

where $\{\eta_g\}_{g \in \mathsf{G}}$ are linearly independent over $\mathbb{F}$. And we define in $\mathbb{F}^\sigma[\mathsf{G}]$ the multiplication which extends by linearity the product $\eta_g \eta_h = \sigma(g,h)\eta_{gh}$, $g, h \in \mathsf{G}$. The algebra $\mathbb{F}^\sigma[\mathsf{G}]$ is called a **twisted group algebra***.*

Notice that by the equality in Definition 1.2.1 we can ensure that $\mathbb{F}^\sigma[\mathsf{G}]$ is an associative algebra. Furthermore, we have that $\mathbb{F}^\sigma[\mathsf{G}]$ is G-graded with the natural grading given by

$$\mathfrak{A} = \mathbb{F}^\sigma[\mathsf{G}] = \bigoplus_{g \in \mathsf{G}} \mathbb{F}\eta_g ,$$

where $\mathfrak{A}_g := \mathsf{span}_{\mathbb{F}}\{\eta_g\}$.

Example 1.2.11 *If σ is the trivial 2-cocycle on G, it is easy to see that $\mathbb{F}^\sigma[\mathsf{G}] = \mathbb{F}\mathsf{G}$.*

Remark 1.2.12 *Note that $\mathbb{F}^\sigma[\mathsf{G}]$ is unitary, where the unity of $\mathbb{F}^\sigma[\mathsf{G}]$ is given by $\sigma(e,e)^{-1}\eta_e$ where $e \in \mathsf{G}$ is the neutral element of G, since $\sigma(e,e) = \sigma(e,g) = \sigma(g,e)$ for any $g \in \mathsf{G}$ (by Proposition 1.2.6).*

Theorem 1.2.13 ([3], Theorem 2, or [13], Theorem 2.13) *Let D be a finite dimensional G-graded algebra over an algebraically closed field $\mathbb{F}$. Then D is a graded division algebra with support $T \subseteq \mathsf{G}$ iff D is isomorphic to the twisted group algebra $\mathbb{F}^\sigma[T]$ (with its natural T-grading regarded as a G-grading) for some $\sigma \in \mathsf{Z}^2(T, \mathbb{F}^*)$, where T is a finite subgroup of G. Two twisted group algebras, $\mathbb{F}^{\sigma_1}[H_1]$ and $\mathbb{F}^{\sigma_2}[H_2]$, are isomorphic as G-graded algebras if and only if $H_1 = H_2$ and $[\sigma_1] = [\sigma_2]$.*

Let $\mathfrak{A}$ be a G-graded algebra, and $\beta : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbb{F}^*$ a 2-cocycle on G. We define the β-**commutator** by

$$[a,b]_\beta = \left[\sum_{g\in\mathsf{G}} a_g, \sum_{h\in\mathsf{G}} b_h\right]_\beta = \sum_{g,h\in\mathsf{G}} [a_g,b_h]_\beta$$

if $a,b \in \mathfrak{A}$, where $[a_g,b_h]_\beta := a_g b_h - \beta(g,h) b_h a_g$, $g,h \in \mathsf{G}$. We say that $\mathfrak{A}$ is β-**commutative** if $[a,b]_\beta = 0$ for any $a,b \in \mathfrak{A}$. Obviously, if $[a,b]_\beta = 0$ for any $a,b \in \bigcup_{g\in\mathsf{G}} \mathfrak{A}_g$, we have that $\mathfrak{A}$ is β-commutative. When $\beta(a,b) = 1$ for any $a,b \in \mathsf{G}$, we write $[\ ,\ ]_\beta = [\ ,\ ]$, and $[\ ,\ ]$ is called **commutator**. Note that $\mathfrak{A}$ is commutative iff $[a,b] = 0$ for any $a,b \in \mathfrak{A}$. Observe that β-commutatively is a partial case of f-commutatively defined in Definition 1.1.4.

Remark 1.2.14 *Consider the algebra $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ of all $n \times n$ matrices over $\mathbb{F}^\sigma[H]$, where H is a subgroup of a group G and $\sigma \in \mathsf{Z}^2(H,\mathbb{F}^*)$. Fix an arbitrary k-tuple $\xi = (g_1,\dots,g_k) \in G^k$ of elements of G. Then the equalities $\mathsf{deg}(E_{ij}\eta_h) = g_i^{-1} h g_j$, for any $h \in H$ and $i,j \in \{1,\dots,n\}$, define a G-grading on $\mathfrak{B}$, i.e.*

$$\mathfrak{B} = \bigoplus_{g\in\mathsf{G}} \mathfrak{B}_g \ ,$$

where $\mathfrak{B}_g = \mathsf{span}_{\mathbb{F}}\{E_{ij}\eta_h \in \mathfrak{B} : g = g_i^{-1} h g_j\}$. This G-grading is called **canonical elementary grading** *corresponding to ξ.*

Definition 1.2.15 *Let $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ be two G-graded algebras and $\psi : \mathfrak{A} \to \tilde{\mathfrak{A}}$ a graded homomorphism. We say that ψ is a* **G-graded immersion** *(or* **G-immersion***) from $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$ if ψ is injective. We denote a G-graded immersion from $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$ by $\mathfrak{A} \overset{\mathsf{G}}{\hookrightarrow} \tilde{\mathfrak{A}}$. And, we write $\mathfrak{A} \overset{\mathsf{G}}{\not\hookrightarrow} \tilde{\mathfrak{A}}$ when $\mathfrak{A}$ can not be G-immersed in $\tilde{\mathfrak{A}}$.*

Note that if ψ is a G-graded immersion from $\mathfrak{A}$ to $\tilde{\mathfrak{A}}$, then there exists a G-graded subalgebra $\hat{\mathfrak{A}}$ of $\tilde{\mathfrak{A}}$, namely $\hat{\mathfrak{A}} = \mathsf{im}(\psi)$, such that $\hat{\mathfrak{A}}$ and $\mathfrak{A}$ are G-graded isomorphic. Therefore, we can see $\mathfrak{A}$ as a G-graded subalgebra of $\tilde{\mathfrak{A}}$ or assume that $\tilde{\mathfrak{A}}$ has a G-graded "copy" of $\mathfrak{A}$. In this case, observe that $\mathsf{Supp}(\Gamma_{\mathfrak{A}}) \subseteq \mathsf{Supp}(\Gamma_{\tilde{\mathfrak{A}}})$.

Remark 1.2.16 *Given a group G, consider a subgroup H of G, and a 2-cocycle $\sigma \in \mathsf{Z}^2(H,\mathbb{F}^*)$. For any $n > 1$ and $\xi = (g_1, g_2, \dots, g_n) \in \mathsf{G}^n$, consider the algebra $\mathfrak{B} =$*

$M_n(\mathbb{F}^\sigma[H])$ of $n \times n$ matrices over $\mathbb{F}^\sigma[H]$ with the canonical elementary G-grading Γ corresponding to ξ. For each $g \in \mathsf{Supp}(\Gamma)$, there exist $i, j \in \{1, \ldots, n\}$ and $h \in H$ such that $0 \neq E_{ij}\eta_h \in \mathfrak{B}_g$. Given $g_0 \in \mathcal{Z}_\mathsf{G}(H)$, we have

$$\begin{aligned} g = g_i^{-1}hg_j &= g_i^{-1}(g_0^{-1}g_0)hg_j = (g_i^{-1}g_0^{-1})(g_0h)g_j \\ &= (g_0g_i)^{-1}(hg_0)g_j = (g_0g_i)^{-1}h(g_0g_j) \ . \end{aligned}$$

Therefore, for any $g \in \mathcal{Z}_\mathsf{G}(H)$, in particular, for any $g \in \mathsf{G}$, when G is abelian, we have that $\xi = (g_1, g_2, \ldots, g_n)$ and $\xi_g = (gg_1, gg_2, \ldots, gg_n)$ determine the same canonical elementary G-gradings on $\mathfrak{B}$. Consequently, when $g_1 \in \mathcal{Z}_\mathsf{G}(H)$ or G is abelian, we can assume that $g_1 = e$, the neutral element of G.

Remark 1.2.17 *Under the assumptions of Remark 1.2.16, fix $\xi = (g_1, g_2, \ldots, g_n)$. Take $\alpha \in S_n$, where S_n is the symmetric group of order n, and consider the n-tuple $\xi_\alpha = (g_{\alpha(1)}, g_{\alpha(2)}, \ldots, g_{\alpha(n)}) \in \mathsf{G}^n$. Consider now the algebra $\tilde{\mathfrak{B}} = M_n(\mathbb{F}^\sigma[H])$ with the canonical elementary G-grading corresponding to ξ_α. Assume that $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ is G-graded with the canonical elementary grading corresponding to ξ. It not is difficult to show that $\mathfrak{B} \cong_\mathsf{G} \tilde{\mathfrak{B}}$ (as G-graded algebras) (the graded isomorphism is given by $\varphi(E_{ij}\eta_h) = E_{\alpha(i)\alpha(j)}\tilde{\eta}_h$, for any $i, j \in \{1, \ldots, n\}$ and $h \in H$).*

Remark 1.2.18 *Let G be a group and $m \leqslant n$. Put $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ and $\tilde{\mathfrak{B}} = M_m(\mathbb{F}^{\tilde{\sigma}}[H])$, where $\sigma, \tilde{\sigma} \in \mathsf{Z}^2(H, \mathbb{F}^*)$ such that $[\sigma] = [\tilde{\sigma}]$. Fix an n-tuple $\xi = (g_1, \ldots, g_n) \in G^n$, and consider a canonical elementary G-grading Γ on $\mathfrak{B}$ corresponding to ξ. Now, fixed $\alpha \in S_m$, consider the m-tuple $\xi_\alpha = (g_{\alpha(1)}, \ldots, g_{\alpha(m)}) \in \mathsf{G}^m$, and assume that $\tilde{\Gamma}$ is the canonical elementary G-grading on $\tilde{\mathfrak{B}}$ corresponding to ξ_α. By Theorem 1.2.13 and Remark 1.2.17, we can conclude that $\tilde{\mathfrak{B}} \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{B}$. Therefore, we can assume, without loss of generality, that $\tilde{\mathfrak{B}}$ is a G-graded subalgebra of $\mathfrak{B}$.*

To conclude this section, let us present the next two important results.

Theorem 1.2.19 (Theorem 3, [3]) *Let $\mathfrak{A} = \bigoplus_{g \in \mathsf{G}} \mathfrak{A}_g$ be a finite dimensional algebra, over an algebraically closed field $\mathbb{F}$ that is graded by a group G. Suppose that either $\mathsf{char}(\mathbb{F}) = 0$ or $\mathsf{char}(\mathbb{F})$ is coprime with the order of each finite subgroup of G. Then $\mathfrak{A}$ is a graded simple algebra if and only if $\mathfrak{A}$ is isomorphic to the tensor product $M_k(\mathbb{F}) \otimes \mathbb{F}^\sigma[H] \cong M_k(\mathbb{F}^\sigma[H])$, that is, if and only if $\mathfrak{A}$ is a matrix algebra over the division graded algebra*

$\mathbb{F}^\sigma[H]$, where H is a finite subgroup of G and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. The G-grading on $M_k(\mathbb{F}^\sigma[H])$ is a canonical elementary grading corresponding to a k-tuple $(g_1, \ldots, g_k) \in \mathsf{G}^k$.

Theorem 1.2.20 (Lemma 2, [45]) *Let G be a finite abelian group, and $\mathbb{F}$ an algebraically closed field of characteristic zero. Any finite dimensional G-graded $\mathbb{F}$-algebra $\mathfrak{A}$ is isomorphic as G-graded algebra to a G-graded $\mathbb{F}$-algebra of the form*

$$\mathfrak{A}' = (M_{k_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{k_p}(\mathbb{F}^{\sigma_p}[H_p])) \oplus \mathsf{J} \ .$$

Here the Jacobson radical $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ of $\mathfrak{A}$ is a graded ideal, and $\mathfrak{B} = M_{k_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{k_p}(\mathbb{F}^{\sigma_p}[H_p])$ (direct product of algebras) is the maximal graded semisimple subalgebra of A', $p \in \mathbb{N} \cup \{0\}$. The G-grading on $\mathfrak{B}_l = M_{k_l}(\mathbb{F}^{\sigma_l}[H_l]) \simeq M_{k_l}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}^{\sigma_l}[H_l]$, the algebra of $n \times n$ matrices over $\mathbb{F}^{\sigma_l}[H_l]$, where H_l is a subgroup G and $\sigma \in \mathsf{Z}^2(H_l, \mathbb{F}^*)$ is a 2-cocycle, is the canonical elementary grading corresponding to some k_l-tuple $(\theta_{l_1}, \ldots, \theta_{l_{k_l}}) \in \mathsf{G}^{k_l}$.

1.3 Properties of graded $\mathfrak{A}$-bimodules

In this section, unless otherwise stated, we denote by G a group (not necessarily finite), by $\mathfrak{A}$ a G-graded unitary algebra and by M a G-graded unitary $\mathfrak{A}$-bimodule.

1.3.1 Posets and chain conditions

Consider a nonempty set P together with a binary relation "$\leqslant$" that satisfies the following axioms: for any $a, b, c \in P$,

i) $a \leqslant a$;

ii) if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$;

iii) if $a \leqslant b$ and $b \leqslant a$, then $a = b$.

The pair $(P, \leqslant)$ is called a **partially ordered set** (also called **poset**). It is important to say that given $a, b \in P$, we do not necessarily have $a \leqslant b$ or $b \leqslant a$. For more details, see [20, 26].

Example 1.3.1 *Let X be a nonempty set and $\mathcal{P}(X) = \{\beta : \beta \text{ is a subset of } X\}$. We have that $(\mathcal{P}(X), \subseteq)$ is a poset. We say that X has the ordering by inclusion.*

Example 1.3.2 *Let* G *be a group and* $\mathbb{F}$ *a field. For each subgroup* $H \leqslant \mathsf{G}$*, consider* $\mathsf{H}^2(H, \mathbb{F}^*) = \{[\sigma] : \sigma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)\}$*, where* $[\sigma] = [\rho]$ *iff there is a 2-coboundary* $\theta \in \mathsf{B}^2(H, \mathbb{F}^*)$ *such that* $\sigma = \rho\theta$ *(see §1.2.1). Define* $\mathcal{H} = \{(H, [\sigma]) : H \leqslant \mathsf{G}, [\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)\}$*. In* $\mathcal{H}$*, define the relation "*$\preceq$*" by the following: given two pairs* $(H_1, [\sigma_1])$ *and* $(H_2, [\sigma_2])$ *in* $\mathcal{H}$*, we have* $(H_1, [\sigma_1]) \preceq (H_2, [\sigma_2])$ *iff* $H_1 \subseteq H_2$ *and* $[\sigma_1](h, g) = [\sigma_2](h, g)$ *for any* $h, g \in H_1$*. It is not difficult to see that* $\preceq$ *is a partial order relation of the elements of* $\mathcal{H}$*. Therefore,* $(\mathcal{H}, \preceq)$ *is a poset. Hence, we say that* G *is ordering by* $\preceq$*.*

Particularly, $(H, [\sigma]) = (\tilde{H}, [\tilde{\sigma}])$ *iff* $(H, [\sigma]) \preceq (\tilde{H}, [\tilde{\sigma}])$ *and* $(\tilde{H}, [\tilde{\sigma}]) \preceq (H, [\sigma])$*, i.e.* $H = \tilde{H}$ *and* $[\sigma] = [\tilde{\sigma}]$*.*

Let X be a poset (possibly by inclusion). If there is some element $a \in X$ such that $a \leqslant x$ for any $x \in X$, then we say that X has a **least element**. Similarly, if there is some element $b \in X$ such that $x \leqslant b$ for any $x \in X$, then we say that X has a **greatest element**. Notice that the least and the greatest elements, when they exist, are unique.

Now, let us define *minimal and maximal elements* of a poset. Let X be a poset (possibly by inclusion). An element $a \in X$ is called a **minimal element** (resp. a **maximal element**) if $x \leqslant a$ (resp. $a \leqslant x$) in X implies $x = a$ (resp. $a = x$). Note that minimal and maximal elements are not necessarily unique. For more details, see [20].

Consider a subset X' of a poset X. An element $a \in X$ is called a **lower bound** (resp. **upper bound**) of X' in X if $a \leqslant x'$ (resp. $x' \leqslant a$) for any $x' \in X'$. Observe that theleast (resp. greatest) element of X is a lower bound (resp. an upper bound) of X' in X.

Given a poset X, a **chain** in X is a family X' of elements of X such that $a \leqslant b$ or $b \leqslant a$ for any $a, b \in X'$.

Lemma 1.3.3 (Zorn's Lemma, [20, 25]) *If* X *is a partially ordered set such that every chain in* X *has an upper bound in* X*, then* X *contains a maximal element.*

Given a poset X, a chain ξ in X is called **ascending chain** if it can be written as $\xi = \{x_1 \leqslant x_2 \leqslant x_3 \leqslant x_4 \leqslant \cdots\}$. Analogously, a chain ζ in X is called **descending chain** if it can be written as $\zeta = \{y_1 \geqslant y_2 \geqslant y_3 \geqslant y_4 \geqslant \cdots\}$, where $a \geqslant b$ means $b \leqslant a$.

Definition 1.3.4 *Let* X *be a partially ordered set. If all ascending (resp. descending) chain in* X *contains the greatest (resp. the least) element, then we say that* X *satisfies* *the* **ascending chain condition** *(resp.* **descending chain condition***). In this case, we say also that X satisfies the ACC (resp. DCC).*

Equivalently, a partially ordered set $(P, \leqslant)$ satisfies ACC (resp. DCC) iff every non-empty family S of P contains a maximal (resp. a minimal) element in the family, that is, an element $Q \in S$ such that if $N \in S$ and $Q \leqslant N$ (resp. $N \leqslant Q$), then $N = Q$.

1.3.2 Group Characters

In this subsection, let us present some definitions and properties of **Group Characters**. Here, let us denote by G an arbitrary finite multiplicative group with identity element 1, $\mathbb{F}$ a field, and $GL_n(\mathbb{F})$ the group of invertible $n \times n$ matrices over $\mathbb{F}$.

Definition 1.3.5 *A* **matrix representation** *of* G *over* $\mathbb{F}$ *of degree* n *is a homomorphism* $T : g \mapsto T(g)$ *of* G *into* $GL_n(\mathbb{F})$*. Two matrix representations* T *and* T' *are* **equivalent** *if they have the same degree, say* n*, and if there exists a fixed* S *in* $GL_n(\mathbb{F})$ *such that*

$$T'(g) = ST(g)S^{-1} \ ,$$

for any $g \in \mathsf{G}$.

To simply notation, we say only "*a representation of* G *of degree* n" to mean "*a matrix representation of* G *of degree* n".

Let $T : \mathsf{G} \to GL_n(\mathbb{F})$ be a representation of a group G. We say that T is **reducible** if there exist representations $T_1 : \mathsf{G} \to GL_{n_1}(\mathbb{F})$, $T_2 : \mathsf{G} \to GL_{n_2}(\mathbb{F})$ of G, with $n = n_1 + n_2$, such that

$$T(g) \text{ and } \begin{pmatrix} T_1(g) & V(g) \\ 0 & T_2(g) \end{pmatrix} \text{ are equivalent },$$

for any $g \in \mathsf{G}$, where $V(g)$ is a matrix over $\mathbb{F}$ of order $n_1 \times n_2$ for each $g \in \mathsf{G}$; if no such reduction exists, then T is an **irreducible representation**. We say that T is **completely reducible** if for any $g \in \mathsf{G}$, the matrix $T(g)$ is equivalent to matrix .

$$\begin{pmatrix} T_1(g) & & 0 \\ & \ddots & \\ 0 & & T_q(g) \end{pmatrix},$$

for some irreducible representations $T_i : \mathsf{G} \to GL_{n_i}(\mathbb{F})$ of G, with $n = n_1 + \cdots + n_q$. For more details, see [9, 27, 41].

Theorem 1.3.6 (Maschke's Theorem, (10.8), p. 41, [9]) *Let $\mathbb{F}$ be a field, G a finite group, and $T : G \to GL_n(\mathbb{F})$ be a representation of G. Assume that $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$. Then T is completely reducible.*

By the previous theorem, given a representation T of a finite group G, when $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$, there exist irreducible representations $T_1, \ldots, T_n$ (not necessarily non-equivalent) of G such that $T = T_1 + \cdots + T_n$.

Theorem 1.3.7 ((27.22), p. 187, [9]) *Let G be a finite group, and $\mathbb{F}$ an algebraically closed field such that $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$. Then the number of non-isomorphic irreducible representations of G is the same as the number of conjugate classes of G.*

Observe that if G is a abelian finite group, then the number of conjugate class of G is equal to $|\mathsf{G}|$.

Consider the $n \times n$ matrices algebra $M_n(\mathbb{F})$. The **trace function** is the linear transformation $\mathsf{tr} : M_n(\mathbb{F}) \to \mathbb{F}$ that satisfies $\mathsf{tr}((a_{ij})_{i,j}) = \sum_{i=1}^{n} a_{ii}$ for any $(a_{ij})_{i,j} \in M_n(\mathbb{F})$, i.e. the trace of $A \in M_n(\mathbb{F})$ is the sum of the elements on the main diagonal of A.

Let $T : \mathsf{G} \to GL_n(\mathbb{F})$ be a representation of G, and $T(g) = (a_{ij}^{(g)})_{i,j} \in GL_n(\mathbb{F})$ for any $g \in \mathsf{G}$. For each $g \in \mathsf{G}$, the **trace** of $T(g)$ is defined by $\mathsf{tr}(T(g)) := \mathsf{tr}((a_{ij}^{(g)})_{i,j})$. Observe that the degree of T is equal to $\mathsf{tr}(T(1)) = \sum_{i=1}^{n} 1_{\mathbb{F}} = n$.

Definition 1.3.8 *Let G be a group, and $T : \mathsf{G} \to GL_n(\mathbb{F})$ a representation of G. The* **character** *of T is a map $\zeta : \mathsf{G} \to \mathbb{F}^*$ defined by $\zeta(g) = \mathsf{tr}(T(g))$ for any $g \in \mathsf{G}$.*

Proposition 1.3.9 ((30.14), p. 214, [9]) *Let T and T_1 be two representations of G over $\mathbb{F}$ with characters ζ and ζ_1 of T and T_1, respectively. If $\mathsf{char}(\mathbb{F}) = 0$, then T and T_1 are equivalent iff $\zeta = \zeta_1$.*

Corollary 1.3.10 *Under the same hypotheses of Theorem 1.3.6, suppose that $\mathsf{char}(\mathbb{F}) = 0$. Then the number of distinct irreducible characters of G is finite.*

Proof: It is immediate of Theorem 1.3.6 and Proposition 1.3.9. ■

The next three propositions are known as **orthogonality relations**.

Proposition 1.3.11 ((31.8), p. 219, [9]) *Let $\mathbb{F}$ be an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$, G *a finite group, and ζ and μ two non-equivalent characters of* G. *Then ζ is irreducible iff*

$$\sum_{g\in\mathsf{G}} \zeta(g)\mu(g^{-1}) = 0 \ .$$

Proposition 1.3.12 ((31.14), p. 221, [9]) *Let $\mathbb{F}$ be an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$, G *a finite group, and $\zeta_1, \ldots, \zeta_s$ all the distinct characters of* G. *Then*

$$\sum_{i=1}^{s} (\zeta_i(1))^2 = |\mathsf{G}| \ .$$

Proposition 1.3.13 ((31.15), p. 221, [9]) *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero,* G *a finite group, and ζ a character of* G. *Then ζ is irreducible iff*

$$\sum_{g\in\mathsf{G}} \zeta(g)\zeta(g^{-1}) = |\mathsf{G}| \ .$$

The next result is a consequence of Theorem 1.3.7 and Proposition 1.3.12.

Theorem 1.3.14 (Theorem 9, [41]) *Let* G *be a finite group, and $\mathbb{F}$ an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) \nmid |\mathsf{G}|$. *The following properties are equivalent:*

i) G *is abelian;*

ii) all the irreducible representations of G *have degree* 1*;*

iii) all the irreducible characters of G *have degree* 1.

A consequence of previous theorem is that if G is finite abelian, then any irreducible character of G is a homomorphism of groups, i.e. if $\zeta : \mathsf{G} \to \mathbb{F}$ is an irreducible character of G, then $\zeta(gh) = \zeta(g)\zeta(h)$ for any $g, h \in \mathsf{G}$.

The proposition below is a fact well-known of theory of group character.

Proposition 1.3.15 (Exercise 3.3, p. 26, [41]) *Let* G *be a finite group, and $\mathbb{F}$ an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) = 0$. *Let* $\hat{\mathsf{G}}$ *be the set of irreducible characters of* G. *We have that the groups* G *and* $\hat{\mathsf{G}}$ *are isomorphic.*

The group $\hat{\mathsf{G}}$ is called the **dual** of the group G.

1.3.3 Some results on graded $\mathfrak{A}$-bimodules

In this subsection, (graded) bimodules are (graded) $\mathfrak{A}$-bimodules.

Definition 1.3.16 *Let* G *be a group,* $\mathfrak{A}$ *a* G*-graded algebra and* M *a* G*-graded* $\mathfrak{A}$*-bimodule. We say that* M *is* **G-Noetherian** *(resp.* **G-Artinian***) if it satisfies the ascending (resp. descending) chain condition for graded subbimodules.*

Equivalently, a G-graded $\mathfrak{A}$-bimodule M is G-Noetherian (resp. G-Artinian) iff every non-empty family S of graded subbimodules of M contains a maximal (resp. a minimal) graded subbimodule in the family, that is, a graded subbimodule $P \in S$ such that if $N \in S$ and $N \supset P$ (resp. $N \subset P$), then $N = P$. In particular, any G-Noetherian (resp. G-Artinian) not irreducible G-graded $\mathfrak{A}$-bimodule has some maximal (resp. minimal) proper graded subbimodule. An example of a bimodule that satisfies both chain conditions is given by a finite dimensional G-graded $\mathfrak{A}$-bimodule.

Proposition 1.3.17 *Let* G *be a group,* $\mathfrak{A}$ *a* G*-graded algebra and* M *a* G*-graded* $\mathfrak{A}$*-bimodule. Let* N *be a graded subbimodule of* M*. Then* M *is* G*-Artinian (resp.* G*-Noetherian) iff* N *and* M/N *are* G*-Artinian (resp.* G*-Noetherian).*

Proof: Suppose M is G-Artinian (resp. G-Noetherian). Let N be a G-graded subbimodule of M. Since each G-graded subbimodule of N is still a G-graded subbimodule of M, we conclude that N is a G-Artinian (resp. G-Noetherian). Moreover, by Theorem 1.1.25, each G-graded subbimodule of M/N is of the form W/N for some G-graded subbimodule W such that $\mathsf{N} \subset \mathsf{W} \subset \mathsf{M}$. Therefore, M/N is G-Artinian (resp. G-Noetherian).

Conversely, suppose N and M/N are G-Artinian $\mathfrak{A}$-bimodules, and consider a descending chain of G-graded subbimodules of M

$$\mathsf{M}_1 \supseteq \mathsf{M}_2 \supseteq \mathsf{M}_3 \supseteq \cdots . \tag{1.2}$$

Now, consider the chain of G-graded submodules of N given by

$$(\mathsf{M}_1 \cap \mathsf{N}) \supseteq (\mathsf{M}_2 \cap \mathsf{N}) \supseteq (\mathsf{M}_3 \cap \mathsf{N}) \supseteq \cdots . \tag{1.3}$$

Since N is G-Artinian, it follows that there exists $i_0 \in \mathbb{N}$ such that $\mathsf{M}_{i_0} \cap \mathsf{N} = \mathsf{M}_j \cap \mathsf{N}$ for any $j \geqslant i_0$. Thus, by (1.3), now consider the chain of G-graded subbimodules of M/N

given bys
$$\frac{\mathsf{M}_{i_0}}{\mathsf{M}_{i_0}\cap\mathsf{N}} \supseteq \frac{\mathsf{M}_{i_0+1}}{\mathsf{M}_{i_0+1}\cap\mathsf{N}} \supseteq \frac{\mathsf{M}_{i_0+2}}{\mathsf{M}_{i_0+2}\cap\mathsf{N}} \supseteq \cdots , \tag{1.4}$$
Since M/N is G-Artinian, we conclude from (1.4) that there exists $j_0 \in \mathbb{N}$, $j_0 \geqslant i_0$, such that
$$\frac{\mathsf{M}_{j_0}}{\mathsf{M}_{i_0}\cap\mathsf{N}} = \frac{\mathsf{M}_{j_0}}{\mathsf{M}_{j_0}\cap\mathsf{N}} = \frac{\mathsf{M}_{l}}{\mathsf{M}_{l}\cap\mathsf{N}} = \frac{\mathsf{M}_{l}}{\mathsf{M}_{i_0}\cap\mathsf{N}} \tag{1.5}$$
for any $l \geqslant j_0 \geqslant i_0$. Hence, again by Theorem 1.1.25, we can conclude from (1.5) that $\mathsf{M}_{j_0} = \mathsf{M}_l$ for any $l \geqslant j_0$. Therefore, the chain in (1.2) stabilizes. Thus M is a G-Artinian $\mathfrak{A}$-bimodule.

Similarly to first part, we can show that if N and M/N are G-Noetherian, then M is G-Noetherian. ■

Using the ideas of the previous proposition and Theorem 1.1.25, we can build a descending chain or an ascending chain of graded subbimodules with a suitable property. In fact, consider a G-graded $\mathfrak{A}$-bimodule M. Suppose M is G-Noetherian (resp. G-Artinian). By previous proposition, given a graded subbimodule N of M, we have that N and M/N are G-Noetherian (resp. G-Artinian). If M is not graded irreducible, then there exists a G-graded maximal (resp. irreducible) subbimodule N_1 in M. If N_1 (resp. M/N_1) is irreducible, then we have $\mathsf{M} = \mathsf{N}_0 \supsetneq \mathsf{N}_1 \supsetneq \mathsf{N}_2 = \{0\}$ (resp. $\{0\} = \mathsf{N}_0 \subsetneq \mathsf{N}_1 \subsetneq \mathsf{N}_2 = \mathsf{M}$) such that N_{i+1} is maximal in N_i (resp. $\mathsf{N}_{i+1}/\mathsf{N}_i$ is irreducible), for $i = 0, 1$. Otherwise, suppose that N_1 (resp. M/N_1) is not irreducible, and hence, there exists a nonzero graded subbimodule N_2 of M such that $\mathsf{N}_2 \subsetneq \mathsf{N}_1$ is maximal (resp. $\mathsf{N}_1 \subsetneq \mathsf{N}_2$ and $\mathsf{N}_2/\mathsf{N}_1$ is irreducible). So, we obtain the chain $\mathsf{M} = \mathsf{N}_0 \supsetneq \mathsf{N}_1 \supsetneq \mathsf{N}_2 \supsetneq \mathsf{N}_3 = \{0\}$ (resp. $\{0\} = \mathsf{N}_0 \subsetneq \mathsf{N}_1 \subsetneq \mathsf{N}_2 \subsetneq \mathsf{N}_3 = \mathsf{M}$). If N_2 (resp. M/N_2) is irreducible, it follows that N_{i+1} is maximal in N_i (resp. $\mathsf{N}_{i+1}/\mathsf{N}_i$ is irreducible), for $i = 0, 1, 2$. Otherwise, using this process inductively we must obtain a descending chain (resp. an ascending chain) of graded subbimodules

$$\mathsf{M} = \mathsf{N}_0 \supsetneq \mathsf{N}_1 \supsetneq \mathsf{N}_2 \supsetneq \cdots \supsetneq \{0\} \quad (\text{resp. } \{0\} = \mathsf{N}_0 \subsetneq \mathsf{N}_1 \subsetneq \mathsf{N}_2 \subsetneq \cdots \subsetneq \mathsf{M}) , \tag{1.6}$$

such that N_{i+1} is maximal in N_i (resp. $\mathsf{N}_{i+1}/\mathsf{N}_i$ is irreducible), for $i = 0, 1, 2, \ldots$. Notice that we use Theorem 1.1.25 to ensure that if M/N_i is not irreducible, then there exists a graded subbimodule $\mathsf{N}_{i+1} \supsetneq \mathsf{N}_i$ such that $\mathsf{N}_{i+1}/\mathsf{N}_i$ is irreducible.

Now, again by Proposition 1.3.17, it is easy to show that the finite direct sum (internal or external) of G-Noetherian (resp. G-Artinian) $\mathfrak{A}$-bimodules is a G-Noetherian (resp. G-Artinian) $\mathfrak{A}$-bimodule.

Notice that given a homogeneous homomorphism of graded $\mathfrak{A}$-bimodules $\varphi : \mathsf{M}_1 \to \mathsf{M}_2$, if M_1 is G-Noetherian (resp. G-Artinian), then $\mathsf{im}(\varphi)$ and $\mathsf{ker}(\varphi)$ are G-Noetherian (resp. G-Artinian) left $\mathfrak{A}$-subbimodules of M_2 and M_1, respectively.

Recall any G-graded $\mathfrak{A}$-bimodule can be generated by homogeneous elements.

Proposition 1.3.18 *Let G be a group, $\mathfrak{A}$ a finite dimensional unitary algebra with a G-grading, and M a G-graded unitary $\mathfrak{A}$-bimodule. If M is finitely generated as an $\mathfrak{A}$-bimodule, then M is G-Noetherian and G-Artinian.*

Proof: Let $m_1, \dots, m_n \in \mathsf{M}$ be homogeneous elements such that

$$\mathsf{M} = \left\{ \sum_{k=1}^{n} r_k m_k s_k : r_i, s_i \in \mathfrak{A} \right\} = \sum_{k=1}^{n} \mathfrak{A} m_k \mathfrak{A} \ .$$

Since $\mathfrak{A}$ is finite dimensional, we can take homogeneous elements $a_1, a_2, \dots, a_m \in \mathfrak{A}$ such that $\mathfrak{A} = \mathsf{span}_{\mathbb{F}}\{a_1, a_2, \dots, a_m\}$. Hence, we have that

$$\begin{aligned} \mathsf{M} &= \left\{ \sum_{k=1}^{n} \left(\sum_{i,j=1}^{m} \lambda_{k,i,j} a_i m_k a_j \right) : \lambda_{k,i,j} \in \mathbb{F} \right\} \\ &= \mathsf{span}_{\mathbb{F}}\{a_i m_k a_j : i, j = 1, \dots, m, k = 1, \dots, n\} \ . \end{aligned}$$

Hence, the $a_i m_k a_j$'s are homogeneous elements which generate M as an $\mathbb{F}$-vector space.

Observe that $\mathsf{dim}_{\mathbb{F}}(\mathsf{M}) \leqslant \#\{a_i m_k a_j : i, j = 1, \dots, m, k = 1, \dots, n\} \leqslant m^2 n$, and thus M is finite dimensional. Therefore, M satisfies both chain conditions for graded subbimodules. ■

It is important to note that Proposition 1.3.18 also is true for G-graded non-unitary $\mathfrak{A}$-bimodules. The proof of this is similar to the proof of Proposition 1.3.18.

Corollary 1.3.19 *Let G be a group, $\mathbb{F}$ a field, $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ the algebra of $n \times n$ matrices over $\mathbb{F}^\sigma[H]$ with a canonical elementary G-grading, where H is a finite subgroup of G and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. Any G-graded $\mathfrak{B}$-bimodule M which is finitely generated (as a $\mathfrak{B}$-bimodule) is G-Noetherian and G-Artinian.*

Proof: We have that $\mathsf{dim}(\mathfrak{B}) = n^2|H| < \infty$. From this, by Proposition 1.3.18, the result is immediate. ∎

In what follows, let us exhibit elements in a unitary G-graded $\mathfrak{B}$-bimodule M, where $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$, whose product with (homogeneous) elements of $\mathfrak{B}$ is similar to the product in $\mathfrak{B}$.

Consider a unitary G-graded $\mathfrak{B}$-bimodule M, where G is a group, H is a finite subgroup of G, $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ with the canonical elementary G-grading defined bay an n-tuple $(g_1, \dots, g_n) \in \mathsf{G}^n$. Recall that $\mathfrak{B}$ is unitary with unity $1_{\mathfrak{B}} = \sigma(e,e)^{-1}\sum_{i=1}^n E_{ii}\eta_e$, and the set $\{E_{ij}\eta_g \in \mathfrak{B} : i, j = 1, dots, n, h \in H\}$ is a homogeneous basis of $\mathfrak{B}$. Fixed a nonzero homogeneous element $w_0 \in \mathsf{M}$. Since M is unitary, it follows that $\eta_e E_{i_0 i_0} w_0 E_{j_0 j_0}\eta_e \neq 0$ for some $i_0, j_0 \in \{1, \dots, n\}$, and hence, $\eta_g E_{r i_0} w_0 E_{j_0 s}\eta_h \neq 0$ for any $r, s \in \{1, \dots, n\}$ and $g, h \in H$. Observe that all elements $\eta_g E_{r i_0} w_0 E_{j_0 s}\eta_h$'s are homogeneous. Without lost of the generality, we can consider the element $\eta_e E_{11} w_0 E_{11}\eta_e \neq 0$ instead of w_0. Given any $g \in H$ and $i, j \in \{1, \dots, n\}$, define the element

$$m_{ij}^g := \sum_{h \in H} \sigma(h, h^{-1}g)^{-1} E_{i1}\eta_h w_0 E_{1j}\eta_{h^{-1}g} \ ,$$

where each $\eta_g E_{r1} w_0 E_{1s}\eta_h \neq 0$ for any $r, s \in \{1, \dots, n\}$ and $g, h \in H$.

When $m_{ij}^g \neq 0$ for some $g \in H$ and $i, j \in \{1, \dots, n\}$, observe that m_{ij}^g has behaviour similar to $E_{ij}\eta_g$ in relation to the product by elements of $\mathfrak{B}$, i.e. for any $r, s \in \{1, \dots, n\}$ and $h \in H$, we have

$$\begin{aligned} E_{rs}\eta_h m_{ij}^g &= \delta_{si}\sigma(h, g) m_{rj}^{hg} \ , \quad \text{and} \\ m_{ij}^g E_{rs}\eta_h &= \delta_{jr}\sigma(g, h) m_{is}^{gh} \ , \end{aligned} \tag{1.7}$$

where $\delta_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases}$ is the Kronecker delta. In fact, take any $t \in H$ and $r, s \in \{1, \dots, n\}$. It is obvious that $E_{rs}\eta_h m_{ij}^g = 0$ when $s \neq i$, and $m_{ij}^g E_{rs}\eta_h = 0$ when $r \neq j$.

Now, if $s = i$, we have

$$\begin{aligned}
E_{ri}\eta_t m_{ij}^g &= E_{ri}\eta_t \left(\sum_{h\in H} \sigma(h, h^{-1}g)^{-1} E_{i1}\eta_h w_0 E_{1j}\eta_{h^{-1}g} \right) \\
&= \sum_{h\in H} \sigma(t,h)\sigma(h,h^{-1}g)^{-1} E_{r1}\eta_{th} w_0 E_{1j}\eta_{h^{-1}g} \\
&= \sum_{h\in H} \sigma(t,h)\sigma(h,(th)^{-1}tg)^{-1} E_{r1}\eta_{th} w_0 E_{1j}\eta_{(th)^{-1}tg} \\
&= \sigma(t,g) \left(\sum_{h\in H} \sigma(th,(th)^{-1}tg)^{-1} E_{r1}\eta_{th} w_0 E_{1j}\eta_{(th)^{-1}tg} \right) \\
&= \sigma(t,g)\ m_{rj}^{tg}\ ,
\end{aligned}$$

since $\sigma(t,h)\sigma(th,(th)^{-1}tg) = \sigma(t,g)\sigma(h,(th)^{-1}tg)$ for any $h, g, t \in H$ (see Definition 1.2.1). And if $r = j$, we have

$$\begin{aligned}
m_{ij}^g E_{js}\eta_t &= \left(\sum_{h\in H} \sigma(h, h^{-1}g)^{-1} E_{i1}\eta_h w_0 E_{1j}\eta_{h^{-1}g} \right) E_{js}\eta_t \\
&= \sum_{h\in H} \sigma(h^{-1}g,t)\sigma(h,h^{-1}g)^{-1} E_{i1}\eta_h w_0 E_{1s}\eta_{h^{-1}gt} \\
&= \sigma(g,t) \left(\sum_{h\in H} \sigma(h,h^{-1}gt)^{-1} E_{i1}\eta_h w_0 E_{1s}\eta_{h^{-1}gt} \right) \\
&= \sigma(g,t)\ m_{is}^{gt}\ ,
\end{aligned}$$

since $\sigma(h,h^{-1}g)\sigma(g,t) = \sigma(h,h^{-1}gt)\sigma(h^{-1}g,t)$ for any $h, g, t \in H$ (see Definition 1.2.1).

Another peculiarity of m_{ij}^g's is that if $m_{ij}^g \neq 0$ for some $i, j \in \{1,\dots,n\}$ and $g \in H$, then $m_{lj}^{hg} = \sigma(h,g)^{-1}E_{li}\eta_h m_{ij}^g \neq 0$, and $m_{il}^{gh} = \sigma(g,h)^{-1} m_{ij}^g E_{jl}\eta_h \neq 0$ for any $h \in H$ and $l \in \{1,\dots,n\}$, since $m_{ij}^g = (\sigma(e,e)^{-1}\sigma(h^{-1},h)^{-1}E_{il}\eta_{h^{-1}}E_{li}\eta_h)m_{ij}^g = (\sigma(e,e)^{-1}\sigma(h^{-1},h)^{-1}E_{il}\eta_{h^{-1}})(E_{li}\eta_h m_{ij}^g)$, and $m_{ij}^g = m_{ij}^g(\sigma(e,e)^{-1}\sigma(h,h^{-1})E_{jl}\eta_h E_{lj}\eta_{h^{-1}}) = (m_{ij}^g E_{jl}\eta_h)(\sigma(e,e)^{-1}\sigma(h,h^{-1})E_{lj}\eta_{h^{-1}})$. From this, we can deduce that $m_{ij}^g \neq 0$ for some $i, j \in \{1,\dots,n\}$ and $g \in H$ iff $m_{rs}^h \neq 0$ for any $r, s \in \{1,\dots,n\}$ and $h \in H$.

Besides that, it is easy to prove that the element $\mathfrak{i} := \sigma(e,e)^{-1}\sum_{i=1}^n m_{ii}^e$ satisfies $b\mathfrak{i} = \mathfrak{i}b$ for any $b \in \mathfrak{B}$.

Notice that when either $\mathsf{deg}(w_0) \in \mathcal{Z}(\mathsf{G})$ or $H \subset \mathcal{Z}(\mathsf{G})$, we have m_{ij}^g is a homogeneous element of M for any $g \in H$, and $i, j \in \{1,\dots,n\}$. Particularly, when $\mathsf{deg}(w_0) \in \mathcal{Z}(\mathsf{G})$, it follows that $m_{ij}^g \in \mathsf{M}_{g_i^{-1}gg_j\mathsf{deg}(w_0)} = \mathsf{M}_{\mathsf{deg}(w_0)g_i^{-1}gg_j}$ for any $g \in H$, $i, j = 1,\dots,n$.

By these observations, it follows that the linear transformation $\psi : \mathfrak{B} \to \mathsf{M}$ which extends the map $E_{ij}\eta_g \mapsto m_{ij}^g$ is a homogeneous homomorphism of $\mathfrak{B}$-bimodules of degree $\mathsf{deg}(w_0)$ when $\mathsf{deg}(w_0) \in \mathcal{Z}(\mathsf{G})$.

Remark 1.3.20 *Let* G *be a group,* H *a finite abelian subgroup of* G*,* $\mathbb{F}$ *a field,* $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$*, and* $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ *with a canonical elementary* G*-grading. Let* M *be a unitary* G*-graded* $\mathfrak{B}$*-bimodule. Fix a nonzero homogeneous element* $m_0 \in \mathsf{M}$*, and define*

$$m_{ij}^g = \sum_{h \in H} \sigma(h, h^{-1}g)^{-1} E_{i1}\eta_h m_0 E_{1j}\eta_{h^{-1}g} \ ,$$

for any $g \in H$*, and* $i, j = 1, \ldots, n$*. Recall that* $E_{rs}\eta_h m_{ij}^g = \delta_{si}\sigma(h, g) m_{rj}^{hg}$ *and* $m_{ij}^g E_{rs}\eta_h = \delta_{jr}\sigma(g, h) m_{is}^{gh}$*, for any* $h, g \in H$ *and* $i, j, r, s \in \{1 \ldots, n\}$*, where* δ_{ij} *is the Kronecker delta.*

Suppose that $m_{ij}^g \neq 0$ *for some* $g \in H$ *and* $i, j \in \{1, \ldots, n\}$*, and thus,* $m_{rs}^h \neq 0$ *for any* $h \in H$ *and* $r, s \in \{1, \ldots, n\}$*. Consider* $N = \mathsf{span}_{\mathbb{F}}\{m_{ij}^g : g \in H, i, j = 1, \ldots, n\}$*. If* $\mathsf{deg}(m_0) \in \mathcal{Z}(\mathsf{G})$*, then we have that* N *is an irreducible* G*-graded* $\mathfrak{B}$*-subbimodule of* M*.*

Indeed, consider the linear transformation $\psi : \mathfrak{B} \longrightarrow N$ *which extends the map* $E_{ij}\eta_g \mapsto m_{ij}^g$*. By (1.7), it follows that* ψ *is a homomorphism of* $\mathfrak{B}$*-bimodules. Notice that* ψ *is surjective, and it is homogeneous of degree* $\mathsf{deg}(m_0)$*. Since* $\mathfrak{B}$ *is an irreducible graded* $\mathfrak{B}$*-bimodule (and also a* G*-graded simple algebra), it follows that* ψ *is injective (*$\mathsf{ker}(\psi)$ *is a graded subbimodule of* $\mathfrak{B}$*). Thus,* ψ *is bijective. Therefore,* N *is an irreducible graded* $\mathfrak{B}$*-bimodule, and* ψ *is a homogeneous isomorphism of* G*-graded* $\mathfrak{B}$*-bimodules. In particular, if* M *is irreducible graded, then* $\mathsf{M} = N$*.*

In the above remark, it is important to comment that not always $m_{ij}^g \neq 0$ for some $g \in H$ and $i, j \in \{1, \ldots, n\}$. Let us present below other cases of irreducible G-graded $\mathfrak{B}$-bimodules (possibly when $m_{ij}^g = 0$ for any $g \in H$ and $i, j \in \{1, \ldots, n\}$), which are not isomorphic to $\mathfrak{B}$ as graded $\mathfrak{B}$-bimodules.

Remark 1.3.21 *Let* G *be a finite abelian group,* $\mathbb{F}$ *an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) = 0$*, and* $\mathfrak{B} = \mathbb{F}^\sigma[\mathsf{G}]$ *a twisted group algebra. Let* M *be a unitary* G*-graded* $\mathfrak{B}$*-bimodule. Let* $\chi_1, \ldots, \chi_s$ *be all distinct irreducible characters of* G *(see the last subsection). Since* G *and* $\hat{\mathsf{G}} = \{\chi_1, \ldots, \chi_s\}$ *are isomorphic groups (see Proposition 1.3.15), we have* *that $s = |\mathsf{G}|$. Fix a nonzero homogeneous element $m_0 \in \mathsf{M}$, and define*

$$w_{\chi_i} = \sum_{h\in\mathsf{G}} \sigma(h,h^{-1})^{-1}\chi_i(h)\eta_h m_0 \eta_{h^{-1}} \ ,$$

for all $i \in \{1,\ldots,s\}$. Let us show that $\mathfrak{B}w_{\chi_i} = w_{\chi_i}\mathfrak{B}$ and $\mathfrak{B}m_0\mathfrak{B} = \sum_{i=1}^{s}\mathfrak{B}w_{\chi_i}$. Observe that each $\mathfrak{B}w_{\chi_i}\mathfrak{B}$ is a graded subbimodule of M, since m_0 is a homogeneous element, and G is abelian.

Take any $\eta_t \in \mathfrak{B}$, and $i = 1,\ldots,s$. We have that

$$\begin{aligned}
\eta_t w_{\chi_i} &= \eta_t \left(\sum_{h\in\mathsf{G}} \sigma(h,h^{-1})^{-1}\chi_i(h)\eta_h m_0\eta_{h^{-1}}\right) = \sum_{h\in\mathsf{G}} \sigma(t,h)\sigma(h,h^{-1})^{-1}\chi_i(h)\eta_{th}w_0\eta_{h^{-1}} \\
&= \sum_{h\in\mathsf{G}} \sigma(t,h)\sigma(h,h^{-1})^{-1}\sigma((th)^{-1},t)^{-1}\chi_i(h)\eta_{th}w_0\eta_{(th)^{-1}}\eta_t \\
&= \left(\sum_{h\in\mathsf{G}} \sigma((th)^{-1},th)^{-1}\chi_i(h)\eta_{th}w_0\eta_{(th)^{-1}}\right)\eta_t = \chi_i(t)^{-1}w_{\chi_i}\eta_t \ ,
\end{aligned}$$

since $\sigma(t,h)\sigma((th)^{-1},th)^{-1} = \sigma(h,h^{-1})\sigma((th)^{-1},t)$ for any $h,t \in \mathsf{G}$ (see Definition 1.2.1), and $\chi_i(ht) = \chi_i(h)\chi_i(t)$ for any $h,t \in \mathsf{G}$ (see Theorem 1.3.14). From this, it follows that $\mathfrak{B}w_{\chi_i} = w_{\chi_i}\mathfrak{B}$ for alli $= 1,\ldots,s$.

Now, write $\mathsf{G} = \{g_1,\ldots,g_s\}$. Since the matrix

$$[\chi] := \begin{pmatrix} \chi_1(g_1) & \chi_1(g_2) & \cdots & \chi_1(g_s) \\ \chi_2(g_1) & \chi_2(g_2) & \cdots & \chi_2(g_s) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_s(g_1) & \chi_s(g_2) & \cdots & \chi_s(g_s) \end{pmatrix}$$

is invertible, with inverse matrix given by

$$[\chi]^{-1} = |\mathsf{G}|^{-1} \begin{pmatrix} \chi_1(g_1^{-1}) & \chi_2(g_2^{-1}) & \cdots & \chi_s(g_s^{-1}) \\ \chi_1(g_1^{-1}) & \chi_2(g_2^{-1}) & \cdots & \chi_s(g_s^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_1(g_1^{-1}) & \chi_2(g_2^{-1}) & \cdots & \chi_s(g_s^{-1}) \end{pmatrix}$$

(this is a consequence of Propositions 1.3.11 and 1.3.13), we have that $\eta_h m_0 \eta_{h^{-1}} \in$

$\sum_{i=1}^{s}\mathfrak{B}w_{\chi_i}$ for any $h \in \mathsf{G}$, since

$$\begin{pmatrix} w_{\chi_1} \\ w_{\chi_2} \\ \vdots \\ w_{\chi_s} \end{pmatrix} = \begin{pmatrix} \chi_1(g_1) & \chi_1(g_2) & \cdots & \chi_1(g_s) \\ \chi_2(g_1) & \chi_2(g_2) & \cdots & \chi_2(g_s) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_s(g_1) & \chi_s(g_2) & \cdots & \chi_s(g_s) \end{pmatrix} \begin{pmatrix} \sigma(g_1, g_1^{-1})^{-1}\eta_{g_1}m_0\eta_{g_1^{-1}} \\ \sigma(g_2, g_2^{-1})^{-1}\eta_{g_2}m_0\eta_{g_2^{-1}} \\ \vdots \\ \sigma(g_s, g_s^{-1})^{-1}\eta_{g_s}m_0\eta_{g_s^{-1}} \end{pmatrix}.$$

Hence, we can conclude that $\eta_g m_0 \eta_h \in \sum_{i=1}^{s}\mathfrak{B}w_{\chi_i}$ for any $g, h \in \mathsf{G}$, and so $\mathfrak{B}m_0\mathfrak{B} = \sum_{i=1}^{s}\mathfrak{B}w_{\chi_i}$.

Observe that if M is irreducible graded, then we have that $\mathsf{M} = \mathfrak{B}w_\chi$ for some irreducible character of G, since each $\mathfrak{B}w_{\chi_i}$ is a graded subbimodule of M.

Below, let us show that, given a simple G-graded finite dimensional algebra $\mathfrak{A}$, any unitary G-graded $\mathfrak{A}$-bimodule M which satisfies both chain conditions can be written as a finite direct sum of the form $\bigoplus_w \mathfrak{A}w$ of irreducible G-graded $\mathfrak{A}$-subbimodules $\mathfrak{A}w$ such that $w\mathfrak{A} = \mathfrak{A}w$.

Proposition 1.3.22 *Let G be a group, H a finite abelian subgroup of G, and $\mathbb{F}$ an algebraically closed field such that $\mathsf{char}(\mathbb{F}) = 0$. Consider $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ with a canonical elementary G-grading, where $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and a unitary G-graded $\mathfrak{B}$-bimodule M. If M is irreducible, then there exits a homogeneous element $w \in \mathsf{M}$ such that $\mathfrak{B}w = w\mathfrak{B}$ and $\mathsf{M} = \mathfrak{B}w$.*

Proof: If $n = 1$, then the result follows of Remark 1.3.21.

Suppose that $n > 1$. Let $\chi_1, \ldots, \chi_s$ be all distinct irreducible characters of H, where $H \cong \hat{H} = \{\chi_1, \ldots, \chi_s\}$. Fix a nonzero homogeneous element $m_0 \in \mathsf{M}$, and define

$$\hat{w}_{\chi_i} = \sum_{i=1}^{s} E_{i1}\eta_e w_{\chi_i} E_{1i}\eta_e$$

for all $i = 1, \ldots, s$, where w_{χ_i}'s was defined in Remark 1.3.21. It is not difficult to see that $E_{pq}\eta_g\hat{w}_{\chi_i} = \chi_i(t)^{-1}\hat{w}_{\chi_i}E_{pq}\eta_g$ for any $g \in H$, $p, q, i \in \{1, \ldots, s\}$, and hence, $\mathfrak{B}\hat{w}_{\chi_i} = \hat{w}_{\chi_i}\mathfrak{B}$, for all $i = 1, \ldots, s$.

Similarly to Remark 1.3.21, we deduce that $\mathsf{M} = \mathfrak{B}m_0\mathfrak{B} = \sum_{i=1}^{s}\mathfrak{B}\hat{w}_{\chi_i}$. Since M is a irreducible G-graded $\mathfrak{B}$-bimodule, we conclude that $\mathsf{M} = \mathfrak{B}w_\chi$ for some irreducible character χ of H. ■

We can rewrite the previous proposition as follows.

Corollary 1.3.23 *Let* G *be a group,* H *a finite abelian subgroup of* G*,* $\mathbb{F}$ *an algebraically closed field such that* $\mathsf{char}(\mathbb{F}) = 0$*, and* $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ *with a canonical elementary* G*-grading defined by an* n*-tuple* $(g_1, \ldots, g_n) \in \mathsf{G}^n$*. Let* M *be a unitary* G*-graded* $\mathfrak{B}$*-bimodule. If* M *is irreducible graded, then there exists a homogeneous element* $m_0 \in \mathsf{M}$*, and a map* $\chi : H \to \mathbb{F}^*$ *such that the element defined as*

$$\hat{w}_\chi = \sum_{i=1}^{n} \left(\sum_{h \in H} \chi(h)\sigma(h, h^{-1})^{-1}\eta_h E_{i1} m_0 E_{1i} \eta_{h^{-1}} \right) ,$$

for all $i, j \in \{1, \ldots, n\}$*, satisfies* $\hat{w}_\chi E_{ij}\eta_g = \chi(g)^{-1}E_{ij}\eta_g\hat{w}_\chi$*, and* $\mathsf{M} = \mathfrak{B}\hat{w}_\chi$*.*

Proof: It is immediate of the proof of Proposition 1.3.22, and of Remark 1.3.21. ■

Observe that the element $\hat{w}_\chi$ in Corollary 1.3.23 is homogeneous, and satisfies $\mathfrak{B}\hat{w}_\chi = \hat{w}_\chi\mathfrak{B}$. Recall that χ is an irreducible character of H.

Corollary 1.3.24 *Let* $\mathbb{F}$ *be an algebraically closed field with* $\mathsf{char}(\mathbb{F}) = 0$*,* G *a abelian group, and* $\mathfrak{A}$ *a finite dimensional algebra over* $\mathbb{F}$ *with a* G*-grading. If* $\mathfrak{A}$ *is graded simple, then any irreducible* G*-graded* $\mathfrak{A}$*-bimodule* M *is isomorphic to* $\mathfrak{A}w$ *as a* G*-graded* $\mathfrak{A}$*-bimodule, for some homogeneous element* $w \in \mathsf{M}$ *satisfying* $\mathfrak{A}w = w\mathfrak{A}$ *and* $\mathsf{M} = \mathfrak{A}w$*.*

Proof: By Theorem 1.2.19, we have that $\mathfrak{A} \cong_{\mathsf{G}} M_n(\mathbb{F}^\sigma[H])$ for some finite subgroup H of G, and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and $M_n(\mathbb{F}^\sigma[H])$ has a canonical elementary G-grading. Hence, we have the same conditions of Proposition 1.3.22. Therefore, the result follows. ■

Proposition 1.3.25 *Let* $\mathbb{F}$ *be an algebraically closed field with* $\mathsf{char}(\mathbb{F}) = 0$*,* G *a group,* H *a finite abelian subgroup of* G*,* $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$*, and* $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ *with a canonical elementary* G*-grading. If* M *is a* G*-graded unitary* $\mathfrak{B}$*-bimodule satisfying both chain conditions for* G*-graded* $\mathfrak{B}$*-subbimodules, then* M *can be written as a finite direct sum of irreducible* G*-graded* $\mathfrak{B}$*-subbimodules* $\mathfrak{B}w$*,* $w \in \mathsf{M}$*, such that* $w\mathfrak{B} = \mathfrak{B}w$*. Moreover,* $wb = \gamma_w(b)wb$ *for any* $b \in \mathfrak{B}$*,* $\gamma_w(b) \in \mathbb{F}$*.*

Proof: Let us denote by $\beta = \{E_{ij}\eta_h : i, j = 1, \ldots, n, h \in H\}$ the canonical homogeneous basis of $\mathfrak{B}$. Since M satisfies the descending chain condition for G-graded $\mathfrak{B}$-subbimodules, we can build an ascending chain as in (1.6). From this, consider a chain of G-graded $\mathfrak{B}$-subbimodules of M given as follows:

$$\{0\} = \mathsf{M}_0 \subseteq \mathsf{M}_1 \subseteq \mathsf{M}_2 \subseteq \cdots \tag{1.8}$$

where $\mathsf{M}_{i+1}/\mathsf{M}_i$ is an irreducible G-graded $\mathfrak{B}$-bimodule for each $i = 0, 1, 2, \dots$. On the other hand, since M is G-Noetherian, it follows that there is $n \in \mathbb{N}$ such that $\mathsf{M}_{n-1} \subsetneq \mathsf{M}$ is maximal in $\mathsf{M} = \mathsf{M}_n$. Hence, we obtain from (1.8) the following finite sequence

$$\{0\} = \mathsf{M}_0 \subsetneq \mathsf{M}_1 \subsetneq \cdots \subsetneq \mathsf{M}_{n-1} \subsetneq \mathsf{M}_n = \mathsf{M}\ , \tag{1.9}$$

where M_i's are G-graded $\mathfrak{B}$-subbimodules of M such that $\mathsf{M}_{i+1}/\mathsf{M}_i$ is an irreducible G-graded $\mathfrak{B}$-bimodule for $i = 0, 1, \dots, n-1$.

Let us show by induction on n that there exist homogeneous elements $w_1, \dots, w_n \in \mathsf{M}$ such that $\mathsf{M} = \bigoplus_{i=1}^n \mathfrak{B}w_i$ with $bw_i = \gamma_i(b)w_i b \neq 0$ for any $b \in \beta$, where $\gamma_i(b) \in \mathbb{F}$, for all $i = 1, \dots, n$ and $b \in \beta$.

Firstly, suppose $n = 1$. Hence, $\mathsf{M} = \mathsf{M}_1$ with $\mathsf{M}/\mathsf{M}_0 \cong_{\mathsf{G}} \mathsf{M}$ irreducible. It follows of Corollary 1.3.23 that there exists a nonzero homogeneous element $w_1 \in \mathsf{M}_1$ such that $w_1 b = \gamma_1(b) b w_1 \neq 0$ for any $b \in \beta$ and $\mathsf{M}_1 = \mathfrak{B}w_1$, where $\gamma_1(b) \in \mathbb{F}$ for any $b \in \beta$.

Now, suppose that the result is valid for all $d \geqslant 1$, i.e. there exist nonzero homogeneous elements $w_1 \in \mathsf{M}_1 - \mathsf{M}_0, w_2 \in \mathsf{M}_2 - \mathsf{M}_1, \dots, w_d \in \mathsf{M}_d - \mathsf{M}_{d-1}$ such that $w_i b = \gamma_i(b) b w_i \neq 0$ for any $b \in \beta$ and $i = 1, \dots, d$ ($\gamma_i(b) \in \mathbb{F}$ for any $b \in \beta$ and $i = 1, \dots, n$), satisfying

$$\mathsf{M}_d = \mathfrak{B}w_1 \oplus \mathfrak{B}w_2 \oplus \cdots \oplus \mathfrak{B}w_d\ ,$$

where each $\mathfrak{B}w_i$ is irreducible graded. Notice that $\mathsf{M}_{d+1} = \mathsf{M}_d + \mathfrak{B}w\mathfrak{B}$ (quotient vector space) for any $w \in \mathsf{M}_{d+1} - \mathsf{M}_d$. Since $\mathsf{M}_{d+1}/\mathsf{M}_d$ is irreducible graded, by Corollary 1.3.23, there exists a nonzero homogeneous element $w_0 \in \mathsf{M}_{d+1} - \mathsf{M}_d$ such that w_{d+1} defined by

$$w_{d+1} = \sum_{i=1}^{n} \left(\sum_{h \in H} \chi(h)\sigma(h, h^{-1})^{-1} \eta_h E_{i1} w_0 E_{1i} \eta_{h^{-1}} \right) \neq 0$$

satisfies $\mathsf{M}_{d+1}/\mathsf{M}_d = \mathfrak{B}(w_{d+1} + \mathsf{M}_d)$, and $b(w_{d+1} + \mathsf{M}_d) = \gamma_{d+1}(b)(w_{d+1} + \mathsf{M}_d)b \neq 0 + \mathsf{M}_d$ for any $b \in \beta$, where $\gamma_{d+1}(b) \in \mathbb{F}$ for any $b \in \beta$. It is immediate of the proof of Proposition 1.3.22 that $E_{ij}\eta_h w_{d+1} = \gamma_{d+1}(b) w_{d+1} E_{ij}\eta_h \notin \mathsf{M}_d$ for any $i, j \in \{1, \dots, n\}$ and $h \in H$.

Hence, we have that $\mathfrak{B}w_{d+1}\mathfrak{B} = \mathfrak{B}w_{d+1} = w_{d+1}\mathfrak{B}$. Consequently, it follows that $\mathsf{M}_{d+1} = \mathsf{M}_d \oplus \mathfrak{B}w_{d+1}$, since $\mathfrak{B}w_{d+1} \cap \mathsf{M}_d = \{0\}$. Let us prove that $\mathsf{M}_{d+1}/\mathsf{M}_d \cong_{\mathsf{G}} \mathfrak{B}w_{d+1}$. In fact, by Isomorphisms Theorem (see Theorem 1.1.24), we have that

$$\frac{\mathsf{M}_{d+1}}{\mathsf{M}_d} = \frac{\mathsf{M}_d \oplus \mathfrak{B}w_{d+1}}{\mathsf{M}_d} \cong_{\mathsf{G}} \frac{\mathfrak{B}w_{d+1}}{\mathfrak{B}w_{d+1} \cap \mathsf{M}_d} = \frac{\mathfrak{B}w_{d+1}}{\{0\}} \cong_{\mathsf{G}} \mathfrak{B}w_{d+1} \ ,$$

as G-graded $\mathfrak{B}$-bimodules. Therefore, we prove that $\mathfrak{B}w_{d+1}$ is an irreducible G-graded $\mathfrak{B}$-subbimodule of M such that $w_{d+1}b = \gamma_{d+1}(b)bw_{d+1}$ for any $b \in \beta$ ($\gamma_{d+1}(b) \in \mathbb{F}$). Hence,

$$\mathsf{M}_{d+1} = \mathsf{M}_d \oplus \mathfrak{B}w_{d+1} = \mathfrak{B}w_1 \oplus \cdots \oplus \mathfrak{B}w_d \oplus \mathfrak{B}w_{d+1},$$

where each $\mathfrak{B}w_i$ is irreducible graded with $bw_i = \gamma_i(b)w_ib \neq 0$ for any $b \in \beta$, and $i = 1, \ldots, d+1$, where $\gamma_i(b) \in \mathbb{F}$, for any $b \in \beta$. Furthermore, by induction, the result is proved. ■

By Corollary 1.3.19 and Proposition 1.3.25, it follows that if M is a unitary G-graded $\mathfrak{B}$-bimodule, where G is a group, $\mathbb{F}$ is an algebraically closed field, H a finite abelian subgroup of G such that $\mathsf{char}(\mathbb{F}) = 0$, and $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$, then M is finitely generated iff M satisfy both chain conditions for graded subbimodules. More general, we have have the following corollary.

Corollary 1.3.26 *Let G be an abelian group, $\mathbb{F}$ an algebraically closed field such that $\mathsf{char}(\mathbb{F}) = 0$, $\mathfrak{A}$ a finite dimensional G-graded $\mathbb{F}$-algebra, and M a G-graded $\mathfrak{A}$-bimodule. Suppose that $\mathfrak{A}$ is graded simple, and M is a G-graded unitary $\mathfrak{A}$-bimodule. Then M is G-Noetherian and G-Artinian iff M is finitely generated.*

Proof: By Theorem 1.2.19, without loss of generality we can assume that $\mathfrak{A} = M_n(\mathbb{F}^\sigma[H])$ with a canonical elementary G-grading, where H is a finite abelian subgroup of G and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. The result follows from the previous proposition.

By Corollary 1.3.19 and Proposition 1.3.25, it follows that M is finitely generated iff M satisfy both chain conditions for graded subbimodules. ■

Corollary 1.3.27 *Let $\mathbb{F}$ be a algebraically closed field, G an abelian group and $\mathfrak{A}$ a finite dimensional algebra over $\mathbb{F}$ with a G-grading. Suppose that $\mathsf{char}(\mathbb{F}) = 0$, and $\mathfrak{A}$ is graded* *simple. If* M *is a* G*-graded unitary* $\mathfrak{A}$*-bimodule satisfying both chain conditions for* G*-graded* $\mathfrak{A}$*-subbimodules, then there exist nonzero homogeneous elements* $w_1, \dots, w_n \in \mathsf{M}$ *such that*

$$\mathsf{M} = \mathfrak{A}w_1 \oplus \cdots \oplus \mathfrak{A}w_n$$

where $w_i\mathfrak{A} = \mathfrak{A}w_i \neq 0$ *for all* $i = 1, \dots, n$*, and* $\mathfrak{A}w_i$ *is irreducible.*

Proof: By Theorem 1.2.19, we have that $\mathfrak{A} \cong_{\mathsf{G}} M_n(\mathbb{F}^\sigma[H])$ for some finite subgroup H of G and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and $M_n(\mathbb{F}^\sigma[H])$ with a canonical elementary G-grading. Hence, we have the same conditions of Proposition 1.3.25. Therefore, the result follows. ■

1.4 Some results on Graded Polynomial Identities

In this section, we present some definitions concerning graded polynomial identities as well as some results about their properties. Also we present the definition of the graded Grassmann envelope $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ of a $\mathsf{G} \times \mathbb{Z}_2$-graded algebra $\mathfrak{A}$, and its main properties. These notions and facts will be our principal tools in the next chapters. Here, $\mathbb{F}$ denotes a field, and G denotes a group.

1.4.1 Free Graded Algebra and Graded Polynomial Identities

Let $\mathcal{F} = \mathbb{F}\langle X^{\mathsf{G}}\rangle$ be the free G-graded associative algebra over $\mathbb{F}$ generated by a countable set $X^{\mathsf{G}} = \bigcup_{g \in \mathsf{G}} X_g$, where $X_g = \{x_1^{(g)}, x_2^{(g)}, \dots\}$, $g \in \mathsf{G}$. The indeterminates of X_g are said to be homogeneous of degree g. Given a monomial $m = x_{i_1}^{(g_1)} x_{i_2}^{(g_2)} \cdots x_{i_s}^{(g_s)} \in \mathcal{F}$, the **homogeneous degree** of m, denoted by $\mathsf{deg}(m)$, is defined by $g_1 g_2 \cdots g_s$. Therefore, it is natural to write $\mathcal{F} = \bigoplus_{g \in \mathsf{G}} \mathcal{F}_g$, where $\mathcal{F}_g$ is the subspace of the algebra $\mathcal{F}$ generated by all monomials having homogeneous degree g. It is easy to check that $\mathcal{F}_g \mathcal{F}_h \subseteq \mathcal{F}_{gh}$ for all $g, h \in \mathsf{G}$. The above decomposition into direct sum makes $\mathcal{F}$ a G-graded algebra. Thus, $\mathcal{F}$ is the free G-graded associative algebra generated by the set X^{G}.

Definition 1.4.1 *A* G*-graded ideal* I *of* $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ *is a* $\mathsf{G}T$**-ideal** *if* $\varphi(I) \subseteq I$ *for any* G*-graded endomorphism* φ *of* $\mathbb{F}\langle X^{\mathsf{G}}\rangle$*.*

Definition 1.4.2 *Let* G *be a group,* $\mathbb{F}$ *a field, and* $\mathfrak{A}$ *an associative* $\mathbb{F}$*-algebra with a* G*-grading* Γ*. Given a graded polynomial* $f = f(x_1^{(g_1)}, x_2^{(g_2)}, \dots, x_n^{(g_n)}) \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$*, we say that*

f is a **graded polynomial identity** *in $\mathfrak{A}$ if $f(a_1, a_2, \ldots, a_n) = 0$ for any $a_1 \in \mathfrak{A}_{g_1}, a_2 \in \mathfrak{A}_{g_2}, \ldots, a_n \in \mathfrak{A}_{g_n}$. In this case, we write $f \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a* $\mathsf{G}PI$**-algebra** *over $\mathbb{F}$ (or simply $\mathsf{G}PI$-algebra) if there exists a nonzero graded polynomial $f \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ such that $f \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}$.*

We denote by $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$ the set of all G-graded polynomial identities of $\mathfrak{A}$. In other words, $\mathsf{T}^{\mathsf{G}}(\mathfrak{A}) = \{f \in \mathbb{F}\langle X^{\mathsf{G}}\rangle : f \equiv_{\mathsf{G}} 0 \text{ in } \mathfrak{A}\}$. It is easy to check that $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$ is a G-graded ideal of $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ closed by all G-graded endomorphisms of $\mathbb{F}\langle X^{\mathsf{G}}\rangle$, it is called the $\mathsf{G}T$-ideal of G-graded identities of $\mathfrak{A}$.

We say that two G-graded algebras $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are $\mathsf{G}PI$-equivalent iff $\mathsf{T}^{\mathsf{G}}(\mathfrak{A}) = \mathsf{T}^{\mathsf{G}}(\tilde{\mathfrak{A}})$. In this case, we denote $\mathfrak{A} \equiv_{\mathsf{G}PI} \tilde{\mathfrak{A}}$.

We say that a graded polynomial $f \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ is a **G-consequence** of a set $S \subset \mathbb{F}\langle X^{\mathsf{G}}\rangle$ if f belongs to the $\mathsf{G}T$-ideal generated by S.

Given $S \subset \mathbb{F}\langle X^{\mathsf{G}}\rangle$, we denote by $\langle S\rangle_{\mathsf{G}T}$ the $\mathsf{G}T$-ideal of $\mathbb{F}\langle X^{\mathsf{G}}\rangle$ generated by S ($\langle S\rangle_{\mathsf{G}T}$ is the least $\mathsf{G}T$-ideal containing S).

Definition 1.4.3 *Given a nonempty set $S \subset \mathbb{F}\langle X^{\mathsf{G}}\rangle$, the class of all G-graded algebras $\mathfrak{A}$ such that $f \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}$ for any $f \in S$ is called* **graded variety** *defined by S, and it is denoted by $\mathsf{var}_{\mathsf{G}}(S)$.*

We can define also a (ordinary) polynomial identity of an algebra. In this case, we define a nongraded polynomial identity. Firstly, let $\mathbb{F}\langle X\rangle$ be the free associative algebra over $\mathbb{F}$ generated by a countable set $X = \{x_1, x_2, \ldots\}$.

Definition 1.4.4 *An ideal I of $\mathbb{F}\langle X\rangle$ is a* **T-ideal** *if $\varphi(I) \subseteq I$ for any endomorphism φ of $\mathbb{F}\langle X\rangle$.*

Definition 1.4.5 *Let $\mathbb{F}$ be a field and $\mathfrak{A}$ an associative $\mathbb{F}$-algebra. Given a polynomial $f = f(x_1, x_2, \cdots, x_n) \in \mathbb{F}\langle X\rangle$, we say that f is a* **polynomial identity** *in $\mathfrak{A}$ if $f(a_1, a_2, \ldots, a_n) = 0$ for any $a_1, a_2, \ldots, a_n \in \mathfrak{A}$. In this case, we write $f \equiv 0$ in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a* PI**-algebra** *over $\mathbb{F}$ (or simply PI-algebra) if there exists a nonzero polynomial $f \in \mathbb{F}\langle X\rangle$ such that $f \equiv 0$ in $\mathfrak{A}$.*

We denote by $\mathsf{T}(\mathfrak{A})$ the set of all polynomial identities of $\mathfrak{A}$. In other words, $\mathsf{T}(\mathfrak{A}) = \{f \in \mathbb{F}\langle X\rangle : f \equiv 0 \text{ in } \mathfrak{A}\}$. It is easy to check that $\mathsf{T}(\mathfrak{A})$ is an ideal of $\mathbb{F}\langle X\rangle$ closed by all endomorphisms of $\mathbb{F}\langle X\rangle$, it is called the **$T$-ideal** of identities of $\mathfrak{A}$. Note that, without loss of generality, $\mathsf{T}(\mathfrak{A}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. For more details, see [10, 11, 17].

We say that two algebras $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ are PI-equivalent iff $\mathsf{T}(\mathfrak{A}) = \mathsf{T}(\tilde{\mathfrak{A}})$. In this case, we denote $\mathfrak{A} \equiv_{PI} \tilde{\mathfrak{A}}$.

We say that a polynomial identity $f \in \mathbb{F}\langle X\rangle$ is a **consequence** of a set $S \subset \mathbb{F}\langle X\rangle$ if f belongs to the T-ideal generated by S.

Given $S \subset \mathbb{F}\langle X\rangle$, we denote by $\langle S\rangle_T$ the T-ideal of $\mathbb{F}\langle X\rangle$ generated by S.

Definition 1.4.6 *Given a nonempty set $S \subset \mathbb{F}\langle X\rangle$, the class of all algebras $\mathfrak{A}$ such that $f \equiv 0$ in $\mathfrak{A}$ for any $f \in S$ is called* **variety** *defined by S, and it is denoted by $\mathsf{var}(S)$.*

Proposition 1.4.7 (Proposition 4.2.3, [10]) *Let*

$$f(x_1, \ldots, x_m) = \sum_{i=0}^{n} f_i \in \mathbb{F}\langle X^{\mathsf{G}}\rangle,$$

where f_i is the homogeneous component of f of degree i in x_1.

i) If the base field $\mathbb{F}$ contains more than n elements (e.g. $\mathbb{F}$ is infinite), then the graded polynomial identities $f_i \equiv 0$, $i = 0, 1, \ldots, n$, follow from $f \equiv 0$;

ii) If the base field is of characteristic 0 (or if $\mathsf{char}(\mathbb{F}) > \mathsf{deg}(f)$), then $f \equiv 0$ is equivalent to a set of multilinear graded polynomial identities.

By the previous proposition, item i), given an $\mathbb{F}$-algebra $\mathfrak{A}$, $\mathbb{F}$ is an infinite field, the graded polynomial identities of $\mathfrak{A}$ can be generated by multihomogeneous graded polynomials.

Proposition 1.4.8 *Let G be a group, $\mathbb{F}$ a characteristic zero field, $\mathfrak{A}$ a $\mathsf{G}PI$-algebra, and N a commutative algebra with the trivial G-grading. If N is not nilpotent, then $\mathsf{T}^{\mathsf{G}}(\mathfrak{A} \otimes_{\mathbb{F}} N) = \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. If $N^d = \{0\}$ and $\mathsf{T}^{\mathsf{G}}(\mathfrak{A}) = \langle f_1, \ldots, f_n\rangle_{\mathsf{G}T}$, where f_i's are graded multilinear polynomials, then*

$$\mathsf{T}^{\mathsf{G}}(\mathfrak{A} \otimes_{\mathbb{F}} N) = \left\langle (x_1^{(e)} x_2^{(e)} \cdots x_d^{(e)}), f_{i_1}, f_{i_2}, \ldots, f_{i_m} : \mathsf{deg}(f_j) < d \right\rangle_{\mathsf{G}T}.$$

Proof: We have that $N = N_e$, where e is the neutral element of G, and thus, $\mathfrak{A} \otimes_{\mathbb{F}} N = \bigoplus_{g\in\mathsf{G}} \mathfrak{A}_g \otimes_{\mathbb{F}} N$. Let $f = f(x_1^{(g_1)}, x_2^{(g_2)}, \ldots, x_r^{(g_r)}) \in \mathsf{T}^{\mathsf{G}}(\mathfrak{A} \otimes_{\mathbb{F}} N)$ be a multilinear polynomial, and $x_{\sigma(1)}^{(g_\sigma(1))}x_{\sigma(2)}^{(g_\sigma(2))}\cdots x_{\sigma(r)}^{(g_\sigma(r))}$ a monomial of f. Since N is commutative, we have that

$$(a_{\sigma(1)}\otimes y_{\sigma(1)})(a_{\sigma(2)}\otimes y_{\sigma(2)})\cdots(a_{\sigma(r)}\otimes y_{\sigma(r)}) = (a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(r)})\otimes y_1y_2\cdots y_r ,$$

for any homogeneous elements $a_1,\ldots,a_r \in \mathfrak{A}$ ($\mathsf{deg}(a_i) = g_i$), $y_1,\ldots,y_r \in N$ and $\sigma \in S_r$. From this, we have that

$$0 = f(a_1\otimes y_1, a_2\otimes y_2,\ldots,a_r\otimes y_r) = f(a_1,a_2,\ldots,a_r)\otimes y_1y_2\cdots y_r ,$$

for any $a_i\otimes y_i \in (\mathfrak{A}\otimes_{\mathbb{F}} N)_{g_i} = \mathfrak{A}_{g_i}\otimes_{\mathbb{F}} N$. Hence, it follows that either $f \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}$ or $x_1^{(e)}x_2^{(e)}\cdots x_r^{(e)} \equiv_{\mathsf{G}} 0$ in N, and so $f \equiv_{\mathsf{G}} 0$ or $x_1^{(g_1)}x_2^{(g_2)}\cdots x_r^{(g_r)} \equiv_{\mathsf{G}} 0$ in $\mathfrak{A}\otimes_{\mathbb{F}} N$. Notice that if $N^d = \{0\}$, and $f \in \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$ such that $\mathsf{deg}(f) \geqslant d$, then f is a consequence of $x_1^{(e)}x_2^{(e)}\cdots x_r^{(e)}$. The result follows. ■

The Proposition 1.4.8 exhibits a tool to build graded nilpotent algebras with another graded polynomial identity.

Theorem 1.4.9 (Theorem 1, [45]) *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and G a finite abelian group. Any $\mathsf{G}T$-ideal of G-graded identities of a finitely generated associative PI-algebra over $\mathbb{F}$ graded by G coincides with the ideal of graded identities of some finite dimensional over the base field $\mathbb{F}$ associative G-graded algebra.*

Remark 1.4.10 *Under the same hypothesis of the Theorems 1.2.20 and 1.4.9, we have that if $\mathfrak{A}$ is a finitely generated associative G-graded PI-algebra over a field $\mathbb{F}$, then there exists a finite dimensional associative G-graded algebra*

$$\mathfrak{A}' = (M_{k_1}(\mathbb{F}^{\sigma_1}[H_1])\times\cdots\times M_{k_p}(\mathbb{F}^{\sigma_p}[H_p]))\oplus \mathsf{J}$$

such that $\mathsf{T}^{\mathsf{G}}(\mathfrak{A}) = \mathsf{T}^{\mathsf{G}}(\mathfrak{A}')$. Here, $\mathfrak{A}'$ satisfies all the claims of Theorem 1.2.20.

One of the central problems in the study of graded algebras is to obtain non-graded (ordinary) properties from the analysis of gradings assumed for a given algebra, and vice versa. In this sense, given a graded algebra, we can determine relationships between its graded identities and its non-graded identities. Let $\mathfrak{A} = \bigoplus_{g\in\mathsf{G}}\mathfrak{A}_g$ be a G-graded algebra, G is a finite group with the neutral element e. In [5], Bergen and Cohen showed that if $\mathfrak{A}_e$

is a PI-algebra, then $\mathfrak{A}$ is also a PI-algebra. They did not exhibit, in the general case, a bound for the degree of the polynomial identity satisfied by $\mathfrak{A}$. On the other hand, in [2], Bahturin, Giambruno and Riley proved the same result. Moreover, they gave a bound on the minimal degree of a polynomial identity satisfied by $\mathfrak{A}$. Namely, the following results were shown:

Theorem 1.4.11 (Corollary 9, [5]) *Suppose an algebra $\mathfrak{A}$ is graded by a group G such that $|\mathsf{G}| = n$. Then $\mathfrak{A}_e$ is a PI-algebra iff $\mathfrak{A}$ is a PI-algebra.*

Theorem 1.4.12 (Theorem 5.3, [2]) *Let $\mathbb{F}$ be an arbitrary field and G a finite group. Suppose that $\mathfrak{A}$ is a G-graded associative $\mathbb{F}$-algebra such that $\mathfrak{A}_e$ satisfies a polynomial identity of degree d. Then $\mathfrak{A}$ satisfies a polynomial identity of degree n, where n is any integer satisfying the inequality*

$$\frac{|\mathsf{G}|^n(|\mathsf{G}|d-1)^{2n}}{(|\mathsf{G}|d-1)!} < n!.$$

In particular, if n is the least integer such that $\mathsf{e}|\mathsf{G}|(|\mathsf{G}|d-1)^2 \leqslant n$, then $\mathfrak{A}$ satisfies a polynomial identity of degree n, where e is the base of the natural logarithm.

The purpose of our work is to examine some concrete cases of the statements of Theorems 1.4.11 and 1.4.12.

1.4.2 The Grassmann Envelope of an Algebra

Let $\mathfrak{A}$ be a $(\mathsf{G}\times\mathbb{Z}_2)$-graded finite dimensional algebra, namely

$$\mathfrak{A} = \bigoplus_{(g,\lambda)\in\mathsf{G}\times\mathbb{Z}_2} \mathfrak{A}_{(g,\lambda)} \ .$$

Notice that $\mathfrak{A} = \bigoplus_{g\in\mathsf{G}}\mathfrak{A}_g$ with $\mathfrak{A}_g = \mathfrak{A}_{(g,0)}\oplus\mathfrak{A}_{(g,1)}$, for any $g\in\mathsf{G}$, is a G-grading on $\mathfrak{A}$, and $\mathfrak{A} = \mathfrak{A}_0\oplus\mathfrak{A}_1$ with $\mathfrak{A}_\lambda = \bigoplus_{g\in\mathsf{G}}\mathfrak{A}_{(g,\lambda)}$, for $\lambda = 0, 1$, is a $\mathbb{Z}_2$-grading on $\mathfrak{A}$. We denote by $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ the **Grassmann Envelope** of $\mathfrak{A}$ which is given by

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) = (\mathfrak{A}_0\otimes\mathsf{E}_0)\oplus(\mathfrak{A}_1\otimes\mathsf{E}_1)\,,$$

where $\mathsf{E} = \mathsf{E}_0 \oplus \mathsf{E}_1$ is an infinitely generated non-unitary Grassmann algebra[1] with its natural $\mathbb{Z}_2$-grading. Notice that if $\left(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\right)_{(g,\lambda)} = \mathfrak{A}_{(g,\lambda)} \otimes_{\mathbb{F}} \mathsf{E}_\lambda$ for any $(g,\lambda) \in \mathsf{G} \times \mathbb{Z}_2$, then

$$\Gamma : \mathsf{E}^{\mathsf{G}}(\mathfrak{A}) = \bigoplus_{(g,\lambda)\in\mathsf{G}\times\mathbb{Z}_2} \left(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\right)_{(g,\lambda)} = \bigoplus_{(g,\lambda)\in\mathsf{G}\times\mathbb{Z}_2} \mathfrak{A}_{(g,\lambda)} \otimes \mathsf{E}_\lambda \tag{1.10}$$

is a $(\mathsf{G} \times \mathbb{Z}_2)$-grading on $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$. For this reason, it follows that

$$\begin{aligned} \mathsf{Supp}\,(\Gamma) &= \{(g,\lambda) \in \mathsf{G} \times \mathbb{Z}_2 : (\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_{(g,\lambda)} \neq 0\} \\ &= \{(g,\lambda) \in \mathsf{G} \times \mathbb{Z}_2 : \mathfrak{A}_{(g,\lambda)} \neq 0\} \\ &= \mathsf{Supp}(\Gamma_{\mathfrak{A}})\ . \end{aligned} \tag{1.11}$$

It is clear that $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ is an $\mathfrak{A}_0 \otimes \mathsf{E}_0$-bimodule. Now, let $\mathfrak{B}$ be a $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebra of $\mathfrak{A}$. By (1.10) and (1.11), it is easy to see that $\mathsf{E}^{\mathsf{G}}(\mathfrak{B})$ is a $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebra of $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$. Observe that

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) = \bigoplus_{g\in\mathsf{G}} (\mathfrak{A}_{(g,0)} \otimes_{\mathbb{F}} \mathsf{E}_0) + (\mathfrak{A}_{(g,1)} \otimes_{\mathbb{F}} \mathsf{E}_1)$$

defines a G-grading on $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$.

The next theorems give the positive answer to the well-known **Specht problem**[2] for graded varieties.

Theorem 1.4.13 (Theorem 2, [45]) *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and G any finite abelian group. Any $\mathsf{G}T$-ideal of graded identities of a G-graded associative PI-algebra over $\mathbb{F}$ coincides with the ideal of G-graded identities of the G-graded Grassmann envelope of some finite dimensional over $\mathbb{F}$ associative $\mathsf{G} \times \mathbb{Z}_2$-graded algebra.*

Theorem 1.4.14 (Theorem 1.3, [1]) *Let G be a finite group and W a $\mathsf{G}PI$-graded algebra over $\mathbb{F}$, $\mathsf{char}(\mathbb{F}) = 0$. Then there exists a field extension $\mathbb{K}$ of $\mathbb{F}$ and a finite-dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded algebra $\mathfrak{A}$ over $\mathbb{K}$ such that $\mathsf{T}^{\mathsf{G}}(W) = \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$.*

[1] Grassmann algebra: $\mathsf{E} = \langle e_1, e_2, e_3, \cdots \mid e_ie_j = -e_je_i, \forall i,j\rangle$ is $\mathbb{Z}_2$-graded with $\mathsf{E}_0 = \mathsf{span}_{\mathbb{F}}\{e_{i_1}e_{i_2}\cdots e_{i_n} : n \text{ is even}\}$, and $\mathsf{E}_1 = \mathsf{span}_{\mathbb{F}}\{e_{j_1}e_{j_2}\cdots e_{j_m} : m \text{ is odd}\}$.

[2] Specht problem was purposed in [44] by W. Specht (1950), and it can be formulated by the following question: given any algebra $\mathfrak{A}$, is any set of polynomial identities of $\mathfrak{A}$ a consequence of a finite number of identities of $\mathfrak{A}$? For more details about Specht Problem, see [4].

1.5 A description of $\mathsf{J}(\mathfrak{A})$

The goal of this section is to present some results concerning the Jacobson radical of the $\hat{\mathsf{G}}$-graded Grassmann Envelope of $\mathfrak{A}$, where $\hat{\mathsf{G}}$ is a group, and $\mathfrak{A}$ is a $\hat{\mathsf{G}}$-graded finite dimensional algebra over a field $\mathbb{F}$. Unless otherwise stated, we assume that $\hat{\mathsf{G}}$ is an abelian finite group and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ is a finite dimensional $\hat{\mathsf{G}}$-graded $\mathbb{F}$-algebra, where $\mathfrak{B} = M_{k_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{k_p}(\mathbb{F}^{\sigma_p}[H_p])$ is a maximal $\hat{\mathsf{G}}$-graded semisimple subalgebra of $\mathfrak{A}$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, $\mathbb{F}$ is an algebraically closed field, $\mathsf{char}(\mathbb{F}) = 0$. Here, $H_s \trianglelefteq \hat{\mathsf{G}}$ and $\sigma_s \in \mathsf{Z}^2(H_s, \mathbb{F}^*)$ (see Theorem 1.2.20). For each $r = 1, \ldots, p$, we denote by $\mathfrak{i}_r = \sigma_r(e,e)^{-1} \sum_{s=1}^{k_r} \mathsf{E}_{ss}\eta_{(e,0)} \in \mathfrak{B}$ the identity matrix of $\mathfrak{B}_r := M_{k_r}(\mathbb{F}^{\sigma_r}[H_r])$. Hence, by Proposition 1.2.6, it follows that $\mathfrak{i} = \sum_{r=1}^{p} \mathfrak{i}_r \in \mathfrak{A}$ is the unity of $\mathfrak{B}$, since $\mathfrak{i}\mathfrak{i}_r = \mathfrak{i}_r\mathfrak{i} = \mathfrak{i}_r$ for all $r = 1, \ldots, p$, and $\mathfrak{i}_s\mathfrak{i}_r = \mathfrak{i}_r\mathfrak{i}_s = 0$ for all $s \neq r$.

Suppose that $\mathfrak{A}$ is a unitary algebra. If $\epsilon \in \mathfrak{A}$ is a central idempotent element, i.e. $\epsilon^2 = \epsilon$, and $\epsilon \in \mathcal{Z}(\mathfrak{A})$, then $1 - \epsilon \in \mathfrak{A}$ is also a central idempotent element of $\mathfrak{A}$ such that ϵ and $(1-\epsilon)$ are orthogonal when $\epsilon \neq 0$, i.e. $\epsilon(1-\epsilon) = (1-\epsilon)\epsilon = 0$. Therefore, given $x \in \mathfrak{A}$, notice that $x = x\epsilon + x(1-\epsilon)$, and hence, it is not difficult to see that $\mathfrak{A} = \mathfrak{A}\epsilon \oplus \mathfrak{A}(1-\epsilon)$. This decomposition is called the **Peirce Decomposition** of $\mathfrak{A}$ relative to ϵ. Naturally, we can extend this definition to n idempotent elements of $\mathfrak{A}$, as follows. Let $\epsilon_1, \ldots, \epsilon_n \in \mathfrak{A}$ be distinct central orthogonal idempotent elements. Without loss of generality, suppose that $1 = \sum_{i=1}^{n} \epsilon_i$. Given $x \in \mathfrak{A}$, we have $x = x1 = \sum_{i=1}^{n} x\epsilon_i$, and hence, $\mathfrak{A} = \mathfrak{A}\epsilon_1 \oplus \cdots \oplus \mathfrak{A}\epsilon_n$ is the Peirce decomposition of $\mathfrak{A}$ relative to $\epsilon_1, \ldots, \epsilon_n$. In the next subsections, we will use Theorem 1.2.20 to give a description of the Jacobson radical J and its Grassmann envelope $\mathsf{E}^{\hat{\mathsf{G}}}(\mathsf{J})$ in terms of the concept of the Peirce decomposition and of the semisimple part $\mathfrak{B}$.

Given a G-graded algebra $\tilde{\mathfrak{A}}$, recall that a G-graded left (resp. right) $\tilde{\mathfrak{A}}$-module M is called a **0-module** if $\tilde{\mathfrak{A}}\mathsf{M} = \{0\}$ (resp. $\mathsf{M}\tilde{\mathfrak{A}} = \{0\}$).

The following lemmas are the graded versions of Lemma 2 in [16].

Lemma 1.5.1 *Let $\hat{\mathsf{G}}$ be a group and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a finite dimensional algebra with a $\hat{\mathsf{G}}$-grading, where $\mathfrak{B}$ is a $\hat{\mathsf{G}}$-graded maximal semisimple subalgebra of $\mathfrak{A}$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical and a graded ideal of $\mathfrak{A}$. Then J can be decomposed as*

$$\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{10} \oplus \mathsf{J}_{01} \oplus \mathsf{J}_{11} \ ,$$

where J_{ij}'s are $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules such that:

i) for $r = 0, 1$, J_{0r} is a left 0-module and J_{1r} is a left $\hat{\mathsf{G}}$-graded faithful $\mathfrak{B}$-module;

ii) for $s = 0, 1$, J_{s0} is a right 0-module and J_{s1} is a right $\hat{\mathsf{G}}$-graded faithful $\mathfrak{B}$-module;

iii) $\mathsf{J}_{rq}\mathsf{J}_{qs} \subseteq \mathsf{J}_{rs}$, and $\mathsf{J}_{rp}\mathsf{J}_{qs} = \{0\}$ for $r, p, q, s \in \{0, 1\}$ with $p \neq q$.

Proof: Let $\mathfrak{i} \in \mathfrak{A}$ be the unity of $\mathfrak{B}$, and consider the applications $R_{\mathfrak{i}}, L_{\mathfrak{i}} : \mathsf{J} \longrightarrow \mathsf{J}$ defined by $R_{\mathfrak{i}}(x) = x\mathfrak{i}$ and $L_{\mathfrak{i}}(y) = \mathfrak{i}y$ for all $x, y \in \mathsf{J}$, respectively. Note that J is a graded ideal of $\mathfrak{A}$, and $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$ are homogeneous homomorphisms of $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules such that $R_{\mathfrak{i}}^2 = R_{\mathfrak{i}}$, and $L_{\mathfrak{i}}^2 = L_{\mathfrak{i}}$, since $\mathfrak{i} \in \mathfrak{B}_e$. Hence, $\mathsf{ker}(R_{\mathfrak{i}})$, $\mathsf{im}(R_{\mathfrak{i}})$, $\mathsf{ker}(L_{\mathfrak{i}})$ and $\mathsf{im}(L_{\mathfrak{i}})$ are $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules. On the other hand, we conclude that $0, 1 \in \mathbb{F}$ are the only eigenvalues of $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$. So, we have that $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$ are diagonalizable, since the minimal polynomials of $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$ can be written as the product of linear factors (see [23], Theorem 6.4.6). Notice that $R_{\mathfrak{i}}L_{\mathfrak{i}} = L_{\mathfrak{i}}R_{\mathfrak{i}}$, and hence, it follows that $\mathsf{ker}(R_{\mathfrak{i}})$ and $\mathsf{im}(R_{\mathfrak{i}})$ are invariant by $L_{\mathfrak{i}}$, and $\mathsf{ker}(L_{\mathfrak{i}})$ and $\mathsf{im}(L_{\mathfrak{i}})$ are invariant by $R_{\mathfrak{i}}$. Thus, it is easy to check that $V_0^R = \mathsf{ker}(R_{\mathfrak{i}})$ and $V_0^L = \mathsf{ker}(L_{\mathfrak{i}})$ are the eigenspaces of $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$ associated with 0, respectively, and $V_1^R = \mathsf{im}(R_{\mathfrak{i}})$ and $V_1^L = \mathsf{im}(L_{\mathfrak{i}})$ are the eigenspaces of $R_{\mathfrak{i}}$ and $L_{\mathfrak{i}}$ associated with 1, respectively.

Let us show that

$$\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{01} \oplus \mathsf{J}_{10} \oplus \mathsf{J}_{11} \ , \tag{1.12}$$

where $\mathsf{J}_{00} = V_0^R \cap V_0^L$, $\mathsf{J}_{10} = V_0^R \cap V_1^L$, $\mathsf{J}_{01} = V_1^R \cap V_0^L$ and $\mathsf{J}_{11} = V_1^R \cap V_1^L$.

Put $\tilde{\mathsf{J}} = \sum_{r,s=0,1} \mathsf{J}_{rs}$. Notice that

$$\mathsf{J}_{10} = \{x \in \mathsf{J} : x = \mathfrak{i}x, x\mathfrak{i} = 0\} \ , \ \mathsf{J}_{01} = \{x \in \mathsf{J} : x = x\mathfrak{i}, \mathfrak{i}x = 0\} \ ,$$
$$\mathsf{J}_{11} = \{x \in \mathsf{J} : x = \mathfrak{i}x\mathfrak{i}\} \ , \ \mathsf{J}_{00} = \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = 0\} \ .$$

From this, if $x_{10} + x_{01} + x_{11} + x_{00} = 0$ with $x_{rs} \in \mathsf{J}_{rs}$, then

$$0 = \mathfrak{i}(x_{10} + x_{01} + x_{11} + x_{00})\mathfrak{i} = \mathfrak{i}x_{11}\mathfrak{i} = x_{11} \ ,$$
$$0 = \mathfrak{i}(x_{10} + x_{01} + x_{11} + x_{00}) = \mathfrak{i}x_{10} + \mathfrak{i}x_{11} = x_{10} + x_{11} \ ,$$
$$0 = (x_{10} + x_{01} + x_{11} + x_{00})\mathfrak{i} = x_{01}\mathfrak{i} + x_{11}\mathfrak{i} = x_{01} + x_{11} \ ,$$

and hence, $x_{11} = x_{10} = x_{01} = x_{00} = 0$, and thus, $\tilde{\mathsf{J}} = \bigoplus_{r,s=0,1} \mathsf{J}_{rs}$. By Rank-Nullity Theorem (see [23], Theorem 3.1.2), it follows that $\mathsf{J} = \ker(R_{\mathfrak{i}}) \oplus \mathrm{im}(R_{\mathfrak{i}}) = \ker(L_{\mathfrak{i}}) \oplus \mathrm{im}(L_{\mathfrak{i}})$, since $\dim_{\mathbb{F}} \mathsf{J} < \infty$ and $\ker(R_{\mathfrak{i}}) \cap \mathrm{im}(R_{\mathfrak{i}}) = \ker(L_{\mathfrak{i}}) \cap \mathrm{im}(L_{\mathfrak{i}}) = \{0\}$. Take $x \in \mathsf{J}$. There exist $x_0 \in \ker(L_{\mathfrak{i}})$ and $x_1 \in \mathrm{im}(L_{\mathfrak{i}})$ such that $x = x_0 + x_1$, and hence, there exist $x_{00}, x_{10} \in \ker(R_{\mathfrak{i}})$ and $x_{01}, x_{11} \in \mathrm{im}(R_{\mathfrak{i}})$ such that $x_0 = x_{00} + x_{01}$ and $x_1 = x_{10} + x_{11}$. Notice that $x_{00}, x_{01} \in \ker(L_{\mathfrak{i}})$, since $0 = \mathfrak{i}x_0 = \mathfrak{i}x_{00} + \mathfrak{i}x_{01}$ and $\ker(L_{\mathfrak{i}}) \cap \mathrm{im}(L_{\mathfrak{i}}) = \{0\}$. Similarly, $x_{10}, x_{11} \in \mathrm{im}(L_{\mathfrak{i}})$, since $x_1 = x_{10} + x_{11}$ and $\ker(R_{\mathfrak{i}}) \cap \mathrm{im}(R_{\mathfrak{i}}) = \{0\}$, hence, $x_1 = \mathfrak{i}x_1 = \mathfrak{i}x_{10} + \mathfrak{i}x_{11}$, and thus $x_{10} - \mathfrak{i}x_{10} = \mathfrak{i}x_{11} - x_{11}$. Therefore,

$$x = x_{00} + x_{01} + x_{10} + x_{11} \in \mathsf{J}_{00} \oplus \mathsf{J}_{01} \oplus \mathsf{J}_{10} \oplus \mathsf{J}_{11} \ .$$

This finishes the proof of (1.12).

To conclude the proof of this lemma, let us show that J_{rs}'s satisfy items *i), ii), iii)*. Indeed, J_{rs} are $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules. Since J is an ideal of $\mathfrak{A}$ and $b = \mathfrak{i}b\mathfrak{i} = \mathfrak{i}b = b\mathfrak{i}$ for any $b \in \mathfrak{B}$, we have $ax = (a\mathfrak{i})x = a(\mathfrak{i}x)$ and $xa = x(\mathfrak{i}a) = (x\mathfrak{i})a$ for any $a \in \mathfrak{B}$ and $x \in \mathsf{J}$. Notice that given a nonzero $x \in \mathsf{J}_{01} \cup \mathsf{J}_{10} \cup \mathsf{J}_{11}$, we have that either $\mathfrak{i}x \neq 0$ or $x\mathfrak{i} \neq 0$. Hence, by definition of J_{rs}, items *i)* and *ii)* follow. Moreover, it follows that

$$\begin{aligned}
&\mathsf{J}_{00}\mathsf{J}_{11} = \mathsf{J}_{00}\mathsf{J}_{10} = \mathsf{J}_{01}\mathsf{J}_{00} = \mathsf{J}_{10}\mathsf{J}_{11} = \mathsf{J}_{11}\mathsf{J}_{01} = \mathsf{J}_{11}\mathsf{J}_{00} = \{0\} \ ,\\
&\mathsf{J}_{00}\mathsf{J}_{00} = \mathrm{span}_{\mathbb{F}}\{xy \in \mathsf{J} : x\mathfrak{i} = 0, \mathfrak{i}x = 0, \mathfrak{i}y = 0, y\mathfrak{i} = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{00} \ ,\\
&\mathsf{J}_{00}\mathsf{J}_{01} = \mathrm{span}_{\mathbb{F}}\{xy\mathfrak{i} \in \mathsf{J} : x\mathfrak{i} = 0, \mathfrak{i}x = 0, \mathfrak{i}y = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{01} \ ,\\
&\mathsf{J}_{10}\mathsf{J}_{00} = \mathrm{span}_{\mathbb{F}}\{\mathfrak{i}xy \in \mathsf{J} : x\mathfrak{i} = 0, \mathfrak{i}y = 0, y\mathfrak{i} = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{10} \ ,\\
&\mathsf{J}_{10}\mathsf{J}_{01} = \mathrm{span}_{\mathbb{F}}\{\mathfrak{i}xy\mathfrak{i} \in \mathsf{J} : x\mathfrak{i} = 0, \mathfrak{i}y = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{11} \ ,\\
&\mathsf{J}_{01}\mathsf{J}_{10} = \mathrm{span}_{\mathbb{F}}\{x\mathfrak{i}y \in \mathsf{J} : \mathfrak{i}x = 0, y\mathfrak{i} = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{00} \ ,\\
&\mathsf{J}_{01}\mathsf{J}_{11} = \mathrm{span}_{\mathbb{F}}\{x\mathfrak{i}y\mathfrak{i} \in \mathsf{J} : \mathfrak{i}x = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{01} \ ,\\
&\mathsf{J}_{11}\mathsf{J}_{10} = \mathrm{span}_{\mathbb{F}}\{\mathfrak{i}x\mathfrak{i}y \in \mathsf{J} : y\mathfrak{i} = 0, x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{10} \ ,\\
&\mathsf{J}_{11}\mathsf{J}_{11} = \mathrm{span}_{\mathbb{F}}\{\mathfrak{i}x\mathfrak{i}y\mathfrak{i} \in \mathsf{J} : x, y \in \mathsf{J}\} \subseteq \mathsf{J}_{11} \ ,
\end{aligned}$$

and this ensures item *iii)*. ■

By previous lemma, it follows that if $\mathfrak{A}$ is a unitary algebra, then $\mathfrak{i} \in \mathfrak{B}$ is still the unity of $\mathfrak{A}$, since J is nilpotent. In this case, we have

$$\mathsf{J}_{00} \oplus \mathsf{J}_{01} \oplus \mathsf{J}_{10} = \mathfrak{i}\,(\mathsf{J}_{00} \oplus \mathsf{J}_{01} \oplus \mathsf{J}_{10})\,\mathfrak{i} = \{0\}\ , \tag{1.13}$$

and so $\mathsf{J} = \mathfrak{i}\mathsf{J}\mathfrak{i} = \mathsf{J}_{11}$ which is described in the next lemma. Observe also that J_{10} and J_{11} are unitary left $\mathfrak{B}$-modules and J_{01} and J_{11} are unitary right $\mathfrak{B}$-modules.

Lemma 1.5.2 *Under the assumptions of Lemma 1.5.1, consider that* G *is a finite abelian group, and* $\mathbb{F}$ *is an algebraically closed field with* $\mathsf{char}(\mathbb{F}) = 0$*. The following statements are true:*

i) $\mathsf{J}_{11} = \bigoplus_{s,r=1}^{p} \mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s$*, where each* $\mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s$ *is a* $\hat{\mathsf{G}}$*-graded* $(\mathfrak{B}_r, \mathfrak{B}_s)$*-bimodule, where* $\mathfrak{B} = \bigoplus_{i=1}^{p} \mathfrak{B}_i$ *with* $\mathfrak{B}_i = M_{k_i}(\mathbb{F}^{\sigma_i}[H_i])$*. In addition,* $\mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s \neq 0$ *implies that* $\mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s$ *is a faithful left* $\mathfrak{B}_i$*-module and a faithful right* $\mathfrak{B}_j$*-module;*

ii) For each $s = 1, \ldots, p$*, there exists a* $\hat{\mathsf{G}}$*-graded vector space* $\mathsf{N}_s = \mathsf{span}_{\mathbb{F}}\{d_{1s}, \ldots, d_{r_s s}\} \subset \mathfrak{i}_s\mathsf{J}_{11}\mathfrak{i}_s$ *such that* $\mathfrak{i}_s\mathsf{J}_{11}\mathfrak{i}_s = \mathfrak{B}_s\mathsf{N}_s$ *and* $bd_{is} = \gamma_{is}(b)d_{is}b \neq 0$ *for any nonzero* $b \in \beta_s$*, and* $i = 1, \ldots, r_s$*, where* $\gamma_{is} \in \mathbb{F}$*, and* $\beta_s = \{E_{l_s j_s}\eta_{h_s} \in \mathfrak{B}_s : l_s, j_s = 1, \ldots, k_s, h_s \in H_s\}$ *is the canonical homogeneous basis of* $\mathfrak{B}_s = M_{k_s}(\mathbb{F}^{\sigma_s}[H_s])$*. Moreover, for each* $i = 1, \ldots, r_s$*, we have that* $\mathfrak{B}_s d_{is}$ *is a* $\hat{\mathsf{G}}$*-simple* $\mathfrak{B}_s$*-bimodule.*

Proof: By Lemma 1.5.1, we have $\mathfrak{i}\mathsf{J}\mathfrak{i} = \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = x\} = \{\mathfrak{i}x\mathfrak{i} : x \in \mathsf{J}\} = \mathsf{J}_{11}$. Notice that $\mathfrak{i}_s\mathsf{J}\mathfrak{i}_r = \mathfrak{i}_s\mathsf{J}_{11}\mathfrak{i}_r$, where $\mathfrak{i} = \sum_{r=1}^{p} \mathfrak{i}_r$. Since $\mathfrak{i}_r\mathfrak{i}_s = 0$ for all $r \neq s$, we have $\mathfrak{i}\mathfrak{i}_q = \mathfrak{i}_q\mathfrak{i} = \mathfrak{i}_q$ for all $q = 1, \ldots, p$.

i) Let us show that $\mathsf{J}_{11} = \bigoplus_{s,r=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_r$. Put $\tilde{\mathsf{J}} = \sum_{s,r=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_r$. Again, since $\mathfrak{i}_r\mathfrak{i}_s = 0$ for all $r \neq s$, it follows that $\tilde{\mathsf{J}} = \bigoplus_{s,r=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_r$ and $\tilde{\mathsf{J}} \subseteq \mathfrak{i}\mathsf{J}\mathfrak{i} = \mathsf{J}_{11}$. On the other hand, for any $x \in \mathsf{J}_{11}$, we have

$$x = \mathfrak{i}x\mathfrak{i} = \left(\sum_{s=1}^{p} \mathfrak{i}_s\right) x \left(\sum_{r=1}^{p} \mathfrak{i}_r\right) = \sum_{s,r=1}^{p} \mathfrak{i}_s x \mathfrak{i}_r \in \tilde{\mathsf{J}}\ .$$

Hence, $\mathsf{J}_{11} \subseteq \tilde{\mathsf{J}}$, and so $\mathsf{J}_{11} = \tilde{\mathsf{J}}$. It is immediate that $\mathfrak{i}_r\mathsf{J}\mathfrak{i}_s$ is a $\hat{\mathsf{G}}$-graded faithful $(\mathfrak{B}_r, \mathfrak{B}_s)$-bimodule for all $r, s = 1, \ldots, p$, since Lemma 1.5.1 ensures that J_{11} is a $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodule, $\mathfrak{i}_r, \mathfrak{i}_s$ are homogeneous elements of degree e for all $r, s = 1, \ldots, p$, and $\mathfrak{i}_r(\mathfrak{i}_r x \mathfrak{i}_s)\mathfrak{i}_s = (\mathfrak{i}_r\mathfrak{i}_r)x(\mathfrak{i}_s\mathfrak{i}_s) = \mathfrak{i}_r x \mathfrak{i}_s$ for any $x \in \mathsf{J}$.

Now, suppose that $\mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s \neq 0$ for some $r, s \in \{1, \ldots, p\}$. Since $\mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s$ is a $(\mathfrak{B}_r, \mathfrak{B}_s)$-bimodule, it follows that there exist $w_0 \in \mathfrak{i}_r\mathsf{J}_{11}\mathfrak{i}_s - \{0\}$, $E_{ii}\eta_e \in \mathfrak{B}_r$ and $E_{jj}\tilde{\eta}_e \in \mathfrak{B}_s$ such that $E_{ii}\eta_e w_0 E_{jj}\tilde{\eta}_e \neq 0$. From this, it is not difficult to see that $m_{lt} := E_{li}\eta_e w_0 E_{jt}\tilde{\eta}_e \neq 0$ for any $l \in \{1, \ldots, k_r\}$ and $t \in \{1, \ldots, k_s\}$. Thus, $E_{nl}\eta_g m_{lt} E_{tm}\tilde{\eta}_h \neq 0$ for any $n, l \in \{1, \ldots, k_r\}$, $t, m \in \{1, \ldots, k_s\}$, $g \in H_r$ and $h \in H_s$. The result follows.

ii) Fix $s \in \{1, \ldots, p\}$. We have that $\mathsf{V}_s := \mathfrak{i}_s \mathsf{J}_{11} \mathfrak{i}_s$ is a finite dimensional $\hat{\mathsf{G}}$-graded $\mathfrak{B}_s$-bimodule, and also a graded subalgebra of J. Since J satisfies both chain conditions, because has a finite dimension, by Proposition 1.3.25, there exist homogeneous elements $d_{s1}, \ldots, d_{sq_s} \in \mathsf{V}_s$ such that $\mathsf{V}_s = \bigoplus_{i=1}^{q_s} \mathfrak{B}_s d_{si}$, where each $\mathfrak{B}_s d_{si}$ is an irreducible $\hat{\mathsf{G}}$-graded $\mathfrak{B}_s$-bimodule, and $d_{si} b = \gamma_{is}(b) b d_{si} \neq 0$ for any $b \in$
$beta_s$. Now, consider $\mathsf{N}_s = \mathsf{span}_{\mathbb{F}}\{d_{s1}, \ldots, d_{sq_s}\} \subset \mathsf{J}$ which is a $\hat{\mathsf{G}}$-graded vector space. Since each d_{si} almost commutes with all the elements base β_s of $\mathfrak{B}_s$, it is easy to check that $\mathfrak{B}_s d_{si} = d_{si}\mathfrak{B}_s$, and $\mathfrak{i}_s \mathsf{J}_{11} \mathfrak{i}_s = \mathfrak{B}_s \mathsf{N}_s$. ■

Observing the proof of Lemma 1.5.2, for all $s = 1, \ldots, p$, we obtain that if $\mathsf{deg}(d_{si}) \in \mathsf{Supp}(\Gamma_{\mathfrak{B}_s})$ for all $i = 1, \ldots, q_s$, then $\mathsf{Supp}(\Gamma_{\mathfrak{A}_s}) = \mathsf{Supp}(\Gamma_{\mathfrak{B}_s})$, where $\mathfrak{A}_s = \mathfrak{B}_s \oplus \mathfrak{i}_s \mathsf{J} \mathfrak{i}_s$. On the other hand, given $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ with $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ (which is a graded nilpotent finite dimensional ideal of $\mathfrak{A}$), assume that $\mathsf{N} = \mathsf{span}_{\mathbb{F}}\{d_1, \ldots, d_m\}$ is a $\hat{\mathsf{G}}$-graded vector space such that $\mathsf{J} = \mathfrak{B}\mathsf{N}$, where d_i's are homogeneous elements such that $d_i\mathfrak{B} = \mathfrak{B} d_i$ for all $i = 1, \ldots, m$. If for some $s = 1, \ldots, m$ we have $d_s b = b d_s \neq 0$, for any $b \in \mathfrak{B}$, then the map

$$\begin{array}{rccc} \psi_s : & \mathfrak{B} & \longrightarrow & \mathsf{J} \\ & b & \longmapsto & b d_s \end{array} \tag{1.14}$$

is a homogeneous monomorphism of $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules such that $\psi_s(\mathfrak{i}) = d_s$. In this case, $\mathfrak{B} d_s \cong_{\mathsf{G}} \mathfrak{B}$ as a $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodules.

Corollary 1.5.3 *Under the assumptions of Lemma 1.5.1, if* $\mathfrak{i} \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h)$ *for some* $h \in \hat{\mathsf{G}}$*, then* $\mathsf{J}_h = (\mathsf{J}_{00})_h \oplus (\mathsf{J}_{11})_h$*. In addition,* $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$ *implies* $\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{11}$.

Proof: By Lemmas 1.5.1 and by Lemma 1.5.2, we can write

$$\mathsf{J} = \bigoplus_{g \in \hat{\mathsf{G}}} ((\mathsf{J}_{00})_g \oplus (\mathsf{J}_{01})_g \oplus (\mathsf{J}_{10})_g \oplus (\mathsf{J}_{11})_g) = \bigoplus_{\substack{g \in \hat{\mathsf{G}} \\ i,j \in \{0,1\}}} (\mathsf{J}_{ij})_g \ .$$

Now, given $h \in \hat{\mathsf{G}}$, suppose $\mathfrak{i} \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h)$, i.e., $[\mathfrak{i}, a_h] = 0$ for all $a_h \in \mathfrak{A}_h$. In particular, for any $x_h \in \mathsf{J}_h \subseteq \mathfrak{A}_h$, we have $\mathfrak{i} x_h = x_h \mathfrak{i}$. Since $\mathsf{J}_h = (\mathsf{J}_{00})_h \oplus (\mathsf{J}_{10})_h \oplus (\mathsf{J}_{01})_h \oplus (\mathsf{J}_{11})_h$, where

$\mathsf{J}_{00} = \{z \in \mathsf{J} : \mathfrak{i}z = 0 = z\mathfrak{i}\}$, $\mathsf{J}_{10} = \{x \in \mathsf{J} : \mathfrak{i}x = x, x\mathfrak{i} = 0\}$ and $\mathsf{J}_{01} = \{y \in \mathsf{J} : y\mathfrak{i} = y, \mathfrak{i}y = 0\}$, it follows that $\mathsf{J}_{10} = \mathsf{J}_{01} = \{0\}$. Therefore, $\mathfrak{i} \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h)$ for $h \in \hat{\mathsf{G}}$ implies $\mathsf{J}_h = (\mathsf{J}_{00})_h \oplus (\mathsf{J}_{11})_h$.

Consequently, if $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$, we conclude that

$$\mathsf{J} = \bigoplus_{g\in\hat{\mathsf{G}}} \mathsf{J}_g = \bigoplus_{g\in\hat{\mathsf{G}}} ((\mathsf{J}_{00})_g \oplus (\mathsf{J}_{11})_g) = \mathsf{J}_{00} \oplus \mathsf{J}_{11} \ ,$$

and the result follows. ■

Corollary 1.5.4 *Under the assumptions of Lemma 1.5.2, if $\mathfrak{i}_s \in \mathcal{Z}(\mathfrak{A})$ for all $s = 1, \ldots, p$, then $\mathfrak{i}\mathsf{J}\mathfrak{i} = \bigoplus_{s=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_s$.*

Proof: Suppose $\mathfrak{i}_s \in \mathcal{Z}(\mathfrak{A})$ for any $s = 1, \ldots, p$. Since $\mathfrak{i} = \sum_{s=1}^{p} \mathfrak{i}_s$, it follows that $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$. By Proposition 1.5.1, we have $\mathfrak{i}\mathsf{J}\mathfrak{i} = \mathsf{J}_{11} = \bigoplus_{s,r=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_r$, and hence, we can conclude that $\mathfrak{i}_s\mathsf{J}\mathfrak{i}_r = \{0\}$ for $r \neq s$, since $\mathfrak{i}_s x\mathfrak{i}_r = (\mathfrak{i}_s x\mathfrak{i}_r)\mathfrak{i}_s = 0$ for all $x \in \mathsf{J}$ and $r \neq s$. From this, $\mathfrak{i}\mathsf{J}\mathfrak{i} = \mathsf{J}_{11} = \bigoplus_{s=1}^{p} \mathfrak{i}_s\mathsf{J}\mathfrak{i}_s$. Therefore, we have the result. ■

Observe that all results of this section we can apply for the cases $\hat{\mathsf{G}} = \mathsf{G}$ or $\hat{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$, G is a given finite abelian group.

1.5.1 Some Conditions on $\mathsf{E}^{\mathsf{G}}(\mathsf{J}(\mathfrak{A}))$

Here, we present important results concerning the Grassmann envelope of a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded algebra. We exhibit some results that help to study the graded polynomials identities of the Grassmann Envelope of a graded finite dimensional algebra. The main result here is that, under suitable conditions, it is sufficient to study the Grassmann Envelope of the subalgebras $\mathfrak{A}_s = \mathfrak{B}_s \oplus \mathfrak{i}_s\mathsf{J}\mathfrak{i}_s$ instead of the Grassmann Envelope of $\mathfrak{A} = \times_{s=1}^{p} \mathfrak{A}_s$.

Given a $\mathsf{G} \times \mathbb{Z}_2$-graded algebra $\mathfrak{A}$, it is easy to check that if $a \otimes x_0 \in \mathfrak{A}_0 \otimes \mathsf{E}_0$ with $x_0 \neq 0$, and $(a \otimes x_0)x = x(a \otimes x_0)$ for any $x \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_h$ for some $h \in \mathsf{G}$, then $a \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h)$, where $\mathfrak{A}_h = \mathfrak{A}_{(h,0)} \oplus \mathfrak{A}_{(h,1)}$, since $0 = [a \otimes x_0, b \otimes y] = [a, b] \otimes x_0 y$ for any $b \otimes y \in (\mathfrak{A}_0 \otimes \mathsf{E}_0) \cup (\mathfrak{A}_1 \otimes \mathsf{E}_1)$. Analogously, for any $h \in \mathsf{G}$, we have $a \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h) \subseteq \mathfrak{A}$ implies $(a \otimes y_0)z = z(a \otimes y_0)$ for any $y_0 \in \mathsf{E}_0$ and $z \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_h$.

Lemma 1.5.5 *Let G be a group, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a $\mathsf{G} \times \mathbb{Z}_2$-graded finite dimensional algebra, where $\mathfrak{B}$ is a $\mathsf{G} \times \mathbb{Z}_2$-graded maximal semisimple subalgebra of $\mathfrak{A}$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is* *the Jacobson radical and a graded ideal of* $\mathfrak{A}$. *Given* $h \in \mathsf{G}$, *if there is a nonzero element* $x_0 \in \mathsf{E}_0$ *such that* $(\mathfrak{i} \otimes x_0)x = x(\mathfrak{i} \otimes x_0)$ *for any* $x \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_h$, *then* $\mathsf{E}^{\mathsf{G}}(\mathsf{J})_h = \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00} \oplus \mathsf{J}_{11})_h$. *Particularly,* $\mathsf{E}^{\mathsf{G}}(\mathsf{J}) = \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}) \oplus \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{11})$ *iff* $(\mathfrak{i} \otimes y_0)y = y(\mathfrak{i} \otimes y_0)$ *for all* $y \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, *for some nonzero* $y_0 \in \mathsf{E}_0$.

Proof: For some $h \in \mathsf{G}$, suppose $(\mathfrak{i} \otimes x_0)x = x(\mathfrak{i} \otimes x_0)$ for any $x \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_h$. By above observations, it follows that $\mathfrak{i} \in \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_h)$, and by Corollary 1.5.3, we have $\mathsf{J}_h = (\mathsf{J}_{00} \oplus \mathsf{J}_{11})_h = (\mathsf{J}_{00})_h \oplus (\mathsf{J}_{11})_h$. Hence, we have that $\mathsf{E}^{\mathsf{G}}(\mathsf{J})_h = \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00} \oplus \mathsf{J}_{11})_h$.

Finally, suppose $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$. Again by Corollary 1.5.3, we have $\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{11}$. Consequently, $\mathsf{E}^{\mathsf{G}}(\mathsf{J}) = \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}) \oplus \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{11})$. Conversely, if $\mathsf{E}^{\mathsf{G}}(\mathsf{J}) = \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}) \oplus \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{11})$, then $\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{11}$, and hence, it is easy to see that $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$, since $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}_{00} \oplus \mathsf{J}_{11}$. The result follows. ■

The following result establishes a good condition for the $\mathsf{G}PI$-equivalence between the Grassmann envelope of an algebra $\mathfrak{A}$ and the Grassmann envelope of a unitary subalgebra of $\mathfrak{A}$, where $\mathfrak{A}$ is finite dimensional and G-graded.

Lemma 1.5.6 *Let* G *be a group, and* $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ *a finite dimensional* $\mathbb{F}$*-algebra with a* $\mathsf{G} \times \mathbb{Z}_2$*-grading, where* $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ *is the maximal graded semisimple subalgebra of* $\mathfrak{A}$, $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ *is the Jacobson radical of* $\mathfrak{A}$. *Suppose* $\mathbb{F}$ *is a field of characteristic zero, and* $\mathfrak{i}$ *is the unity of* $\mathfrak{B}$. *If* $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$, *then*

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathfrak{B} \oplus \mathsf{J}_{11}) \times \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}) \ ,$$

where $\mathsf{J}_{11} = \mathfrak{i}\mathsf{J}\mathfrak{i}$, *and* $\mathsf{J}_{00} = \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = 0\}$. *In particular, if* $\mathfrak{A}$ *is unitary, then* $\mathsf{J}_{00} = \{0\}$, *and* $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathfrak{B} \oplus \mathsf{J}_{11})$.

Proof: Firstly, write $\tilde{\mathfrak{A}} = \mathfrak{B} \oplus \mathfrak{i}\mathsf{J}\mathfrak{i}$. Since $\mathfrak{i}\mathsf{J}\mathfrak{i} \subseteq \mathsf{J}$ is a $\mathsf{G} \times \mathbb{Z}_2$-graded ideal of $\tilde{\mathfrak{A}}$ (by Lemma 1.5.2), it follows that $\tilde{\mathfrak{A}}$ and J_{00} are $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebras of $\mathfrak{A}$, and hence, $\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}), \mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}) \subseteq \mathsf{E}^{\mathsf{G}}(\mathfrak{A})$. Thus, $\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subseteq \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})) \cap \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}))$.

Suppose $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$. Hence, by Lemma 1.5.5, $\mathsf{J} = \mathsf{J}_{00} \oplus \mathsf{J}_{11}$. Let us show that $f \in \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})) \cap \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}))$ implies $f \in \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$. In fact, take $f \notin \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$, where $f = f(x_1^{(g_1)}, \ldots, x_k^{(g_k)}) \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ is a polynomial in G-graded variables. By Proposition 1.4.7 we can assume that f is multilinear. Let $a_1 \otimes y_1, \ldots, a_k \otimes y_k \in \mathsf{E}^{\mathsf{G}}(\mathfrak{B}) \cup \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{11}) \cup \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})$ be homogeneous elements such that $\mathsf{deg}(a_i \otimes y_i) = g_i$, $a_1, \ldots, a_k \in \mathfrak{B} \cup \mathsf{J}_{00} \cup \mathsf{J}_{11}$ and

$f(a_1 \otimes y_1, \ldots, a_k \otimes y_k) \neq 0$, such elements exists because f is multilinear. Hence, there exists a monomial $m = x_{i_1} x_{i_2} \cdots x_{i_k}$ in f such that $(a_{i_1} \otimes y_{i_1})(a_{i_2} \otimes y_{i_2}) \cdots (a_{i_k} \otimes y_{i_k}) \neq 0$. From this, $a_{i_1} a_{i_2} \cdots a_{i_k} \neq 0$, where $a_1, \ldots, a_k \in \mathfrak{B} \cup \mathsf{J}_{00} \cup \mathsf{J}_{11}$ are homogeneous elements. If $a_{i_j} \in \mathfrak{B} \cup \mathsf{J}_{11}$ for some $j \in \{1, \ldots, k\}$, it follows that

$$0 \neq a_{i_1} \cdots a_{i_j} \cdots a_{i_k} = a_{i_1} \cdots (\mathfrak{i} a_{i_j} \mathfrak{i}) \cdots a_{i_k} = (\mathfrak{i} a_{i_1} \mathfrak{i}) \cdots (\mathfrak{i} a_{i_j} \mathfrak{i}) \cdots (\mathfrak{i} a_{i_k} \mathfrak{i}), \tag{1.15}$$

since $\mathfrak{i} \in \mathcal{Z}(\mathfrak{A})$, and $\mathsf{J}_{11} = \mathfrak{i}\mathsf{J}\mathfrak{i}$ (by Proposition 1.5.1). So, since $\mathsf{J}_{00} = \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = 0\}$, it follows from the expression in (1.15) that $\{a_{i_1}, \ldots, a_{i_k}\} \cap \mathsf{J}_{00} = \varnothing$. Reciprocally, if $\{a_{i_1}, \ldots, a_{i_k}\} \cap \mathsf{J}_{00} \neq \varnothing$, then $a_{i_j} \notin \mathfrak{B} \cup \mathsf{J}_{11}$ for all $i = 1, \ldots, k$. Hence, we conclude that either $\{a_{i_1}, \ldots, a_{i_k}\} \subset \mathsf{J}_{00}$ or $\{a_{i_1}, \ldots, a_{i_k}\} \subset \mathfrak{B} \cup \mathsf{J}_{11}$, exclusively. Therefore, we have that either $f \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\tilde{\mathfrak{A}}))$ or $f \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathsf{J}_{00}))$, and consequently, $f \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\tilde{\mathfrak{A}})) \cap \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathsf{J}_{00}))$.

Suppose that $\mathfrak{A}$ is unitary. Take $x \in \mathsf{J}_{00}$. Since J is nilpotent, we have that the unity of $\mathfrak{A}$ must be $\mathfrak{i}$, the unity of $\mathfrak{B}$. From this, $x = \mathfrak{i}x = 0$, and so $\mathsf{J}_{00} = \{0\}$. Therefore, the result follows. ■

Theorem 1.5.7 *Let G be a group, $\mathbb{F}$ a field of characteristic zero, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a finite dimensional $\mathbb{F}$-algebra with a $\mathsf{G} \times \mathbb{Z}_2$-grading, where $\mathfrak{B} = \bigtimes_{s=1}^{p} \mathfrak{B}_s$ is the maximal graded semisimple subalgebra of $\mathfrak{A}$, with $\mathfrak{B}_s = M_{k_s}(\mathbb{F}^{\sigma_s}[H_s])$, $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{i}_s$ is the unity of $\mathfrak{B}_s$. If $\mathfrak{i}_s \in \mathcal{Z}(\mathfrak{A})$ for any $s = 1, \ldots, p$, then*

$$\mathsf{E}^\mathsf{G}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^\mathsf{G}(\mathfrak{A}_1) \times \cdots \times \mathsf{E}^\mathsf{G}(\mathfrak{A}_p) \times \mathsf{E}^\mathsf{G}(\mathsf{J}_{00}) ,$$

where $\mathfrak{A}_s = \mathfrak{B}_s + \mathfrak{i}_s \mathsf{J} \mathfrak{i}_s$, $s = 1, \ldots, p$, and $\mathsf{J}_{00} = \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = 0\}$. Moreover, if $\mathfrak{A}$ is unitary, then $\mathsf{J}_{00} = \{0\}$.

Proof: Firstly, observe that

$$\begin{aligned} \mathsf{J}_{00} &= \{x \in \mathsf{J} : \mathfrak{i}x = x\mathfrak{i} = 0\} \\ &= \{x \in \mathsf{J} : \mathfrak{i}_s x = x\mathfrak{i}_s = 0, \forall s = 1, \ldots, p\} , \end{aligned}$$

where $\mathfrak{i} = \sum_{r=1}^{p} \mathfrak{i}_r$. By Lemma 1.5.6, we have that

$$\mathsf{E}^\mathsf{G}(\mathfrak{A}_s \oplus \mathsf{J}_{00}) = \mathsf{E}^\mathsf{G}(\mathfrak{B}_s \oplus \mathfrak{i}_s \mathsf{J} \mathfrak{i}_s \oplus \mathsf{J}_{00}) \equiv_{\mathsf{G}PI} \mathsf{E}^\mathsf{G}(\mathfrak{B}_s \oplus \mathfrak{i}_s \mathsf{J} \mathfrak{i}_s) \times \mathsf{E}^\mathsf{G}(\mathsf{J}_{00}) = \mathsf{E}^\mathsf{G}(\mathfrak{A}_s) \times \mathsf{E}^\mathsf{G}(\mathsf{J}_{00})$$

for all $s = 1, \ldots, p$.

Since $\mathsf{iJi} = \bigoplus_{s,r=1}^{p} \mathsf{i}_r\mathsf{J}\mathsf{i}_s$ (see Lemma 1.5.2), and $\mathsf{i}_s \in \mathscr{Z}(\mathfrak{A})$, it follows that $\mathsf{iJi} = \bigoplus_{s=1}^{p} \mathsf{i}_s\mathsf{J}\mathsf{i}_s$. Hence, it is immediate that

$$\mathfrak{A} = (\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_p) \oplus \left(\bigoplus_{s=1}^{p} \mathsf{i}_s\mathsf{J}\mathsf{i}_s \right) \oplus \mathsf{J}_{00} \cong_{\mathsf{G}\times\mathbb{Z}_2} (\mathfrak{B}_1 \oplus \ \mathsf{i}_1\mathsf{J}\mathsf{i}_1) \times \cdots \times (\mathfrak{B}_p \oplus \mathsf{i}_p\mathsf{J}\mathsf{i}_p) \times \mathsf{J}_{00} \ ,$$

and consequently, $\mathfrak{A} \equiv_{(\mathsf{G}\times\mathbb{Z}_2)PI} \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_p \times \mathsf{J}_{00}$. From this, we have that $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_1) \times \cdots \times \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_p) \times \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})$.

Note that $\mathsf{J}_{00} = \{0\}$ when $\mathfrak{A}$ is unitary (see Lemma 1.5.6). The result follows. ■

By Theorems 1.2.13 and 1.2.20 and Remark 1.2.17, we can determine when two matrix algebras over twisted algebras with canonical elementary G-gradings are G-graded isomorphic. Consequently, we can have $\mathfrak{B}_r \overset{\mathsf{G}\times\mathbb{Z}_2}{\hookrightarrow} \mathfrak{B}_s$ for some of the algebras $\mathfrak{B}_i$'s in Theorem 1.5.7, but not necessarily we have that $\mathfrak{A}_r \overset{\mathsf{G}\times\mathbb{Z}_2}{\hookrightarrow} \mathfrak{A}_s$ (see the definition of a graded immersion in 1.2.15). From this, when $\mathfrak{B}_r \cong_{\mathsf{G}\times\mathbb{Z}_2} \mathfrak{B}_s$, let us build below a $\mathsf{G}\times\mathbb{Z}_2$-graded algebra $\tilde{\mathfrak{A}} = \mathfrak{B}_s \oplus \mathsf{J}(\tilde{\mathfrak{A}})$ such that $\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})) = \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_r)) \cap \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_s))$, where $\mathfrak{A}_i = \mathfrak{B}_i \oplus \mathsf{J}_i$'s are given in Theorem 1.5.7.

The next construction was presented by I. Sviridova in [45]. It is a construction, for any finite dimensional $\hat{\mathsf{G}}$-graded algebra $\mathfrak{A}$, of a graded algebra with a graded "free" Jacobson radical.

Let $\hat{\mathsf{G}}$ be a finite abelian group, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}(\mathfrak{A})$ a finite dimensional $\hat{\mathsf{G}}$-graded algebra, where $\mathfrak{B}$ is a maximal $\hat{\mathsf{G}}$-graded semisimple subalgebra of $\mathfrak{A}$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, which is a nilpotent graded ideal of $\mathfrak{A}$. Consider a $\hat{\mathsf{G}}$-graded semisimple subalgebra $\tilde{\mathfrak{B}} \subseteq \mathfrak{B}$, and $q \in \mathbb{Z}$ with $q > 0$. For any $q \in \mathbb{N}$, consider the set $X_q^{\hat{\mathsf{G}}} = \bigcup_{g\in\hat{\mathsf{G}}}\{x_1^{(g)}, x_2^{(g)}, \ldots, x_q^{(g)}\}$ of graded indeterminates. Now, consider the free product $\tilde{\mathfrak{B}}^{\#} *_{\mathbb{F}} \mathbb{F}\langle X_q^{\hat{\mathsf{G}}}\rangle^{\#}$, and define on it the $\hat{\mathsf{G}}$-grading by the equalities $\mathsf{deg}_{\hat{\mathsf{G}}}(u_1 \cdots u_s) = (\mathsf{deg}_{\hat{\mathsf{G}}} u_1) \cdots (\mathsf{deg}_{\hat{\mathsf{G}}} u_s)$, where $u_i \in \tilde{\mathfrak{B}}^{\#} \bigcup \mathbb{F}\langle X_q^{\hat{\mathsf{G}}}\rangle^{\#}$ are homogeneous elements, and $C^{\#}$ denotes an algebra with the adjoint unity. Let $\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})$ be the graded subalgebra of $\tilde{\mathfrak{B}}^{\#} *_{\mathbb{F}} \mathbb{F}\langle X_q^{\hat{\mathsf{G}}}\rangle^{\#}$ generated by the set $\tilde{\mathfrak{B}} \bigcup \mathbb{F}\langle X_q^{\hat{\mathsf{G}}}\rangle$. Denote by $(X_q^{\hat{\mathsf{G}}})$ the graded (two-sided) ideal of $\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})$ generated by the set of variables $X_q^{\hat{\mathsf{G}}}$. Particularly, we have that $\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}}) = \tilde{\mathfrak{B}} \oplus (X_q^{\hat{\mathsf{G}}})$.

Given a $\hat{\mathsf{G}}T$-ideal Γ of $\mathbb{F}\langle X^{\hat{\mathsf{G}}}\rangle$, denote by $\Gamma(\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}}))$ the $\hat{\mathsf{G}}$-graded ideal

$$\Gamma(\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})) = \{f(h_1,\ldots,h_n) : f \in \Gamma, h_i \in \tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}}), \mathsf{deg}_{\hat{\mathsf{G}}}(h_i) = \mathsf{deg}_{\hat{\mathsf{G}}}(x_i), \forall i\} \trianglelefteq \tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}}),$$

which is called a **verbal ideal** of $\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})$ corresponding to Γ. Now, for all $s \in \mathbb{N}$, we can consider the quotient algebra

$$\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma) = \frac{\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})}{(\Gamma(\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})) + (X_q^{\hat{\mathsf{G}}})^s)}\ . \tag{1.16}$$

Denote also $\mathcal{R}_{q,s}(\mathfrak{A}) = \mathcal{R}_{q,s}(\mathfrak{B},\mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}))$ for $\Gamma = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A})$ and $\tilde{\mathfrak{B}} = \mathfrak{B}$.

Lemma 1.5.8 (Lemma 16, [45]) *Let $\hat{\mathsf{G}}$ be a finite abelian group. For any $q, s \in \mathbb{N}$ and a $\hat{\mathsf{G}}T$-ideal $\Gamma \subseteq \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A})$, the algebra $\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma)$ is a finite dimensional $\hat{\mathsf{G}}$-graded algebra with the ideal of graded identities $\mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma)) \supseteq \Gamma$. Moreover, $\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma) = \overline{\mathfrak{B}} \oplus \mathsf{J}(\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma))$, where $\overline{\mathfrak{B}}$ is a maximal semisimple $\hat{\mathsf{G}}$-graded subalgebra of $\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma)$, and $\overline{\mathfrak{B}} \cong_{\hat{\mathsf{G}}} \tilde{\mathfrak{B}}$. The Jacobson radical of $\mathcal{R}_{q,s}(\tilde{\mathfrak{B}},\Gamma)$ is equal to $(X_q^{\hat{\mathsf{G}}})/(\Gamma(\tilde{\mathfrak{B}}(X_q^{\hat{\mathsf{G}}})) + (X_q^{\hat{\mathsf{G}}})^s)$, and it is nilpotent of degree less than or equal to s. In addition, if $q \geqslant \mathsf{max}_{g\in\hat{\mathsf{G}}}(\mathsf{dim}_{\mathbb{F}}(\mathsf{J}(\mathfrak{A}))_g)$ and $s \geqslant \mathsf{nd}(\mathsf{J}(\mathfrak{A}))$, then $\mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{A})) = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A})$.*

Corollary 1.5.9 (Absorption Lemma) *Let $\hat{\mathsf{G}}$ be a finite abelian group, and $\mathbb{F}$ a field of characteristic zero. Consider any two finite dimensional $\hat{\mathsf{G}}$-graded algebras $\mathfrak{A}_1 = \mathfrak{B}_1 \oplus \mathsf{J}_1$ and $\mathfrak{A}_2 = \mathfrak{B}_2 \oplus \mathsf{J}_2$, where $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are maximal semisimple $\hat{\mathsf{G}}$-graded subalgebras of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and J_1 and J_2 are the Jacobson radicals of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively. If $\mathfrak{B}_1 \cong_{\hat{\mathsf{G}}} \mathfrak{B}_2$, then there exists a finite dimensional $\hat{\mathsf{G}}$-graded algebra $\tilde{\mathfrak{A}} = \overline{\mathfrak{B}}_1 \oplus \tilde{\mathsf{J}}$ such that $\mathsf{T}^{\hat{\mathsf{G}}}(\tilde{\mathfrak{A}}) = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_1 \times \mathfrak{A}_2)$. Here, $\overline{\mathfrak{B}}_1 \cong_{\hat{\mathsf{G}}} \mathfrak{B}_1$, and it is the maximal graded semisimple subalgebra of $\tilde{\mathfrak{A}}$, $\tilde{\mathsf{J}}$ is the Jacobson radical of $\tilde{\mathfrak{A}}$, which is a nilpotent graded ideal of $\tilde{\mathfrak{A}}$, and $\mathsf{nd}(\tilde{\mathsf{J}}) = \mathsf{max}\{\mathsf{nd}(\mathsf{J}_i) : i = 1,2\}$.*

Proof: Let us denote $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$. Let $q = \mathsf{max}\{\mathsf{dim}_{\mathbb{F}}(\mathsf{J}_i) : i = 1,2\}$, $s = \mathsf{max}\{\mathsf{nd}(\mathsf{J}_i) : i = 1,2\}$, and $\Gamma = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}) = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_1) \cap \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_2)$. Consider the algebra $\mathcal{R}_{q,s}(\mathfrak{B}_1,\Gamma)$ as in (1.16). By Lemma 1.5.8, we have that $\Gamma \subseteq \mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{B}_1,\Gamma))$, where $\mathcal{R}_{q,s}(\mathfrak{B}_1,\Gamma) = \overline{\mathfrak{B}}_1 \oplus \tilde{\mathsf{J}}$ is a finite dimensional $\hat{\mathsf{G}}$-graded algebra, such that $\overline{\mathfrak{B}}_1 \cong_{\hat{\mathsf{G}}} \mathfrak{B}_1$, and it is the maximal graded semisimple subalgebra of $\mathcal{R}_{q,s}(\mathfrak{B}_1,\Gamma)$, and $\tilde{\mathsf{J}}$ is the Jacobson radical of $\mathcal{R}_{q,s}(\mathfrak{B}_1,\Gamma)$, which is a nilpotent graded ideal of $\tilde{\mathfrak{A}}$ with $\mathsf{nd}(\tilde{\mathsf{J}}) = s$.

Now, since $\mathfrak{B}_2 \cong_{\hat{\mathsf{G}}} \mathfrak{B}_1$, there exists a $\hat{\mathsf{G}}$-isomorphism $\psi_2 : \mathfrak{B}_1 \to \mathfrak{B}_2$, and hence, we have $\psi_2(\mathfrak{B}_1) = \mathfrak{B}_2$. Denote $I = (\Gamma(\mathfrak{B}_1(X_q^{\hat{\mathsf{G}}})) + (X_q^{\hat{\mathsf{G}}})^s)$, which is a graded ideal of $\mathfrak{B}_1(X_q^{\hat{\mathsf{G}}})$ (see (1.16)). Suppose that the set $\{r_1, \ldots, r_{q_2}\}$ is an $\mathbb{F}$-basis of J_2. Then we have $r_i = \sum_{\theta \in \hat{\mathsf{G}}} r_{i\theta}$, where $r_{i\theta} \in \mathsf{J}_2 \cap (\mathfrak{A}_2)_\theta$, for all $i = 1, \ldots, q_2$. Consider the map $\bar{\varphi}_2 : \overline{x}_i^{(\theta)} = x_i^{(\theta)} + I \mapsto r_{i\theta}$, for all $i = 1, \ldots, q_2$, and $\bar{\varphi}_2 : \bar{x}_i^{(\theta)} == x_i + I \mapsto 0$ for all $i = q_2 + 1, \ldots, q$ $(q_2 \leqslant q)$. Assume that $\bar{\varphi}_2(b + I) = \psi_2(b)$ for any $b \in \mathfrak{B}_1$. Then $\bar{\varphi}_2$ can be extended to a graded epimorphism $\bar{\varphi}_2 : \mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma) \to \mathfrak{A}_2$ of graded algebras. In fact, consider the homomorphism of $\hat{\mathsf{G}}$-graded algebras $\varphi_2 : \mathfrak{B}_1(X_q^{\hat{\mathsf{G}}}) \to \mathfrak{A}_2$ such that $\varphi_2(b) = \psi_2(b)$ for any $b \in \mathfrak{B}_1$, and $\varphi_2(x_j^{(\theta)}) = r_{j\theta}$, for all $j = 1, \ldots, q_2$, and $\varphi_2(x_j^{(\theta)}) = 0$ for $j = q_2 + 1, \ldots, q$. Then $\varphi_2 = \bar{\varphi}_2 \pi$, where $\pi : \mathfrak{B}_1(X_q^{\hat{\mathsf{G}}}) \to \mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)$ is the natural $\hat{\mathsf{G}}$-graded homomorphism (quotient), since $\mathsf{ker}(\pi) \subseteq \mathsf{ker}(\varphi_2)$. In fact, $\mathsf{ker}(\pi) = I$, and $\varphi_2(I) \subseteq \Gamma(\mathfrak{A}_2) + \mathsf{J}_2^s = \{0\}$, since $\Gamma \subseteq \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_2)$, and $\mathsf{nd}(\mathsf{J}_2) \leqslant s$. Hence, the following diagram commutes:

$$\begin{tikzcd} \mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma) \arrow[rr, "\bar{\varphi}_2"] & & \mathfrak{A}_2 \\ & \mathfrak{B}_1(X_q^{\hat{\mathsf{G}}}) \arrow[ul, "\pi"] \arrow[ur, "\varphi_2"'] & \end{tikzcd} .$$

Take any multilinear $f = f(x_1^{(g_1)}, x_2^{(g_2)}, \ldots, x_n^{(g_n)}) \in \mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma))$. Let us show that $f \equiv_{\hat{\mathsf{G}}} 0$ in $\mathfrak{A}_2$. In fact, for any homogeneous elements $a_1, a_2, \ldots, a_n \in \mathfrak{B}_2 \cup \mathsf{J}_2$ such that $\mathsf{deg}(a_i) = \mathsf{deg}(x_i^{(g_i)}) = g_i$, for all $i = 1, \ldots, n$, we have that there exist $b_1, b_2, \ldots, b_n \in \mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)$ homogeneous elements such that $\bar{\varphi}_2(b_i) = a_i$, with $\mathsf{deg}(b_i) = g_i$, for all $i = 1, \ldots, n$. Hence, we have that

$$f(a_1, a_2, \ldots, a_n) = f(\bar{\varphi}_2(b_1), \bar{\varphi}_2(b_2), \ldots, \bar{\varphi}_2(b_n)) = \bar{\varphi}_2(f(b_1, b_2, \ldots, b_n)) = \varphi_2(\overline{0}) = 0 .$$

From this, we have that $f \equiv_{\hat{\mathsf{G}}} 0$ in $\mathfrak{A}_2$, and so $f \in \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_2)$. Consequently, $\mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)) \subseteq \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_2)$.

Analogously, we can build a graded epimorphism of $\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)$ to $\mathfrak{A}_1$ (it is sufficient to consider $\psi_1 : \mathfrak{B}_1 \to \mathfrak{B}_1$ such that $\psi_1(b) = b$ for any $b \in \mathfrak{B}_1$, and to process as in the first part of this proof), and hence, we conclude also that $\mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)) \subseteq \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_1)$. Therefore, it follows that $\mathsf{T}^{\hat{\mathsf{G}}}(\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma)) \subseteq \left(\mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_1) \cap \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}_2)\right) = \mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A})$, and consequently, we have that $\mathcal{R}_{q,s}(\mathfrak{B}_1, \Gamma) \equiv_{\hat{\mathsf{G}}PI} \mathfrak{A}$. ■

Lemma 1.5.10 (Lemma 31, [45]) *Given $\hat{\mathsf{G}}\times\mathbb{Z}_2$-graded algebras $\mathfrak{A}$, and $\mathfrak{B}$ over a field $\mathbb{F}$ of characteristic zero, where $\hat{\mathsf{G}}$ is a finite abelian group, we have $\mathfrak{A}\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathfrak{B}$ iff $\mathsf{E}^{\hat{\mathsf{G}}\times\mathbb{Z}_2}(\mathfrak{A})\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathsf{E}^{\hat{\mathsf{G}}\times\mathbb{Z}_2}(\mathfrak{B})$ (as $\hat{\mathsf{G}}\times\mathbb{Z}_2$-graded algebras).*

Lemma 1.5.11 *Let $\hat{\mathsf{G}}$ be a finite abelian group, $\mathfrak{A}$ and $\mathfrak{B}$ two $\hat{\mathsf{G}}\times\mathbb{Z}_2$-graded algebras over a field $\mathbb{F}$ of characteristic zero, which are PI-algebras. If $\mathfrak{A}\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathfrak{B}$, then $\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{A})\equiv_{\hat{\mathsf{G}}PI}\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{B})$ (as $\hat{\mathsf{G}}$-graded algebras).*

Proof: Any $\hat{\mathsf{G}}$-graded polynomial identity $f\in\mathbb{F}\langle X^{\hat{\mathsf{G}}}\rangle$ can be seen as a $(\hat{\mathsf{G}}\times\mathbb{Z}_2)$-graded polynomial identity. More precisely, $f=f(x_1^{(g_1)},\ldots,x_n^{(g_n)})\in\mathbb{F}\langle X^{\hat{\mathsf{G}}}\rangle$ corresponds to the set of polynomials $W_f:=\{f(x_1^{(g_1,\lambda_1)},\ldots,x_n^{(g_n,\lambda_n)}):\forall(\lambda_1,\ldots,\lambda_n)\in(\mathbb{Z}_2)^n\}\subset\mathbb{F}\langle X^{\hat{\mathsf{G}}\times\mathbb{Z}_2}\rangle$.

Let $\mathfrak{A}=\bigoplus_{(g,\lambda)\in\hat{\mathsf{G}}\times\mathbb{Z}_2}\mathfrak{A}_{(g,\lambda)}$ be a $(\hat{\mathsf{G}}\times\mathbb{Z}_2)$-graded algebra, where the $\hat{\mathsf{G}}$-grading is induced, i.e. $\mathfrak{A}=\bigoplus_{g\in\hat{\mathsf{G}}}\mathfrak{A}_g$ with $\mathfrak{A}_g=\mathfrak{A}_{(g,0)}\oplus\mathfrak{A}_{(g,1)}$ for any $g\in\hat{\mathsf{G}}$. We have that $\mathfrak{A}$ satisfies a $\hat{\mathsf{G}}$-graded polynomial identity $f\in\mathbb{F}\langle X^{\hat{\mathsf{G}}}\rangle$ iff $\mathfrak{A}$ satisfies all the $(\hat{\mathsf{G}}\times\mathbb{Z}_2)$-graded polynomial identities of W_f, i.e. $f\in\mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A})$ iff $W_f\subseteq\mathsf{T}^{\hat{\mathsf{G}}\times\mathbb{Z}_2}(\mathfrak{A})$.

Consider two $(\hat{\mathsf{G}}\times\mathbb{Z}_2)$-graded algebras $\mathfrak{A}$ and $\mathfrak{B}$, and their Grassmann envelope $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ and $\mathsf{E}^{\mathsf{G}}(\mathfrak{B})$, respectively. By Lemma 1.5.10, we have that $\mathfrak{A}\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathfrak{B}$ iff $\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{A})\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{B})$, where we consider $\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{A})$ and $\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{B})$ with their $(\hat{\mathsf{G}}\times\mathbb{Z}_2)$-gradings. By above reason, we obtain that if $\mathfrak{A}\equiv_{(\hat{\mathsf{G}}\times\mathbb{Z}_2)PI}\mathfrak{B}$, then $\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{A})\equiv_{\hat{\mathsf{G}}PI}\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{B})$. Therefore, we conclude that if $\mathsf{T}^{\hat{\mathsf{G}}\times\mathbb{Z}_2}(\mathfrak{A})=\mathsf{T}^{\hat{\mathsf{G}}\times\mathbb{Z}_2}(\mathfrak{B})$, then $\mathsf{T}^{\hat{\mathsf{G}}}(\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{A}))=\mathsf{T}^{\hat{\mathsf{G}}}(\mathsf{E}^{\hat{\mathsf{G}}}(\mathfrak{B}))$. ■

By Corollary 1.5.9, it is not difficult to check that $\mathfrak{B}_r\cong_{\hat{\mathsf{G}}}\mathfrak{B}_1$ for all $r=2,\ldots,n$ implies that $\mathfrak{A}=(\mathfrak{A}_1\times\cdots\times\mathfrak{A}_n)\equiv_{\hat{\mathsf{G}}PI}\hat{\mathfrak{A}}$, where $\mathfrak{A}_i=\mathfrak{B}_i\oplus\mathsf{J}(\mathfrak{A}_i)$, and $\hat{\mathfrak{A}}=\mathcal{R}_{q,s}(\mathfrak{B}_1,\mathsf{T}^{\hat{\mathsf{G}}}(\mathfrak{A}))$. In this sense, assuming $\hat{\mathsf{G}}=\mathsf{G}\times\mathbb{Z}_2$, we can use Corollary 1.5.9 and Lemma 1.5.11 to improve Theorem 1.5.7 up to $\mathsf{G}\times\mathbb{Z}_2$-isomorphisms of semisimple parts of $\mathfrak{A}_i$. Thus, we have the following result.

Theorem 1.5.12 *Let G be a finite abelian group, $\mathbb{F}$ a field of characteristic zero, and $\mathfrak{A}=\mathfrak{B}\oplus\mathsf{J}$ a finite dimensional $\mathbb{F}$-algebra with a $\mathsf{G}\times\mathbb{Z}_2$-grading, where $\mathfrak{B}=\bigtimes_{s=1}^{p}\mathfrak{B}_s$ is the maximal graded semisimple subalgebra of $\mathfrak{A}$, with $\mathfrak{B}_s=M_{k_s}(\mathbb{F}^{\sigma_s}[H_s])$, $\mathsf{J}=\mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{i}_s$ is the unity of $\mathfrak{B}_s$. If $\mathfrak{i}_s\in\mathcal{Z}(\mathfrak{A})$ for any $s=1,\ldots,p$, then there exist finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded unitary algebras $\tilde{\mathfrak{A}}_1,\tilde{\mathfrak{A}}_2,\ldots,\tilde{\mathfrak{A}}_p,\tilde{\mathsf{J}}_{00}$ such that*

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\equiv_{\mathsf{G}PI}\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}_1)\times\cdots\times\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}_q)\times\mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}}_{00})\ ,$$

where $\tilde{\mathfrak{A}}_s = \overline{\mathfrak{B}}_s \oplus \tilde{\mathsf{J}}_s$ *are* $\mathsf{G} \times \mathbb{Z}_2$*-graded algebras satisfying* $\overline{\mathfrak{B}}_s \ncong_{\mathsf{G}\times\mathbb{Z}_2} \overline{\mathfrak{B}}_r$ *for all* $s \neq r$, $\overline{\mathfrak{B}}_s \cong_{\mathsf{G}\times\mathbb{Z}_2} \mathfrak{B}_{i_s}$ *for some* $i_s \in \{1, \ldots, p\}$, $\tilde{\mathsf{J}}_s$ *is the Jacobson radical of* $\tilde{\mathfrak{A}}_s$, *and* $\tilde{\mathsf{J}}_{00}$ *is a finite dimensional* $\mathsf{G} \times \mathbb{Z}_2$*-graded nilpotent algebra.*

Proof: By Theorem 1.5.7, we have $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_1) \times \cdots \times \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_p) \times \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})$. Suppose that $\mathfrak{B}_r \cong_{\mathsf{G}\times\mathbb{Z}_2} \mathfrak{B}_s$ for some $r \neq s$, and hence, by Corollary 1.5.9 and Lemma 1.5.11, we have that $\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_r) \times \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_s)) = \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}_s))$ for some finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded algebra $\tilde{\mathfrak{A}}_s = \mathfrak{B}_s \oplus \mathsf{J}(\tilde{\mathfrak{A}}_s)$. From this, the result follows. ■

Assume that $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ is a finite dimensional $\mathbb{F}$-algebra with a $\mathsf{G} \times \mathbb{Z}_2$-grading, where $\mathsf{J} = \mathsf{J}(\mathfrak{A})$, and $\mathfrak{B} = \times_{i=1}^{p} \mathbb{F}^{\sigma_i}[H_i]$, with $H_i \leqslant \mathsf{G} \times \mathbb{Z}_2$, $\sigma \in \mathsf{Z}^2(H_i, \mathbb{F}^*)$. Observe that by Theorem 1.2.13 and Example 1.3.2, the previous theorem can be rewritten in terms of partial order "$\preceq$", since $(H_i, [\sigma_i]) = (H_j, [\sigma_j])$ (i.e. $H_i = H_j$, and $[\sigma_i]) = [\sigma_j]$) implies $\mathbb{F}^{\sigma_i}[H_i] \cong_{\mathsf{G}\times\mathbb{Z}_2} \mathbb{F}^{\sigma_j}[H_j]$.

CHAPTER 2

SECOND COHOMOLOGY GROUP

In this chapter we present some notions and properties of the cohomology theory of groups. We will use these concepts as tools in Chapter 4. Our goal in this chapter is to determine suitable conditions to ensure that the restriction homomorphism from $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ into $\mathsf{H}^2(H,\mathsf{M})$ is surjective, where H is a subgroup of a group G. Unless otherwise stated, G denotes a multiplicative group and M denotes an abelian additive group that has a structure of a left G-module. All modules in this chapter are assumed to be left modules.

2.1 Definitions and Properties

Let us define the **Second Cohomology Group**. Posteriorly, we will exhibit some important results. The following definition is a generalization of Definition 1.2.1. For more details, see [39], Section 9.1.2.

Definition 2.1.1 *Let* $(\mathsf{G},\cdot)$ *be a multiplicative group, and* $(\mathsf{M},+)$ *an additive abelian group. We say that* M *is a* **left G-module** *if there is a well-defined map from* $\mathsf{G}\times\mathsf{M}$ *into* M *which satisfies*

i) $r(m_1+m_2)=rm_1+rm_2$,

ii) $(r_1+r_2)m=r_1m+r_2m$,

iii) $(r_1r_2)m=r_1(r_2m)$,

for any $r, r_1, r_2 \in \mathbb{Z}\mathsf{G}$ and $m, m_1, m_2 \in \mathsf{M}$, where $\mathbb{Z}\mathsf{G}$ is a group ring.

Recall that we denote by η_g the element of $\mathbb{Z}\mathsf{G}$ which corresponds to an element $g \in \mathsf{G}$. For convenience we assume that $\eta_e m = m$ for any $m \in \mathsf{M}$, where e is the neutral element of G (M is a unitary left $\mathbb{Z}\mathsf{G}$-module).

Definition 2.1.2 *Let G be a group and M a (left) G-module. A map $\sigma : \mathsf{G} \times \mathsf{G} \to \mathsf{M}$ is said to be a 2-**cocycle**[1] if it satisfies the following relation:*

$$\sigma(g,h) + \sigma(gh,t) = \eta_g \sigma(h,t) + \sigma(g,ht) \ ,$$

*for any $g, h, t \in \mathsf{G}$. We say that a 2-cocycle $\rho : \mathsf{G} \times \mathsf{G} \to \mathsf{M}$ is a 2-**coboundary** if there exists a function $f : \mathsf{G} \to \mathsf{M}$ such that*

$$\rho(g,h) = \eta_g f(h) - f(gh) + f(g)$$

for any $g, h \in \mathsf{G}$.

Notice that if M is a trivial (left) G-module (also we say "G acts trivially on M"), i.e. $\eta_g m = m$ for any $g \in \mathsf{G}$ and $m \in \mathsf{M}$, it follows that

$$\sigma(g,h) + \sigma(gh,t) = \sigma(h,t) + \sigma(g,ht)$$

for any 2-cocycle σ and $g, h, t \in \mathsf{G}$. Similarly for a 2-coboundary ρ, we have: $\rho(g,h) = f(h) - f(gh) + f(g)$ for any $g, h \in \mathsf{G}$.

Definition 2.1.3 *Let G be a group and M a G-module. We define*

$$\mathsf{Z}^2(\mathsf{G},\mathsf{M}) = \{\textit{all the 2-cocycles } \sigma : \mathsf{G} \times \mathsf{G} \to \mathsf{M}\} \ , \ \textit{and}$$

$$\mathsf{B}^2(\mathsf{G},\mathsf{M}) = \{\textit{all the 2-coboundaries } \rho : \mathsf{G} \times \mathsf{G} \to \mathsf{M}\} \ .$$

Given $\sigma, \rho \in \mathsf{Z}^2(\mathsf{G},\mathsf{M})$, we define[2]

$$\sigma + \rho : (g,h) \mapsto \sigma(g,h) + \rho(g,h)$$

[1] If we assume that M has a multiplicative notation, then $\sigma \in \mathsf{Z}^2(\mathsf{G},\mathsf{M})$ if $\sigma(g,h)\sigma(gh,t) = (\eta_g \cdot \sigma(h,t))\sigma(g,ht)$, and $\varrho \in \mathsf{B}^2(\mathsf{G},\mathsf{M})$ if $\varrho(g,h) = \frac{(\eta_g \cdot f(h))f(g)}{f(gh)}$ for some map $f : \mathsf{G} \to \mathsf{M}$.

[2] If we assume that M has a multiplicative notation, then $\sigma\rho : (g,h) \mapsto \sigma(g,h)\rho(g,h)$ for any $g, h \in \mathsf{G}$.

for any $g, h \in \mathsf{G}$. In [39], Proposition 9.11 ensures that $\mathsf{Z}^2(\mathsf{G},\mathsf{M})$ is an abelian group, and $\mathsf{B}^2(\mathsf{G},\mathsf{M})$ is a subgroup of $\mathsf{Z}^2(\mathsf{G},\mathsf{M})$, with respect to this operation. Observe that, since $\mathsf{B}^2(\mathsf{G},\mathsf{M})$ is a group, the inverse element of a 2-coboundary ρ defined by $\rho(g,h) = \eta_g f(h) - f(gh) + f(g)$ for some $f : \mathsf{G}\times\mathsf{G} \to \mathsf{M}$ is given by

$$(-\rho)(g,h) = -f(g) + f(gh) - \eta_g f(h)\ .$$

Definition 2.1.4 *The* **second cohomology group** *of* G *is defined as a quotient group*

$$\mathsf{H}^2(\mathsf{G},\mathsf{M}) := \frac{\mathsf{Z}^2(\mathsf{G},\mathsf{M})}{\mathsf{B}^2(\mathsf{G},\mathsf{M})}\ .$$

The elements of $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ are denoted by $[\sigma]$, where $\sigma \in \mathsf{Z}^2(\mathsf{G},\mathsf{M})$. Hence, $[\sigma] = [\rho]$ in $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ if there exists $\xi \in \mathsf{B}^2(\mathsf{G},\mathsf{M})$ such that $\sigma = \xi + \rho$.

Let us present some basic results, which relate the second cohomology group and the orders of groups and subgroups. These results and some other facts can be found in [7, 19, 25, 39, 46].

Theorem 2.1.5 (Theorem 6.14, [25]) *Let* G *be a finite group and* M *a* G*-module. Every element of* $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ *has finite order, which is a divisor of* $|\mathsf{G}|$*.*

In [39], it is shown that (see Corollary 9.41) if M is a finitely generated G-module, then $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ is finite. Still in [39], it is proved (see Corollary 9.90) the following items:

i) There is an injection $\theta : \mathsf{H}^2(\mathsf{G},\mathsf{M}) \longrightarrow \bigoplus_p \mathsf{H}^2(\mathsf{G}_p,\mathsf{M})$, where G_p is a Sylow p-subgroup of G, p is a prime divisor of $|\mathsf{G}|$;

ii) If $\mathsf{H}^2(\mathsf{G}_p,\mathsf{M}) = \{0\}$ for all Sylow p-subgroups, then $\mathsf{H}^2(\mathsf{G},\mathsf{M}) = \{0\}$.

Already in [46], it is proved (see Theorem 11.8.18) that if $H \trianglelefteq \mathsf{G}$ with index $[\mathsf{G} : H] = m$ coprime to the order of H, then $\mathsf{H}^2(\mathsf{G},\mathsf{M}) \cong \mathsf{H}^2(H,\mathsf{M})^{\mathsf{G}} \oplus \mathsf{H}^2(\mathsf{G}/H,\mathsf{M}^H)$, where $\mathsf{M}^H = \{m \in \mathsf{M} : \eta_h m = m, \forall h \in H\}$, and $\mathsf{H}^2(H,\mathsf{M})^{\mathsf{G}} = \{\sigma \in \mathsf{H}^2(H,\mathsf{M}) : \eta_g\sigma = \sigma, \forall g \in \mathsf{G}\}$. It is also proved there (see Theorem 12.1.3) that if H is a subgroup of G and M is an H-module, then $\mathsf{H}^2(\mathsf{G},\mathsf{Hom}_{\mathbb{Z}H}(\mathbb{Z}\mathsf{G},\mathsf{M}))$ and $\mathsf{H}^2(H,\mathsf{M})$ are isomorphic, where $\mathsf{Hom}_{\mathbb{Z}H}(\mathbb{Z}\mathsf{G},\mathsf{M})$ is the group of $\mathbb{Z}H$-homomorphisms from $\mathbb{Z}\mathsf{G}$ into M. This last result is known as Shapiro Lemma (see Theorem 6.3.2 in [19]).

The proposition below improves Exercise 6.10.3, in [25]. Roughly speaking, for all $\lambda \in \mathbb{F}$, a field $\mathbb{F}$ contains $\sqrt[n]{\lambda}$ iff $\mathbb{F}$ contains a root of the polynomial $p_\lambda(x) = x^n - \lambda$, i.e. p_λ has a solution in $\mathbb{F}$. In particular, any algebraically closed field $\mathbb{F}$ contains $\sqrt[n]{\lambda}$ for all $\lambda \in \mathbb{F}$. Hence, we write $\gamma = \sqrt[n]{\lambda}$ to denote that $\gamma^n = \lambda$.

Proposition 2.1.6 *Let* G *be a finite group of order* n*,* $\mathbb{F}$ *a field such that* $\sqrt[n]{\lambda} \in \mathbb{F}$ *for any* $\lambda \in \mathbb{F}$*, and* $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$ *the second cohomology group of* G *with coefficients in the multiplicative group* $\mathbb{F}^*$*, where* G *acts trivially on* $\mathbb{F}^*$*. For any* $[\gamma] \in \mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$*, the representative cocycle* γ *can be chosen to have values that are* n*-th roots of unity. Therefore,* $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$ *is finite.*

Proof: Let us assume that $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$ is a group with multiplicative notation. Let $[\sigma] \in \mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$, where $\sigma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ is a 2-cocycle. Since G is a group of order n, by Theorem 2.1.5, it follows that $[\sigma]^n$ is the neutral element of $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$, and so $[\sigma]^n = [\sigma^n] = [1]$ which implies $\sigma^n \in \mathsf{B}^2(\mathsf{G}, \mathbb{F}^*)$. Let $f : \mathsf{G} \to \mathbb{F}^*$ be a map, and $\xi \in \mathsf{B}^2(\mathsf{G}, \mathbb{F}^*)$ a 2-coboundary such that

$$\xi(g,h) = \frac{f(gh)}{f(g)f(h)} \quad \text{and} \quad \xi(g,h) = \sigma^n(g,h) = (\sigma(g,h))^n$$

for any $g, h \in \mathsf{G}$. For each $g \in \mathsf{G}$, by hypothesis, it follows that $\sqrt[n]{f(g)} \in \mathbb{F}^*$, and thus, we can consider the map $\hat{f} : \mathsf{G} \to \mathbb{F}^*$ defined by $\hat{f}(g) = \sqrt[n]{f(g)}$ for any $g \in \mathsf{G}$. Put $\hat{\xi}(g,h) = \dfrac{\hat{f}(g)\hat{f}(h)}{\hat{f}(gh)}$. It is not difficult to see that $\hat{\xi} \in \mathsf{B}^2(\mathsf{G}, \mathbb{F}^*)$, and

$$(\hat{\xi}\sigma(g,h))^n = \hat{\xi}^n(g,h)\sigma^n(g,h) = (\xi(g,h))^{-1}\sigma^n(g,h) = 1 \tag{2.1}$$

for any $g, h \in \mathsf{G}$. Notice that $\hat{\xi}\sigma$ is an element of $\mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$, and $[\hat{\xi}\sigma] = [\sigma]$, since $\hat{\xi} \in \mathsf{B}^2(\mathsf{G}, \mathbb{F}^*)$. Consider now $\hat{\sigma} = \hat{\xi}\sigma$. By (2.1), for each $g, h \in \mathsf{G}$, it follows that $\hat{\sigma}(g,h)$ is an n-th root of unit. Therefore, we have $\hat{\sigma}$ is a representative of $[\sigma]$ which has values that are n-th roots of the unit.

Finally, since $\mathbb{F}$ has only n roots of unit and G is a finite group, it follows that $\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)$ is a finite group. ■

Notice that there exists at most n^{n^2} possibilities for functions from $\mathsf{G} \times \mathsf{G}$ into {all n-th roots of unity}, and hence, under the assumptions of the previous proposition, it follows that $|\mathsf{H}^2(\mathsf{G}, \mathbb{F}^*)| \leqslant n^{n^2}$. But, by proof of the previous proposition and Proposition

1.2.6, we can improve this estimate as follows

$$|\mathsf{H}^2(\mathsf{G},\mathbb{F}^*)| \leqslant n^{\frac{n(n-1)}{2}+1} ,$$

We have $\sigma(g,e) = \sigma(e,h) = \sigma(e,e)$, and $\sigma(g,g^{-1}) = \sigma(g^{-1},g)$ for any $g,h \in \mathsf{G}$. And hence, being $\mathsf{G} = \{e, x_1, \ldots, x_{n-1}\}$, fix $\sigma(e,e) = \lambda_0 \in \mathbb{F}^*$, and $\sigma(x_i,x_j) = \lambda_{ij} \in \mathbb{F}^*$ for all $1 \leqslant i \leqslant j \leqslant n-1$. Thus, by definition of a 2-cocycle (see Definition 2.1.2), we have that

$$\sigma(x_i,x_j) = \frac{\sigma(x_j,(x_ix_j)^{-1})\sigma(x_i,x_i^{-1})}{\sigma(x_ix_j,(x_ix_j)^{-1})} = \frac{\sigma(x_j,(x_ix_j)^{-1})\sigma(x_i^{-1},x_i)}{\sigma((x_ix_j)^{-1},x_ix_j)} ,$$

where we put $g = x_i$, $h = x_j$ and $t = (x_ix_j)^{-1}$ in Definition 1.2.17, and we consider that G acts trivially on $\mathbb{F}^*$. Let us fix a choice for values of $\sigma(e,e)$, and $\sigma(x_i,x_j)$ for all $1 \leqslant j \leqslant i \leqslant n-1$. Consequently, we can build the following table

σ	e	x_1	$\cdots$	x_{n-1}
e	n	1	$\cdots$	1
x_1	1	n	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	1
x_{n-1}	1	n	$\cdots$	n

which represents the number of possibilities (possible combinations) for the function $\sigma(g,h)$, $g,h \in \mathsf{G}$.

2.2 Restriction $\mathsf{res}_H^{\mathsf{G}}$

In the previous section we present some results which relate the second cohomology groups of G and its subgroups. Here, we study a way to ensure an existence of a surjective homomorphism from $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ onto $\mathsf{H}^2(H,\mathsf{M})$, for a subgroup H of G. Here, to simplify notation, we write $gm := \eta_g m$ for any $g \in \mathsf{G}$ and $m \in \mathsf{M}$.

Consider the pair (H,M), where H is a subgroup of a group G and M is a G-module. Given an element $g \in \mathsf{G}$, we denote by $c(g)$ the bijection

$$\begin{array}{rccc} c(g) := (\gamma_g, f_g) : & (H,\mathsf{M}) & \longrightarrow & (gHg^{-1},\mathsf{M}) \\ & (h,m) & \mapsto & (\gamma_g(h), f_g(m)) \end{array} ,$$

where $\gamma_g : h \mapsto ghg^{-1}$ and $f_g : m \mapsto g^{-1}m$. Observe that γ_g is an isomorphism of groups, and f_g is an isomorphism of abelian groups. This application is an action of G on (H, M), called **conjugation action**. Note that for any $h \in H$ and $m \in \mathsf{M}$, we have:

$$f_g(\gamma_g(h)m) = f_g(ghg^{-1}m) = g^{-1}ghg^{-1}m = hg^{-1}m = hf_g(m) \ ,$$

In this case, we say that the pair (γ_g, f_g) is a **compatible pair**.

It is well known (see §III.8 in [7], or §9.5 in [39], or §11.8 in [46]) that there exists an isomorphism of groups

$$c(g)^* : \mathsf{H}^2(gHg^{-1}, \mathsf{M}) \longrightarrow \mathsf{H}^2(H, \mathsf{M})$$

induced by the conjugation map $c(g) = (\gamma_g, f_g)$. For more details how $c(g)$ induces a homomorphism, see §9.5 in [39].

Define a map from $\mathsf{G} \times \mathsf{H}^2(H, \mathsf{M})$ to $\mathsf{H}^2(gHg^{-1}, \mathsf{M})$ by

$$g \cdot \sigma = (c(g)^*)^{-1}(\sigma) \in \mathsf{H}^2(gHg^{-1}, \mathsf{M}) \ , \tag{2.2}$$

which is induced by the conjugation action of G on (H, M). Unless otherwise stated, we denote by $g\sigma$ the product defined in (2.2).

Now, by (2.2), H acts trivially on $\mathsf{H}^2(H, \mathsf{M})$, and if H is a normal subgroup of G, then the conjugation action of G on (H, M) defined above induces an action of G/H on $\mathsf{H}^2(H, \mathsf{M})$. For more details, see §9.5 in[39], or §III.9 in [7]. More precisely, we have the following result:

Proposition 2.2.1 (Corollaries 3.8.3 and 3.8.4, [7]) *Let H be a subgroup of G. Then the conjugation action of H on (H, M) induces an action of H on $\mathsf{H}^2(H, \mathsf{M})$, which is trivial. In addition, if $H \lhd \mathsf{G}$ and M is a G-module, then the conjugation action of G on (H, M) induces an action of G/H on $\mathsf{H}^2(H, \mathsf{M})$.*

Proposition 2.2.2 (Exercise 3.8.1, [7]) *If H is central in G and M is an abelian group with the trivial G-action, then G/H acts trivially on $\mathsf{H}^2(H, \mathsf{M})$.*

Other proofs of Propositions 2.2.1 and 2.2.2 can also be found in [39], written as Lemma 9.82.

Let us show now a relation between $\mathsf{H}^2(H,\mathsf{M})$ and $\mathsf{H}^2(\mathsf{G},\mathsf{M})$ for any subgroup H of a group G. But firstly, we need some definitions.

Definition 2.2.3 *Let H be a subgroup of a group G, M a G-module, $\mathfrak{i}$ the inclusion map from H into G, and 1_{M} is the identity map on M. The pair $(\mathfrak{i},1_{\mathsf{M}})$ is compatible, i.e. $1_{\mathsf{M}}(\mathfrak{i}(h)m) = hm = h1_{\mathsf{M}}(m)$ for any $h \in H$ and $m \in \mathsf{M}$. The homomorphism induced by the pair $(\mathfrak{i},1_{\mathsf{M}})$ is denoted by $\mathsf{res}_H^{\mathsf{G}} : \mathsf{H}^2(\mathsf{G},\mathsf{M}) \longrightarrow \mathsf{H}^2(H,\mathsf{M})$ and is called a* **restriction**.

The above definition can be founded in [39], page 566.

It is well known that the restriction homomorphism is defined as follows: if $\sigma : \mathsf{G}\times\mathsf{G} \longrightarrow \mathsf{M}$ is a 2-cocycle, then σ restricted to $H\times H$ is also a 2-cocycle, namely $\sigma_H(h) := \sigma(h)$ for any $h \in H$, and $\mathsf{res}_H^{\mathsf{G}}([\sigma]) = [\sigma_H]$ in $\mathsf{H}^2(H,\mathsf{M})$ (see [46], §11.8, or [39], §9.5).

Proposition 2.2.4 (Lemma 11.8.15, [46]) *Let G be a group, H a subgroup of G and M a G-module. If $H \lhd \mathsf{G}$, then*

$$\mathsf{res}_H^{\mathsf{G}}(\mathsf{H}^2(\mathsf{G},\mathsf{M})) \subseteq \mathsf{H}^2(H,\mathsf{M})^{\mathsf{G}} \ ,$$

where $\mathsf{H}^2(H,\mathsf{M})^{\mathsf{G}} = \{\sigma \in \mathsf{H}^2(H,\mathsf{M}) : g\cdot\sigma = \sigma, \forall g \in \mathsf{G}\}$, and $g\cdot\sigma$ is the action defined in (2.2).

Another proof of the previous proposition is given in Corollary 9.83, [39].

Now, let us use the next proposition to define a homomorphism of $\mathsf{H}^2(H,\mathsf{M})$ to $\mathsf{H}^2(\mathsf{G},\mathsf{M})$, called **corestriction** (also called **transfer**). For more details about this homomorphism, see §III.9 in [7], or §9.6 in [39], or §11.8 in [46].

Proposition 2.2.5 *Let H be a subgroup of finite index in a group G, and M a G-module. There exists a homomorphisms of groups*

$$\mathsf{cores}_H^{\mathsf{G}} : \mathsf{H}^2(H,\mathsf{M}) \longrightarrow \mathsf{H}^2(\mathsf{G},\mathsf{M})$$

satisfying

i) $\mathsf{cores}_H^{\mathsf{G}}\mathsf{res}_H^{\mathsf{G}}\sigma = [\mathsf{G}:H]\sigma$, *for all $\sigma \in \mathsf{H}^2(\mathsf{G},\mathsf{M})$;*

ii) If $H \lhd \mathsf{G}$, then $\mathsf{res}_H^{\mathsf{G}}\mathsf{cores}_H^{\mathsf{G}}\rho = \sum_{g\in\mathsf{G}/H} g\rho$, *for all $\rho \in \mathsf{H}^2(H,\mathsf{M})$,* *where $g\rho = g \cdot \rho$ is the action defined in (2.2).*

Proof: The existence of cores is ensured by Proposition 9.87 in [39] (see also §III.9 in [7] or §11.8 in [46]). The properties *i)* and *ii)* are proved in Proposition (3.9.5), [7] (see also Theorem 9.88 in [39], or Theorem 11.8.6 in [46]). ∎

The homomorphism cores in Proposition 2.2.5 is called a **corestriction** (or **transfer**). Observe that Proposition 2.2.1 ensures that $\sum_{g\in \mathsf{G}/H} g\rho$, for all $\rho \in \mathsf{H}^2(H, \mathsf{M})$ is well defined.

Suppose that M is a G-module. We say that $\lambda \in \mathbb{Z}$, $\lambda > 0$, is **invertible** in M if the multiplication by λ is an automorphism of M, i.e., the map given by

$$\begin{array}{rcll} d_\lambda : & \mathsf{M} & \longrightarrow & \mathsf{M} \\ & m & \mapsto & d_\lambda(m) = \underbrace{m + \cdots + m}_{\lambda-times} \end{array}$$

is an isomorphism of G-modules. In this case, by *item i)* in the previous proposition, if $[\mathsf{G} : H] < \infty$ and $[\mathsf{G} : H]$ is invertible in M, then $\mathsf{res}^{\mathsf{G}}_H$ is an injective map. Let us show now that, under suitable conditions, $\mathsf{res}^{\mathsf{G}}_H$ is surjective.

Corollary 2.2.6 *If $[\mathsf{G} : H] < \infty$, H is central in G, and M is an abelian group with the trivial G-action, then $\mathsf{res}^{\mathsf{G}}_H \mathsf{cores}^{\mathsf{G}}_H \rho = [\mathsf{G} : H]\rho$ for all $\rho \in \mathsf{H}^2(H, \mathsf{M})$. In addition, if $[\mathsf{G} : H]$ is invertible in M, then $\mathsf{res}^{\mathsf{G}}_H$ is a surjection from $\mathsf{H}^2(\mathsf{G}, \mathsf{M})$ into $\mathsf{H}^2(H, \mathsf{M})$.*

Proof: Since H is central in G, it follows that $H \lhd \mathsf{G}$ and, by *item ii)* in Proposition 2.2.5, $\mathsf{res}^{\mathsf{G}}_H \mathsf{cores}^{\mathsf{G}}_H \rho = \sum_{g\in\mathsf{G}/H} g\rho$ for all $\rho \in \mathsf{H}^2(H, \mathsf{M})$. On the other hand, since G/H acts trivially on $\mathsf{H}^2(H, \mathsf{M})$ (Proposition 2.2.2), it follows that

$$\sum_{g\in\mathsf{G}/H} g\rho = \sum_{g\in\mathsf{G}/H} \rho = [\mathsf{G} : H]\rho \ , \ \ \rho \in \mathsf{H}^2(H, \mathsf{M}) \ .$$

Therefore, $\mathsf{res}^{\mathsf{G}}_H \mathsf{cores}^{\mathsf{G}}_H \rho = [\mathsf{G} : H]\rho$ for all $\rho \in \mathsf{H}^2(H, \mathsf{M})$.

Suppose now that $[\mathsf{G} : H]$ is invertible in M. Fixed $\sigma \in \mathsf{H}^2(H, \mathsf{M})$, we have $d^{-1}_{[\mathsf{G}:H]}(\sigma(a, b)) \in \mathsf{M}$ for any $a, b \in H$. Consider the map

$$\xi := d^{-1}_{[\mathsf{G}:H]}\sigma : (a, b) \mapsto d^{-1}_{[\mathsf{G}:H]}(\sigma(a, b)) \ ,$$

from $H \times H$ to M. Since G/H acts trivially on $\mathsf{H}^2(H, \mathsf{M})$ and any 2-cocycle satisfies the equality in Definition 2.1.2, we have

$$\xi(a, b) + \xi(ab, c) = \xi(b, c) + \xi(a, bc) \ ,$$

for any $a, b, c \in H$, since M is an abelian group, and so $d_{[\mathsf{G}:H]}^{-1}$ is a homomorphism of groups. Hence, it follows that $d_{[\mathsf{G}:H]}^{-1}\sigma = \xi \in \mathsf{H}^2(H, \mathsf{M})$. From the first part of this proof, it follows that

$$\mathsf{res}_H^{\mathsf{G}}\mathsf{cores}_H^{\mathsf{G}}\xi = [\mathsf{G} : H]\xi = [\mathsf{G} : H][\mathsf{G} : H]^{-1}\sigma = \sigma \ ,$$

and so $\sigma = \mathsf{res}_H^{\mathsf{G}}(\mathsf{cores}_H^{\mathsf{G}}\xi) \in \mathsf{res}_H^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right)$. Therefore, $\mathsf{res}_H^{\mathsf{G}}$ is surjective. ■

Let H be a central subgroup of G, and M an abelian group. Observe that when M is a multiplicative group, then the last result means that $\mathsf{res}_H^{\mathsf{G}}\mathsf{cores}_H^{\mathsf{G}}\rho = \rho^{[\mathsf{G}:H]}$ for all $\rho \in \mathsf{H}^2(H, \mathsf{M})$, and in the case when $[\mathsf{G} : H]$ is invertible in M, then we have that $d_{[\mathsf{G}:H]}^{-1}(\rho(a, b)) = \sqrt[[\mathsf{G}:H]]{\rho(a, b)} \in \mathsf{M}$ for any $a, b \in \mathsf{G}$. In particular, when $\mathsf{M} = \mathbb{F}^*$ is the multiplicative group of a field $\mathbb{F}$, where G acts trivially on M, then $p = [\mathsf{G} : H]$ invertible in $\mathbb{F}$ means that $\sqrt[p]{\lambda} \in \mathbb{F}$ for any $\lambda \in \mathbb{F}$.

Now. by Propositions 2.2.5 and 2.2.6, it follows that

$$|\mathsf{H}^2(H, \mathsf{M})| = \left|\mathsf{res}_{\mathsf{H}}^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right)\right| \ ,$$

when $[\mathsf{G} : H]$ is invertible in M. We have

$$\{[\sigma_H] : [\sigma] \in \mathsf{H}^2(\mathsf{G}, \mathsf{M})\} \subseteq \mathsf{H}^2(H, \mathsf{M}) \ . \tag{2.3}$$

The previous corollary gives enough conditions to ensure the equality in (2.3), since $\mathsf{res}_H^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right) \subseteq \mathsf{H}^2(H, \mathsf{M})$. Thus, we have the following result.

Corollary 2.2.7 *Let H be a central subgroup of G, $[\mathsf{G} : H] < \infty$, and M an abelian group with the trivial G-action. If $[\mathsf{G} : H]$ is invertible in M, then*

$$\mathsf{H}^2(H, \mathsf{M}) = \mathsf{res}_{\mathsf{H}}^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right) \ .$$

Proof: By Corollary 2.2.6, it follows that $\mathsf{H}^2(H, \mathsf{M}) \subseteq \mathsf{res}_{\mathsf{H}}^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G}, \mathsf{M})\right)$. Therefore, the result follows, since by definition $\mathsf{res}_{\mathsf{H}}^{\mathsf{G}}(\mathsf{H}^2(\mathsf{G},\mathsf{M})) \subseteq \mathsf{H}^2(H,\mathsf{M})$. ■

The previous corollary means that, under the assumptions of Corollary 2.2.7, given $[\sigma] \in \mathsf{H}^2(H,\mathsf{M})$, there exists $[\xi] \in \mathsf{H}^2(\mathsf{G},\mathsf{M})$ such that $\xi_H = \sigma$. Therefore,

$$\mathsf{H}^2(H,\mathsf{M}) = \left\{[\xi_H] : \xi \in \mathsf{Z}^2(\mathsf{G},\mathsf{M})\right\} \ .$$

In particular, given $\sigma \in \mathsf{Z}^2(H,\mathsf{M})$, there exists $\xi \in \mathsf{Z}^2(\mathsf{G},\mathsf{M})$ such that $\xi_H = \sigma$.

Consider a finite abelian group G, and a field $\mathbb{F}$. Suppose that G acts trivially on $\mathbb{F}^*$. If $\mathbb{F}$ is algebraically closed, by Corollary 2.2.7, for any subgroup H of G, we have that

$$\mathsf{H}^2(H,\mathbb{F}^*) = \mathsf{res}_{\mathsf{H}}^{\mathsf{G}}\left(\mathsf{H}^2(\mathsf{G},\mathbb{F}^*)\right) \ .$$

In particular, by Proposition 2.1.6, if $|\mathsf{G}|$=n, then $\mathsf{H}^2(\mathsf{G},\mathbb{F}^*)$ is finite, and $\mathsf{H}^2(\mathsf{G},\mathbb{F}^*) = \{[\sigma_1],\dots,[\sigma_r]\}$, where $\sigma_i \in \mathsf{Z}^2(\mathsf{G},\sqrt[n]{1_{\mathbb{F}}})$, i.e. $(\sigma_i(g,h))^n = 1_{\mathbb{F}}$ for any $g,h \in \mathsf{G}$, and $i = 1,\dots,r$.

CHAPTER 3

GRADED RINGS WITH THE NIL NEUTRAL COMPONENT

In this chapter, we study a concrete case of Theorem 1.4.12 and we have answered the following question: what can we say about $\mathfrak{R}$ when $\mathfrak{R}_e$ is nil/nilpotent, where $\mathfrak{R}$ is an associative ring with an S-grading, S is a monoid and e its neutral element?

Therefore, we consider an associative ring with a finite grading by a left cancellative monoid, and we prove that if its neutral component is nil and f-commutative, then the whole ring is also nil. Among other results, we have given various counterexamples showing that our hypotheses are necessary. Consequently, using Nagata-Higman Theorem, we have exhibited some important applications of our results (see Theorem 3.3.3 and Corollary 3.3.4). Besides that, we have exhibited a considerable relation between graded rings and Köthe's Problem (see Theorem 3.3.7).

3.1 Graded rings with the nil neutral component

Let S be a left cancellative monoid, i.e. $gh = gt$ implies $h = t$ for any $g, h, t \in \mathsf{S}$. Let $\mathfrak{R}$ be an associative ring with a finite S-grading Γ. In this chapter, we have studied an important class of rings: nil rings. Our principal goal in this chapter is to present some results which are direct implications of the case "$\mathfrak{R}_e$ is nilpotent" or "$\mathfrak{R}_e$ is nil", where e is the neutral element of S. Here, we are interested in studying associative rings with an S-grading, whose neutral component is nil. We also we interested to find conditions providing the nilpotency of the whole ring $\mathfrak{R}$. In this case, we have given some upper bounds for $\mathsf{nd}(\mathfrak{R})$, the nilpotency index of $\mathfrak{R}$. Unless otherwise stated, $\mathfrak{R}$ is an associative ring, and S is a left cancellative monoid, with the neutral element e.

3.2 Main Results

In this section, we present some important results concerning S-graded rings with the nil neutral component. Unless otherwise stated, in this section we denote by $\mathfrak{R}$ an associative ring with an S-grading given by $\Gamma : \mathfrak{R} = \bigoplus_{g\in\mathsf{S}} \mathfrak{R}_g$, where S is an arbitrary left cancellative monoid. Observe that any group is a left cancellative monoid. Thus, all the results here presented are valid for rings graded by groups. We also assume that Γ has a finite support, namely $|\mathsf{Supp}(\Gamma)| = d < \infty$.

Let $\mathfrak{R}$ be an S-graded ring. Note that to prove that $\mathfrak{R}$ is nil/nilpotent, it is sufficient to analyse only products of its homogeneous elements. In fact, given $a_1, a_2, \ldots, a_k \in \mathfrak{R}$, we can write $a_i = \sum_{j=1}^{d} a_{ig_j}$, where $a_{ig_j} \in \mathfrak{R}_{g_j}$ and $\mathsf{Supp}(\Gamma) = \{g_1, \ldots, g_d\}$. Hence, we have

$$\begin{aligned} a_1a_2\cdots a_k &= \left(\sum_{j_1=1}^{d} a_{1g_{j_1}}\right)\left(\sum_{j_2=1}^{d} a_{2g_{j_2}}\right)\cdots\left(\sum_{j_k=1}^{d} a_{kg_{j_k}}\right) \\ &= \sum_{j_1,j_2,\ldots,j_k=1}^{d} a_{1g_{j_1}}a_{2g_{j_2}}\cdots a_{kg_{j_k}}. \end{aligned} \tag{3.1}$$

Therefore, without loss of generality we study only the products of homogeneous elements in the grading of $\mathfrak{R}$.

Remark 3.2.1 *Let $a_1, a_2, \ldots, a_n \in \mathfrak{R}$ be homogeneous elements. Note that if $\mathsf{deg}(a_i) \notin \mathsf{Supp}(\Gamma)$ for some $i = 1, \ldots, n$, then $a_1a_2\cdots a_n = 0$. Moreover, put $\mathsf{deg}(a_i) = g_i$ for $i = 1, 2, \ldots, n$, and consider the set $\Lambda_{(g_1,\ldots,g_n)} := \{g_ig_{i+1}\cdots g_{i+m} : i = 1, \ldots, n, 0 \leqslant m \leqslant n-i\}$. If $\Lambda_{(g_1,\ldots,g_n)} \nsubseteq \mathsf{Supp}(\Gamma)$, then $a_1a_2\cdots a_n = 0$, since $\mathfrak{R}$ is an associative ring. Therefore, if $a_1a_2\cdots a_n \neq 0$ with $a_1 \in \mathfrak{R}_{g_1}, a_2 \in \mathfrak{R}_{g_2}, \ldots, a_n \in \mathfrak{R}_{g_n}$, then $\Lambda_{(g_1,\ldots,g_n)} \subseteq \mathsf{Supp}(\Gamma)$.*

Recall that $\mathsf{Supp}(\Gamma) = d < \infty$. Observe that if $g \in \mathsf{Supp}(\Gamma)$, then either $(\mathfrak{R}_g)^{d+1} = \{0\}$ or $e \in \mathsf{Supp}(\Gamma)$, where e is the neutral element of S. In fact, suppose that $e \notin \mathsf{Supp}(\Gamma)$. By contradiction, suppose also that there exist $a_1, a_2, \ldots, a_{d+1} \in \mathfrak{R}_g$ such that $a_1a_2\cdots a_{d+1} \neq 0$. Hence, $\{g, g^2, \ldots, g^{d+1}\} \subset \mathsf{Supp}(\Gamma)$, since $\mathfrak{R}$ is an associative ring. But

$|\mathsf{Supp}(\Gamma)| = d$, and thus, there exist $1 \leqslant l < t \leqslant d+1$ such that $g^t = g^l$, and hence, $g^{t-l} = e \notin \mathsf{Supp}(\Gamma)$, because S is a left cancellative monoid, where $1 \leqslant t-l \leqslant d$. From this, we obtain a contradiction. Therefore, for any $g \in \mathsf{Supp}(\Gamma)$, it follows that $(\mathfrak{R}_g)^{d+1} = \{0\}$ when $e \notin \mathsf{Supp}(\Gamma)$.

The following result ensures that any S-graded non-nilpotent ring has necessarily some nonzero homogeneous element of degree e.

Proposition 3.2.2 *Let $\mathfrak{R}$ be a ring with a finite S-grading Γ, where S is a left cancellative monoid. If $\mathfrak{R}_e = \{0\}$, then $\mathfrak{R}^{d+1} = \{0\}$, where $d = |\mathsf{Supp}(\Gamma)|$.*

Proof: Suppose that $e \notin \mathsf{Supp}(\Gamma)$, and write $n := d+1$. We will show that $\mathfrak{R}^n = \{0\}$. For this purpose, it is sufficient to prove that $a_1 a_2 \cdots a_n = 0$ for all homogeneous elements $a_1, a_2, \ldots, a_n \in \mathfrak{R}$ (see (3.1)).

By contradiction, suppose that there exist homogeneous elements $a_1, a_2, \ldots, a_n \in \mathfrak{R}$ such that $a_1 a_2 \cdots a_n \neq 0$. Put $\mathsf{deg}(a_i) = g_i$ for $i = 1, \ldots, n$, and define $\Lambda := \Lambda_{(g_1, g_2, \ldots, g_n)}$ (as in Remark 3.2.1). Hence, by Remark 3.2.1, we have $\Lambda \subseteq \mathsf{Supp}(\Gamma)$, and since $|\mathsf{Supp}(\Gamma)| = d$, it follows that $|\Lambda| \leqslant d < n$. Notice that $\{g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_n\} \subseteq \Lambda$, and hence, we conclude that there exist $1 \leqslant l < t \leqslant n$ such that

$$g_1 g_2 \cdots g_l = (g_1 g_2 \cdots g_l) g_{(l+1)} \cdots g_t.$$

Thus, since S is left cancellative, we conclude that $e = g_{l+1} \cdots g_t \in \Lambda \subseteq \mathsf{Supp}(\Gamma)$. This contradicts our assumption. Therefore, we prove that $a_1 a_2 \cdots a_n = 0$ for all homogeneous elements $a_1, a_2, \ldots, a_n \in \mathfrak{R}$, and hence, by (3.1), we conclude that $\mathfrak{R}^n = \{0\}$. Consequently, $\mathfrak{R}$ is nilpotent of index at most $n = d+1$. ∎

Besides ensuring that any non-nilpotent S-graded ring has a nonzero neutral homogeneous element, the previous proposition provides an upper bound for the nilpotency index $\mathsf{nd}(\mathfrak{R})$, when $\mathfrak{R}$ is an S-graded ring with a finite support, whose neutral component is zero. The following example exhibits a nilpotent ring whose nilpotency index is not less than that predicted by Proposition 3.2.2.

Example 3.2.3 *Let $\mathbb{F}$ be an arbitrary field and $n \in \mathbb{N}$, $n > 1$. Consider $\mathfrak{R} = SUT_n(\mathbb{F})$, the ring of the strictly upper triangular matrices of order $n \times n$ over $\mathbb{F}$. The family of subspaces $(\mathfrak{R}_\gamma)_{\gamma \in \mathbb{Z}_n}$, where $\mathfrak{R}_\gamma = \mathsf{span}_{\mathbb{F}}\{E_{ij} : \overline{j-i} = \gamma\}$, defines a $\mathbb{Z}_n$-grading on $\mathfrak{R}$ (called* *an* **elementary** $\mathbb{Z}_n$**-grading** *corresponding to* $(\bar{0},\bar{1},\ldots,\overline{n-1})$*), namely* Γ*. It is easily seen that* $\mathsf{Supp}(\Gamma)=\mathbb{Z}_n-\{\bar{0}\}$ *and* $nd(\mathfrak{R})=n=|\mathsf{Supp}(\Gamma)|+1$*. Therefore, we conclude that the previous proposition provides a good upper bound for the nilpotency index of graded rings (with a finite support) whose neutral component is zero.*

On the other hand, the following example shows that the "finite support" condition is required, in particular.

Example 3.2.4 *Consider the ring* $\mathfrak{R}=\mathbb{R}[x]$ *of all the real polynomials in one variable. We have that* $\mathfrak{R}$ *is naturally* $\mathbb{Z}$*-graded with the infinite support. Now, consider the subset* $\tilde{\mathfrak{R}}=\{p(x)\in\mathfrak{R}:p(0)=0\}$ *of* $\mathfrak{R}$*. Notice that* $\tilde{\mathfrak{R}}$ *is a* $\mathbb{Z}$*-graded ring (with the* $\mathbb{Z}$*-grading induced by the* $\mathbb{Z}$*-grading of* $\mathfrak{R}$*) such that* $\tilde{\mathfrak{R}}_0=\{0\}$*, but its support is not finite and* $(\tilde{\mathfrak{R}})^n\neq\{0\}$ *for all* $n\in\mathbb{N}$*, since* $x^n\in\tilde{\mathfrak{R}}_n$*.*

In the proof of the previous proposition, we have used combinatorial arguments. Evidently, the techniques used in Proposition 3.2.2 can be extended to answer the following question: "what can we say about $\mathfrak{R}$ when $\mathfrak{R}_e$ is nil?". Thus, one of the most natural question is the following:

Problem 3.2.5 *Given an* S*-graded ring* $\mathfrak{R}$ *with a finite support, does* $\mathfrak{R}_e$ *being nil imply that* $\mathfrak{R}$ *is nil?*

Below we present some results concerning this question. Before it, observe that the following example ensures the existence of an S-graded ring with an infinite support, which is not nil, although its neutral component is nil, and hence, a problem similar to Problem 3.2.5 for infinite support is not valid.

Example 3.2.6 (Theorem 2.7, [42]) *For every countable field* $\mathbb{K}$ *there is an associative nil* $\mathbb{K}$*-algebra* N *such that the polynomial ring in one indeterminate over* N *(which is naturally* $\mathbb{Z}$*-graded with the neutral component equal to* N*) is not nil.*

An S-graded ring $\mathfrak{R}$ is called S-nil if all its homogeneous components are nil, i.e. for any $g\in\mathsf{S}$, we have that any $x\in\mathfrak{R}_g$ is nilpotent. Analogously, $\mathfrak{R}$ is called S-nilpotent if for each $g\in\mathsf{S}$ there exists $n_g\in\mathbb{N}$ such that $a_{1g}a_{2g}\cdots a_{n_gg}=0$ for any $a_{1g},a_{2g},\ldots,a_{n_gg}\in\mathfrak{R}_g$. Notice that if the support of S is finite and $\mathfrak{R}$ is S-nilpotent, then necessarily there exists

$n \in \mathbb{N}$ such that the product of any n homogeneous elements of the same homogeneous degree is zero.

Let S be a monoid. Given an element $g \in \mathsf{S} - \{e\}$, if there exists an $m \in \mathbb{N}$ such that $g^m = e$, then we say that the **order** of g in S is the smallest number $n \in \mathbb{N}$ such that $g^n = e$, and in this case we denote $\mathsf{o}(g) = n$. If there is not $m \in \mathbb{N}$ such that $g^m = e$, then we say that g has infinite order, and we denote $\mathsf{o}(g) = \infty$. Note that when S is finite and S is a left cancellative monoid, we have that all the elements of S have finite orders.

Proposition 3.2.7 *Let S be a left cancellative monoid and $\mathfrak{R}$ a ring with an S-grading Γ of finite support, namely $|\mathsf{Supp}(\Gamma)| = d$. Suppose that $\mathfrak{R}_e$ is a nonzero nil ring. Then the following items are true:*

i) $\mathfrak{R}$ is an S-nil ring;

ii) Suppose $\mathfrak{R}_e$ is nil of bounded index, namely $\mathsf{nd}_{nil}(\mathfrak{R}_e) = s$. Then:

1. *$\left(a_1 a_2 \cdots a_{k_g}\right)^s = 0$ for any $g \in \mathsf{Supp}(\Gamma)$, and any $a_1, a_2, \ldots, a_{k_g} \in \mathfrak{R}_g$, where $k_g := \min\{\mathsf{o}(g), d\}$;*
2. *there exists $k \in \mathbb{N}$ such that $(a_1 a_2 \cdots a_k)^s = 0$ for any homogeneous elements $a_1, a_2, \ldots, a_k$ of the same homogeneous degree ($\deg(a_1) = \cdots = \deg(a_k)$);*
3. *$\mathfrak{R}$ is S-nil of bounded index.*

Proof: *i)* Firstly, since $\mathfrak{R}_e \neq \{0\}$, $e \in \mathsf{Supp}(\Gamma)$. Without loss of generality we can take any $g \in \mathsf{Supp}(\Gamma) - \{e\}$, since $\mathfrak{R}_e$ is nil by the claim. Put $s = \min\{\mathsf{o}(g), d\}$ and consider the subset $\beta = \{g, g^2, \ldots, g^s\}$ of S. Notice that if $\beta \nsubseteq \mathsf{Supp}(\Gamma)$, then $a^s = 0$ for any $a \in \mathfrak{R}_g$, since $\mathfrak{R}$ is an associative ring. For this reason, we can assume $\beta \subseteq \mathsf{Supp}(\Gamma)$. It follows that either $g^s = e$ or $e \notin \beta$. In fact, $e \in \beta$ implies that $g^r = e$ for some $r \in \{1, \ldots, s\}$. By definition of $\mathsf{o}(g)$, we have $\mathsf{o}(g) \leqslant r$. Thus, $r = s$ and $g^s = e$, since $s \leqslant \mathsf{o}(g)$ and $r \leqslant s$. Note that $e \notin \beta$ iff $g^r \neq e$ for all $1 \leqslant r \leqslant s$.

If $g^s = e$, then for any $x \in \mathfrak{R}_g$, we have $x^s \in \mathfrak{R}_e$, and hence, x is nilpotent, since $\mathfrak{R}_e$ is nil. Otherwise, we have that $e \notin \beta$, and $s = d < \mathsf{o}(g)$. Hence, $\beta = \mathsf{Supp}(\Gamma)$, since $|\beta| = s$ (all the elements of β are different). From this, it follows that $e \notin \mathsf{Supp}(\Gamma)$ in this case, otherwise we would have $\mathsf{o}(g) \leqslant s$, since S is left cancellative, that contradicts to the claims.

Anyway, we show that any element of $\mathfrak{R}_g$ is nilpotent, for all $g \in \mathsf{S}$. Therefore, $\mathfrak{R}$ is S-nil.

ii-1) Since $\mathfrak{R}_e \neq \{0\}$, we have $\mathsf{nd}(\mathfrak{R}_e) > 1$. Fix any $g \in \mathsf{Supp}(\mathfrak{R})$. If $g = e$, the result is obvious. Assume that $g \neq e$. Notice that $g^l \notin \mathsf{Supp}(\Gamma)$ for some $l \in \mathbb{N}$ implies that $(\mathfrak{R}_g)^l = \{0\}$. Consider $k_g := \min\{\mathsf{o}(g), d\}$, where $d = |\mathsf{Supp}(\Gamma)|$. Take arbitrary elements $a_1, a_2, \ldots, a_{k_g} \in \mathfrak{R}_g$.

Consider $\gamma = \{g, g^2, \ldots, g^{k_g}\}$. Let us show that either $k_g = \mathsf{o}(g)$ or $\gamma \nsubseteq \mathsf{Supp}(\Gamma)$. Suppose that $k_g \neq \mathsf{o}(g)$, and hence, $d = k_g < \mathsf{o}(g)$. Since $k_g < \mathsf{o}(g)$, it is easy to see that $e \notin \gamma$, and all elements of the set γ are different, because S is left cancellative. Then $|\gamma| = d = |\mathsf{Supp}(\Gamma)|$, and for this reason, we can conclude that $\gamma \nsubseteq \mathsf{Supp}(\Gamma)$, since $e \in \mathsf{Supp}(\Gamma)$.

If $k_g = \mathsf{o}(g)$, then $a_1a_2\cdots a_{k_g} \in \mathfrak{R}_e$. If $\gamma \nsubseteq \mathsf{Supp}(\Gamma)$, then there exists $g^l \in \gamma - \mathsf{Supp}(\Gamma)$, and consequently, $a_1a_2\cdots a_l = 0$, $1 \leqslant l \leqslant k_g$.

Therefore, we have shown that for any $g \in \mathsf{Supp}(\Gamma)$ and any $a_1, a_2, \ldots, a_{k_g} \in \mathfrak{R}_g$, where $k_g = \min\{\mathsf{o}(g), |\mathsf{Supp}(\Gamma)|\}$, we have either $a_1a_2\cdots a_{k_g} \in \mathfrak{R}_e$ or $a_1a_2\cdots a_{k_g} = 0$. Thus, in any case, we conclude that $\left(a_1a_2\cdots a_{k_g}\right)^s = 0$, since $\mathsf{nd}_{nil}(\mathfrak{R}_e) = s$.

ii-2) By the arguments of *(ii-1)*, it is sufficient to take $k := \mathsf{lcm}\{k_g : g \in \mathsf{Supp}(\Gamma)\}$, since $\mathsf{Supp}(\Gamma)$ is finite.

ii-3) Let k be the integer defined in *(ii-2)*. Given $g \in \mathsf{Supp}(\Gamma)$, we have that $k = k_gp_g$ for some integer p_g, where k_g is the integer defined in *(ii-1)*. Hence,

$$a^{ks} = a^{(k_gp_g)s} = a^{k_g(sp_g)} = (a^{k_g})^{sp_g} = ((a^{k_g})^s)^{p_g} = 0$$

for any $a \in \mathfrak{R}_g$. Therefore, we conclude that $b^{ks} = 0$ holds for any homogeneous element $b \in \mathfrak{R}$. The result follows. ∎

Notice that, when we assume that $\mathfrak{R}_e$ is nil, the previous proposition exhibits consequences only for homogeneous components. From now on we will show more general results, i.e. not only for homogeneous components.

Lemma 3.2.8 *Let S be a left cancellative monoid and $\mathfrak{R}$ be a ring with an S-grading Γ with a finite support, $|\mathsf{Supp}(\Gamma)| = d$. For any integer $r > 1$ and any homogeneous elements $a_1, a_2, \ldots, a_{rd} \in \mathfrak{R}$, we have that either $a_1a_2\cdots a_{rd} = 0$ or there exist $0 \leqslant s_0 < s_1 < \cdots <$*

$s_r \leqslant rd$ *satisfying*

$$e = \mathsf{deg}(a_{s_0+1} \cdots a_{s_1}) = \mathsf{deg}(a_{s_1+1} a_{s_1+2} \cdots a_{s_2}) = \cdots = \mathsf{deg}(a_{s_{r-1}+1} \cdots a_{s_r}). \tag{3.2}$$

Proof: By Proposition 3.2.2, if $e \notin \mathsf{Supp}(\Gamma)$, then $\mathfrak{R}^{d+1} = \{0\}$. From this, the result follows, since $d+1 \leqslant dr$ for all $r > 1$ in $\mathbb{N}$. Observe that in this case we always have the first alternative.

Now, assume that $e \in \mathsf{Supp}(\Gamma)$. Let $a_1, a_2, \ldots, a_{rd} \in \mathfrak{R}$ be homogeneous elements, such that $a_1 a_2 \cdots a_{rd} \neq 0$. Let us show that there exist $0 \leqslant s_0 < s_1 < \cdots < s_r \leqslant rd$ such that (3.2) holds. Put $\mathsf{deg}(a_i) = g_i$ for each $i = 1, 2, \ldots, rd$. For all $1 \leqslant l \leqslant k \leqslant rd$, define $b_{l,k} = a_l a_{l+1} \ldots a_k$, $b_k = b_{1,k}$, and $b_{l,l} = a_l$. It is easy to see that

$$\mathsf{deg}(b_{l,k}) = \mathsf{deg}(a_l)\mathsf{deg}(a_{l+1}) \cdots \mathsf{deg}(a_k) = g_l g_{l+1} \cdots g_k, \tag{3.3}$$

for all $1 \leqslant l \leqslant k \leqslant rd$. Since $a_1 a_2 \ldots a_{rd} \neq 0$, it follows that $\Lambda := \{\mathsf{deg}(b_{l,k}) : 1 \leqslant l \leqslant k \leqslant rd\} = \Lambda_{(g_1, g_2, \ldots, g_{rd})}$ is contained in $\mathsf{Supp}(\Gamma)$ (see Remark 3.2.1). Now, consider the subset $\tilde{\Lambda} := \{\mathsf{deg}(b_i) : i = 1, 2, \ldots, rd\}$ of Λ, and notice that

$$|\tilde{\Lambda}| \leqslant \begin{cases} d-1, & \text{if } e \notin \tilde{\Lambda} \\ d, & \text{if } e \in \tilde{\Lambda} \end{cases}, \tag{3.4}$$

since $\tilde{\Lambda} \subseteq \mathsf{Supp}(\Gamma)$, and $|\mathsf{Supp}(\Gamma)| = d$. For each $g \in \tilde{\Lambda}$, consider the integer $\lambda_g := |\{i : \mathsf{deg}(b_i) = g\}|$, and assume $\lambda_g = 0$ for any $g \notin \tilde{\Lambda}$. Take $g_0 \in \tilde{\Lambda}$ such that $\lambda_{g_0} = \mathsf{max}\{\lambda_g : g \in \tilde{\Lambda}, g \neq e\}$. Let us show that either $\lambda_e \geqslant r$ or $\lambda_{g_0} \geqslant r+1$.

Firstly, note that $\{i : \mathsf{deg}(b_i) = g\} \cap \{j : \mathsf{deg}(b_j) = h\} = \varnothing$ for any $g \neq h$, and hence,

$$rd = \left| \bigcup_{g \in \tilde{\Lambda}} \{i : \mathsf{deg}(b_i) = g\} \right| = \sum_{g \in \tilde{\Lambda}} |\{i : \mathsf{deg}(b_i) = g\}| = \sum_{g \in \tilde{\Lambda}} \lambda_g \,. \tag{3.5}$$

Then by (3.4), we have

$$rd = \lambda_e + \sum_{g \in \tilde{\Lambda} - \{e\}} \lambda_g \leqslant \lambda_e + \sum_{g \in \tilde{\Lambda} - \{e\}} \lambda_{g_0} \leqslant \lambda_e + (d-1)\lambda_{g_0} \,. \tag{3.6}$$

If $e \notin \tilde{\Lambda}$, then $\lambda_e = 0$, and hence, by (3.6), it follows that $rd \leqslant (d-1)\lambda_{g_0}$ which implies that $\lambda_{g_0} > r$.

Suppose now that $e \in \tilde{\Lambda}$. Assume that $\lambda_g < (r+1)$ for any $g \in \tilde{\Lambda} - \{e\}$. Hence, $\lambda_{g_0} \leqslant r$, and by (3.6), we have $rd \leqslant \lambda_e + (d-1)r$, and thus, $\lambda_e \geqslant rd - (d-1)r = r$. From this, we deduce that $\lambda_{g_0} < (r+1)$ implies $\lambda_e \geqslant r$.

Therefore, we show that either $\lambda_e \geqslant r$ or there exists at least one $g_0 \in \tilde{\Lambda} - \{e\}$ such that $\lambda_{g_0} \geqslant r+1$.

Finally, suppose that $\lambda_e \geqslant r$, and take $1 \leqslant i_1 < \cdots < i_r \leqslant rd$ such that $e = \mathsf{deg}(b_{i_1}) = \cdots = \mathsf{deg}(b_{i_r})$. Hence, it follows that

$$\begin{aligned} b_{i_r} &= a_1 a_2 \cdots a_{i_r} \\ &= (a_1 a_2 \cdots a_{i_1})(a_{(i_1+1)} a_{(i_1+2)} \cdots a_{i_2}) \cdots (a_{(i_{r-1}+1)} a_{(i_{r-1}+2)} \cdots a_{i_r}) \\ &= b_{i_1} b_{(i_1+1),i_2} \cdots b_{(i_{r-1}+1),i_r}. \end{aligned} \tag{3.7}$$

We deduce from (3.3) and (3.7) that $e = \mathsf{deg}(b_{i_1}) = \mathsf{deg}(b_{(i_1+1),i_2}) = \cdots = \mathsf{deg}(b_{(i_{r-1}+1),i_r})$. Thus, we obtain $s_0 = 0$, $s_j = i_j$ for $j = 1, \ldots, r$.

Assume now that $\lambda_{g_0} \geqslant r+1$ for some $g_0 \in \tilde{\Lambda} - \{e\}$. Let us take $1 \leqslant i_1 < \cdots < i_r < i_{(r+1)} \leqslant rd$ such that $g_0 = \mathsf{deg}(b_{i_1}) = \cdots = \mathsf{deg}(b_{i_r}) = \mathsf{deg}(b_{i_{(r+1)}})$. Similarly to (3.7), we have $b_{i_{(r+1)}} = b_{i_1} b_{(i_1+1),i_2} \cdots b_{(i_{r-1}+1),i_r} b_{(i_r+1),i_{(r+1)}}$. From this, and by (3.3), we conclude that $\mathsf{deg}(b_{(i_1+1),i_2}) = \cdots = \mathsf{deg}(b_{(i_{r-1}+1),i_r}) = \mathsf{deg}(b_{(i_r+1),i_{(r+1)}}) = e$, since $\mathsf{deg}(b_{i_l}) = \mathsf{deg}(b_{i_{l+1}}) = g_0$, $b_{i_{l+1}} = b_{i_l} b_{i_l+1,i_{l+1}}$, $i = 1, \ldots, r$, and S is left cancellative. Therefore, we obtain (3.2) for $s_0 = i_1, s_1 = i_2, \ldots, s_r = i_{r+1}$. ■

Let us recall the notion of f-commutativity, defined in Definition 1.1.4.

Let us consider a semigroup $\mathfrak{S}$ (that is, a set together with a binary operation from $\mathfrak{S} \times \mathfrak{S}$ to $\mathfrak{S}$ which is associative), and an associative ring $\mathfrak{R}$. A **left action** of $\mathfrak{S}$ on $\mathfrak{R}$ is a mapping $\cdot : \mathfrak{S} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ satisfying

$$(\lambda\gamma) \cdot x = \lambda(\gamma \cdot x) \quad \text{and} \quad \lambda \cdot (xy) = (\lambda \cdot x)y \ ,$$

for any $\lambda, \gamma \in \mathfrak{S}$ and $x, y \in \mathfrak{R}$, it is called an **action by semigroup**.

Consider any application $\mathsf{f} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{S}$, and define the **$\mathsf{f}$-commutator** of $\mathfrak{R}$ by

$$[a, b]_{\mathsf{f}} = ab - \mathsf{f}(a, b) \cdot ba \ ,$$

for any $a, b \in \mathfrak{R}$. Particularly, we can consider some concrete cases.

Example 3.2.9 *Given any subring P of $\mathfrak{R}$, and any map f of $\mathfrak{R} \times \mathfrak{R}$ to P, we have that*

$$[a,b]_{\mathsf{f}} = ab - \mathsf{f}(a,b)ba \ ,$$

for any $a, b \in \mathfrak{R}$, defines an f-commutator of $\mathfrak{R}$.

Example 3.2.10 *Given any ring $\mathfrak{R}$, we have that $\mathbb{Z}$ acts on the left of $\mathfrak{R}$ naturally, assuming that*

$$\lambda a = \underbrace{a + \cdots + a}_{\lambda - times} \ , \quad \gamma a = (-\gamma)(-a) = \underbrace{(-a) + \cdots + (-a)}_{(-\gamma) - times} \ , \quad and \ \ 0a = 0a \ ,$$

for any $a \in \mathfrak{R}$ and $\lambda, \gamma \in \mathbb{Z}$, with $\lambda > 0$ and $\gamma < 0$. We can consider for each $\lambda \in \mathbb{Z}$ the mapping λ satisfying $\lambda(a,b) = \lambda$ for any $a, b \in \mathfrak{R}$, and hence, the λ-commutative $[\ ,\ ]_\lambda$ is well defined. In particular, for $\lambda = 1$, take $\mathsf{1}(a,b) = 1$ for any $a, b \in \mathfrak{R}$, and thus, the $\mathsf{1}$-commutator $[\ ,\ ]_1$ is given by $[a,b]_1 = ab - 1 \cdot ba = ab - ba = [a,b]$, and so $[\ ,\ ]_1 = [\ ,\ ]$. On the other hand, when $\lambda = 0$, take $\mathsf{0}(a,b) = 0$ for any $a, b \in \mathfrak{R}$, and hence, the $\mathsf{0}$-commutator $[\ ,\ ]_0$ is given by $[a,b]_0 = ab - 0 \cdot ba = ab$, and so $[\ ,\ ]_0$ is the product of $\mathfrak{R}$.

Definition 3.2.11 *A an associative ring $\mathfrak{R}$ is called a* **f-commutative ring** *if there exist a semigroup $\mathfrak{S}$ that acts on the left of $\mathfrak{R}$, and a mapping $\mathsf{f} : \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{S}$ such that $[a,b]_{\mathsf{f}} = 0$ for any $a, b \in \mathfrak{R}$, then $\mathfrak{R}$ is said to be* **f-commutative**.

Example 3.2.12 *Given any ring $\mathfrak{R}$, we have that $\mathfrak{R}$ is $\mathsf{1}$-commutative iff $\mathfrak{R}$ is commutative. Also, $\mathfrak{R}$ is $\mathsf{0}$-commutative iff $\mathfrak{R}$ has the trivial product.*

Let us denote by Υ the class of all f-commutative rings.

In particular, all commutative rings, anticommutative rings, and the nilpotent rings of index 2 belong to the class Υ. An interesting question is whether every ring is f-commutative for some f. To answer this question, we need some tools. In fact, Example 3.2.23 gives a ring, which is not f-commutative for any f. In general, if for any $a, b \in \mathfrak{R}$, the equation $xba = ab$ has a solution in some semigroup $\mathfrak{S}$, which acts on $\mathfrak{R}$ from the left, then $\mathfrak{R}$ belongs to Υ.

Remark 3.2.13 *Under the conditions of Lemma 3.2.8, consider any integer $r > 1$. Let $0 \leqslant s_0 < s_1 < \cdots < s_r \leqslant rd$ be integers such that (3.2) holds. Consider the set $\xi = \{i \in \{1, \ldots, r\} : s_i - s_{i-1} > 2d\}$. We have*

$$\begin{aligned} rd = s_0 + \sum_{i=1}^{r}(s_i - s_{i-1}) + (rd - s_r) \geqslant \sum_{i=1}^{r}(s_i - s_{i-1}) \\ \geqslant \sum_{i \in \xi}(s_i - s_{i-1}) \geqslant \sum_{i \in \xi}(2d+1) \geqslant |\xi|(2d+1) \ . \end{aligned} \tag{3.8}$$

Consider the integer $\hat{r} \in \mathbb{Z}$, $\hat{r} \geqslant 1$, such that $r \in \{2\hat{r}, 2\hat{r}+1\}$. Observe that $s_i - s_{i-1} \leqslant 2d$ for at least $\hat{r}+1$ integers $i \in \{1, \ldots, r\}$, that is, $r - |\xi| \geqslant \hat{r}+1$. In fact, firstly suppose $r = 2\hat{r}$. Let us show that $|\xi| < \hat{r}$. By contradiction, suppose that $|\xi| \geqslant \hat{r} \geqslant 1$. By (3.8), it follows that

$$rd \geqslant |\xi|(2d+1) \geqslant \hat{r}(2d+1) \geqslant 2\hat{r}d + \hat{r} \geqslant rd + 1 \ ,$$

and hence, we obtain a contradiction.

Now, suppose $r = 2\hat{r}+1$. By contradiction, assume that $|\xi| \geqslant \hat{r}+1 \geqslant 1$. By (3.8), we have that

$$\begin{aligned} rd &\geqslant |\xi|(2d+1) \geqslant (\hat{r}+1)(2d+1) \\ &\geqslant 2\hat{r}d + \hat{r} + 2d + 1 = (2\hat{r}d + d) + 1 + \hat{r} + d \\ &\geqslant rd + 1 + \hat{r} + d \geqslant rd + 1 \ , \end{aligned}$$

which is impossible. Therefore, we conclude that $r - |\xi| \geqslant \hat{r}+1$, for any integer $r \in \{2\hat{r}, 2\hat{r}+1\}$, for any integer $\hat{r} \geqslant 1$.

Let us use the previous remark to prove that any ring with a finite S-grading is nil if its neutral component is nil and f-commutative.

Suppose that $\mathfrak{R}$ is an f-commutative ring. Consider any monomials $m_1, m_2, m_3 \in \mathfrak{R}$ (i.e. m_i is the product of elements of $\mathfrak{R}$). For any $x, y, z, t \in \mathfrak{R}$, we have

$$\begin{aligned} xm_1ym_2zm_3t &= (\mathsf{f}(x, m_1)m_1)xym_2zm_3t \\ &= (\mathsf{f}(x, m_1)m_1)(\mathsf{f}(xy, m_2)m_2)xyzm_3t \\ &= (\mathsf{f}(x, m_1)m_1)(\mathsf{f}(xy, m_2)m_2)(\mathsf{f}(xyz, m_3)m_3)xyzt, \end{aligned} \tag{3.9}$$

where $\mathsf{f}(x, m_1)m_1, \mathsf{f}(xy, m_2)m_2, \mathsf{f}(xyz, m_3)m_3 \in \mathfrak{R}$. We can write also

$$\begin{aligned} xm_1ym_2zm_3t &= x(\mathsf{f}(m_1, y)y)m_1m_2zm_3t \\ &= x(\mathsf{f}(m_1, y)y)(\mathsf{f}(m_1m_2, z)z)m_1m_2m_3t, \end{aligned} \tag{3.10}$$

where $\mathsf{f}(m_1, y)y, \mathsf{f}(m_1m_2, z)z \in \mathfrak{R}$. We will use (3.9) and (3.10) to prove Theorems 3.2.14 and 3.2.19.

Theorem 3.2.14 *Let S be a left cancellative monoid with the neutral element e, and $\mathfrak{R}$ an S-graded ring with a finite support Γ. If $\mathfrak{R}_e$ is nil and f-commutative, then $\mathfrak{R}$ is nil. In addition, if $\mathfrak{R}_e$ is nil of bounded index, then $\mathfrak{R}$ is nil of bounded index.*

Proof: Let $\Gamma : \mathfrak{R} = \bigoplus_{i=1}^{d} \mathfrak{R}_{g_i}$ be an S-grading on $\mathfrak{R}$ with $\mathsf{Supp}(\Gamma) = \{g_1, g_2, \ldots, g_d\} \subseteq \mathsf{S}$. Assume that $\mathfrak{R}_e$ is an f-commutative nil ring. If $e \notin \mathsf{Supp}(\Gamma)$, by Proposition 3.2.2, it follows that $\mathfrak{R}^{d+1} = \{0\}$, and the result follows.

Assume now that $e \in \mathsf{Supp}(\Gamma)$. Let $a = \sum_{i=1}^{d} a_{g_i} \in \mathfrak{R}$ be an arbitrary element, with $a_{g_i} \in \mathfrak{R}_{g_i}$. Let us show that a is nilpotent, i.e. there exists $n \in \mathbb{N}$ such that $a^n = 0$. By (3.1), it is sufficient to consider only the products of n homogeneous components of a. Consider the set

$$\Lambda = \{b_1b_2 \cdots b_k : 1 \leqslant k \leqslant 2d,\ b_1, \ldots, b_k \in \{a_{g_1}, \ldots, a_{g_d}\}\} ,$$

which is finite, and its subset $\tilde{\Lambda} = \{b \in \Lambda : \mathsf{deg}(b) = e\}$. By Lemma 3.2.8, for any $b_1, b_2, \ldots, b_{2d} \in \{a_{g_1}, \ldots, a_{g_d}\}$, we have that $b_1b_2 \cdots b_{2d} = 0$ or there exist $0 \leqslant s_0 \leqslant s_1 \leqslant s_2 \leqslant 2d$ such that $e = \mathsf{deg}(b_{s_0+1} \cdots b_{s_1}) = \mathsf{deg}(b_{s_1+1} \cdots b_{s_2})$. In this last case, we have that $(b_{s_0+1} \cdots b_{s_1}), (b_{s_1+1} \cdots b_{s_2}) \in \tilde{\Lambda}$. Thus, if $b_1b_2 \cdots b_{2d} \neq 0$ for some $b_1, b_2, \ldots, b_{2d} \in \{a_{g_1}, \ldots, a_{g_d}\}$, then $\tilde{\Lambda} \neq \varnothing$. We have that $\tilde{\Lambda}$ contains all elements of the neutral degree formed by the products of at most $2d$ elements of the set $\{a_{g_1}, \ldots, a_{g_d}\}$. Note that $\tilde{\Lambda}$ is contained in $\mathfrak{R}_e$. Hence, since $\tilde{\Lambda}$ is finite and $\mathfrak{R}_e$ is nil, we can take $r = \mathsf{min}\{m \in \mathbb{N} : b^m = 0, \forall b \in \tilde{\Lambda}\}$. Put $n = r|\tilde{\Lambda}|$, and fix any $b_1, b_2, \ldots, b_{2nd} \in \{a_{g_1}, \ldots, a_{g_d}\}$. Let us show that the monomial $m = b_1b_2 \cdots b_{2nd}$ is equal to zero.

To obtain a contradiction, suppose that $m \neq 0$. By Lemma 3.2.8, since $m \neq 0$, there exist $0 \leqslant s_0 < s_1 < \cdots < s_{2n} \leqslant 2nd$ such that

$$c_1 = b_{s_0+1} \cdots b_{s_1},\ c_2 = b_{s_1+1}b_{s_1+2} \cdots b_{s_2}, \ldots,\ c_{2n} = b_{s_{(2n-1)}+1}b_{s_{(2n-1)}+2} \cdots b_{s_{2n}} \in \mathfrak{R}_e\,. \tag{3.11}$$

By Remark 3.2.13, it follows that there exist $i_1, \ldots, i_n \in \{1, \ldots, 2n\}$ such that $s_{i_j} - s_{i_j-1} \leqslant 2d$ for all $j \in \{1, \ldots, n\}$, and hence,

$$\tilde{c}_1 = (b_{s_{i_1-1}+1} \cdots b_{s_{i_1}}),\ \tilde{c}_2 = (b_{s_{i_2-1}+1} \cdots b_{s_{i_2}}), \ldots,\ \tilde{c}_n = (b_{s_{i_n-1}+1} \cdots b_{s_{i_n}}) \in \tilde{\Lambda}\ .$$

Observe that $\tilde{c}_k = c_{i_k}$ for all $k = 1, \ldots, n$.

Since $\tilde{c}_1, \ldots, \tilde{c}_n \in \tilde{\Lambda}$, and $n = r|\tilde{\Lambda}|$, where $\tilde{\Lambda}$ is a finite set, it follows that there exist $1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n$ such that $\tilde{c}_{j_1} = \tilde{c}_{j_2} = \cdots = \tilde{c}_{j_r} = c \in \tilde{\Lambda}$, and thus, $\tilde{c}_{j_1}\tilde{c}_{j_2} \cdots \tilde{c}_{j_r} = c^r = 0$. From this, by (3.9), (3.10) and (3.11), since $\mathfrak{R}_e$ is f-commutative, it follows that

$$\begin{aligned} m = b_1 \cdots b_{2nd} &= b_1 \cdots b_{s_0}(c_1 c_2 \cdots c_{2n}) b_{s_{2n}+1} \cdots b_{2nd} \\ &= (b_1 \cdots b_{s_0})(c_1 \cdots c_{i_1-1})\tilde{c}_1(c_{i_1+1} \cdots c_{i_2-1})\tilde{c}_2 \cdots \tilde{c}_n(c_{i_n+1} \cdots c_{2n})(b_{s_{2n}+1} \cdots b_{2nd}) \\ &= (b_1 \cdots b_{s_0}) m_1 \tilde{c}_{j_1} m_2 \tilde{c}_{j_2} m_3 \cdots m_r \tilde{c}_{j_r} m_{r+1} (b_{s_{2n}+1} \cdots b_{2nd})\ , \end{aligned}$$

where $m_1 = (c_1 \cdots c_{i_{j_1}-1})$, $m_2 = (c_{i_{j_1}+1} \cdots c_{i_{j_2}-1})$, $\ldots$, $m_r = (c_{i_{j_{r-1}}+1} \cdots c_{i_{j_r}-1})$, $m_{r+1} = (c_{i_{j_r}+1} \cdots c_{2n}) \in \mathfrak{R}_e$. Put $\tilde{m}_1 = (b_1 \cdots b_{s_0}) m_1$, and $\tilde{m}_{r+1} = m_{r+1}(b_{s_{2n}+1} \cdots b_{2nd})$, it follows that

$$\begin{aligned} m &= \tilde{m}_1 \tilde{c}_{j_1} m_2 \tilde{c}_{j_2} m_3 \cdots m_r \tilde{c}_{j_r} \tilde{m}_{r+1} = \tilde{m}_1(\tilde{c}_{j_1} m_2 \tilde{c}_{j_2} m_3 \cdots m_r \tilde{c}_{j_r})\tilde{m}_{r+1} \\ &= \tilde{m}_1 \mathsf{f}(\tilde{c}_{j_1}, m_2) m_2 \mathsf{f}(\tilde{c}_{j_1}\tilde{c}_{j_2}, m_3) m_3 \cdots \mathsf{f}(\tilde{c}_{j_1}\tilde{c}_{j_2} \cdots \tilde{c}_{j_{r-1}}, m_r) m_r (\tilde{c}_{j_1}\tilde{c}_{j_2} \cdots \tilde{c}_{j_r})\tilde{m}_{r+1}\ . \end{aligned}$$

Since $\tilde{c}_{j_1}\tilde{c}_{j_2} \cdots \tilde{c}_{j_r} = 0$, we have that $m = b_1 \cdots b_{2nd} = 0$. Evidently, this is a contradiction. Thus, we have that $b_1 \cdots b_{2nd} = 0$ if $\tilde{\Lambda} = \varnothing$, and also $b_1 \cdots b_{2nd} = 0$ if $\tilde{\Lambda} \neq \varnothing$, for any $b_1, \ldots, b_{2nd} \in \{a_{g_1}, \ldots, a_{g_d}\}$. Anyway, we conclude that $\mathfrak{R}$ is a nil ring.

To prove the second part of the theorem it is sufficient to take $r = \mathsf{nd}_{nil}(\mathfrak{R}_e)$ and to proceed as in first part of this proof. ■

By the proof of the previous theorem, if $\mathfrak{R}$ is an S-graded ring whose neutral component is nil of bounded index, we can exhibit an upper bound for $\mathsf{nd}_{nil}(\mathfrak{R})$. Indeed, it is easy to see that $\mathsf{nd}_{nil}(\mathfrak{R}) \leqslant 2rd^2\left(\dfrac{d^{2d}-1}{d-1}\right)$, where $\mathsf{nd}_{nil}(\mathfrak{R}_e) = r < \infty$ and $d = |\mathsf{Supp}(\mathfrak{R})|$, since $|\tilde{\Lambda}| \leqslant |\Lambda| \leqslant d + d^2 + \cdots + d^{2d} = \dfrac{d(d^{2d}-1)}{d-1}$.

Observe also that a proof similar to proof of Theorem 3.2.14 ensures a positive answer to Problem 3.2.5 in the class of f-commutative rings. Beside that, Theorem 3.2.14

provides that Problem 3.2.5 has a positive solution in the class of all associative rings with a finite grading whose neutral component belongs to the class Υ of all the f-commutative rings.

Notice that we can weaken the definition of an f-commutator and still obtain that the previous theorem is true. In fact, it is sufficient to assume that a semigroup $\mathfrak{S}$ acts on $\mathfrak{R}$ if $(\lambda\gamma)x = \lambda(\gamma x)$ for any $\lambda, \gamma \in \mathfrak{S}$, and $x \in \mathfrak{R}$, and hence, to define $[a, b]_{\mathsf{f}} = ab - (\mathsf{f}(a, b)b)a$ for any $a, b \in \mathfrak{R}$, where f is a map from $\mathfrak{R} \times \mathfrak{R}$ into $\mathfrak{S}$. Therefore, (3.9) and (3.10) are still true, and thus, Theorem 3.2.14 can also be verified in this case.

Theorem 3.2.15 *Let* S *be a left cancellative monoid and* $\mathfrak{R}$ *be a ring with a finite* S-*grading* Γ. *If* $\mathfrak{R}_e$ *is nilpotent of index* $\mathsf{nd}(\mathfrak{R}_e) = r \geqslant 1$, *then* $\mathfrak{R}$ *is a nilpotent ring with* $r \leqslant \mathsf{nd}(\mathfrak{R}) \leqslant dr$, *where* $d = |\mathsf{Supp}(\Gamma)|$, *and* $r > 1$; *or* $r \leqslant \mathsf{nd}(Ri) \leqslant d + 1$ *if* $r = 1$.

Proof: Suppose that $\mathfrak{R}_e$ is a nilpotent ring with $\mathsf{nd}(\mathfrak{R}_e) = r > 1$. We will show that $a_1 a_2 \cdots a_{rd} = 0$ for any homogeneous elements $a_1, a_2, \ldots, a_{rd} \in \mathfrak{R}$ (see (3.1)), where $d = |\mathsf{Supp}(\Gamma)|$.

Taking into account Lemma 3.2.8, suppose that there exist $0 \leqslant s_0 < s_1 < \cdots < s_r \leqslant rd$ satisfying

$$e = \mathsf{deg}(a_{s_0+1} \cdots a_{s_1}) = \mathsf{deg}(a_{s_1+1} a_{s_1+2} \cdots a_{s_2}) = \cdots = \mathsf{deg}(a_{s_{r-1}+1} \cdots a_{s_r}).$$

Hence, $(a_{s_0+1} \cdots a_{s_1}), (a_{s_1+1} a_{s_1+2} \cdots a_{s_2}), \ldots, (a_{s_{r-1}+1} \cdots a_{s_r}) \in \mathfrak{R}_e$, and thus, it follows that $(a_{s_0+1} \cdots a_{s_r}) \in (\mathfrak{R}_e)^r = \{0\}$. By this reason, we have that $a_1 \ldots a_{rd} = 0$.

Thus, by Lemma 3.2.8, for any $a_1, \ldots, a_{rd} \in \mathfrak{R}$ we always have that $a_1 \ldots a_{rd} = 0$. Therefore, we conclude that $\mathfrak{R}$ is a nilpotent ring with $\mathsf{nd}(\mathfrak{R}) \leqslant dr$.

Observe that for $\mathsf{nd}(\mathfrak{R}_e) = r = 1$, the result holds by Proposition 3.2.2. ■

It easily follows from the previous theorem that Problem 3.2.5 has positive solution in the class of graded rings, whose support is finite and the neutral component is a nilpotent ring.

Example 3.2.16 *Let* $\mathfrak{R}$ *be a commutative nilpotent* $\mathbb{F}$-*algebra, whose nilpotency index is* $\mathsf{nd}(\mathfrak{R}) = 2p$, $\mathbb{F}$ *an algebraically closed field and* $\mathsf{char}(\mathbb{F}) = p > 0$. *Consider the algebra* *given by*

$$\mathfrak{A} = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & (a_{33})^p \end{pmatrix} : a_{ij} \in \mathfrak{R} \right\}.$$

Notice that $\mathfrak{A}$ *is a subalgebra of* $M_3(\mathfrak{R})$*, the* $\mathbb{F}$*-algebra of* 3×3 *matrices over* $\mathfrak{R}$*, and* $\mathfrak{A}^2 \subseteq \mathsf{SUT}_3(\mathfrak{R})$*. Now, consider the* $\mathbb{F}$*-algebra* M *such that*

$$M = \left\{ \begin{pmatrix} \mathfrak{A} & 0_{3\times 3} \\ 0_{3\times 3} & \mathfrak{A} \end{pmatrix} \right\}.$$

We have that M *is* $\mathbb{Z}_6$*-graded with the elementary grading* Γ *defined by* $(\overline{0}, \overline{1}, \ldots, \overline{5}) \in (\mathbb{Z}_6)^6$*, with support of order equal to* 3*. It is easy to see that* $(M_{\overline{0}})^2 = \{0\}$*, and hence, by Theorem 3.2.15, it follows that* $M^6 = \{0\}$*. Observe that* $\mathsf{nd}(M) = 4$*, and hence,* $\mathsf{nd}(M) \leqslant \mathsf{nd}(M_{\overline{0}})|\mathsf{Supp}(\Gamma)| < \mathsf{nd}(M_{\overline{0}})|\mathbb{Z}_6|$*, i.e the previous theorem provides an upper bound better than if we look only at the order of the group.*

Remark 3.2.17 *Let* $\mathfrak{R}$ *be an* f*-commutative finitely generated ring. Suppose that* $\mathfrak{R}$ *is nil. Let* $n \in \mathbb{N}$ *be the smallest number of generators of* $\mathfrak{R}$*. Fix a set* β *of generators of* $\mathfrak{R}$ *with* n *elements. Let* $s \in \mathbb{N}$ *be the largest nilpotency index of the elements of* β*. By* (3.1)*,* (3.9) *and* (3.10)*, it is easy to check that* $a_1a_2\cdots a_{(s-1)n+1} = 0$ *for any* $a_1, a_2, \ldots, a_{(s-1)n+1} \in \mathfrak{R}$*. Thus, we can see that* $\mathfrak{R}$ *is a nilpotent ring with nilpotency index* $s \leqslant \mathsf{nd}(\mathfrak{R}) \leqslant (s-1)n+1$*.*

Let us consider some classes of graded rings, such that the condition "the neutral component is nil" provides the nilpotency of the whole ring.

Example 3.2.18 *Let* $\mathfrak{R}$ *be a ring with a finite* S*-grading* Γ *such that* $\mathfrak{R}_e$ *is nil of index* 2*. Suppose that* $\mathsf{char}(\mathfrak{R}_e) \neq 2$*. Given* $a, b \in \mathfrak{R}_e$*, we have*

$$0 = (a+b)^2 = a^2 + b^2 + ab + ba = ab + ba \ ,$$

and hence, $ab = -ba$ *for any* $a, b \in \mathfrak{R}_e$*. Now, considering any* $a, b, c \in \mathfrak{R}_e$*, it follows that*

$$\begin{aligned} 0 &= (ab+c)^2 = (ab)^2 + c^2 + abc + cab = abc + cab = abc + (ca)b \\ &= abc - (ac)b = abc - a(cb) = abc - a(-bc) = 2abc \ , \end{aligned}$$

and so $abc = 0$, since $\mathsf{char}(\mathfrak{R}_e) \neq 2$. Therefore, $(\mathfrak{R}_e)^3 = 0$. By Theorem 3.2.15, it follows that $\mathfrak{R}$ is a nilpotent ring with $\mathsf{nd}(\mathfrak{R}) \leqslant 3d$, where $d = |\mathsf{Supp}(\Gamma)|$.

Theorem 3.2.19 *Let S be a left cancellative monoid and $\mathfrak{R}$ a ring with a finite S-grading Γ. If $\mathfrak{R}_e$ is nil, f-commutative and finitely generated, then $\mathfrak{R}$ is a nilpotent ring. Moreover, if $\{a_1, \ldots, a_n\}$ is a generator set of $\mathfrak{R}_e$ and $d = |\mathsf{Supp}(\Gamma)|$, then $s \leqslant \mathsf{nd}(\mathfrak{R}) \leqslant d((s-1)n+1)$, where $s = \min\{m \in \mathbb{N} : a_i^m = 0, i = 1, \ldots, n\}$ $(s > 1)$. If $\mathfrak{R}_e = \{0\}$ $(s = 1)$, then $1 \leqslant \mathsf{nd}(\mathfrak{R}) \leqslant d + 1$.*

Proof: In fact, by Remark 3.2.17, it follows that $\mathfrak{R}_e$ is nilpotent with $s \leqslant \mathsf{nd}(\mathfrak{R}_e) \leqslant r$, where $r = (s-1)n + 1$, s and n are as in Remark 3.2.17. Thus, by Theorem 3.2.15, we conclude that $\mathfrak{R}$ is nilpotent with $s \leqslant \mathsf{nd}(\mathfrak{R}) \leqslant dr$.

If $\mathfrak{R}_e = \{0\}$, then by Proposition 3.2.2, we have $\mathfrak{R}^{d+1} = \{0\}$. ■

The following examples ensures that the assumptions of previous theorems are necessary. The first three examples present graded rings or algebras, in which the neutral component is not finitely generated. And the last example concerns the case $\mathfrak{R}_e$ is not f-commutative.

Example 3.2.20 *If $\mathfrak{R}_e$ can not be finitely generated, the previous theorem does not hold. To see this, a counterexample is given below. Let $\mathfrak{R} = \mathbb{Z}[x_1, x_2, x_3, \ldots]/I$ be the quotient ring of the polynomial ring over $\mathbb{Z}$ in the variables $x_1, x_2, x_3, \ldots$ by its ideal $I = \langle x_1^2, x_2^3, x_3^4, \ldots \rangle$, with the trivial grading ($\mathfrak{R}_e = \mathfrak{R}$). We have that $\mathfrak{R}$ is a commutative ring which is nil but it is not nilpotent.*

Example 3.2.21 (5. Remark (I), [33]) *Let $\mathbb{K}$ be a field of characteristic $p \neq 0$. Let $\mathfrak{A}_k$ be the algebra over $\mathbb{K}$ with the generating elements $x_1, \ldots, x_k$ with the fundamental relations $x_i^p = 0$, $x_i x_j = x_j x_i$ for $i, j = 1, 2, \ldots, k$; and put $\mathfrak{A} = \sum_{k=1}^{\infty} \mathfrak{A}_k$. Then $\mathfrak{A}$ is a commutative algebra which is nil of bounded index, with the trivial grading for any left cancellative monoid S, but $\mathfrak{A}$ is not nilpotent.*

Example 3.2.22 (Lemma 8(5.6), [38]) *Let E be the infinite dimensional Grassmann algebra over a field of characteristic $p \neq 0$ and let us consider $\mathsf{E}^* = \mathsf{E} - \{1\}$, then E^* satisfies the identity $x^p \equiv 0$, i.e. E^* is nil of degree p. We have that E^* is a $\mathbb{Z}_2$-graded ring, such that E_0 is a nil commutative algebra (ring), but E^* is not nilpotent.*

Example 3.2.23 *In [18] (text in Russian), Example* 1 *exhibits a construction of a nil ring which is finitely generated but it is not nilpotent (see also [8] Exercise 5, Section 6.3, page 179, or [21], Chapter 8: The Golod-Shafarevitch Theorem). Then by Remark 3.2.14, this ring can not be* f*-commutative for any semigroup* $\mathfrak{S}$ *and map* f*. This ring with the trivial grading also gives an example which shows the necessity of the condition "*$\mathfrak{R}_e$ *is* f*-commutative" to be required in Theorem 3.2.19.*

Corollary 3.2.24 *Let* S *be a group and* $\mathfrak{R}$ *be a ring with an* S*-grading* Γ*, not necessary finite. Let* H *be a normal subgroup of* S *and* $\overline{\Gamma} : \mathfrak{R} = \bigoplus_{\bar{g}\in\mathsf{S}/H}\mathfrak{R}_{\bar{g}}$ *be the* S/H*-grading induced by* Γ*, i.e.* $\mathfrak{R}_{\bar{g}} = \bigoplus_{h\in H}\mathfrak{R}_{gh}$ *for any* $\bar{g} \in \mathsf{G}/H$*. Suppose* $\overline{\Gamma}$ *has the finite support of order* d*. The following assumptions are true:*

i) If $H \cap \mathsf{Supp}(\Gamma) = \varnothing$*, then* $\mathfrak{R}^{d+1} = \{0\}$*;*

ii) If $\mathfrak{R}_{\bar{e}} = \bigoplus_{h\in H}\mathfrak{R}_h$ *is* f*-commutative and nil (resp. nil of bounded index), then* $\mathfrak{R}$ *is nil (resp. nil of bounded index).*

iii) $\mathfrak{R}_{\bar{e}} = \bigoplus_{h\in H}\mathfrak{R}_h$ *is nilpotent iff* $\mathfrak{R}$ *is nilpotent.*

Proof: Considering $\mathfrak{R}$ with its S/H-grading $\overline{\Gamma}$ (induced by Γ), it is sufficient to apply Proposition 3.2.2, Theorem 3.2.14 and Theorem 3.2.15. The result follows. ∎

It is important to note that, in general, the previous corollary ensures that for a graded ring with a support not necessarily finite we can obtain the same results as in the first part of this chapter. In addition, if support of Γ is finite, then $d \leqslant |\mathsf{Supp}(\Gamma)|$.

Observe nevertheless that, in Corollary 3.2.24, $\mathfrak{R}_{\bar{e}} = \bigoplus_{h\in H}\mathfrak{R}_h$, and hence, the initial claim must be true for the major part of $\mathfrak{R}$.

3.3 Applications

This part of the 3rd chapter is important. We present here two considerable applications of the results of the previous section: one of them generalizes the Dubnov-Ivanov-Nagata-Higman Theorem, and another one shows a relation between graded rings and Köthe's Problem.

3.3.1 Graded Algebras and Dubnov-Ivanov-Nagata-Higman Theorem

Let $\mathbb{F}$ be a field, S a left cancellative monoid, and $\mathfrak{R}$ be an (associative) $\mathbb{F}$-algebra with a finite S-grading (grading of finite support).

Let us now introduce the Dubnov-Ivanov-Nagata-Higman Theorem. Under suitable conditions, it ensures the equivalence between nil algebras of bounded degree, and nilpotent algebras. Besides that, an upper bound is given to the nilpotency index, depending only on the nil index of the algebra. In 1953, Nagata proved that any nil algebra of bounded degree over a field of characteristic zero is nilpotent. Afterwards, in 1956, Higman generalized the result of Nagata to any field. Posteriorly, it was discovered that this result was firstly published in [12], in 1943, by Dubnov and Ivanov.

Theorem 3.3.1 (Dubnov-Ivanov-Nagata-Higman, [12, 33, 22]) *Let $\mathfrak{R}$ be an associative algebra over a field $\mathbb{F}$. Assume $\mathsf{char}(\mathbb{F}) = p$. Suppose $x^n \equiv 0$ in $\mathfrak{R}$. If $p = 0$ or $n < p$, then $x_1x_2\cdots x_{2^n-1} \equiv 0$ in $\mathfrak{R}$.*

In [30], E. N. Kuzmin exhibited a lower bound for the nilpotency index of a nil algebra of bounded index $\mathfrak{R}$ over a field of characteristic zero. He showed that $\mathsf{nd}(\mathfrak{R}) \geqslant \dfrac{n(n+1)}{2}$, where $n = \mathsf{nd}_{nil}(\mathfrak{R})$. Later, in [36], Razmyslov proposed a smaller estimate than that given by Higman in [22], the proof can be founded in [37].

Theorem 3.3.2 (Theorem 33.1, [37]) *In any associative algebra over a field of characteristic zero in which the identity $y^n \equiv 0$ is valid, the identity $x_1x_2\cdots x_{n^2} \equiv 0$ is valid.*

Finally, we deduce an immediate consequence from Theorem 3.2.15 and the previous theorem. Therefore, we have answered Problem 3.2.5 for S-graded algebras over a field of characteristic zero, if $\mathbb{R}_e$ is nil of bounded index.

Theorem 3.3.3 *Let S be a left cancellative monoid and $\mathfrak{R}$ an associative algebra over a field $\mathbb{F}$ with an S-grading of finite support, $\mathsf{char}(\mathbb{F}) = p$. Suppose $\mathfrak{R}_e$ is a nil algebra of bounded index $s = \mathsf{nd}_{nil}(\mathfrak{R}_e) > 1$. If $p = 0$ or $p > s$, then $\mathfrak{R}$ is a nilpotent algebra. In addition, if $d = |\mathsf{Supp}(\Gamma_{\mathsf{S}})|$, we have*

i) if $p > s$, then $\mathsf{nd}(\mathfrak{R}) \leqslant d(2^s - 1)$;

ii) if $p = 0$ *, then* $\mathsf{nd}(\mathfrak{R}) \leqslant dq$ *where* $q = \begin{cases} 2^s - 1, & \text{if} \quad s = 2,3,4 \\ s^2, & \text{if} \quad s \geqslant 5 \end{cases}$.

Is $s = \mathsf{nd}(\mathfrak{R}_e) = 1$ *then* $\mathfrak{R}$ *is nilpotent for any field* $\mathbb{F}$*, and* $\mathsf{nd}(\mathfrak{R}) \leqslant d+1$.

Proof: The first part follows directly from Theorem 3.3.1 and from Theorem 3.2.15. Already the items *i)* and *ii)* follow from Theorem 3.3.2 and again from Theorem 3.3.1, and also by $2^n - 1 \leqslant n^2$ in $\mathbb{N}$ iff $n = 1,2,3,4$.

The case $s = 1$ follows from Theorem 3.2.15 (or Proposition 3.2.2). ∎

The corollary below is an immediate consequence of Theorem 3.3.3.

Corollary 3.3.4 *Let* S *be a left cancellative monoid and* $\mathfrak{R}$ *an associative algebra over a field* $\mathbb{F}$ *with an* S*-grading of finite support. If* $\mathfrak{R}_e$ *is nil,* $\mathsf{char}(\mathbb{F}) \neq 2,3$ *and* $s \in \{2,3,4\}$*, then* $x_1x_2\cdots x_{d(2^s-1)} \equiv 0$ *in* $\mathfrak{A}$.

Note that the upper bound for the nilpotency index obtained in Theorem 3.2.19 can be smaller for the case of a little number n of generators of $\mathfrak{R}_e$ than the limitation given by Theorem 3.3.3. Nevertheless, in Theorem 3.3.3, $\mathfrak{R}_e$ is not necessarily f-commutative, and the bound of the nilpotency degree does not depend on the number of generators of $\mathfrak{R}$ (inclusively $\mathfrak{R}_e$ can be infinitely generated).

3.3.2 Graded Rings and Köthe's Problem

As in the previous sections, here, all the rings are associative, not necessarily with unity.

In [29], Köthe conjectured that if a ring $\mathfrak{R}$ has no nonzero nil ideals, then $\mathfrak{R}$ has no nonzero one-sided nil ideals. The question if this conjecture is true is known as Köthe's Problem, and is still unsolved in the general case. For some equivalences of this problem, see [14], [43], [42]. In this section, we present a relation between graded rings and Köthe's Problem. Firstly, below, we exhibit some equivalences of Köthe's Problem, which are the basic tools for our study.

Theorem 3.3.5 (Some equivalences of Köthe Problem, [43]) *The following assumptions are equivalent:*

i) If a ring has no nonzero nil ideals, then it has no nonzero one-sided ideals (Köthe's conjecture);

ii) The sum of two right nil ideals in any ring is nil;

iii) For every nil ring $\mathfrak{R}$, the ring of 2×2 matrices over $\mathfrak{R}$ is nil;

iv) For every nil ring $\mathfrak{R}$, the ring of $n \times n$ matrices over $\mathfrak{R}$ is nil.

Various other equivalences of Köthe's Problem have been exhibited since 1930. Also the problem was solved positively in some classes of rings, but no answer in the general case. Now we present one more class of rings which issues a positive solution Köthe's Problem.

Corollary 3.3.6 *The Köthe's Problem has a positive solution for any* f*-commutative ring.*

Proof: Let $\mathfrak{R}$ be a nil f-commutative ring. Let us show that $M_2(\mathfrak{R})$ is nil. We have $M_2(\mathfrak{R}) = M_0 \oplus M_1$, with $M_0 = \left\{\begin{pmatrix} \mathfrak{R} & 0 \\ 0 & \mathfrak{R} \end{pmatrix}\right\}$, and $M_1 = \left\{\begin{pmatrix} 0 & \mathfrak{R} \\ \mathfrak{R} & 0 \end{pmatrix}\right\}$, defines the elementary $\mathbb{Z}_2$-grading on $M_2(\mathfrak{R})$. Since

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

for any $a, b \in \mathfrak{R}$ and $n \in \mathbb{N}$, we have that M_0 is nil. Suppose that $f : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{S}$, where $\mathfrak{S}$ is a semigroup acting on the left of $\mathfrak{R}$. Then define a semigroup $\tilde{\mathfrak{S}} = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathfrak{S}\right\}$ with the usual product of diagonal of matrices. Observe that $\tilde{\mathfrak{S}}$ acts on $\mathfrak{M}_0$ from the left naturally:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ 0 & \beta b \end{pmatrix},$$

for any $alpha, \beta \in \mathfrak{S}$, and $a, b \in \mathfrak{R}$. Consider the map $\tilde{\mathsf{f}}$ of $M_0 \times M_0$ to $M_2(\mathfrak{R})$ defined by

$$\tilde{\mathsf{f}}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} \mathsf{f}(a, c) & 0 \\ 0 & \mathsf{f}(b, d) \end{pmatrix}.$$

Observe that M_0 is $\tilde{\mathsf{f}}$-commutative, and hence, by Theorem 3.2.14, it follows that $M_2(\mathfrak{R})$

is a nil ring. By Theorem 3.3.5, we conclude that for $\mathfrak{R}$ the solution of Köthe's Problem is positive, and thus, the result follows. ■

Let $\mathfrak{A}$ be an associative algebra over a field of characteristic zero graded by a left cancellative monoid S. If $\mathfrak{A}$ is nil of bounded index, then, by Theorem 3.3.3, and similarly to the proof of Corollary 3.3.6, it follows that the Köthe's Problem has a positive solution for $\mathfrak{A}$.

Now, suppose that Köthe's Problem has a negative solution. By Theorem 3.3.5, it follows that the 2×2 matrices ring $M_2(\mathfrak{R})$ is not a nil ring for some nil ring $\mathfrak{R}$. Considering the elementary $\mathbb{Z}_2$-grading on $\mathfrak{A} = M_2(\mathfrak{R})$, we have, that $\mathfrak{A}_0 \cong \mathfrak{R}\times\mathfrak{R}$ is also nil. Thus, we can conclude that $\mathfrak{A}$ is a counterexample to Problem 3.2.5.

More general, let $\mathfrak{R}$ be a ring and $M_n(\mathfrak{R})$ be the ring of all $n\times n$ matrices over $\mathfrak{R}$. For each $\lambda\in\mathbb{Z}_n$, consider the subgroups of $(\mathfrak{R},+)$ given by

$$M_\lambda = \{E_{ij}(a)\in M_n(\mathfrak{R}) : a\in\mathfrak{R}, \overline{j-i}=\lambda\} \ .$$

We have that $\Gamma : M_n(\mathfrak{R}) = \bigoplus_{\lambda\in\mathbb{Z}_n} M_\lambda$ defines a $\mathbb{Z}_n$-grading on $M_n(\mathfrak{R})$. This grading is called **elementary $\mathbb{Z}_n$-grading** on $M_n(\mathfrak{R})$. Notice that

$$M_{\overline{0}} = \{E_{11}(a_1)+E_{22}(a_2)+\cdots E_{nn}(a_n) \ : \ a_1,a_2,\ldots,a_n\in\mathfrak{R}\}\,, \tag{3.12}$$

and hence, $M_{\overline{0}}$ is nil iff $\mathfrak{R}$ is nil.

In what follows, we show a relation between graded rings and Köthe's Problem.

Theorem 3.3.7 *A positive answer to Problem 3.2.5 implies that the Köthe's Problem has a positive solution. In particular, being $\mathsf{S}=\mathbb{Z}_n$, a positive solution of Problem 3.2.5 for S-graded rings implies Köthe's conjecture.*

Proof: Let us apply the item *v)* of Theorem 3.3.5 for the S-graded matrix ring $\mathfrak{A} = M_n(\mathfrak{R})$ over a ring $\mathfrak{R}$ with the elementary S-grading defined above. The positive answer of Problem 3.2.5 for the ring $\mathfrak{A}$ gives the positive solution of Köthe's problem for a nil ring $\mathfrak{R}$. ■

The previous theorem shows a connection between graded rings and Köthe's Problem. We present below a question still unanswered in the general case.

Problem 3.3.8 *Are Köthe's Problem and Problem 3.2.5 equivalent?*

CHAPTER 4

GRADED ALGEBRAS WITH THE CENTRAL NEUTRAL COMPONENT

In this chapter, we study the variety of G-graded algebras over an algebraically closed field of characteristic zero defined by G-graded polynomial identities $[x^{(e)}, y^{(g)}]$ for all $g \in \mathsf{G}$, where G is an abelian finite group.

On other words, here we exhibit results concerning to the variety $\mathfrak{V}^{\mathsf{G}}$ of all G-graded algebras whose neutral component is central, i.e. $\mathfrak{V}^{\mathsf{G}} := \mathsf{var}^{\mathsf{G}}\left(\{[x^{(e)}, y^{(g)}] : g \in \mathsf{G}\}\right)$. We present some properties of algebras which belong to the variety $\mathfrak{V}^{\mathsf{G}}$, and in suitable conditions, we give a description of $\mathfrak{V}^{\mathsf{G}}$, in the language of a carrier.

The first section also contains results concerning associative rings graded by a two-sided cancellative monoid with the central neutral component.

4.1 Graded rings with the central neutral components

In this section, we present some general results involving associative rings graded by a (two-sided) cancellative monoid, i.e. a monoid which satisfies $gh = th$ iff $g = t$, and $h'g' = h't'$ iff $g' = t'$, for any $g, g', h, h', t, t' \in \mathsf{S}$, whose neutral component is central. Here, let us denote by S a cancellative monoid, $\mathfrak{R}$ is an associative ring with an S-grading Γ. Let us assume also that Γ has a finite support, namely $|\mathsf{Supp}(\Gamma)| = d < \infty$.

Theorem 4.1.1 *Let S be a cancellative monoid, and $\mathfrak{R}$ an associative ring with a finite S-grading Γ. If $|\mathsf{Supp}(\Gamma)| = d$ and $\mathfrak{R}_e$ is central in $\mathfrak{R}$, then $[x_1, \ldots, x_{d+1}] \equiv 0$ in $\mathfrak{R}$ for $d \in \{1, 2, 3\}$.*

Proof: Firstly, by Proposition 3.2.2, if $\mathfrak{R}_e = \{0\}$, then $\mathfrak{R}^{d+1} = \{0\}$. In particular, $[x_1, \ldots, x_{d+1}] \equiv 0$ in $\mathfrak{R}$ in this case.

Assume that $\mathfrak{R}_e \neq \{0\}$. For $d = 1$, we have $\mathfrak{R} = \mathfrak{R}_e$, and hence, if $\mathfrak{R}_e \subseteq \mathcal{Z}(\mathfrak{R})$, then $\mathfrak{R}$ is commutative, i.e. $[x_1, x_2] \equiv 0$ in $\mathfrak{R}$.

Suppose $d = 2$ and put $\mathsf{Supp}(\Gamma) = \{e, g\}$, where $g \neq e$, then, either $g^2 = e$ or $g^2 \notin \mathsf{Supp}(\Gamma)$, because S is cancellative. Anyway, $(\mathfrak{R}_g)^2 \subseteq \mathfrak{R}_e$. Given $a, b, c \in \mathfrak{R}$, we can write $a = a_e + a_g$, $b = b_e + b_g$. Since $\mathfrak{R}_e \subseteq \mathcal{Z}(\mathfrak{R})$, it follows that

$$[a, b, c] = [a_e + a_g, b_e + b_g, c] = [a_g, b_g, c] = [[a_g, b_g], c] = 0.$$

Therefore, $[a, b, c] = 0$ for any $a, b, c \in \mathfrak{R}$.

Now, assume $d = 3$, and put $\mathsf{Supp}(\Gamma) = \{e, g, h\}$. Consider the elements $gh, hg \in \mathsf{S}$. Observe that either $hg = gh = e$ or $hg, gh \notin \mathsf{Supp}(\Gamma)$, since S is cancellative. In fact, since S is cancellative, we have $gh, hg \notin \{h, g\}$. Hence, if $hg \in \mathsf{Supp}(\Gamma)$, then $hg = e$, and hence, $hgh = h$, and by cancellation law, it follows that $gh = e$. Similarly, $gh \in \mathsf{Supp}(\Gamma)$ implies $gh = hg = e$. Anyway, we have $\mathfrak{R}_g\mathfrak{R}_h, \mathfrak{R}_h\mathfrak{R}_g \subseteq \mathfrak{R}_e$.

Given $a, b, c \in \mathfrak{R}$, we can write $a = a_e + a_g + a_h$, $b = b_e + b_g + b_h$, and $c = c_e + c_g + c_h$. Hence, since $\mathfrak{R}_e \subseteq \mathcal{Z}(\mathfrak{R})$ and $\mathfrak{R}_g\mathfrak{R}_h, \mathfrak{R}_h\mathfrak{R}_g \subseteq \mathfrak{R}_e$, we have that

$$\begin{aligned}
[a, b, c] &= [a_e + a_g + a_h, b_e + b_g + b_h, c] = [a_g + a_h, b_g + b_h, c] \\
&= [a_g, b_g, c] + [a_h, b_h, c] + [a_h, b_g, c] + [a_g, b_h, c] \\
&= [a_g, b_g, c] + [a_h, b_h, c] = [a_g, b_g, c_e + c_g + c_h] + [a_h, b_h, c_e + c_g + c_h] \\
&= [a_g, b_g, c_g + c_h] + [a_h, b_h, c_g + c_h] \\
&= [a_g, b_g, c_g] + [a_g, b_g, c_h] + [a_h, b_h, c_g] + [a_h, b_h, c_h] \\
&= ([a_g, b_g, c_g] + [a_h, b_h, c_h]) + ([a_g, b_g, c_h] + [a_h, b_h, c_g]).
\end{aligned}$$

Notice that

$$
\begin{aligned}
&[a_g, b_g, c_h] + [a_h, b_h, c_g] = [a_g b_g - b_g a_g, c_h] + [a_h b_h - a_h b_h, c_g] \\
&= a_g(b_g c_h) - (c_h a_g) b_g - b_g(a_g c_h) + (c_h b_g) a_g + a_h(b_h c_g) - (c_g a_h) b_h - a_h(b_h c_g) + (c_g a_h) b_h \\
&= (b_g c_h) a_g - (b_g c_h) a_g - (a_g c_h) b_g + a_g(c_h b_g) + (b_h c_g) a_h - b_h(c_g a_h) - (b_h c_g) a_h + b_h(c_g a_h) \\
&= 0\ .
\end{aligned}
$$

Hence, $[a, b, c] = [a_g, b_g, c_g] + [a_h, b_h, c_h]$. Observe that $g^2 \neq g$ and $h^2 \neq h$, by cancellation law. If $\mathfrak{R}_g\mathfrak{R}_g \subseteq \mathfrak{R}_e$ (resp. $\mathfrak{R}_h\mathfrak{R}_h \subseteq \mathfrak{R}_e$), i.e. $g^2 = e$ or $g^2 \notin \mathsf{Supp}(\Gamma)$ (resp. $h^2 = e$ or $h^2 \notin \mathsf{Supp}(\Gamma)$), then $[a, b, c] = [a_h, b_h, c_h]$ (resp. $[a, b, c] = [a_g, b_g, c_g]$) for any $a, b, c \in \mathfrak{R}$. Now, if $\mathfrak{R}_g\mathfrak{R}_g \subseteq \mathfrak{R}_h$ (resp. $\mathfrak{R}_h\mathfrak{R}_h \subseteq \mathfrak{R}_g$), i.e. $g^2 = h$ or $g^2 \notin \mathsf{Supp}(\Gamma)$ (resp. $h^2 = g$ or $h^2 \notin \mathsf{Supp}(\Gamma)$), then $g^3 = hg$ (resp. $h^3 = gh$) which is equal to e or does not belong to $\mathsf{Supp}(\Gamma)$, since $gh, hg \notin \{h, g\}$. Consequently, we deduce that either $(\mathfrak{R}_g)^2 \subseteq \mathfrak{R}_e$ or $(\mathfrak{R}_g)^3 \subseteq \mathfrak{R}_e$, and either $(\mathfrak{R}_h)^2 \subseteq \mathfrak{R}_e$ or $(\mathfrak{R}_h)^3 \subseteq \mathfrak{R}_e$, and thus, $[a_g, b_g, c_g], [a_h, b_h, c_h] \in \mathfrak{R}_e$ in any case. Therefore, $[a, b, c] \in \mathfrak{R}_e$ for $a, b, c \in \mathfrak{R}$, and thus, $[a, b, c, d] = 0$ for any $a, b, c, d \in \mathfrak{R}$. The result follows. ■

By Theorem 4.1.1, if $\mathsf{S} = \mathbb{Z}_2$ (resp. $\mathsf{S} = \mathbb{Z}_3$), then any S-graded ring $\mathfrak{R}$ with the central neutral component satisfies the polynomial identity $[x_1, x_2, x_3] = 0$ (resp. $[x_1, x_2, x_3, x_4] = 0$).

We exhibit below two counterexamples to the previous theorem for the case of the support of $\mathfrak{R}$ with more than 3 elements. Anyway, we show that Theorem 4.1.1 does not work when $d \geqslant 4$.

Example 4.1.2 *Let $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{F}$ be an arbitrary field, and $M_2(\mathbb{F})$ the algebra of matrices of order 2 over $\mathbb{F}$. Consider the G-grading on $M_2(\mathbb{F})$ given by $M_2(\mathbb{F}) = M_{(0,0)} \oplus M_{(0,1)} \oplus M_{(1,0)} \oplus M_{(1,1)}$, where*

$$
\begin{aligned}
M_{(0,0)} &= \mathsf{span}_{\mathbb{F}}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad M_{(1,1)} = \mathsf{span}_{\mathbb{F}}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}, \\
M_{(0,1)} &= \mathsf{span}_{\mathbb{F}}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, \quad M_{(1,0)} = \mathsf{span}_{\mathbb{F}}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}.
\end{aligned}
$$

Notice that $M_2(\mathbb{F})$ satisfies the G-graded polynomial identities $[x^{(e)}, y^{(g)}]$ for any $g \in \mathsf{G}$, *where $e = (0,0)$ is the neutral element of G, but $[x_1, x_2, \ldots, x_n]$ is not a (ordinary) polynomial identity of $M_2(\mathbb{F})$, for all $n \in \mathbb{N}$, since*

$$[E_{12}, \underbrace{E_{22}, \ldots, E_{22}}_{(n-1)-times}] = E_{12} \neq 0$$

for all $n \in \mathbb{N}$, where E_{ij} is an elementary matrix.

Example 4.1.3 *Consider $\mathfrak{A}$ a* **Quaternion algebra** *over a field $\mathbb{F}$, $\mathsf{char} \neq 2$, i.e. $\mathfrak{A} = \{a1 + b\mathsf{i} + c\mathsf{j} + d\mathsf{k} : a, b, c, d \in \mathbb{F}\} = \mathbb{F}(\mathsf{i}, \mathsf{j}, \mathsf{k})$, where $\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1$, and $\mathsf{ij} = -\mathsf{ji} = \mathsf{k}$, and 1 is the unity. We have that $\mathfrak{A}$ has a natural $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. In fact, considering $\mathfrak{A}_{(0,0)} = \mathsf{span}_{\mathbb{F}}\{1\}$, $\mathfrak{A}_{(0,1)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{i}\}$, $\mathfrak{A}_{(1,0)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{j}\}$ and $\mathfrak{A}_{(1,1)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{k}\}$, we have*

$$\mathfrak{A} = \mathfrak{A}_{(0,0)} \oplus \mathfrak{A}_{(0,1)} \oplus \mathfrak{A}_{(1,0)} \oplus \mathfrak{A}_{(1,1)} \ .$$

Notice that $\mathfrak{A}_{(0,0)}$ is central in $\mathfrak{A}$, but $\mathfrak{A}$ is not a nilpotent algebra, since $\mathfrak{A}$ is a division algebra, and it is not a Lie nilpotent algebra, i.e. $\mathfrak{A}$ does not satisfy the identity $[x_1, x_2, \ldots, x_n]$ for all $n \in \mathbb{N}$, since $[\mathsf{i}, \underbrace{\mathsf{j}, \mathsf{j}, \ldots, \mathsf{j}}_{(n-1)-times}] \in \{\lambda\mathsf{i}, \lambda\mathsf{k} : \lambda \in \{-2^{n-1}, 2^{n-1}\}\}$.

From the example above, we can build, for all $d \geqslant 4$, an S-graded ring with the central neutral component and the support of grading of order d such that the polynomial $[x_1, \ldots, x_{d+1}] \not\equiv 0$ in $\mathfrak{A}$. In fact, consider the a Quaternion algebra $\mathfrak{A}$ as in the previous example. Now, suppose $\mathfrak{B} = \mathbb{F}x$ is a nilpotent algebra, such that $x \neq 0$, and $x^2 = 0$. Let $\mathsf{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider now the algebra $\mathfrak{A}_1 = \mathfrak{A} \times \mathfrak{B}$ (the direct product of the algebras $\mathfrak{A}$ and $\mathfrak{B}$), and the G-grading Γ_1 on $\mathfrak{A}_1$ given by $(\mathfrak{A}_1)_{(0,0,0)} = \mathsf{span}_{\mathbb{F}}\{1\}$, $(\mathfrak{A}_1)_{(0,1,0)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{i}\}$, $(\mathfrak{A}_1)_{(1,0,0)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{j}\}$, $(\mathfrak{A}_1)_{(1,1,0)} = \mathsf{span}_{\mathbb{F}}\{\mathsf{k}\}$, and $(\mathfrak{A}_1)_{(0,0,1)} = \mathfrak{B}$. We have that Γ has the support of order 5, and $(\mathfrak{A}_1)_e$ is central in $\mathfrak{A}_1$. Since $\mathfrak{A} \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{A}_1$, it follows that $[x_1, \ldots, x_n] \not\equiv 0$ in $\mathfrak{A}_1$ for all $n \in \mathbb{N}$. In particular, $[x_1, \ldots, x_6] \not\equiv 0$ in $\mathfrak{A}_1$. By this process, we can build a ring $\mathfrak{A}_n$ which is not Lie nilpotent, such that $\mathfrak{A}_n$ is $(\mathbb{Z}_2)^{n+2}$-graded with support of order $n+4$, and $(\mathfrak{A}_n)_e$ is central in $\mathfrak{A}_n$. It is sufficient to consider, for any $n \in \mathbb{N}$, the algebra $\mathfrak{A}_n = \mathfrak{A} \times \mathfrak{B}^n = \mathfrak{A} \times \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n-times}$ with $(\mathbb{Z}_2)^{n+2}$-grading Γ_n induced by gradings of $\mathfrak{A}$ and $\mathfrak{B}$. Since $\mathfrak{A} \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{A}_n$, it follows that $\mathfrak{A}_n$ is not Lie nilpotent. Observe that $(\mathfrak{A}_n)_e$ is central in $\mathfrak{A}_n$, and $|\mathsf{Supp}(\Gamma_n)| = 4 + n$. Furthermore, our affirmation follows.

Corollary 4.1.4 *Let* S *be a group and* $\mathfrak{R}$ *a ring with an* S*-grading* Γ*. Let* P *be a normal subgroup of* S*, and* $\overline{\Gamma} : \mathfrak{R} = \bigoplus_{\bar{g}\in \mathsf{S}/P} \mathfrak{R}_{\bar{g}}$ *the* S/P*-grading induced by* Γ*. Suppose* $\overline{\Gamma}$ *has a finite support of order* d*. If* $\mathfrak{R}_{\bar{e}} = \bigoplus_{p\in P} \mathfrak{R}_p \subseteq \mathcal{Z}(\mathfrak{R})$ *and* $d \in \{1,2,3\}$*, then* $[x_1,\ldots,x_{d+1}] \equiv 0$ *in* $\mathfrak{R}$*.*

Proof: Considering $\mathfrak{R}$ with its induced S/P-grading $\overline{\Gamma}$, by Theorem 4.1.1, it follows that $[x_1,\ldots,x_{d+1}] \equiv 0$ in $\mathfrak{R}$. ■

It is important to note that Γ in the previous corollary is not necessarily a finite S-grading.

4.2 The Variety $\mathfrak{V}^{\mathsf{G}}$

Let G be a finite group, $\mathbb{F}$ a field, and $\mathfrak{V}^{\mathsf{G}}$ the variety of all G-graded associative $\mathbb{F}$-algebras with the central neutral component. Let $\mathfrak{A}$ be a G-graded algebra which belongs to $\mathfrak{V}^{\mathsf{G}}$. Hence, $\mathfrak{A}_e$ is central in $\mathfrak{A}$, where e is the neutral element of G. In particular, $\mathfrak{A}_e$ is commutative, and so a PI-algebra. By Theorem 1.4.11 (or Theorem 1.4.12), we conclude that $\mathfrak{A}$ is a PI-algebra. From this, $\mathfrak{V}^{\mathsf{G}}$ is a G-graded variety of PI-algebras, and if G is a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, then we can apply Theorems 1.2.20 and 1.4.13. We have that there exists a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded algebra $\mathfrak{A}$ such that

$$\mathfrak{V}^{\mathsf{G}} = \mathsf{var}_{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) := \mathsf{var}_{\mathsf{G}}(\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))) \ ,$$

where $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ with

$$\mathfrak{B} = M_{n_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{n_k}(\mathbb{F}^{\sigma_k}[H_k]), \quad \text{and} \quad \mathsf{J} = \mathsf{J}(\mathfrak{A}) \ ,$$

where $H_i \trianglelefteq \mathsf{G}\times\mathbb{Z}_2$, $\sigma_i \in \mathsf{Z}^2(H_i,\mathbb{F}^*)$, $M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ is a subalgebra with a canonical elementary $G\times\mathbb{Z}_2$-grading defined by some n_i-tuple $(g_1,\ldots,g_{n_i}) \in (G\times\mathbb{Z}_2)^{n_i}$. Here, $\mathfrak{B}$ is a maximal semisimple $\mathsf{G}\times\mathbb{Z}_2$-graded subalgebra of $\mathfrak{A}$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, which is a finite dimensional graded ideal.

Notice that $\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subsetneq \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{J}))$, since $\mathsf{J} \subseteq \mathfrak{A}$ is $\mathsf{G}\times\mathbb{Z}_2$-graded, and in Chapter 3, we exhibit some results which ensure that $\mathfrak{A}_e \neq \mathsf{J}_e$. Particularly, by Theorem 3.2.15, $\mathfrak{A}$

is nilpotent when $\mathfrak{A}_e = \mathsf{J}_e$, and consequently, $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ is nilpotent, but $\mathfrak{V}^{\mathsf{G}}$ is not a nilpotent variety, because $\mathbb{F}\mathsf{G}$ belongs to $\mathfrak{V}^{\mathsf{G}}$. Therefore, we conclude that $\mathfrak{B}_e \neq \{0\}$, namely $\mathfrak{i} \in \mathfrak{B}_e$, where $\mathfrak{i}$ is the unity of $\mathfrak{B}$ (see Sections 1.1 and 1.2 in Chapter 1).

In the next sections, we will study the variety $\mathfrak{V}^{\mathsf{G}}$, as well as some subvarieties of $\mathfrak{V}^{\mathsf{G}}$.

4.3 On the $\mathfrak{A}$ when $\mathfrak{A}_e$ is central

In this section, let us consider a finite dimensional G-graded $\mathbb{F}$-algebra $\mathfrak{A}$, whose neutral component is central, for a finite abelian group G. This means that $\mathfrak{A}$ is a G-graded algebra which belongs to the variety $\mathfrak{V}^{\mathsf{G}}$. We assume also that the base field $\mathbb{F}$ is algebraically closed of characteristic zero. In the results below, we use the following reason: by Theorem 1.2.20, there exist $k_1, \ldots, k_p \in \mathbb{N}$, $H_1, \ldots, H_p \trianglelefteq \mathsf{G}$, $\sigma_1 \in \mathsf{Z}^2(H_1, \mathbb{F}^*), \ldots, \sigma_p \in \mathsf{Z}^2(H_p, \mathbb{F}^*)$ such that

$$\mathfrak{A} \cong_{\mathsf{G}} (M_{k_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{k_p}(\mathbb{F}^{\sigma_p}[H_p])) \oplus \mathsf{J} \ ,$$

where $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$. By Lemma 1.5.6 and Theorem 1.5.7, we can write

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_p \times \mathsf{J}_{00} \ , \tag{4.1}$$

where $\mathfrak{A}_r = \mathfrak{B}_r \oplus \mathsf{J}_r$, $\mathsf{J}_r = \mathfrak{i}_r \mathsf{J} \mathfrak{i}_r$ and $\mathfrak{i}_r$ is the unity of $\mathfrak{B}_r$, where $\mathfrak{B}_r = M_{k_r}(\mathbb{F}^{\sigma_r}[H_r])$. By Theorem 1.5.12, we can assume that $\mathfrak{B}_{s_i} \not\cong_{\mathsf{G}} \mathfrak{B}_{s_j}$ for all $i \neq j$.

Fix $r \in \{1, \ldots, p\}$. Since $(\mathfrak{B}_r)_e \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{A}_e$ and $\mathfrak{A}_e \subseteq \mathcal{Z}(\mathfrak{A})$, it follows that $k_r = 1$ for all $r = 1, \ldots, p$. In fact, since $E_{ii}\eta_e \in (\mathfrak{B}_r)_e$, we have that $0 = [E_{11}\eta_e, E_{1i}\eta_e] = \sigma(e,e)(E_{1i} - E_{1i}E_{11})\eta_e$, which is only possible when $i = 1$. Hence, $\mathfrak{A}_r = \mathbb{F}^{\sigma_r}[H_r] \oplus \mathsf{J}_r$ for all $r = 1, \ldots, p$. Therefore, let us study the unitary algebras $\mathfrak{A} = \mathbb{F}^{\sigma}[H] \oplus \mathsf{J}$.

Observe that by Theorem 1.2.13 and Example1.3.2, the immersion $\mathbb{F}^{\sigma_i}[H_i] \overset{\mathsf{G}\times\mathbb{Z}_2}{\hookrightarrow} \mathbb{F}^{\sigma_j}[H_j]$ can be expressed in terms of the partial order "$\preceq$", i.e. $(H_i, [\sigma_i]) \preceq (H_j, [\sigma_j])$. More precisely, we have the next lemma.

Lemma 4.3.1 (Immersion Lemma) *Let $\mathbb{F}$ be an algebraically closed field, G a group, and H_1, H_2 two finite abelian subgroups of G. Consider two 2-cocycles $\sigma_1 \in \mathsf{Z}^2(H_1, \mathbb{F}^*)$ and $\sigma_2 \in \mathsf{Z}^2(H_2, \mathbb{F}^*)$, and two twisted group algebras $\mathfrak{B}_1 = \mathbb{F}^{\sigma_1}[H_1]$ and $\mathfrak{B}_2 = \mathbb{F}^{\sigma_2}[H_2]$.*

Then $\mathfrak{B}_1 \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{B}_2$ *iff* $(H_1, [\sigma_1]) \preceq (H_2, [\sigma_2])$.

Proof: Firstly, suppose that $H_1 \leqslant H_2$ and $[\sigma_1] = [\sigma_2]_{H_1}$. Hence, there exists some $\theta \in \mathsf{B}^2(H_1, \mathbb{F}^*)$ such that $(\sigma_2)_{H_1} = \theta\sigma_1$. By Theorem 1.2.13, it is immediate that $\mathfrak{B}_1 \cong \mathbb{F}^{\tilde{\sigma}}[H_1]$, where $\tilde{\sigma}$ is the 2-cocycle of $\mathsf{Z}^2(H_1, \mathbb{F}^*)$ defined by $\tilde{\sigma}(g,h) = \theta(g,h)\sigma_1(g,h) = \sigma_2(g,h)$ for any $g, h \in H_1$.

Observe that $\mathbb{F}^{\tilde{\sigma}}[H_1]$ is a graded subspace of $\mathbb{F}^{\sigma_2}[H_2]$, since $H_1 \leqslant H_2$, and by definition of twisted group algebra. Let us now show that $\mathbb{F}^{\tilde{\sigma}}[H_1]$ is an H_2-graded subalgebra of $\mathbb{F}^{\sigma_2}[H_2]$. In fact, let us denote by "$*$" the multiplication of $\mathbb{F}^{\sigma_2}[H_2]$ and by "$\star$" the multiplication of $\mathbb{F}^{\tilde{\sigma}}[H_1]$. Given $\eta_h, \eta_g \in \mathbb{F}^{\tilde{\sigma}}[H_1]$, it follows that

$$\eta_h \star \eta_g = \tilde{\sigma}(h,g)\eta_{hg} = (\theta\sigma_1)(h,g)\eta_{hg} = \theta(h,g)\sigma_1(h,g)\eta_{hg} = \sigma_2(h,g)\eta_{hg} = \eta_h * \eta_g \ .$$

This shows that "$*$" and "$\star$" are equal in $\mathbb{F}^{\sigma_2}[H_2]$, and so we conclude that $\mathbb{F}^{\tilde{\sigma}}[H_1]$ is an H_2-graded subalgebra of $\mathbb{F}^{\sigma_2}[H_2]$. Hence, we have that $\mathfrak{B}_1 \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{B}_2$, because $\mathfrak{B}_1 \cong_{\mathsf{G}} \mathbb{F}^{\tilde{\sigma}}[H_1]$.

On other hand, suppose that $\mathfrak{B}_1 \overset{\mathsf{G}}{\hookrightarrow} \mathfrak{B}_2$. Hence, by definition of "$\overset{\mathsf{G}}{\hookrightarrow}$" ($\mathsf{G}$-immersion), there exists a graded homomorphism ψ of $\mathfrak{B}_1$ to $\mathfrak{B}_2$ which is injective. Notice that $\psi(\eta_g) \in (\mathfrak{B}_2)_g$ is different to zero for any $g \in H_1$, because ψ is injective. Hence, $H_1 = \mathsf{Supp}(\Gamma_{\mathfrak{B}_1}) \subseteq \mathsf{Supp}(\Gamma_{\mathfrak{B}_2}) = H_2$. In particular, $H_1 \leqslant H_2$. Since $\mathfrak{B}_1 \cong_{\mathsf{G}} \mathsf{im}(\psi)$, by Theorem 1.2.13, there exits $\hat{\sigma} \in \mathsf{Z}^2(H_1, \mathbb{F}^*)$ such that $\mathsf{im}(\psi) = \mathbb{F}^{\hat{\sigma}}[H_1]$, and $[\sigma_1] = [\hat{\sigma}]$. Now, observe that $(\sigma_2)_{H_1} \in \mathsf{Z}^2(H_1, \mathbb{F}^*)$, and $\mathbb{F}^{\sigma_2}[H_1]$ is a graded subalgebra of $\mathfrak{B}_2$. Hence, $\mathbb{F}^{\hat{\sigma}}[H_1]$ and $\mathbb{F}^{\sigma_2}[H_1]$ are graded subalgebras of $\mathfrak{B}_2$. For any $g, h \in H_1$, we have

$$\hat{\sigma}(g,h)\eta_{gh} = \eta_g\eta_h = \sigma_2(g,h)\eta_{gh} \text{ (in } \mathfrak{B}_2).$$

Thus, $\hat{\sigma} = (\sigma_2)_{H_1}$, that is, $[\hat{\sigma}] = [\sigma_2]_{H_1}$. From this, we conclude that $[\sigma_1] = [\sigma_2]_{H_1}$. Therefore, we conclude that $(H_1, [\sigma_1]) \preceq (H_2, [\sigma_2])$. The result follows. ■

In the proof of Lemma 4.3.1, we ensures that $\mathbb{F}^{\hat{\sigma}}[H_1]$ and $\mathbb{F}^{\sigma_2}[H_1]$ are graded subalgebras of $\mathbb{F}^{\sigma_2}[H_2]$. Observe that Corollary 2.2.7 ensures that the restriction $[\sigma_2]_{H_1}$ is unique, and hence, we must have $[\hat{\sigma}] = [\sigma_2]_{H_1}$ in $\mathsf{H}^2(H_1, \mathbb{F}^*)$. This is another proof for $[\sigma_{H_1}] = [\sigma_2]_{H_1}$ in Lemma 4.3.1.

Observe that we can rewrite Theorem 1.5.12, for the case $k_1 = \cdots = k_p = 1$.

Recall that if $H, \tilde{H}$ are subgroups of a group G, and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$ and $\tilde{\sigma} \in$

$\mathsf{Z}^2(\tilde{H},\mathbb{F}^*)$, then $(H,[\sigma]) = (\tilde{H},[\tilde{\sigma}])$ iff $H = \tilde{H}$ and $[\sigma] = [\tilde{\sigma}]$ (see Example 1.3.2).

Lemma 4.3.2 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field with $\mathsf{char}(\mathbb{F}) = 0$, and $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_p \times \mathsf{J}_{00}$ a finite dimensional G-graded algebra, where $\mathfrak{A}_i = \mathfrak{B}_i \oplus \mathsf{J}_i$ are finite dimensional G-graded unitary algebras, where $\mathfrak{B}_i = \mathbb{F}^{\sigma_i}[H_i]$ with $H_i \trianglelefteq \mathsf{G}$, $\sigma_i \in \mathsf{Z}^2(H_i,\mathbb{F}^*)$, and $\mathsf{J}_i = \mathsf{J}(\mathfrak{A}_i)$ is the Jacobson radical of $\mathfrak{A}_i$. If $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$, then*

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \tilde{\mathfrak{A}}_1 \times \cdots \times \tilde{\mathfrak{A}}_q \times \tilde{\mathsf{J}}_{00} \ , \tag{4.2}$$

where $\tilde{\mathfrak{A}}_j = \tilde{\mathfrak{B}}_j \oplus \tilde{\mathsf{J}}_j$ are finite dimensional G-graded unitary algebras, with $\tilde{\mathfrak{B}}_j \cong_{\mathsf{G}} \mathfrak{B}_l$, for some $l \in \{1,\ldots,p\}$, and $(H_i,[\sigma_i]) \neq (H_j,[\sigma_j])$ for all $i \neq j$. Moreover, $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}_1 \times \cdots \times \tilde{\mathfrak{A}}_q \times \tilde{\mathsf{J}}_{00}$ belongs to $\mathfrak{V}^{\mathsf{G}}$, and $\mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}}_e)) \leqslant \mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}})) \leqslant |\mathsf{G}|\mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}}_e))$, where $\mathsf{J}(\tilde{\mathfrak{A}}) = \tilde{\mathsf{J}}_1 \times \cdots \times \tilde{\mathsf{J}}_q \times \tilde{\mathsf{J}}_{00}$.

Proof: The first part is immediate from Corollary 1.5.9 and Lemma 4.3.1, similarly to the proof of Theorem 1.5.12. Since $\tilde{\mathfrak{A}} \equiv_{\mathsf{G}PI} \mathfrak{A}$, we have that $\tilde{\mathfrak{A}} \in \mathfrak{V}^{\mathsf{G}}$. It is clear that $\tilde{\mathsf{J}}_1 \times \cdots \times \tilde{\mathsf{J}}_q \times \tilde{\mathsf{J}}_{00}$ is the major nilpotent ideal of $\tilde{\mathfrak{A}}$, since $\tilde{\mathsf{J}}_{00}$ is a finite dimensional \$$G$-graded nilpotent algebra (as in Theorem 1.5.12). Hence $\tilde{\mathfrak{A}}$ is also finite dimensional. The inequality $\mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}}_e)) \leqslant \mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}})) \leqslant |\mathsf{G}|\mathsf{nd}(\mathsf{J}(\tilde{\mathfrak{A}}_e))$ follows of Theorem 3.2.15. ■

Let $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ be an algebra of the list $\{\tilde{\mathfrak{A}}_1,\ldots,\tilde{\mathfrak{A}}_q\}$ in (4.2). We have that $\mathfrak{A}$ satisfies all the claims of Lemma 4.3.2, i.e. $\mathfrak{B} = \mathbb{F}^{\sigma}[H]$, $H \leqslant \mathsf{G}$, $\sigma \in \mathsf{Z}^2(H,\mathbb{F}^*)$, J is a finite dimensional graded nilpotent ideal of $\mathfrak{A}$. Now, by Lemma 1.5.2, we have that $\mathsf{J} = \mathfrak{B}\mathsf{N}$ for some G-graded vector space $\mathsf{N} \subseteq \mathsf{J}$, where $\mathsf{N} = \mathsf{span}_{\mathbb{F}}\{d_1,\ldots,d_n\}$ with $d_i\mathfrak{B} = \mathfrak{B}d_i$, where d_i's are homogeneous elements. Moreover, if $\beta = \{eta_h : h \in H\}$ is the canonical homogeneous basis of $\mathfrak{B}$, then $d_ib = \gamma_i(b)bd_i$for any $b \in \beta$, for some $\gamma_i(b) \in \mathbb{F}$ (see Remark 1.3.21).

Now, given a subgroup H of the a group $\mathsf{G} \times \mathbb{Z}_2$, by Theorem 1.2.13, we have that $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]) \cong_{\mathsf{G}} \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\gamma}[H])$ for any $[\sigma] = [\gamma]$ in $\mathsf{H}^2(H,\mathbb{F}^*)$. Hence, we deduce that the $\mathsf{G}T$-ideal of graded identities of $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H])$ does not depend on a representative element of $[\sigma] \in \mathsf{H}^2(H,\mathbb{F}^*)$. If $\mathbb{F}$ is an algebraically closed field, and G is a finite abelian group, then the values of σ can be chosen in $\sqrt[|\mathsf{G}|]{1} := \{\lambda \in \mathbb{F}^* : \lambda^{|\mathsf{G}|} = 1\}$. In this sense, we have the following result.

Lemma 4.3.3 *Let $\mathbb{F}$ be an algebraically closed field, $\tilde{\mathsf{G}}$ a finite abelian group, and $\mathfrak{B} = \mathbb{F}^\sigma[H]$ a $\tilde{\mathsf{G}}$-graded simple finite dimensional algebra. If $\hat{H}$ is a subgroup of $\tilde{\mathsf{G}}$ such that $H \leqslant \hat{H}$, then there exists a $\tilde{\mathsf{G}}$-graded simple finite dimensional algebra $\hat{\mathfrak{B}}$ such $\mathfrak{B} \overset{\tilde{\mathsf{G}}}{\hookrightarrow} \hat{\mathfrak{B}}$, and $\mathsf{T}^{\tilde{\mathsf{G}}}(\hat{\mathfrak{B}}) \subseteq \mathsf{T}^{\tilde{\mathsf{G}}}(\mathfrak{B})$, $\hat{\mathfrak{B}} = \mathbb{F}^{\hat{\sigma}}[\hat{H}]$ for some $\hat{\sigma} \in \mathsf{Z}^2(\hat{H}, \mathbb{F}^*)$ which extends σ. Moreover, if $\mathsf{char}(\mathbb{F}) = 0$ and $\tilde{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$, then $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{B})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{B}}))$.*

Proof: By Corollary 2.2.7, given $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, there is $\hat{\sigma} \in \mathsf{Z}^2(\hat{H}, \mathbb{F}^*)$ such that $\hat{\sigma}(g,h) = \hat{\sigma}_H(g,h) = \sigma(g,h)$ for any $g, h \in H$. Consider $\hat{\mathfrak{B}} = \mathbb{F}^{\hat{\sigma}}[\hat{H}]$. By Lemma 4.3.1, it follows that $\mathfrak{B} \overset{\tilde{\mathsf{G}}}{\hookrightarrow} \hat{\mathfrak{B}}$. Consequently, $\mathsf{T}^{\tilde{\mathsf{G}}}(\hat{\mathfrak{B}}) \subseteq \mathsf{T}^{\tilde{\mathsf{G}}}(\mathfrak{B})$.

Therefore, assuming that $\tilde{\mathsf{G}} = \mathsf{G} \times \mathbb{Z}_2$, by Lemma 1.5.11, we can conclude that $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}))$. ∎

Remark 4.3.4 *A good application of the previous lemma is given when $H \leqslant \mathsf{G} \times \{0\}$, i.e. $H \cong \pi(H) \times \{0\}$, where "$\pi$" is the* **projection map** *$\pi : \mathsf{G} \times \mathbb{Z}_2 \longrightarrow \mathsf{G}$ defined by $\pi(g, \lambda) = g$ for any $g \in \mathsf{G}$ and $\lambda \in \mathbb{Z}_2$. Assume $\mathfrak{B} = \mathbb{F}^\sigma[H]$. Naturally, we have that H is a subgroup of $\mathsf{G} \times \{0\}$ and, by Lemma 4.3.3, there is $\hat{\sigma} \in \mathsf{Z}^2(\mathsf{G} \times \{0\}, \mathbb{F}^*)$ such that*

$$\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{B})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\hat{\sigma}}[\mathsf{G} \times \{0\}])) \ ,$$

where $\hat{\sigma}_H = \sigma$. Notice that $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\hat{\sigma}}[\mathsf{G} \times \{0\}]) \equiv_{\mathsf{G}PI} \mathbb{F}^{\check{\sigma}}[\mathsf{G}]$, where $\check{\sigma} \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ defined by $\check{\sigma}(h_1, h_2) = \hat{\sigma}((h_1, 0), (h_2, 0))$ for any $h_1, h_2 \in \mathsf{G}$. Therefore, we deduce that $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{B})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathbb{F}^\gamma[\mathsf{G}])$ for some $\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$.

Remark 4.3.5 *In the whole work, let us assume that "π" is always the projection map of $\mathsf{G} \times \mathbb{Z}_2$ on G.*

Lemma 4.3.6 *Let $\hat{\mathsf{G}}$ be a group, $\mathbb{F}$ a algebraically closed field of characteristic zero,$\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a $\hat{\mathsf{G}}$-graded finite dimensional unitary algebra, where $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{B} = \mathbb{F}^\sigma[H]$ with $H \leqslant \hat{\mathsf{G}}$ and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. If H is finite abelian, then J_H can be generated as a graded $\mathfrak{B}$-bimodule by a graded nilpotent subalgebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$ such that $\mathfrak{B}d = d\mathfrak{B}$ for any $d \in \hat{\mathsf{N}}$, where $\mathsf{J}_H = \bigoplus_{h \in H} \mathsf{J}_h$. In addition, if $\mathfrak{A}_e$ is central in $\mathfrak{A}$, then $\mathsf{J}_H = \mathfrak{B}\hat{\mathsf{N}} \cong_{\mathsf{G}} \mathfrak{B} \otimes_{\mathbb{F}} \hat{\mathsf{N}}$ as graded $\mathfrak{B}$-bimodules, and as graded algebras.*

Proof: Firstly, since $\mathfrak{A}$ is unitary, as observed in (1.13), in Section 1.5, Chapter 1, we have that $\mathsf{J} = \mathsf{iJi}$, where i is the unity of $\mathfrak{A}$ (which is also the unity of $\mathfrak{B}$).

Consider $\mathfrak{A}_H = \mathfrak{B} \oplus \mathsf{J}_H$, where $\mathsf{J}_H = \bigoplus_{g\in H} \mathsf{J}_h$. Since H is a subgroup of $\hat{\mathsf{G}}$, it is easy to see that $\mathfrak{A}_H$ and J_H are graded subalgebras of $\mathfrak{A}$. Notice that J_H is the greatest nilpotent ideal of $\mathfrak{A}_H$, and so $\mathsf{J}_H = \mathsf{J}(\mathfrak{A}_H)$. By Lemma 1.5.2, we can write $\mathsf{J}_H = \mathfrak{B}\mathsf{N}$ for some graded vector space $\mathsf{N} = \mathsf{span}_{\mathbb{F}}\{d_1, \ldots, d_r\} \subset \mathsf{J}$, where $d_1, \ldots, d_r \in \mathsf{J}_H$ are homogeneous elements such that $bd_i = \gamma_i(h)d_ib \neq 0$ for any homogeneous element $b \in \mathfrak{B}_{h,}$, $h \in H$, and $i = 1, \ldots, r$, and some $\gamma_i(b) \in \mathbb{F}$; and $\mathfrak{B}d_i$ is an irreducible G-graded $\mathfrak{B}$-bimodule. We have that the set $\{\eta_e d_1, \ldots, \eta_e d_r\}$ generate J_H as a $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodule (or as H-graded $\mathfrak{B}$-bimodule, more precisely). Put $h_i = \mathsf{deg}(d_i)$ for all $i \in \{1, \ldots, r\}$. Since $\mathsf{Supp}(\Gamma_{\mathsf{J}_H}) \subseteq H$, it follows that $h_1, \ldots, h_r \in H$, and hence,

$$\eta_e d_i = (\sigma(h_i, h_i{}^{-1})^{-1}\eta_{h_i}\eta_{h_i{}^{-1}})d_i = \sigma(h_i, h_i{}^{-1})^{-1}\eta_{h_i}(\eta_{h_i{}^{-1}}d_i)$$

for all $i \in \{1, \ldots, r\}$. For any $i \in \{1, \ldots, r\}$, write $\hat{d}_i := \eta_{h_i{}^{-1}}d_i$, and put $\hat{\mathsf{N}} = \mathsf{span}_{\mathbb{F}}\{\hat{d}_1, \ldots, \hat{d}_r\}$. It is obvious that $b\hat{d}_i \neq 0$ and $\hat{d}_ib \neq 0$ for any nonzero $b \in \mathfrak{B}$, and $i = 1, \ldots, r$. Since $\hat{d}_i \in \mathsf{J}_e$, it follows that $\hat{\mathsf{N}} \subset \mathsf{J}_e$. For any $g \in H$ (because H is abelian), and $i \in \{1, \ldots, r\}$, by Proposition 1.2.6 and Corollary 1.3.23, we have that

$$\begin{aligned}
\eta_g\hat{d}_i &= \eta_g(\eta_{h_i{}^{-1}}d_i) = \sigma(g, h_i^{-1})\eta_{gh_i{}^{-1}}d_i = \sigma(g, h_i^{-1})\eta_{h_i{}^{-1}g}d_i \\
&= \sigma(g, h_i^{-1})\sigma(h_i^{-1}, g)^{-1}(\eta_{h_i{}^{-1}}\eta_g)d_i = \sigma(g, h_i^{-1})\sigma(h_i^{-1}, g)^{-1}\eta_{h_i{}^{-1}}(\eta_g d_i) \\
&= \gamma_i(g)\sigma(g, h_i^{-1})\sigma(h_i^{-1}, g)^{-1}(\eta_{h_i{}^{-1}}d_i)\eta_g = \gamma_i(g)\sigma(g, h_i^{-1})\sigma(h_i^{-1}, g)^{-1}\hat{d}_i\eta_g \ ,
\end{aligned}$$

for some $\gamma_i(g) \in \mathbb{F}$. Hence, we obtain that $\mathfrak{B}d = d\mathfrak{B}$ for any $d \in \hat{\mathsf{N}}$. We have still that $\hat{\mathsf{N}}$ is subalgebra of J_e. In fact, for all $i, j \in \{1, \ldots, r\}$, we have

$$\hat{d}_j\hat{d}_i = (\eta_{h_j{}^{-1}}d_j)(\eta_{h_i{}^{-1}}d_i) = \gamma_j(h_i^{-1})(\eta_{h_j{}^{-1}}\eta_{h_i{}^{-1}})(d_jd_i) = \gamma_j(h_i^{-1})\sigma(h_j^{-1}, h_i^{-1})(\eta_{(h_jh_i)^{-1}})(d_jd_i) \ .$$

Since $\mathsf{deg}(d_jd_i) = \mathsf{deg}(d_j)\mathsf{deg}(d_i) = h_jh_i \in H$, it follows that $d_jd_i \in \mathfrak{B}\mathsf{N} = \bigoplus_{k=1}^{r} \mathfrak{B}d_k$, and hence, there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{F}$, and $g_1, \ldots, g_r \in H$ such that $d_jd_i = \sum_{k=1}^{r} \lambda_k\eta_{g_k}d_k$. Hence, by Corollary 1.3.23

$$\begin{aligned}
\hat{d}_j\hat{d}_i &= \gamma_j(h_i^{-1})\sigma(h_j, h_i)(\eta_{(h_jh_i)^{-1}})(d_jd_i) = \gamma_j(h_i^{-1})\sigma(h_j^{-1}, h_i^{-1})(\eta_{(h_jh_i)^{-1}})\left(\sum_{k=1}^{r}\lambda_k\eta_{g_k}d_k\right) \\
&= \gamma_j(h_i^{-1})\sigma(h_j^{-1}, h_i^{-1})\sum_{k=1}^{r}\lambda_k\sigma((h_jh_i)^{-1}, g_k)\left(\eta_{(h_jh_i)^{-1}g_k}d_k\right) \ .
\end{aligned}$$

Observe that if $\lambda_k \neq 0$ for some $k = 1, \ldots, r$, then

$$e = \mathsf{deg}(\hat{d}_j \hat{d}_i) = \mathsf{deg}(\eta_{(h_j h_i)^{-1} g_k} d_k) = ((h_j h_i)^{-1} g_k)\mathsf{deg}(d_k) \ ,$$

and so $\mathsf{deg}(d_k)^{-1} = ((h_j h_i)^{-1} g_k)$, that is, $\eta_{(h_j h_i)^{-1} g_k} d_k = \hat{d}_k$. We deduce that $\hat{d}_j \hat{d}_i = \sum_{k=1}^{r} \tilde{\lambda}_k \hat{d}_k \in \hat{\mathsf{N}}$, $\tilde{\lambda}_k \in \mathbb{F}$, for all $i, j = 1, \ldots, r$. We conclude that $\hat{\mathsf{N}}$ is a graded subalgebra of J_e such that $\mathfrak{B} d = d \mathfrak{B}$ for any $d \in \hat{\mathsf{N}}$. Therefore, it follows that $\hat{\mathsf{N}}$ is a graded nilpotent algebra which generates J_H as a graded $\mathfrak{B}$-bimodule.

Suppose that $\mathfrak{A}_e$ is central in $\mathfrak{A}$. Hence, we have that $bd = db$ for any $b \in \mathfrak{B}$ and $d \in \hat{\mathsf{N}}$ ($\hat{\mathsf{N}}_e \subset \mathfrak{A}_e$). It is clear that $\mathfrak{B}\hat{d}_k = \mathfrak{B} d_k$, for all $k = 1, \ldots, r$, and it is an irreducible $\hat{\mathsf{G}}$-graded $\mathfrak{B}$-bimodule, and $\mathsf{J}_H = \mathfrak{B}\hat{\mathsf{N}} = \bigoplus_{k=1}^{r} \mathfrak{B}\hat{d}_k$. From this, it is not difficult to see that the linear transformation ψ of J_H to $\mathfrak{B} \otimes_{\mathbb{F}} \hat{\mathsf{N}}$ that extends the map $\eta_g \hat{d}_k \mapsto \eta_g \otimes \hat{d}_k$, for any $g \in H$ and $i = 1, \ldots, r$, is a homogeneous isomorphism of graded $\mathfrak{B}$-bimodules, and of graded algebras. ■

By conditions of the previous lemma, we have that H is a finite abelian subgroup of $\hat{\mathsf{G}}$. Observe again that J_H is a nilpotent graded subalgebra of J and $\mathfrak{A}$.

Another consequence of the above lemma is that if $\mathfrak{A}_e$ is central in $\mathfrak{A}$, then J_H can be generated as a graded $\mathfrak{B}$-bimodule by a graded nilpotent algebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$ which is central in $\mathfrak{A}$, where $\mathsf{J}_H = \bigoplus_{h \in H} \mathsf{J}_h$. Consequently, J can be generated as a G-graded $\mathfrak{B}$-bimodule by a graded vector space $\tilde{\mathsf{N}}$ such that $\tilde{\mathsf{N}}_H = \hat{\mathsf{N}} = \hat{\mathsf{N}}_e$, and hence, $\tilde{\mathsf{N}}_H$ is a graded nilpotent subalgebra of $\mathfrak{A}$ which is central in $\mathfrak{A}$. To prove these facts, it is enough to apply Lemma 1.5.2 for J, and to proceed as in the proof of Lemma 4.3.6.

Remark 4.3.7 *Let $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ be a finite dimensional G-graded unitary algebra, with $\mathfrak{B} = \mathbb{F}^{\sigma}[H]$ and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$. Suppose G is a finite abelian group, and $\mathbb{F}$ is an algebraically closed field with $\mathsf{char}(\mathbb{F}) = 0$. Then $\mathfrak{A}$ is $\mathsf{G}PI$-equivalent to $\mathfrak{B}\hat{\mathsf{N}}^{\#} \oplus \mathfrak{B}d_1 \oplus \cdots \oplus \mathfrak{B}d_s$, for some nilpotent graded algebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$, and homogeneous elements $d_1, \ldots, d_s \in \mathsf{J}$ such that $d_i \mathfrak{B} = \mathfrak{B} d_i \neq 0$ for any $\mathsf{deg}(d_i) \notin H$. Really, we have that $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J} = \mathfrak{B} \oplus \mathsf{J}_H \oplus (\mathfrak{B}d_1 \oplus \cdots \oplus \mathfrak{B}d_s)$, where $d_i \in \mathsf{J}$, $\mathfrak{B}d_i = d_i\mathfrak{B}$, and $\mathsf{deg}(d_i) \notin H$. By Lemma 4.3.6, we have that $\mathfrak{B} \oplus \mathsf{J}_H = \mathfrak{B} \oplus \mathfrak{B}\hat{\mathsf{N}} = \mathfrak{B}\hat{\mathsf{H}}^{\#}$, where $\hat{\mathsf{H}}^{\#} = \mathbb{F} \oplus \hat{\mathsf{N}}$ is the nilpotent subalgebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$ with the adjoint unity.*

Theorem 4.3.8 *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, G a finite* *abelian group, H a subgroup of G, $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and $\mathfrak{A} = \mathbb{F}^\sigma[H] \oplus \mathsf{J}$ a finite dimensional G-graded unitary $\mathbb{F}$-algebra. Suppose that $\mathsf{Supp}(\Gamma_{\mathfrak{A}}) = H$. If $\mathfrak{A}_e$ is central in $\mathfrak{A}$, then*

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \mathbb{F}^\sigma[H]\ .$$

Moreover, $\mathfrak{A}$ belongs to $\mathsf{var}^\mathsf{G}(\mathbb{F}^\gamma[\mathsf{G}])$ for some $\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ which extends σ.

Proof: Firstly, note that $\mathsf{J}_H = \mathsf{J}$. Put $\mathfrak{B} = \mathbb{F}^\sigma[H]$. It follows from Lemma 4.3.6 that $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ with $\mathsf{J} = \mathfrak{B}\hat{\mathsf{N}} \cong_\mathsf{G} \mathfrak{B} \otimes_\mathbb{F} \hat{\mathsf{N}}$ for some nilpotent graded algebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$ such that $bd = db$ for any $b \in \mathfrak{B}$ and $d \in \hat{\mathsf{N}}$. We can conclude that $\hat{\mathsf{N}}$ is a commutative algebra, and $\hat{\mathsf{N}} \subset \mathcal{Z}(\mathfrak{A})$. Write $\hat{\mathsf{N}} = \mathsf{span}_\mathbb{F}\{d_1, \ldots, d_n\}$ with homogeneous nonzero $d_i \in \mathcal{Z}(\mathfrak{A})$.

Now, since $\mathfrak{B} = \mathbb{F}^\sigma[H]$ is a G-graded subalgebra of $\mathfrak{A}$, it follows that $\mathsf{T}^\mathsf{G}(\mathfrak{A}) \subseteq \mathsf{T}^\mathsf{G}(\mathfrak{B})$. Conversely, take a graded multilinear polynomial $f \notin \mathsf{T}^\mathsf{G}(\mathfrak{A})$. Since f is multilinear, we can take homogeneous elements $a_1, \ldots, a_n \in \mathfrak{B} \cup \mathsf{J}$ such that $f(a_1, \ldots, a_n) \neq 0$. If $a_i \in \mathfrak{B}$, it follows that $f \notin \mathsf{T}^\mathsf{G}(\mathfrak{B})$, and the result follows. Suppose that $a_j \in \mathsf{J}$ for some $j \in \{1, \ldots, n\}$. For each $i = 1 \ldots, n$, write $a_i = b_i c_i$ where $c_i = 1_\mathbb{F}$ for $a_i \in \mathfrak{B}$, and $c_i \in \hat{\mathsf{N}}$ if $a_i \in \mathsf{J}$. From this, since $\hat{\mathsf{N}} \subseteq \mathfrak{A}_e \subseteq \mathcal{Z}(\mathfrak{A})$, we have for any $\alpha \in S_n$ that

$$a_{\alpha(1)} \cdots a_{\alpha(n)} = (b_{\alpha(1)} c_{\alpha(1)})(b_{\alpha(2)} c_{\alpha(2)}) \cdots (b_{\alpha(n)} c_{\alpha(n)}) = (b_{\alpha(1)} b_{\alpha(2)} \cdots b_{\alpha(n)}) c_1 c_2 \cdots c_n,$$

and consequently, $0 \neq f(a_1, \ldots, a_n) = f(b_1, b_2, \ldots, b_n) c_1 c_2 \cdots c_n$. Therefore, $f \notin \mathsf{T}^\mathsf{G}(\mathfrak{B})$, and so $\mathsf{T}^\mathsf{G}(\mathfrak{B}) = \mathsf{T}^\mathsf{G}(\mathfrak{A})$.

To finish the proof, observe that, by Lemma 4.3.3, there exists $\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ which extends σ such that $\mathfrak{B} \overset{\tilde{\mathsf{G}}}{\hookrightarrow} \mathbb{F}^\gamma[\mathsf{G}]$, and so $\mathsf{var}^\mathsf{G}(\mathfrak{B}) \subseteq \mathsf{var}^\mathsf{G}(\mathbb{F}^\gamma[\mathsf{G}])$. Consequently, we have

$$\mathfrak{A} \in \mathsf{var}^\mathsf{G}(\mathfrak{A}) = \mathsf{var}^\mathsf{G}(\mathfrak{B}) \subseteq \mathsf{var}^\mathsf{G}(\mathbb{F}^\gamma[\mathsf{G}]),$$

since $\mathfrak{A} \equiv_{\mathsf{G}PI} \mathfrak{B}$ (the first part of this proof). ■

Observe that if $H = \mathsf{G}$ in Theorem 4.3.8, then $\mathsf{Supp}(\Gamma_{\mathfrak{A}}) = H$, and hence, $\mathsf{J}_H = \mathsf{J}$. Hence, the corollary below is immediate.

Corollary 4.3.9 *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, G a finite abelian group, $\sigma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$, and $\mathfrak{A} = \mathbb{F}^\sigma[\mathsf{G}] \oplus \mathsf{J}$ a finite dimensional G-graded unitary $\mathbb{F}$-algebra. If $\mathfrak{A}_e$ is central in $\mathfrak{A}$, then $\mathfrak{A} \equiv_{\mathsf{G}PI} \mathbb{F}^\sigma[\mathsf{G}]$.*

4.3.1 Finitely generated graded algebras of the variety $\mathfrak{V}^{\mathsf{G}}$

The next theorems are obvious consequences of the Lemmas of Section 4.3 of this chapter. Let us denote by $\mathfrak{V}^{\mathsf{G}}$ the G-graded variety of G-graded algebras with the central neutral component.

Theorem 4.3.10 *Let G be a finite group, and $\mathbb{F}$ a field. Let N be a G-graded $\mathbb{F}$-algebra (not necessarily finitely generated) which belongs to the variety $\mathfrak{V}^{\mathsf{G}}$. If N_e is nil (resp. nil of bounded index), then N is nil (resp. nil of bounded index). If N_e is finitely generated and nil, then N is nilpotent. In particular, in characteristic zero, if N_e is nil of bounded index, then N is nilpotent. Moreover, if N_e is nilpotent, then N is nilpotent with $\mathsf{nd}(\mathsf{N}) \leqslant |\mathsf{G}|\mathsf{nd}(\mathsf{N}_e)$.*

Proof: The theorem is immediate consequence of the Theorems 3.2.14, 3.2.15 and 3.2.19, and of the fact that N_e is commutative is $\mathsf{N} \in \mathfrak{V}^{\mathsf{G}}$. Observe also $|\mathsf{Supp}(\Gamma_{\mathsf{N}})| \leqslant |\mathsf{G}|$, and $|\mathsf{Supp}(\Gamma_{\mathsf{N}})| + 1 \leqslant |\mathsf{G}|$ if $e \notin \mathsf{Supp}(\Gamma_{\mathsf{N}})$. ■

Example 4.3.11 *Let $\mathbb{F}$ be a field, and G a group. Consider $\mathsf{N}_s = \mathbb{F}[x_1, x_2, \ldots, x_k : x_{i_1}x_{i_2}\cdots x_{i_s} = 0, \forall 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s \leqslant k]$, the free commutative nilpotent k-generated $\mathbb{F}$-algebra. We have that N_s with its trivial G-grading belongs to $\mathfrak{V}^{\mathsf{G}}$, and $\mathsf{nd}(\mathsf{N}) = s$. Thus, $\mathfrak{V}^{\mathsf{G}}$ contains a nilpotent algebra of index s for all $s \in \mathbb{N}$, i.e. in general, the nilpotency index of algebras in $\mathfrak{V}^{\mathsf{G}}$ can not be limited.*

Theorem 4.3.12 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A}$ a finitely generated G-graded algebra. If $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$, then*

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \tilde{\mathfrak{A}}_1 \times \cdots \times \tilde{\mathfrak{A}}_q \times \tilde{\mathsf{J}}_{00} \ ,$$

such that for any $j = 1, \ldots, q$, $\tilde{\mathfrak{A}}_j = \mathbb{F}^{\sigma_j}[H_j] \oplus \tilde{\mathsf{J}}_i$ is a finite dimensional G-graded algebra, that satisfies all the claims of Lemma 4.3.2, $H_j \leqslant \mathsf{G}$, $\sigma_j \in \mathsf{Z}^2(H_j, \mathbb{F}^*)$, $(H_i, [\sigma_i]) \neq (H_j, [\sigma_j])$ for all $i \neq j$, i.e. $H_i \neq H_j$ or $[\sigma_i] \neq [\sigma_j]$ when $i \neq j$. Besides that, $\tilde{\mathsf{J}}_j, \tilde{\mathsf{J}}_{00}$ are G-graded nilpotent finite dimensional algebras which belong to $\mathfrak{V}^{\mathsf{G}}$, and $\mathsf{nd}(\tilde{\mathsf{J}}_j) \leqslant \mathsf{nd}((\tilde{\mathsf{J}}_j)_e)|\mathsf{G}|$, and $\mathsf{nd}(\tilde{\mathsf{J}}_{00}) \leqslant \mathsf{nd}((\tilde{\mathsf{J}}_{00})_e)|\mathsf{G}|$.

Proof: The statement follows of Theorem 1.4.9 (or Remark 1.4.10, Lemma 4.3.2, and Theorem 4.3.10. ■

Definition 4.3.13 *Let G be a group, and $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ two G-graded PI-algebras. We say that $\mathfrak{A}$ satisfies asymptotically all the G-graded identities of $\tilde{\mathfrak{A}}$ if there exists some $n \in \mathbb{N}$ such that $\mathfrak{A}$ satisfies all the G-graded identities of $\tilde{\mathfrak{A}}$ of degree $m \geqslant n$. We write in this case*

$$\mathsf{T}^{\mathsf{G}}(\tilde{\mathfrak{A}}) \subseteq_{\infty} \mathsf{T}^{\mathsf{G}}(\mathfrak{A}) \ .$$

Theorem 4.3.14 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A}$ a finitely generated G-graded algebra. If $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$, there exists a finite dimensional G-graded algebra*

$$\mathsf{C}_{\mathsf{G},\mathfrak{A}} = \bigtimes_{H \trianglelefteq \mathsf{G}} \left(\bigtimes_{[\sigma] \in \mathsf{H}^2(H,\mathbb{F}^*)} \left(\mathbb{F}^{\sigma}[H] \oplus \mathsf{J}_{(H,[\sigma])}\right) \right) , \tag{4.3}$$

where each $\mathsf{J}_{(H,[\sigma])}$ is a finite dimensional G-graded nilpotent algebra ($\mathsf{J}_{(H,[\sigma])}$ is the Jacobson radical of $\mathfrak{A}_{(H,[\sigma])} := \mathbb{F}^{\sigma}[H] \oplus \mathsf{J}_{(H,[\sigma])}$), satisfying

$$\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_{\infty} \mathsf{T}^{\mathsf{G}}(\mathfrak{A}) \ .$$

Moreover, if $\mathfrak{A}$ is unitary, then $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$.

Proof: Fix any finitely generated G-graded algebra $\mathfrak{A}$, such that $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$. Observe that $\mathsf{T}^{G}(\mathfrak{A})$ is the $\mathsf{G}T$-ideal of graded identities of a finitely generated PI-algebra, since $\mathfrak{A}_e$ is central in $\mathfrak{A}$, and hence, $\mathfrak{A}_e$ is commutative, and so, by Theorem 1.4.12, $\mathfrak{A}$ is a PI-algebra. By Theorems 1.4.9 and 1.2.20, and Lemma 4.3.2, there exist finite dimensional G-graded algebras $\tilde{\mathfrak{A}}_i$'s such that

$$\mathfrak{A} \equiv_{\mathsf{G}PI} \tilde{\mathfrak{A}}_1 \times \cdots \times \tilde{\mathfrak{A}}_q \times \tilde{\mathsf{J}}_{00} \ ,$$

where $\tilde{\mathfrak{A}}_j = \tilde{\mathfrak{B}}_j \oplus \tilde{\mathsf{J}}_j$, with $\tilde{\mathfrak{B}}_j = \mathbb{F}^{\sigma_j}[H_j]$, for some subgroup H_j of G and $\sigma_j \in \mathsf{Z}^2(H_j, \mathbb{F}^*)$, where $(H_i, [\sigma_i]) \neq (H_j, [\sigma_j])$ for all $i \neq j$.

Now, consider $\mathsf{C}_{\mathsf{G}} = \bigtimes_{H \trianglelefteq \mathsf{G}} \left(\bigtimes_{[\sigma] \in \mathsf{H}^2(H,\mathbb{F}^*)} (\mathbb{F}^{\sigma}[H]) \right)$. Observe that $\mathsf{C}_{\mathsf{G}} \in \mathfrak{V}^{\mathsf{G}}$. By Absorption Lemma (Corollary 1.5.9), for any $H \trianglelefteq \mathsf{G}$, and $[\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)$, there exist finite dimensional G-graded nilpotent algebras $\mathsf{J}_{(H,[\sigma])}$'s satisfying

$$\left(\bigtimes_{j=1}^{q} \tilde{\mathfrak{A}}_j \right) \times \tilde{\mathsf{J}}_{00} \times \mathsf{C}_{\mathsf{G}} \equiv_{\mathsf{G}PI} \mathsf{C}_{\mathsf{G},\mathfrak{A}} \times \tilde{\tilde{\mathsf{J}}}_{00} \ ,$$

where $\mathsf{C}_{\mathsf{G},\mathfrak{A}}$ is defined in (4.3), and $\tilde{\tilde{\mathsf{J}}}_{00}$ is some nilpotent G-graded algebra which belongs to $\mathfrak{V}^{\mathsf{G}}$. Consequently, $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}} \times \tilde{\tilde{\mathsf{J}}}_{00}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. Since $\mathsf{nd}(\tilde{\tilde{\mathsf{J}}}_{00}) < \infty$, and $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}} \times \tilde{\tilde{\mathsf{J}}}_{00}) = \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \cap \mathsf{T}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00})$, it follows that any multilinear polynomial identity $f \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$ of degree $m \geqslant \mathsf{nd}(\tilde{\tilde{\mathsf{J}}}_{00})$ belongs to $\mathsf{T}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00})$. Therefore, we conclude that $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_{\infty} \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$.

Notice that $\mathsf{C}_{\mathsf{G},\mathfrak{A}} \times \tilde{\tilde{\mathsf{J}}}_{00} \in \mathfrak{V}^{\mathsf{G}}$ (since C_{G} and $\mathfrak{A}$ belong to $\mathfrak{V}^{\mathsf{G}}$), hence, we also have that $\mathsf{C}_{\mathsf{G},\mathfrak{A}} \in \mathfrak{V}^{\mathsf{G}}$.

Finally, suppose that $\mathfrak{A}$ is a unitary algebra. Take any $g = g(x_1^{(\theta_1)}, \dots, x_r^{(\theta_r)}) \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$. Since $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_{\infty} \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$, fix $n \in \mathbb{N}$ such that any polynomial identity $w \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$ of degree $m \geqslant n$ belongs to $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. Consider the graded polynomial identity $\tilde{g} = \tilde{g}(x_1^{(\theta_1)}, \dots, x_r^{(\theta_r)}, y_1^{(e)}, \dots, y_n^{(e)}) = g(x_1^{(\theta_1)}, \dots, x_r^{(\theta_r)}) y_1^{(e)} \cdots y_n^{(e)}$. Observe that $\tilde{g} \in \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$, since $\mathsf{deg}(\tilde{g}) = n + \mathsf{deg}(g) \geqslant n$, and $\tilde{g} \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$ (because $\tilde{g}$ is a consequence of g). Being $1_{\mathfrak{A}}$ the unity of $\mathfrak{A}$ (where $1_{\mathfrak{A}} \in \mathfrak{A}_e$), for any homogeneous elements $a_1, \dots, a_r \in \mathfrak{A}$, with $\mathsf{deg}(a_i) = \theta_i$, we have that

$$0 = \tilde{g}(a_1, \dots, a_r, \underbrace{1_{\mathfrak{A}}, \dots, 1_{\mathfrak{A}}}_{n}) = g(a_1, \dots, a_r) \underbrace{1_{\mathfrak{A}} \dots 1_{\mathfrak{A}}}_{n} = g(a_1, \dots, a_r) \ ,$$

and consequently, $g \in \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. Therefore, we conclude that $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. ■

Corollary 4.3.15 *Let $\mathsf{G} = \mathbb{Z}_n$ be a cyclic finite group of order n. Then any finitely generated G-graded algebra $\mathfrak{A}$ which belongs to $\mathfrak{V}^{\mathsf{G}}$ satisfies the ordinary (non-graded) identity*

$$[x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}] \in \mathbb{F}\langle X \rangle \ ,$$

for some $k \in \mathbb{N}$.

Proof: Fix any finitely generated G-graded algebra $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$. By Theorem 4.3.14, we have that $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_{\infty} \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. Let us analyse each component $\mathbb{F}^{\sigma}[H] \oplus \mathsf{J}_{(H,[\sigma])}$ of $\mathsf{C}_{\mathsf{G},\mathfrak{A}}$.

Given $H \trianglelefteq \mathsf{G}$ and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, consider $\mathfrak{A}_{(H,[\sigma])} = \mathbb{F}^{\sigma}[H] \oplus \mathsf{J}_{(H,[\sigma])}$. Observe that $\mathbb{F}^{\sigma}[H]$ is commutative, because H is cyclic, and σ is symmetric (by Corollary 1.2.8). Hence $[a + x, b + y] = [x, b] + [a, y] + [x, y] \in \mathsf{J}_{(H,[\sigma])}$ for any $a, b \in \mathbb{F}^{\sigma}[H]$, and $x, y \in \mathsf{J}_{(H,[\sigma])}$. Take $a_1 = b_1 + y_1, \dots, a_{2n} = b_{2n} + y_{2n} \in \mathfrak{A}_{(H,[\sigma])}$ with $b_1, \dots, b_{2n} \in \mathbb{F}^{\sigma}[H]$, and $y_1, \dots, y_{2n} \in \mathsf{J}_{(H,[\sigma])}$. We have

$$[a_1, a_2][a_3, a_4] \cdots [a_{2n-1}, a_{2n}] \in (\mathsf{J}_{(H,[\sigma])})^n \ ,$$

for all $n \in \mathbb{N}$.

Let us choose $\tilde{n} = \mathsf{max}\{\mathsf{nd}(\mathsf{J}_{(H,[\sigma])} : H \trianglelefteq \mathsf{G}, [\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)\}$. Then we have that $[a_1, a_2] \cdots [a_{2\tilde{n}-1}, a_{2\tilde{n}}] \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$. Since $\mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}}) \subseteq_\infty \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$, then there exists $m \in \mathbb{N}$ such that any multilinear polynomial $f \in \mathsf{T}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G},\mathfrak{A}})$ of degree greater than or equal to m alsobelongs to $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$.

Then, for $k = \mathsf{max}\{m, \tilde{n}\}$ we have that $[x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}] \in \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$. ■

4.4 On the algebra $\mathfrak{A}$ when $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e$ is central

Recall that we assume that G is a finite abelian group, and $\mathbb{F}$ is an algebraically closed field of characteristic zero.

The following result is basic for our study.

Lemma 4.4.1 *Let $\mathfrak{A}$ be a (arbitrary) $\mathsf{G} \times \mathbb{Z}_2$-graded algebra. Then $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ iff $\mathfrak{A}_{(e,0)} \subseteq \mathcal{Z}(\mathfrak{A})$, $\mathfrak{A}_{(e,1)} \subseteq \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_0)$ and $ab + ba = 0$ for any $a \in \mathfrak{A}_{(e,1)}$ and $b \in \mathfrak{A}_1$.*

Proof: Suppose $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e \subseteq \mathcal{Z}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$. Take any $a \in \mathfrak{A}_{(e,0)}$ and $b \in \mathfrak{A}$. Put $b = b_0 + b_1$, $b_0 \in \mathfrak{A}_0$ and $b_1 \in \mathfrak{A}_1$. We have

$$0 = [a \otimes x_0, b_0 \otimes y_0 + b_1 \otimes y_1] = [a, b_0] \otimes x_0 y_0 + [a, b_1] \otimes x_0 y_1$$

for any $x_0, y_0 \in \mathsf{E}_0$, and $y_1 \in \mathsf{E}_1$. It follows that $[a, b_0] = [a, b_1] = 0$, since $[a, b_0] \otimes x_0 y_0 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_0$ and $[a, b_1] \otimes x_0 y_1 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_1$. Hence, $[a, b] = [a, b_0 + b_1] = [a, b_0] + [a, b_1] = 0$. From this, we conclude that $\mathfrak{A}_{(e,0)} \subseteq \mathcal{Z}(\mathfrak{A})$.

Now, take any $c \in \mathfrak{A}_{(e,1)}$. Let $d_0 \in \mathfrak{A}_0$ and $d_1 \in \mathfrak{A}_1$, then we have

$$0 = [c \otimes x_1, d_0 \otimes z_0 + d_1 \otimes z_1] = [c, d_0] \otimes x_1 z_0 + (cd_1 + d_1 c) \otimes x_1 z_1$$

for any $z_0 \in \mathsf{E}_0$ and $x_1, z_1 \in \mathsf{E}_1$. It follows that $[c, d_0] = 0$ and $cd_1 + d_1 c = 0$, since $[c, d_0] \otimes x_1 z_0 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_1$ and $(cd_1 + d_1 c) \otimes x_1 z_1 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})_0$. Therefore, we deduce that $\mathfrak{A}_{(e,1)} \subseteq \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_0)$, and $ab + ba = 0$ for any $a \in \mathfrak{A}_{(e,1)}$ and $b \in \mathfrak{A}_1$.

Reciprocally, suppose that $\mathfrak{A}_{(e,0)} \subseteq \mathcal{Z}(\mathfrak{A})$, $\mathfrak{A}_{(e,1)} \subseteq \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_0)$ and $ab + ba = 0$ for any $a \in \mathfrak{A}_{(e,1)}$ and $b \in \mathfrak{A}_1$. Take any $a_{(e,0)} \otimes x_0 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{(e,0)})$ and $a_{(e,1)} \otimes x_1 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{(e,1)})$. For any

$(b_0 \otimes y_0 + b_1 \otimes y_1) \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, we have that

$$
\begin{aligned}
[a_{(e,0)} \otimes x_0 + a_{(e,1)} \otimes x_1, b_0 \otimes y_0 + b_1 \otimes y_1] &= [a_{(e,0)} \otimes x_0, b_0 \otimes y_0] + [a_{(e,0)} \otimes x_0, b_1 \otimes y_1]+ \\
&+ [a_{(e,1)} \otimes x_1, b_0 \otimes y_0] + [a_{(e,1)} \otimes x_1, b_1 \otimes y_1] = [a_{(e,0)}\otimes, b_0]x_0y_0+ \\
&+ [a_{(e,0)}, b_1] \otimes x_0y_1 + [a_{(e,1)}, b_0] \otimes x_1y_0 + (a_{(e,1)}b_1 + b_1a_{(e,1)}) \otimes x_1y_1 = 0 \ .
\end{aligned}
$$

Since $a_{(e,0)} \otimes x_0, a_{(e,1)} \otimes x_1, b_0 \otimes y_0, b_1 \otimes y_1 \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ can be chosen as basic elements, the result follows. ■

The above result motivate the following definition.

Definition 4.4.2 *A $\mathbb{Z}_2$-graded algebra $\mathfrak{A}$ is called a* **super-commutative algebra** *if $\mathfrak{A}_0 \subseteq \mathcal{Z}(\mathfrak{A})$, and $\mathfrak{A}_1$ is anti-commutative. Now, given a graded subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we say that $\mathfrak{B}$ is a* **super-central algebra** *in $\mathfrak{A}$ if $\mathfrak{B}_0 \subseteq \mathcal{Z}(\mathfrak{A})$, $\mathfrak{B}_1 \subseteq \mathcal{Z}_{\mathfrak{A}}(\mathfrak{A}_0)$, and $ab + ba = 0$ for any $a \in \mathfrak{A}_1$ and $b \in \mathfrak{B}_1$.*

In the above definition, notice that $\mathfrak{B}$ is also super-commutative. Also, in Lemma 4.4.1, we can conclude that $\mathfrak{A}_e$ is super-central in $\mathfrak{A}$ when $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$.

By previous lemma, supposing $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e \subseteq \mathcal{Z}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$, it follows that $a^2 = 0$ for any $a \in \mathfrak{A}_{(e,1)}$ if $\mathsf{char}(\mathbb{F}) \neq 2$, and $\mathfrak{A}_e$ is central in $\mathfrak{A}$ when $\mathsf{char}(\mathbb{F}) = 2$ (and so $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$ if $\mathsf{char}(\mathbb{F}) = 2$). Anyway, we conclude that $\mathfrak{A}_e$ is super-central iff $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$.

Now assume $\mathsf{char}(\mathbb{F}) = 0$. Let us show that we can assume that $\mathfrak{A}$ is unitary if $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$. In fact, consider $\mathfrak{A}^{\#} = \mathfrak{A} \oplus \mathbb{F} \cdot 1$, the algebra derived from the algebra $\mathfrak{A}$ by adjoining the unit "1". The product in $\mathfrak{A}^{\#}$ is defined in (1.1, Section 1.1 of Chapter 1). We have that $\mathfrak{A}^{\#}$ is $G \times \mathbb{Z}_2$-graded with the grading induced from $\mathfrak{A}$, i.e. $\mathfrak{A}^{\#}_{(g,\lambda)} = \mathfrak{A}_{(g,\lambda)}$ if $(g, \lambda) \neq (e, 0)$, and $\mathfrak{A}^{\#}_{(e,0)} = \mathfrak{A}_{(e,0)} \oplus \mathbb{F} \cdot 1$. Hence, we can see $\mathfrak{A}$ as a $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebra of $\mathfrak{A}^{\#}$. Then $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ is a $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebra of $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#})$. It follows that $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#}))$.

Reciprocally, to show that $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$, let us show that $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#}) \in \mathfrak{V}^{\mathsf{G}}$. By Lemma 4.4.1, it follows that

$$
\begin{aligned}
[a_{(e,0)} + \lambda, b + \gamma] &= (a_{(e,0)} + \lambda)(b + \gamma) - (b + \gamma)(a_{(e,0)} + \lambda) \\
&= a_{(e,0)}b + \lambda b + \gamma a_{(e,0)} + \lambda\gamma - (ba_{(e,0)} + \gamma a_{(e,0)} + \lambda b + \gamma\lambda) \\
&= a_{(e,0)}b - ba_{(e,0)} = [a_{(e,0)}, b] = 0 \ ,
\end{aligned}
$$

for any $a_{(e,0)} \in \mathfrak{A}_{(e,0)}$, $b \in \mathfrak{A}$ and $\lambda, \gamma \in \mathbb{F} \cdot 1$. Hence, and again by Lemma 4.4.1, we have

$$
\begin{aligned}
&[(a_{(e,0)} + \lambda) \otimes x_0 + a_{(e,1)} \otimes x_1, (b_0 + \gamma) \otimes y_0 + c_1 \otimes y_1] = \\
&\qquad =[a_{(e,0)} + \lambda, b_0 + \gamma] \otimes x_0 y_0 + [a_{(e,0)} + \lambda, c_1] \otimes x_0 y_1 + \\
&\qquad\quad + [a_{(e,1)}, b_0 + \gamma] \otimes x_1 y_0 + [a_{(e,1)} \otimes x_1, c_1 \otimes y_1] \\
&\qquad =0 \otimes x_0 y_0 + 0 \otimes x_0 y_1 + [a_{(e,1)}, b_0] \otimes x_1 y_0 + (a_{(e,1)} c_1 + a_{(e,1)} c_1) \otimes x_1 y_1 = 0,
\end{aligned}
$$

for any $a_{(e,0)} + \lambda \in \mathfrak{A}^{\#}_{(e,0)}$, $\lambda \in \mathbb{F} \cdot 1$, $a_{(e,1)} \in \mathfrak{A}^{\#}_{(e,1)} = \mathfrak{A}_{(e,1)}$, $b_0 + \gamma \in \mathfrak{A}^{\#}_0$, $\gamma \in \mathbb{F} \cdot 1$, $c_1 \in \mathfrak{A}^{\#}_1 = \mathfrak{A}_1$, $x_0, y_0 \in \mathsf{E}_0$, and $x_1, y_1 \in \mathsf{E}_1$. Therefore, $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#})_e$ is a central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#})$. Thus $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#}) \in \mathfrak{V}^{\mathsf{G}}$. We conclude that $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}^{\#}))$.

Therefore, let $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ be a finite dimensional $\hat{\mathsf{G}}$-graded unitary algebra, where $\mathfrak{B}$ is the maximal semisime subalgebra of $\mathfrak{A}$, J is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{i}$ is the unity of $\mathfrak{A}$. Without loss of generality, we can assume that $\mathfrak{A} = \mathfrak{i}\mathfrak{A}\mathfrak{i} = \mathfrak{B} \oplus \mathfrak{i}\mathsf{J}\mathfrak{i} = \mathfrak{B} \oplus \mathsf{J}_{11}$, where J_{11} is described by Lemma 1.5.1. Let $\mathfrak{B} = \times_{s=1}^{p} \mathfrak{B}_s$, where $\mathfrak{B}_s$'s are graded simple subalgebras of $\mathfrak{A}$, and $\mathfrak{i}_s$ the unity of $\mathfrak{B}_s$, where $\mathfrak{i} = \sum_{s=1}^{p} \mathfrak{i}_s$. From this, since $\mathfrak{i}_s \in \mathcal{Z}(\mathfrak{A})$, by Theorem 1.5.12, $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_1) \times \cdots \times \mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_k) \times \mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}))\right)$, where $\mathfrak{B}_{s_i} \not\cong_{\mathsf{G} \times \mathbb{Z}_2} \mathfrak{B}_{s_j}$ for any $i \neq j$. Hence

$$
\mathsf{T}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\right) = \left(\bigcap_{j=1}^{k} \mathsf{T}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_j)\right)\right) \bigcap \left(\mathsf{T}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})\right)\right), \tag{4.4}
$$

where $\hat{\mathfrak{A}}_j = \hat{\mathfrak{B}}_j \oplus \hat{\mathsf{J}}_j$ is a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra, $\hat{\mathfrak{B}}_j$ is a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded simple algebra, and $\hat{\mathsf{J}}_j = \mathfrak{i}_j \hat{\mathsf{J}}_j \mathfrak{i}_j$ is the Jacobson radical of $\hat{\mathfrak{A}}_j$, for $j = 1, \ldots, k$. Therefore, to describe the variety $\mathfrak{V}^{\mathsf{G}}$ it is sufficient to study the varieties $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_j))$ for all $j = 1, \ldots, k$, and $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}))$.

In the next subsections, let us study the graded polynomial identities of $\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_j)$ for all $j = 1, \ldots, k$. For this, let us describe the $\mathsf{G} \times \mathbb{Z}_2$-graded subalgebras $\hat{\mathfrak{B}}_j$ and $\hat{\mathsf{J}}_j$ when $\mathsf{E}^{\mathsf{G}}(\hat{\mathfrak{A}}_j)$ has the central neutral component.

4.4.1 On $\mathfrak{B}$ and J when $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e$ is central

As it observed in the previous section, to determine $\mathfrak{V}^{\mathsf{G}}$ is sufficient to describe $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, where $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ is a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra, and $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, where $\mathfrak{B} = M_n(\mathbb{F}^{\sigma}[H])$, $\mathsf{J} = \mathfrak{i}\mathsf{J}(\mathfrak{A})\mathfrak{i}$ is the Jacobson radical of $\mathfrak{A}$, with $H \leqslant \mathsf{G} \times \mathbb{Z}_2$, $\sigma : H \times H \longrightarrow \mathbb{F}^*$, $n \in \mathbb{N}$, and $\mathfrak{i}$ is the unity of $\mathfrak{A}$. Notice that

$$(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_g = (\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_{(g,0)} + (\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_{(g,1)} = \mathfrak{A}_{(g,0)} \otimes \mathsf{E}_0 + \mathfrak{A}_{(g,1)} \otimes \mathsf{E}_1 = \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_g),$$

for any $g \in \mathsf{G}$. We consider a canonical elementary G-grading on $\mathfrak{B}$ determined by an n-tupla $(g_1, g_2, \ldots, g_n) \in (\mathsf{G} \times \mathbb{Z}_2)^n$.

In the following, we present a characterization for $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, where $\mathfrak{A} = M_n(\mathbb{F}^\sigma[H]) \oplus \mathsf{J}$ denotes a finite dimensional unitary $\mathsf{G} \times \mathbb{Z}_2$-graded algebra over a field $\mathbb{F}$, and $\mathsf{J} = \mathfrak{i}\mathsf{J}(\mathfrak{A})\mathfrak{i}$, which is a nilpotent graded ideal of $\mathfrak{A}$.

Lemma 4.4.3 *Suppose* $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ *is central in* $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, *and* $\mathsf{char}(\mathbb{F}) \neq 2$. *Then* $\mathfrak{B} = \mathbb{F}^\sigma[H]$, *and* $(e, 1) \notin H$. *In particular, we have* $\mathfrak{B}_e = \mathfrak{B}_{(e,0)} = \mathsf{span}_{\mathbb{F}}\{\eta_{(e,0)}\}$.

Proof: Assume $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e \subseteq \mathscr{Z}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))$. To obtain a contradiction, suppose that $n \geqslant 2$. By Lemma 4.4.1, we have that $\mathfrak{A}_{(e,0)}$ is central in $\mathfrak{A}$. Consider $a = E_{22}\eta_{(e,0)} \in \mathfrak{B}$. Note that $E_{22}\eta_{(e,0)} \in \mathfrak{B}_{(e,0)} \subseteq \mathfrak{A}_{(e,0)}$, since $\mathsf{deg}(E_{jj}\eta_{(e,0)}) = g_j^{-1}(e,0)g_j = (e,0)$ for any $j \in \{1, \ldots, n\}$. From this, it follows that

$$\begin{aligned} 0 = [a, E_{12}\eta_{(e,0)}] &= [E_{22}\eta_{(e,0)}, E_{12}\eta_{(e,0)}] \\ &= \sigma((e,0),(e,0)) \left(\delta_{21}E_{22}\eta_{(e,0)} - \delta_{22}E_{12}\eta_{(e,0)}\right) \\ &= -\sigma((e,0),(e,0))E_{12}\eta_{(e,0)} \ , \end{aligned}$$

and hence, we have a contradiction, since $\sigma((g,\lambda),(h,\gamma)) \neq 0$, and $E_{ij}\eta_{(g,\lambda)} \neq 0$ for any $i, j \in \{1, \ldots, n\}$, $g, h \in \mathsf{G}$ and $\lambda, \gamma \in \mathbb{Z}_2$. Therefore, we have showed that $n = 1$. Since $\mathfrak{A}$ is unitary, and hence, $\mathfrak{i}\eta_{(e,0)} \in \mathfrak{B}_{(e,0)}$. From this, $\mathfrak{B} = \mathbb{F}^\sigma[H]$.

Now, to obtain a contradiction, assume that $(e,1) \in H$. Hence, we have $\eta_{(e,1)} \in \mathfrak{B}_{(e,1)}$. By Lemma 4.4.1, it follows that

$$0 = \eta_{(e,1)}\eta_{(e,1)} + \eta_{(e,1)}\eta_{(e,1)} = 2(\eta_{(e,1)})^2 = 2\sigma((e,1),(e,1))\eta_{(e,0)} \ .$$

Since $\mathsf{char}(\mathbb{F}) \neq 2$ and $\eta_{(e,0)} \neq 0$, we deduce that $\sigma((e,1),(e,1)) = 0$, which obviously is a contradiction. Therefore, we conclude that $(e,1) \notin H$ and $\mathfrak{B}_e = \mathsf{span}_{\mathbb{F}}\{\eta_{(e,0)}\}$. ■

It follows from Lemmas 4.4.3 and 1.5.2 that, when $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e \subseteq \mathscr{Z}\left(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\right)$, $\mathsf{J} = \mathbb{F}^\sigma[H]\mathsf{N}$ for a suitable finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded vector subspace $\mathsf{N} \subset \mathsf{J}$. Hence, it follows that $\mathfrak{B} \in \mathfrak{V}^{\mathsf{G}}$ by Lemma 4.4.3. Hence, it follows from Lemma 4.4.3 that $\left[x^{(e,0)}, y^{(h)}\right], x^{(e,1)}$ belong to $\mathsf{T}^{\mathsf{G}\times\mathbb{Z}_2}(\mathfrak{B})$, for any $h \in \mathsf{G} \times \mathbb{Z}_2$ ($\mathsf{char}(\mathbb{F}) \neq 2$).

Remark 4.4.4 *Assume that $\mathfrak{A} = \mathbb{F}^\sigma[H] \bigoplus \mathsf{J}$ is a unitary $\mathsf{G}\times\mathbb{Z}_2$-graded finite dimensional algebra, where $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$, and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. Being $\mathfrak{B} = \mathbb{F}^\sigma[H]$, as in Lemma 1.5.2, put $\mathsf{J} = \mathfrak{B}\mathsf{N}$ for some $\mathsf{G} \times \mathbb{Z}_2$-graded vector space N.*

Under these conditions, we have that $\mathsf{J}_{(e,0)} = \{0\}$ iff $\mathsf{Supp}(\Gamma_\mathsf{N}) \bigcap H = \varnothing$, where $\mathsf{Supp}(\Gamma_\mathfrak{B}) = H$ and $\mathsf{Supp}(\Gamma_\mathsf{N})$ is the support of N.

Indeed, suppose firstly that $\mathsf{J}_{(e,0)} \neq \{0\}$. Hence, there exist homogeneous elements $b \in \mathfrak{B}$ and $x \in \mathsf{N}$ such that $bx \in \mathsf{J}_{(e,0)}$. Thus, $\mathsf{deg}(x)^{-1} = \mathsf{deg}(b) \in H$, because $(e,0) = \mathsf{deg}(bx) = \mathsf{deg}(b)\mathsf{deg}(x)$. Since H is a subgroup of $\mathsf{G} \times \mathbb{Z}_2$, it follows that $\mathsf{deg}(x) \in H$, and so $\mathsf{Supp}(\Gamma_\mathsf{N}) \bigcap H \neq \varnothing$.

Conversely, suppose that $\mathsf{J}_{(e,0)} = \{0\}$. To obtain a contradiction, take a nonzero homogeneous $d \in \mathsf{N}$ such that $\mathsf{deg}(d) \in H$. Put $\mathsf{deg}(d) = (g, \lambda) \in \mathsf{G} \times \mathbb{Z}_2$ for some $g \in \mathsf{G}$ and $\lambda \in \mathbb{Z}_2$. Since H is a subgroup of $\mathsf{G} \times \mathbb{Z}_2$, it follows that $\mathsf{deg}(d)^{-1} \in H$. Hence, there exists $h \in H$ such that $h = \mathsf{deg}(d)^{-1} = (g^{-1}, \lambda)$, and so $\eta_h \in \mathfrak{B}_{(g^{-1},\lambda)} - \{0\}$. From this, we have that $\eta_h d \neq 0$ is a homogeneous element of J such that its degree is

$$\mathsf{deg}(\eta_h d) = \mathsf{deg}(\eta_h)\mathsf{deg}(d) = h(g,\lambda) = (g^{-1},\lambda)(g,\lambda) = (e,0) \ ,$$

that contradicts our hypothesis $\mathsf{J}_{(e,0)} = \{0\}$.

The above observations motivate the next results. Let us exhibit some conditions to ensure that J can be generated (as a graded $\mathfrak{B}$-bimodule) by homogeneous elements of degree $(e,0)$ or $(e,1)$.

Remark 4.4.5 *Let $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ be a $\mathsf{G} \times \mathbb{Z}_2$-graded algebra, where $\mathfrak{B} = \mathbb{F}^\sigma[H]$, and J is the Jacobson radical of $\mathfrak{A}$. It is obvious that $\pi(\mathsf{Supp}(\Gamma_\mathfrak{A})) = \pi(H)$ if, and only if, $\pi(\mathsf{Supp}(\Gamma_\mathsf{J})) \subseteq \pi(H)$, where $\pi : \mathsf{G} \times \mathbb{Z}_2 \to \mathsf{G}$ is the projection map. Particularly, if $\mathsf{J} = \mathfrak{B}\mathsf{N} = \bigoplus_{i=1}^r \mathfrak{B}d_i$ for some $\mathsf{G}\times\mathbb{Z}_2$-graded subspace $\mathsf{N} = \mathsf{span}_\mathbb{F}\{d_1, \dots, d_r\} \subseteq \mathsf{J}$, we have taht $\pi(\mathsf{Supp}(\Gamma_\mathsf{J})) \subseteq \pi(H)$ iff $\pi(\mathsf{supp}(\Gamma_\mathsf{N})$. In fact,*

$$\begin{aligned}\mathsf{Supp}(\Gamma_\mathsf{J}) &= \{g \in \mathsf{G} \times \mathbb{Z}_2 : \mathsf{J}_g \neq 0\} \\ &= \{g \in \mathsf{G} \times \mathbb{Z}_2 : (\mathfrak{B}\mathsf{N})_g \neq 0\} \\ &= \{th \in \mathsf{G} \times \mathbb{Z}_2 : \mathfrak{B}_t\mathsf{N}_h \neq 0\} \ .\end{aligned}$$

Let us consider $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a finite dimensional $(\mathsf{G} \times \mathbb{Z}_2)$-graded algebra, where $\mathfrak{B} = \mathbb{F}^\sigma[H]$ is a twisted group algebra, with $H \leqslant \mathsf{G} \times \mathbb{Z}_2$, $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and J is the Jacobson radical of $\mathfrak{A}$. Consider a graded subspace N of $\mathfrak{A}$. Recall that π denotes the projection map of $\mathsf{G} \times \mathbb{Z}_2$ to G. Let us consider $\mathsf{N}_{\pi(H)} = \bigoplus_{\substack{g \in \mathsf{Supp}(\Gamma_\mathsf{N}) \\ \pi(g) \in \pi(H)}} \mathsf{N}_g$, and $\mathsf{N}_H = \bigoplus_{h \in H} \mathsf{N}_h$.

Lemma 4.4.6 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a $(\mathsf{G} \times \mathbb{Z}_2)$-graded finite dimensional unitary algebra, where J is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{B} = \mathbb{F}^\sigma[H]$ with $H \leqslant \mathsf{G} \times \mathbb{Z}_2$ and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$.*

i) If $\pi(\mathsf{Supp}(\Gamma_\mathsf{J}))$ is contained in $\pi(H)$, then there exists a nilpotent $\mathbb{Z}_2$-graded algebra $\tilde{\mathsf{N}} \subseteq \mathsf{J}_e = \mathsf{J}_{(e,0)} \oplus \mathsf{J}_{(e,1)}$, which generates J as a $(\mathsf{G} \times \mathbb{Z}_2)$-graded $\mathfrak{B}$-bimodule;

ii) J can be generated as a $\mathsf{G} \times \mathbb{Z}_2$-graded $\mathfrak{B}$-bimodule by a graded vector space $\hat{\mathsf{N}} \subseteq \mathsf{J}$, such that $\hat{\mathsf{N}}_{\pi(H)} \subseteq \mathsf{J}_e$, $\hat{\mathsf{N}}_{\pi(H)}$ is a $\mathbb{Z}_2$-graded subalgebra, with $\hat{\mathsf{N}}_H = \hat{\mathsf{N}}_{(e,0)}$ which is a subalgebra of $\mathfrak{A}$.

In addition, if $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, then $\hat{\mathsf{N}}_H$ is central in $\mathfrak{A}$, and $\hat{\mathsf{N}}_{\pi(H)}$ and $\tilde{\mathsf{N}}$ are super-central in $\mathfrak{A}$.

Proof: Firstly, let us apply Lemma 1.5.2 for J. Write $\mathsf{J} = \mathfrak{B}\mathsf{N}$ for some $\mathsf{G} \times \mathbb{Z}_2$-graded vector space $\mathsf{N} = \mathsf{span}_\mathbb{F}\{d_1, \ldots, d_r\}$ where the d_i's are homogeneous elements of J such that $bd_i = \gamma_i(h) d_i b \neq 0$ for any $b \in \mathfrak{B}_h$ and $i = 1, \ldots, r$, $\gamma_i(h) \in \mathbb{F}$ (see Corollary 1.3.23). Besides that, $\mathsf{J} = \mathfrak{B}\mathsf{N} = \mathfrak{B}d_1 \oplus \cdots \oplus \mathfrak{B}d_r$ as a $(\mathsf{G} \times \mathbb{Z}_2)$-graded $\mathfrak{B}$-bimodule, and $\mathfrak{B}d_i$ is an irreducible graded $\mathfrak{B}$-bimodule, for all $i = 1, \ldots, r$.

i) By Remark 4.4.5, we have that $\pi(\mathsf{Supp}(\Gamma_\mathfrak{A})) = \pi(H)$. Let us show that J can be generated (as a $\mathfrak{B}$-bimodule) by homogeneous elements of degree $(e, 0)$ or $(e, 1)$.

Fix $i \in \{1, \ldots, r\}$, and put $g_i = (\pi(g_i), \lambda) = \mathsf{deg}(d_i) \in \mathsf{G} \times \mathbb{Z}_2$. Since $\pi(\mathsf{Supp}(\Gamma_\mathsf{J})) \subseteq \pi(H)$, we have that $g_i \in H$ or $(\pi(g_i), \lambda + 1) \in H$. If $g_i \in H$, then take $h_i = g_i^{-1}$, and $\tilde{d_i} = \eta_{h_i} d_i$, and hence, $\tilde{d_i} \in \mathsf{J}_{(e,0)}$ when $d_i \in \mathsf{J}_H$. Otherwise, if $g_i \notin H$, then take $h_i = (\pi(g_i)^{-1}, \lambda + 1) \in H$, and $\tilde{d_i} = \eta_{h_i} d_i$, and in this case, $\tilde{d_i} \in \mathsf{J}_{(e,1)}$. Thus, for all $i = 1, \ldots, r$, we have

$$\tilde{d_i} \in \begin{cases} \mathsf{J}_{(e,0)} & , \text{ if } \mathsf{deg}(d_i) \in H \\ \mathsf{J}_{(e,1)} & , \text{ if } \mathsf{deg}(d_i) \notin H \end{cases} . \tag{4.5}$$

Observe that if $\mathsf{deg}(\tilde{d_i}) = (e, 1)$ for some $i = 1, \ldots, r$, then we must have $(e, 1) \notin H$. In fact, suppose that $(e, 1) \in H$. Hence, for any $(g, \gamma) \in H$, we have that $(g, \gamma + 1) =$

$(g,\gamma)(e,1) \in H$. We conclude that $\mathsf{deg}(d_i) \in H$ when $(e,1) \in H$, for all $r = 1,\ldots,r$. So we must have $(e,1) \in H$ implies that $\tilde{d}_i \in \mathsf{J}_{(e,0)}$, for all $i = 1,\ldots,r$.

For any $g \in H$ and $i = 1,\ldots,r$, since H is abelian, and by Corollary 1.3.23, we have

$$\begin{aligned}\eta_g\tilde{d}_i &= \eta_g(\eta_{h_i}d_i) = \eta_g\eta_{h_i}d_i = \sigma(g,h_i)\eta_{gh_i}d_i = \sigma(g,h_i)\eta_{h_ig}d_i\\ &= \sigma(g,h_i)\sigma(h_i,g)^{-1}\eta_{h_i}\eta_g d_i = \gamma_i(g)\sigma(g,h_i)\sigma(h_i,g)^{-1}(\eta_{h_i}d_i)\eta_g\\ &= \gamma_i(g)\sigma(g,h_i)\sigma(h_i,g)^{-1}\tilde{d}_i\eta_g\ ,\end{aligned}$$

and hence, $\mathfrak{B}\tilde{d}_i = \tilde{d}_i\mathfrak{B}$, for all $i = 1,\ldots,r$. Since $\mathfrak{B}d_i$ is irreducible, and $bd_i = \gamma_i(h)d_ib$ for any $b \in \mathfrak{B}_h$, $h \in H$, it follows that $\mathfrak{B}\tilde{d}_i = \mathfrak{B}d_i$ is an irreducible $\mathsf{G}\times\mathbb{Z}_2$-graded $\mathfrak{B}$-bimodule, for all $i = 1,\ldots,r$. Observe that $\mathsf{J} = \mathfrak{B}\mathsf{N} = \bigoplus_{i=1}^r \mathfrak{B}d_i = \bigoplus_{i=1}^r \mathfrak{B}\tilde{d}_i = \mathfrak{B}\tilde{\mathsf{N}}$, where $\tilde{\mathsf{N}} = \mathsf{span}_{\mathbb{F}}\{\tilde{d}_1,\ldots,\tilde{d}_r\}$.

Consider $\tilde{\mathsf{N}} = \mathsf{span}_{\mathbb{F}}\{\tilde{d}_1,\ldots,\tilde{d}_r\} \subseteq \mathsf{J}_e = \mathsf{J}_{(e,0)}\oplus\mathsf{J}_{(e,1)}$, which is a $\mathbb{Z}_2$-graded subspace of J. Let us show that $\tilde{\mathsf{N}}$ is a graded subalgebra of J. Indeed, for all $i,j = 1,\ldots,r$, we have $\tilde{d}_i\tilde{d}_j \in \mathsf{J} = \bigoplus_{k=1}^r \mathfrak{B}\tilde{d}_k$. Hence, there exist $\lambda_1,\ldots,\lambda_r \in \mathbb{F}$ and $t_1,\ldots,t_r \in H$ such that $\tilde{d}_i\tilde{d}_j = \sum_{k=1}^r \lambda_k\eta_{t_k}\tilde{d}_k$ (since all elements $\tilde{d}_i$ are homogeneous). If $\lambda_i \neq 0$ for some $i = 1,\ldots,r$, then

$$t_k\mathsf{deg}(\tilde{d}_k) = \mathsf{deg}(\eta_{t_k}\tilde{d}_k) = \mathsf{deg}(\tilde{d}_i)\mathsf{deg}(\tilde{d}_j)\ .$$

Hence, we have $t_k = \mathsf{deg}(\tilde{d}_k)\mathsf{deg}(\tilde{d}_i)\mathsf{deg}(\tilde{d}_j) \in \{(e,0),(e,1)\}$, since $\mathsf{deg}(\tilde{d}_k), \mathsf{deg}(\tilde{d}_i), \mathsf{deg}(\tilde{d}_j) \in \{(e,0),(e,1)\}$. Recall that $(e,1) \in H$ implies that $\tilde{d}_i \in \mathsf{J}_{(e,0)}$, for all $i = 1,\ldots,r$. If $(e,1) \in H$, then $t_k = (e,0)$ when $\lambda_k \neq 0$, because $\mathsf{deg}(\tilde{d}_k)\mathsf{deg}(\tilde{d}_i)\mathsf{deg}(\tilde{d}_j) = (e,0)$. If $(e,1) \notin H$, then we conclude also that $t_k = (e,0)$ ($t_k \in H$). Anyway, when $\lambda_k \neq 0$, we have that $t_k = (e,0)$, and hence, $\tilde{d}_i\tilde{d}_j = \sum_{k=1}^r \lambda_k\eta_{(e,0)}\tilde{d}_k = \sum_{k=1}^r \sigma((e,0),h_k)\lambda_k\tilde{d}_k \in \tilde{\mathsf{N}}$. Therefore, $\tilde{\mathsf{N}}$ is a nilpotent graded subalgebra of J_e which generates J as a $\mathsf{G}\times\mathbb{Z}_2$-graded $\mathfrak{B}$-bimodule.

ii) Without loss of generality, we can assume $\mathsf{N}_{\pi(H)} = \mathsf{span}_{\mathbb{F}}\{d_1,\ldots,d_s\}$, for some $0 \leqslant s \leqslant r$, such that $\pi(\mathsf{deg}(d_j)) \notin \pi(H)$, for all $s < j \leqslant r$. By item *i)* of this lemma, there exists $\tilde{d}_1,\ldots,\tilde{d}_s \in \mathsf{J}_e = \mathsf{J}_{(e,0)}\oplus\mathsf{J}_{(e,1)}$, such that $\mathsf{J}_{\pi(H)} = \mathfrak{B}\tilde{\mathsf{N}}$, where $\tilde{\mathsf{N}} = \mathsf{span}_{\mathbb{F}}\{\tilde{d}_1,\ldots,\tilde{d}_s\}$ satisfies the claims of the item *i)*.

Observe that $\hat{\mathsf{N}} = \mathsf{span}_{\mathbb{F}}\{\tilde{d}_1,\ldots,\tilde{d}_s,d_{s+1},\ldots,d_r\}$ is a graded vector space, which generates J as a graded $\mathfrak{B}$-bimodule such that $\hat{\mathsf{N}}_{\pi(H)} \subseteq \hat{\mathsf{N}}_e$, and $\hat{\mathsf{N}}_H = \hat{\mathsf{N}}_{(e,0)}$ (by proof of the item *i)*).

Now, let us show the second part of lemma. Supposing $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ to be central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, it is easy to show that $\hat{\mathsf{N}}_H$ is central in $\mathfrak{A}$, and $\tilde{\mathsf{N}}$ and $\hat{\mathsf{N}}_{\pi(H)}$ are super-central in $\mathfrak{A}$, it is enough to apply Lemma 4.4.1. ∎

Similarly to Remark 4.3.7, using Lemma 4.4.6 instead of Lemma 4.3.6, we obtain the following observation.

Remark 4.4.7 *Let $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ be a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra, with $\mathfrak{B} = \mathbb{F}^\sigma[H]$ and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$. Suppose that $\mathbb{F}$ is an algebraically closed field with $\mathsf{char}(\mathbb{F}) = 0$, and G is an finite abelian group. Then $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ is $\mathsf{G}PI$-equivalent to $\mathsf{E}^{\mathsf{G}}(\mathfrak{B}\hat{\mathsf{N}}^{\#} \oplus \mathfrak{B}d_1 \oplus \cdots \oplus \mathfrak{B}d_s)$ for some nilpotent $\mathbb{Z}_2$-graded algebra $\hat{\mathsf{N}} \subset \mathsf{J}_e$, and homogeneous elements $d_1, \ldots, d_s \in \mathsf{J}$ such that $d_i b = \gamma_i(h) b d_i \neq 0$ for any nonzero homogeneous $b \in \mathfrak{B}_h$, $h \in H$, $\gamma(h) \in \mathbb{F}$, and $\deg(d_i) \notin \pi(H)$.*

Next, we exhibit a result similar to Corollary 4.3.15.

Proposition 4.4.8 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field with $\mathsf{char}(\mathbb{F}) = 0$, $\mathfrak{A} = \mathbb{F}^\gamma[H] \oplus \mathsf{J}$ a finite dimensional unitary algebra, where H is subgroup of a group G, $\gamma \in \mathsf{Z}^2(H, \mathbb{F}^*)$, and J is the Jacobson radical of $\mathfrak{A}$. If γ is symmetric, and $\mathsf{nd}(\mathsf{J}) = n$, we have that*

$$[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}] \in \mathsf{T}(\mathfrak{A}) \ .$$

In general, assume that $\tilde{H} \lhd \mathsf{G} \times \mathbb{Z}_2$ has an odd order, $\sigma \in \mathsf{Z}^2(\tilde{H}, \mathbb{F}^*)$ is symmetric, and $\tilde{\mathfrak{A}} = \mathbb{F}^\sigma[\tilde{H}] \oplus \tilde{\mathsf{J}}$ is a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra. If $\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})_e$ is central in $\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})$, then

$$[x_1, x_2][x_3, x_4] \cdots [x_{2m-1}, x_{2m}] \in \mathsf{T}(\mathsf{E}(\tilde{\mathfrak{A}}))$$

where $m = \mathsf{nd}(\tilde{\mathsf{J}})$.

Proof: Similarly to Corollary 4.3.15, if H is abelian, and γ is symmetric, then $\mathbb{F}^\gamma[H]$ is commutative. Hence, for any $a_1 = b_1 + y_1, a_2 = b_2 + y_2, \ldots, a_n = b_n + y_n \in \mathfrak{A}$ with $b_1, b_2, \ldots, b_n \in \mathbb{F}^\gamma[H]$, and $y_1, y_2, \ldots, y_n \in \mathsf{J}$, we obtain $[a_i, a_{i+1}] = [b_i + y_i, b_{i+1} + y_{i+1}] = [b_i, y_{i+1}] + [y_i, b_{i+1}] + [y_i, y_{i+1}] \in \mathsf{J}$. hence, we have that $[a_1, a_2][a_3, a_4] \cdots [a_{2n-1}, a_{2n}] \in \mathsf{J}^n$. Thus, $[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}] \in \mathsf{T}(\mathfrak{A})$.

If $\tilde{H} \leqslant \mathsf{G}\times\mathbb{Z}_2$ has an odd order, then it is clear that $\tilde{H} = H\times\{0\}$, for some $H \leqslant \mathsf{G}$. In fact, an element $(h,1) \in \mathsf{G}\times\mathbb{Z}_2$, $h \in \mathsf{G}$ can not have an odd order, hence, $(h,1) \notin \tilde{H}$, for any $h \in \mathsf{G}$. Since $\tilde{H}$ is abelian, and σ is symmetric, then $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[\tilde{H}]) \cong_{\mathsf{G}} \mathbb{F}^{\tilde{\sigma}}[H] \otimes_{\mathbb{F}} \mathsf{E}_0$ is commutative, since $\tilde{\sigma} \in \mathsf{Z}^2(H,\mathbb{F}^*)$ defined by $\tilde{\sigma}(h_1,h_2) = \sigma((h_1,0),(h_2,0))$, for any $h_1,h_2 \in H$, is also symmetric, and hence, $\mathbb{F}^{\tilde{\sigma}}[H]$ and $\mathbb{F}^{\tilde{\sigma}}[H] \otimes_{\mathbb{F}} \mathsf{E}_0$ are commutative algebras. It is clear that $\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}) = \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\tilde{\sigma}}[H] \oplus \tilde{\mathsf{J}}) = \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\tilde{\sigma}}[H]) \oplus \mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})$ ($\mathbb{F}^{\tilde{\sigma}}[H]$ and $\tilde{\mathsf{J}}$ are $\mathsf{G}\times\mathbb{Z}_2$-graded algebras). Hence, for any $a_1 = b_1+y_1, a_2 = b_2+y_2, \ldots, a_n = b_n+y_n \in \mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}})$ with $b_1,b_2,\ldots,b_n \in \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\gamma}[H])$, and $y_1,y_2,\ldots,y_n \in \mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})$, we have $[a_i,a_{i+1}] = [b_i + y_i, b_{i+1}+y_{i+1}] = [b_i,y_{i+1}]+[y_i,b_{i+1}]+[y_i,y_{i+1}] \in \mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})$. Observe that $\mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})$ is a nilpotent algebra with $\mathsf{nd}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})) = \mathsf{nd}(\tilde{\mathsf{J}})$, since $\tilde{\mathsf{J}}$ is nilpotent. Hence $[a_1,a_2][a_3,a_4]\cdots[a_{2n-1},a_{2n}] \in \mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}})^n = \{0\}$. Therefore, we conclude that $[x_1,x_2][x_3,x_4]\cdots[x_{2n-1},x_{2n}] \in \mathsf{T}(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}))$.

■ Observe that if G is cyclic, then any its subgoup H is also cyclic, and any $\gamma \in \mathsf{Z}^2(H,\mathbb{F}^*)$ is symmetric. Hence, in this case Proposition 4.4.8 holds.

4.4.2 Some Informations on H

In this section, we present some immediate consequences of Lemma 4.4.3.

Firstly, notice that, under the hypothesis of Lemma 4.4.3, if $(g,\lambda) \in H$, then either $\lambda = \bar{0}$ and $(g,1) \notin H$ or 2 divides $\mathsf{o}(g)$. Indeed, fixed $(g,\lambda) \in H$, suppose 2 does not divide $\mathsf{o}(g)$. Thus, $\mathsf{o}(g) = 2n+1$ for some $n \in \mathbb{N}$. It follows that

$$(g,\lambda)^{2(n+1)} = (g^{2(n+1)},\lambda^{2(n+1)}) = (g^{2n+1}g,0) = (eg,0) = (g,0) \ ,$$

and hence, $(g,0) \in H$. Consequently, $(g,1) \notin H$, since $(e,\lambda) = (g,0)^{-1}(g,\lambda) \in H$, which is only possible for $\lambda = \bar{0}$ (by Lemma 4.4.3, $(e,1) \notin H$). The following lemma shows that if $H \trianglelefteq \mathsf{G}\times\mathbb{Z}_2$ with $(e,1) \notin H$, then $H \cong \pi(H) \trianglelefteq \mathsf{G}$. In particular, if G is a cyclic group, then H is cyclic.

Lemma 4.4.9 *Let G be a finte abelian group, H a subgroup of $\mathsf{G}\times\mathbb{Z}_2$, and $\pi : \mathsf{G}\times\mathbb{Z}_2 \to \mathsf{G}$ the projection map. Suppose $(e,1) \notin H$. The restriction map $\pi_H : H \longrightarrow \pi(H)$ given by $\pi_H(h) = \pi(h)$ for any $h \in H$ is an isomorphism of groups. In addition, $|H| = |\pi(H)|$, and $[\mathsf{G}\times\mathbb{Z}_2 : H] = 2[\mathsf{G} : \pi(H)]$; and $[\mathsf{G}\times\mathbb{Z}_2 : H] = 2$ implies $\pi(H) = \mathsf{G}$.*

Proof: Since H is a group and π is a homomorphism of groups, we have that $\pi(H)$ is a subgroup of G. Let $(h_1, \lambda_1), (h_2, \lambda_2) \in H$ be some elements such that $\pi(h_1, \lambda_1) = \pi(h_2, \lambda_2)$. Hence, it follows that $h_1 = h_2$, and thus we must have $\lambda_1 = \lambda_2$, otherwise we can obtain

$$(e, 1) = (h_1, \lambda_1)^{-1}(h_2, \lambda_2) \in H$$

which is a contradiction. Therefore, π_H is an injective map. It is clear that π_H is a surjection. Hence, π_H is an isomorphism of groups. By this reason, $|H| = |\pi(H)|$, and

$$[\mathsf{G} \times \mathbb{Z}_2 : H] = \left|\frac{\mathsf{G} \times \mathbb{Z}_2}{H}\right| = \frac{|\mathsf{G} \times \mathbb{Z}_2|}{|H|} = \frac{2|\mathsf{G}|}{|\pi(H)|} = 2\left|\frac{\mathsf{G}}{\pi(H)}\right| = 2[\mathsf{G} : \pi(H)]\ .$$

Additionally, if $[\mathsf{G} \times \mathbb{Z}_2 : H] = 2$, from the last equality, it follows that $2 = [\mathsf{G} \times \mathbb{Z}_2 : H] = 2[\mathsf{G} : \pi(H)]$, and hence, we have $[\mathsf{G} : \pi(H)] = 1$. ■

Lemma 4.4.10 *Let G be an abelian finite group, H a subgroup of $\mathsf{G} \times \mathbb{Z}_2$, and $\pi : \mathsf{G} \times \mathbb{Z}_2 \longrightarrow \mathsf{G}$ the projection map. Suppose $|\mathsf{G}| = n$, and $(e, 1) \notin H$.*

i) If n is odd, then $H = \pi(H) \times \{0\}$. In addition, $\tilde{\mathsf{G}} = \mathsf{G} \times \{0\}$ is a subgroup of $\mathsf{G} \times \mathbb{Z}_2$ such that $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{\mathsf{G}}] = 2$, $(e, 1) \notin \tilde{\mathsf{G}}$, and $H \subseteq \tilde{\mathsf{G}}$;

ii) If n is even, and G has not an element of order 4, then there exists a subgroup $\tilde{\mathsf{G}}$ of $\mathsf{G} \times \mathbb{Z}_2$ such that $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{\mathsf{G}}] = 2$, $(e, 1) \notin \tilde{\mathsf{G}}$ and $H \subseteq \tilde{\mathsf{G}}$.

Proof: *i)* Let $h \in \pi(H)$ and $\lambda \in \mathbb{Z}_2$ such that $(h, \lambda) \in H$. Since G has an odd order, it follows that $h^{2s+1} = e$ for some $s \in \mathbb{N}$. Hence, $(h, \lambda)^{2s+1} = (h^{2s+1}, \lambda^{2s+1}) = (e, \lambda)$, and thus, $\lambda \neq 1$ since $(e, 1) \notin H$. Therefore, it follows that $H = \pi(H) \times \{0\}$.

For $\tilde{\mathsf{G}} = \mathsf{G} \times \{0\}$, it is obvious that $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{\mathsf{G}}] = 2$, $(e, 1) \notin \tilde{\mathsf{G}}$, and $H \subseteq \tilde{\mathsf{G}}$.

ii) Without loss of generality, we can assume that $\mathsf{G} = \hat{\mathsf{G}} \times (\mathbb{Z}_2)^m$ for some $m \in \mathbb{N}$ and some subgroup of odd order $\hat{\mathsf{G}}$. Put $e = (\hat{e}, 0) \in \mathsf{G}$ where $\hat{e}$ is the neutral element of $\hat{\mathsf{G}}$. Fix any $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$. Since $\mathsf{gcd}(|\hat{\mathsf{G}}|, |(\mathbb{Z}_2)^m|) = 1$, we have $H = \hat{\mathsf{G}}_1 \times H_1$ for some $\hat{\mathsf{G}}_1 \triangleleft \hat{\mathsf{G}}$, and $H_1 \triangleleft (\mathbb{Z}_2)^m \times \mathbb{Z}_2$. Notice that $H \trianglelefteq \hat{\mathsf{G}} \times H_1$, $(e, 1) = ((\hat{e}, 0), 1) \notin \hat{\mathsf{G}} \times H_1$ and $[\mathsf{G} \times \mathbb{Z}_2 : \hat{\mathsf{G}} \times H_1]$ divides 2^{m+1}. From this, it is sufficient to show that there exists $\hat{H}_1 \triangleleft (\mathbb{Z}_2)^m \times \mathbb{Z}_2$ of index 2 such that $(0, 1) \notin \hat{H}_1$, and $H_1 \subseteq \hat{H}_1$. Then, $\hat{H} = \hat{\mathsf{G}} \times \hat{H}_1$ is a subgroup of index 2 of $\mathsf{G} \times \mathbb{Z}_2$, which contains H and $(e, 1) = ((\hat{e}, 0), 1) \notin \hat{H}$. Put $n_1 = [(\mathbb{Z}_2)^m \times \mathbb{Z}_2 : H_1]$. If $n_1 = 2$, the result follows ($\hat{H}_1 = H_1$). Suppose that $n_1 > 2$, and consider the quotient group $\dfrac{(\mathbb{Z}_2)^m \times \mathbb{Z}_2}{H_1}$. Hence, there exists an element $g_1 \in ((\mathbb{Z}_2)^m \times \mathbb{Z}_2) - H_1$ such that

$g_1H_1 \neq (0,1)H_1$ and has the order 2 in the quotient. And so the subgroup H_2 generated by H_1 and g_1 has not the element $(0,1)$, and $[(\mathbb{Z}_2)^m \times \mathbb{Z}_2 : H_2] < n_1$. This process can be applied until we obtain a subgroup $\hat{H}_1$ of $(\mathbb{Z}_2)^m \times \mathbb{Z}_2$, which does not contain $(0,1)$, and has index 2 in $(\mathbb{Z}_2)^m \times \mathbb{Z}_2$. The result follows. ■

Under the assumptions of Lemma 4.4.10, we can always determine a subgroup of $\mathsf{G} \times \mathbb{Z}_2$ of index 2 which does not contain $(e,1)$ and contain H if G does not contain an element of order 4.

In the proof of item *ii)* Lemma 4.4.10, we could use arguments on $\mathbb{Z}_2$-vector spaces (we consider $\mathbb{Z}_2$ as a 2-element field). Indeed, supposing $\mathsf{G} = (\mathbb{Z}_2)^d$ and $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ which does not contain $(e,1)$, we have that H is a subspace of $\mathbb{Z}_2$-vector space $\mathsf{G} \times \mathbb{Z}_2$ of a finite dimension. Since any vector space over a field has a well defined basis, we can consider a basis of $\mathsf{G} \times \mathbb{Z}_2$ formed by a basis of H, the vector $(e,1) \notin H$, and other vectors. The detail is that we can complete a linearly independent set until a basis of space. Let β be a basis of H and $\gamma \cup \{(e,1)\}$ a basis of $\mathsf{G} \times \mathbb{Z}_2$ which contains β. Consider the $\mathbb{Z}_2$-space $\tilde{H}$ generated by γ. Note that $(e,1) \notin \tilde{H}$, H is a subspace of $\tilde{H}$ and $\mathsf{dim}_{\mathbb{Z}_2}(\tilde{H}) = d$. Consequently, $|\tilde{H}| = 2^d$, and so $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{H}] = 2$. Let us now study the cases when G has an element of order 4.

Lemma 4.4.11 *For $d > 1$, consider $\mathsf{G} = \mathbb{Z}_{2^d}$. If $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ is such that $(e,1) \notin H$, then H is a subgroup of H_m, where H_m is one of the following subgroups: i) $H_{-1} = \mathsf{G} \times \{0\}$; ii) $H_0 = \langle(\bar{1},\bar{1})\rangle$; iii) $H_n = \langle(\bar{2}^n,\bar{1})\rangle$ for $n = 1,\ldots,d-1$. In addition, given $r \neq s$, we have that neither $H_s \subset H_r$ nor $H_r \subset H_s$.*

Proof: Firstly, if $H = \pi(H)\times\{0\}$, we have $H \trianglelefteq \mathsf{G}\times\{0\}$. Suppose now that $H \neq \pi(H)\times\{0\}$, i.e. there exists $x \in H$ such that $x = (h,\bar{1})$ for some $h \in \pi(H)$. Since G is a cyclic group and $(e,\bar{1}) \notin H$, by Lemma 4.4.9, since $\pi(H) \leqslant \mathsf{G}$, and $\pi(H) \cong H$, it follows that H is cyclic, and hence, there exists $y \in \mathsf{G} \times \mathbb{Z}_2$ such that $H = \langle y \rangle$ with $y = (n\bar{1},\bar{1}) \in \mathsf{G} \times \mathbb{Z}_2$ for some $n \in \{1,\ldots,2^{d-1}\}$. We have two cases to study: 1) n is odd and 2) n is even. Supposing n odd, namely $n = 2m+1$, we have $n\bar{1} = (2m+1)\bar{1} = \bar{1}$ in $\mathbb{Z}_2$, and so $y = (n\bar{1},\bar{1}) = (n\bar{1},n\bar{1}) = n(\bar{1},\bar{1}) \in H_0$. In this case, we conclude that $\langle(n\bar{1},\bar{1})\rangle \subseteq H_0$ for all n odd. On the other hand, supposing n even, take $r \in \{1,\ldots,d-1\}$ such that

$\mathsf{gcd}(2^r, n/2^r) = 1$. Put $n = 2^r m'$ for some $m' \in \mathbb{N}$, m' odd. We have

$$y = (n\bar{1}, \bar{1}) = (2^r m'\bar{1}, \bar{1}) = m'(\bar{2^r}, \bar{1}) \in H_r \ ,$$

and hence $H \trianglelefteq H_r$ for some $r \in \{1, \ldots, d-1\}$.

Finally, fix distinct $r \neq s$, namely $r = s+p$ for some $p \in \mathbb{N}$. We must have $H_r \nsubseteq H_s$ and $H_s \nsubseteq H_r$, otherwise, we would have one of the following situations: $(\bar{2}^r, \bar{1}) \in H_s$ or $(\bar{2}^s, \bar{1}) \in H_r$. Thus, we could find $k \in \mathbb{N}$ such that $(\bar{2}^r, \bar{1}) = k(\bar{2}^s, \bar{1})$ (resp. $(\bar{2}^s, \bar{1}) = k(\bar{2}^r, \bar{1})$), which implies k odd, and $\bar{2}^{s+p} = k\bar{2}^s$ (resp. $\bar{2}^s = k\bar{2}^{s+p}$) in $\mathbb{Z}_{2^d}$, and consequently, $(2^p - k)\bar{1} = \bar{0}$ (resp. $(k2^p - 1)\bar{1} = \bar{0}$) in $\mathbb{Z}_{2^d}$. This generates a contradiction because $2^p - k$ (resp. $k2^p - 1$) is odd and $\bar{1}$ has even order in $\mathbb{Z}_{2^d}$. Therefore, it follows that neither $H_s \subset H_r$ nor $H_r \subset H_s$ for $r \neq s$. ■

Differently of Lemma 4.4.10, given a group G under the conditions of Lemma 4.4.11, always there exists some subgroup H of $\mathsf{G} \times \mathbb{Z}_2$ which does not contain $(e, 1)$ and if $H \trianglelefteq \tilde{H} \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ with $(e, 1) \notin \tilde{H}$, then $H = \tilde{H}$. It is important to note that, for G as in Lemma 4.4.11, we have

$$\begin{aligned} &[\mathsf{G} \times \mathbb{Z}_2 : H_{-1}] = [\mathsf{G} \times \mathbb{Z}_2 : H_0] = 2, \\ &[\mathsf{G} \times \mathbb{Z}_2 : H_n] = 2^{n+1} \ \text{ for } \ n = 1, \ldots, d-1. \end{aligned}$$

And hence, by Lemma 4.4.9, it follows that $\pi(H_{-1}) = \pi(H_0) = \mathsf{G}$ and $[\mathsf{G} : \pi(H_n)] = 2^n$ for $n = 1, \ldots, d-1$. Notice that, fixed any $i \in \{-1, 0, 1, \ldots, d-1\}$, there is no $\hat{H} \subseteq \mathsf{G} \times \mathbb{Z}_2$ such that $H_i \subset \hat{H}$ and $(e, 1) \notin \hat{H}$. Therefore, the H_i's are maximal in the family of subgroups of $\mathsf{G} \times \mathbb{Z}_2$ which do not contain $(e, 1)$.

Finally, given G an abelian finite group and $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ which does not contain the element $(e, 1)$, using Lemmas 4.4.10 and 4.4.11, we can exhibit a subgroup $\tilde{H} \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ such that $H \subseteq \tilde{H}$, $(e, 1) \notin \tilde{H}$, and $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{H}]$ is the smallest for a subgroup $\tilde{H}$ under these conditions.

4.5 The variety $\mathsf{var}^{\mathsf{G}}\left([x^{(e)}, y^{(g)}] : g \in G\right)$

In this section we present the main results of this chapter. Here, let us denote by G a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{V}^{\mathsf{G}}$ the G-

graded variety defined by the set of graded polynomial identities $\{[x^{(e)}, y^{(g)}],\ \text{for } g \in \mathsf{G}\}$. In the results of this section we strongly use Remark 4.4.5, as well as the results of Subsection 4.4.2.

Let $\mathbb{F}$ be a field, G a finite abelian group, $H \trianglelefteq \mathsf{G}\times\mathbb{Z}_2$, and $\sigma \in \mathsf{Z}^2(H,\mathbb{F}^*)$. Consider a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded unitary algebra $\mathfrak{A} = \mathbb{F}^\sigma[H] \oplus \mathsf{J}$, with the Jacobson radical $\mathsf{J} = \mathbb{F}^\sigma[H]\mathsf{N}$ (as is described in Lemma 1.5.2), where N is a graded subspace of J. In the lemmas below, recall that $\mathfrak{A}_{\pi(H)} = \bigoplus\limits_{\substack{g\in\mathsf{Supp}(\Gamma_{\mathfrak{A}})\\ \pi(g)\in\pi(H)}} \mathfrak{A}_g$, which is a $\mathsf{G}\times\mathbb{Z}_2$-graded subalgebra of $\mathfrak{A}$.

Lemma 4.5.1 *Let H be a finite abelian subgroup of a group G, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A} = \mathfrak{B}\oplus\mathsf{J}$ a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded unitary algebra, where $\mathfrak{B} = \mathbb{F}^\sigma[H]$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$. If $(\mathsf{E}^\mathsf{G}(\mathfrak{A}))_e$ is central in $\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)})$, then*

$$\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)}) \equiv_{\mathsf{G}PI} \mathsf{E}^\mathsf{G}(\mathbb{F}^\sigma[H]).$$

Proof: By Lemma 4.4.6, there exists a nilpotent $\mathsf{G}\times\mathbb{Z}_2$-graded algebra $\hat{\mathsf{N}}$ contained in $\mathsf{J}_e = \mathsf{J}_{(e,0)}\oplus\mathsf{J}_{(e,1)}$, such that $\mathsf{J}_{\pi(H)} = \mathfrak{B}\hat{\mathsf{N}}$. Since $(\mathsf{E}^\mathsf{G}(\mathfrak{A}))_e \subseteq \mathcal{Z}(\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)}))$, it follows from Lemma 4.4.6 (also Lemma 4.4.1) that $\hat{\mathsf{N}}$ is super-central in $\mathfrak{A}_{\pi(H)}$.

It is immediate that $\mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)})) \subseteq \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathfrak{B}))$, because $\mathsf{E}^\mathsf{G}(\mathfrak{B})$ is a graded subalgebra of $\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)})$. Thus, to prove that $\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)}) \equiv_{GPI} \mathsf{E}^\mathsf{G}(\mathbb{F}^\sigma[H])$ is sufficient to show that if $f \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)}))$, then $f \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathfrak{B}))$.

Let $f = f(z_1,\ldots,z_k) \notin \mathsf{T}^\mathsf{G}(\mathsf{E}^\mathsf{G}(\mathfrak{A}_{\pi(H)}))$ be a G-graded polynomial. If $\mathsf{char}(\mathbb{F}) = 0$, we can assume, without loss of generality, that f is multilinear, and hence, we can take homogeneous elements $a_1\otimes x_1,\ldots,a_k\otimes x_k \in \mathsf{E}^\mathsf{G}(\mathfrak{B})_0 \cup \mathsf{E}^\mathsf{G}(\mathfrak{B})_1 \cup \mathsf{E}^\mathsf{G}(\mathsf{J}_{\pi(H)})_0 \cup \mathsf{E}^\mathsf{G}(\mathsf{J}_{\pi(H)})_1$ such that $f(a_1\otimes x_1,\ldots,a_k\otimes x_k) \neq 0$. If $a_1,\ldots,a_k \in \mathfrak{B}$, the result is obvious. If for some $i = 1,\ldots,k$ we have that $a_i \in \mathsf{J}_{\pi(H)} = \mathfrak{B}\hat{\mathsf{N}}$, then $a_i = \sum_{j=1}^r b_{ij}c_{ij}$, where $b_{ij} \in \mathfrak{B}$, $c_{ij} \in \hat{\mathsf{N}}$. Since f is multilinear, without loss of generality, we can assume that $a_i = b_ic_i$, where $b_i \in \mathfrak{B}$, and $c_i \in \hat{\mathsf{N}}$. For all $i = 1,\ldots,k$, write $a_i = b_ic_i$ with $b_i \in \mathfrak{B}$, and
$$c_i = \begin{cases} \sigma((e,0),(e,0))^{-1}\eta_{(e,0)}, & \text{if } a_i \notin \mathsf{J}_{\pi(H)} \\ c_i \in \hat{\mathsf{N}}, & \text{if } a_i \in \mathsf{J}_{\pi(H)} \end{cases}.$$
Note that $\mathsf{deg}_\mathsf{G}(b_i) = \mathsf{deg}_\mathsf{G}(a_i)$, since $\mathsf{deg}_\mathsf{G}(c_i) = e$ in any case. We always can assume that $x_i = \tilde{x}_iy_i$, where $\mathsf{deg}_{\mathbb{Z}_2}(y_i) = \mathsf{deg}_{\mathbb{Z}_2}(c_i)$, $\mathsf{deg}_{\mathbb{Z}_2}(\tilde{x}_i) = \mathsf{deg}_{\mathbb{Z}_2}(b_i)$, and $x_i,\tilde{x}_i,y_i \in \mathsf{E}_0\cup\mathsf{E}_1$. Hence $a_i\otimes x_i = (b_ic_i)\otimes(\tilde{x}_iy_i) = (b_i\otimes\tilde{x}_i)(c_i\otimes y_i)$ for all $i = 1,\ldots,k$. It is clear

that $b_i \otimes \tilde{x}_i, c_i \otimes y_i \in \mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{\pi(H)})$. Since $\hat{\mathsf{N}} \subseteq \mathsf{J}_e$ is super-central in $\mathfrak{A}$, we have that $\mathsf{E}^{\mathsf{G}}(\hat{\mathsf{N}})$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{\pi(H)})$. It follows that

$$\begin{aligned}(a_{\alpha(1)} \otimes x_{\alpha(1)}) \cdots (a_{\alpha(k)} \otimes x_{\alpha(k)}) &= ((b_{\alpha(1)}c_{\alpha(1)}) \otimes x_{\alpha(1)}) \cdots ((b_{\alpha(k)}c_{\alpha(k)}) \otimes x_{\alpha(k)}) \\ &= (b_{\alpha(1)} \otimes \tilde{x}_{\alpha(1)})(c_{\alpha(1)} \otimes y_{\alpha(1)}) \cdots (b_{\alpha(k)} \otimes \tilde{x}_{\alpha(k)})(c_{\alpha(k)} \otimes y_{\alpha(k)}) \\ &= \left((c_{\alpha(1)} \otimes y_{\alpha(1)}) \cdots (c_{\alpha(k)} \otimes y_{\alpha(k)})\right) \left((b_{\alpha(1)} \otimes \tilde{x}_{\alpha(1)}) \cdots (b_{\alpha(k)} \otimes \tilde{x}_{\alpha(k)})\right) \\ &= ((c_1 \otimes y_1) \cdots (c_k \otimes y_k)) \left((b_{\alpha(1)} \otimes \tilde{x}_{\alpha(1)}) \cdots (b_{\alpha(k)} \otimes \tilde{x}_{\alpha(k)})\right) \\ &= ((c_1 \cdots c_k) \otimes (y_1 \cdots y_k)) \left((b_{\alpha(1)} \otimes \tilde{x}_{\alpha(1)}) \cdots (b_{\alpha(k)} \otimes \tilde{x}_{\alpha(k)})\right),\end{aligned}$$

for all $\alpha \in S_k$. From this, we deduce that

$$0 \neq f(a_1 \otimes x_1, \ldots, a_k \otimes x_k) = ((c_1 \cdots c_k) \otimes (y_1 \cdots y_k)) \, (f(b_1 \otimes \tilde{x}_1, \ldots, b_k \otimes \tilde{x}_k)), \quad (4.6)$$

where $\mathsf{deg}_{\mathsf{G}}(a_i \otimes x_i) = \mathsf{deg}_{\mathsf{G}}(b_i \otimes \tilde{x}_i)$, for all $i = 1, \ldots, k$. By (4.6), it follows that $f \notin \mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{B}))$, and the result follows. ■

An immediate consequence of the previous lemma holds when $\pi(\mathsf{Supp}(\Gamma_{\mathsf{N}})) \subseteq \pi(H)$, in particular, if $\mathsf{N} = \mathsf{N}_{(e,0)} \oplus \mathsf{N}_{e,1)}$. In this case, it follows from Lemma 4.5.1 that $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{GPI} \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H])$.

Corollary 4.5.2 *Let H be a subgroup of a finite abelian group G, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ a finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra, where $\mathfrak{B} = \mathbb{F}^{\sigma}[H]$, and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$. If $\mathfrak{A} = \mathfrak{A}_{\pi(H)}$ and $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, then*

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]) \ .$$

In addition, if $H \leqslant (\mathsf{G} \times \{0\})$, then $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathbb{F}^{\tilde{\sigma}}[\pi(H)]$, for some $\tilde{\sigma} \in \mathsf{Z}^2(\pi(H), \mathbb{F}^*)$.

Proof: The first part follows of Lemma 4.5.1, since $\mathfrak{A} = \mathfrak{A}_{\pi(H)}$.

Now, let us prove the second part of the lemma. It is easy to see that

$$\mathfrak{B} = \mathbb{F}^{\sigma}[H] = \mathbb{F}^{\sigma}[\pi(H) \times \{0\}] \cong_{\mathsf{G}} \mathbb{F}^{\tilde{\sigma}}[\pi(H)] \ ,$$

where $\tilde{\sigma} \in \mathsf{Z}^2(\pi(H), \mathbb{F}^*)$ is defined by map $\tilde{\sigma}(h, g) = \sigma((h, 0), (g, 0))$ for any $g, h \in \pi(H)$

(see Remark 1.2.9). The result follows. ■

It is important to note that if G has an odd order, then any subgroup H of G, which does not contain $(e,1)$ is necessarily of the form $\pi(H)\times\{0\}$. For more details, see Lemma 4.4.10, in Section 4.4.2.

The next result is a combination (immediate consequence) of Lemma 4.5.1 and Corollary 4.5.2.

Corollary 4.5.3 *Suppose that $\mathbb{F}$ is an algebraically closed field with $\mathsf{char}(\mathbb{F})=0$, and $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$. If $H\leqslant(\mathsf{G}\times\{0\})$, then*

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{\pi(H)})\equiv_{\mathsf{G}PI}\mathbb{F}^{\sigma}[H]\equiv_{\mathsf{G}PI}\mathbb{F}^{\tilde{\sigma}}[\pi(H)]$$

for some $\tilde{\sigma}\in\mathsf{Z}^2(\pi(H),\mathbb{F}^*)$.

Proof: The $\mathsf{G}PI$-equivalence $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{\pi(H)})\equiv_{\mathsf{G}PI}\mathbb{F}^{\sigma}[H]$ follows from Lemma 4.5.1, and the fact that $H=\pi(H)\times\{0\}$ (because $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H])=\mathbb{F}^{\sigma}[H]\otimes_{\mathbb{F}}\mathsf{E}_0$, where E_0 is commutative non-nilpotent). Now, the $\mathsf{G}PI$-equivalence $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_{\pi(H)})\equiv_{\mathsf{G}PI}\mathbb{F}^{\tilde{\sigma}}[\pi(H)]$ follows from Corollary 4.5.2. ■

Finally, let us now combine all the above results into a unique theorem.

Theorem 4.5.4 *Let G be a finite abelian group, H a subgroup of $\mathsf{G}\times\mathbb{Z}_2$, $\sigma\in\mathsf{Z}^2(H,\mathbb{F}^*)$, and $\mathbb{F}$ an algebraically closed field of characteristic zero. Let $\mathfrak{A}=\mathbb{F}^{\sigma}[H]\oplus\mathsf{J}$ be a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded unitary algebra, with the semisimple part $\mathfrak{B}=\mathbb{F}^{\sigma}[H]$, and J is the Jacobson radical of $\mathfrak{A}$. Suppose that one of the following hypotheses is true:*

1) $\mathfrak{A}=\mathfrak{A}_{\pi(H)}$;

2) $H=\mathsf{G}\times\{0\}$;

3) $\pi(\mathsf{Supp}(\Gamma_{\mathsf{J}}))\subseteq\pi(H)$;

4) $H\leqslant\mathsf{G}\times\{0\}$ and $(\mathsf{Supp}(\Gamma_{\mathsf{J}}))\subseteq\pi(H)$;

If $(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}))_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$, then J is generated as a $\mathsf{G}\times\mathbb{Z}_2$-graded $\mathfrak{B}$-bimodule by a nilpotent subalgebra $\hat{\mathsf{N}}$ of J, which is super-central in $\mathfrak{A}$, and

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A})\equiv_{\mathsf{G}PI}\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H])\ .$$

In particular, in the cases 2) and 4) we have that

$$\mathsf{E}^{\mathsf{G}}(\mathfrak{A}) \equiv_{\mathsf{G}PI} \mathbb{F}^{\sigma}[H] \equiv_{\mathsf{G}PI} \mathbb{F}^{\tilde{\sigma}}[\pi(H)] \ ,$$

for some $\tilde{\sigma} \in \mathsf{Z}^2(\pi(H), \mathbb{F}^*)$.

Proof: All the four cases follow of Lemma 4.5.1 (or Corollary 4.5.2), because all them imply that $\mathfrak{A}_{\pi(H)} = \mathfrak{A}$.

The existence of $\hat{\mathsf{N}}$ satisfying the claims of theorem is ensured by Lemma 4.4.6.

Finally, the last affirmation, about the items *2)* and *4)*, is ensured by Corollary 4.5.3. The result follow. ■

Under the hypotheses of Theorem 4.5.4, we have that $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]))$.

It is important to note that $\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]) \in \mathfrak{V}^{\mathsf{G}}$, where $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}([x^{(e)}, y^{(g)}] : g \in \mathsf{G})$, for any $H \leqslant \mathsf{G} \times \{0\}$, and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$. And thus, $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H])) \subseteq \mathfrak{V}^{\mathsf{G}}$, for any $H \leqslant \mathsf{G} \times \{0\}$, and $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$.

Also by the Theorem 4.5.4, supposing that the item *4)* (or item *2)*) is true, $\mathbb{F}$ is an algebraically closed field, and G is finite abelian, by Lemma 4.3.3, there is $\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ which extends $\tilde{\sigma}$ such that

$$\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathbb{F}^{\gamma}[\mathsf{G}]) \subseteq \mathfrak{V}^{\mathsf{G}} \ .$$

Applying Lemma 4.3.3 and Theorem 4.5.4 together with the results of Subsection 4.4.2, we can complete the above observation. In fact, supposing that the item *1)* (or item *3)*) is true, it follows from the previous theorem that $\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]))$. If there exists a subgroup $\tilde{H}$ of $\mathsf{G} \times \mathbb{Z}_2$ such that $H \subseteq \tilde{H}$ and $(e, 1) \notin \tilde{H}$ (for more details, see Subsection 4.4.2), then

$$\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\tilde{\sigma}}[\tilde{H}])) \subseteq \mathfrak{V}^{\mathsf{G}}$$

for some $\tilde{\sigma} \in \mathsf{Z}^2(\tilde{H}, \mathbb{F}^*)$. It is worth to note that $[\mathsf{G} \times \mathbb{Z}_2 : \tilde{H}]$ is at least 2, since $(e, 1) \notin \mathsf{Supp}(\Gamma_{\mathfrak{A}})$.

The last observation is important because in various classes of groups we can always extend a group $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$, which does not contain $(e, 1)$ to a subgroup $\tilde{H} \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ of index

2, which also does not contain $(e,1)$. For more details, see Lemma 4.4.10 and Subsection 4.4.2 of this chapter.

As observed at the beginning of this chapter, module graded polynomial identity, to study the graded variety $\mathfrak{V}^{\mathsf{G}}$, of all G-graded algebras whose neutral component is central, is equivalent to studying a graded variety generated by the Grassmann envelope of a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded algebra $\mathfrak{A}$ which can be written as

$$(\mathbb{F}^{\sigma_1}[H_1]\oplus\mathsf{J}_1)\times\cdots\times(\mathbb{F}^{\sigma_k}[H_k]\oplus\mathsf{J}_k)\times\mathsf{J}_{00}\ ,$$

where for all $r=1,\ldots,k$, H_r is a subgroup of $\mathsf{G}\times\mathbb{Z}_2$, $\sigma_r\in\mathsf{Z}^2(H_r,\mathbb{F}^*)$, and $\mathsf{J}_r=(\mathbb{F}^{\sigma_r}[H_r])\mathsf{N}_r$ is the Jacobson radical of $\mathfrak{A}_r=\mathbb{F}^{\sigma_r}[H_r]\oplus\mathsf{J}_r$ for some graded subspace $\mathsf{N}_r\subset\mathsf{J}_r$, and J_{00} is a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded nilpotent algebra, such that $\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})$ belongs to $\mathfrak{V}^{\mathsf{G}}$. We have that $\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})$ is also nilpotent, and $\mathsf{nd}(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00}))=\mathsf{nd}(\mathsf{J}_{00})$. Therefore, we have

$$\mathsf{var}^{\mathsf{G}}\mathfrak{V}=\bigcap_{r=1}^{k}\mathsf{var}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_r)\right)\cap\mathsf{var}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\mathsf{J}_{00})\right)\ .$$

The results obtained before describe the algebras $\mathfrak{A}_r$'s when $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_r)_e$ is central in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A}_r)$.

Theorem 4.5.5 *Let G be a finite abelian group, and $\mathbb{F}$ an algebraically closed field of characteristic zero. There exists a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded unitary algebra*

$$\mathsf{C}_{\mathsf{G}}=\bigtimes_{\substack{H\trianglelefteq\mathsf{G}\times\mathbb{Z}_2\\(e,1)\notin H}}\left(\bigtimes_{[\sigma]\in\mathsf{H}^2(H,\mathbb{F}^*)}\left(\mathbb{F}^{\sigma}[H]\oplus\mathsf{J}_{(H,[\sigma])}\right)\right)\ ,\tag{4.7}$$

such that $\mathsf{J}_{(H,[\sigma])}$ is a finite dimensional $\mathsf{G}\times\mathbb{Z}_2$-graded nilpotent algebra ($\mathsf{J}_{(H,[\sigma])}$ is the Jacobson radical of $\mathfrak{A}_{(H,[\sigma])}:=\mathbb{F}^{\sigma}[H]\oplus\mathsf{J}_{(H,[\sigma])}$), satisfying

$$\mathfrak{V}^{\mathsf{G}}=\mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}}))\ .$$

Proof: The idea of the proof is similar to the proof of Theorem 4.3.14.

Observe that $\mathfrak{V}^{G}$ is a variety of PI-algebras, since if $\mathfrak{A}\in\mathfrak{V}^{\mathsf{G}}$, then $\mathfrak{A}_e$ is central in $\mathfrak{A}$, and hence, $\mathfrak{A}_e$ is commutative, and so, by Theorem 1.4.12, $\mathfrak{A}$ is a PI-algebra. By Theorems 1.4.13 and 1.2.20 and expression in (4.4), consider the $\mathsf{G}\times\mathbb{Z}_2$-graded algebras

$\tilde{\mathfrak{A}}_1, \tilde{\mathfrak{A}}_2, \ldots, \tilde{\mathfrak{A}}_p, \tilde{\mathsf{J}}_{00}$ such that

$$\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}\left(\mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}_1) \times \cdots \times \mathsf{E}^{\mathsf{G}}(\tilde{\mathfrak{A}}_k) \times \mathsf{E}^{\mathsf{G}}(\tilde{\mathsf{J}}_{00})\right) \ ,$$

where $\tilde{\mathfrak{A}}_s = \overline{\mathfrak{B}}_s \oplus \tilde{\mathsf{J}}_s$ are finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebras such that $\overline{\mathfrak{B}}_s \not\cong_{\mathsf{G}\times\mathbb{Z}_2} \overline{\mathfrak{B}}_r$ for all $s \neq r$, where $\overline{\mathfrak{B}}_s = \mathbb{F}^{\sigma_s}[H_s]$, with $H_s \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$, and $\sigma_s \in \mathsf{Z}^2(H_s, \mathbb{F}^*)$, for all $s \in \{1, \ldots, k\}$, $\tilde{\mathsf{J}}_s$ is the Jacobson radical of $\tilde{\mathfrak{A}}_s$, and $\tilde{\mathsf{J}}_{00}$ is a finite dimensional nilpotent $\mathsf{G} \times \mathbb{Z}_2$-graded algebra.

Now, consider $\tilde{\mathsf{C}}_{\mathsf{G}} = \bigtimes_{\substack{H \trianglelefteq \mathsf{G}\times\mathbb{Z}_2 \\ (e,1)\notin H}} \left(\bigtimes_{[\sigma]\in\mathsf{H}^2(H,\mathbb{F}^*)} (\mathbb{F}^{\sigma}[H]) \right)$. By Absorption Lemma (Corollary 1.5.9), for any $H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2$ such that $(e, 1) \notin H$, and $[\sigma] \in \mathsf{H}^2(H, \mathbb{F}^*)$, there exists a finite dimensional G-graded nilpotent algebra $\mathsf{J}_{(H,[\sigma])}$ such that

$$\left(\bigtimes_{j=1}^{k} \tilde{\mathfrak{A}}_j \right) \times \tilde{\mathsf{J}}_{00} \times \tilde{\mathsf{C}}_{\mathsf{G}} \equiv_{(\mathsf{G}\times\mathbb{Z}_2)PI} \mathsf{C}_{\mathsf{G}} \times \tilde{\tilde{\mathsf{J}}}_{00} \ ,$$

where C_{G} is defined in (4.7), and $\tilde{\tilde{\mathsf{J}}}_{00}$ is some finite dimensional $\mathsf{G} \times \mathbb{Z}_2$-graded nilpotent algebra. Consequently, it follows from Lemma 1.5.11 that

$$\mathfrak{V}^{\mathsf{G}} \subseteq \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}} \times \tilde{\tilde{\mathsf{J}}}_{00})) \ .$$

Observe that $\tilde{\mathsf{C}}_{\mathsf{G}} \in \mathfrak{V}^{\mathsf{G}}$, and hence, we have that $\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}}) \times \mathsf{E}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00}) = \mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}} \times \tilde{\tilde{\mathsf{J}}}_{00}) \in \mathfrak{V}^{\mathsf{G}}$. Particularly, $\mathsf{E}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00}, \mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}}) \in \mathfrak{V}^{\mathsf{G}}$. Therefore, we conclude that $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}}) \times \mathsf{E}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00}))$.

Since $\mathfrak{V}^{\mathsf{G}}$ can be generated by the Grassmann envelope of a $\mathsf{G} \times \mathbb{Z}_2$-graded unitary algebra, $\mathsf{nd}(\mathsf{E}^{\mathsf{G}}(\tilde{\tilde{\mathsf{J}}}_{00}))$ is nilpotent, similarly to the proof of Theorem 4.3.14, we deduce that $\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}}))$. ■

Remark 4.5.6 *In (4.7) we can suppose that either* $\pi(\mathsf{Supp}(\Gamma_{\mathsf{J}_{(H,[\sigma])}})) \nsubseteq \pi(H)$ *or* $\mathsf{J}_{(H,[\sigma])} = \{0\}$ *(see Lemma 4.5.4). Besides that, if* $\mathsf{J}_{(H,[\sigma])} = \{0\}$ *for some* $H \trianglelefteq \mathsf{G} \times \{0\}$*, namely* $H = H_1 \times \{0\}$*, and* $\sigma \in \mathsf{Z}^2(H, \mathbb{F}^*)$*, then*

$$\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\sigma}[H]) \times \mathsf{E}^{\mathsf{G}}(\mathbb{F}^{\hat{\sigma}}[\mathsf{G} \times \{0\}]) = \mathsf{T}^{\mathsf{G}}(\mathbb{F}^{\sigma_1}[H_1] \times \mathbb{F}^{\tilde{\sigma}_1}[\mathsf{G}]) = \mathsf{T}^{\mathsf{G}}(\mathbb{F}^{\tilde{\sigma}_1}[\mathsf{G}]) \ ,$$

where $\hat{\sigma} \in \mathsf{Z}^2(\mathsf{G} \times \{0\}, \mathbb{F}^*)$ *extends* σ *(see Lemma 4.3.3),* $\sigma_1 \in \mathsf{Z}^2(H_1, \mathbb{F}^*)$ *is defined by* $\sigma_1(h, h_1) = \sigma((h, 0), (h_1, 0))$ *for any* $h, h_1 \in H$*, and* $\tilde{\sigma}_1 \in \mathsf{Z}^2(\mathsf{G}, \mathbb{F}^*)$ *extends* σ_1*.*

Corollary 4.5.7 *Under all the conditions of Theorem 4.5.5, if* $\mathfrak{A} \in \mathfrak{V}^{\mathsf{G}}$*, then* $\mathsf{T}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathsf{C}_{\mathsf{G}})) \subseteq \mathsf{T}^{\mathsf{G}}(\mathfrak{A})$*.*

Corollary 4.5.8 *Under all the conditions of Theorem 4.5.5, if* G *is a finite cyclic group, then* $\mathsf{C}_{\mathsf{G}} = \bigtimes_{\substack{H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2 \\ (e,1) \notin H \\ H \neq \mathsf{G} \times \{0\}}} \left(\mathbb{F}[H] \oplus \mathsf{J}_{(H,[1])}\right) \times \mathbb{F}[\mathsf{G} \times \{0\}]$*, and*

$$\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}} \left(\bigtimes_{\substack{H \trianglelefteq \mathsf{G} \times \mathbb{Z}_2 \\ (e,1) \notin H \\ H \neq \mathsf{G} \times \{0\}}} \mathsf{E}^{\mathsf{G}} \left(\mathbb{F}[H] \oplus \mathsf{J}_{(H,[1])}\right) \right) \cap \mathsf{var}^{\mathsf{G}}(\mathbb{F}\mathsf{G}) \ ,$$

where $[1]$ *is the class of the trivial 2-cocycle of* $\mathsf{Z}^2(H, \mathbb{F}^*)$*, and* $\mathsf{J}_{(H,[1])}$ *is a finite dimensional nilpotent* $\mathsf{G} \times \mathbb{Z}_2$ *algebra.*

Corollary 4.5.9 *Under all the conditions of Theorem 4.5.5, if* G *is a finite cyclic group of a prime order* $p > 2$*, then* $\mathsf{C}_{\mathsf{G}} = \mathsf{J}^{\#}_{(\{(e,0)\},[1])} \times \mathbb{F}[\mathsf{G} \times \{0\}]$*, and*

$$\mathfrak{V}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}} \left(\mathsf{E}^{\mathsf{G}} \left(\mathsf{J}^{\#}_{(\{(e,0)\},[1])} \right) \right) \bigcap \mathsf{var}^{\mathsf{G}}(\mathbb{F}\mathsf{G}) \ ,$$

where $[1]$ *is the class of the trivial 2-cocycle of* $\mathsf{Z}^2(H, \mathbb{F}^*)$*, and* $\mathsf{J}^{\#}_{(\{(e,0)\},[1])}$ *is a finite dimensional nilpotent* $\mathsf{G} \times \mathbb{Z}_2$ *algebra with adjoint unity.*

If $\mathsf{G} = \mathbb{Z}_2$ $(p = 2)$, then $\mathsf{C}_{\mathbb{Z}_2} = \mathsf{J}^{\#}_{(\{(e,0)\},[1])} \times \mathbb{F}[\mathbb{Z}_2 \times \{0\}] \times \mathbb{F}[\{(0,0),(1,1)\}]$, and $\mathfrak{V}^{\mathbb{Z}_2} = \mathsf{var}^{\mathbb{Z}_2}(\mathsf{E}^{\mathbb{Z}_2}(\mathsf{J}^{\#}_{(\{(e,0)\},[1])})) \cap \mathsf{var}^{\mathbb{Z}_2}(\mathbb{F}[\mathbb{Z}_2]) \cap \mathsf{var}^{\mathbb{Z}_2}(\mathsf{E})$, where $\mathsf{E} = \mathsf{E}_0 \oplus \mathsf{E}_1$ is the infinite dimensional Grassmann algebra with the canonical $\mathbb{Z}_2$-grading.

4.6 Graded algebras with the neutral component satisfying a polynomial identity of degree 2

Let G be a finite group, and $\mathfrak{A}$ a G-graded algebra. In this section, we present a study about the general case when the neutral component $\mathfrak{A}_e$ satisfies a polynomial identity of degree 2. A polynomial f of degree 2 in the variables $x_1, \dots, x_n$ is a polynomial in $\mathbb{F}\langle X \rangle$ of the form

$$f(x_1, \dots, x_n) = \sum_{r,s,k=1}^{n} \lambda_{rs} x_r x_s + \gamma_k x_k \ ,$$

where $\lambda_{rs}, \gamma_k \in \mathbb{F}$.

Let G be a finite group, and $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}(\mathfrak{A})$ a finite dimensional G-graded algebra, where $\mathfrak{B} = \times_{i=1}^{p} M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ and $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$. Assume that $\mathfrak{B}_i = M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ is graded with a canonical elementary G-grading. Suppose that f is a polynomial identity of degree 2 of $\mathfrak{A}_e$. We must analyse the two following situations:

i) $\mathfrak{A}_e$ is nilpotent;

ii) $\mathfrak{A}_e$ is not nilpotent.

Firstly, suppose that $\mathfrak{A}_e$ is a nilpotent algebra. By Theorem 3.2.15, in Chapter 3, we have that $\mathfrak{A}$ is nilpotent with $\mathsf{nd}(\mathfrak{A}_e) \leqslant \mathsf{nd}(\mathfrak{A}) \leqslant |G|\mathsf{nd}(\mathfrak{A}_e)$, and thus, $\mathfrak{A} = \mathsf{J}$. Let us assume that $\mathfrak{A}_e$ is not nilpotent (and also $\mathfrak{A}$). Hence, observe that $(\mathfrak{B}_i)_e \subseteq \mathfrak{A}_e$ for all $i = 1, \dots, p$. From this, if $\mathfrak{A}_e$ satisfies $f \equiv 0$, then $(\mathfrak{B}_i)_e$ satisfies $f \equiv 0$ for all $i = 1, \dots, p$.

Lemma 4.6.1 *Let G be a group, $\mathbb{F}$ a field, $f = f(x_1^{(e)}, \dots, x_n^{(e)}) \in \mathbb{F}\langle X^{\mathsf{G}} \rangle$ a polynomial of degree 2, and $\mathfrak{B} = M_n(\mathbb{F}^\sigma[H])$ the algebra of $n \times n$ matrices over $\mathbb{F}^\sigma[H]$ with a canonical elementary G-grading. Suppose that $\mathsf{char}(\mathbb{F}) \neq 2$, and $\mathfrak{B}_e$ satisfies $f \equiv 0$. Then f and $[x^{(e)}, y^{(e)}]$ generate the same $\mathsf{G}T$-ideal.*

Proof: Write $f(x_1^{(e)}, \dots, x_n^{(e)}) = \sum_{r,s,k=1}^{n} \lambda_{rs} x_r^{(e)} x_s^{(e)} + \gamma_k x_k^{(e)}$. We have that $E_{11}\eta_e \in \mathfrak{B}_e$, and so for $x_i = E_{11}\eta_e$, $x_j = 0$ for all $j \neq i$, we obtain

$$\begin{aligned} 0 = f(0, \dots, 0, E_{11}\eta_e, 0, \dots, 0) &= \lambda_{ii}(E_{11}\eta_e)^2 + \gamma_i(E_{11}\eta_e) \\ &= \lambda_{ii}\sigma(e,e)E_{11}\eta_e + \gamma_i E_{11}\eta_e = (\lambda_{ii}\sigma(e,e) + \gamma_i)E_{11}\eta_e \ . \end{aligned}$$

And for $x_i = -E_{11}\eta_e$, $x_j = 0$, for all $j \neq i$, we obtain

$$\begin{aligned} 0 = f(0, \dots, 0, -E_{11}\eta_e, 0, \dots, 0) &= \lambda_{ii}(-E_{11}\eta_e)^2 + \gamma_i(-E_{11}\eta_e) \\ &= \lambda_{ii}\sigma(e,e)E_{11}\eta_e - \gamma_i E_{11}\eta_e = (\lambda_{ii}\sigma(e,e) - \gamma_i)E_{11}\eta_e \ , \end{aligned}$$

and hence, $\lambda_{ii}\sigma(e,e)+\gamma_i=0$ and $\lambda_{ii}\sigma(e,e)-\gamma_i=0$ for all $i=1,\ldots,p$. Thus, it follows that $\lambda_{ii}=0$ for all $i=1,\ldots,p$, and consequently, $\gamma_i=0$ for all $i=1,\ldots,p$. From this, we have that $f(x_1^{(e)},\ldots,x_n^{(e)})=\sum\limits_{\substack{r,s=1\\ r\neq s}}^{n}\lambda_{rs}x_r^{(e)}x_s^{(e)}$.

Finally, we have for the evaluation $x_i=E_{11}\eta_e$, $x_j=E_{11}\eta_e$, $x_k=0$, for all $k\neq i,j$ (for any pair $i,j\in\{1,\ldots,n\}$), $i\neq j$

$$\begin{aligned}0=f(0,\ldots,0,E_{11}\eta_e,0,\ldots,0,E_{11}\eta_e,0,\ldots,0)&=\lambda_{ij}(E_{11}\eta_e)(E_{11}\eta_e)+\lambda_{ji}(E_{11}\eta_e)(E_{11}\eta_e)\\ &=(\lambda_{ij}+\lambda_{ji})\sigma(e,e)E_{11}\eta_e\ ,\end{aligned}$$

and hence, $\lambda_{ij}+\lambda_{ji}=0$ for all $i,j=1,\ldots,p$ distinct. Therefore, we conclude that

$$f(x_1^{(e)},\ldots,x_n^{(e)})=\sum_{1\leqslant r<s\leqslant n}\lambda_{rs}[x_r^{(e)},x_s^{(e)}]\ .$$

The result follows. ■

By the previous lemma, given a finite dimensional G-graded algebra $\mathfrak{A}=\mathfrak{B}\oplus\mathsf{J}$, where $\mathfrak{B}=\times_{i=1}^{p}M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ and $\mathsf{J}=\mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, and $\mathfrak{B}_i=M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ has a canonical elementary G-grading, if $\mathfrak{A}_e$ satisfies a graded polynomial identity $f=f(x_1^{(e)},\ldots,x_n^{(e)})$ of degree 2, then either $\mathfrak{A}$ is nilpotent, or f and $[x^{(e)},y^{(e)}]$ generate the same $\mathsf{G}T$-ideal.

Theorem 4.6.2 *Let G be a finite abelian group, $\mathbb{F}$ an algebraically closed field of characteristic zero, and $\mathfrak{A}$ a finitely generated G-graded algebra $\mathfrak{A}$. If $\mathfrak{A}_e$ satisfies a polynomial identity $f=f(x_1^{(e)},\ldots,x_n^{(e)})\in\mathbb{F}\langle X^{\mathsf{G}}\rangle$ of degree 2, then either $\mathfrak{A}$ is a nilpotent algebra, with $\mathsf{nd}(\mathfrak{A}_e)\leqslant\mathsf{nd}(\mathfrak{A})\leqslant\mathsf{nd}(\mathfrak{A}_e)|\mathsf{G}|$, or $\mathfrak{A}$ satisfies $[x^{(e)},y^{(e)}]$.*

Proof: Observe that since $\mathfrak{A}_e$ is PI-algebra, then $\mathfrak{A}$ is also PI-algebra (see Theorems 1.4.11 and 1.4.12). By Remark 1.4.10, there exists a finite dimensional G-graded algebra $\mathfrak{A}'=\mathfrak{B}\oplus\mathsf{J}$ such that $\mathsf{T}^{\mathsf{G}}(\mathfrak{A})=\mathsf{T}^{\mathsf{G}}(\mathfrak{A}')$, where $\mathsf{J}=\mathsf{J}(\mathfrak{A}')$ is the Jacobson radical of $\mathfrak{A}'$ (which is a graded ideal of $\mathfrak{A}'$), and

$$\mathfrak{B}=M_{n_1}(\mathbb{F}^{\sigma_1}[H_1])\times\cdots\times M_{n_p}(\mathbb{F}^{\sigma_p}[H_p])\ ,$$

with $H_i\leqslant\mathsf{G}$, $\sigma_i\in\mathsf{Z}^2(H_i,\mathbb{F}^*)$, $M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ is equipped with a canonical elementary G-grading, for any $i=1,\ldots,p$. Suppose that $\mathfrak{A}'$ is not nilpotent, i.e. $\mathfrak{B}\neq\{0\}$. Then

$(\mathfrak{B}_i)_e \subseteq \mathfrak{A}'_e$ for all $i = 1, \ldots, p$. The result follows from Lemma 4.6.1. ■

Now, assume that $\mathbb{F}$ is an algebraically close field of characteristic zero, and G is a finite abelian group, $f = f(x_1^{(e)}, \ldots, x_n^{(e)}) \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ is a polynomial identity of degree 2, and $\mathfrak{W}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(f)$. By Theorems 1.2.20 and 1.4.13, it follows that there exists a $\mathsf{G} \times \mathbb{Z}_2$-graded finite dimensional algebra $\mathfrak{A} = \mathfrak{B} \oplus \mathsf{J}$ such that

$$\mathfrak{W}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(\mathsf{E}^{\mathsf{G}}(\mathfrak{A})) ,$$

where $\mathsf{J} = \mathsf{J}(\mathfrak{A})$ is the Jacobson radical of $\mathfrak{A}$, and

$$\mathfrak{B} = M_{n_1}(\mathbb{F}^{\sigma_1}[H_1]) \times \cdots \times M_{n_p}(\mathbb{F}^{\sigma_p}[H_p]) ,$$

with $H_i \leqslant \mathsf{G} \times \mathbb{Z}_2$, $\sigma_i \in \mathsf{Z}^2(H_i, \mathbb{F}^*)$, $M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$ has a canonical elementary G-grading. Observe that $f \equiv 0$ is satisfied in $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})_e = \mathfrak{A}_{(e,0)} \otimes_{\mathbb{F}} \mathsf{E}_0 + \mathfrak{A}_{(e,1)} \otimes_{\mathbb{F}} \mathsf{E}_1$. In particular, $f \equiv 0$ is satisfied in $\mathfrak{A}_{(e,0)} \otimes_{\mathbb{F}} \mathsf{E}_0$. Hence, either $\mathfrak{A}_{(e,0)} = \mathsf{J}_{(e,0)}$, and it is nilpotent, or $(\mathfrak{B}_i)_{(e,0)} \subseteq \mathfrak{A}_{(e,0)}$ for all $i = 1, \ldots, p$, where $\mathfrak{B}_i = M_{n_i}(\mathbb{F}^{\sigma_i}[H_i])$. Observe that if $\mathfrak{B} \neq \{0\}$ ($\mathfrak{A} \neq \mathsf{J}$, i.e. $\mathfrak{A}$ is not nilpotent), then $\mathfrak{B}_{(e,0)} \neq \{0\}$, since $1_{\mathfrak{B}} \in \mathfrak{B}_{(e,0)}$. It means $\mathfrak{A} = \mathsf{J}$ is nilpotent if $\mathfrak{A}_{(e,0)} = \mathsf{J}_{(e,0)}$. In this case, $\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$ is also a nilpotent algebra.

Theorem 4.6.3 *Let G be a finite abelian group, $\mathbb{F}$ a algebraically closed field of characteristic zero, $f = f(x_1^{(e)}, \ldots, x_n^{(e)}) \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$ a polynomial degree 2, and $\mathfrak{W}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(f)$ the G-graded variety defined by f. Then either $\mathfrak{W}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}([x^{(e)}, y^{(e)}])$ or $\mathfrak{W}^{\mathsf{G}} = \mathsf{var}^{\mathsf{G}}(N)$ for some nilpotent algebra N.*

Proof: The result follows from above observations and Theorem 4.6.2. ■

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NOTATION

$< \infty$	finite order (or bounded upper)
$[\mathsf{G} : H]$	index of H in G
$[\sigma]$	$= \{\gamma \in \mathsf{Z}^2(\mathsf{G}, \mathsf{M}) : \sigma\gamma \in \mathsf{B}^2(\mathsf{G} : \mathsf{M})\}$
$[a, b]$	commutator of a and b — $[a, b] = ab - ba$
$[a, b]_{\mathsf{f}}$	f-commutator of a and b — $[a, b] = ab - f(a, b)ba$
$[a_1, a_2, \ldots, a_n]$	$= [[a_1, a_2], a_3, \ldots, a_n]$
$\#\{\beta\}$	number of elements of the set β
$\mathfrak{A}S$	left submodule of M
$\mathfrak{A}S\tilde{\mathfrak{A}}$	subbimodule of M
$\mathfrak{A}$	associative algebra
$\mathfrak{A} \otimes_{\mathbb{F}} \mathfrak{B}$	tensor product of $\mathbb{F}$-algebras
$\mathfrak{A}^{\#}$	algebra $\mathfrak{A}$ with adjoint unity
$\mathfrak{A}_g$	homogeneous component of $\mathfrak{A}$ of degree $g \in \mathsf{G}$
$\mathfrak{A}_{\pi(H)}$	$= \bigoplus_{\substack{g \in \mathsf{Supp}(\Gamma_{\mathfrak{A}}) \\ \pi(g) \in \pi(H)}} \mathfrak{A}_g$
$\bar{m}$	element $m + N$ which belongs to M/N
$\mathbb{C}$	Complex field
$\cap, \bigcap$	intersection of sets
$\cong$	isomorphism of algebras (or modules, or groups)

$\cong_{\mathsf{G}}$	G-graded isomorphism of algebras (or modules)
$\cup, \bigcup$	union of sets
δ_{ij}	Kronecker delta
$\mathsf{E} = \mathsf{E}_0 \oplus \mathsf{E}_1$	Grassmann algebra with its natural $\mathbb{Z}_2$-grading
$\mathsf{E}^{\mathsf{G}}(\mathfrak{A})$	G-graded Grassmann envelope of $\mathfrak{A}$
$\varnothing$	empty set
$\equiv_{\mathsf{G}PI}$	$\mathsf{G}PI$-equivalence of $\mathsf{G}PI$-algebras
$\equiv_{PI}$	PI-equivalence of PI-algebras
η_g	basic element of $\mathbb{F}\mathsf{G}$ (and also of $\mathbb{F}^\sigma[\mathsf{G}]$)corresponding to $g \in \mathsf{G}$
$\mathbb{F}$, $\mathbb{K}$	fields
$\mathbb{F}[x_1, x_2, x_3, \dots]$	polynomial ring over $\mathbb{F}$
$\mathbb{F}\mathsf{G}$	group algebra
$\mathbb{F}\langle X\rangle$	free associative algebra over $\mathbb{F}$ generated by X
$\mathbb{F}\langle X^{\mathsf{G}}\rangle$	free G-graded associative algebra over $\mathbb{F}$ generated by X^{G}
$\mathbb{F}^*$	$= \mathbb{F} - \{0\}$ — multiplicative group of $\mathbb{F}$
$\mathbb{F}^\sigma[\mathsf{G}]$	twisted group algebra
$\mathsf{Ann}_{\mathfrak{A}}(\mathsf{M})$	$= \{a \in \mathfrak{A} : am = 0, \forall m \in \mathsf{M}\}$
$\mathsf{B}^2(\mathsf{G}, \mathsf{M})$	group of all the 2-coboundaries of G with coefficients in M $(= \mathbb{F}^*)$
$\mathsf{char}(\mathbb{F})$	characteristic of field $\mathbb{F}$
$\mathsf{cores}_H^{\mathsf{G}}$	corestriction map
$\mathsf{deg}(a)$	homogeneous degree of $a \in \mathfrak{A}$
$\mathsf{deg}(r)$	homogeneous degree of $r \in \mathfrak{R}$
$\mathsf{dim}_{\mathbb{F}}(\mathsf{V})$	dimension over $\mathbb{F}$ of vector space V
e	Euler number
gcd	greatest common divisor
$\mathsf{H}^2(\mathsf{G}, \mathsf{M})$	second cohomology group of G with coefficients in M $(= \mathbb{F}^*)$
$\mathsf{H}^2(H, \mathsf{M})^{\mathsf{G}}$	subgroup of $\mathsf{H}^2(H, \mathsf{M})$ of points fixed by G
$\mathsf{im}(\psi)$	image of homormophism ψ
$\mathsf{ker}(\psi)$	kernel of homormophism ψ

lcm	least common multiple
$\mathsf{max}\{x \in X\}$	$= y \in X$ such that $x \leqslant y$ for any $x \in X$
$\mathsf{min}\{x \in X\}$	$= y \in X$ such that $x \geqslant y$ for any $x \in X$
M	left $\mathfrak{A}$-module, right $\mathfrak{A}$-module, $(\mathfrak{A}, \tilde{\mathfrak{A}})$-bimodule, or $\mathfrak{A}$-bimodule
M/N	quotient module
M^H	submodule of M of points fixed by H
$\mathsf{nd}(\mathfrak{R})$	nilpotency order of $\mathfrak{R}$
$\mathsf{nd}_{nil}(\mathfrak{R})$	the smallest number $r \in \mathbb{N}$ such that $a^r = 0$ for any $a \in \mathfrak{R}$
$\mathsf{o}(g)$	order of element $g \in \mathsf{G}$
$\mathsf{res}_H^{\mathsf{G}}$	restriction map
$\mathsf{res}_H^{\mathsf{G}}(\mathsf{H}^2(\mathsf{G}, \mathsf{M}))$	subset of $\mathsf{H}^2(H, \mathsf{M})$ of all σ_H, where $\sigma \in \mathsf{H}^2(\mathsf{G}, \mathsf{M})$
$\mathsf{res}_H^{\mathsf{G}}(\sigma)$	$= \sigma_H \in \mathsf{H}^2(H, \mathsf{M})$, where $\sigma \in \mathsf{H}^2(\mathsf{G}, \mathsf{M})$
$\mathsf{span}_{\mathbb{F}}\{v \in N\}$	vector $\mathbb{F}$-subspace generated by N
$\mathsf{Supp}(\Gamma)$	support of G-grading Γ
S	left cancellative monoid
S/P	quotient group
tr	trace function
$\mathsf{var}^{\mathsf{G}}(f)$	G-graded variety defined by $f \subset \mathbb{F}\langle X^{\mathsf{G}}\rangle$
$\mathsf{var}^{\mathsf{G}}(S)$	G-graded variety defined by $S \subset \mathbb{F}\langle X^{\mathsf{G}}\rangle$
$\mathsf{Z}^2(\mathsf{G}, \mathsf{M})$	group of all the 2-cocycles of G with coefficients in M $(= \mathbb{F}^*)$
G	group
$\mathsf{G} \times \mathbb{Z}_2$	direct product of G and $\mathbb{Z}_2$
G^n	$= \underbrace{\mathsf{G} \times \cdots \times \mathsf{G}}_{n-times}$
$\Gamma, \Gamma_{\mathfrak{A}}$	G-gradings on $\mathfrak{A}$
$\hat{\mathsf{G}}$	$= \{\chi_1, \ldots, \chi_n\}$ — group of irreducible characters of G
$\mathfrak{i}$	unity of algebra of matrices $M_n(\mathbb{F}^\sigma[H])$
$\mathsf{J} = \mathsf{J}(\mathfrak{A})$	Jacobson radical of an algebra $\mathfrak{A}$
J_{00}	nilpotent algebra, and a 0-bimodule
λ, γ	elements of $\mathbb{F}$

$\langle S\rangle_{\mathsf{G}T}$	$\mathsf{G}T$-ideal generated by the subset $S \in \mathbb{F}\langle X^{\mathsf{G}}\rangle$
$\langle S\rangle_T$	T-ideal generated by the subset $S \in \mathbb{F}\langle X\rangle$
$\mathcal{Z}(X)$	center of X
$\mathcal{Z}_X(Y)$	center of Y in X
$\mathfrak{S}$	semigroup
$\mid$	$a \mid b$ means b is divisible by a
$\mathbb{N}$	Natural numbers
$\nmid$	$a \nmid b$ means b is not divisible by a
$\ncong$	there is no an isomorphism
$\ncong_{\mathsf{G}}$	there is no a G-graded isomorphism
$\oplus$, $\bigoplus$	direct sum of vector subspaces
$\otimes$, $\otimes_{\mathbb{F}}$	tensor products of $\mathbb{F}$-algebras
π	the projection map of $\mathsf{G} \times \mathbb{Z}_2$ to G
$\pi(H)$	$= \{g \in \mathsf{G} : g \in \mathsf{im}(\pi)\}$
$\mathfrak{R}$	associative ring
$\mathfrak{R}_s$	homogeneous component of $\mathfrak{R}$ of degree $s \in \mathsf{S}$
σ	2-cocycle
$\sigma(g,h)$	2-cocycle σ applied in $(g,h) \in \mathsf{G} \times \mathsf{G}$
σ_H	restriction of σ to H
$\sqrt[n]{1}$	$= \{a \in \mathbb{F} : a^n = 1\}$
$\sqrt[n]{\lambda}$	$= a \in \mathbb{F}$ such that $a^n = \lambda$
$\overset{\mathsf{G}}{\hookrightarrow}$	G-graded immersion
$\overset{\mathsf{G}}{\not\hookrightarrow}$	there is no a G-graded immersion
$\subseteq_\infty$	asymptotically contained
$\subsetneq$	$X \subsetneq Y$ means $X \subset Y$ and $X \neq Y$
$\sum$	sum
$\mathsf{T}(\mathfrak{A})$	T-ideal of all polynomial identities of $\mathfrak{A}$
$\mathsf{T}^{\mathsf{G}}(\mathfrak{A})$	$\mathsf{G}T$-ideal of all G-graded polynomial identities of $\mathfrak{A}$
$\times$, $\bigtimes$	direct products of algebras (or groups)

$\trianglelefteq$	$H \trianglelefteq \mathsf{G}$ means H is a normal subgroup of G
Υ	class of all f-commutative rings
$\mathfrak{V}^{\mathsf{G}}, \mathfrak{W}^{\mathsf{G}}$	varieties of G-graded algebras
φ, ψ	(graded) homomorphisms of algebras (or modules)
$\mathbb{Z}$	Integer ring
$\mathbb{Z}[x_1, x_2, x_3, \ldots]$	polynomial ring over $\mathbb{Z}$
$\mathbb{Z}_n$	quotient group $\mathbb{Z}/n\mathbb{Z}$
${}_{\mathfrak{A}}S$	left submodule of M generated by $S \subset \mathsf{M}$
${}_{\mathfrak{A}}S_{\tilde{\mathfrak{A}}}$	subbimodule of M generated by $S \subset \mathsf{M}$
e	neutral element of G (or of S)
E_{ij}	elementary matrix of $M_n(\mathbb{F})$
$E_{ij}\eta_g$	elementary matrix of $M_n(\mathbb{F}^\sigma[H])$
$f \equiv 0$	polynomial identity
$f \equiv_{\mathsf{G}} 0$	G-graded polynomial identity
g^{-1}	inverse element of $g \in \mathsf{G}$, i.e. $g^{-1}g = gg^{-1} = e$
H	subgroup of G
$Hom_{\mathbb{Z}H}(\mathbb{Z}\mathsf{G}, \mathsf{M})$	$\mathbb{Z}H$-homomorphisms from $\mathbb{Z}\mathsf{G}$ into M
$M_n(\mathbb{F})$	algebra of matrices over $\mathbb{F}$
$M_n(\mathbb{F}^\sigma[H])$	algebra of matrices over twisted group algebra
$S\mathfrak{A}$	right submodule of M
$S_{\mathfrak{A}}$	right submodule of M generated by $S \subset \mathsf{M}$
S_n	symmetric group of order n
X	a countable set $X = \{x_1, x_2, \ldots\}$
X^{G}	a countable set $\bigcup_{g \in \mathsf{G}} X_g$, where $X_g = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$

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