Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Rigidity theorems for submanifolds and GQY-manifolds

by

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Advisor: Xia Changyu

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Hudson Pina de Oliveira

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À minha *família* e *amigos*.

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Abstract

Using Kato-type inequality for n-dimensional minimal submanifold of $\mathbb{H}^{n+m}$, we obtain necessary conditions so that a complete minimal submanifold immersed in $\mathbb{H}^{n+m}$ to be totally geodesic and using the Simons' inequality to get complete non-compact hypersurface immersed in $\mathbb{H}^{n+1}$ with constant mean curvature to be totally umbilical. If M n-dimensional complete spacelike CMC hypersurfaces is immersed in $M_1^{n+1}(c)$, where $c = \{-1, 0, 1\}$, using the norm $\mathbb{L}^d$ of the tracelles second fundamental form and the first eigenvalue of M, we prove that M is isometric to $\mathbb{H}(c - H^2)$, where H is the constant mean curvature of M.

Taking a *generalized quasi-Einstein manifold* (GQY-manifold), in certain directions for $\nabla\mu$, we have μ constant.

Lastly, considering $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$, the warped product of M with $\mathbb{R}$, be a static space-time, where $(M^n,\ g)$, $n \geq 3$, is a noncompact, connected and oriented Riemannian manifold and use the Einstein equation with perfect fluid as a matter field to show that the energy density in M is zero. Using known techniques, we gave estimates of the volume growth of the geodesic balls and the validity of the weak maximum principle.

Keywords: totally geodesic, totally umbilical, de Sitter space, Lorentz space, anti-de Sitter space, generalized quasi-Einstein manifold, static space-time, static vacuum space.

Contents

Introduction

The work was divided in four parts. The first is a preliminary with results that we found interesting and were important to obtain some results. The following chapters contain results obtained during the PhD program.

The celebrated Bernstein theorem states that if $u(x,y)$ is a C^2 function on $\mathbb{R}^2$ which solves the nonparametric minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0,$$

then u is a linear function of x, y, i.e. the graph of u is a plane. It was proved in the works of Fleming [46], Almgren [7] and Simons [90] that an entire minimal graph in $\mathbb{R}^{n+1}$ is a hyperplane provided $n \leq 7$. When $n > 7$ counterexamples were found by Bombieri [15] et al. do Carmo and Peng [41] and Fisher-Colbrie and Schoen [45] proved independently that any complete oriented stable minimal surface in $\mathbb{R}^3$ must be a plane, which is an important generalization of the Bernstein theorem. Recall that a minimal submanifold in a Riemannian manifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. For the higher dimensional case, it is interesting to know if a complete oriented stable minimal hypersurface in $\mathbb{R}^{n+1}(3 \leq n \leq 7)$ is a hyperplane. With respect to this problem, do Carmo and Peng proved the following result.

Theorem A.([42]) *Let M^n be a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$. Suppose that*

$$\lim_{R\to\infty}\frac{\int_{B_p(R)}|A|^2}{R^{2q+2}}=0, \quad q<\sqrt{\frac{2}{n}},$$

then M is a hyperplane. Here, $B_p(R)$ denotes the geodesic ball of radius R centered *at $p \in M$ and A is the second fundamental form of M.*

The proof of this result relies on Simons' formula for the Laplacian of $|A|^2$ which is a fundamental tool in studying rigidity of Riemannian submanifolds. Many interesting gap theorems for Riemannian submanifolds have been proved by using Simons' formula during the past years. In Chapter 2, we shall use Simons' formula, the technique developed in do Carmo-Peng's paper [42], the estimates for first eigenvalue obtained in Cheng-Yau [43] and Cheung-Leung [39] and the Sobolev inequality in [58] to prove rigidity theorems for minimal submanifolds in a hyperbolic space. By definition, the hyperbolic space $\mathbb{H}^l$ is a simply connected complete l-dimensional Riemannian manifold with a constant negative sectional curvature -1.

In Chapter 2, still using the ideas of Do Carmo-Peng, we study rigidity phenomenon for complete non-compact hypersurfaces with constant mean curvature (CMC hypersurfaces) in a hyperbolic space and space-like CMC hypersurfaces in a Lorentz space forms. A hypersurface in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite. Let M be a CMC hypersurface immersed in $\mathbb{H}^{n+1}(-1)$ or a space-like CMC hypersurface immersed in $M_1^{n+1}(c)$, $c = \{-1, 0, 1\}$. According to $c = 1$; $c = 0$ or $c = -1$; $M_1^{n+1}(c)$ is called a de Sitter space, a Minkowski space or an anti-de Sitter space, respectively.

A complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi-Einstein manifold*, if there exist three smooth functions f, μ and β on M such that

$$Ric + \nabla^2 f - \mu df \otimes df = \beta g,$$

where Ric and ∇^2 denotes, respectively, the Ricci tensor and Hessian of the metric g. This concept, introduced by Catino in [27], generalizes the m-quasi-Einstein manifolds (see, for instance [11, 63]). Inspired by [27], we will introduce a class of Riemannian manifolds (see [28]).

In Chapter 3 consider a *generalized quasi Yamabe gradient soliton* (GQY manifold), let us point out that if $\mu = 0$, (3.1) becomes the fundamental equation of gradient Yamabe soliton. For $\lambda = 0$ the Yamabe soliton is *steady*, for $\lambda < 0$

is *expanding* and for $\lambda > 0$ is *shrinking*. Daskalopoulos and Sesum [44] proved that locally conformally flat gradient Yamabe solitons with positive sectional curvature are rotationally symmetric. Then in [26], they proved that a gradient Yamabe soliton admits a warped product structure without any additional hypothesis. They also proved that a locally conformally flat gradient Yamabe solitons has a more special warped product structure. Inspired by the Generalized quasi-Einstein metrics (see [27, 63]), they started to consider the quasi Yamabe gradient solitons (see [51, 62, 100]). In [51], they introduced the concept of quasi Yamabe gradient soliton and showed that locally conformally flat quasi Yamabe gradient solitons with positive sectional curvature are rotationally symmetric. Moreover, they proved that a compact quasi Yamabe gradient soliton has constant scalar curvature. Leandro [62] investigated the quasi Yamabe gradient solitons on four-dimensional case and proved that half locally conformally flat quasi Yamabe gradient solitons with positive sectional curvature are rotationally symmetric. And he proved that half locally conformally flat gradient Yamabe solitons admit the same warped product structure proved in [26]. Wang [100] gave several estimates for the scalar curvature and the potential function of the quasi Yamabe gradient solitons. He also proved that a quasi Yamabe gradient solitons carries a warped product structure. In [28], they define and study the geometry of gradient Einstein-type manifolds. This metric generalizes the GQY manifolds. In chapter 3, together with Professor Benedito Leandro, to prove, certain conditions, that μ is constant in the GQY manifolds.

In Chapter 4, together with Professors Benedito Leandro and Ernani Ribeiro, we prove that a Riemannian manifold that satisfies the equations (4.3) and (4.4) it has density energy equal to zero ($\mu = 0$) and implying in a volume growth of polynomial geodesic balls and thereby validating a version of the Omori-Yau maximum principle, introduced by Rigoli and Setti [87].

Chapter 1

Preliminary

Assuming that the reader has a certain level of understanding about the issues approached, we started the work with concepts and equations that will be fundamental for a better understanding of the covered subjects.

1.1 Concepts and fundamental equations

1.1.1 Tensors

Definition 1.1 (Tensors and Tensors fields). *A tensor A of order s, briefly $(0, s)$-tensor, at a point p on a differentiable n-dimensional manifold M^n is a multilinear mapping*

$$A_p : (\underbrace{T_pM \times \cdots \times T_pM}_{s}) \to \mathbb{R}$$

Similarly, a $(1, s)$-tensor at a point p is a multilinear mapping

$$A_p : (\underbrace{T_pM \times \cdots \times T_pM}_{s}) \to T_pM$$

Fix a point $p \in M$ and let Ω be a neighborhood of $p \in M$ on which it is possible to define vectors fields $E_1, \cdots, E_n \in \chi(M)$, in such a fashion that at each $q \in \Omega$, the vectors $\{E_i(q)\}_{i=1}^n$ form a basis of T_pM. We say that $\{E_i\}_{i=1}^n$ is a moving frame on Ω and

$$A_{j_1,\cdots,j_s} = A_p\left(E_{j_1}, \cdots, E_{j_s}\right)$$

are called the components of A in the frame $\{E_i\}$. The similar notation $A^i_{j_1,\cdots,j_s}$ for a $(1,s)$-tensors, we have

$$A^i_{j_1,\cdots,j_s} E_i = A_p\left(E_{j_1},\cdots,E_{j_s}\right)$$

Remark 1.2. *In this work we will use basically only tensors of the type $(0,s)$ and $(1,s)$. More generally one considers also mixed tensors, for more details see [75].*

Example 1.3. *A Riemannian metric g, $(0,2)$-tensor, yields an isomorphism of T_pM and your dual T_pM^* by*

$$T_pM \ni X \to g(\cdot, X) \in T_pM^*.$$

Example 1.4 (THE CURVATURE TENSOR). *The curvature tensor is a $(1,3)$-tensor define by*

$$X,Y,Z \to R(X,Y,Z) := \nabla_Y\nabla_X Z + \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z. \tag{1.1}$$

where $X,Y,Z \in T_pM$ and ∇ is the Levi-Civita connection of M.

The components of the curvature tensor are given by

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} &= \sum_l R^l_{ijk}\frac{\partial}{\partial x_l} \\ R^l_{ijk} &= \frac{\partial \Gamma^l_{ij}}{\partial x_k} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_r\left(\Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj}\right). \end{aligned}$$

By lowering the remaining upper index, we get the corresponding $(0,4)$-tensor

$$X,Y,Z,W \to g(R(X,Y)Z,W)$$

with components

$$\left\langle R\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_l}\right\rangle = \sum_s R^s_{ijk}g_{sl} = R_{ijkl}.$$

From the definition of the tensor curvature, given $X,Y,Z \in T_pM$

$$\begin{aligned} & R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_Y\nabla_X Z\nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\ & +\nabla_Z\nabla_Y Z\nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X + \nabla_X\nabla_Z Y\nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y \\ & = \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] - \nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X \\ & = [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] = 0. \end{aligned} \tag{1.2}$$

This equation above is known as first Bianchi identity and also, given $X, Y, Z, T \in T_pM$:

i) $\langle R(X,Y)Z,T\rangle + \langle R(Y,Z)X,T\rangle + \langle R(Z,X)Y,T\rangle = 0$

ii) $\langle R(X,Y)Z,T\rangle = -\langle R(Y,X)Z,T\rangle$

iii) $\langle R(X,Y)Z,T\rangle = -\langle R(X,Y)T,Z\rangle$

iv) $\langle R(X,Y)Z,T\rangle = \langle R(Z,Y)X,T\rangle$.

Showing the symmetries of the curvature tensor.

Definition 1.5. *Let A be a $(0,s)$-tensor field (resp. a $(1,s)$-tensor field), and let X be a fixed vector field. Then we define the covariant derivative of A in the direction X by the formula*

$$\begin{aligned}(\nabla_X A)(Y_1,\cdots,Y_s) &:= \nabla_X(A(Y_1,\cdots,Y_s)) \\ &\quad -\sum_{i=1}^{s}(Y_1,\cdots,Y_{i-1},\nabla_X Y_i, Y_{i+1},\cdots,Y_s)\end{aligned}$$

$\nabla_X A$ is then also a $(0,s)$-tensor (resp. $(1,s)$-tensor), and ∇A is a $(0,s+1)$-tensor (resp. $(1,s+1)$-tensor) by means of the formula

$$(\nabla A)(X,Y_1,\cdots,Y_s) := (\nabla_X A)(Y_1,\cdots,Y_s).$$

Since A is an $(1,s)$-tensor, then for every $i \in \{1,\cdots,s\}$ and fixed vectors X_j, $j \neq i$, whose contraction (or trace) is denoted by $tr(A)$

$$trA = \sum_{j=1}^{n}\langle A(X_1,\cdots,X_{i-1},E_j,X_{i+1},\cdots,X_s),E_j\rangle \tag{1.3}$$

trA is then a $(1,s-1)$-tensor.

Example 1.6 (Ricci tensor, scalar curvature)**.** *The first contraction of the curvature tensor $R(X,Y,Z)$ is give by the expression*

$$(trR)(X,Z) = tr(Y \to R(X,Y,Z)) = \sum_i \langle R(X,E_i)Z,E_i\rangle$$

and is called the Ricci tensor $Ric(X, Z)$*. The trace of the Ricci tensor is called the scalar curvature* S*. One has*

$$S = \sum_{i,j} \langle R(E_j, E_i)E_j, E_i \rangle.$$

Let $\sigma \subset T_pM$ be a two-dimensional subspace and $x, y \in \sigma$ be two linearly independent vectors the real number

$$K(\sigma) = K(x, y) = \frac{\langle R(x, y)z, w \rangle}{|x \wedge y|^2}$$

is called the sectional curvature of σ at p. Where $|x \wedge y|$ represents the area of a two-dimensional parallelogram determined by the pair of vectors $x, y \in \sigma$.

Using the notation given above, if $X, Y, Z, W \in T_pM \subset T_p\overline{M}$ are linearly independent, denote by R and $\overline{R}$ the Ricci tensor of the M and $\overline{M}$, respectively, we have

Theorem 1.7 (Gauss Equation, see [16])**.** *Let* $p \in M \subset \overline{M}$ *and* X, Y *be orthonormal vectors in* T_pM*. Then*

$$\langle \overline{R}(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle + \langle A(X, Z), A(Y, W) \rangle - \langle A(Y, Z), A(X, W) \rangle \tag{1.4}$$

where A *is the 2nd fundamental form.*

A result well known about the subject

Theorem 1.8 ([16])**.** *Let* M *be a* n*-dimensional Riemannian manifold,* p *a point of* M *and* $\{e_1, \cdots, e_n\}$ *an orthonormal basis of* T_pM*. Then, since* $K(p, \sigma) = K_0$ *for all* $\sigma \subset T_pM$*, if and only if*

$$R_{ijkl} = k_0(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Theorem 1.9 (Ricci equation, [76])**.** *Let* M *be an* n*-dimensional Riemannian manifold and* $f \in C^3(M)$ *with orthonormal frame* $\{e_i\}_{i=1}^n$*, to any* $1 \leq i, j, k \leq n$ *equality is worth:*

$$f_{ijk} - f_{ikj} = \sum_{l=1}^{n} f_l R_{ijkl}$$

or simply

$$f_{ijk} - f_{ikj} = f_l R_{ijk}{}^l$$

with $f_{ijk} = \nabla^3 f(e_i, e_j, e_k)$ *and using the Einstein notation.*

The differential Bianchi Identity is, see [14], using the Einstein notation,

$$\nabla_i R_{jkm}{}^l + \nabla_j R_{kim}{}^l + \nabla_k R_{ijm}{}^l = 0. \tag{1.5}$$

Contract on the indices i and l

$$0 = \nabla_l R_{jkm}{}^l + \nabla_j R_{klm}{}^l + \nabla_k R_{ljm}{}^l = \nabla_l R_{jkm}{}^l - \nabla_j R_{km} + \nabla_k R_{jm}$$

and then

$$\nabla_l R_{jkm}{}^l = \nabla_j R_{km} - \nabla_k R_{jm}. \tag{1.6}$$

The above equation is also known as Bianchi identity. Now, trace on the indices k and m,

$$g^{km} \nabla_l R_{jkm}{}^l = g^{km} \nabla_j R_{km} - g^{km} \nabla_k R_{jm}$$

Since the metric is parallel, we can move the g^{km} terms inside,

$$\nabla_l g^{km} R_{jkm}{}^l = \nabla_j g^{km} R_{km} - \nabla_k g^{km} R_{jm}.$$

The left hand side is

$$\begin{aligned} \nabla_l g^{km} R_{jkm}{}^l &= \nabla_l g^{km} g^{lp} R_{jkpm} \\ &= \nabla_l g^{lp} g^{km} R_{jkpm} \\ &= \nabla_l g^{lp} R_{jp} = \nabla_l R_j^l. \end{aligned}$$

Since $\nabla_k g^{km} R_{jm} = \nabla_k R_j^k$, so we have the second Bianchi identity:

$$2\nabla_l R_j^l = \nabla_j R. \tag{1.7}$$

It is normal in the literature to find the following situation as well;

$$\sum_{l=1}^{n} R_{ilkl} = R_{ilkl}$$

that is, repeated indices means that it is adding up in these indices.

1.1.2 The second fundamental

Let $f : M^n \to \overline{M}^{n+m}$ be an immersion , ∇ and $\overline{\nabla}$ the Riemannian connection of the M and $\overline{M}$ respectively. If X, Y are local vector fields on M, we defined

$$A(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$$

where $\overline{X}, \overline{Y}$ are local extensions to $\overline{M}$.

Let ν belongs to the orthogonal complement of T_pM, ($\nu \in (T_pM)^{\perp}$), the mapping $H_\nu : T_pM \times T_pM \to \mathbb{R}$ given by

$$H_\nu = \langle A(X,Y), \nu\rangle, \qquad X,\, Y \in T_pM$$

is a symmetric bilinear form.

Definition 1.10. *The quadratic form II_ν defined on T_pM by*

$$II_\nu(X) = H_\nu(X,X) \tag{1.8}$$

is called the second fundamental form of f at p along the normal vector ν.

We have that the bilinear form H_ν is associated to a linear self-adjoint operator $A_\nu : T_pM \to T_pM$ by

$$\langle A_\nu(X), Y\rangle = H_\nu(X,Y) = \langle A(X,Y), \nu\rangle.$$

and also, taking $X, Y \in T_pM$ and $\nu \in (T_pM)^{\perp}$. Then

$$\begin{aligned} \langle A_\nu(X), Y\rangle &= \langle A(X,Y), \nu\rangle = \langle \overline{\nabla}_X Y, \nu\rangle \\ &= \langle Y, -\overline{\nabla}_X \nu\rangle, \end{aligned}$$

that is,

$$A_\nu = -(\overline{\nabla}_X \nu)^{\top}.$$

In Chapter 2 we will study a particular case of immersions for submanifolds, justifying our next definitions.

Definition 1.11. *An immersion $f : M \to \overline{M}$ is called to be geodesic at $p \in M$ if for every $\nu \in (T_pM)^{\perp}$ the second fundamental form is identically zero at p. An immersion f is called tottally geodesic if it is geodesic for all $p \in M$.*

Definition 1.12. *An immersion $f : M \to \overline{M}$ is minimal if every $p \in T_pM$ and every $\nu \in (T_pM)^{\perp}$ we have the trace of the A_ν is zero, that is, $trA_\nu = 0$.*

Now, since $f : M^n \to \overline{M}^{n+m}$ is an immersion, taking $p \in M$ and $\{E_i\}_{i=1}^m$ an orthonormal frame of the $(T_pM)^{\perp}$, with $\Omega \subset M$ is a neighborhood of the p, where f is an embedding. We can write, at p,

$$\begin{aligned} A(X,Y) = \sum_{i=1}^m H_i(X,Y)E_i &= \sum_{i=1}^m \langle A(X,Y), E_i\rangle E_i \\ &= \sum_{i=1}^m \langle A_i(X), Y\rangle E_i \end{aligned} \tag{1.9}$$

where $H_i = H_{E_i}$ and $A_i = A_{E_i}$. We defined the mean curvature vector as

$$\begin{aligned} \overrightarrow{H}(p) &= \overrightarrow{H} = \frac{1}{n}\sum_{i=1}^m (trA_i)E_i \\ &= \frac{1}{n}\sum_{i=1}^m \left[\sum_{j=1}^n \langle A_i(e_j), e_j\rangle\right] E_i \\ &= \frac{1}{n}\sum_{i=1}^m \left[\sum_{j=1}^n \langle A(e_j,e_j), E_i\rangle\right] E_i \\ &= \frac{1}{n}\sum_{j=1}^n \left[\sum_{i=1}^m H_i(e_j,e_j)E_i\right] = \frac{1}{n}\sum_{j=1}^n A(e_j,e_j) \end{aligned}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame of the T_pM. Of course that f is minimal if and only if $H(p) = 0$

The next definition will be explored in the Chapter 2, Section 2.2.

Definition 1.13. *Let $f : M^n \to \overline{M}^{n+m}$ an isometric immersion and $p \in M$. An immersion f is said to be umbilical at p if exist $Z \in (T_pM)^{\perp}$ such that*

$$A(X,Y) = \langle X,Y\rangle Z, \quad \forall X,Y \in T_pM \tag{1.10}$$

An immersion f is totally umbilical if f is umbilical at every $p \in M$

Here, it's interesting to define an important operator that involves the second fundamental form.

Let M^n be an n-dimensional Riemannian hypersurface of $\overline{M}^{n+1}$, with denote the trace-free second fundamental form, the operator defined by

$$\phi = A - HI,$$

where A is second fundamental form and H mean curvature of the M.

This operator is massively studied for problems involving rigidity and classification of hyperfurfaces in Riemannian and Semi-Riemannian manifods, for example, see [6, 18, 53, 54, 98]. The square of the trace-free operator norm is given in the form $|\phi|^2 = |A|^2 - nH^2$. In Chapter 2 using a limitation for the trace-free second fundamental form and a condition of Do Carmo, Peng (see [42]), it was possible to show that hypersurfaces of the hyperbolic space with this property are totally umbilical, that is, $\phi \equiv 0$.

In the Section 1.3, we will show an Simons' inequality for M^n mean constant curvature hypersurfaces immersed in $\mathbb{Q}^{n+1}(\kappa)$ (space form with sectional curvature κ), involving the trace-free second fundamental form (see eq. 1.43) to show the results of the Chapter 2, Section 2.2.

1.1.3 The Index Lemma

Let J be a differentiable vector field along geodesic $\gamma : [0, a] \to M$. We called J of the Jacobi field if it satisfies the Jacobi Equation

$$J'' + R(\gamma', J)\gamma' = 0 \tag{1.11}$$

where $J'' = \frac{D^2}{dt^2}J$ and $J(0) = 0$. If exist $t_0 \in (0, a]$ such that $J(0) = J(t_0) = 0$, so the point $\gamma(t_0)$ is said to be conjugate to $\gamma(0)$ along γ. The maximum number of such linearly independent fields is called the multiplicity of the conjugate point $\gamma(t_0)$.

Definition 1.14 (Index form). *Let $\gamma : [0, a] \to M$ be a geodesic, V be a piecewise differentiable vector field along γ. For $t_0 \in (0, a]$ let*

$$I_{t_0}(V, V) = \int_0^{t_0} \left\{ \langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \right\} dt \tag{1.12}$$

where $V'(t) = \frac{D}{dt}V(t)$.

Lemma 1.15 (Index Lemma, see [16]). *Let* $\gamma : [0, a] \to M$ *be a geodesic without conjugate points to* $\gamma(0)$ *in the interval* $(0, a]$. *Let* J *a Jacobi field along* γ, *with* $\langle J, \gamma' \rangle = 0$, *and let* V *a piecewise differentiable vector field along* γ, *with* $\langle V, \gamma' \rangle = 0$. *Suppose that* $J(0) = V(0) = 0$ *and that* $J(t_0) = V(t_0)$, $t_0 \in (0, a]$. *Then*

$$I_{t_0}(J, J) \leq I_{t_0}(V, V) \tag{1.13}$$

and equality occurs if and only if $V = J$ *on* $[0, t_0]$

This Lemma is very important for the study of the estimates of the curvature and comparison of geodesics in Riemannian manifolds, for example Rauch Theorem (see pg. 215 in [16]). In Chapter 4 we use this lemma for to proof the Theorem 4.2.

1.2 Kato-type inequality

Let M^n be an n-dimensional hypersurface in a space form $\mathbb{Q}^{n+1}(\kappa)$. We choose a local field of orthonormal frame $\{e_A\}$ in $\mathbb{Q}^{n+1}(\kappa)$, with dual coframe ω_A, such that, at each point of M^n, $e_1, \cdots, e_n$ are tangent to M^n and e_{n+1} is normal to M^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \cdots \leq n+1, \quad 1 \leq i, j, k, \cdots \leq n.$$

In this setting, denoting by ω_{AB} the connection forms of $\mathbb{Q}^{n+1}(\kappa)$, we have that the structure equations of $\mathbb{Q}^{n+1}(\kappa)$ are given by:

$$d\omega_A = \sum_{i=1}^{n} \omega_{Ai} \wedge \omega_i + \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0 \tag{1.14}$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D=1}^{n+1} K_{ABCD}\omega_C \wedge \omega_D \tag{1.15}$$

$$K_{ABCD} = \kappa(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{1.16}$$

Remember that $\omega_{n+1} = 0$ on M, so $\sum_{i=1}^{n} \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and using *Cartan Lemma [30]*, we can write

$$\omega_{n+1i} = \sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{1.17}$$

This gives the second fundamental form of M, $A = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}$. Furthermore, the mean curvature H of M is defined by $H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$.

The structure equations of M are given by

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0 \tag{1.18}$$

$$d\omega_{ij} = \sum_{C=1}^{n+1} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n+1} R_{ijkl}\omega_k \wedge \omega_l. \tag{1.19}$$

Using the structure equations we obtain the Gauss Equation

$$R_{ijkl} = \kappa(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}) \tag{1.20}$$

where R_{ijkl} are the components of the curvature tensor of M. The components h_{ijk} of the covariant derivative ∇A, by definition, satisfy

$$\sum_{k=1}^{n} h_{ijk}\omega_k = dh_{ij} + \sum_{k=1}^{n} h_{kj}\omega_{ki} + \sum_{k=1}^{n} h_{ik}\omega_{kj}. \tag{1.21}$$

The Codazzi equation and the Ricci identity are, respectively, given by

$$h_{ijk} = h_{ikj} \tag{1.22}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{mj}R_{mikl} + \sum_{m=1}^{n} h_{mi}R_{mjkl} \tag{1.23}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij} (more details see [76]) .

The Laplacian Δh_{ij} of h_{ij} is defined by $\Delta h_{ij} = \sum_{k=1}^{n} h_{ijkk}$. From equations (1.22) and (1.23), we obtain that

$$\Delta h_{ij} = \sum_{k=1}^{n} h_{kkij} + \sum_{k,l=1}^{n} h_{kl}R_{lijk} + \sum_{k,l=1}^{n} h_{li}R_{lkjk}. \tag{1.24}$$

Since $\Delta|A|^2 = 2(\sum_{i,j=1}^{n} h_{ij}\Delta h_{ij} + \sum_{i,j,k=1}^{n} h_{ijk}^2)$, from (1.24) we get

$$\frac{1}{2}\Delta|A|^2 = \sum_{i,j,k=1}^{n} h_{ijk}^2 + \sum_{k=1}^{n} h_{ij}h_{kkij} + \sum_{k,l=1}^{n} h_{ij}h_{kl}R_{lijk} + \sum_{k,l=1}^{n} h_{ij}h_{li}R_{lkjk}. \tag{1.25}$$

Taking a (local) orthonormal frame $\{e_i\}_{i=1}^n$ on M such that $h_{ij} = \mu_i \delta_{ij}$

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_{i=1}^{n} \mu_i (nH)_{ii} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\mu_i - \mu_j)^2. \tag{1.26}$$

Using the Gauss equation (1.20) in the frame above, $R_{ijij} = \kappa + \mu_i\mu_j$, and the facts of easy verification

$$\sum_{i,j=1}^{n} (\mu_i - \mu_j)^2 = 2n|A|^2$$

$$\sum_{i,j=1}^{n} \mu_i\mu_j(\mu_i - \mu_j)^2 = -2|A|^4$$

we have

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \kappa n|A|^2 - |A|^4 + \sum_{i=1}^{n} \mu_i (nH)_{,ii} \tag{1.27}$$

To conclude the demonstration, we need the following lemma

Lemma 1.16. *([104]) Let M be n-dimensional immersed submanifold with parallel mean curvature in $\mathbb{Q}^{n+m}(\kappa)$, then*

$$|\nabla A|^2 - |\nabla |A||^2 \geq \frac{2}{mn}|\nabla |A||^2. \tag{1.28}$$

So, we get the following Kato-type inequality for n-dimensional minimal hypersurfaces of $\mathbb{Q}^{n+1}(\kappa)$,

$$|A|\Delta|A| + |A|^4 + n|A|^2 \geq \frac{2}{n}|\nabla |A||^2. \tag{1.29}$$

□

In the Chapter 2, Section 2.1 the steps for demonstrations Kato-type inequality are given for minimal submanifolds of the $\mathbb{Q}^{n+m}(\kappa)$, for more details see [40, 90, 103]. Placing conditions on the length of the second fundamental form and using Kato-type inequality for minimal submanifolds of the $\mathbb{H}^{n+m}(-1)$, it was possible to show that immersions with such properties are totally geodesic in $\mathbb{H}^{n+m}(-1)$ and we produced an article with these results, see [81].

1.3 Simons' inequality: The traceless second fundamental form

Let $\mathbb{Q}^{n+1}(\kappa)$ be the space form of constant sectional curvature κ and M^n a hypersurface in $\mathbb{Q}^{n+1}(\kappa)$ with constant mean curvature H. Choose $\{\omega_i\}_{i=1}^n$ be a orthonormal frame field defined on M. Then the structure equations of M are given by

$$\begin{aligned} d\omega_i &= \sum_{j=1}^n \omega_{ij}\wedge\omega_i, \qquad \omega_{ij}+\omega_{ji}=0 \\ d\omega_{ij} &= \sum_{k=1}^n \omega_{ik}\wedge\omega_{kj}+\Omega_{ij}, \end{aligned} \tag{1.30}$$

where

$$\Omega_{ij} = -\frac{1}{2}\sum_{k,l=1}^n R_{ijkl}\omega_k\wedge\omega_l, \tag{1.31}$$

the functions R_{ijkl} are called the components of the tensor curvature with $R_{ijkl}+R_{ijlk}=0$. For any $f\in C^2(M)$, we define its gradient and hessian by the following formulas

$$\begin{aligned} df &= \sum_{i=1}^n f_i\omega_i \\ \sum_{j=1}^n f_{ij}\omega_j &= df_i+\sum_{j=1}^n f_j\omega_{ji} \end{aligned}$$

f_i and f_{ij} are the components of the gradient and hessian, respectively.

Since $\phi=\sum_{i,j=1}^n \phi_{ij}\omega_i\otimes\omega_j$ be a symmetric tensor defined on M, written on the frame $\{\omega_i\}_{i=1}^n$. Note that the covariant derivative of ϕ_{ij} is defined by

$$\sum_{k=1}^n \phi_{ijk}\omega_k = d\phi_{ij}+\sum_{k=1}^n \phi_{kj}\omega_{ki}+\sum_{k=1}^n \phi_{ik}\omega_{kj}. \tag{1.32}$$

The second covariant derivative of ϕ_{ij} is defined by

$$\sum_{l=1}^n \phi_{ijkl}\omega_l = d\phi_{ijk}+\sum_{m=1}^n \phi_{mjk}\omega_{mi}+\sum_{m=1}^n \phi_{imk}\omega_{mj}+\sum_{m=1}^n \phi_{ijm}\omega_{mk}. \tag{1.33}$$

Taking exterior differentiate of (1.32), we obtain

$$\sum_{l,k=1}^{n} \phi_{ijkl}\omega_l \wedge \omega_k = \sum_{k=1}^{n} \phi_{kj}\Omega_{ki} + \sum_{k=1}^{n} \phi_{ik}\Omega_{kj}.$$

Therefore,

$$\begin{aligned}\sum_{k,l=1}^{n} (\phi_{ijkl} - \phi_{ijlk})\,\omega_l \wedge \omega_k &= \sum_{k,l=1}^{n} \phi_{ijkl}\omega_l \wedge \omega_k + \sum_{l,k=1}^{n} \phi_{ijlk}\omega_k \wedge \omega_l \\ &= \sum_{k=1}^{n} \phi_{kj}\Omega_{ki} + \sum_{k=1}^{n} \phi_{ik}\Omega_{kj} \\ &\quad + \sum_{l=1}^{n} \phi_{lj}\Omega_{ki} + \sum_{l=1}^{n} \phi_{il}\Omega_{lj}\end{aligned}$$

with (1.31)

$$(\phi_{ijkl} - \phi_{ijlk}) = -\sum_{m=1}^{n} \phi_{mj}R_{milk} - \sum_{m=1}^{n} \phi_{im}R_{mjlk}. \tag{1.34}$$

The Laplacian of the tensor ϕ_{ij} is defined to be $\sum_{k=1}^{n} \phi_{ijkk}$ and so

$$\begin{aligned}\Delta\phi_{ij} &= \sum_{k=1}^{n} \phi_{ijkk} \\ &= \sum_{k=1}^{n}(\phi_{ijkk} - \phi_{ikjk}) + \sum_{k=1}^{n}(\phi_{ikjk} - \phi_{ikkj}) + \sum_{k=1}^{n}(\phi_{ikkj} - \phi_{kkij}) \\ &\quad + \left(\sum_{k=1}^{n} \phi_{kk}\right)_{ij}\end{aligned} \tag{1.35}$$

Since ϕ_{ij} satisfying "Codazzi equation"

$$\phi_{ijk} = \phi_{ikj},$$

we have from (1.34) and (1.35)

$$\Delta\phi_{ij} = \left(\sum_{k=1}^{n} \phi_{kk}\right)_{ij} - \sum_{m,k=1}^{n} \phi_{mk}R_{mikj} - \sum_{m,k=1}^{n} \phi_{im}R_{mkkj}. \tag{1.36}$$

Being $|\phi|^2 = \sum_{i,j=1}^{n} \phi_{ij}^2$ and $tr\phi = \sum_{i=1}^{n} \phi_{ii}$. Then equation (1.36) give us

$$\begin{aligned}
\frac{1}{2}\Delta|\phi|^2 &= |\nabla|\phi||^2 + |\phi|\Delta|\phi| \\
&= \sum_{i,j,k=1}^{n} \phi_{ijk}^2 + \sum_{i,j=1}^{n} \phi_{ij}(tr\phi)_{ij} \\
&\quad - \sum_{i,j,k,m=1}^{n} \phi_{ij}\phi_{mk}R_{mikj} - \sum_{i,j,k,m=1}^{n} \phi_{ij}\phi_{im}R_{mkkj}. \qquad (1.37)
\end{aligned}$$

Choose a frame field $\{\omega_i\}_{i=1}^n$ which diagonalizes ϕ at each fixed point on M^n, i.e. $\phi_{ij} = \lambda_i\delta_{ij}$. Then (1.37) simplifies to

$$\frac{1}{2}\Delta|\phi|^2 = \sum_{i,j,k=1}^{n} \phi_{ijk}^2 + \sum_{i=1}^{n} \lambda_i(tr\phi)_{ii} + \frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\lambda_i - \lambda_j)^2 \qquad (1.38)$$

where ϕ_{ijk} are components of the covariant derivative of the tensor ϕ, and R_{ijij} is the sectional curvature of the plane spanned by $\{e_i, e_j\}$. Since $\phi = A - HI$, therefore, $\phi e_i = \lambda_i e_i = (\mu_i - H)e_i$, where $Ae_i = \mu_i e_i$, $i = 1, \cdots, n$, are they the eigenvalues of the second form operator A. By Gauss formula, we conclude that

$$\begin{aligned}
\frac{1}{2}\sum_{i,j=1}^{n} R_{ijij}(\lambda_i - \lambda_j)^2 &= \frac{1}{2}\sum_{i,j=1}^{n} (\kappa - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 - \frac{H}{2}\sum_{i,j=1}^{n} (\lambda_i + \lambda_j)(\lambda_i - \lambda_j)^2 \\
&\quad + \frac{H^2}{2}\sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2. \qquad (1.39)
\end{aligned}$$

Remember that $0 = tr\phi = \sum_{i=1}^n \lambda_i$, it is easy to check that

$$\begin{aligned}
\sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2 &= 2n|\phi|^2, \\
\sum_{i,j=1}^{n} (\lambda_i + \lambda_j)(\lambda_i - \lambda_j)^2 &= 2n\sum_{i,j=1}^{n} \lambda_i^3, \\
\sum_{i,j=1}^{n} \lambda_i\lambda_j(\lambda_i - \lambda_j)^2 &= -2|\phi|^4.
\end{aligned}$$

From the above, it follows that

$$\begin{aligned}
\frac{1}{2}\Delta|\phi|^2 &= |\nabla|\phi||^2 + |\phi|\Delta|\phi| \\
&= \sum_{i,j,k=1}^{n} \phi_{ijk}^2 - |\phi|^4 - nH\sum_{i=1}^{n} \lambda^3 + n(H^2 + \kappa)|\phi|^2 \qquad (1.40)
\end{aligned}$$

In this case, it follows from ([17] (2.3), (2.4)) that

$$\sum_{i,j,k=1}^{n} \phi_{ijk}^2 \geq \frac{2}{n}|\nabla|\phi||^2 + |\nabla|\phi||^2$$

To conclude the demonstration we need the following lemma

Lemma 1.17. *([4]) Let λ_i, $i = 1, \cdots, n$, be real numbers such that $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = \beta^2$, where $\beta = const > 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_{i=1}^{n} \lambda_i \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3$$

Proof of the lemma. We can assume that $\beta > 0$, and use the method of Lagrange's multipliers to find the critical points of $g = \sum_{i=1}^{n} \lambda_i^3$ subject to the conditions: $\sum_{i=1}^{n} \lambda_i = 0$, $\sum_{i=1}^{n} \lambda_i^2 = \beta^2$. It follows that the critical points are given by the values of λ_i that satisfy the quadratic equation

$$\lambda_i^2 - \mu\lambda_i - \alpha = 0, \qquad i = 1, \cdots, n.$$

Therefore, after reenumeration if necessary, the critical points are given by:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_p = a > 0, \qquad \lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n = -b < 0. \tag{1.41}$$

Since, at the critical points,

$$\begin{aligned} \beta^2 &= \sum_i \lambda_i^2 = pa^2 + (n-p)b^2 \\ 0 &= \sum_i \lambda_i = pa - (n-p)b \\ g &= \sum_i \lambda_i^3 = pa^3 + (n-p)b^3. \end{aligned}$$

Solving the system, we have

$$a^2 = \frac{n-p}{pn}\beta^2, \qquad b^2 = \frac{p}{n(n-p)}\beta^2, \qquad g = \left(\frac{n-p}{n}a - \frac{p}{n}b\right)\beta^2.$$

It follows that g decreases when p increases. Hence g reaches a maximum when $p = 1$, and the maximum of g is given by

$$\begin{aligned} max(g) &= \sum_{i=1}^{n} \lambda_i^3 = a^3 - (n-1)b^3 \\ &= ((n-1)b)^3 - (n-1)b^3 = n(n-2)(n-1)b^3 \\ &= \frac{n-2}{\sqrt{n(n-1)}}\beta^3. \end{aligned} \tag{1.42}$$

$\square$

So we have finally the following Simons' inequality for M^n, mean constant curvature hypersurfaces immersed in $\mathbb{Q}^{n+1}(\kappa)$

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\phi|^3 + n(H^2+\kappa)|\phi|^2. \tag{1.43}$$

$\square$

In the Chapter 2, section 2.2. Placing conditions on the length of the trace-free second fundamental form and using Simons' inequality for M^n mean constant curvature hypersurfaces immersed in $\mathbb{H}^{n+1}(-1)$, it was possible to show that immersions with such properties are totally umbilical in $\mathbb{H}^{n+1}(-1)$.

1.4 Warped Product

Let's turn our attention to a class of metrics on the product variety $B \times F$. Let's define the warped product (see [14, 75]).

Definition 1.18. Let be (B, g_B) and (F, g_F) be Riemannian manifolds and $f > 0$ a function on B. The *warped product* $M = B \times_f F$ is a product manifold $B \times F$ with metric

$$g = \pi^* g_B + (\pi^* f)^2 \sigma^* g_F,$$

where π and σ are projections of $B \times F$ in B and F, respectively. Explicitly, for $u, v \in T_{(p,q)}M$, we have

$$g(u, v) = g_B(d\pi(u), d\pi(v)) + (f \circ \pi)^2 g_F(d\sigma(u), d\sigma(v)).$$

Remark 1.19. If f is a constant equal to 1, we say that M is a *Riemannian product* and g the *product metric*. When a manifold M can not be written as Riemannian product of the others two manifolds we say that it is *irreducible*.

The fibers $p \times F = \pi^{-1}(p)$ and the leaves $B \times q = \sigma^{-1}(q)$, with $p \in B$ and $q \in F$ are submanifolds of M. The warped product metric is characterized by

1. For each $q \in F$, the mapping $\pi|(B \times q)$ is an isometry onto B,
2. For each $p \in B$, the mapping $\sigma|(p \times F)$ is a positive homothety onto F, with scale factor $1/f(p)$.
3. For each $(p, q) \in M$, the leaf $B \times q$ and the fiber $p \times F$ are orthogonal in (p, q), so we can decompose $T_{(p,q)}M$ in direct sum
$$T_{(p,q)}M = T_{(p,q)}(B \times q) \oplus T_{(p,q)}(p \times F).$$
We will call the vectors tangent to leaves of the *horizontal* and tangent to the fibers are *vertical*. If $v \in T_{(p,q)}M$ denoted by $hor(v)$ and $ver(v)$ the components horizontal e vertical de v, respectivamente.

Remark 1.20. For a product manifold $B \times F$, denoted by $\mathfrak{F}(B)$ the set of differentiable functions on B, We have the following notions of lifting

1. If $h \in \mathfrak{F}(B)$, the lift h for $B \times F$ is $\tilde{h} = h \circ \pi \in \mathfrak{F}(B \times F)$.
2. If $v \in T_pB$ and $q \in F$ so the lift $\tilde{v}$ do the v in (p, q) is the only vector in $T_{(p,q)}(B \times q)$, such that $d\pi(\tilde{v}) = v$.
3. If $X \in \mathfrak{X}(B)$, the lift of X to $B \times F$ is the only vector field $\tilde{X}$ whose value at each point (p, q) is the lift of $X(p)$ to (p, q). This field is differentiable and is the only element of $\mathfrak{X}(B \times F)$, such that $d\pi(\tilde{X}) = X$ e $d\sigma(\tilde{X}) = 0$. Denoted by $\mathfrak{L}(B)$ the *lift set* elements of the $\mathfrak{X}(B)$ to $B \times F$.

Functions, tangents vectors and differentiable fields onto F can be lift to $B \times F$ similarly using the projection σ.

Here, are some results about the warped product that will be very important throughout the text. These results can be found in [75].

Lemma 1.21. *If $h \in \mathfrak{F}(B)$, so the gradient of the lift $h \circ \pi$ of h to $M = B \times_f F$ is the lift to M of the gradient of h on B.*

Denoting the Riemannian connections of the M, B e F by ∇, ∇^B e ∇^F , respectively, we can relate them as follows:

Proposition 1.22. *Let $M = B \times_f F$ a warped product. If $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, so*

1. *$\nabla_X Y \in \mathfrak{L}(B)$ is the lift of $\nabla^B_X Y$ from B,*
2. *$\nabla_X V = \nabla_V X = \frac{X(f)}{f} V$,*
3. *$hor(\nabla_V W) = -\frac{\langle V,W \rangle}{f} \nabla f$,*
4. *$ver(\nabla_V W) \in \mathfrak{L}(F)$ is the lift of $\nabla^F_V W$ from F,*

where ∇f is the gradient of f in the metric g.

We will now present a result that relates the curvatures of M with the base curvatures B, at the leaves F.

Proposition 1.23. *Let $M = B \times_f F$ be a warped product with tensor curvature R. Let R^B and R^F the pullback of the tensor curvature of B and F, respectively. If $X, Y, Z \in \mathfrak{L}(B)$ and $U, V, W \in \mathfrak{L}(F)$, so*

1. *$R(X, Y)Z \in \mathfrak{L}(M)$ is the lift of $R^B(X, Y)Z \in \mathfrak{L}(B)$ from B,*
2. *$R(V, X)Y = \frac{\nabla^2 f(X,Y)}{f} V$, where ∇^2 is Hessian of the warped product M, Which coincides with the Hessian of B in horizontal vector,*
3. *$R(X, Y)V = R(V, W)X = 0$,*
4. *$R(X, V)W = \frac{\langle V,W \rangle}{f} \nabla_X \nabla f$,*
5. *$R(U, V)W = R^F(U, V)W - \frac{\langle \nabla f, \nabla f \rangle}{f^2} \{\langle W, U\rangle V - \langle W, V \rangle U\}$.*

As a result of the above result, we'll show how the Ricci tensor of the warped product, Ric. We denoted Ric^B and Ric^F the pullback of the Ricci tensor of B and F, respectively.

Corollary 1.24. *About a warped product $M = B \times_f F$ with $n = dim(F) > 1$, if X, Y are horizontal and V, W vertical, so*

1. *$Ric(X, Y) = Ric^B(X, Y) - n\frac{\nabla^2 f(X,Y)}{f}$,*

2. $Ric(X,V)=0$,

3. $Ric(V,W)=Ric^F(V,W)-\langle V,W\rangle\{\frac{\Delta_B f}{f}+\frac{1}{f^2}(n-1)\langle\nabla f,\nabla f\rangle\}$,

onde $\Delta_B f$ *is the laplacian of the* f *on* B.

The above mentioned results demonstrations can be found in [14] and [75].

1.5 Weighted Manifolds and the Index

Let $(M^{n+1},\overline{g},e^f d\mu)$ be a smooth metric measure space, which is a $(n+1)$-dimensional Riemannian manifold with a weighted volume form $e^f d\mu$ on M, where f is a smooth function on M and $d\mu$ is the volume element induced by the metric $\overline{g}$. In this work, we denote by "bar" all quantities on $(M^{n+1},\overline{g})$, for instance $\overline{\nabla}$ and $\overline{Ric}$ are the Levi-Civita connection and the Ricci curvature tensor for $\overline{g}$, respectively. In $(M^{n+1},\overline{g},e^f d\mu)$, the *Bakry-Émery-Ricci curvature tensor* will be defined by

$$\overline{Ric}_f:=\overline{Ric}+\overline{\nabla}^2 f.$$

where $\overline{\nabla}^2 f$ is the Hessian of f for $\bar{g}$.

Remark 1.25. *In literature it is common to find the definition of the weighted manifold as* $(M^{n+1},\overline{g},e^{-f}d\mu)$. *At the end of this section we will better understand that defining a weight manifold is directly related to the manifold in question.*

Now, consider an n-dimensional smooth immersion $h:\Sigma^n\to M^{n+1}$, we know that h induces a metric $g=h^*\overline{g}$ on Σ, thus $h:(\Sigma^n,g)\to(M^{n+1},\overline{g})$ is an isometric immersion. Here ∇, Ric, Δ and $d\sigma$ denote, respectively, the Levi- Civita connection, the Ricci curvature tensor, the Laplacian, and the element volume form of (Σ,g).

The restriction of the function f on Σ give us a weighted measure $e^f d\sigma$ on Σ, and hence $(\Sigma,g,e^f d\sigma)$ is a smooth metric measure space. Associated with this metric we have the *weighted Laplacian*, or *drift* Laplacian Δ_f on Σ, defined by

$$\Delta_f u:=\Delta u+\langle\nabla f,\nabla u\rangle.$$

The second fundamental form $A : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ is given by

$$A(X,Y) = \langle \overline{\nabla}_X Y, \nu \rangle \nu$$

where $p \in \Sigma,\ \ X, Y \in T_p\Sigma$, ν is a unit normal vector at p. Taking a local orthonormal frame $\{e_i\}_{i=1}^n$ of Σ, the components of A are $a_{ij} = A(e_i, e_j) = \langle \overline{\nabla}_{e_i}\nu, e_j \rangle$, and the shape operator is

$$AX = \overline{\nabla}_X \nu, \quad X \in T_p\Sigma.$$

Moreover, the Mean curvature H of Σ is

$$H = tr(A) = \sum_{i=1}^{n} a_{ii}.$$

It is well known that in $(\Sigma, g, e^f d\sigma)$ the *weighted mean curvature* H_f of the hypersurface Σ is defined by

$$H_f = H + \langle \overline{\nabla} f, \nu \rangle, \quad \text{with} \quad \nu \in \Sigma^{\perp}.$$

Σ is a f-*minimal hypersurface* if

$$H + \langle \overline{\nabla} f, \nu \rangle = 0. \tag{1.44}$$

We define the *weighted volume* of Σ by

$$V_f(\Sigma) = \int_{\Sigma} e^f d\sigma. \tag{1.45}$$

Let $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a variation of Σ, i.e., F is a map with compact support such that $F(x, 0) = x$ for all $x \in \Sigma$. An immersed hypersurface Σ in $(M, \overline{g}, e^f d\mu)$ is called f-minimal if

$$\frac{d}{dt} \mid_{t=0} V_f(F(\Sigma, t)) = 0 \tag{1.46}$$

for all variations F of Σ. Therefore, Σ is a f-minimal hypersurfaces of M if and only if it is a critical point of the weighted volume functional.

Moreover, an immersed hypersurface $\Sigma \subset M$ is called L_f-stable if f-minimal and

$$\frac{d^2}{dt^2} \mid_{t=0} V_f(F(M, t)) \geq 0 \tag{1.47}$$

for all variations F of Σ.

We consider the following useful lemma.

Lemma 1.26. *[70,86] For a L_f-stable hypersurface Σ of M the following inequality holds for any smooth function $\eta \in C_0^\infty(\Sigma)$:*

$$\int_\Sigma [|\nabla\eta|^2 - (|A|^2 + \overline{Ric}_f(\nu,\nu))\eta^2]e^f d\sigma \geq 0, \tag{1.48}$$

or equivalently

$$\int_\Sigma -\eta[\Delta_f\eta + (|A|^2 + \overline{Ric}_f(\nu,\nu))\eta]e^f d\sigma = -\int_\Sigma \eta L_f\eta e^f d\sigma \geq 0. \tag{1.49}$$

Now we assume that Σ is a *two-sided hypersurface*, that is, there is a globally-defined unit normal ν on Σ. The L_f operator on Σ is given by

$$L_f := \Delta_f + (|A|^2 + \overline{Ric}_f(\nu,\nu)). \tag{1.50}$$

The operator L_f is called L_f-*stability operator* of Σ. Therefore, we can associate the problem of L_f-stability with the Index Problem. Since Δ_f is self-adjoint in the weighted space $L^2(e^f d\sigma)$, we may define a symmetric bilinear form B_f on $C_0^\infty(\Sigma)$ by

$$\begin{aligned} B_f(\phi,\psi) &:= -\int_\Sigma \phi L_f\psi e^f d\sigma \\ &= \int_\Sigma [\langle\nabla\phi,\nabla\psi\rangle - (|A|^2 + \overline{Ric}_f(\nu,\nu))\phi\psi]e^f d\sigma. \end{aligned} \tag{1.51}$$

Definition 1.27. *The L_f-index of Σ, denoted by L_f-ind(Σ), is defined to be the maximum of the dimensions of negative definite subspaces for B_f, that is*

$$L_f\text{-}ind(\Sigma) = \sup_{\Omega\subset\subset\Sigma} L_f\text{-}ind(\Omega),$$

In particular, using Lemma 1.26, Σ is L_f-stable if and only if L_f-ind$(\Sigma) = 0$. Furthermore, from Lemma 1.26 we may define the Dirichlet problem for L_f on a compact domain $\Omega \subset \Sigma$:

$$L_f u = -\lambda u, \quad u \in C_0^\infty(\Omega); \quad u|_{\partial\Omega} = 0.$$

and that the first eigenvalue of the Dirichlet problem for L_f operator is given by

$$\lambda_1(\Omega) = \inf_{\eta \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Sigma [|\nabla \eta|^2 - (|A|^2 + \overline{Ric}_f(\nu,\nu))\eta^2] e^f d\sigma}{\int_\Sigma \eta^2 e^f d\sigma}.$$

Therefore, if the first eigenvalue of the Dirichlet problem for the stability operator L_f is non-negative for all compact, $\Omega \subset\subset \Sigma$, we have that Σ is L_f-stable, or yet, if the number of negative (Dirichlet) eigenvalues of L_f over supremum compact domains of Σ is zero, L_f-ind$(\Sigma) = 0$, which implies that Σ is L_f-stable .

Following we present a small motivation for the study of weighted manifolds.

1.5.1 Motivation

Weighted volume measures arise naturally from the study of conformal deformation of a Riemannian metric. Let (M, g) be an n-dimensional and complete Riemannian manifold, and let Δ_g and μ_g denote the Laplace-Beltrami operator and the Riemannian volume measure respectively. In a local chart, write $g = (g_{ij})$. Then

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \qquad \text{and} \qquad d\mu_g = \sqrt{g} dx$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = det(g_{ij})$. Suppose the metric g is conformally deformed by a positive smooth function φ on M, that is, let $\overline{g}$ be a new metric on M defined by, $X, Y \in TM$

$$\overline{g}(X,Y) = \varphi g(X,Y)$$

Then the volume measure μ_g is the weighted volume measure $\varphi^{\frac{n}{2}} \mu_g$ and

$$\Delta_g = \varphi^{-1} \left\{ \Delta_g + \left(\frac{n}{2} - 1 \right) \nabla ln\varphi \right\}$$

and the transformed formula of Ricci curvature is given by (See [14])

$$\begin{aligned} Ric_{\overline{g}} &= Ric_g - \frac{n-2}{n} Hess(ln\varphi) + \frac{n-2}{4} \nabla ln\varphi \otimes \nabla ln\varphi \\ &\quad - \frac{1}{2} \left\{ \Delta_g ln\varphi + \frac{n-2}{n} |\nabla ln\varphi|^2 \right\} g \end{aligned}$$

where the hessian and gradient are computed using the metric g.

We want to establish geometric results relating to the metric $\overline{g}$ by using the data associated to the original metric g. To this end we need a concept of curvature associated to a weighted Laplacian. Such a concept has been introduced by Bakry and Emery [13]. Let L be a diffusion operator. Then the metric Γ and the curvature operator Γ_2 are defined by

$$\begin{aligned}\Gamma(f,g) &= \frac{1}{2}\{L(fg) - fLg - gLf\}\\ \Gamma_2(f,g) &= \frac{1}{2}\{L\Gamma(fg) - \Gamma(Lf,g) - \Gamma(Lg,f)\}\end{aligned}$$

respectively.

Then if $L = \Delta_h$ we have (cf. [13] Proposition 3, p. 187)

$$\Gamma(f,g) = \langle \nabla f, \nabla g\rangle$$

and

$$\Gamma_2(f,g) = \langle Hess(f), Hess(g)\rangle + (Ric + Hess(h))(\nabla f, \nabla g) \;\; \forall\, f,g \in C^2(M)$$

We call $Ric_h = Ric - Hess(h)$ the "Ricci curvature" of the weighted Laplacian Δ_h. It is natural to generalize known results for Δ to Δ_h using $Ric - Hess(h)$. For example we have the following several well known results using Ricci curvature.

Theorem 1.28 ([99]). *Every non-compact manifold with non-negative Ricci curvature possesses infinite volume.*

Theorem 1.29 ([71]). *Let M be an n-dimensional and complete Riemannian manifold, and $Ric \geq k^2$ for some positive constant k. Then M is compact and the diameter $d(M) \leq \sqrt{n}k^{-1}$.*

The following simple example shows that such results are no longer true for a weighted Laplacian if we replace Ricci curvature by $Ric - Hess(h)$.

Example 1.30. *Let $M = \mathbb{R}^2$ be the Euclidean space with the standard metric, and let $h(x,y) = -(x^2+y^2)$. Then $Ric - Hess(h) = 2$ but*

$$Vol_h(M) = \int_M e^h dxdy < \infty$$

although M is non-compact, where $Vol_h(M) < \infty$, the volume associated to Δ_h. Moreover, Theorem 1.29 does not hold if we replace Ricci curvature by $Ric - Hess(h)$.

Chapter 2

Minimal submanifolds and CMC hypersurfaces

In this Chapter we present results obtained from studies related to do Carmo-Peng's theorem, thus achieving conditions for having totally geodesic immersions or totally umbilical immersions in submanifolds of the hyperbolic space and conditions for a complete non compact CMC hypersurface in $M_1^{n+1}(c)$, where $c = \{-1, 0, 1\}$, is isometric hyperbolic space $\mathbb{H}^n(-r^2)$.

The first section is the result of studies with Prof. Dr. Xia and is based on work of the H. Pina and C. Xia [81].

2.1 Rigidity of complete minimal submanifolds in a hyperbolic space

We shall use Simons' formula, the technique developed in do Carmo-Peng's paper [17], the estimates for first eigenvalue obtained in Cheng-Yau [43] and CheungLeung [39] and the Sobolev inequality in [58] to prove rigidity theorems for minimal submanifolds in a hyperbolic space. Our results are as follows

Theorem 2.1. *Let M be an n-dimensional complete immersed minimal submanifold in $\mathbb{H}^{n+m}$ such that $(n^2 - 6n + 1) + 8/m > 0$ and let d be a constant satisfying*

1. *if* $m = 1$ *and* $n = 2$, *then*
$$d \in \left(0, \frac{1}{2}\right);$$

2. *if* $m = 1$ *and* $n > 3$, *then*
$$d \in \left(\frac{(n-1)}{n}, \frac{(n-2)(n-1)}{n}\right);$$

3. *if* $m \geq 2$ *and* $n > 5$, *then*
$$d \in \frac{(n-1)^2}{2n}\left(1 - \sqrt{1 - \frac{4n}{(n-1)^2}\left(1 - \frac{2}{mn}\right)}, 1 + \sqrt{1 - \frac{4n}{(n-1)^2}\left(1 - \frac{2}{mn}\right)}\right).$$

Suppose that
$$\limsup_{R\to\infty} \frac{\int_{B_p(R)} |A|^d}{R^2} = 0, \tag{2.1}$$
and
$$\sup_{x\in M} |A|^2(x) < D(n,m,d) = \begin{cases} \left(d - 1 + \frac{2}{n}\right)\frac{(n-1)^2}{d^2} - n, & if \quad m = 1, \\ \frac{2}{3}\left(\left(1 - \frac{mn-2}{dmn}\right)\frac{(n-1)^2}{d} - n\right), & if \quad m \geq 2, \end{cases} \tag{2.2}$$
then M *is totally geodesic.*

Theorem 2.2. *Let* M *be an* n*-dimensional complete immersed minimal submanifold in* $\mathbb{H}^{n+m}$ *such that* $(n^2 - 6n + 1) + 8/m > 0$. *Suppose that*
$$\limsup_{R\to\infty} \frac{\int_{B_p(R)} |A|^d}{R^2} = 0, \tag{2.3}$$
where d *is a constant satisfying*

1. *if* $m = 1$ *and* $n > 3$, *then*
$$d \in \left(\frac{(n-1)}{n}, \frac{(n-2)(n-1)}{n}\right);$$

2. *if* $m > 1$ *and* $n > 5$, *then*
$$d \in \frac{(n-1)^2}{2n}\left(1 - \sqrt{1 - \frac{4}{(n-1)^2}\left(n - \frac{2}{m}\right)}, 1 + \sqrt{1 - \frac{4}{(n-1)^2}\left(n - \frac{2}{m}\right)}\right).$$

There exists a positive constant C *which depends only on* n, m, *and* d *such that if*
$$\int_M |A|^n < C, \tag{2.4}$$
then M *is totally geodesic.*

2.1.1 Proof of the main theorems

Before proving the results, let us recall some known facts we need.

Let M be a complete submanifold immersed in a simply connected space form $M^{n+m}(\kappa)$ of constant curvature κ. We adopt the usual convention on the range of the indices

$$1 \leq A, B, C, \cdots \leq n+m, \quad 1 \leq i,\, j,\, k,\, \cdots \leq n, \quad n+1 \leq \alpha,\, \beta,\, \gamma,\, \cdots \leq n+m.$$

Choose a local orthonormal adapted frame $\{e_a\}$ in $M^{n+m}(\kappa)$, so that, when restricted to M^n, the vectors e_α, are perpendicular to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual basis to $\{e_A\}$ and the connection forms on $M^{n+m}(\kappa)$, respectively. Restricting these forms to M^n, we have

$$\omega_{i\alpha} = h^\alpha_{ij}\omega^j, \quad h^\alpha_{ij} = h^\alpha_{ji},$$

and

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}$$
$$A = h^\alpha_{ij}\omega^i \otimes \omega^j \otimes e_\alpha, \quad \overrightarrow{H} = \frac{1}{n}\sum_i h^\alpha_{ii}e_\alpha$$
$$\nabla A = h^\alpha_{ijk}\omega^i \otimes \omega^j \otimes \omega^k \otimes e_\alpha, \quad h^\alpha_{ijk} = h^\alpha_{ikj},$$

where we have used the Einstein's summation convention, A is the second fundamental form, R_{ijkl} are the components of the Riemannian curvature tensor, $\overrightarrow{H}$ is the mean curvature vector of M and h^α_{ijk} are the components of the covariant derivative of h^α_{ij}. Let

$$|A|^2 = \sum_{i,\,j,\,\alpha}(h^\alpha_{ij})^2, \quad H = \frac{1}{n}\sqrt{\sum_\alpha\left(\sum_i h^\alpha_{ii}\right)^2}$$

be the squared length of the second fundamental form and the mean curvature of M, respectively. With these notations, the Gauss Equation has the shape:

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_t h^\alpha_{tj}R_{tikl} - \sum_t h^\alpha_{it}R_{tjkl}.$$

When M is minimal, that is, $H \equiv 0$, using the definition of the Laplacian of h_{ij}^{α}, the Gauss and Codazzi equations, we can obtain the well-known Simons' formula (cf. [40], [90]):

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + nc|A|^2 + \sum_{\alpha,\beta} tr(A^{\alpha}A^{\beta} - A^{\beta}A^{\alpha})^2 - \sum_{\alpha,\beta} tr(A^{\alpha}A^{\beta}), \tag{2.5}$$

where $|\nabla A|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2$ and $A^{\alpha} = (h_{ij}^{\alpha})_{n\times n}$.

The last terms in the above expression can be estimated as (cf. [40], [90], [103])

$$\sum_{\alpha,\beta} tr(A^{\alpha}A^{\beta} - A^{\beta}A^{\alpha})^2 - \sum_{\alpha,\beta} tr(A^{\alpha}A^{\beta}) \leq b(m)|A|^4, \tag{2.6}$$

with $b(1) = 1$, and $b(m) = \frac{3}{2}$ if $m \geq 2$

Recalling that $\Delta|A|^2 = 2|A|\Delta|A| + 2|\nabla|A||^2$, using (2.6), Lemma 1.16 and taking $\kappa = -1$, we get the following Kato-type inequality for n-dimensional minimal submanifold of $\mathbb{H}^{n+m}(-1)$:

$$|A|\Delta|A| + b(m)|A|^4 + n|A|^2 \geq \frac{2}{mn}|\nabla|A||^2 \tag{2.7}$$

Setting $\eta = \frac{d}{2}$, we have

$$\Delta|A|^{\eta} = \eta(\eta-1)|A|^{\eta-2}|\nabla|A||^2 + \eta|A|^{\eta-1}\Delta|A|. \tag{2.8}$$

Multiplying (2.8) by $|A|^{\eta}$ and using (2.7), we have

$$\begin{aligned} |A|^{\eta}\Delta|A|^{\eta} &= |A|^{\eta}(\eta(\eta-1)|A|^{\eta-2}|\nabla|A||^2 + \eta|A|^{\eta-1}\Delta|A|) \\ &= \frac{\eta-1}{\eta}|\nabla|A|^{\eta}|^2 + \eta|A|^{2\eta-2}|A|\Delta|A| \\ &\geq \left(1 - \frac{nm-2}{\eta nm}\right)|\nabla|A|^{\eta}|^2 - (n\eta + \eta b(m)|A|^2)|A|^{2\eta} \end{aligned} \tag{2.9}$$

Let ϕ be a function in $C_0^{\infty}(M)$. Multiplying (2.9) by ϕ^2 and integrating on M, we get

$$\begin{aligned} \left(1 - \frac{mn-2}{\eta mn}\right)\int_M \phi^2|\nabla|A|^{\eta}|^2 &\leq \int_M (n\eta + \eta b(m)|A|^2)\phi^2|A|^{2\eta} \\ &\quad + \int_M \phi^2|A|^{\eta}\Delta|A|^{\eta}. \end{aligned} \tag{2.10}$$

It then follows from divergence theorem that

$$\begin{aligned}\left(1-\frac{mn-2}{\eta mn}\right)\int_M \phi^2|\nabla|A|^\eta|^2 &\le -\int_M \phi^2|\nabla|A|^\eta|^2 - 2\int_M \phi|A|^\eta\langle\nabla\phi,\nabla|A|^\eta\rangle + \\ &\quad + \int_M (n\eta+\eta b(m)|A|^2)\phi^2|A|^{2\eta}.\end{aligned}$$

That is,

$$\begin{aligned}\left(2-\frac{mn-2}{\eta mn}\right)\int_M \phi^2|\nabla|A|^\eta|^2 &\le -2\int_M \phi|A|^\eta\langle\nabla\phi,\nabla|A|^\eta\rangle \\ &\quad + \int_M (n\eta+\eta b(m)|A|^2)\phi^2|A|^{2\eta}.\end{aligned} \tag{2.11}$$

The following estimates for the first eigenvalue are important tools for us.

Lemma 2.3. *([43]) Let M be a complete Riemannian manifold. Suppose that there are numbers a and c such that, for all geodesic balls $B_p(r)$ of radius r around some point p, $Vol(B_p(r)) \le cr^a$. Then $\liminf_{i\to\infty} 4^i\lambda_1(B(2^i))$ is bounded, where $\lambda_1(B_p(2^i)$ is the first Dirichlet eigenvalue of the Laplacian of $B_p(2^i)$. In particular, $\liminf_{r\to\infty} r^2\lambda_1(B_p(r))$ is bounded and*

$$\lambda_1(M) := \inf_{f\in H_0^1(M), f\neq 0} \frac{\int_M |\nabla f|^2}{\int_M f^2} \tag{2.12}$$

Lemma 2.4. *([101]) Let M be a complete simply connected Riemannian manifold with sectional curvature $K_M \le -1$ and let N be an n-dimensional complete non-compact submanifold immersed in M. Assume that the mean curvature vector of N satisfies $|H|(x) \le (n-1)/n < 1$, $\forall x \in N$. Then*

$$\lambda_1(M) \ge \frac{(n-1-nl)^2}{4}. \tag{2.13}$$

The Sobolev inequality below is needed in the proof of Theorem 2.2.

Lemma 2.5. *(See [58]) Let M be an n-dimensional complete hypersurface in a Hadarmard manifold $\overline{M}$ with mean curvature H. There exists a positive constant a which depends only on n such that*

$$\left(\int_M |\psi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le a\int_M (|\nabla\psi| + \psi H), \tag{2.14}$$

for any $\psi \in H_0^1(M)$.

Now we are ready to prove the main results in this section.

Proof of Theorem 2.1. From (2.2), we can find a sufficiently small $\epsilon > 0$ such that

$$\eta b(m)|A|^2(x) + \eta n \quad \leq \quad \left(2 - \frac{nm-2}{\eta nm}\right)\frac{(n-1)^2}{4} - \epsilon, \, \forall\, x \in M \tag{2.15}$$

We have from Lemma 2.4 that

$$\int_M |A|^{2\eta}\phi^2 \leq \frac{4}{(n-1)^2}\int_M |\nabla(\phi|A|^\eta)|^2. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.11), we get

$$\begin{aligned} &\left(2 - \frac{mn-2}{\eta mn}\right)\int_M \phi^2|\nabla|A|^\eta|^2 \\ \leq\ & -2\int_M \phi|A|^\eta\langle\nabla\phi, \nabla|A|^\eta\rangle + \left(2 - \frac{mn-2}{\eta mn} - \frac{4\epsilon}{(n-1)^2}\right)\int_M |\nabla(\phi|A|^\eta)|^2. \end{aligned} \tag{2.17}$$

That is,

$$\begin{aligned} &\frac{4\epsilon}{(n-1)^2}\int_M |\nabla|A|^\eta|^2\phi^2 \\ &\leq 2\delta\int_M \phi|A|^\eta\langle\nabla\phi, \nabla|A|^\eta\rangle + (1+\delta)\int_M |A|^{2\eta}|\nabla\phi|^2, \end{aligned} \tag{2.18}$$

where $\delta = 1 - \frac{mn-2}{\eta mn} - \frac{4\epsilon}{(n-1)^2}$. Combining (2.18) with Young's inequality

$$2\delta\int_M \phi|A|^\eta\langle\nabla\phi, \nabla|A|^\eta\rangle \quad \leq \quad \frac{\epsilon}{(n-1)^2}\int_M |\nabla|A|^\eta|^2\phi^2 + \frac{(n-1)^2}{\epsilon}\delta^2\int_M |A|^{2\eta}|\nabla\phi|^2,$$

we infer

$$\frac{3\epsilon}{(n-1)^2}\int_M |\nabla|A|^\eta|^2\phi^2 \leq \left(1 + \delta + \frac{(n-1)^2}{\epsilon}\delta^2\right)\int_M |A|^{2\eta}|\nabla\phi|^2 \tag{2.19}$$

Fix a point $p \in M$ and choose ϕ to be a cut-off function with the properties

$$0 \leq \phi \leq 1, \quad |\nabla\phi| \leq \frac{1}{R}, \quad \phi = \begin{cases} 1 & on \quad B_p(R) \\ 0 & on \quad M \setminus B_p(2R) \end{cases} \tag{2.20}$$

One can then easily get from (2.19) that

$$\begin{aligned} \int_{B_p(R)} |\nabla|A|^\eta|^2\phi^2 \quad &\leq \quad \int_M |\nabla|A|^\eta|^2\phi^2 \\ &\leq \quad \frac{(n-1)^2}{3\epsilon}\left(1 + \delta + \frac{(n-1)^2}{\epsilon}\delta^2\right)\frac{\int_{B_p(2R)}|A|^{2\eta}}{R^2}. \end{aligned} \tag{2.21}$$

Taking $R \to \infty$ and using (2.1), we conclude that $\nabla|A| = 0$, that is, $|A| = c = const$. If $c \neq 0$, we know from (2.1) that

$$\limsup_{R\to\infty} \frac{Vol[B_p(R)]}{R^2} = 0. \tag{2.22}$$

It then follows from Lemma 2.3 the $\lambda_1(M) = 0$ which contradicts with (2.13). Hence $|A| = 0$.

$\square$

Proof of Theorem 2.2**.** Replacing ψ by $\psi^{\frac{2(n-1)}{n-2}}$ in (2.14), we get

$$\left(\int_M |\psi^{\frac{2(n-1)}{n-2}}|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq a \int_M |\nabla \psi^{\frac{2(n-1)}{n-2}}|$$

using the Hölder's inequality, we have

$$\left(\int_M |\psi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \quad \leq \quad a_1 \int_M |\nabla\psi|^2 \tag{2.23}$$

where $a_1 = \left[a\frac{2(n-1)}{n-2}\right]^2$. Taking $\eta = \frac{d}{2}, \psi = |A|^\eta\phi$, with $\phi \in C_0^\infty(M)$, we get

$$\left(\int_M (|A|^\eta\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \quad \leq \quad a_1 \int_M |\nabla(|A|^\eta\phi)|^2. \tag{2.24}$$

Setting $\gamma = \left(\int_M |A|^n\right)^{\frac{2}{n}}$ and using Hölder's inequality again we obtain

$$\begin{aligned}\int_M |A|^{2\eta+2}\phi^2 \quad &\leq \quad \left(\int_M |A|^n\right)^{\frac{2}{n}} \left(\int_M (|A|^\eta\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq \quad a_1\gamma \int_M |\nabla(|A|^\eta\phi)|^2. \end{aligned}\tag{2.25}$$

Combining (2.11), (2.13) and (2.25) we have

$$\begin{aligned}\left(2 - \frac{mn-2}{\eta mn}\right)\int_M \phi^2|\nabla|A|^\alpha|^2 \quad &\leq \quad -2\int_M \phi|A|^\eta\langle\nabla\phi, \nabla|A|^\eta\rangle + \\ &\left(\eta b a_1\gamma + \frac{4\eta n}{(n-1)^2}\right)\int_M |\nabla(|A|^\eta\phi)|^2. \end{aligned}\tag{2.26}$$

that is,

$$\begin{aligned}\left(2 - \frac{mn-2}{\eta mn} - l\right)\int_M \phi^2|\nabla|A|^\eta|^2 \quad &\leq \quad 2(l-1)\int_M \phi|A|^\eta\langle\nabla\phi, \nabla|A|^\eta\rangle + \\ &+l\int_M |A|^{2\eta}|\nabla\phi|^2. \end{aligned}\tag{2.27}$$

where

$$l = \eta b a_1 \gamma + \frac{4\eta n}{(n-1)^2}.$$

Observe that the condition on the number $d = 2\eta$ in Theorem 2.2 implies that

$$2 - \frac{mn-2}{\eta mn} - \frac{4\eta n}{(n-1)^2} > 0.$$

Now let us take the constant C in Theorem (2.2) as

$$C = \left(\frac{2 - \frac{mn-2}{\eta mn} - \frac{4\eta n}{(n-1)^2}}{\eta b a_1} \right)^{\frac{n}{2}}. \tag{2.28}$$

With this choice for C, it is easy see that if (2.4) holds then

$$2 - \frac{mn-2}{\eta mn} - l > 0.$$

Hence, we can find a $\xi > 0$ so that

$$\left(2 - \frac{mn-2}{\eta mn} - l \right) \geq \xi. \tag{2.29}$$

Consequently, we have

$$\xi \int_M \phi^2 |\nabla |A|^\eta|^2 \leq 2(l-1) \int_M \phi |A|^\eta \langle \nabla \phi, \nabla |A|^\eta \rangle + l \int_M |A|^{2\eta} |\nabla \phi|^2. \tag{2.30}$$

For any $\sigma > 0$, it holds

$$2(l-1) \int_M \phi |A|^\eta \langle \nabla \phi, \nabla |A|^\eta \rangle \leq |l-1|\sigma \int_M \phi^2 |\nabla |A|^\eta|^2 + \frac{|l-1|}{\sigma} \int_M |A|^{2\eta} |\nabla \phi|^2. \tag{2.31}$$

Making an appropriate choice for σ so that $|l-1|\sigma \leq \frac{\xi}{2}$, we can deduce from (2.30) and (2.31) that there exists a constant $\theta > 0$ such that

$$\int_M \phi^2 |\nabla |A|^\eta|^2 \leq \theta \int_M |A|^{2\eta} |\nabla \phi|^2. \tag{2.32}$$

One can now define the cut-off function as in (2.20) and use the same arguments as in the proof of the final part of Theorem 2.1 to show that M is totally geodesic.

□

2.2 CMC hypersurfaces in hyperbolic space and semi-Riemannian manifolds

This section is the result of studies with Prof. Dr Xia, Prof. Dr. Wang and Prof. Dr. Adriano and is based on work H. Pina and C. Xia [81]. I would like to thank you for your contributions.

An important issue in differential geometry is to investigate relations between the geometric structure and the geometric invariants of submanifolds. A pioneering work in this direction due to Simons [90] states that if M is an n-dimensional closed minimal submanifold in an $(n+m)$-dimensional unit sphere with squared norm of the second fundamental form less than $n/(2-1/m)$, then M is totally geodesic. The proof of this result is based on so-called Simons' formula about the Laplacian of the squared norm of the second fundamental form of the minimal submanifolds. The appearance of Simons' formula is a landmark in the theory of submanifolds. The generalizations of Simons' formula have been widely used to prove rigidity theorems for submanifolds. Many interesting gap results have been proven during the past years.

In this section, we study rigidity phenomenon for complete non-compact hypersurfaces with constant mean curvature (CMC hypersurfaces) in a hyperbolic space and space-like CMC hypersurfaces in a Lorentz space form. Before stating our results, let us fix some notations. Let $\mathbb{H}^{n+1}(-1)$ be the $(n+1)$-dimensional complete Riemannian manifold with constant sectional curvature -1 and let $M_1^{n+1}(c)$ denote the Lorentzian space form with constant sectional curvature $c \in \{-1, 0, 1\}$. According to $c = 1, c = 0$ or $c = -1$, $M_1^{n+1}(c)$ is called a de Sitter space, a Minkowski space or anti-de Sitter space, respectively. A hypersurface in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite. Let M be an n-dimensional complete CMC hypersurface immersed in $\mathbb{H}^{n+1}(-1)$ or an n-dimensional space-like CMC hypersurface immersed in $M_1^{n+1}(c)$. In both cases, we denote by A and $H = \frac{1}{n} trA$ the second fundamental form and the mean curvature of M, respectively. Without loss of generality, we will assume throughout this paper that $H \geq 0$. Let $\langle , \rangle$ be the Riemannian metric on M and ϕ the traceless second fundamental form of M which is defined by

$$\langle \phi X, Y \rangle = \langle AX, Y \rangle - H\langle X, Y \rangle, \quad \forall X, Y \in T_pM, \; p \in M.$$

In the first part of this paper, we consider CMC hypersurfaces in a hyperbolic space.

Theorem 1. *Let M be a $n(\geq 2, \neq 3)$-dimensional complete non-compact CMC hypersurface immersed in $\mathbb{H}^{n+1}(-1)$ such that*

$$H < \frac{n(n-1) - 2\sqrt{(n-2)(6n-9)}}{n^2 + 4n - 8}. \tag{2.33}$$

Suppose that

$$\limsup_{R \to +\infty} \frac{\int_{B_p(R)} |\phi|^d}{R^2} = 0, \tag{2.34}$$

for some d satisfying

$$\frac{2n(1-H^2)}{(n-1-nH)^2} d \in \left(1 - \sqrt{1 + \frac{4(n-2)(H^2-1)}{(n-1-nH)^2}}\,, \right. \tag{2.35}$$
$$\left. 1 + \sqrt{1 + \frac{4(n-2)(H^2-1)}{(n-1-nH)^2}} \right),$$

where $B_p(R)$ denotes the geodesic ball of radius R centered at $p \in M$. If

$$\sup_{x \in M} \left(|\phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| \right) < \left(1 - \frac{n-2}{nd}\right) \frac{(n-1-nH)^2}{d} - n(1-H^2), \tag{2.36}$$

then M is totally umbilical.

In next result we replace the point-wise condition (1.10) by a global condition, that is, the L^n-norm of $|\phi|$ on M.

Theorem 2. *Let M be a $n(> 3)$-dimensional complete non-compact CMC hypersurface immersed in $\mathbb{H}^{n+1}(-1)$. Suppose that (2.34) is satisfied for a constant d such that*

$$\frac{2n(1-H^2)}{(n-1-nH)^2} d \in \left(1 - \sqrt{1 + 4\frac{(n-2)(1-H^2)}{(n-1-nH)^2}}, 1 + \sqrt{1 + 4\frac{(n-2)(1-H^2)}{(n-1-nH)^2}}\right) \tag{2.37}$$

with

$$H < \frac{n(n-1) - 2\sqrt{(n-2)(6n-9)}}{n^2 + 4n - 8}. \tag{2.38}$$

If there exists a positive constant C which depends only on n, H, and d such that

$$\int_M |\phi|^n < C, \tag{2.39}$$

then M is totally umbilical.

In the next section, we will cover CMC hypersurfaces in semi-Riemannian manifolds $M_1^{n+1}(c)$, with constant curvature $c \in \{-1, 0, 1\}$.

CMC Hypersurfaces in $M_1^{n+1}(c)$

When the ambient spacetime is Lorentz-Minkowski space $M_1^{n+1}(0) = \mathbb{L}^{n+1}$ and the spacelike hypersurface is given as a graph of a certain function u, the condition of constant mean curvature H is written in terms of u as follows:

$$(1 - |\nabla u|^2)\Delta u + (\nabla^2 u)(\nabla u, \nabla u) = nH(1 - |\nabla u|^2)^{\frac{3}{2}}, \quad |\nabla u|^2 < 1$$

where, ∇, ∇^2 and Δ denote the gradient, Hessian and Laplacian of M, respectively. Cheng and Yau [38], to show that if M be a maximal, so the only entire solutions to that equation are linear. The case $H \neq 0$, which has a completely different behaviour, was extensively studied by [2, 92].

In 1977 Goddard, conjectured the following: Every complete spacelike CMC hypersurface in $\mathbb{M}_1^{n+1}(1) = \mathbb{S}_1^{n+1}(1)$ must be totally umbilical. The first result in this direction was obtained by J. Ramanathan [84], 1987, he proved that if a complete spacelike CMC hypersurface in $\mathbb{M}_1^3(1)$ with $H^2 < 1$, then the surface is totally umbilical. Akutagawa [2], 1987, has proved that Goddard's conjecture is true in $\mathbb{M}_1^{n+1}(c)$, with $c > 0$, when $n = 2$ and $H^2 < c$ or $n \geq 3$ and $H^2 < 4\frac{(n-1)}{n^2}$. Montiel [68], 1988, exhibited examples of complete spacelike CMC hypersurfaces in $\mathbb{M}_1^{n+1}(1)$ with $H^2 \geq 4\frac{(n-1)}{n^2}$ and being non totally umbilical, the so-called hyperbolic cylinders, which are isometric to the Riemannian product $M_1^{n-1}(c_1) \times M_1^1(c_2)$, where

$c_1 > 0$, $c_2 < 0$ and $\frac{1}{c_1} + \frac{1}{c_2} = 1$. Montiel to showed example: Consider the spacelike hypersurface embedded into $\mathbb{S}_1^{n+1}$ given by

$$M^n = \left\{x \in \mathbb{S}_1^{n+1} | -x_0^2 + x_1^2 + \cdots + x_k^2 = -\sinh^2 r\right\}$$

with $r > 0$ and $1 \leq k \leq n$. M is isometric to the Riemannian product $\mathbb{H}^k(1 - \coth^2 r) \times \mathbb{S}^{n-k}(1 - \tanh^2 r)$ with mean curvature

$$H^2 \geq \frac{1}{n^2} (\coth r + (n-1)\tanh r)^2 \geq \frac{4(n-1)}{n^2}$$

Then the Goddard's Conjecture is not always true. The following theorem is generalizations of the [38] and can be seen as an extension of Goddard's conjecture for complete non-compact spacelike CMC hypersurfaces in $M_1^{n+1}(c)$, where $c = \{-1, 0, 1\}$.

Theorem 3. *Let M be an $n(\geq 3)$-dimensional non-compact complete spacelike CMC hypersurface immersed in $M_1^{n+1}(c)$, $c \in \{-1, 0, 1\}$. Suppose that (2.34) is satisfied for constant d such that*

$$\begin{aligned} &\frac{(n-1)(H^2-c) - \sqrt{(n-1)^2(H^2-c)^2 - H^2(n^2-4n+5)(H^2-c)}}{nH^2(n^2-4n+5)} \\ &< \frac{d}{2(n-1)(n-2)} < \\ &\frac{(n-1)(H^2-c) + \sqrt{(n-1)^2(H^2-c)^2 - H^2(n^2-4n+5)(H^2-c)}}{nH^2(n^2-4n+5)}, \end{aligned} \tag{2.40}$$

with

$$\inf_{x \in M} \left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|\phi| + n(c - H^2)\right) > -\left(\frac{n^2-4n+5}{n-2}\right)\frac{nH^2}{4}. \tag{2.41}$$

If the first eigenvalue of Laplacian of M satisfies

$$\lambda_1(M) > \frac{n^2H^2d^2}{16(n-2)}\left(\frac{n^2-4n+5}{n(d-1)+2}\right), \tag{2.42}$$

in additional for $c = 1$ if $H > \frac{(n-1)^2}{2(n-2)}$. Then $M = \mathbb{H}^n(c - H^2)$, $c = \{-1, 0, 1\}$.

The next theorem extends the result obtained by J. Ramanathan [84] for surfaces in $M_1^3(c)$.

Theorem 4. *Let M be a non-compact complete space-like CMC surface immersed in $M_1^3(c)$, $c = \{-1, 0, 1\}$. Suppose that* (2.34) *is satisfied with a constant d satisfying*

$$d \in \left(0, \frac{1}{2}\right) \tag{2.43}$$

and the first eigenvalue of M bounded by

$$\lambda_1(M) > \frac{d(H^2 - c)}{2}, \tag{2.44}$$

with $H > 1$ if $c = 1$. Then $M = \mathbb{H}^2(c - H^2)$.

Remark 1.1 An important result due to Cheng-Yau states that the first eigenvalue of a complete non-compact Riemannian manifold with polynomial volume growth is zero (Cf. [43], [66]). Combining this result with the Calabi- Cheng-Yau theorem (Cf. [21], [34]) we know that the mean curvature H of M in Theorems 1.3, 1.4 and 1.5 is not zero since $\lambda_1(M) > 0$. The following theorem considers maximal immersions in anti de-Sitter space $M_1^{n+1}(-1)$.

Theorem 5. *Let M be an n (≥ 2)-dimensional complete maximal spacelike hypersurface immersed in $M_1^{n+1}(-1)$. Suppose that*

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_p(R)} |A|^d = 0, \tag{2.45}$$

where d is a positive constant such that

$$\frac{n-1}{n} < d < \frac{(n-1)(n-2)}{n} \tag{2.46}$$

where $B_p(R)$ is the geodesic ball centred in $p \in M$. If the first eigenvalue of Laplacian of M bounded lower by

$$\lambda_1(M) > \frac{d^2 n^2}{4(n(d-1)+2)}, \tag{2.47}$$

then M is totally geodesic.

2.3 Proof of the main theorems - $\mathbb{H}^{n+1}(-1)$

From the definition of the traceless part of second fundamental form of M, that is, $\phi = A - HI$, we have $|\phi|^2 = |A|^2 - nH^2$. Both parts of this work we will consider

$H > 0$, otherwise simply reverse the orientation of M. Cheung and Zhou [19] get the following Simons' type inequality for traceless second fundamental form:

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 + n(H^2-1)|\phi|^2. \tag{2.48}$$

Taking $\theta = \frac{n(n-2)}{\sqrt{n(n-1)}}H$ and $\beta = n(H^2-1)$ then the above inequality is rewritten as

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \theta|\phi|^3 + \beta|\phi|^2. \tag{2.49}$$

By (2.49), we compute

$$\begin{aligned}
|\phi|^\sigma\Delta|\phi|^\sigma &= |\phi|^\sigma \mathrm{div}(\nabla|\phi|^\sigma) \\
&= \frac{\sigma-1}{\sigma}|\nabla|\phi|^\sigma|^2 + \sigma|\phi|^{2\sigma-2}|\phi|\Delta|\phi| \\
&\geq \frac{\sigma-1}{\sigma}|\nabla|\phi|^\sigma|^2 + \frac{2\sigma}{n}|\phi|^{2\sigma-2}|\nabla|\phi||^2 \\
&\quad -\sigma\left(|\phi|^{2\sigma+2} - \theta|\phi|^{2\sigma+1} + \beta|\phi|^{2\sigma}\right) \\
&= \left(1 - \frac{n-2}{n\sigma}\right)|\nabla|\phi|^\sigma|^2 - \sigma\left(|\phi|^2 + \theta|\phi| - \beta\right)|\phi|^{2\sigma},
\end{aligned} \tag{2.50}$$

where σ is a nonnegative constant. Let $f \in C_0^\infty(M)$. Multiplying (2.50) by f^2 and integrating on M, we obtain

$$\left(1 - \frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq \int_M f^2|\phi|^\sigma\Delta|\phi|^\sigma + \sigma\int_M \left(|\phi|^2 + \theta|\phi| - \beta\right) f^2|\phi|^{2\sigma}.$$

Applying the divergence theorem in inequaliy above we obtain

$$\begin{aligned}
\left(2 - \frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 &\leq -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma, \nabla f\rangle + \\
&\quad +\sigma\int_M \left(|\phi|^2 + \theta|\phi| - \beta\right) f^2|\phi|^{2\sigma}.
\end{aligned} \tag{2.51}$$

Proof of Theorem 1. From (2.33), (2.35) and (2.36) taking $d = 2\sigma$, we can find $\varepsilon > 0$ sufficiently small in

$$\varepsilon \in \left(0, \left(2 - \frac{n-2}{n\sigma}\right)\frac{(n-1-nH)^2}{4\sigma}\right) \tag{2.52}$$

such that

$$|\phi|^2 + \theta|\phi| - \beta \leq \left(2 - \frac{n-2}{n\sigma}\right)\frac{(n-1-nH)^2}{4\sigma} - \varepsilon. \tag{2.53}$$

Substituting in (2.51) we have

$$\begin{aligned}\left(2-\frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle \\ +\left(\left(2-\frac{n-2}{n\sigma}\right)\frac{(n-1-nH)^2}{4}-\sigma\varepsilon\right)\int_M f^2|\phi|^{2\sigma}.\end{aligned} \tag{2.54}$$

Taking $\gamma=\frac{(n-1-nH)^2}{4}$, by Lemmas 2.3 and 2.4, we get

$$\int_M f^2 \leq \frac{4}{(n-1-nH)^2}\int_M |\nabla f|^2 = \frac{1}{\gamma}\int_M |\nabla f|^2.$$

We can taking $f=|\phi|^\sigma f$ in inequality above and substituting in (2.54) we have

$$\begin{aligned}\left(2-\frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \quad &\leq \quad -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle \\ &\quad +\left(\left(2-\frac{n-2}{n\sigma}\right)-\frac{\sigma\varepsilon}{\gamma}\right)\int_M |\nabla(|\phi|^\sigma f)|^2.\end{aligned} \tag{2.55}$$

So we got the following inequality

$$\frac{\sigma\varepsilon}{\gamma}\int_M |\nabla|\phi|^\sigma|^2 f^2 \quad \leq \quad 2\delta\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle+(\delta+1)\int_M |\phi|^{2\sigma}|\nabla f|^2, \tag{2.56}$$

where $\delta=1-\frac{n-2}{n\sigma}-\frac{\sigma\varepsilon}{\gamma}$. Using Young's inequality in (2.56) we have

$$\frac{3\sigma\varepsilon}{4\gamma}\int_M |\nabla|\phi|^\sigma|^2 f^2 \quad \leq \quad (\delta+1+\frac{4\gamma}{\sigma\varepsilon}\delta^2)\int_M |\phi|^{2\sigma}|\nabla f|^2. \tag{2.57}$$

Fix a point $p\in M$ we can choose f to be a cut-off function with the properties

$$0\leq f\leq 1, \quad |\nabla f|\leq \frac{1}{R}, \quad f=\begin{cases} 1 & on \quad B_p(R) \\ 0 & on \quad M\setminus B_p(2R).\end{cases} \tag{2.58}$$

One can then easily get from (2.57) that

$$\int_{B_p(R)} |\nabla|\phi|^\sigma|^2 f^2 \leq \int_M |\nabla|\phi|^\sigma|^2 f^2 \leq \frac{4\gamma}{3\sigma\varepsilon}(\delta+1+\frac{4\gamma}{\sigma\varepsilon}\delta^2)\int_M |\phi|^{2\sigma}|\nabla f|^2. \tag{2.59}$$

Taking $R\to+\infty$, by (2.34) we conclude that $\nabla|\phi|=0$, that is, $|\phi|=c$, where c is constant. If $c\neq 0$, we know again from (2.34) that

$$\limsup_{R\to\infty}\frac{Vol[B_p(R)]}{R^2}=0. \tag{2.60}$$

It then follows from Lemma 2.3 the $\lambda_1(M)=0$, which contradicts with (2.13). Hence $|\phi|=0$. $\square$

Proof of Theorem 2. Let $\psi \in H_0^1(M),\ \psi \geq 0$. Replacing ψ by $\psi^{\frac{2(n-1)}{n-2}}$ in (2.14) and using Hölder's inequality, we get

$$\left(\int_M \psi^{\frac{2n}{n-2}}\right)^{\frac{n-1}{n}} \leq a_1 \int_M \left(|\nabla\psi| + \psi H\right)^2, \tag{2.61}$$

where $a_1 = \left[a\frac{2(n-1)}{n-2}\right]^2$. Taking $\psi = |\phi|^\sigma f$, with $f \in C_0^\infty(M),\ f \geq 0$ in (2.61), we get

$$\begin{aligned}\left(\int_M (|\phi|^\sigma f)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq a_1 \int_M \left(|\nabla(|\phi|^\sigma f)| + |\phi|^\sigma f H\right)^2 \\ &\leq 2a_1 \int_M \left(|\nabla(|\phi|^\sigma f)|^2 + |\phi|^{2\sigma} f^2 H^2\right).\end{aligned} \tag{2.62}$$

Setting $\Lambda = \left(\int_M |\phi|^n\right)^{\frac{2}{n}}$, we then get from Hölder's inequality that

$$\begin{aligned}\int_M |\phi|^{2\sigma+2} f^2 &\leq \left(\int_M |\phi|^n\right)^{\frac{2}{n}} \left(\int_M (|\phi|^\sigma f)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq 2a_1\Lambda \int_M \left(|\nabla(|\phi|^\sigma f)|^2 + |\phi|^{2\sigma} f^2 H^2\right).\end{aligned} \tag{2.63}$$

Taking $\varepsilon > 0$, returning in (2.51) and using the Young's inequality

$$\theta|\phi| \leq \frac{\theta^2|\phi|^2}{2\varepsilon} + \frac{\varepsilon}{2},$$

we get

$$\begin{gathered}\left(2 - \frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma, \nabla f\rangle + \\ \sigma\left(1 + \frac{\theta^2}{2\varepsilon}\right)\int_M f^2|\phi|^{2\sigma+2} + \sigma\left(\frac{\varepsilon}{2} - \beta\right)\int_M f^2|\phi|^{2\sigma}.\end{gathered} \tag{2.64}$$

Using (2.61), (2.62) and (2.63) in (2.64) we have

$$\begin{gathered}\left(2 - \frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma, \nabla f\rangle + \\ 2a_1\Lambda\sigma\left(1 + \frac{\theta^2}{2\varepsilon}\right)\int_M |\nabla(f|\phi|^\sigma)|^2 + \left[2a_1\Lambda\sigma\left(1 + \frac{\theta^2}{2\varepsilon}\right)H^2 + \sigma\left(\frac{\varepsilon}{2} - \beta\right)\right]\int_M f^2|\phi|^{2\sigma}.\end{gathered} \tag{2.65}$$

Taking $\gamma = \frac{(n-1-nH)^2}{4}$, we have from Lemma 2.4 and (2.65) that

$$\begin{gathered}\left(2 - \frac{n-2}{n\sigma}\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq -2\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma, \nabla f\rangle \\ +\sigma\left[2a_1\Lambda\left(1 + \frac{\theta^2}{2\varepsilon}\right)\left(1 + \frac{H^2}{\gamma}\right) + \frac{1}{\gamma}\left(\frac{\varepsilon}{2} - \beta\right)\right]\int_M |\nabla(f|\phi|^\sigma)|^2.\end{gathered} \tag{2.66}$$

We soon have

$$\left(2-\frac{n-2}{n\sigma}-\sigma\kappa\right)\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq 2\,(\sigma\kappa-1)\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle + \sigma\kappa\int_M |\phi|^{2\sigma}|\nabla f|^2, \tag{2.67}$$

where

$$\kappa = 2a_1\Lambda\left(1+\frac{\theta^2}{2\varepsilon}\right)\left(1+\frac{H^2}{\gamma}\right)+\frac{1}{\gamma}\left(\frac{\varepsilon}{2}-\beta\right).$$

Then for each $\varepsilon>0$, using (2.37) and (2.38) we can take the constant C in Theorem 2 as

$$C=\left(\frac{2-\frac{n-2}{n\sigma}+\frac{\sigma}{\gamma}\left(\beta-\frac{\varepsilon}{2}\right)}{2a_1\sigma\left(1+\frac{\theta^2}{2\varepsilon}\right)\left(1+\frac{H^2}{\gamma}\right)}\right)^{\frac{n}{2}}. \tag{2.68}$$

So that if (2.39) holds then

$$\left(2-\frac{n-2}{n\sigma}\right)-\sigma\kappa>0. \tag{2.69}$$

Hence, we can find a $\rho>0$ such that

$$\left(2-\frac{n-2}{n\sigma}\right)-\sigma\kappa\geq\rho. \tag{2.70}$$

Consequently,

$$\rho\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq 2(\sigma\kappa-1)\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle + \sigma\kappa\int_M |\phi|^{2\sigma}|\nabla f|^2. \tag{2.71}$$

For any $\delta>0$, it holds

$$2(\sigma\kappa-1)\int_M |\phi|^\sigma f\langle\nabla|\phi|^\sigma,\nabla f\rangle \leq |\sigma\kappa-1|\delta\int_M |\nabla|\phi|^\sigma|^2 f^2 + \frac{|\sigma\kappa-1|}{\delta}\int_M |\phi|^{2\sigma}|\nabla f|^2. \tag{2.72}$$

Making an appropriate choice for δ so that $|\sigma\kappa-1|\delta\leq\rho$, we can deduce from (2.72) that there exists a constant $C_2>0$ such that

$$\int_M |\nabla|\phi|^\sigma|^2 f^2 \leq C_2\int_M |\phi|^{2\sigma}|\nabla f|^2. \tag{2.73}$$

One can now use the same arguments as in the proof of the final part of Theorem 1 to show that M is totally umbilical. □

2.4 Proof of the main theorems - $\mathbb{M}_1^{n+1}(c)$

Consider the inequality deduced by Montiel in [69] for spacelike CMC hypersurfaces in $M_1^{n+1}(c)$:

$$\frac{1}{2}\Delta|\phi|^2 \geq |\nabla\phi|^2 + |\phi|^2\left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c-H^2)\right). \qquad (2.74)$$

Since that H is constant, in [60] we have the following Kato's inequality:

Lemma 2.6. *Let M be an spacelike hypersurface immersed in $M_1^{n+1}(c)$ with parallel mean curvature, then*

$$|\nabla\phi|^2 - |\nabla|\phi||^2 \geq \frac{2}{n}|\nabla|\phi||^2. \qquad (2.75)$$

On the other hand, H constant provides us with $\nabla A = \nabla\phi$ and $\nabla|A|^2 = \nabla|\phi|^2$. So, using 2.75 we can rewrite (2.74) as follows

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 + |\phi|^2\left(|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi| + n(c-H^2)\right). \qquad (2.76)$$

Taking $\theta = \frac{n(n-2)}{\sqrt{n(n-1)}}H$ and $\overline{\beta} = n(c-H^2)$ then the above inequality is rewritten as

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 + |\phi|^2\left(|\phi|^2 - \theta|\phi| + \overline{\beta}\right). \qquad (2.77)$$

Similarly to what was done in (2.50), by (2.77) we compute

$$|\phi|^\alpha\Delta|\phi|^\alpha \leq \left(1 - \frac{n-2}{n\alpha}\right)|\nabla|\phi|^\alpha|^2 + \alpha\left(|\phi|^2 - \theta|\phi| + \overline{\beta}\right)|\phi|^{2\alpha}, \qquad (2.78)$$

where α is a nonnegative constant. The following Lemma is a important tools for us.

Lemma 2.7. *[1] - Let M^n a complete spacelike hypersurface in $\overline{M}_1^{n+1}(c)$. If M has $\{\mu\}$, $\{\mu,\nu\}$ as a set of its main curvatures, with μ and ν constants, then M is isometric*

(i) Umbilical hypersurface

$$\mathbb{R}^n = \{x \in \overline{M}_1^{n+1}(c); x_{n+1} = 0\}$$

or

$$\mathbb{H}^n(-r^2) = \left\{x \in \overline{M}_1^{n+1}(c); \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -\frac{1}{r^2}\right\}$$

(ii) Euclidean product space of $\mathbb{R}^m$ *and hyperbolic space* $\mathbb{H}^{n-m}(-r^2)$*, i.e.,*

$$\mathbb{R}^m \times \mathbb{H}^{n-m}(-r^2) = \left\{ x \in \overline{M}_1^{n+1}(c); \sum_{i=m+1}^{n} x_i^2 - x_{n+1}^2 = -\frac{1}{r^2} \right\}$$

Proof Theorem 3. Let $f \in C_0^\infty(M)$. Multiplying (2.78) by f^2 and integrating on M, we obtain

$$\begin{aligned} \left(1 - \frac{n-2}{n\alpha}\right) \int_M |\nabla|\phi|^\alpha|^2 f^2 \quad &\leq \quad \alpha \int_M \left(-|\phi|^2 + \theta|\phi| - \overline{\beta}\right) f^2 |\phi|^{2\alpha} \qquad (2.79) \\ &\quad + \int_M f^2 |\phi|^\alpha \Delta |\phi|^\alpha. \end{aligned}$$

By (2.41) exists $\varepsilon > 0$ such that

$$-|\phi|^2 + \theta|\phi| - \overline{\beta} < \left(\frac{n^2 - 4n + 5}{n-2}\right) \frac{nH^2}{4} - \varepsilon.$$

Applying the Divergence Theorem in (2.79) and using Young's inequality we have

$$\begin{aligned} \left(2 - \frac{n-2}{n\alpha} - \varepsilon\right) \int_M |\nabla|\phi|^\alpha|^2 f^2 \quad &\leq \quad \frac{1}{\varepsilon} \int_M |\nabla f|^2 |\phi|^{2\alpha} \qquad (2.80) \\ &\quad + \alpha \left(\left(\frac{n^2 - 4n + 5}{n-2}\right) \frac{nH^2}{4} - \varepsilon \right) \int_M f^2 |\phi|^{2\alpha}. \end{aligned}$$

Taking $f = f|\phi|^\alpha$ in (2.11) and using Young's inequality we obtain

$$\lambda_1 \int_M f^2 |\phi|^{2\alpha} \leq \frac{1+\varepsilon}{\varepsilon} \int_M |\nabla f|^2 |\phi|^{2\alpha} + (1+\varepsilon) \int_M |\nabla|\phi|^\alpha|^2 f^2. \qquad (2.81)$$

Note that it is possible to obtain $\varepsilon > 0$ such that $2 - \frac{n-2}{n\alpha} - \varepsilon > 0$. Multiplying (2.80) by $(1+\varepsilon)$ and (2.81) by $2 - \frac{n-2}{n\alpha} - \varepsilon$ and joining these inequalities we get

$$\begin{aligned} &\left[\lambda_1 \left(2 - \frac{n-2}{n\alpha} - \varepsilon\right) - \alpha \left(\left(\frac{n^2 - 4n + 5}{n-2}\right) \frac{nH^2}{4} - \varepsilon \right) (1+\varepsilon) \right] \int_M f^2 |\phi|^{2\alpha} \leq \\ &\qquad \leq (1+\varepsilon) \left(2 - \frac{n-2}{n\alpha} - \varepsilon + \frac{1}{\varepsilon}\right) \int_M |\nabla f|^2 |\phi|^{2\alpha}. \qquad (2.82) \end{aligned}$$

We can taking $d = 2\alpha$. Rearranging the therms in (2.82) we have

$$\begin{aligned} &\left[\lambda_1 \left(\frac{nd - (n-2)}{n\alpha}\right) - \frac{d}{8} \left(\frac{n^2 - 4n + 5}{n-2}\right) nH^2 \right. \qquad (2.83) \\ &\left. -\varepsilon \left(\alpha \left(\frac{n^2 - 4n + 5}{n-2}\right) \frac{nH^2}{4} + \lambda_1 + 1 + \varepsilon \right) \right] \int_M f^2 |\phi|^d \\ &\leq (1+\varepsilon) \left(2 - \frac{n-2}{n\alpha} - \varepsilon + \frac{1}{\varepsilon}\right) \int_M |\nabla f|^2 |\phi|^d. \end{aligned}$$

By (2.40) and (2.42) for $c = \{-1, 0, 1\}$, case $c = 1$ consider $H^2 > \frac{(n-1)^2}{2(n-2)}$, we have

$$\lambda_1\left(\frac{nd-(n-2)}{n\alpha}\right) - \frac{d}{8}\left(\frac{n^2-4n+5}{n-2}\right)nH^2$$
$$-\varepsilon\left(\alpha\left(\frac{n^2-4n+5}{n-2}\right)\frac{nH^2}{4} + \lambda_1 + 1 + \varepsilon\right) > 0.$$

Therefore, exists a constant C such that (2.83) give us

$$\int_M f^2|\phi|^d \leq C\int_M |\nabla f|^2|\phi|^d. \tag{2.84}$$

One can then easily get from (2.84) and using the cut-off function (2.58) that

$$\int_{B_p(R)} |\phi|^d \leq \int_M f^2|\phi|^d \leq C_1\int_{B_p(R)} |\nabla f|^2|\phi|^d. \tag{2.85}$$

Taking $R \to +\infty$ and using (2.34), we conclude that $|\phi| = 0$ in M, that is, M is totally umbilical. Since that H is constant, we can observe that M is isoparametric. By Lemma 2.7, M is isometric to $\mathbb{R}^n$ or $\mathbb{H}^n(-r^2)$. By hypothesis $\lambda_1(M) > 0$, which shows us that M is isometric to $\mathbb{H}^n(-r^2)$. The fact that M has constant mean curvature and the eigenvalues of the second fundamental form are all equal, implies that $-r^2 = c - H^2$. □

Proof Theorem 4. When $n = 2$ the inequality (1.28)becomes

$$|\phi|^\alpha\Delta|\phi|^\alpha \geq |\nabla|\phi|^\alpha|^2 + \alpha|\phi|^{2\alpha+2} + 2\alpha(c - H^2)|\phi|^{2\alpha}. \tag{2.86}$$

Let q a nonegative constant and $f \in C_0^\infty(M)$. Multiplying (2.86) by $f^2|\phi|^{2q\alpha}$ and integrating on M, we obtain

$$\begin{aligned}\int_M |\nabla|\phi|^\alpha|^2|\phi|^{2q\alpha}f^2 \quad &\leq \quad 2\alpha(H^2-c)\int_M f^2|\phi|^{2(q+1)\alpha} - \alpha\int_M f^2|\phi|^{2(q+1)\alpha+2} \\ &+ \int_M f^2|\phi|^{(2q+1)\alpha}\Delta|\phi|^\alpha.\end{aligned} \tag{2.87}$$

Applying the Divergence Theorem and by Young's inequality in(2.87) we get

$$\begin{aligned}(2(q+1)-\varepsilon)\int_M |\nabla|\phi|^\alpha|^2|\phi|^{2q\alpha}f^2 &\leq \frac{1}{\varepsilon}\int_M |\nabla f|^2|\phi|^{2(q+1)\alpha} + \\ 2\alpha(H^2-c)\int_M f^2|\phi|^{2(q+1)\alpha} &- \alpha\int_M f^2|\phi|^{2(q+1)\alpha+2}.\end{aligned} \tag{2.88}$$

Since $d := 2(q+1)\alpha$, it's possible to obtain $\varepsilon > 0$ such that $\dfrac{d}{\alpha} - \varepsilon = 2(q+1) - \varepsilon > 0$. Multiplying (2.88) by $(q+1)(q+1+\varepsilon)$ and (2.31) by $\left(\dfrac{d}{\alpha} - \varepsilon\right)$ and joining these inequalities we have

$$\begin{aligned}\left(\frac{d}{\alpha} - \varepsilon\right)\lambda_1 \int_M f^2|\phi|^d \quad \leq \quad & \left(\frac{d}{\alpha} - \varepsilon\right)\frac{(q+1+\varepsilon)}{\varepsilon}\int_M |\nabla f|^2|\phi|^d \\ & + \frac{(q+1)}{\varepsilon}(q+1+\varepsilon)\int_M |\nabla f|^2|\phi|^d \\ & + 2\alpha(H^2 - c)(q+1)(q+1+\varepsilon)\int_M f^2|\phi|^d \\ & - \alpha(q+1)(q+1+\varepsilon)\int_M f^2|\phi|^{d+2}.\end{aligned}$$

For a quick calculation in expression above we can obtain

$$\begin{aligned}\left(\frac{d}{\alpha}\lambda_1 + \frac{d^2}{2\alpha}(c - H^2) - \varepsilon\left(\lambda_1 + d(H^2 - c)\right)\right)\int_M f^2|\phi|^d & + \\ \alpha(q+1)(q+1+\varepsilon)\int_M f^2|\phi|^{d+2} \leq \frac{(q+1+\varepsilon)}{\varepsilon}\left(\frac{d}{\alpha} + q + 1 + \varepsilon\right)\int_M |\nabla f|^2|\phi|^d. &\end{aligned} \tag{2.89}$$

By (2.43) and (2.44) we get

$$\frac{d}{\alpha}\lambda_1 + \frac{d^2}{2\alpha}(c - H^2) - \varepsilon\left(\lambda_1 + d(H^2 - c)\right) > 0,$$

with $H > 1$ if $c = 1$. So exists a positive constant C_1 such that (2.89) can be rewritten as

$$\int_M f^2|\phi|^d \leq C_1 \int_M |\nabla f|^2|\phi|^d. \tag{2.90}$$

Using the same argument in the final of the Theorem 3 we have that M is totally umbilical, as the eigenvalues of the second fundamental form are all equal and $\lambda_1(M) > 0$, by Lemma 2.7 $M = \mathbb{H}^2(c - H^2)$. □

Proof of Theorem 5. Since that $c = -1$ and $H = 0$ in (2.6) we get

$$|A|\Delta|A| \geq \frac{2}{n}|\nabla|A||^2 + |A|^2\left(|A|^2 - n\right). \tag{2.91}$$

Let α a nonnegative constant. As already done in previous results, by (2.91) we compute

$$|A|^\alpha\Delta|A|^\alpha \geq \left(1 - \frac{n-2}{n\alpha}\right)|\nabla|A|^\alpha|^2 + \alpha\left(|A|^2 - n\right)|A|^{2\alpha}. \tag{2.92}$$

Let $f \in C_0^\infty(M)$. Multiplying (2.92) by f^2 and integrating over M, we obtain

$$\begin{aligned}\left(1-\frac{n-2}{n\alpha}\right)\int_M |\nabla|A|^\alpha|^2 f^2 &\leq -\alpha\int_M (|A|^2-n)f^2|A|^{2\alpha}\\ &\quad+\int_M f^2|A|^\alpha\Delta|A|^\alpha.\\ &\leq \int_M f^2|A|^\alpha\Delta|A|^\alpha + n\alpha\int_M f^2|A|^{2\alpha}\end{aligned}$$

Applying the Divergence Theorem and Young's inequality in inequality above, we get

$$\left(2-\frac{n-2}{n\alpha}-\varepsilon\right)\int_M |\nabla|A|^\alpha|^2 f^2 \leq \left(1+\frac{1}{\varepsilon}\right)\int_M |\nabla f|^2|A|^{2\alpha} + n\alpha\int_M f^2|A|^{2\alpha}. \tag{2.93}$$

Taking $f = f|A|^\alpha$ in (2.11) and using Young inequality

$$\lambda_1\int_M f^2|A|^{2\alpha} \leq \frac{1+\varepsilon}{\varepsilon}\int_M |\nabla f|^2|A|^{2\alpha} + (1+\varepsilon)\int_M |\nabla|A|^\alpha|^2 f^2. \tag{2.94}$$

Note that it is possible to obtain $\varepsilon > 0$ such that $2-\dfrac{n-2}{n\alpha}-\varepsilon > 0$. Multiplying 2.93 by $(1+\varepsilon)$ and (2.94) by $2-\dfrac{n-2}{n\alpha}-\varepsilon$ and joining these inequalities we have

$$\begin{aligned}\left[\lambda_1\left(2-\frac{n-2}{n\alpha}-\varepsilon\right)-n\alpha(1+\varepsilon)\right]\int_M f^2|A|^{2\alpha} &\leq\\ \leq \frac{(1+\varepsilon)}{\varepsilon}\left(3-\frac{n-2}{n\alpha}\right)\int_M |\nabla f|^2|A|^{2\alpha}.\end{aligned} \tag{2.95}$$

Since $d = 2\alpha$, by (2.46) and (2.47) exist a constant $C > 0$ such that

$$\int_M f^2|A|^d \leq C\int_M |\nabla f|^2|A|^d. \tag{2.96}$$

Set the cut-off function as in (2.58). As already done in previous results, provided that (2.45) is satisfied, we concluded that $|A| = 0$ on M, that is, M is totally geodesic. Therefore isometric to a hyperbolic space $\mathbb{H}^n(-1)$. □

Chapter 3

Generalized quasi Yamabe gradient Solitons

This section is the result of studies with Prof. Dr. Benedito L. N. and is based on work of the B.L. Neto and H. Pina [64]. Thank you for your contribution.

A complete Riemannian manifold (M^n, g), $n \geq 3$, is a *generalized quasi Yamabe gradient soliton* (GQY manifold), if there exist a constant λ and two smooth functions, f and μ, on M, such that

$$(R - \lambda)g = \nabla^2 f - \mu df \otimes df \tag{3.1}$$

where R denotes the scalar curvature of the metric g and df is the dual 1-form of ∇f. In a local coordinates system, we have

$$(R - \lambda)g_{ij} = \nabla_i \nabla_j f - \mu \nabla_i f \nabla_j f. \tag{3.2}$$

When f is a constant function, we say that (M^n, g) is a *trivial* generalized quasi Yamabe gradient soliton. Otherwise, it will be called *nontrivial*.

As it was said in the introduction, the essence of this chapter is to demonstrate the following Theorem. We will see that it has several consequences

Theorem 3.1. *Let $(M^n,\, g)$, $n \geq 3$, be a nontrivial complete generalized quasi Yamabe gradient soliton satisfying (3.1). Then,*

$$\mu \textit{ must be constant on each connected component of } M \textit{ or} \tag{3.3}$$

$$\nabla\mu \textit{ and } \nabla f \textit{ are parallel.} \tag{3.4}$$

Catino, Mastrolia, Monticella and Rigoli [28] showed that a complete generalized quasi Yamabe gradient soliton (M^n, g) has a warped product structure without any hypothesis over g (we recommend Theorem 5.1 on [28] to reader (see also [26])).

As a consequence of Theorem 3.1, we have

Theorem 3.2. *[51] Let $(M^n,\, g)_{n\geq 3}$, be a nontrivial complete connected generalized quasi Yamabe gradient soliton satisfying (3.1) and (3.3), with positive sectional curvature. Then*

(a) if $n = 3$, (M^n, g) is rotationally symmetric;

(b) if $n \geq 5$ and $W = 0$, (M^n, g) is rotationally symmetric.

Theorem 3.3. *[62] Let $(M^4,\, g)$ be a nontrivial complete connected half locally conformally flat generalized quasi Yamabe gradient soliton satisfying (3.1) and (3.3), with positive sectional curvature. Then, M^4 is rotationally symmetric.*

Theorem 3.4. *[51] Let $(M^n,\, g)$, $n \geq 3$, be a nontrivial compact connected generalized quasi Yamabe gradient soliton satisfying (3.1) and (3.3). Then, the scalar curvature R of the metric g is constant.*

From Theorem 3.1, we show that a nontrivial complete connected generalized quasi Yamabe gradient soliton admits a warped product structure (see Proposition 3.7). In the special case when (M^n, g) is locally conformally flat, we can say more about the warped product structure (see [26, 29, 44, 51, 100]).

Theorem 3.5. *Let $(M^n,\, g)$, $n \geq 3$, be a nontrivial complete connected generalized quasi Yamabe gradient soliton satisfying (3.1) and (3.3). Suppose f has no critical point and is locally conformally flat, then (M^n, g) is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla u|} (N^{n-1}, \bar{g}_N)$$

where $u = e^{-\mu f}$, and $(N^{n-1}, \bar{g})$ is a space of constant sectional curvature.

Therefore, when μ is constant on equation (3.1), from the above theorems we also have a classification to the gradient Yamabe solitons.

3.1 Proof of Theorem 3.1

In this section we first recall some basic facts on tensors that will be useful in the proof of our main results. We then prove our Theorem 3.1. For operators $S, T : \mathcal{H} \to \mathcal{H}$ defined over an n-dimensional Hilbert space $\mathcal{H}$, the *Hilbert-Schmidt inner product* is defined according to

$$\langle S, T \rangle = \mathrm{tr}\big(\mathrm{ST}^{\star}\big), \tag{3.5}$$

where tr and $\star$ denote, respectively, the *trace* and the *adjoint* operation.

For a Riemannian manifold $(M^n,\, g)$, $n \geq 3$, the *Weyl tensor W* is defined by the following decomposition formula

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}\big(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}\big) \\ &\quad - \frac{R}{(n-1)(n-2)}\big(g_{jl}g_{ik} - g_{il}g_{jk}\big), \end{aligned} \tag{3.6}$$

where R_{ijkl} stands for the Riemannian curvature operator. In [25], Cao and Chen introduced a covariant 3-tensor D given by

$$\begin{aligned} D_{ijk} &= \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) + \frac{1}{(n-1)(n-2)}(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) \\ &- \frac{R}{(n-1)(n-2)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}). \end{aligned} \tag{3.7}$$

The tensor D is skew-symmetric in its first two indices and trace-free, i.e.,

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij}D_{ijk} = g^{ik}D_{ijk} = g^{jk}D_{ijk} = 0.$$

We will show how these two tensors are related.

In order to set the stage for the proof that follows let us recall some equations for any dimension. Moreover, since

$$\nabla_i|\nabla f|^2 = 2\nabla_i\nabla_j f\nabla^j f, \quad |\nabla f|^2 = g^{ij}\nabla_i f\nabla_j f \quad \text{and} \quad \Delta f = g^{ij}\nabla_i\nabla_j f$$

the trace of (3.2) is given by

$$\Delta f - \mu|\nabla f|^2 = n(R - \lambda) \tag{3.8}$$

and

$$(R - \lambda)\nabla_i f = \frac{1}{2}\nabla_i|\nabla f|^2 - \mu|\nabla f|^2\nabla_i f. \tag{3.9}$$

Taking the covariant derivative of (3.8) we get

$$n\nabla_i R = \nabla_i(\Delta f) - (\nabla_i\mu|\nabla f|^2 + \mu\nabla_i|\nabla f|^2). \tag{3.10}$$

Now, taking the covariant derivative in (3.2) we get

$$\nabla_i R g_{jk} = \nabla_i\nabla_j\nabla_k f - [\nabla_i\mu\nabla_j f\nabla_k f + \mu(\nabla_i\nabla_j f\nabla_k f + \nabla_j f\nabla_i\nabla_k f)]. \tag{3.11}$$

Contracting (3.11) over i and k, and using the Ricci equation we obtain

$$\nabla_j R = R_{jl}\nabla^l f + \nabla_j(\Delta f) - \left[g^{ik}\nabla_i\mu\nabla_k f\nabla_j f + \mu\left(\frac{1}{2}\nabla_j|\nabla f|^2 + \Delta f\nabla_j f\right)\right].$$

From (3.8) and (3.10) and the above equation one has

$$\begin{aligned}\nabla_j R &= R_{jl}\nabla^l f + n\nabla_j R + \nabla_j\mu|\nabla f|^2 + \frac{\mu}{2}\nabla_j|\nabla f|^2 \\ &- g^{ik}\nabla_i\mu\nabla_k f\nabla_j f - n\mu(R-\lambda)\nabla_j f - \mu^2|\nabla f|^2\nabla_j f.\end{aligned}$$

Then, from (3.9) we can infer

$$\begin{aligned}(n-1)\nabla_j R &= -R_{jl}\nabla^l f - |\nabla f|^2\nabla_j\mu \\ &+ [g^{ik}\nabla_i\mu\nabla_k f + \mu(n-1)(R-\lambda)]\nabla_j f.\end{aligned} \tag{3.12}$$

Lemma 3.6. *Let (M^n, g) be an n-dimensional generalized quasi Yamabe gradient soliton satisfying (3.2). Then we have:*

$$\begin{aligned}W_{ijkl}\nabla^l f &= D_{ijk} + (\nabla_i\mu\nabla_j f\nabla_k f - \nabla_j\mu\nabla_i f\nabla_k f) + \left(\frac{|\nabla f|^2}{n-1}\right)(g_{ik}\nabla_j\mu - g_{jk}\nabla_i\mu) \\ &+\frac{g(\nabla\mu,\nabla f)}{n-1}(g_{jk}\nabla_i f - g_{ik}\nabla_j f).\end{aligned}$$

where D_{ijk} is defined from (3.7).

Proof. We may use equation (3.2) to obtain

$$\begin{aligned}\nabla_i R g_{jk} - \nabla_j R g_{ik} &= \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f + \mu(\nabla_i f\nabla_j\nabla_k f - \nabla_j f\nabla_i\nabla_k f)\\ &+ (\nabla_j\mu\nabla_i f\nabla_k f - \nabla_i\mu\nabla_j f\nabla_k f).\end{aligned}$$

Then, by Ricci identity, we get

$$\begin{aligned}\nabla_i R g_{jk} - \nabla_j R g_{ik} &= R_{ijkl}\nabla^l f + \mu(\nabla_i f\nabla_j\nabla_k f - \nabla_j f\nabla_i\nabla_k f)\\ &+ (\nabla_j\mu\nabla_i f\nabla_k f - \nabla_i\mu\nabla_j f\nabla_k f).\end{aligned}$$

Now, from (3.2) we have

$$\begin{aligned}\nabla_i R g_{jk} - \nabla_j R g_{ik} &= R_{ijkl}\nabla^l f + \mu(R-\lambda)(\nabla_i f g_{jk} - \nabla_j f g_{ik})\\ &+ (\nabla_j\mu\nabla_i f\nabla_k f - \nabla_i\mu\nabla_j f\nabla_k f).\end{aligned}$$

It then follows from (3.6) that

$$\begin{aligned}\nabla_i R g_{jk} - \nabla_j R g_{ik} &= W_{ijkl}\nabla^l f + \frac{1}{(n-2)}(R_{ik}\nabla_j f - R_{jk}\nabla_i f)\\ +\frac{1}{(n-2)}(R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) &- \frac{R}{(n-1)(n-2)}(\nabla_j f g_{ik} - \nabla_i f g_{jk})\\ +\mu(R-\lambda)(\nabla_i f g_{jk} - \nabla_j f g_{ik}) &+ (\nabla_j\mu\nabla_i f\nabla_k f - \nabla_i\mu\nabla_j f\nabla_k f). \qquad (3.13)\end{aligned}$$

From (3.12), we obtain

$$\begin{aligned}\nabla_i R g_{jk} - \nabla_j R g_{ik} &= \frac{1}{(n-1)}(R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) + \frac{|\nabla f|^2}{(n-1)}(\nabla_j\mu g_{ik} - \nabla_i\mu g_{jk})\\ &+ \frac{1}{(n-1)}(\nabla_j\mu\nabla_i f\nabla_k f - \nabla_i\mu\nabla_j f\nabla_k f) +\\ &+ \mu(R-\lambda)(\nabla_i f g_{jk} - \nabla_j f g_{ik}) +\\ &+ \frac{g(\nabla\mu,\nabla f)}{n-1}(g_{jk}\nabla_i f - g_{ik}\nabla_j f). \qquad (3.14)\end{aligned}$$

Combining (3.13) and (3.14), we finish the proof of Lemma 3.6. □

We define the 3-tensor E as follows

$$\begin{aligned}E_{ijk} &= (\nabla_i\mu\nabla_j f\nabla_k f - \nabla_j\mu\nabla_i f\nabla_k f) + \left(\frac{|\nabla f|^2}{n-1}\right)(g_{ik}\nabla_j\mu - g_{jk}\nabla_i\mu)\\ &+\frac{g(\nabla\mu,\nabla f)}{n-1}(g_{jk}\nabla_i f - g_{ik}\nabla_j f). \qquad (3.15)\end{aligned}$$

Taking into account this definition, we deduce from Lemma 3.6 that

$$W_{ijkl}\nabla^l f = D_{ijk} + E_{ijk}. \tag{3.16}$$

Proof of Theorem 3.1. Since the Weyl tensor and the 3-tensor D are trace free, i.e. $g^{jk}W_{ijkl} = g^{jk}D_{ijk} = 0$ contracting (3.16) over j and k, we get

$$g^{jk}E_{ijk} = 0. \tag{3.17}$$

from (3.17) we have

$$0 = \left(\frac{n-2}{n-1}\right)|\nabla f|^2\left[|\nabla\mu|^2, |\nabla f|^2 - g(\nabla\mu, \nabla f)^2\right]. \tag{3.18}$$

Considering f nontrivial, from the above equation we can conclude that:

I) If $g(\nabla\mu, \nabla f) = 0$, i.e., ∇f and $\nabla\mu$ are orthogonal, Theorem 3.1 it is true.

II) On the other hand, if $g(\nabla\mu, \nabla f) \neq 0$ from (3.18) we obtain

$$|\nabla\mu|^2|\nabla f|^2 - g(\nabla\mu, \nabla f)^2 = 0$$

which means that $\nabla\mu$ and ∇f are parallel.

□

3.2 The warped product structure

Following the steps in [26], we can prove that a GQY manifold admits a warped product structure without any additional hypothesis over M. From Theorem 3.1 by using a conformal change of variable on (3.1) ($u = e^{-\mu f}$), we get

$$\mu u(R - \lambda)g = \nabla^2 u. \tag{3.19}$$

Cheeger and Colding [29] characterized the warped product structure of (3.19). We will sketch the proof of such warped product structure here for completeness.

Consider the level surface $\Sigma = f^{-1}(c)$ where c is any regular value of the potential function f. Suppose that I is an open interval containing c such that f has no critical point. Let $U_I = f^{-1}(I)$. Fix a local coordinates system

$$(x_1, x_2, \cdots, x_n) = (r, \theta_2, \cdots, \theta_n)$$

in U_I, where $(\theta_2, \cdots, \theta_n)$ is any local coordinates system on the level surface Σ_c, and indices $a, b, c, \cdots$ range from 2 to n. Then we can express the metric g as

$$ds^2 = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta_a d\theta_b,$$

where $g_{ab}(f, \theta) d\theta_a d\theta_b$ is the induced metric and $\theta = (\theta_2, \cdots, \theta_n)$ is any local coordinates system on Σ_c. From (3.9)

$$\frac{1}{2} \nabla_a |\nabla f|^2 = [(R - \lambda) + \mu |\nabla f|^2] \nabla_a f = 0.$$

Since $|\nabla f|^2$ is constant on Σ_c, we can make a change of variable

$$r(x) = \int \frac{df}{|\nabla f|}$$

so that we can express the metric g in U_I as

$$ds^2 = dr^2 + g_{ab}(r, \theta) d\theta_a d\theta_b.$$

Let $\nabla r = \frac{\partial}{\partial r}$, then $|\nabla r| = 1$ and $\nabla f = f'(r) \frac{\partial}{\partial r}$ on U_I. Then,

$$\nabla_{\partial r} \partial r = 0. \tag{3.20}$$

Now, by (3.20) and (3.1), it follows that

$$(R - \lambda) = \nabla^2 f(\partial r, \partial r) - \mu (df \otimes df)(\partial r, \partial r) = f''(r) - \mu (f'(r))^2. \tag{3.21}$$

Whence, from Theorem 3.1 and (3.21), we can see that R is also constant on Σ_c. Moreover, since $g(\nabla f, \partial_a) = 0$, from (3.1) the second fundamental formula on Σ_c is given by

$$h_{ab} = -g(\partial r, \nabla_a \partial_b) = \frac{\nabla_a \nabla_b f}{|\nabla f|} = \frac{(R - \lambda)}{|\nabla f|} g_{ab}. \tag{3.22}$$

Therefore, from (3.21) and (3.22) we have

$$h_{ab} = \frac{f''(r) - \mu (f'(r))^2}{f'(r)} g_{ab}. \tag{3.23}$$

From (3.23) the mean curvature is given by

$$H = (n - 1) \frac{f''(r) - \mu (f'(r))^2}{f'(r)} \tag{3.24}$$

wich is also constant on Σ_c.

Furthermore, from the second fundamental formula on Σ_c, we have that

$$h_{ab} = -g(\partial_r, \nabla_a \partial_b) = -g(\partial_r, \Gamma^l_{ab}\partial_l) = -\Gamma^1_{ab}. \tag{3.25}$$

On the other hand,

$$\Gamma^1_{ab} = -\frac{1}{2} g^{11} \frac{\partial}{\partial r} g_{ab}. \tag{3.26}$$

Therefore, from (3.23), (3.25) and (3.26) we get

$$2\frac{f''(r) - \mu(f'(r))^2}{f'(r)} g_{ab} = \frac{\partial}{\partial r} g_{ab}. \tag{3.27}$$

Hence, it follows from (3.27) that

$$g_{ab}(r,\theta) = (f' e^{-\mu f})^2 g_{ab}(r_0,\theta),$$

where the level set $\{r = r_0\}$ corresponds to $\Sigma_{r_0} = f^{-1}(r_0)$, for any regular value r_0 of the potential function f.

Therefore we can announce the following result analogous to the Proposition 2.1 in [26] (we also recommend [44, 100]).

Proposition 3.7. *Let (M^n, g) be a nontrivial complete connected generalized quasi Yamabe gradient Yamabe soliton, satisfying the GQY equation (3.1), and let $\Sigma_c = f^{-1}(c)$ be a regular level surface. Then*

(1) The scalar curvature R and $|\nabla f|^2$ are constants on Σ_c.

(2) The second fundamental form of Σ_c is given by

$$h_{ab} = \frac{H}{n-1} g_{ab}. \tag{3.28}$$

(3) The mean curvature $H = (n-1)\frac{(R-\lambda)}{|\nabla f|}$ is constant on Σ_c.

(4) In any open neighborhood $U^\beta_\alpha = f^{-1}\big((\alpha,\beta)\big)$ of Σ_c in which f has no critical points, the GQY metric g can be expressed as

$$ds^2 = dr^2 + (f'(r)e^{-\mu f})^2 \bar{g}_{ab}$$

where $(\theta_2, \cdots, \theta_n)$ is any local coordinates system on Σ_c and $\bar{g}(r,\theta) = g_{ab}(r_0,\theta)d\theta_a d\theta_b$ is the induced metric on $\Sigma_c = r^{-1}(r_0)$.

Proof of Theorem 3.5. Consider the warped product manifold, by Proposition (3.7)

$$(M^n, g) = (I, dr^2) \times \phi(N^{n-1}, \bar{g}), \tag{3.29}$$

where $ds^2 = dr^2 + (\phi)^2\bar{g}$. Fix any local coordinates system $\theta = (\theta_2, \cdots, \theta_n)$ on N^{n-1}, and choose $(x_1, x_2, \cdots, x_n) = (r, \theta_2, \cdots, \theta_n)$. Now (see [14, 26, 75]) the scalar curvature formulas of (M^n, g) and $(N^{n-1}, \bar{g})$ are related by

$$R = \phi^{-2}\bar{R} - (n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2 - 2(n-1)\frac{\phi''}{\phi}.$$

Therefore, since $\phi = f'e^{-\mu f}$, from Theorem 3.1 and Proposition 3.7 we have that $\bar{R}$ does not depend on θ. Then $\bar{R}$ is constant.

Moreover, the Weyl tensor W for an arbitrary warped product manifold (3.29) is given by (see [14, 26, 75]):

$$W_{1a1b} = -\frac{1}{n-2}\bar{R}_{ab} + \frac{\bar{R}}{(n-1)(n-2)}\bar{g}_{ab}, \tag{3.30}$$

$$W_{1abc} = 0, \tag{3.31}$$

and

$$W_{abcd} = \phi\bar{W}_{abcd}. \tag{3.32}$$

Where $\bar{W}$ denotes the Weyl tensor of $(N^{n-1}, \bar{g})$. Therefore, since the warped product manifold (3.29) is locally conformally flat, i.e. $W = 0$, from (3.30) and (3.32) we see that N is Einstein and $\bar{W} = 0$. Then, from (3.6) we have

$$\bar{R}_{abcd} = \frac{\bar{R}}{(n-1)(n-2)}(\bar{g}_{bd}\bar{g}_{ac} - \bar{g}_{bc}\bar{g}_{ad}).$$

Since $\bar{R}$ is constant, we get that $\bar{R}_{abcd}$ is also constant. Thus N is a space form.

□

Chapter 4

Bounds on volume growth in static vacuum space

This section is the result of studies with Prof. Dr. Benedito L.N., Prof. Dr. Ernani B. [65]. Thank you for your contributions.

Let $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$, the warped product of M with $\mathbb{R}$, be a static space-time endowed with

$$\hat{g} = -f^2 dt^2 + g, \tag{4.1}$$

where $(M^n,\, g)$, $n \geq 3$, is a noncompact, connected and oriented Riemannian manifold, and $f : M^n \to (0, +\infty)$ is a positive smooth warped function. In this approach, the Einstein equation with perfect fluid as a matter field is given by

$$\hat{R}ic - \frac{\hat{R}}{2}\hat{g} = (\mu + \rho)\eta \otimes \eta + \rho\hat{g}, \tag{4.2}$$

where $\hat{R}ic$ and $\hat{R}$ stand for the Ricci tensor and the scalar curvature with respect to $\hat{g}$, respectively; whereas η is a 1-form with $\hat{g}(\eta, \eta) = -1$ whose associated vector field represents the flux of the fluid. Moreover, μ and ρ are nonnegative smooth functions, namely the energy density and pressure, respectively; for more details, we refer the reader to [55] and [57]. At the same time, it follows from Proposition 1.23, equations (4.1) and (4.2) that

$$f\mathring{R}ic = \mathring{\nabla}^2 f \tag{4.3}$$

as well as

$$\mu = \frac{R}{2} \quad \text{and} \quad \rho f = \frac{n-1}{n}\left(\Delta f - \frac{n-2}{2(n-1)} R f\right), \tag{4.4}$$

where $\mathring{Ric}$ stands for the traceless of Ric. Besides, Δ denotes the Laplacian and R is the scalar curvature with respect to g (cf. [14, 61, 75]).

In the sequel, we shall present a simple proof that the energy density μ vanishes on the boundary Σ of manifolds M^n satisfying (4.3) and (4.4). Here, Σ is compact (possibly with boundary). More precisely, we have established the following result.

Theorem 4.1. *Let (M^n, g, f) be a Riemannian manifold satisfying (4.3) and (4.4). Then the energy density $\mu = 0$ on Σ.*

In order to proceed, we remember that when μ and ρ vanish in (4.2) we obtain the well-known *static vacuum Einstein equations*. Indeed, following the terminology used in [8, 9, 55], we deduce from (4.3) and (4.4) that a semi-Riemannian manifold $(\widehat{M}, \hat{g})$ is Ricci-flat (i.e., $\hat{Ric} = 0$) if and only if the (positive) warped function f and the metric g satisfy the *static vacuum equations*

$$f Ric = \nabla^2 f \quad \text{and} \quad \Delta f = 0. \tag{4.5}$$

In this case, it is easy to check that the scalar curvature R is identically zero. These equations have been extensively studied in classical general relativity. Some explicit examples can be found in [8, 9, 57] and [61].

It should be point out that if a manifold $(M^n,\ g)$ satisfying (4.5) is geodesically complete, then the warped function f must be constant. Therefore, in the spirit of [8, 9] and [55], throughout this article *we consider non trivial solution to (4.5), which are connected and complete up to the boundary*, or equivalently, *complete away from the horizon* (cf. Theorem 3.2 in [8], or Theorem 4.6 in Section 4.1, see also [9] p. 996). In addition, we assume that the boundary Σ is compact, non-empty and such that g and f *extends smoothly* to Σ. We remember that the set $\Sigma = \{f = 0\}$ is called the *horizon*. We further highlight that Σ may be defined as the set of limit points of Cauchy sequences on $(M^n,\ g)$ on which f converges to 0 (cf. [9]). In general relativity, the horizon is closely related with the event horizon, i.e., the boundary of a black hole. For a comprehensive reference on such a subject, we indicate, for instance [8, 9, 14, 55, 57] and [59].

Before proceeding it is important to recall the definition of quasi-Einstein manifolds which is closely related to the problem of building Einstein manifolds. To be precise, a complete Riemannian manifold (M^n, g), $n \geq 2$, will be called *m-quasi-Einstein manifold*, or simply *quasi-Einstein manifold*, if there exist a smooth potential function f on M^n and a constant λ satisfying the following fundamental equation

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g. \tag{4.6}$$

It is easy to check that 1-quasi-Einstein manifolds with $\lambda = 0$ are static vacuum spaces, that is, it becomes Eq. (4.5). Indeed, considering the function $u = e^{-f}$ on M^n, we immediately get $\nabla u = -u\nabla f$ as well as $\nabla^2 f - df \otimes df = -\frac{1}{u}\nabla^2 u$ and these equations confirm our remark. Moreover, it is easy to see that a ∞-quasi-Einstein manifold means a gradient Ricci soliton. Ricci solitons model the formation of singularities in the Ricci flow and correspond to self-similar solutions; for more details see, for instance, [23]. Following the terminology of Ricci solitons, a quasi-Einstein metric g on a manifold M^n will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. For more details see, for instance, [10,14,22,83] and [85].

A classical theorem due to Calabi [20] and Yau [99] asserts that the geodesic balls of complete non-compact manifolds with non negative Ricci tensor have at least linear growth, that is,

$$Vol(B_p(r)) \geq cr,$$

for any $r > r_0$ where r_0 is a positive constant and $B_p(r)$ is the geodesic ball of radius r centered at $p \in M^n$ and c is a constant that does not depend on r. In [72], Munteanu and Sesum obtained the same type of growth for steady gradient Ricci soliton. While Barros et al. [10] were able to prove the same type of growth for steady m-quasi-Einstein manifold (with $m \neq 1$). The classical Bishop volume comparison theorem guarantees that the geodesic balls of complete non-compact manifolds with non negative Ricci tensor must have the following growth rates

$$c_1 r^n \geq Vol(B_p(r)),$$

for some positive constant c_1 and $r > 0$ sufficiently large. In [24], H.-D. Cao and D. Zhou proved an analog of Bishop's theorem for gradient shrinking solitons. While Munteanu and Sesum [72] showed that the geodesic balls of steady gradient Ricci soliton have at most exponential volume growth, namely, there exist uniform constants c, a and r_0 so that for any $r > r_0$ we have

$$Vol(B_p(r)) \leq ce^{a\sqrt{r}}.$$

As well known, volume growth rate is an important piece of geometric information. In this spirit, we obtain an upper bound on the growth of volume of geodesic balls for spatial factor of a static space which is similar to Bishop's estimate. More precisely, we have established the following result.

Theorem 4.2. *Let (M^n, g, f), $n \geq 3$, be a Riemmanian manifold satisfying (4.5). Then there exist uniform constants a and r_0 so that for any $r > r_0$*

$$Vol(B_p(r)) \leq c\, r^{n+a},$$

where c is the volume of the unitary ball.

For what follows, we remember that the Omori-Yau maximum principle (at infinity) is a very powerful tool in Geometric Analysis and it is related to a number of properties of the underlying Riemannian manifold, ranging from the realm of stochastic analysis to that of geometry and PDEs. In [79], Pigola, Rigoli and Setti extended the Omori-Yau maximum principle to a larger class of manifolds and operators, see [79, Remark 1.2 & Examples 1.13, 1.14]. In particular, Pigola, Rigoli and Setti [78, 79] introduced the concept of *weak Omori-Yau maximum principle* as follows.

Definition 4.3. *Let (M^n, g) be a Riemannian manifold. We say that the weak Omori-Yau maximum principle holds if for every function $u \in C^2(M)$ with*

$u^*\colon = \sup_M u < +\infty$, *there exists a sequence* $\{x_k\} \subset M$, $k = 1, 2, \ldots,$ *such that, for every* k.

$$u(x_k) > u^* - 1/k \qquad \textit{and} \qquad \Delta u(x_k) < 1/k.$$

It is worthwhile to remark that the validity of the Weak maximum principle (at infinity) implies, for instance, stochastic completeness (cf. [79,80]). Recall that the $\mathcal{L}_\varphi$-Laplacian operator, or simply φ-Laplacian, is given by

$$\mathcal{L}_\varphi u = \operatorname{div}(|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u), \tag{4.7}$$

for some function $u \in C^1(M)$. Notice that if the vector field in brackets is not C^1, then the divergence in (4.7) must be considered in distributional sense. The φ-Laplacian arises from the Euler-Lagrange equation associated to the energy functional

$$\Lambda(u) = \int \phi(|\nabla u|),$$

where $\phi(t) = \int_0^t \varphi(s)ds$. Notice that when $\varphi(t) = t$ in (4.7) the φ-Laplacian reduces to Laplace-Beltrami operator Δu. On the other hand, when $\varphi(t) = t^{p-1}$ in (4.7) the φ-Laplacian become the p-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$. Further, when $\varphi(t) = 1/(1+t^2)^\alpha$, it becomes the generalized mean curvature operator, $\operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^\alpha}\right)$, $\alpha > 0$.

Inspired in the weak maximum principle to the Laplacian Δ, Rigoli and Setti [87] studied its validity for the φ-Laplacian operator. They were able to prove under suitable geometric assumptions a weak version of the Omori-Yau maximum principle for the φ-Laplacian. For more details on this subject, we refer the reader to [87].

Next, as an application of Theorem 4.2, we have established a weak maximum principle at infinity for the φ-Laplacian Riemmanian manifold satisfying (4.5).

Theorem 4.4. *Let* (M^n, g, f), $n \geq 3$, *be a Riemmanian manifold satisfying (4.5). Let* u *be a smooth function on* M^n *with* $u^* = \sup_M u < +\infty$ *and such that the vector field* $|\nabla u|^{-1}\varphi(|\nabla u|)\nabla u$ *is of class at least* C^1. *Then the weak maximum principle at infinity holds for* φ*-Laplacian on* M^n.

It is interesting to observe that the regularity condition in the statement of Theorem 4.4 is certainly satisfied in the case of the Laplacian, p-Laplacian or the generalized mean curvature operator once u is assumed to be at least C^2.

4.1 Background

In order to set the stage for the proof of the main results we shall present some lemmas which will be useful for the establishment of the desired results. To start with, we recall a lemma that can be found in [8, 55].

Lemma 4.5 ([8,55]). *Let (M, g, f) be a Riemannian manifold satisfying (4.3) and (4.4). Then there is no critical point of f in Σ.*

Proof. Since the proof is short, we include its proof here for the sake of completeness. To begin with, from (4.4) we obtain that $\Delta f = 0$ in Σ. Next, consider $Crit(f) = \{x \in M; (\nabla f)(x) = 0\}$. Further, since $\Delta f = 0$ in Σ we may use Hopf's lemma (cf. [48]) to conclude that $|\nabla f| > 0$ for any $p \in \Sigma$. Whence, $Crit(f) \cap \Sigma = \emptyset$, which finishes the proof of the lemma. $\square$

In the sequel we recall a well known result due to Anderson [8] which plays a crucial role in the to prove of Theorem 4.2.

Theorem 4.6 ([8], Theorem 3.2). *Let (M^n, g, f) be a solution to the static vacuum equations (4.5).*

1. *Suppose that (M^n, g) is a complete Riemannian manifold and $f > 0$ on M^n, then M^n must be flat, and f is constant.*

2. *Let $U \subset M$ be any domain with smooth boundary on which $f > 0$. Assume that $r(x) = dist_M(x, \partial U)$, for $x \in U$. Then there is an absolute constant $K < +\infty$ such that*

$$\frac{|\nabla f|}{f}(x) \leq \frac{K}{r(x)}, \tag{4.8}$$

where the constant K does not depend on the domain U (since $f > 0$ on U), or on the static vacuum solution (M^n, g).

To conclude this section we recall a result due to Rigoli and Setti [87] which will be used in the proof of Theorem 4.4.

Theorem 4.7 ([87]). *Let M^n be a complete Riemannian manifold. Suppose that*

$$\liminf_{r\to+\infty}\frac{\log Vol(B_p(r))}{r^{1+\delta}}<+\infty, \tag{4.9}$$

for some $\delta>0$, and let u be a smooth function on M^n such that $u^*=\sup u<+\infty$. In addition, assume that the vector field $X=|\nabla u|\varphi(|\nabla u|)\nabla u$ is of class at least C^1. Then there exists a sequence $\{x_k\}\subset M$, $k=1,2,\ldots,$ such that

- $u(x_k)\to u^*$,
- $\mathcal{L}_\varphi u(x_k)\leq 1/k$.

Now we are ready to prove the main results.

4.1.1 Proof of the Main Results

Proof of Theorem 4.1 To begin with, we rewrite (4.3) in local coordinates as follows

$$f\left(R_{ij}-\frac{R}{n}g_{ij}\right)=\nabla_i\nabla_j f-\frac{\Delta f}{n}g_{ij}. \tag{4.10}$$

In particular, it is not difficult to check from (4.10) that

$$f\mathring{R}_{ij}\nabla_j f=\frac{1}{2}\nabla_i|\nabla f|^2-\frac{\Delta f}{n}\nabla_i f, \tag{4.11}$$

where $\mathring{R}_{ij}=R_{ij}-\frac{R}{n}g_{ij}$.

On the other hand, take the covariant derivative of (4.10) and then use the well-known Ricci identity $\nabla_i\nabla_j\nabla_k f-\nabla_j\nabla_i\nabla_k f=R_{ijkl}\nabla_l f$ to achieve

$$\begin{aligned} f(\nabla_i\mathring{R}_{jk}-\nabla_j\mathring{R}_{ik})+(\mathring{R}_{jk}\nabla_i f-\mathring{R}_{ik}\nabla_j f) &= R_{ijkl}\nabla_l f\\ &\quad+\frac{1}{n}\Big(\nabla_j(\Delta f)g_{ik}-\nabla_i(\Delta f)g_{jk}\Big). \end{aligned} \tag{4.12}$$

Hereafter, by using the twice contracted second Biachi identity (eq. (1.7)), we get

$$\begin{aligned}\frac{1}{2}\nabla_j R &= g^{ik}\nabla_i R_{jk}\\ &= g^{ik}\nabla_i\left\{R_{jk}-\frac{R}{n}g_{jk}+\frac{R}{n}g_{jk}\right\}\\ &= g^{ik}\nabla_i\mathring{R}_{jk}+g^{ik}\left(g_{jk}\nabla_i\frac{R}{n}\right).\end{aligned}$$

Therefore,

$$g^{ik}\nabla_i\mathring{R}_{jk}=\frac{n-2}{2n}\nabla_j R. \tag{4.13}$$

The trace over i and k of (4.12) gives

$$\begin{aligned}g^{ik}\left\{(\nabla_i\mathring{R}_{jk}-\nabla_j\mathring{R}_{ik})+(\mathring{R}_{jk}\nabla_i f-\mathring{R}_{ik}\nabla_j f)\right\} &= g^{ik}\Big\{R_{ijkl}\nabla_l f+\\ &+\frac{1}{n}\Big(\nabla_j(\Delta f)g_{ik} - \nabla_i(\Delta f)g_{jk}\Big)\Big\}.\end{aligned} \tag{4.14}$$

Using (4.13) and the fact $g^{ik}\mathring{R}_{ik}=0$, we have

$$\frac{1}{2(n-1)}\Big[(n-2)f\nabla_j R-2R\nabla_j f\Big]=\nabla_j(\Delta f). \tag{4.15}$$

In the sequel, it is easy to check that, from (4.11) and (4.15), on $\Sigma=\{f=0\}$, we have

$$\frac{1}{2}\nabla_j|\nabla f|^2-\frac{\Delta f}{n}\nabla_j f=0 \tag{4.16}$$

and

$$-\frac{R}{n-1}\nabla_j f=\nabla_j(\Delta f). \tag{4.17}$$

Whence, it follows from Eq. (4.17) that

$$-\frac{R}{n-1}|\nabla f|^2=g(\nabla(\Delta f),\nabla f). \tag{4.18}$$

Upon integrating (4.18) over Σ we may use Lemma 4.5 jointly with Stoke's formulae to infer

$$-\int_\Sigma\frac{R}{n-1}|\nabla f|^2=\int_\Sigma g(\nabla(\Delta f),\nabla f)=-\int_\Sigma(\Delta f)^2+\int_{\partial\Sigma}\Delta f\frac{\partial f}{\partial\eta}, \tag{4.19}$$

where $\eta=\frac{\nabla f}{|\nabla f|}$. But, from (4.4) we have $\Delta f=0$ on Σ and then Eq. (4.16) jointly with Lemma 4.5 guarantees that $|\nabla f|^2$ is a non null constant on Σ. Next, since $2\mu=R$ is nonnegative, it suffices to substitute these informations into (4.19) to conclude that $\mu=0$ on Σ. This gives the requested result.

□

In the sequel we shall present the proof of Theorem 4.2, which was mainly inspired in the trend of Munteanu and Wang [73] as well as Munteanu and Sesum [72].

Proof of Theorem 4.2 Firstly, we denote by

$$dV|_{exp_x(r,\theta)} = J(x,r,\theta)drd\theta$$

the volume form in the geodesic polar coordinates centered at x as for $r > 0$ and θ a tangent vector field on $x \in M$ and

$$r = d(x_0, x), \quad \text{for} \quad x_0 \in M.$$

From now on we omit the dependence on θ. In this approach, it is known that

$$\Delta r = \frac{J'}{J}(r).$$

Now, let $\gamma(s)$ be a minimizing geodesic starting from x_0, such that $\gamma(0) = x_0$. In particular, given any orthonormal basis $\{E_i\}_{i=1}^{n-1}$ such that $\gamma' \perp E_i$ at $\gamma(s)$, Y_i is the Jacobi fields along γ with $Y_i(0) = 0$ and $Y_i(s) = E_i$. From this, it is known that

$$\Delta r = \sum_{i=1}^{n-1} I_t(Y_i, Y_i).$$

By using the Index lemma, for any piecewise differentiable vector field W_i along γ, we get

$$I_t(Y_i, Y_i) \le \int_0^t \{g(W_i, W_i)(s) - g(R(\gamma', W_i)\gamma', W_i)(s)\}ds.$$

Now, if E_i is the parallel unit field generated by $Y_i(t)$ and μ is a piecewise differentiable function, taking $W_i = \mu E_i$, we deduce

$$I_t(Y_i, Y_i) \le \int_0^t \{(\mu')^2 - \mu^2 g(R(\gamma', E_i)\gamma', E_i)\}(s)ds.$$

Whence, by taking the trace over i, with $X = \gamma'(s)$, we arrive at

$$\Delta r \le \int_0^t \{(n-1)(\mu')^2 - \mu^2 Ric(X, X)\}ds.$$

Proceeding, since $\Delta r = \frac{J'}{J}$, we may express $\mu(s) = \frac{s}{t}$ to infer

$$\frac{J'}{J}(t) \leq \frac{(n-1)}{t} - \frac{1}{t^2}\int_0^t s^2 Ric(X,X)ds \tag{4.20}$$

on the other hand, since f is positive, we use the static vacuum equation (4.5) to obtain

$$\begin{aligned} Ric(X,X) &= \frac{1}{f}\nabla^2 f(X,X) \\ &= \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2 \geq \left(\frac{f'}{f}\right)'. \end{aligned}$$

This substituted into (4.20) yields

$$\frac{J'}{J}(t) \leq \frac{n-1}{t} - \frac{1}{t^2}\int_0^t s^2 \left(\frac{f'}{f}\right)'(s)ds.$$

where $f(s) = f(\gamma(s))$ and $0 \leq s \leq t$. Therefore, upon integrating by parts we achieve

$$\frac{J'}{J}(t) \leq \frac{n-1}{t} + \frac{2}{t^2}\int_0^t s\frac{f'}{f}(s)ds - \frac{f'}{f}(t). \tag{4.21}$$

Moreover, we already know from Eq. (4.8) (see also Theorem 3.2 in [8]) that $\left(\frac{f'}{f}\right)$ is bounded from above. This combined with (4.21) gives

$$\begin{aligned} \frac{J'}{J}(t) &\leq \frac{n-1}{t} + \frac{2}{t^2}\int_0^t s\frac{f'}{f}(s)ds - \frac{f'}{f}(t) \\ &\leq \frac{n-1}{t} + \frac{2}{t^2}\int_0^t s\left(\frac{K}{s}\right)ds + \frac{K}{t} \\ &\leq \frac{n-1}{t} + \frac{2K}{t} + \frac{K}{t} = \frac{n-1+3K}{t}, \end{aligned}$$

which can be rewritten as

$$\Big(\log J(t)\Big)' \leq \frac{n-1+3K}{t}.$$

Hereafter, upon integrating from $s = 1$ to $s = s_0$ we infer

$$\frac{J(t)}{J(1)} \leq t^{n-1+3K}.$$

From here it follows that

$$Area\left(B_{x_0}(r)\right) \leq Area\left(B_{x_0}(1)\right) r^{n-1+3K}. \tag{4.22}$$

To conclude, it suffices to integrate (4.22) to obtain

$$Vol(B_{x_0}(r)) \leq cr^{n+a},$$

where c is the volume of the unitary ball and a is constant.

□

Proof of Theorem 4.4 We start invoking Theorem 4.2 to deduce that there exists positive constants a, c and r_0 so that for any $r > r_0$

$$Vol(B_p(r)) \leq cr^{n+a}.$$

Easily one verifies that

$$\begin{aligned} \frac{\log Vol(B_p(r))}{r^{1+\delta}} &\leq \frac{\log cr^{n+a}}{r^{1+\delta}} \\ &= \frac{\log c}{r^{1+\delta}} + (n+a)\frac{\log r}{r^{1+\delta}}. \end{aligned}$$

Letting $r \to +\infty$ we obtain

$$\liminf_{r\to+\infty} \frac{\log Vol(B_p(r))}{r^{1+\delta}} < +\infty.$$

Therefore, it suffices to apply Theorem 4.7 to conclude that there exists a sequence $(x_n) \subset M$, $n = 1, 2, \ldots$, such that

$$u(x_n) \to u^*$$

and

$$\mathcal{L}_\varphi u(x_n) < \frac{1}{n}.$$

This is what we wanted to prove

□

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