



**UNSCENTED KALMAN FILTERING
ON EUCLIDEAN AND RIEMANNIAN MANIFOLDS**

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**TESE DE DOUTORADO EM ENGENHARIA ELÉTRICA
DEPARTAMENTO DE ENGENHARIA ELÉTRICA**

**FACULDADE DE TECNOLOGIA
UNIVERSIDADE DE BRASÍLIA**

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**FITLRAGEM DE KALMAN UNSCENTED
NAS VARIEDADES EUCLIDIANA E RIEMANNIANA**

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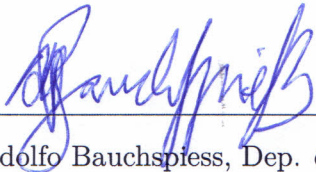
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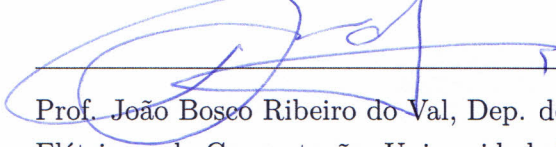
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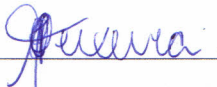
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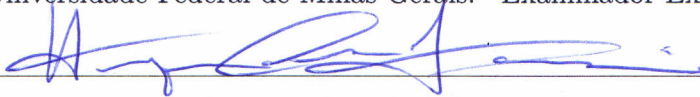
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To my lovely wife and children.

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I register my most sincere acknowledgements to my wife, Laryssa Menegaz; to my parents, Menandro and Mary Menegaz; to my brothers, Felipe and Gabriel Menegaz; to Rafael Moraes and Homero Píccolo; and to all my friends, who are many, fortunaly.

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ABSTRACT

Title: Unscented Kalman Filtering on Euclidean and Riemannian Manifolds

Author: Henrique Marra Taira Menegaz

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Program: Graduate Program in Engineering of Electronic and Automation Systems–PGEA

Keywords: Dual Quaternion, Quaternion, Riemannian Manifold, Riemannian Unscented Kalman Filter (RiUKF), Unscented Kalman Filter (UKF), Unscented Transformation (UT).

In this thesis, we take an in-depth study of an increasingly popular estimation technique known as Unscented Kalman Filter (UKF). We consider theoretical and practical aspects of the unscented filtering.

In the first part of this work, we propose a systematization of the Unscented Kalman filtering theory on Euclidean spaces. In this systematization, we i) gather all available UKF's in the literature, ii) present corrections to theoretical inconsistencies, and iii) provide a tool for the construction of new UKF's in a consistent way. Mainly, this systematization is done by revisiting the concepts of sigma set (SS), Unscented Transformation (UT), Scaled Unscented Transformation (SUT), Square-Root Unscented Transformation (SRUT), UKF, and Square-Root Unscented Kalman Filter (SRUKF). We introduce continuous-time and continuous-discrete-time UKF's. We illustrate the results in i) some analytical and numerical examples, and ii) a practical experiment consisting of estimating the position of an automotive electronic throttle valve using UKF's developed in this work; this valve's position estimation is also, from a technological perspective, a contribution on its own.

In the second part, first, we i) unfold some consistence issues in the theory behind the UKF's and SRUKF's for unit quaternion systems of the literature—such as definitions of random quaternions and additive-noise quaternion systems—, ii) propose an UKF embodying all these UKF's, and iii) propose an SRUKF with better computational properties than all these SRUKF's. Second, we propose an extension of some results of the literature concerning statistics on Riemannian manifolds. Third, we use these statistical results to present an extension to Riemannian systems of the Euclidean systematization developed in the first part. In this Riemannian systematization, we propose i) additive-noise Riemannian systems; and ii) Riemannian versions of the concepts of SS, UT, SUT, SRUT, UKF, and SRUKF. Several new consistent UKF's are introduced. Afterwards, we present closed forms of almost all the operations contained in the Unscented-type Riemannian filters for unit quaternion systems. We also introduce consistent i) UKF's for systems of unit dual quaternions, and ii) continuous-time and continuous-discrete-time UKF's for Riemannian manifolds.

RESUMO

Título: Filtragem de Kalman Unscented nas Variedades Euclideana e Riemanniana

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Orientador: Prof. João Yoshiyuki Ishihara

Programa: Programa de Pós-graduação em Engenharia de Sistemas Eletrônicos e de Automação – PGEA

Keywords: Quatérnio Dual, Quatérnio, Variedade Riemanniana, Filtro de Kalman Unscented Riemanniano, Filtro de Kalman Unscented, Transformação Unscented.

Nesta tese, nós estudamos com profundidade uma técnica cada vez mais popular conhecida como Filtro de Kalman Unscented (FKU). Consideremos tanto aspectos teóricos como práticos da filtragem Unscented.

Na primeira parte deste trabalho, propomos uma sistematização da teoria de filtragem de Kalman Unscented. Nessa sistematização nós i) agrupamos todos os FKUs da literatura, ii) apresentamos correções para inconsistências teóricas detectadas, e iii) propomos uma ferramenta para a construção de novos FKU's de forma consistente. Essencialmente, essa sistematização é feita mediante a revisão dos conceitos de conjunto sigma (SS), Transformação Unscented (TU), Transformação Unscented Escalada (TUE), Transformação Unscented Raiz-Quadrada (TURQ), FKU, e Filtro de Kalman Unscented Raiz-Quadrada (FKURQ). Introduzimos FKUs tempo-contínuo e tempo-contínuo-discreto. Ilustramos os resultados em i) alguns exemplos analíticos e numéricos, e ii) um experimento prático que consiste em estimar a posição de uma válvula de aceleração eletrônica utilizando FKUs desenvolvidos neste trabalho; essa estimação da posição de válvula é também uma contribuição por si só desde um ponto de vista tecnológico.

Na segunda parte, primeiro, nós i) revelamos inconsistência na teoria por trás dos FKUs e FKURQs para sistemas de quatérnios unitários da literatura — tais como definições de quatérnios aleatórios e de sistemas quaterniônicos com ruídos aditivos —, ii) propomos um FKU englobando todos esses FKU's, e iii) propomos um FKURQ com propriedades numéricas superiores a esses FKURQs. Segundo, propomos uma extensão de alguns resultados da literatura relativos a estatísticas em variedades Riemannianas. Terceiro, usamos esses resultados estatísticos para apresentar uma extensão para sistemas riemannianos da sistematização euclidiana desenvolvida na primeira parte. Nessa sistematização riemanniana, introduzimos i) sistemas riemannianos com ruídos aditivos; e versões riemannianas dos conceitos de SS, TU, TUE, TURQ, FKU, e FKURQ. Diversos novos FKUs são introduzidos. Depois, apresentamos formas fechadas para quase todas as operações contidas nos filtros riemannianos para sistemas de quatérnios unitários. Também introduzimos consistentes i) FKUs para sistemas de quatérnios unitários duais, e ii) FKUs tempo-contínuo e tempo-contínuo-discreto.

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List of Acronyms and Abbreviations

GENERAL

pdf	probability density function
po	problematic operation
NU	numerically unstable
flops	floating points operations
RMSD	Root-Mean-Square Deviation
RMST	Root-Mean-Square Trace

NON-UNSCENTED FILTERS

KF	Kalman Filter
EKF	Extended Kalman Filter
CKF	Cubature Kalman Filter
SOEKF	Second Order extended Kalman Filter
SMCF	Sequential Monte Carlo Filter
MCMCF	Markov Chain Monte Carlo based filter
GHF	Gauss-Hermite Filter
CDF	Central Difference Filter
DDF	Divided Difference filter

SIGMA-REPRESENTATIONS

For each of these abbreviations, we also have an associated function providing the σ R's that each abbreviation represent. For instance, we have the function $\text{RhoMi}\sigma\text{R}(\bullet)$ mapping a given domain to a Rho Minimum σ -representation. Besides, there are variants of the abbreviations below with a prefix Ri-, meaning that the word Riemannian

should be added at the left side of the associated name (e.g. Ri σ R stands for Riemannian σ -representation).

l th $N\sigma$ R	l th order N points σ -representation
σ R	σ -representation (it is a 2th $N\sigma$ R)
HoMiSy σ R	(normalized) Homogeneous (odd) Minimum Symmetric σ -representation
RhoMi σ R	Rho Minimum σ -representation
MiSy σ R	Minimum symmetric σ -representation
Ri-	a prefix meaning Riemannian

UNSCENTED TRANSFORMATIONS

For each of these abbreviations, we also have an associated function providing the UT's that each abbreviation represent. For instance, we have the function RhoMiUT(\bullet) mapping a given domain to a Rho Minimum Unscented Transformation. Besides, there are variants of the abbreviations below with a prefix Ri-, meaning that the word Riemannian should be added at the left side of the associated name (e.g. RiUT stands for Riemannian Unscented Transformation).

l th $N\sigma$ R	l th order N points σ -representation
l UT	l th order Unscented Transformation
UT	Unscented Transformation (it is a 2UT)
SUT	Scaled Unscented Transformation from the literature
ScUT	Scaled Unscented Transformation defined in this work
AuxUT	Auxiliary Unscented Transformation
SiScUT	Simplex Scaled Unscented Transformation
SyInScUT	Symmetric Intrinsically-Scaled Unscented Transformation
SRUT	Square-Root Unscented Transformation
ScSRUT	Scaled Square-Root Unscented Transformation
SiScSRUT	Simplex Scaled Square-Root Unscented Transformation

SyInScSRUT	Symmetric Intrinsically-Scaled Square-Root Unscented Transformation
PaUT	Parameterizing Unscented Transformation
PaSRUT	Parameterizing Square-Root Unscented Transformation
Ri-	a prefix meaning Riemannian

UNSCENTED FILTERS

There are variants of the abbreviations below with a prefix Ri-, meaning that the word Riemannian should be added at the left side of the associated name (e.g. RiUT stands for Riemannian Unscented Transformation).

UF's	Unscented filters; it refers to the class of all Unscented-based filters with a KF structure.
UKF	Unscented Kalman Filter
SRUKF	Square-Root Unscented Kalman Filter
AdUKF	Additive Unscented Kalman Filter
AdSRUKF	Additive Square-Root Unscented Kalman Filter
CdUKF	Continuous-discrete (Augmented) UKF
CdAdUKF	Continuous-discrete Additive UKF
CoUKF	Continuous (Augmented) UKF
CoAdUKF	Continuous Additive UKF
MiAdUKF	Minimum Additive Unscented Kalman Filter
HoMiSyAdUKF	Homogeneous Minimum Symmetric Additive Unscented Kalman Filter
RhoMiAdUKF	Rho Minimum Additive Unscented Kalman Filter
MiAdSRUKF	Minimum Additive Square-Root Unscented Kalman Filter
SyAdSRUKF	Homogeneous Minimum Symmetric Additive Square-Root Unscented Kalman Filter
RhoMiAdSRUKF	Rho Minimum Additive Square-Root Unscented Kalman Filter

ECUKF	Equality-Constrained Unscented Kalman Filter
PrUKF	Projected Unscented Kalman Filter
MAUKF	Measurement-Augmentation Unscented Kalman Filter
PaSRUKF	Parameterizing Square-Root Unscented Kalman Filter
Ri-	a prefix meaning Riemannian
RiUF's	Riemannian Unscented filters; it refers to the class of all Unscented-based Riemannian filters with a KF structure.
RiSAdUKF	Riemannian-Sphere Additive Unscented Kalman Filter
RiSAdSRUKF	Riemannian-Sphere Additive Square-Root Unscented Kalman Filter
DqRiUKF	Dual-Quaternion Riemannian Unscented Kalman Filter
DqRiSRUKF	Dual-Quaternion Riemannian Square-Root Unscented Kalman Filter
DqRiAdUKF	Dual-Quaternion Riemannian Additive Unscented Kalman Filter
DqRiAdSRUKF	Dual-Quaternion Riemannian Additive Square-Root Unscented Kalman Filter

RELATIVE TO QUATERNIONS

RoV	rotation vector
GeRV	generalized Rodrigues vector
QuV	quaternion vector
FN	Forced Normalization
DPPSE	Direct Propagation of the Previous State's Estimate
GDA	Gradient Descent Algorithm
MQVCF	Minimization of a Quaternion Vector Cost Function
MAMCF	Minimization of an Attitude-Matrix Cost Function

List of Symbols and Notations

SOME NOTATIONS AND FUNCTIONS

Tab X [p:q,n:m]	the rows p to q and the columns n to m of Table X
Tab X [* ,n:m]	the columns n:m of Table X
Tab X [n:m,*]	the rows n:m of Table X
$y^{[c,n]}$	the Taylor series of y around c truncated at the term of order n , inclusive
$\mathcal{O}(n)$	computational complexity of order n
$gen(u)$	gives a set composed of all different permutations of the scalar elements of u
$\langle \bullet, \bullet \rangle$	a inner product
$\ \bullet\ $	a norm function
$dist(\bullet, \bullet)$	a distance function
$grad f$	the gradient of a function f
$Hess f$	the Hessian of a function f
$vec(\bullet)$	the vector operator
μ_ϵ	an mean square error

RELATED TO SETS AND GROUPS

\mathbb{R}	set of real numbers
\mathbb{R}^n	set of composed of the n tuples of real numbers (x_1, x_2, \dots, x_n) . In matrix notation, (x_1, x_2, \dots, x_n) is written as $[x_1, x_2, \dots, x_n]^T$
S^n	unit sphere in the \mathbb{R}^{n+1} with center at the origin
$SO(n)$	special orthogonal group of dimension n
\mathbb{H}	set of quaternions

\mathcal{H}	the set of dual quaternions
$\mathcal{H}^{\ 1\ }$	set of unit dual quaternions
Φ^n	set of all random vectors taking values in the \mathbb{R}^n
$\mathcal{M}^m, \mathcal{N}^n$	differentiable manifolds with dimension m and n , respectively
$\Phi_{\mathcal{M}}$	the set of all Riemannian random points taking values in the Riemannian manifold \mathcal{M}
$\mathbb{B}_c(r)$	an open ball centered at c with radius r

RELATED TO VECTORS AND MATRICES

\mathcal{V}	a vector space
$a, b, c, r, \alpha, \beta, \rho$	constants of a vector space; often of the \mathbb{R}^n
x, y, z, u, v	variables of a vector space; often of the \mathbb{R}^n
A, B, C, E, S, U, V	matrices
A^T	the transpose matrix of a matrix A
I_n	$n \times n$ identity matrix
$\text{Tr}(A)$	the trace of a matrix A
\otimes	the Kronecker product operator or the quaternion product
$\bigotimes_{i=1}^n A_i, A^{\otimes n}$	$\bigotimes_{i=1}^n A_i := A_1 \otimes \cdots \otimes A_n$, and $A^{\otimes n} := \bigotimes_{i=1}^n A$
$\sqrt{A}, A^{1/2}$	square-root matrix of a matrix A ($\sqrt{A}\sqrt{A}^T = A$ and $A^{1/2}(A^{1/2})^T = A$)
$[A]_{p \times q}$	block matrix consisting of the matrix A being repeated p times on the rows and q on the columns
$(A)_{(i_1:i_2), (j_1:j_2)}$	sub-matrix of the matrix A formed by the rows i_1 to i_2 and the columns j_1 to j_2
$(A)_{i,j}, A_{ij}$	both are the i th row and j th column element of a matrix A
$(A)_{*j}, (A)_{i*}$	$(A)_{*j}$ is the j th column of a matrix A , and $(A)_{i*}$ the i th row

$ A $	the matrix such that $ A _{ij} := A_{ij} $, where $ \cdot $ represents the absolute value operator and A is a matrix
$(A)(\diamond)^T, [A][\diamond]^T$	equal to $(A)(A)^T$ and $[A][A]^T$, respectively
$\text{diag}([a_1, \dots, a_l]^T)$	diagonal matrix with the diagonal elements being the scalars a_1, \dots, a_l , in this order
$\text{diag}(A_1, \dots, A_l)$	block diagonal matrix with the diagonal blocks being the square matrices A_1, \dots, A_l , in this order
$\text{sign}(A)$	the matrix such that $(\text{sign}(A))_{i,j} = 1$, if $A_{i,j} \geq 0$; and $(\text{sign}(A))_{i,j} = -1$, if $A_{i,j} < 0$, where A is a matrix.
$\text{tria}\{A\}$	a lower triangularization of A (e.g. QR decomposition with Q being lower triangular)
$\text{cdown}\{A, B\}$	Cholesky downdate of A by B , that is, the Cholesky factor of $AA^T - BB^T > 0$
exp	exponential function
log	(natural) logarithm function
$\text{rank}(A)$	the rank of the matrix A
$\text{min}\{a, b\}$	the minimum between a and b
cu	a function performing a Cholesky update; this function uses few or even none Cholesky downdatings

RELATIVE TO QUATERNIONS, DUAL QUATERNIONS AND ROTATIONS

$\mathbf{x}, \mathbf{y}, \mathbf{q}, \mathbf{p}$	points of a differentiable manifold, or particularly a quaternion
$\underline{\mathbf{x}}, \underline{\mathbf{y}}, \underline{\mathbf{q}}, \underline{\mathbf{p}}$	dual-quaternions
$1, \hat{i}, \hat{j}, \hat{k}$	the basis elements of the quaternion algebra
\mathbf{v}_m	the Imaginary vector unit; $\mathbf{v}_m := [\hat{i}, \hat{j}, \hat{k}]$
quaternion vector	for a quaternion $q_1 + \mathbf{v}_m q$, q is the quaternion vector
$\text{Re}(\mathbf{q})$	the real part of a quaternion \mathbf{q}
$\text{Im}(\mathbf{q})$	the imaginary of a quaternion \mathbf{q}

\mathbf{q}^*	the conjugate of a quaternion \mathbf{q}
\mathbf{q}^{-1}	the inverse of a quaternion \mathbf{q}
unit quaternion	a quaternion whose norm is equal to 1
unit dual-quaternion	a dual quaternion whose pseudo norm is equal to 1
$\text{QtoRoV}(\bullet)$	the function mapping a unit quaternion to a RoV
$\text{RoVtoQ}(\bullet)$	the function mapping a RoV to a unit quaternion
$\text{QtoGeRV}(\bullet)$	the function mapping a unit quaternion to a GeRV
$\text{GeRVtoQ}(\bullet)$	the function mapping a GeRV to a unit quaternion
$\text{QtoQuV}(\bullet)$	the function mapping a unit quaternion to a QuV
$\text{QuVtoQ}(\bullet)$	the function mapping a QuV to a unit quaternion
$\text{QtoV}(\bullet)$	an \mathbb{R}^3 parameterization of the S^3
$\text{VtoQ}(\bullet)$	the inverse of $\text{QtoV}(\bullet)$
q^v	an \mathbb{R}^3 parameterization of the unit quaternion \mathbf{q}
$R(\theta, n)$	a rotation by an angle θ in turn of the unit vector n
$\psi(\underline{\mathbf{q}})$	the function mapping a dual quaternion $\underline{\mathbf{q}} = \mathbf{q} + \varepsilon \frac{1}{2} \mathbf{q} \mathbf{q}$ to $[\mathbf{q}, \mathbf{q}]^T$
$\mathbf{q} \oplus \mathbf{p}$	an “addition” of two “random unit quaternion” ($\overline{\mathbf{q} \oplus \mathbf{p}} := \bar{\mathbf{q}} \oplus \bar{\mathbf{p}}$ and $P_{\mathbf{q} \oplus \mathbf{p}} := P_{\bar{\mathbf{q}}} \oplus P_{\bar{\mathbf{p}}}$)
$\underline{\mathbf{q}} \boxplus p$	an “addition” of a random unit dual quaternion $\underline{\mathbf{q}}$ such that $\psi(\underline{\mathbf{q}}) \sim (\bar{\mathbf{q}}, \mathbf{P}_{\mathbf{q}})_{S^3 \times \mathbb{R}^3}$ with a random vector $p \sim (\bar{p}, P_p)_{\mathbb{R}^6}$ [$\psi(\underline{\mathbf{q}} \boxplus p) \sim (\exp_{\bar{\mathbf{q}}} \bar{p}, \mathbf{P}_{\mathbf{q}} + P_p)$]

RELATIVE TO DIFFERENTIABLE MANIFOLDS

$\mathbf{x}, \mathbf{y}, \mathbf{q}, \mathbf{p}$	points of a differentiable manifold, or particularly a quaternion
(U, φ)	a parameterization of a differentiable manifold
\mathcal{A}	an atlas
v	a vector of a tangent space of a differentiable manifold

$\gamma(t, \mathbf{q}, v)$	a geodesic (or any curve) in differentiable manifold with $\gamma(0, \mathbf{q}, v) = \mathbf{q}$ and $\dot{\gamma}(0, \mathbf{q}, v) = v$
$\exp_{\mathbf{q}}$	the Riemannian exponential mapping at \mathbf{q}
$\log_{\mathbf{q}}$	the Riemannian logarithm mapping at \mathbf{q}
$C(\mathbf{p})$	the <i>cut locus</i> at \mathbf{q}
$\mathcal{C}(\mathbf{p})$	the tangential <i>cut locus</i> at \mathbf{q}
$\mathcal{D}(\mathbf{p})$	the maximum definition domain of $\exp_{\mathbf{q}}$
$\text{PT}(A)$	the parallel transport of a bilinear mapping A in a tangent space of a Riemannian manifold
$\mathbb{L}(\gamma)$	arc length of the curve γ .

RELATIVE TO RANDOM VECTORS AND RIEMANNIAN RANDOM POINTS

Variants typed in boldface are the Riemannian cases, or particularly the quaternions cases

$(\Omega, \mathcal{B}(\Omega), \text{Pr})$	a probability space. Ω is the sample space, $\mathcal{B}(\Omega)$ the Borel σ -algebra of Ω , and Pr a measure $\mathcal{B}(\Omega)$ such that $\text{Pr}(\Omega) = 1$
X, Y, Z	random vectors. Often we have $Y = f(X)$
$X Y$	the rv X conditioned to Y
pdf_X	the pdf of an rv
$\mathcal{E}_X, \mathcal{E}$	i) \mathcal{E}_X the expected value relative to X , and ii) \mathcal{E} is a expected value relative to a rv known from the context
\bar{X}	the mean of an rv X
$\mathbb{E}(\mathbf{X})$	the set of all the means of a Riemannian random point \mathbf{X} .
σ_X^2	the variance of an rv X
P_{XX}, P_{XY}	i) P_{XX} is the covariance (matrix) of an rv X , and iii) P_{XY} the cross-covariance of a joint rv (X, Y)
M_X^l	the l th order central moment of an rv X

$X \sim (m, M^2, \dots, M^l)^n$	an rv $X \in \Phi^n$ with mean m and i th central moments $M_X^i = M_i, i = 2, \dots, k$
\hat{A}	an estimate of A ; A can be any element (e.g. an rv, a covariance matrix, a Riemannian random point, etc)
$X \sim N(m, P)$	X is a joint normal rv with mean m and covariance P
$\mathbf{q} \oplus \mathbf{p}$	an “addition” of two “random unit quaternion” ($\overline{\mathbf{q} \oplus \mathbf{p}} := \bar{\mathbf{q}} \oplus \bar{\mathbf{p}}$ and $P_{\mathbf{q} \oplus \mathbf{p}} := P_{\bar{\mathbf{q}}} \oplus P_{\bar{\mathbf{p}}}$)

RELATIVE TO WEIGHTED SETS, σ -REPRESENTATIONS AND UNSCENTED TRANSFORMATIONS

Variants typed in boldface are the Riemannian cases, or particularly the quaternions cases

$\{\chi_i\}_{i=b}^c$	a set with elements $\chi_b, \chi_{b+1}, \dots, \chi_c$
χ, γ, ξ, ζ	(weighted) sets. Often, we have $\gamma = f(\chi)$ (meaning that each point of γ is a transformation by f of each point of χ)
χ_i	a point of of a set χ
χ', γ'	Scaled sets of a SiScUT. In this case $\gamma' = f(\chi')$
$\tilde{\chi}, \tilde{\gamma}$	Scaled sets of a SyInScUT. In this case $\tilde{\gamma} = f(\tilde{\chi})$
w_i, w'_i	weight, respectively, of i) a set, and ii) of a scaled set
w, W	the vector $w := [w_1, \dots, w_N]^T$ and the matrix $W = \text{diag}(w_1, \dots, w_N)^T$, respectively
w_i^m, w_i^c, w_i^{cc}	weights used to define, respectively, the i) mean, ii) covariance, and iii) cross-covariance of given sets
$\tilde{w}_i^m, \tilde{w}_i^c, \tilde{w}_i^{cc}$	analogous for the scaled sets of a SyInScUT
μ_χ	the sample mean of a set χ
$\mathcal{E}(\chi)$	the set of all the sample means of a Riemannian set χ .
$\Sigma_{\chi\chi}, \Sigma_{\chi\gamma}$	i) $\Sigma_{\chi\chi}$ is the sample covariance of a set χ , ii) $\Sigma_{\chi\gamma}$ the sample cross-covariance between two sets χ and γ

$\Sigma_{\gamma\gamma}^\alpha, \Sigma_{\chi\gamma}^\alpha$	in a ScUT, i) $\Sigma_{\gamma\gamma}^\alpha$ is the scaled sample covariance of γ , and ii) $\Sigma_{\chi\gamma}^\alpha$ is the scaled sample cross-covariance between χ and γ
$\Sigma_{\gamma'\gamma'}^\alpha$	in a SiScUT, $\Sigma_{\gamma'\gamma'}^\alpha$ is the modified sample covariance of γ'
\mathbb{M}_X^l	the l th order sample central moment of a set χ
α, α', λ	scaling parameters of scaled sets
κ, ρ, v	κ is the tuning parameter of the classical UKF, ρ of the RhoMi σ R, and v of the Mi σ R
$SS(\bullet)$	a sigma set
$\sigma R(\bullet)$	a σ -representation
$g(\bullet)$	the scaling function

RELATIVE TO DYNAMIC SYSTEMS AND FILTERS

Variants typed in boldface are the Riemannian cases, or particularly the quaternions cases.

t, k, x_k	t is the time; for a time sequence t_0, t_1, \dots , k is the index of t_k ; and $x_k := x(t_k)$
$x_k, \varpi_k, f_k, y_k, \vartheta_k, h_k$	x_k is the internal state, ϖ_k the process noise, f_k the process function, y_k the measurement, ϑ_k the measurement noise, and h_k the measurement function for a <i>discrete-time</i> dynamic system
$x(t), \varpi(t), f_t, y_k, \vartheta_k, h_k$	analogous for a <i>continuous-discrete-time</i> dynamic system
$x(t), \varpi(t), f_t, y(t), \vartheta(t), h_t$	analogous for a <i>continuous-time</i> dynamic system
$n_x, n_y, n_\varpi, n_\vartheta$	the length of the i) state, ii) measurement, iii) process noise, and iv) measurement noise
n_a	the length of an augmented vector in an augmented UKF
$Q_k, Q(t)$	the covariances of ϖ_k and $d\varpi(t)/dt$, respectively
$R_k, R(t)$	the covariances of ϑ_k and $d\vartheta(t)/dt$, respectively

$\underline{y}_k, \underline{y}(t)$	the acquired measurements, that is, a realization of y_k and of $y(t)$, respectively
$x_{k k}, x_{k k-1}, y_{k k-1}$	the conditioned random vectors $x_k \underline{y}_1, \dots, \underline{y}_k, x_k \underline{y}_1, \dots, \underline{y}_{k-1}$, and $y_k \underline{y}_1, \dots, \underline{y}_{k-1}$, respectively
$P_{xx}^{k k}, P_{xx}^{k k-1}, P_{yy}^{k k-1}, P_{xy}^{k k-1}$	the covariance matrices of $x_{k k}, x_{k k-1}$, and $y_{k k-1}$; and the cross-covariance matrix of $(x_{k k-1}, y_{k k-1})$ respectively
$x^+(t), x^-(t)$,	the conditioned random vectors $x(t) \underline{y}_1, \dots, \underline{y}_k$, and $x(t) \underline{y}_1, \dots, \underline{y}_{k-1}$, respectively
$G_k, G(t)$	Kalman gains

1. INTRODUCTION

Unscented Kalman filtering has become extremely popular in the control community. According to the IEEE Xplore Digital Library (an website of Institute of Electrical and Electronics Engineers [IEEE])¹, the work [1] reached the impressive numbers of 8222 reads; and 1279 citations on the IEEE, 2735 on the Scopus (<http://www.scopus.com>), and 1564 on the Web of Science (<http://apps.webofknowledge.com>) catalogs.

Since the seminal work [2], Unscented Kalman Filters (UKF's) have been used in numerous applications. For instance, we can find them being used to estimate variables related to batteries [3–7], wind generators [8], frequency control of power systems [9], integrated circuits [10], sigma-delta modulators [11], inertial navigation systems [12], satellites [13], medical imagings [14], computer-assisted surgeries [15], plasma insulins [16], endoscopy capsules [17], microphones [18], acoustic tomographies of the atmosphere [19], mobile robots [20–22], among others.

Some UKF's properties can be well understood when these filters are put in relation with the widely known Extended Kalman Filter (EKF). In many applications—e.g. [7, 16, 21], and [22], among others—, the UKF's performed better than the EKF. This superior performance can be explained, at least, by the following two reasons:

- the computational complexities of the UKF's and the EKF are of the same order, but UKF's tend to attain better estimation performance [23];
- the UKF is derivative-free (no need to compute Jacobian matrices), while the EKF requires the dynamics to be differentiable. Thus, unlike the EKF, UKF's can be used with systems where Jacobian matrices may not exist, such as systems with discontinuities (cf. [1]).

A great part of the Unscented-theory researchers' efforts has been devoted to find extensions of the first UKF. The direction of these extensions are similar to the directions taken by the already proposed EKF variants in the literature. There are EKF extensions toward diverse classes of state spaces and dynamic systems (cf. [24–26]), such as toward the following ones:

1. different classes of *states spaces regarding their algebraic structure*, such as state

¹In [http://ieeexplore.ieee.org/xpl/abstractMetrics.jsp?arnumber=1271397&action=search&sortType=&rowsPerPage=&searchField=Search_All&matchBoolean=true&queryText=\(julier%20unscented%20kalman%20filtering%20for%20nonlinear%20estimation\)](http://ieeexplore.ieee.org/xpl/abstractMetrics.jsp?arnumber=1271397&action=search&sortType=&rowsPerPage=&searchField=Search_All&matchBoolean=true&queryText=(julier%20unscented%20kalman%20filtering%20for%20nonlinear%20estimation)), accessed at 21:00, on February the 15th, 2016.

spaces composed of unit quaternions [27], unit dual quaternions [28], Lie Groups [29], etc;

2. different classes of *dynamic systems regarding the forms of their sets-of-time*—the sets composed of the time parameters—, such as discrete-time systems, continuous-time systems, continuous-discrete-time systems [24].

In this work, we make an extensive study of the Unscented Kalman filtering literature considering different aspects such as algebraic structures of the state-space and forms of the sets-of-time. We show strong and weak points, make comparisons, propose corrections, and present one attempt of a systematic theory.

1.1 UNSCENTED FILTERING PROBLEM

Broadly, filters can be viewed as algorithms that extract information from sets of acquired data. When we want to know the value of some variables of a given system—e.g. the position and velocity of a car, the position and attitude of a satellite, the temperature of a boil, etc—we use instruments to acquire *measurements* from this system. However, only with these measurements (the data), we most often can not determine exactly the value of the desired variables. This can be explained at least by the following two reasons:

1. *Measurements are corrupted by noise.* The sources of noise may vary in each case; beside others, we can point out i) the limited resolution, precision and accuracy of real instruments, which make the measurements certain only to a limited precision—e.g. if the minimum divisor of the scale of a given rulers is 1 cm, the measurements of this instrument are certain only to the precision of centimeters, but not to millimeters—; and ii) the limited knowledge of the real process being investigated, since there are always events influencing the measurements that are difficult to account for.

Therefore, given an acquired *signal* (a data set looked as a sequence ordered by time), we can develop techniques that are able to, at least up to a certain precision, “separate” from the noise the information within this data set that is important to determine the desired variable. We call these techniques *estimators*, and the value of the desired variable given by the estimators, *estimate*.

Estimators for noisy signals can rely, for example, on analyzing i) the frequency of the acquired signal—usually, at least some part of the noise have particular frequency components—such as the so called low-pass filters, band-pass filters, Butterworth filters, among others [30]; ii) the entropy of the acquired signal, such

as the algorithms based on theory of Chaos [31].

2. We might *not be able to measure the desired variables directly*, but only other ones; for instance, we might want the temperature of a given boil, but it may happen that we can measure only its pressure. In this case, we must developed mathematical models relating the desired variables with the measured variables. Let us call a list of the desired variables (*internal*) *state* and denoted it by x ; a list of the measured variables simply *measurement* and denote it by y . Consider also that noises are corrupting the measurement; call a list of these noises *measurement noise* and denote it by ϑ . Since real problems are *dynamic* (they change in time), often it is necessary to consider x, y and ϑ as time varying. In this case, we can write the following equation relating these lists:

$$y(t) = h_t(x(t), \vartheta(t)), \quad (1.1)$$

where $\{x(t)\}$, $\{y(t)\}$, $\{\vartheta(t)\}$, and $\{h_t\}$ —each h_t is an well-defined function called *measurement function*—are sequences parameterized by the time $\{t; t \geq t_0, t \in \mathbb{R}\}$ — \mathbb{R}^n stands for the Euclidean space of dimension n , and $\mathbb{R} := \mathbb{R}^1$. We will denote by $\hat{x}(t)$ an *estimate* of $x(t)$.

Suppose that an estimate $\hat{x}(t^*)$ is provided by an estimator using the history of measurements (a sequence of measurements over time) $y_{t_0:t_1} := \{y(t); t_0 \leq t \leq t_1\}$. We can distinguish three classes of estimators depending on t^* . If $t^* < t_1$, we call the estimator a *smoother* (and the associated problem of finding an estimate of $x(t^*)$ with $y_{t_0:t_1} := \{y(t); t_0 \leq t \leq t_1\}$ a *smoothing*); if $t^* > t_1$, we call the estimator a *predictor* (and the associated problem a *prediction*); and if $t^* = t_1$, we call the estimator a *filter* (and the associated problem a *filtering*). In this work, *we consider only filters*.

For models like (1.1), it is desirable to develop *recursive filters*. Suppose that i) we have an estimate $\hat{x}(t_1)$ that was generated by a given filter φ using the sequence of measurements $y_{t_0:t_1} := \{y(t); t_0 \leq t \leq t_1\}$; ii) we have a sequence of measurements $y_{t_1:t_2} := \{y(t); t_1 \leq t \leq t_2\}$; and iii) we want to estimate $x(t_2)$. We can apply the same filter φ to estimate $x(t_2)$ based on the history $y_{t_1:t_2}$, but we would not use the information of $y_{t_0:t_1}$. On the other hand, we could use the filter φ to estimate $x(t_2)$ based on all the history $y_{t_0:t_2} := \{y(t); t_0 \leq t \leq t_2\}$, but the computational cost would be higher than the previous option. Another solution would be to estimate $x(t_2)$ by “updating” $\hat{x}(t_1)$ with the information of $y_{t_1:t_2}$. This last filter is *recursive*; recursive filters provide online estimates as functions of previous estimates. They are computationally more efficient.

Equation (1.1) describes how the state relates with the measurement, but does not model *how the state evolves in time*. We can, for some problems, develop equations

describing this evolution over time. Since mathematical models describe real processes imperfectly, we should also include a variable accounting for the errors in this model; we will call this error variable the *process noise*, and denote it by $\varpi(t)$ —and its sequence over time by $\{\varpi(t); t \geq t_0\}$. One form of modeling the evolution of $x(t)$ overtime including the noise $\varpi(t)$, is by the following *differential equation*:

$$\frac{d}{dt}x(t) = f_t(x(t), \varpi(t)), \quad (1.2)$$

where f_t is called the *process function*, and $\{f_t; t \geq t_0\}$ its sequence over time. The *pair* of equations (1.1)-(1.2) is called a *dynamic system*. Equation (1.2) models how the internal state evolves *over time*, and (1.1) how the internal state relates with the acquired measurements *at a time instant*.

Since the system (1.1)-(1.2) is corrupted by noises, we have to choose a way of dealing with *non-deterministic variables*. The theories of probability and statistics are often used for this purpose. In this approach, we consider $x(t)$, $y(t)$, $\varpi(t)$, and $\vartheta(t)$ to be *random vectors*, and their sequences over time ($\{x(t)\}$, $\{y(t)\}$, $\{\varpi(t)\}$, and $\{\vartheta(t)\}$), *stochastic processes*. In this case, the system (1.1)-(1.2) is called a *stochastic dynamic system*.

The classical Kalman-Bucy Filter (KF) provides *the optimal solution* with respect to diverse criteria to the problem of filtering system (1.1)-(1.2) when the following two conditions are satisfied: i) each f_t and h_t is *linear*; and ii) the initial state $x(t_0)$, and each noise ϖ_t and ϑ_t are *Gaussian* distributed and mutually *independent* [24, 32]. However, when these conditions are not satisfied, optimal solutions for the filtering problem tend to be computationally intractable. Therefore, sub-optimal approaches must be sought, and the *UKF is one of these sub-optimal filtering solutions*.

There are *variants of the UKF*, and usually they are associated with variants of the considered stochastic dynamic system. Different forms of (1.1)-(1.2) can be considered by varying 1) the form of the set-of-time $\mathcal{T} := \{t; t \geq t_0, t \in \mathbb{R}\}$, and/or 2) the topological space in which $x(t)$, $y(t)$, $\varpi(t)$, and $\vartheta(t)$ take values.

1. Variants of (1.1)-(1.2) respective to \mathcal{T} are the *discrete-time* and *continuous-discrete-time* stochastic dynamic systems.

In (1.1)-(1.2), the time parameter belongs to a *continuous set* $\{t; t \geq t_0, t \in \mathbb{R}\}$; for this reason, we say that (1.1)-(1.2) is *time continuous* (thus the system can be named *continuous-time stochastic dynamic system*). Nonetheless, measurements are usually not acquired continuously, but in instants of time shifted by a fixed interval; this interval is called the *sampling time*, and we say that the signal is *sampled*. Thus, it might be advantageous to write (1.1) parameterized by a discrete set-of-time $\{t_k; k \in \mathbb{N}\}$ — \mathbb{N} stands for the set of natural numbers—where

each t_k is an instant in which the signal is sampled. In this case, we can write the (discrete-time) measurement equation as follows:

$$y_k = h_k(x_k, \vartheta_k), \quad (1.3)$$

where $x_k := x(t_k)$, $y_k := y(t_k)$, $\vartheta_k := \vartheta(t_k)$, $h_k := h_{t_k}$. The pair of equations (1.1)-(1.3) is called a *continuous-discrete-time stochastic dynamic system*.

Because filters are usually implemented in computers, and computers can not perform calculations of continuous variables, we can also consider discrete-time variants of the process equation (1.2). A discrete-time variant of (1.2) is the following *difference equation*:

$$x_k = f_k(x_{k-1}, \varpi_k), \quad (1.4)$$

where $\varpi_k := \varpi(t_k)$, $f_k := f_{t_k}$. In this case, the pair of equations (1.3)-(1.4) is called a *discrete-time stochastic dynamic system*.

2. We can distinguish variants of all these three systems by considering *different topological spaces* in which $x(t)$, $y(t)$, $\varpi(t)$, and $\vartheta(t)$ take values. In the three stochastic dynamic systems above, $x(t)$, $y(t)$, $\varpi(t)$, and $\vartheta(t)$ are considered to be random *vectors*; this means that they take values in Euclidean spaces, but we can also consider these random elements taking values in other spaces. In this thesis, we work with the following three topological spaces:

- (a) the set *unit quaternions*. Unit quaternions are *quaternions* whose norms are equal to 1; quaternions are a 4-dimensional extension of complex numbers [33]—we present the unit quaternions with more details in Section 7.1. Unit quaternions can represent *rotations of 3-dimensional rigid bodies*, and present advantages comparative with other representations of rotations [34].
- (b) the set of *unit dual quaternions*. Unit dual quaternions are *dual quaternions* whose pseudo-norms are equal to 1; dual quaternions are dual numbers whose primary and secondary parts are quaternions [35] (unit dual quaternions are explained in Section 9.6). Inasmuch as unit quaternions are a good choice to represent rotations of 3-dimensional rigid bodies, unit dual quaternions are a good choice to represent (full) *displacements (rotations and translations, simultaneously)* of such bodies.
- (c) *Riemannian manifolds*. In a wide sense, Riemannian manifolds are spaces *locally* resembling Euclidean spaces—we review Riemannian manifolds briefly in Chapter 8. Examples of Riemannian manifolds include i) Euclidean spaces, ii) n -dimensional spheres— S^n ; it is the set of all points distanced

(in the usual sense of distances in Euclidean spaces) by 1 from the origin of the \mathbb{R}^{n+1} ; the set of unit quaternions is the S^3 —, iii) the set of orthogonal matrices, among others. Among other applications, the theory of Riemannian manifolds was used by Albert Einstein to develop the general theory of relativity [36].

In this work, we study Unscented Kalman filtering theory for each of the aforementioned systems: systems composed of i) different sets-of-time (continuous-time, continuous-discrete-time, discrete-time systems), and ii) different spaces for the variables (Euclidean, unit quaternions, unit dual quaternions, and Riemannian manifolds). UKF's on Euclidean spaces are considered in Part I, and UKF's on Riemannian manifolds, the set of unit quaternions, and the set of unit dual quaternions are considered in Part II.

1.2 HISTORICAL NOTES

In 1995, in the American Control Conference work [2], Simon J. Julier, Jeffrey K. Uhlmann, and Hugh F. Durrant-Whyte proposed the first variant of a stochastic filter that later would be called the *Unscented Kalman Filter*². To the best of our knowledge, the first use of the word "Unscented" was in the 1997 papers [37, 39] by Julier and Uhlmann. This word choice is attributed to Uhlmann; he himself narrates the story of this choice in an interview given to the Engineering and Technology History Wiki³. In the following years, the UKF theory would grow up rapidly with numerous scientific contributions.

In 1997, a key concept of the UKF theory was introduced by [37]: the *Unscented Transformation (UT)*. In that work, the UT is presented as an efficient mechanism for computing means and covariances of transformed random vectors. It is also in [37] that an augmented variant of the UKF—the state vector is augmented with the process noise vector; the most important concepts to the UKF theory enunciated in this chapter will be explained in the next one (e.g. augmented UKF, Scaled UT, etc)—is proposed for the first time.

The first *journal* paper on UKF was [40] in 2000. In that work, the UKF theory takes the first steps toward a formal systematized theory. To the best of our knowledge, so far the research on this topic was carried out mainly by the authors of [2], but from

²Apparently, from [2] and [37], some UKF's key ideas are already from [38], but we could not get access to this work.

³Available in http://www.ieeeeghn.org/wiki/index.php/First-Hand:The_Unscented_Transform.

2000 onwards, other authors would contribute to the topic of Unscented filtering.

In the following years, Rudolph van der Merwe and Eric A. Wan presented three conference works regarding the theory of UK filtering:

1. in [41], in 2000, they proposed a variant of the UKF which would become as popular as the original UKF of [2];
2. in [42], in 2001, they proposed the first square-root variant of the UKF; and
3. in [43], also in 2001, along with Arnaud Doucet and Nando de Freitas, they proposed the first use of the UT in more general filtering settings, namely the Unscented Particle Filter.

The scaled variant of the UKF was proposed in [44], in 2002; arguably, this variant would increase the estimation quality of an Unscented filter without increasing its computational cost.

The UKF is composed of a set of weighted points; this set became known as *sigma set*, and its weighted points, *sigma points*. Until 2002, all UKF's were composed of—for n being the length of the state vector—at least, $2n$ sigma points, but in that year, [45] proposed a set composed of $n+2$ sigma points, and in the following year, [46] introduced a set composed of $n+1$ sigma points. On the other hand, [47] introduced an $2n^2+1$ UKF with increased estimation properties.

In 2003, [48] proposed an UKF designed for attitude estimation of systems being modeled with unit quaternions—unit quaternions are efficient to represent rotations, but present some challenges to work with UKF's (see Chapter 7). This UKF, named Unscented Quaternion Estimator (USQUE), became very popular, specially in the aerospace community.

A milestone of this theory has been reached in 2004 with [1], a work by Julier and Uhlmann in the Proceedings of the IEEE that became very popular. Essentially, that work gathered and presented many the results on the UKF theory developed to that date in an didactic fashion.

In the following years, other important results were introduced, such as

1. a comparison between the augmented and the additive UKF variants in 2005 by [49];
2. stability and error analyses for linear measurements in 2006 by [50], and for nonlinear measurements in 2007 by [51];
3. UKF variants for continuous-time and for continuous-discrete-time systems in 2007 by [52];

4. Unscented Rauch-Tung-Striebel Smoother in 2008 by [53];
5. UKF variants for gain-constrained systems in 2008 by [54], for equality-constrained systems in 2009 by [55], and interval-constrained systems in 2010 by [56];
6. a minimum UKF variant in 2011 by our work [57];
7. the Truncated UKF in 2012 by [58]; vii) a study of the scaling parameter of an UKF in 2012 by [59]; and
8. revelation of an important inconsistency in the UKF theory developed so far in 2012 by [60] (see Section 2.4.1).

1.3 OUTLINE OF THIS WORK

This thesis is, in part, a systematization and, as a result, we could not write our own contributions in chapters *different* from those containing analyses of the literature. Usually, separating these contributions in a chapter level facilitates assessment of a thesis. However, this work provides a *large number new results*; additionally, many of these results are related to *different topics* of the theory considered here (e.g. results relative to sigma sets, to UT's, to UKF's, to SRUKF's, to statistics on Riemannian manifolds, and so on). In consequence, if we have chosen to separate our contributions from the literature ones in different chapters, the text would lack in cohesion. Nevertheless, we separate these contributions *in sections*; generally, each section is composed either uniquely of novelties or literature's results; there are *a few* exceptions to this rule, but their are stated expressly.

In Chapter 2, through a detailed analysis of the present Unscented's theory state-of-the-art, we unfold some inconsistencies within the Unscented theory. The results in Sections 2.1 and 2.2 are *not* novelties of this work. In these sections, we *present* the literature's theory regarding UF's for Euclidean manifolds; therefore, regarding the content of these sections, we can only claim contributions in the sense of *gathering* these results. On the contrary, the results in Sections 2.3 to 2.8 are all novelties. In these sections, we *analyze* the literature's theory regarding UF's for Euclidean manifolds, and *show gaps and inconsistencies* within this theory.

Willing to rectify these inconsistencies, we propose a systematization of the Unscented Kalman Filtering theory; this is done constructively in the three subsequent chapters: i) in Chapter 3, we introduce the concept of a σ -representation of a random vector, and establish some results related to this new concept; in Chapter 4, using the new results of Chapter 3, we propose results concerning the Unscented Transformation,

the Scaled Unscented Transformation, and the Square-Root Unscented Transformation; and iii) in Chapter 5, using results of the two preceding chapters, we propose new definitions for the Unscented Kalman Filter and the Square-Root Unscented Kalman Filter. All the results of Chapter 3, 4, and 5 are novelties of this work.

The results of all these three chapters are illustrated in numerical simulations, and afterwards, in Chapter 6, we introduce an experimental/technological innovation using some of the new UKF's: these filters are used to estimate the position of an automotive electronic throttle valve. Part I ends with this application. Also, all the results in Chapter 6 are novelties.

In Part II, we are interested in extending the systematization of Part I to different kind of dynamic systems. In fact, the UKF was firstly defined for systems whose variables belong to Euclidean spaces, and developing similar filters for systems composed of other elements, such as *unit quaternions*, might be challenging.

Systems composed of unit quaternions are important when considering applications in which rotations are considered. Indeed, every element of S^3 —the sphere of radius 1 centered at the origin of the Euclidean space \mathbb{R}^4 ; it is isomorphic to the set of all unit quaternions—can be associated with an element of $SO(3)$ —the special group of orthogonal matrices; actions (with the usual matrix product) of these matrices on three-dimensional vectors are *rotations*) [33]. Unit quaternions can be found modeling rotations and attitudes of elements in aerospace applications involving satellites [61], inertial navigation systems [62], unmanned aircraft vehicles [63]; and also in other areas, such as vision [64], robotics [65], and others.

In Chapter 7, in Sections 7.1 and 7.2, we provide an extensive review of the Unscented Kalman filtering theory for systems composed of unit quaternions—the results in these sections are *not* novelties of this work—. With this review, we get to two main conclusions related to the UKF's for quaternion systems: i) in a considerable amount of UKF's, the norm constraint of the unit quaternions is not respected; and ii) some fundamental concepts and results necessary for developing a consistent Unscented theory for these systems—mainly concepts from probability and statistic theories, such as quaternion random variable, quaternion mean, etc—has not been established yet for these UKF's.

Also in Chapter 7, in Section 7.3, we present i) a single filter gathering all the UKF's for quaternion systems of the literature completely preserving the norm constraint, and ii) a square-root variant of this filter that outperforms all the square-root UKF's of the literature. Numerical examples of these filters are presented in Section 7.4. All the results in Sections 7.3 and 7.4 are novelties of this work.

The set of the unit quaternions is a Riemannian manifold—essentially, a Riemannian

nian manifold is a differentiable manifold with a metric induced by its tangent space; we provide a brief review of Riemannian manifolds in Appendix A—. By using and extending these results of [66], we move toward extending the systematization of Part I to Riemannian manifolds. The results of i) Sections 8.1 and 8.2 are *not* novelties, ii) Sections 8.3, 8.4, 8.5, 8.6 *are* novelties; nevertheless, the results of Sections 8.3 and 8.6 are novelties only in the sense of being *extensions* of some literature’s results.

In Chapter 9, we provide the whole systematization of UKF for Riemannian manifolds. We are able to provide analogous of the σ -representation, UT, and UKF’s and SRUKF’s of Part I to the Riemannian case. These Unscented Filters are either the first in the literature, or, when a similar UKF already exists (which happens only in one case), our UKF is endowed with better properties than the literature’s one. Continuous-time and Continuous-discrete-time variants are also introduced. Almost closed forms of these filters for unit quaternions are obtained. Except for Section 9.4, all the sections of Chapter 9 are composed uniquely of new results.

As already mentioned, unit quaternions plays an important role in diverse areas for modeling rotations; an extension of these numbers, the unit dual quaternions, plays an analog role for modeling full rigid body motions (rotations and translations, simultaneously). In Chapter 9, we propose Unscented filters for this set extending the Riemannian Unscented filters for the set of unit spheres developed in this work. These Unscented filters for unit dual quaternions are the first consistent ones in the literature.

Part I

Unscented Kalman Filtering on Euclidean manifolds

2. ANALYSIS OF THE LITERATURE OF UNSCENTED FILTERING ON EUCLIDEAN MANIFOLDS

In this part (Part I), we present a systematization of the theory of Unscented Kalman filtering on Euclidean spaces. Even though some works in the literature, such as [38] and [67], already provided, in some degree, systematic views of this theory, these views differ from our systematization. Comparative with these literature's views, we can say that the following contributions are part *only* of our systematization:

- New inconsistencies and gaps in the theory are identified (Chapter 2) and corrected (Chapters 3, 4, and 5);
- All the variants of Unscented Kalman filters are treated, including variants regarding i) the composition of the state variables (discrete-time, continuous-time, continuous-discrete-time forms); ii) the used UT (scaled and non-scaled forms); iii) the structure of the filter (square-root and covariance forms, augmented and additive forms); iv) the composition of the sigma sets (all the sigma sets of the literature are considered).
- New concepts and results are introduced such as the concept of σ -representation (Chapter 3), a new definition for the UT generalizing all the other variants of UT's (Chapter 4), new UKF's (Chapter 5), among others.

UKF's have become extremely popular in the past few years. However, all known UKF formulations have had their algorithms originated by *ad hoc* reasoning, and this lack of rigor might have lead to misleading interpretations and inconsistencies.

These inconsistencies are related to multiple UKF definitions (Section 2.3); the matching order of the transformed covariance and cross-covariances of both the Unscented Transformation and the Scaled Unscented Transformation (Section 2.4); issues with some reduced sets of sigma points described in the literature (Section 2.5); the conservativeness of the Scaled Unscented Transformation, and the scaling effect of the Scaled Unscented Transformation on both its transformed covariance and cross-covariances (Section 2.6); possibly ill-conditioned results in Square-Root Unscented Kalman Filters (Section 2.7); and definitions of some Additive UKF's (Section 2.8).

In the following section, we review the basics of the theory of nonlinear Kalman filtering. Then, in Section 2.2, we review the main concepts of the theory of Unscented Kalman filtering theory in the literature. Afterwards, from Section 2.3 to 2.8, we describe the inconsistencies in the literature's theory Unscented Kalman filtering mentioned above.

Remark 2.1. For now on, we will use the term *Unscented filter (UF)* referring to a general *Unscented-based filter with a KF structure*. There are numerous *Unscented-based filters with KF structures* such as Unscented Kalman Filter (UKF's), Square-Root Unscented Kalman Filters (SRUKF's), continuous-time UKF's, among others; and UF will stand for a general filter of the class composed of all these filters. Unscented-based *smoothers* and *predictors* are *not* UF's; neither *non-KF-structured* Unscented-based filters, such as Unscented Particle Filters. Note that if we use, for example, UKF's in the place of UF's, we would not be able to make the distinction of i) UKF's in the strict sense of non SRUKF's from ii) UKF's in the broader sense of all Unscented-based filters with a KF structure.

Notation 1. The set of all random vectors taking values in \mathbb{R}^n is denoted by Φ^n . For a random vector $X \in \Phi^n$, $\text{pdf}_X(x)$ stands for its probability density function (pdf), and

$$\mathcal{E}_X\{x\} := \int_{\mathbb{R}^n} x \text{pdf}_X(x) dx$$

or $\bar{X} := \mathcal{E}_X\{x\}$, for its expected value. For the random vectors X and Y , $X|Y$ stands for the random variable X conditioned to Y .

2.1 NONLINEAR KALMAN FILTERING

Discrete-time Unscented Filters are *suboptimal solutions* for the stochastic filtering problem of a discrete-time, dynamical system described either in the additive form

$$\begin{aligned} x_k &= f_k(x_{k-1}) + \varpi_k, \\ y_k &= h_k(x_k) + \vartheta_k; \end{aligned} \tag{2.1}$$

or, more generally, in the form

$$\begin{aligned} x_k &= f_k(x_{k-1}, \varpi_k), \\ y_k &= h_k(x_k, \vartheta_k), \end{aligned} \tag{2.2}$$

where k is the time step; $x_k \in \Phi^{n_x}$ is the internal state; $y_k \in \Phi^{n_y}$ is the measured output; and $\varpi_k \in \Phi^{n_\varpi}$ and $\vartheta_k \in \Phi^{n_\vartheta}$ are the process and measurement noises, respectively.

The noise terms ϖ_k and ϑ_k are assumed to be uncorrelated with, respectively, means $\bar{\varpi}_k = [0]_{n_\varpi \times 1}$ and $\bar{\vartheta}_k = [0]_{n_\vartheta \times 1}$, and covariances Q_k and R_k .

The stochastic filtering problem consists of finding estimates of the state x_k as new measurements y_k are acquired (see Section 1.1). Based on the output history $y_{1:k} := \{y_i | 1 \leq i \leq k\}$, the conditional mean

$$\mathcal{E}\{x_k | y_{1:k}\} = \int_{\mathbb{R}^n} x_k \text{pdf}(x_k | y_{1:k}) dx_k$$

is, in general, *chosen* to be the estimate of x_k because $\mathcal{E}\{x_k | y_{1:k}\}$ is i) *unbiased*—meaning that $\mathcal{E}\{x_k - \mathcal{E}\{x_k | y_{1:k}\} | y_{1:k}\} = 0$ —, and ii) an *optimal solution* with respect to diverse criteria such as the Minimum Variance criterion [24, 68]. For *linear dynamical systems*, the *Kalman Filter (KF)* provides an *optimal* value for $\mathcal{E}\{x_k | y_{1:k}\}$ with respect to the Minimum Variance criterion, as well as other criteria, when *independent Gaussian noise and initial state* are considered [24, 32]. However, in the case of non-linear systems, computing optimal values for $\mathcal{E}\{x_k | y_{1:k}\}$ tends to be computationally intractable [24, 26, 69]. Therefore, *suboptimal* approaches must be sought.

Suboptimal, non-linear filters can be classified under four different criteria, at least.

A first classification distinguishes the filters *approximating* the system's functions f_k and h_k (called *local filters*, cf. [59]) from those *not approximating* these functions (called *global filters, idem*). Examples of i) local filters are the EKF, and Second Order extended Kalman Filter (SOEKF) [26, 70]; and of ii) global filters are the Gaussian Mixture filters [71], point-mass filters [72], Sequential Monte Carlo Filters (SMCF's) [73–76]—e.g. Particle Filters, Bootstrap Filters—, and Markov Chain Monte Carlo based filters (MCMCF's) [77]—e.g. filters using Metropolis-Hastings or Gibbs sampling.

A second classification is based on whether there is the necessity of calculating derivatives of the system functions f_k and h_k , or not (cf. [59, 60]); i.e., whether the filter is *derivative-free* or not. Examples of i) derivative-free filters are the UF's [1, 2], GHF [78], Central Difference Filter (CDF) [78], Divided Difference filter (DDF) [79], and CKF [80, 81]; and of ii) non derivative-free filters are the EKF and SOEKF.

A third classification considers filters for which statistics of the *posterior* random vectors of f_k and h_k are obtained by *sampling* the pdf's of the *previous* random vectors of f_k and h_k . The samplings can be *random* or *deterministic*. Examples of i) random-sampling filters are the so called Monte Carlo (MC) filters such as SMCF's and MCMCF's; and of ii) deterministic-sampling filters are the sigma point filters such as the UKF's and the DDF. Essentially, MC filters consist of taking a *very large quantity* of samples *randomly* [73–77], while sigma point filters consist of choosing *some*

weighted samples *analytically* [67].

A fourth classification takes into account whether the state's estimates of a filter are based on *Gaussian assumptions* or not. Examples of i) Gaussian filters are the UF's, EKF, SOEKF, GHF, CDF, DDF, CKF; and of ii) non-Gaussian filters are SMCF's, MCMCF's, and the point-mass filter.

Among all non-linear filters, the EKF is the most widely-known and implemented in practical applications [1, 24, 26]. It is obtained as the *first order* truncation of the Taylor series of the system's non-linear functions f_k and h_k while retaining the same prediction-correction structure as the (linear) KF. Although several filters in the literature have been proposed in order to improve upon computational aspects related to the EKF, it was just recently that UF's have become noticeable as a competitive and preferable alternative [1, 67].

The good properties related to the first UF's have become well-known since its introduction (see more details in Section 2.3). However, later, some UF's have been reported to be inconsistent (see Sections 2.4 to 2.7) and, until our work [23], it was difficult to assess whether these inconsistencies are present in all UF's. Seeking to provide clarifications, we first review all main UF's in the next section.

2.2 UNSCENTED FILTERING

In this section, we provide a *broad* view over the main concepts of the Unscented Kalman Filtering theory as it is in the *literature*. Later, as we develop *our* theory, we provide *more details* of these concepts.

All UF's, as in the EKF, keep the (linear) KF's structure composed of one *prediction step* (or *a priori* estimation) and one *correction step* (or *a posteriori* estimation, or *update* step). This can be seen, for instance, in the Unscented Kalman Filter (UKF) of [82]: consider (2.1) and suppose that, at time step k , $\hat{x}_{k-1|k-1}$ and $\hat{P}_{xx}^{k-1|k-1}$ are given; then this UKF is given by the following algorithm. For a matrix A , $(A)(\diamond)^T$ and $[A][\diamond]^T$ stand for $[A][A]^T$; and $(A)_{*j}$ and $(A)_{i*}$ stand, respectively, for the j th column and i th row of A .

Algorithm 1 (UKF of [82]). *Perform the following steps:*

1. *Prediction.*

(a) *Choose a real $\kappa > -n_x$ and define, for $1 \leq i \leq n_x$, the weights and points*

$$w_0 := \frac{\kappa}{n_x + \kappa},$$

$$\begin{aligned}
w_i = w_{i+n_x} &:= \frac{1}{2(n_x + \kappa)} \\
\chi_0^{k-1|k-1} &:= \hat{x}_{k-1|k-1}, \\
\chi_i^{k-1|k-1} &:= \hat{x}_{k-1|k-1} + \left(\sqrt{(n_x + \kappa) \hat{P}_{xx}^{k-1|k-1}} \right)_{*i}, \\
\chi_{i+n_x}^{k-1|k-1} &:= \hat{x}_{k-1|k-1} - \left(\sqrt{(n_x + \kappa) \hat{P}_{xx}^{k-1|k-1}} \right)_{*i}.
\end{aligned} \tag{2.3}$$

(b) For $0 \leq i \leq 2n_x$, define the transformed sigma points

$$\begin{aligned}
\chi_i^{k|k-1} &:= f_k \left(\chi_i^{k-1|k-1} \right), \\
\gamma_i^{k|k-1} &:= h_k \left(\chi_i^{k|k-1} \right);
\end{aligned} \tag{2.4}$$

and their associated statistics

$$\begin{aligned}
\hat{x}_{k|k-1} &:= \sum_{i=0}^{2n_x} w_i \chi_i^{k|k-1}, \\
\hat{y}_{k|k-1} &:= \sum_{i=0}^{2n_x} w_i \gamma_i^{k|k-1}, \\
\hat{P}_{xx}^{k|k-1} &:= \sum_{i=0}^{2n_x} w_i \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1} \right) \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1} \right)^T + Q_k, \\
\hat{P}_{xy}^{k|k-1} &:= \sum_{i=0}^{2n_x} w_i \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1} \right) \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right)^T;
\end{aligned} \tag{2.5}$$

along with the innovation's covariance

$$\hat{P}_{yy}^{k|k-1} := \sum_{i=0}^{2n_x} w_i \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right) \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right)^T + R_k. \tag{2.6}$$

2. Correction.

(a) Instantiate the KF's correction equations

$$G_k := \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \tag{2.7}$$

$$\hat{x}_{k|k} := \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \tag{2.8}$$

$$\hat{P}_{xx}^{k|k} := \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \tag{2.9}$$

All UF's are based on *prediction-correction structures* like the one in Algorithm 1, but they can vary in their form.

Other two important concepts upon which UF's are built upon are the ones of *sigma sets* and *Unscented Transformations (UT's)*.

Roughly, an UT approximates the joint pdf of 2 random vectors by 2 sets of weighted points; these points are called *sigma points*; and these sets, *sigma sets*. For two random vectors $X \sim (\bar{X}, P_{XX})^n$ and $Y \sim (\bar{Y}, P_{YY})^{n_y}$ with cross-covariance given by— $X \sim (m, M_2, \dots, M_k)^n$ stands for a random variable $X \in \Phi^n$ with mean m and i th central moment $M_X^i = M_i, i = 2, \dots, k$; $P_{XX} := M_2$ is the covariance of X —

$$P_{XY} := \mathcal{E}_{(X,Y)} \left\{ (X - \bar{X}) (Y - \bar{Y})^T \right\};$$

suppose that X and Y are related by a given function F by

$$Y = F(X); \tag{2.10}$$

and consider i) the *previous* sigma set [the notation $\{\xi_i\}_{i=b}^c$ stands for the set $\{\xi_b, \xi_{b+1}, \dots, \xi_c\}$]

$$\chi = \{\chi_i, w_i^m, w_i^c : \chi_i \in \mathbb{R}^n; w_i^m, w_i^c \in \mathbb{R}\}_{i=1}^N \tag{2.11}$$

where χ_i 's are sigma points, and w_i^m 's as well as w_i^c 's are weights; and ii) the *posterior* (or *transformed*) sigma set

$$\gamma = \{\gamma_i, w_i^m, w_i^c : \gamma_i = F(\chi_i)\}_{i=1}^N. \tag{2.12}$$

Define the *sample means* of these sets by

$$\mu_\chi := \sum_{i=1}^N w_i^m \chi_i, \tag{2.13}$$

$$\mu_\gamma := \sum_{i=1}^N w_i^m \gamma_i; \tag{2.14}$$

their *sample covariances* by

$$\Sigma_{\chi\chi} := \sum_{i=1}^N w_i^c (\chi_i - \mu_\chi) (\diamond)^T, \tag{2.15}$$

$$\Sigma_{\gamma\gamma} := \sum_{i=1}^N w_i^c (\gamma_i - \mu_\gamma) (\diamond)^T; \tag{2.16}$$

and their *sample cross-covariance* by

$$\Sigma_{\chi\gamma} := \sum_{i=1}^N w_i^c (\chi_i - \mu_\chi) (\gamma_i - \mu_\gamma)^T. \tag{2.17}$$

Then, an UT approximates the joint pdf of (X, Y) in the following way:

1. the sample mean μ_γ is an approximation of the mean \bar{Y} ,

2. the sample covariance $\Sigma_{\gamma\gamma}$ is an approximation of the covariance P_{YY} , and
3. the sample cross-covariance $\Sigma_{\chi\gamma}$ is an approximation of the cross-covariance P_{XY} .

If the points and weights in the sigma set χ are such that $\mu_\chi = \bar{X}$ and $\Sigma_{\chi\chi} = P_{XX}$, then μ_γ and $\Sigma_{\gamma\gamma}$ are expected to be, respectively, equal to \bar{Y} and P_{YY} up to their *second order* Taylor approximations [40]—in the literature, these requirements are said to be properties of UT’s, but, in Section 2.4, we present some counter-examples to these claims; the approximation’s quality of P_{XY} by $\Sigma_{\chi\gamma}$ is discussed in Section 2.4.2; further, in the development of our theory of Unscented Filtering, we provide precise results regarding these approximations (see Section 4, for example)—. In consequence, these approximations should be better than the one provide by a linearization [40].

In Algorithm 1, for instance, two UT’s are performed, namely: one UT for

$$F(X) = f_k(X) + \varpi_k$$

with i) X being the previous state

$$x_{k-1}|y_{1:k-1} \sim \left(\hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right),$$

ii) χ being the set

$$\left\{ \chi_i^{k-1|k-1}, w_i, w_i \right\},$$

and iii) γ being the set

$$\left\{ \chi_i^{k|k-1}, w_i, w_i \right\};$$

and another UT for

$$H(X) := h_k(X) + \vartheta_k$$

with i) X being the predicted state

$$x_k|y_{1:k-1} \sim \left(\hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1} \right),$$

ii) χ being the set

$$\left\{ \chi_i^{k|k-1}, w_i, w_i \right\},$$

and iii) γ being the set

$$\left\{ \gamma_i^{k|k-1}, w_i, w_i \right\}.$$

Summing up, we have 3 fundamental concepts in an UF, namely: the *prediction-correction structure*, *UT*, and *sigma set*. By varying the forms of each of these elements, we have different UF’s. Let us first consider different sigma sets.

2.2.1 Unscented Filter variants considering different sigma sets

All sigma sets in the literature are presented in Table 2.1, and examples are given afterwards; these sigma sets are considered associated with a random vector $X \sim (\bar{X}, P_{XX})^n$.

Note that the sigma sets of [2] (Tab 2.1 [1,1]) and [1] (Tab 2.1 [1,2]) are equivalent—Tab X [p:q,n:m] refers to the rows p to q and the columns n to m of Table X; Tab X [* ,n:m] refers to the columns n:m of Table X, and Tab X [n:m,*] to the rows n:m—. Indeed by choosing $\kappa = w_0 n / (1 - w_0)$ in the sigma set of [2], we have the sigma set of [1]; conversely, by choosing $w_0 = \kappa / (\kappa + n)$ in the sigma set of [1] (cf. Tab 2.1 [1,2]), we have the the sigma set of [2] (cf. Tab 2.1 [1,1]). Hence, we can say that UKF's of [2] and [1] are *equivalent*; the difference is only in their choice of the tuning parameter w_0 or κ .

In the Fifth order set of [47] (Tab 2.1 [4,2]) we use the function *gen* defined as follows: for a vector $[u_1, \dots, u_r]$ with $u_1, \dots, u_r \in \mathbb{R}$, we define the function

$$\text{gen}([u_1, \dots, u_r, [0]_{1 \times (n-r)}]) := \{\chi_i\}$$

where $\{\chi_i; \chi_i \in \mathbb{R}^n, n \geq r\}$ is the set composed of all permutations of the scalar elements

$$[u_1, \dots, u_r, [0]_{1 \times (n-r)}]^T.$$

Example 2.1 (Sigma Sets). Consider a random vector $X \sim ([0]_{2 \times 1}, I_2)^2$. We have that, from Tab 2.1 [1,1], for the Symmetric set of [2] with $\kappa = 1$, the weights are given by

$$w_0 := \frac{\kappa}{n + \kappa} = \frac{1}{2 + 1} = \frac{1}{3},$$

$$w_1 = w_2 = w_3 = w_4 = \frac{1}{2(n + \kappa)} = \frac{1}{2(2 + 1)} = \frac{1}{6};$$

and the sigma points by

$$\chi_0 = [0]_{2 \times 1},$$

$$\chi_1 = \bar{X} + \left(\sqrt{(n + \kappa) P_{XX}} \right)_{*1} = \left(\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \right)_{*1} = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix},$$

$$\chi_2 = \bar{X} + \left(\sqrt{(n + \kappa) P_{XX}} \right)_{*2} = \left(\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \right)_{*2} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix},$$

$$\chi_3 = \bar{X} - \left(\sqrt{(n + \kappa) P_{XX}} \right)_{*1} = - \left(\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \right)_{*1} = \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix};$$

Table 2.1: Literature's sigma sets.

<p>Symmetric set of [2] ($N = 2n + 1$) Choose $\kappa > -n$. 1 Set $w_0 = \frac{\kappa}{n+\kappa}$ and, for $i = 1, \dots, n$: $w_i = w_{i+n} = \frac{1}{2(n+\kappa)}$, $\chi_0 = \bar{X}$, $\chi_i = \bar{X} + \left(\sqrt{(n+\kappa)P_{XX}}\right)_{*i}$, $\chi_{i+n} = \bar{X} - \left(\sqrt{(n+\kappa)P_{XX}}\right)_{*i}$.</p>	<p>Symmetric set of [1] ($N = 2n + 1$) Choose $w_0 < 1$. Set, for $i = 1, \dots, n$: $w_i = w_{i+n} = \frac{1-w_0}{2n}$, $\chi_0 = \bar{X}$, $\chi_i = \bar{X} + \left(\sqrt{\frac{n}{1-w_0}P_{XX}}\right)_{*i}$, $\chi_{i+n} = \bar{X} - \left(\sqrt{\frac{n}{1-w_0}P_{XX}}\right)_{*i}$.</p>
<p>Reduced set of [45] ($N = n + 1$) Choose $0 \leq w_0 \leq 1$. Set $w_2 = w_1 = \frac{1-w_0}{2^n}$, and: $w_i = 2^{i-1}w_1$, for $i = 3, \dots, n + 1$; 2 $\chi_0^1 = 0$, $\chi_1^1 = -1/\sqrt{2w_1}$, $\chi_2^1 = -\chi_1^1$; for $j = 1, \dots, n - 1$; : and $i = 1, \dots, j$; $\chi_0^{j+1} = [\chi_0^j, 0]^T$, $\chi_i^{j+1} = \left[\chi_i^j, \frac{-1}{\sqrt{2w_j}}\right]^T$, $\chi_{j+1}^{j+1} = \left[[0]_{1 \times j}, \frac{1}{\sqrt{2w_j}}\right]^T$, $\chi_i := \sqrt{P_{XX}}\chi_i^n + \bar{X}$.</p>	<p>Spherical simplex set of [46] ($N = n + 2$) Choose $0 \leq w_0 \leq 1$. Set: $w_i = \frac{1-w_0}{n}$, $\forall i = 1, \dots, n + 1$; $\chi_0^1 = 0$, $\chi_1^1 = -1/\sqrt{2w_1}$, $\chi_2^1 = -\chi_1^1$; for $j = 2, \dots, n$; and $i = 1, \dots, j$: $\chi_0^j = [\chi_0^{j-1}, 0]^T$, $\chi_i^j = \left[\chi_i^{j-1}, \frac{-1}{\sqrt{j(j+1)w_1}}\right]^T$, $\chi_{j+1}^j = \left[[0]_{1 \times (j-1)}, \frac{1}{\sqrt{j(j+1)w_1}}\right]^T$, $\chi_i := \sqrt{P_{XX}}\chi_i^n + \bar{X}$.</p>
<p>Simplex set of [83] ($N = n + 1$) $w_i = 1/(n + 1)$, $i = 1, \dots, n + 1$, 3 $\xi = [\xi_{1*}, \dots, \xi_{n*}]^T$ where $\xi_{j*} = \sqrt{n+1} \times \left[[\sqrt{j(j+1)}]_{1 \times j}, -\sqrt{(j+1)/j}, [0]_{1 \times (n-j)}\right]^T$, $[\chi_1, \dots, \chi_{n+1}] := \sqrt{P_{XX}}\gamma + [\bar{X}]_{1 \times (n+1)}$.</p>	<p>Minimum set of [57] ($N = n + 1$) Choose $0 < w_p < 1$. Set $w_{(n+1)} = w_p$ and $\rho := \sqrt{\frac{1-w_p}{n}}$, $C := \sqrt{I_n - \rho^2 [1]_{n \times n}}$, $w_i = \left(C^{-1}w_p\rho^2[1]_{n \times n}(C^T)^{-1}\right)_{i,i}$, $W = \text{diag}(w_1, \dots, w_n)$, $\chi_i := \left(\sqrt{P_{XX}}C(\sqrt{W})^{-1}, -\frac{\rho}{\sqrt{w_p}}\sqrt{P_{XX}}[1]_{n \times 1}\right)_{*i} + \bar{X}$, $i = 1, \dots, n$.</p>
<p>Symmetric set of [41] ($N = 2n + 1$) Choose $\alpha \in (0, 1]$ and $\kappa \in \mathbb{R}$, such that $\lambda = \alpha^2(n + \kappa) - n > -n$. Set $\chi_0 = \bar{X}$, $w_0^m = \frac{\lambda}{n+\lambda}$, 4 $w_0^c = \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta)$; for $1 \leq i \leq n$: $w_i = w_{i+n}^m = w_i = w_{i+n}^c = \frac{\lambda}{n+\lambda}$, $\chi_i = \bar{X} + \left(\sqrt{(n+\lambda)P_{XX}}\right)_{*i}$, $\chi_{i+n} = \bar{X} - \left(\sqrt{(n+\lambda)P_{XX}}\right)_{*i}$.</p>	<p>Fifth order set of [47] ($N = 2n^2 + 1$) Set $w_i = \frac{1}{36}$, for $2n + 1 \leq i \leq 2n^2$; $w_i = \frac{4-n}{18}$, for $1 \leq i \leq 2n$; $w_{2n^2+1} = \frac{n^2-7n}{18} + 1$ Set $\{\xi_i\}_{i=1}^{2n} = \text{gen}([\pm\sqrt{3}])$, $\{\xi_i\}_{i=2n+1}^{2n^2} = \text{gen}([\pm\sqrt{3}, \pm\sqrt{3}])$, and $\xi_{2n^2+1} = [0]_{n \times 1}$. For $i = 1, \dots, 2n^2 + 1$, set $\chi_i = \bar{X} + \sqrt{P_{XX}}\xi_i$.</p>
<p>Set of [84] ($N = \kappa^n$)</p>	
<p>5 Choose $\kappa \in \mathbb{N}$ and, for $j_i = 1, \dots, \kappa$ and $i = 1, \dots, n$, set: $\Xi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$; $b_i = \Xi^{-1}\left(\frac{i+1}{2\kappa}\right)$; $c_{j_i} = (\sum_{l=1}^{\kappa} b_l^2)^{1/2} \kappa^{-1/2} b_i$; $w_{j_1, \dots, j_n} = \frac{1}{\kappa^n}$; $\chi_{j_1, \dots, j_n} = \bar{X} + \sum_{i=1}^n c_{j_i} \sqrt{\lambda_i} v_i$, for where λ_i is an eigenvalue and v_i an eigenvector of P_{XX}.</p>	

$$\chi_4 = \bar{X} - \left(\sqrt{(n+\kappa)P_{XX}}\right)_{*2} = - \left(\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}\right)_{*2} = \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}.$$

From Tab 2.1 [1,2], for the Symmetric set of [1] with $w_0 = 1/3$, the weights are

given by

$$w_1 = w_2 = w_3 = w_4 = \frac{1 - w_0}{2n} = \frac{1 - (1/3)}{2 \times 2} = \frac{1}{6};$$

and the sigma points by

$$\chi_0 = [0]_{2 \times 1},$$

$$\begin{aligned}\chi_1 &= \bar{X} + \left(\sqrt{\frac{n}{1 - w_0} P_{XX}} \right)_{*1} = \left(\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \right)_{*1} = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \\ \chi_2 &= \bar{X} + \left(\sqrt{\frac{n}{1 - w_0} P_{XX}} \right)_{*2} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}, \\ \chi_3 &= \bar{X} - \left(\sqrt{\frac{n}{1 - w_0} P_{XX}} \right)_{*2} = \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix} \\ \chi_4 &= \bar{X} - \left(\sqrt{\frac{n}{1 - w_0} P_{XX}} \right)_{*1} = \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}.\end{aligned}$$

From Tab 2.1 [2,1], for the Reduced set of [45] with $w_0 = 1/3$, the weights are given by

$$w_1 = w_2 = \frac{1 - w_0}{2n} = \frac{1 - (1/3)}{2^2} = \frac{2}{3 \times 4} = \frac{1}{6}.$$

Define

$$\begin{aligned}\chi_0^1 &:= 0, \\ \chi_1^1 &:= \frac{-1}{\sqrt{2w_1}} = \frac{-1}{\sqrt{2 \frac{1}{6}}} = -\sqrt{3}, \\ \chi_2^1 &:= -\chi_1^1 = \sqrt{3};\end{aligned}$$

for $j = 1$, define

$$\begin{aligned}\chi_0^{j+1} = \chi_0^2 &:= \begin{bmatrix} \chi_0^j \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \chi_1^{j+1} = \chi_1^2 &:= \begin{bmatrix} \chi_1^j \\ \frac{-1}{\sqrt{2w_j}} \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ \frac{-1}{\sqrt{2w_1}} \end{bmatrix} \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix}, \\ \chi_2^{j+1} = \chi_2^2 &:= \begin{bmatrix} \chi_2^j \\ \frac{-1}{\sqrt{2w_j}} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ \frac{-1}{\sqrt{2w_1}} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}.\end{aligned}$$

The sigma points are given by

$$\begin{aligned}\chi_0 &= \sqrt{P_{XX}}\chi_0^2 + \bar{X} = I_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \chi_1 &= \sqrt{P_{XX}}\chi_1^2 + \bar{X} = I_2 \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix}, \\ \chi_2 &= \sqrt{P_{XX}}\chi_2^2 + \bar{X} = I_2 \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}.\end{aligned}$$

From Tab 2.1 [2,2], for the Spherical simplex set of [46] with $w_0 = 1/3$, the weights are given by

$$w_1 = w_2 = w_3 = \frac{1 - w_0}{n} = \frac{1 - (1/3)}{2} = \frac{1}{3}.$$

Define

$$\begin{aligned}\chi_0^1 &= 0, \\ \chi_1^1 &= \frac{-1}{\sqrt{2w_1}} = \frac{-1}{\sqrt{2\frac{1}{3}}} = -\sqrt{\frac{3}{2}}, \\ \chi_2^1 &= -\chi_1^1 = \sqrt{\frac{3}{2}};\end{aligned}$$

for $j = 2$, define

$$\begin{aligned}\chi_0^j &= \chi_0^2 := \begin{bmatrix} \chi_0^{j-1} \\ \frac{-1}{0} \end{bmatrix} = \begin{bmatrix} \chi_0^1 \\ \frac{-1}{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \chi_1^j &= \chi_1^2 := \begin{bmatrix} \chi_1^{j-1} \\ \frac{-1}{\sqrt{j(j+1)w_1}} \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix}, \\ \chi_2^j &= \chi_2^2 := \begin{bmatrix} \chi_2^{j-1} \\ \frac{-1}{\sqrt{j(j+1)w_1}} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix}, \\ \chi_3^j &= \chi_3^2 := \begin{bmatrix} [0]_{(j-1) \times 1} \\ \frac{1}{\sqrt{j(j+1)w_1}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{3}{2}} \end{bmatrix}.\end{aligned}$$

The sigma points are given by

$$\chi_0 = \sqrt{P_{XX}}\chi_0^2 + \bar{X} = I_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned}
\chi_1 &= \sqrt{P_{XX}}\chi_1^2 + \bar{X} = I_2 \begin{bmatrix} -\sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix}, \\
\chi_2 &= \sqrt{P_{XX}}\chi_2^2 + \bar{X} = I_2 \begin{bmatrix} \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} \end{bmatrix}; \\
\chi_3 &= \sqrt{P_{XX}}\chi_3^2 + \bar{X} = I_2 \begin{bmatrix} 0 \\ \sqrt{\frac{3}{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{3}{2}} \end{bmatrix}.
\end{aligned}$$

From Tab 2.1 [3,1], for the Simplex set of [83], the weights are given by

$$w_1 = w_2 = w_3 = \frac{1}{n+1} = \frac{1}{3}.$$

Define

$$\begin{aligned}
\xi_1 &= \sqrt{n+1} \begin{bmatrix} [\sqrt{1(1+1)}]_{1 \times 1} \\ -\sqrt{\frac{1+1}{1}} \\ 0 \end{bmatrix} := \sqrt{3} \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 0 \end{bmatrix}, \\
\xi_2 &= \sqrt{n+1} \begin{bmatrix} [\sqrt{2(2+1)}]_{1 \times 2} \\ -\sqrt{\frac{2+1}{2}} \end{bmatrix} := \sqrt{3} \begin{bmatrix} \sqrt{6} \\ \sqrt{6} \\ -\sqrt{\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{18} \\ \sqrt{18} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}, \\
\xi &= [\xi_1, \xi_2]^T = \begin{bmatrix} \sqrt{6} & -\sqrt{6} & 0 \\ \sqrt{18} & \sqrt{18} & -\sqrt{\frac{9}{2}} \end{bmatrix}.
\end{aligned}$$

The sigma points are given by

$$\begin{aligned}
\chi_0 &= \sqrt{P_{XX}}\chi_0^2 + \bar{X} = I_2 \begin{bmatrix} \sqrt{6} \\ \sqrt{18} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6} \\ \sqrt{18} \end{bmatrix}, \\
\chi_1 &= \sqrt{P_{XX}}\chi_1^2 + \bar{X} = I_2 \begin{bmatrix} -\sqrt{6} \\ \sqrt{18} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{6} \\ \sqrt{18} \end{bmatrix}, \\
\chi_2 &= \sqrt{P_{XX}}\chi_2^2 + \bar{X} = I_2 \begin{bmatrix} 0 \\ -\sqrt{\frac{9}{2}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{\frac{9}{2}} \end{bmatrix}.
\end{aligned}$$

From Tab 2.1 [3,2], for the Minimum set of [57] with $w_p = 1/3$, define

$$\begin{aligned}
\rho &:= \sqrt{\frac{1-w_p}{n}} = \sqrt{\frac{1-\frac{1}{3}}{2}} = \sqrt{\frac{1}{3}}, \\
C &:= \sqrt{I_n - \rho^2 [1]_{n \times n}} = \sqrt{\begin{bmatrix} 1-\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1-\frac{1}{3} \end{bmatrix}} = \sqrt{\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}}, \\
A &:= C^{-1}w_p\rho^2[1]_{n \times n}(C^T)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^{-1} \frac{1}{3} \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^{-1} \right)^T \\
&= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix};
\end{aligned}$$

hence the weights are

$$\begin{aligned}
w_1 &= (A)_{11} = \left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \right)_{11} = \frac{1}{3}; \\
w_2 &= (A)_{22} = \left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \right)_{22} = \frac{1}{3}; \\
w_3 &= w_p = \frac{1}{3}.
\end{aligned}$$

Define

$$\begin{aligned}
W &:= \text{diag}(w_1, \dots, w_n) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \\
e &:= -\frac{\rho}{\sqrt{w_p}} \sqrt{P_{XX}} [1]_{n \times 1} = -\frac{\sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}}} \sqrt{I_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \\
E &:= \sqrt{P_{XX}} C (\sqrt{W})^{-1} = \sqrt{I_2} \sqrt{\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}} \left(\sqrt{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}} \right)^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix}.
\end{aligned}$$

The the sigma points are given by

$$\begin{aligned}
\chi_1 &= (E)_{*1} + \bar{X} = \left(\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix} \right)_{*1} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \end{bmatrix}, \\
\chi_2 &= (E)_{*2} + \bar{X} = \left(\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix} \right)_{*2} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix}, \\
\chi_3 &= e + \bar{X} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.
\end{aligned}$$

From Tab 2.1 [4,1], for the Symmetric set of [41] with $\alpha = 1$, $\kappa = 1$, and $\beta = 1$, define

$$\lambda = \alpha^2 (n + \kappa) - n = 1^2(2 + 1) - 1 = 2.$$

The weights are given by

$$w_0^m = \frac{\lambda}{n + \lambda} = \frac{2}{2 + 2} = \frac{1}{2},$$

$$w_0^c = \frac{\lambda}{n + \lambda} + (1 - \alpha^2 + \beta) = \frac{2}{2 + 2} + (1 - 1^2 + 1) = \frac{3}{2}$$

$$w_i^m = w_i^c = \frac{\lambda}{n + \lambda} = \frac{2}{2 + 2} = \frac{1}{2}, \quad \text{for } i = 1, 2, 3, 4;$$

and the sigma points by

$$\begin{aligned} \chi_1 &= \bar{X} + \left(\sqrt{(n + \lambda) P_{XX}} \right)_{*1} = \left(\begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{4} \end{bmatrix} \right)_{*1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \\ \chi_2 &= \bar{X} + \left(\sqrt{(n + \lambda) P_{XX}} \right)_{*2} = \left(\begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{4} \end{bmatrix} \right)_{*2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \\ \chi_3 &= \bar{X} - \left(\sqrt{(n + \lambda) P_{XX}} \right)_{*1} = - \left(\begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{4} \end{bmatrix} \right)_{*1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}; \\ \chi_4 &= \bar{X} - \left(\sqrt{(n + \lambda) P_{XX}} \right)_{*2} = - \left(\begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{4} \end{bmatrix} \right)_{*2} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}. \end{aligned}$$

From Tab 2.1 [4,2], for the fifth order set of [47], the weights are given by

$$\begin{aligned} w_i &= \frac{4 - n}{18} = \frac{4 - 2}{18} = \frac{1}{9}, \quad \text{for } i = 1, \dots, 4; \\ w_i &= \frac{1}{36} \quad \text{for } i = 5, \dots, 8; \\ w_9 &= \frac{n^2 - 7n}{18} + 1 = \frac{2^2 - 7 \times 2}{18} + 1 = \frac{4}{9}. \end{aligned}$$

Define

$$\begin{aligned} \xi_1 &:= \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \quad \xi_2 := \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix}, \quad \xi_3 := \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}, \quad \xi_4 := \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}; \\ \xi_5 &:= \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \end{bmatrix}, \quad \xi_6 := \begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \quad \xi_7 := \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}, \quad \xi_8 := \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix}; \\ \xi_9 &:= \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \end{aligned}$$

hence the sigma points are given by

$$\begin{aligned} \chi_1 &= \bar{X} + \sqrt{P_{XX}} \xi_1 = [0]_{2 \times 1} + I_2 \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \\ \chi_2 &= \bar{X} + \sqrt{P_{XX}} \xi_2 = [0]_{2 \times 1} + I_2 \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix}, \\ \chi_3 &= \bar{X} + \sqrt{P_{XX}} \xi_3 = [0]_{2 \times 1} + I_2 \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
\chi_4 &= \bar{X} + \sqrt{P_{XX}}\xi_4 = [0]_{2 \times 1} + I_2 \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}, \\
\chi_5 &= \bar{X} + \sqrt{P_{XX}}\xi_5 = [0]_{2 \times 1} + I_2 \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \end{bmatrix}, \\
\chi_6 &= \bar{X} + \sqrt{P_{XX}}\xi_6 = [0]_{2 \times 1} + I_2 \begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \end{bmatrix}, \\
\chi_7 &= \bar{X} + \sqrt{P_{XX}}\xi_7 = [0]_{2 \times 1} + I_2 \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}, \\
\chi_8 &= \bar{X} + \sqrt{P_{XX}}\xi_8 = [0]_{2 \times 1} + I_2 \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \end{bmatrix}, \\
\chi_9 &= \bar{X} + \sqrt{P_{XX}}\xi_{19} = [0]_{2 \times 1} + I_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

From Tab 2.1 [5,*], for the set of [84] with $\kappa = 2$, define

$$w_{1,1} = w_{1,2} = w_{2,1} = w_{2,2} = \frac{1}{\varphi^n} = \frac{1}{2^2} = \frac{1}{4};$$

thence the weights are given by

$$\begin{aligned}
w_1 &= w_{1,1} = \frac{1}{4}, \\
w_2 &= w_{1,2} = \frac{1}{4}, \\
w_3 &= w_{2,1} = \frac{1}{4}, \\
w_4 &= w_{2,2} = \frac{1}{4}.
\end{aligned}$$

Define

$$F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du;$$

$$\begin{aligned}
b_1 &:= F^{-1}\left(\frac{i+1}{2\kappa}\right) = F^{-1}\left(\frac{1+1}{2 \times 2}\right) = F^{-1}\left(\frac{1}{2}\right) = 0, \\
b_2 &:= F^{-1}\left(\frac{i+1}{2\kappa}\right) = F^{-1}\left(\frac{2+1}{2 \times 2}\right) = F^{-1}\left(\frac{3}{4}\right) = 0.6745,
\end{aligned}$$

$$\sum_{l=1}^{\kappa} b_l^2 = b_1^2 + b_2^2 = 0.4549$$

$$c_1 := \sqrt{\frac{\kappa}{\sum_{l=1}^{\kappa} b_l^2}} b_i = \sqrt{\frac{2}{0.4549}} 0 = 0$$

$$c_2 := \sqrt{\frac{\kappa}{\sum_{l=1}^{\kappa} b_l^2}} b_i = \sqrt{\frac{2}{0.4549}} b_2 = 1.4142$$

The eigenvalues λ_1, λ_2 and eigenvalues v_1, v_2 of P_{XX} are

$$\lambda_1 = 1, \lambda_2 = 1;$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, the sigma points are given by

$$j_1 = 1, j_2 = 1: \chi_{1,1} = \bar{X} + c_1 \sqrt{\lambda_1} v_1 + c_1 \sqrt{\lambda_2} v_2 = 0\sqrt{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\sqrt{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$j_1 = 1, j_2 = 2: \chi_{1,2} = \bar{X} + c_1 \sqrt{\lambda_1} v_1 + c_2 \sqrt{\lambda_2} v_2 = \begin{bmatrix} 0 \\ 1.4142 \end{bmatrix}$$

$$j_1 = 2, j_2 = 1: \chi_{2,1} = \bar{X} + c_2 \sqrt{\lambda_1} v_1 + c_1 \sqrt{\lambda_2} v_2 = \begin{bmatrix} 1.4142 \\ 0 \end{bmatrix}$$

$$j_1 = 2, j_2 = 2: \chi_{2,2} = \bar{X} + c_2 \sqrt{\lambda_1} v_1 + c_2 \sqrt{\lambda_2} v_2 = \begin{bmatrix} 1.4142 \\ 1.4142 \end{bmatrix}$$

Remark 2.2. Sigma sets can be composed of i) *only* positive weights, or ii) *both* positive and negative weights (never *only* negative weights). However, using UF's composed of both positive and negative weights [option i)] should be avoided; this practice may result in some numerical problems such as non-positive sample covariances [1] or a large amount of round-off errors [85].

Regarding *sigma sets*, UF's can be classified according to 3 different criteria.

A *first* criterion considers the *geometrical distribution* of the sigma points within each sigma set in an UF. In the UF's of [1, 2, 41, 47], the sigma points of every sigma set are distributed *symmetrically*; and in the reduced set of [45], spherical simplex set of [46], simplex set of [83], and minimum set of [57], the sigma points of every sigma set are distributed *asymmetrically*.

A *second* criterion considers the *number* of sigma points in each sigma set. Every sigma set $\chi = \{\chi_i, w_i^m, w_i^c\}_{i=1}^N$ is associated with a random vector $X \in \Phi^n$, and the number of sigma points N depends on n , the dimension of the space in which X takes value. As can be seen in Table 2.1, the number of sigma points is i) $N = n + 1$ in the UKF's of [45, 57, 83], ii) $N = n + 2$ in the UKF of [46], iii) $N = 2n + 1$ in the UKF's of [1, 2, 41], iv) $N = 2n^2 + 1$ in the UKF of [47], and v) $N = \kappa^n$ ($\kappa \in \mathbb{N}$ is a parameter) in the UKF of [84].

A *third* criterion considers which *sample moments* of each sigma set χ are equal to *the moments* of their associated random vector. *Generally*, this matching occur with the moments of *order 1 (the mean) and 2 (the covariance)* in the UKF's of [1, 2, 41, 45, 47, 57, 83]. If the random vector is *symmetric*—a random vector $X \in \Phi^n$ is symmetric if $\text{pdf}_X(\bar{X} + x) = \text{pdf}_X(\bar{X} - x)$ for every $x \in \mathbb{R}^n$ —, then this matching occur with the moments of order 1, 2, and also *all odd-order* moments in the UKF's of [1, 2, 41, 47]. If the random vector is *Gaussian*, then this matching occur with the moments of order 1, 2, with all *odd-order* moments, and also with the *moment of order 4* in the UKF's of [47]. Not all UKF's have their sigma sets matching the first and second moments of their associated random vector (see Section 2.5).

Let us now consider different UF's regarding UT's.

2.2.2 Unscented Filter variants considering different Unscented Transformations

Table 2.2 presents all different UT's in the literature, namely i) the *(ordinary) UT* (first column of Table 2.2), ii) the *scaled UT of [44]* (second column of Table 2.2), and iii) the *Auxiliary form of the UT (AuxUT) of [44]* (third column of Table 2.2).

Comparative with the ordinary UT, the scaled UT of [44] (second column of Table 2.2), essentially, have two different steps: i) the *previous sigma set* χ is transformed by a *scaling transformation* with scaling parameter α (Tab 2.2 [2,2])—note that the transformation of the sigma points $\chi'_i = \chi_1 + \alpha(\chi_i - \chi_1)$ is a *convex* transformation—; and ii) the transformed sample covariance is $\Sigma_{\gamma\gamma}^*$ in (Tab 2.2 [7,2]).

Comparative with the ordinary UT, the AuxUT of [44] (third column of Table 2.2), essentially, has only one different step: the transformed sigma set γ (Tab 2.2 [3,3]) is transformed by the scaling function g (Tab 2.2 [2,3]); this function also has a scaling parameter α .

Since both the scaled UT of [44] and the AuxUT of [44] are composed of scaling transformations, we call the set of these two UT's by the name of *scaling UT's*. In Section 2.6, we show an inconsistency regarding the Scaled UT of [44]. We point out that [86] presented an embryonic form of these scaling UT's.

Let us now consider different UF's regarding different prediction-correction structures.

Table 2.2: Literature's Unscented Transformations.

	(Ordinary) UT	Scaled UT of [44]	Auxiliary form of the UT (AuxUT) of [44]
1	<i>Previous sigma set</i> $\{\chi_i, w_i^m, w_i^c\}_{i=1}^N$	<i>Previous sigma set</i> $\{\chi_i, w_i^m, w_i^c\}_{i=1}^N$	<i>Previous sigma set</i> $\{\chi_i, w_i^m, w_i^c\}_{i=1}^N$
2	—	<i>Scaled sigma set</i> Choose $\alpha \geq 0$; Set, for $2 \leq i \leq N$: $\chi'_i = \chi_1 + \alpha(\chi_i - \chi_1)$; $w_1^m = \alpha^{-2} w_i^m + 1 - \alpha^{-2}$, $w_i^m = \alpha^{-2} w_i^m$; $w_1^c = \alpha^{-2} w_i^c + 1 - \alpha^{-2}$, $w_i^c = \alpha^{-2} w_i^c$.	<i>Scaled sigma set</i> Choose $\alpha \geq 0$; and define $g(X, c, \alpha, \kappa) := \frac{1}{\kappa} F(c + \alpha(X - c)) - \frac{1}{\kappa} F(c) + F(c)$.
3	<i>Transformed set</i> $\{\gamma_i, w_i^m, w_i^c \gamma_i = F(\chi_i)\}_{i=1}^N$.	<i>Transformed set</i> $\{\gamma_i, w_i^m, w_i^c \gamma_i = F(\chi'_i)\}_{i=1}^N$.	<i>Transformed set</i> $\{\gamma_i, w_i^m, w_i^c \gamma_i = g(\bar{X}, \mu_X, \alpha, \alpha^2)\}_{i=1}^N$.
4	<i>Previous sample mean</i> $\mu_X := \sum_{i=1}^N w_i^m \chi_i$.	<i>Previous sample mean</i> $\mu_{X'} := \sum_{i=1}^N w_i^m \chi'_i$.	<i>Previous sample mean</i> $\mu_X := \sum_{i=1}^N w_i^m \chi_i$.
5	<i>Previous sample covariance</i> $\Sigma_{XX} := \sum_{i=1}^N w_i^c (\chi_i - \mu_X) (\chi_i - \mu_X)^T$.	<i>Previous sample covariance</i> $\Sigma_{XX} := \sum_{i=1}^N w_i^c (\chi'_i - \mu_{X'}) (\chi'_i - \mu_{X'})^T$.	<i>Previous sample covariance</i> $\Sigma_{XX} := \sum_{i=1}^N w_i^c (\chi_i - \mu_X) (\chi_i - \mu_X)^T$.
6	<i>Transformed sample mean</i> $\mu_\gamma := \sum_{i=1}^N w_i^m \gamma_i$.	<i>Transformed sample mean</i> $\mu_\gamma := \sum_{i=1}^N w_i^m \gamma_i$.	<i>Transformed sample mean</i> $\mu_\gamma := \sum_{i=1}^N w_i^m \gamma_i$.
7	<i>Transformed sample covariance</i> $\Sigma_{\gamma\gamma} := \sum_{i=1}^N w_i^c (\gamma_i - \mu_\gamma) (\gamma_i - \mu_\gamma)^T$.	<i>Transformed sample covariance</i> $\Sigma_{\gamma\gamma}^* := \sum_{i=0}^N w_i^c (\gamma_i - \mu_\gamma) (\gamma_i - \mu_\gamma)^T + (\gamma_i - \mu_\gamma) (\gamma_i - \mu_\gamma)^T$.	<i>Transformed sample covariance</i> $\Sigma_{\gamma\gamma}^* := \alpha^2 \sum_{i=0}^N w_i^c (\gamma_i - \mu_\gamma) (\gamma_i - \mu_\gamma)^T$.
8	<i>Sample cross-covariance</i> $\Sigma_{X\gamma} := \sum_{i=1}^N w_i^c (\chi_i - \mu_X) (\gamma_i - \mu_\gamma)^T$.	<i>Sample cross-covariance</i> $\Sigma_{X\gamma} := \sum_{i=1}^N w_i^c (\chi'_i - \mu_{X'}) (\gamma_i - \mu_\gamma)^T$.	<i>Sample cross-covariance</i> $\Sigma_{X\gamma} := \sum_{i=1}^N w_i^c (\chi_i - \mu_X) (\gamma_i - \mu_\gamma)^T$.

2.2.3 Unscented Filter variants considering different prediction-correction structures

UF's can be classified relative to their prediction-correction structures according to 2 criteria.

A *first* criterion is related to the form of the *underlying dynamical system*, i.e., whether the system is described in the *additive form* (2.1), or in the more *general form* (2.2). UF's designed for systems in the additive form (2.1) are called *Additive UF's*, and UF's designed for systems in the more general form (2.2) are called *Augmented UF's*.

The *a priori* random vector at each time step k is different in each of these filters. In Additive UF's, the *a priori* random vector is the previous state $x_{k-1}|y_{1:k-1}$ with mean $\hat{x}_{k-1|k-1}$ and covariance $\hat{P}_{xx}^{k-1|k-1}$ (as in the Algorithm 1). On the other hand, in the Augmented UF's, the *a priori* random vector is the previous *augmented* vector $x_{k-1}^a|y_{1:k-1}$ (cf. [67]) defined by (recall that ϖ_k and ϑ_k are the noises of the systems (2.1) and (2.1))

$$x_{k-1}^a|y_{1:k-1} := (x_{k-1}, \varpi_k, \vartheta_k) | y_{1:k-1};$$

the dimension of $x_{k-1}^a|y_{1:k-1}$ is $n_a = n_x + n_\varpi + n_\vartheta$; its mean is

$$\hat{x}_{k-1}^a|y_{1:k-1} := \left[\hat{x}_{k-1|k-1}^T, [0]_{1 \times n_\varpi}, [0]_{1 \times n_\vartheta} \right]^T; \quad (2.18)$$

and its covariance and square-root covariance are, respectively,

$$\hat{P}_{xx}^{a,k-1|k-1} := \text{diag} \left(\hat{P}_{xx}^{k-1|k-1}, Q_k, R_k \right), \quad (2.19)$$

$$\sqrt{\hat{P}_{xx}^{a,k-1|k-1}} := \text{diag} \left(\sqrt{\hat{P}_{xx}^{k-1|k-1}}, \sqrt{Q_k}, \sqrt{R_k} \right). \quad (2.20)$$

Although it is always possible to use Augmented UF's for either (2.1) or (2.2), *Additive UF's are preferable for* (2.1), because n Additive UF's are computationally cheaper than their corresponding Augmented UF's (Augmented UF's with the same sigma sets and UT's).

Remark 2.3. Filters for some system descriptions besides (2.1) and (2.2) can be easily obtained. For partially-*additive* systems, where (2.2) is considered either with f_k with additive ϖ_k or h_k with additive ϑ_k , the augmented state vector $\hat{x}_{k-1}^a|y_{1:k-1}$ is composed of only by the noise of whatever function is in general form [68]. For partially-*nonlinear* systems, where f_k or h_k is linear, the linear KF equations usually can be used in the parts of the UF referring to the linear equation [87].

A *second* criterion for classifying UF's regarding their prediction-correction struc-

tures is the *propagation form of covariances*; covariances within the UF's can be propagated in their *covariance forms, themselves*, (e.g. $\hat{P}_{xx}^{k|k-1}$, $\hat{P}_{xy}^{k|k-1}$, and $\hat{P}_{xx}^{k|k}$ in Algorithm 1), or in their *square-root forms* (e.g. $\sqrt{\hat{P}_{xx}^{k|k-1}}$, $\sqrt{\hat{P}_{xy}^{k|k-1}}$, and $\sqrt{\hat{P}_{xx}^{k|k}}$ in the filter of [42]). UF's whose covariances are propagated in their i) covariance forms are called *UKF's*, and in their ii) square-root forms are called *Square-Root Unscented Kalman Filter's (SRUKF's)*.

SRUKF's are usually preferred over UKF's in *computationally ill-conditioned situations*; for example, applications where the machine precision of the used computer is such that rounding errors can cause UKF's to diverge. In such ill-conditioned situations, usually SRUKF's are less likely to diverge than UKF's—indeed, generally, for *any* KF-based filter, a square-root form is less likely to diverge than a covariance form [88].

In SRUKF's, algorithms of *QR* decomposition and Cholesky factor update are used in order to propagate square-root covariances (cf. [42]). To date, we are aware of five variants for the Square-Root Unscented Kalman Filters (SRUKF's): SRUKF of [42] (system in additive form (2.1) with the sigma set of [41], Tab 2.1 [4,1], and statistics calculation (2.12)-(2.17)); SRUKF of [67] (general form (2.2) with the set of [41]); SRUKF of [89] (additive form (2.1) with the spherical simplex set of [46], Tab 2.1 [2,2]); SRUKF of [21] (general form (2.2) with the set of [2], Tab 2.1 [1,1]); the Improved SRUKF of [90] (additive form (2.1) with the reduced set of [45], Tab 2.1 [2,1]).

All additive UKF's (AdUKF's) in the literature are represented in Table 2.3—this table refers to Tables 2.1 and 2.2 for the expressions of each element in the presented AdUKF's. The (Additive) UKF of [82] (Algorithm 1), for example, can be obtained by taking the first row of Table 2.3; the previous set of this filter is the symmetric set of [2] calculated for $\bar{X} = \hat{x}_{k-1|k-1}$ and $P_{XX} = \hat{P}_{xx}^{k-1|k-1}$. Augmented UKF's and SRUKF's can be obtained with Tables 2.1 and 2.2 with corresponding, slightly modified versions of Table 2.3.

Remark 2.4. Due to the difficulty of describing UKF's as presented in the original formulations in a simple and systematized way, the forms of the UKF's shown in Table 2.3 are not necessarily the ones introduced by their corresponding authors. Nevertheless, the forms contained in this table, if different from the original ones, are trivial extensions (e.g., the additive form for the symmetric UKF of [1] in Table 2.1 is slightly more general). Moreover, some of these extensions have already been explicitly proposed (e.g., the additive form of the symmetric UKF of [2] was modified in [82]).

Table 2.3: Literature’s most known Additive Unscented Kalman Filters.

	Additive Unscented Kalman Filters	Previous Set	Transformed Sets	Sample Statistics	corrected estimates
1	AdUKF of [82] ($N = 2n + 1$)	Tab 2.1 [1,1]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
2	AdUKF of [1] ($N = 2n + 1$)	Tab 2.1 [1,2]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
3	Reduced AdUKF of [45] ($N = n + 1$)	Tab 2.1 [2,1]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
4	Spherical Simplex AdUKF of [46] ($N = n + 2$)	Tab 2.1 [2,2]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
5	AdUKF of [41] ($N = 2n + 1$)	Tab 2.1 [4,1]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
6	AdUKF of [84] ($N = m^n, m \in \mathbb{N}$)	Tab 2.1 [5,*]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
7	Simplex AdUKF of [83] ($N = n + 1$)	Tab 2.1 [3,1]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
8	Minimum AdUKF of [57] ($N = n + 1$)	Tab 2.1 [3,2]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)
9	Scaled AdUKF [44] (N depends on the set)	Tab 2.2 [1-2,2]	Tab 2.2 [3,2]	Tab 2.2 [4-8,2]	(2.7),(2.8),(2.9)
10	AdUKF with Auxiliary form of the UT [44] (N depends on the set)	Tab 2.2 [1-2,3]	Tab 2.2 [3,3]	Tab 2.2 [4-8,3]	(2.7),(2.8),(2.9)
11	Fifth order AdUKF of [47] ($N = 2n^2 + 1$)	Tab 2.1 [4,2]	Tab 2.2 [3,1]	Tab 2.2 [4-8,1]	(2.7),(2.8),(2.9)

Remark 2.5. There are 3 other UKF's that are not presented in Table 2.3: [37] describes a symmetric UKF matching up to the 4th central moment of the previous random vector; [86], an asymmetric UKF matching up to the 3rd central moment of the previous random vector; and [91], a symmetric UKF matching up to the 8th central moment of a scalar Gaussian random vector. Table 2.3 does not show these UKF's because, instead of presenting their expressions, these works only show procedures from which these UKF's can be obtained.

Next, we present general comments about UKF variants and analyze some of their properties.

2.3 DEFINITIONS FOR UKF'S

In this section, we point out some problems concerning *definitions* of some UKF's of the literature.

2.3.1 Variations on UKF definitions

From Section 2.2, it is clear that there are many UKF variants. Given that, in general, these variants are not equivalent, we cannot properly point out which one is *the definition* for the UKF. Nonetheless, most works in the literature use the term *UKF* when referring to either the UKF of [2]—as can be seen in [59,80]—or to the UKF of [41]—as can be seen in [52,60]. By comparing their sets of sigma points (cf. Tab 2.1 [1,1] with Tab 2.1 [4,1]), we can see that there are two main differences between these filters. First, the UKF of [2] uses the factor κ to calculate the weights and the sigma points, while the one of [41] uses a term $\lambda = \alpha^2(n + \kappa) - n$ to do so. Second, in the UKF of [41], w_0^m and w_0^c are distinct objects, while in the UKF of [2], $w_0 = w_0^m = w_0^c$.

2.3.2 Variation on scaled UKF definitions

Although the UKF of [41] (Tab 2.3 [5,*]) is described and widely referred to as a non-scaled UKF (cf. [52,60]), Merwe himself, one of the authors in [41], describes this filter as a scaled UKF form (cf. [67])—it has a scaling parameter α (cf. Tab 2.1 [4,1]). Apart from that, one should notice that this scaled UKF form differs from the ones proposed by [44] (the ones using the UT's of Table 2.2).

2.4 ACCURACY OF THE UT'S

In this section, we point out some problems concerning the *accuracy* of the UT.

2.4.1 Transformed covariance

Consider equations (2.10) to (2.10). As [60] states, a large number of papers repeat the statement of [1] that if $\mu_\chi = \bar{X}$ and $\Sigma_{\chi\chi} = P_{XX}$, then μ_γ and $\Sigma_{\gamma\gamma}$ are equal to \bar{Y} and P_{YY} up to their *second order* Taylor approximations. However, that is not true for all UT's. Indeed, [60] has already pointed out this issue for the UT in the symmetric UKF of [41] by providing a counter-example: for

$$X \sim N(0_{n \times 1}, I_n) \text{ and } Y := F(X) = X^T X,$$

the analytical result for the covariance of Y is

$$P_{YY} = 2n.$$

but the UT of [41] provides different results (see Table II in [60])—note that, since $F(X) = X^T X$ is a *second order* polynomial the UT should provide the same result as the analytical one if the second order approximation claim above was true.

2.4.2 Transformed cross covariance

The transformed cross-covariance is necessary for the UKF, but before our work [23]—a result of this thesis—an estimation quality for it was *not* provided.

2.5 SMALL SIGMA SETS

In this section, we point out some problems related to the *sigma sets of the literature composed of less than $2n$ sigma points*—for n being the dimension of the associated random vector (cf. Table 2.1).

The reduced set of [45] has two drawbacks. First, it can be numerically unstable for great values of n due to the fact that the weights are composed by fractions of 2^n [46]. Second, neither the sample mean, μ_χ , nor the sample covariance, $\Sigma_{\chi\chi}$, are equal to the mean and covariance of the prior distribution when n is greater than one [57]. In fact, from Tab 2.1 [2,1], for $n = 2$, $X \sim ([0]_{2 \times 1}, I_2)^2$, $w_0 = 0.5$, and using (2.13) and (2.15)

with $w_i = w_i = w_i$, we have that

$$\mu_\chi := \sum_{i=0}^n w_i \chi_i = \frac{1}{4} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \neq [0]_{2 \times 1} = \bar{X},$$

and

$$\Sigma_{\chi\chi} := \sum_{i=0}^n w_i (\chi_i - \mu_\chi) (\diamond)^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \neq I_2 = P_{XX}.$$

The spherical simplex set of [46] does not present the instability problem of the set of [45], but still has the same problem that neither μ_χ nor $\Sigma_{\chi\chi}$ is equal to the mean and the covariance of X , respectively, when n is greater than one [57]. In fact, from Tab 2.1 [2,2], for

$$n = 2, X \sim ([0]_{2 \times 1}, I_2)^2, w_0 = 0.5,$$

and using (2.13) and (2.15) with $w_i = w_i = w_i$, we have that

$$\mu_\chi := \sum_{i=0}^{n+1} w_i \chi_i = \frac{1}{2\sqrt{6}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq [0]_{2 \times 1} = \bar{X},$$

and

$$\Sigma_{\chi\chi} := \sum_{i=0}^{n+1} w_i (\chi_i - \mu_\chi) (\diamond)^T = \begin{bmatrix} 1 & 0 \\ 0 & \frac{53}{96} \end{bmatrix} \neq I_2 = P_{XX}.$$

For the minimum set of [83], the sample covariance does not match with the covariance of the considered random vector. Using Tab 2.1 [3,1] and (2.13), it can be shown that

$$\Sigma_{\chi\chi} := \sum_{i=1}^{n+1} w_i (\chi_i - \mu_\chi) (\diamond)^T = P_{XX} + \frac{1}{n+1} \bar{X} \bar{X}^T$$

for $X \sim (\bar{X}, P_{XX})^n$, which is not equal to P_{XX} if $\bar{X} \neq [0]_{(n+1) \times 1}$. In fact, for $X \in \Phi^1$ with mean $\bar{X} = 3$ and covariance $P_{XX} = 4$, the sample covariance of the set of [83] is

$$\Sigma_{\chi\chi} := \sum_{i=1}^{n+1} w_i (\chi_i - \mu_\chi) (\diamond)^T = 13 \neq 4 = P_{XX}.$$

Finally, our minimum set in [57] is the only sigma set composed by less than $2n$ points matching the mean and covariance of X .

2.6 SCALING TRANSFORMATIONS

In this section, we point out some problems related to the *scaling transformations* of the literature.

2.6.1 Scalable sigma sets

The scaled UT was proposed by [44]. In this work, it is stated that the “*scaled unscented transformation* [...] allows **any** set of sigma points to be scaled by an arbitrary scaling factor” (the italic is of [44], and the bold was added by us). However, suppose that a random vector $X \in \Phi^2$ has mean $\bar{X} = [0]_{2 \times 1}$ and covariance $P_{XX} = I_2$, and that the previous set $\chi = \{\chi_i, w_i\}_{i=1}^4$ is composed by the sigma points

$$\chi_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \chi_2 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \chi_3 = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}, \chi_4 = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix},$$

and the weights

$$w_1 = w_2 = w_3 = w_4 = \frac{1}{4}.$$

For $\alpha = 0.5$ and choosing

$$\chi'_1 = \chi_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

one can see that, from Tab 2.2 [*], the sample mean ($\mu_{\chi'} := \sum_{i=1}^4 w'_i \chi'_i$) and the sample covariance ($\Sigma_{\chi'\chi'} := \sum_{i=1}^4 w'_i (\chi'_i - \mu_{\chi'}) (\chi'_i - \mu_{\chi'})^T$) of the scaled sigma set $\chi' = \{\chi'_i, w'_i\}_{i=1}^4$ are

$$\mu_{\chi'} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\Sigma_{\chi'\chi'} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \neq I_2.$$

This example shows that the sample mean and the sample covariance of χ' are not equal to the mean and covariance of X , respectively. In fact, as one can see from the following theorem, this property is not guaranteed to hold for any sigma set, except for those having one sigma point equal to the mean of X .

Theorem 2.1. Consider $X \sim (\bar{X}, P_{XX})^n$ and a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n_y}$ defining a new random vector $Y := F(X)$ and consider a set of sigma points $\chi = \{\chi_i, w_i\}_{i=1}^N$ for X . Consider also the set of scaled sigma points $\chi' = \{\chi'_i, w'_i\}_{i=1}^N$ obtained from the scaled UT of [44] (second column of Table 2.2). We have that:

1. if $\sum_{i=1}^N w_i = 1$, then $\sum_{i=1}^N w'_i = 1$;
2. $\mu_{\chi'} = \alpha^{-1}\mu_\chi + \chi_1(1 - \alpha^{-1})$;
3. if $\chi_1 = \bar{X}$, then $\mu_{\chi'} = \mu_\chi$;
4. if $\alpha \neq 1$, then $\chi_1 = \bar{X} \Leftrightarrow \mu_{\chi'} = \mu_\chi$;
5. if $\chi_1 = \bar{X}$, then $\Sigma_{\chi'\chi'} = \Sigma_{\chi\chi}$.

Proof. Suppose $\sum_{i=1}^N w_i = 1$, then

$$\sum_{i=1}^N w'_i = \frac{1}{\alpha^2} \left(-1 + \alpha^2 + \sum_{i=1}^N w_i \right) = 1.$$

For the second and third assertion, note that, from the definition,

$$\mu_{\chi'} := \sum_{i=1}^N w'_i \chi'_i = \frac{1}{\alpha} \mu_\chi + \chi_1 \left(1 - \frac{1}{\alpha} \right),$$

which, supposing $\chi_1 = \mu_\chi$, gives $\mu_{\chi'} = \mu_\chi$ and, supposing $\mu_{\chi'} = \mu_\chi$, $\alpha \neq 1$, gives $\chi_1 = \mu_\chi$. The last assertions can be proven by the fact that, from the definition,

$$\begin{aligned} \Sigma_{\chi'\chi'} &:= \sum_{i=1}^N w'_i (\chi'_i - \mu_{\chi'}) (\chi'_i - \mu_{\chi'})^T \\ &= w'_1 (\chi'_1 - \mu_\chi) (\diamond)^T + \sum_{i=2}^N w'_i (\chi_1 + \alpha(\chi_i - \chi_1) - \mu_\chi) (\diamond)^T, \end{aligned}$$

which, for $\chi_1 = \mu_\chi$, gives $\Sigma_{\chi'\chi'} = \Sigma_{\chi\chi}$. □

Therefore, the scaled UT of [44] is restrictive in the sense that this UT does not provide the mentioned results for any previous set of sigma points. For instance, the SUT of cannot be used with the sigma set of [57] (Tab 2.1 [3,2]) because

$$w_{n+1} \neq 1 \Rightarrow \rho \neq 0$$

and

$$C \neq 0$$

imply that $\chi_i \neq \bar{X}$, $\forall i = 1, \dots, n+1$.

2.6.2 Covariance

Consider $X \sim N([0]_{3 \times 1}, I_3)$ and

$$Y := F(X) = X^T X.$$

Then, $\bar{Y} = 3$ and $P_{YY} = 6$. Using the scaled UT of [44] with the symmetric sigma set of [1], we get, from (2.14) and (2.16),

$$\mu_\gamma = 3 = \bar{Y},$$

and

$$\Sigma_{\gamma\gamma}^* = 3\alpha^2 - 8 \neq P_{YY}.$$

This result shows two problems involving the matching of the covariance. First, the transformed covariance for this scaled UT is not matched up to the order 2, but only to the order 1. Second, the scaling factor modifies the covariance even for second order polynomial approximation.

2.6.3 Cross-covariance

Similar to the case for the non-scaled UT's, the estimation quality of cross-covariances for the scaled UT of [44] and for the AuxUT of [44] has not been presented in the literature yet. Moreover, there is no mention of the influence of the scaling factor on the transformed cross-covariance for the UKF of [41] (recall from Section 2.3.2 that this UKF has to be investigated whether it is a scaled UKF or not). Since it is desirable to match the first and the second moments, the free parameter α should modify only the third and higher terms. However, consider $X \sim N([0]_{3 \times 1}, I_3)$ and

$$Y := F(X) = X^T X.$$

Then, from Tab 2.1 [4,1], (2.14) and (2.17), we have

$$\mu_\gamma := \sum_{i=1}^N w_i \gamma_i = 3 = \bar{Y},$$

and

$$\Sigma_{\chi\gamma} := \sum_{i=1}^N w_i (\chi_i - \mu_\chi) (\gamma_i - \mu_\gamma)^T = 6\alpha [I_3]_{*i} - \frac{9}{2\alpha} [I_3]_{*i}.$$

Therefore, the second order term of $\Sigma_{\chi\gamma}$ is also modified .

2.7 SQUARE-ROOT FORMS OF THE UKF'S

In this section, we point out some problems related to *SRUKF's* of the literature.

2.7.1 DOWNDATING THE CHOLESKY FACTOR

For an equation in the form

$$AA^T = RR^T - SS^T,$$

where A, R, S are Cholesky factors, we say that A is a *downdated Cholesky factor* of R by S . There are three parts within the SRUKF algorithms in the literature where Cholesky factors are downdated: in the calculations of the square-root matrices of the predicted state's covariance, of the innovation's covariance, and of the corrected state's covariance. In the first two steps, the downdating steps are performed only for the sigma points with negative weights, while, in the last, they are always performed.

Since the direct downdating of a Cholesky factor is “inherently more *ill-conditioned* than if Q (the Q matrix of a QR decomposition) is also available” [92] (the comment within parentheses and the emphasis is ours), filters resulting from the substitution of downdating steps by QR decompositions—or, more generally, by any triangulation technique [80]—should be computationally more stable. In fact, [93] has developed such a technique for calculating the square-root matrix of the corrected state's covariance for quadrature Kalman filters and [80] for the CKF.

2.7.2 Square-Root Scaled UKF

The literature does not present any filter conjugating the SRUKF with the scaled UT of [44] (second column of Table 2.2) nor with the AuxUT (third column of Table 2.2).

2.7.3 Square-Root UT

Although there are definitions for filters in square-root forms using the UT, we have not been able to find any definition for a Square-Root Unscented Transformation (SRUT). Explicitly defining an SRUT can be justified by three reasons, at least: 1) it gives SRUKF's better mathematical formal principle; 2) it is possible to study a SRUKF's by focusing on its respective SRUT, since it is the core difference between SRUKF's relative to other nonlinear SR KF-based filters; and 3) an SRUT can be applied not only within the KF framework, but in any framework or application that

requires uncertainty propagation (e.g. [94]) or within other stochastic filter (e.g. [95]). In Section 4.3, we provide a definition for the SRUT.

2.8 ADDITIVE UNSCENTED KALMAN FILTERS

When UKF's are solutions to the filtering problem of systems in the form (2.1), we call them AdUKF's. There is a great number of AdUKF's in the literature, such as [1, 2, 39, 40, 42, 44–47, 83, 84, 86, 96, 97].

From Section 2.2, we saw that AdUKF's can vary from each other by different criteria; in this section, we analyze every AdUKF of the literature distinct from each other according to the following three criteria (Section 2.8.1):

1. *in which equation* the process noise's covariance Q_k is considered,
2. whether the predicted state sigma set $\{\chi_{i,\{j\}}^{k|k-1}, w_{i,\{j\}}\}$ is *regenerated or not*, and
3. *how* this regeneration is done if it is the case.

Four different classes are found.

From this analysis, we show that *only one* of these classes of AdUKF's, namely the AdUKF 1, provides *the same estimates as the (linear) KF when the system (2.1) is linear (Section 2.8.3)*. By the facts that i) the UKF's are extensions of the (linear) Kalman Filter (KF) to nonlinear system and ii) that the KF provides the minimum variance estimate of the state of a linear system with Gaussian noise and initial state [24, 26], it is expected [from i)] and desirable [from ii)] that the estimates of the AdUKF's are equal to the ones of the KF when linear systems are considered.

Numerical simulations *indicates* that this *linear* property of the AdUKF 1—of providing the same estimates as the KF when the system is *linear*—is related with a superior performance of this AdUKF 1—comparative with the other classes of AdUKF's—when *nonlinear* systems are considered. In Section 2.8.2, we compare the performance of all classes of AdUKF's in a numerical example, and the AdUKF 1 *outperformed* all the other classes of AdUKF's. Later, in Chapter 5.1, endowed with the results developed in Chapters 3 and 4, we will be able to develop stronger conclusions.

2.8.1 Additive Unscented Kalman Filters of the literature

Each class is represented by a particular AdUKF; in this way, we can analyze their algorithms. We chose the AdUKF's of [2, 42, 59, 67].

However, in order to not lose generality, the algorithms described below are in a more generalized form than the ones presented in [2, 42, 59, 67]; each AdUKF is defined with particular sigma sets in their original work, but here we consider general sigma sets. For this, we define the function

$$SS(\bar{X}, P_{XX}) := \{\chi_i, w_i\}_{i=1}^N$$

mapping the mean \bar{X} and the covariance P_{XX} of a random vector X to a sigma set.

For easy reference, we named i) AdUKF 1 the class gathering the AdUKF of [59], ii) AdUKF 2 the class gathering the AdUKF of [42], iii) AdUKF 3 the class gathering the AdUKF of [67], and iv) AdUKF 4 the class gathering the AdUKF of [2]. We point out that, broadly, as long as our knowledge go, AdUKF's 1, 2, and 4 are used in an approximately-equal number of works.

Below, some variables are written with a subscript $\{j\}$ as in $A_{\{j\}}$, for $j = 1, 2, 3$ and 4; this notation associates the element A to the AdUKF j . For example, $\chi_{i,\{1\}}^{k-1|k-1}$ is sigma point of the AdUKF 1, $\chi_{i,\{2\}}^{k-1|k-1}$ of the AdUKF 2, $\chi_{i,\{3\}}^{k-1|k-1}$ of the AdUKF 3, and $\chi_{i,\{4\}}^{k-1|k-1}$ of the AdUKF 4.

Algorithm 2 (*AdUKF 1 (in [59])*). Perform the following steps:

1. $\hat{x}_{k-1|k-1,\{1\}}$, $\hat{P}_{xx,\{1\}}^{k-1|k-1}$, Q_k , R_k and a measurement y_k are given.

2. *State's prediction.*

(a) *Predicted statistics.*

$$\begin{aligned} \{\chi_{i,\{1\}}^{k-1|k-1}, w_{i,\{1\}}\}_{i=1}^{N_{\{1\}}} &= SS(\hat{x}_{k-1|k-1,\{1\}}, \hat{P}_{xx,\{1\}}^{k-1|k-1}); \\ \chi_{i,*,\{1\}}^{k|k-1} &= f_k(\chi_{i,\{1\}}^{k-1|k-1}), \quad 1 \leq i \leq N_{\{1\}}; \\ \hat{x}_{k|k-1,\{1\}} &= \sum_{i=1}^{N_{\{1\}}} w_{i,\{1\}} \chi_{i,*,\{1\}}^{k|k-1}; \\ \hat{P}_{xx,\{1\}}^{k|k-1} &= \sum_{i=1}^{N_{\{1\}}} w_{i,\{1\}} (\chi_{i,*,\{1\}}^{k|k-1} - \hat{x}_{k|k-1,\{1\}}) (\diamond)^T + Q_k. \end{aligned} \quad (2.21)$$

(b) *Regeneration of predicted state sigma points.*

$$\{\chi_{i,\{1\}}^{k|k-1}, w_{i,\{1\}}\}_{i=1}^{N_{\{1\}}} = SS(\hat{x}_{k|k-1,\{1\}}, \hat{P}_{xx,\{1\}}^{k|k-1}). \quad (2.22)$$

3. *Measurements prediction.*

$$\gamma_{i,\{1\}}^{k|k-1} = h_k(\chi_{i,\{1\}}^{k-1|k}), \quad 1 \leq i \leq N_{\{1\}};$$

$$\begin{aligned}
\hat{y}_{k|k-1,\{1\}} &= \sum_{i=1}^{N_{\{1\}}} w_{i,\{1\}} \chi_{i,\{1\}}^{k|k-1}; \\
\hat{P}_{yy,\{1\}}^{k|k-1} &= \sum_{i=1}^{N_{\{1\}}} w_{i,\{1\}} \left(\chi_{i,\{1\}}^{k|k-1} - \hat{y}_{k|k-1,\{1\}} \right) (\diamond)^T + R_k; \\
\hat{P}_{xy,\{1\}}^{k|k-1} &= \sum_{i=1}^{N_{\{1\}}} w_{i,\{1\}} \left(\chi_{i,\{1\}}^{k|k-1} - \hat{x}_{k|k-1,\{1\}} \right) \left(\chi_{i,\{1\}}^{k|k-1} - \hat{y}_{k|k-1,\{1\}} \right)^T.
\end{aligned}$$

4. *State's correction.*

$$\begin{aligned}
G_{k,\{1\}} &= \hat{P}_{xy,\{1\}}^{k|k-1} \left(\hat{P}_{yy,\{1\}}^{k|k-1} \right)^{-1}; \\
\hat{x}_{k|k,\{1\}} &= \hat{x}_{k|k-1,\{1\}} + G_{k,\{1\}} \left(y_k - \hat{y}_{k|k-1,\{1\}} \right); \\
\hat{P}_{xx,\{1\}}^{k|k} &= \hat{P}_{xx,\{1\}}^{k|k-1} - G_{k,\{1\}} \hat{P}_{yy,\{1\}}^{k|k-1} G_{k,\{1\}}^T.
\end{aligned}$$

Algorithm 3 (*AdUKF 2* (in [42])). Perform the following steps:

1. $\hat{x}_{k-1|k-1,\{2\}}$, $\hat{P}_{xx,\{2\}}^{k-1|k-1}$, Q_k , R_k and a measurement y_k are given.

2. *Prediction of the state.*

(a) *Predicted statistics.*

$$\begin{aligned}
\left\{ \chi_{i,\{2\}}^{k-1|k-1}, w_{i,\{2\}} \right\}_{i=1}^{N_{\{2\}}} &= SS \left(\hat{x}_{k-1|k-1,\{2\}}, \hat{P}_{xx}^{k-1|k-1} \right); \\
\chi_{i,\{2\}}^{k|k-1} &= f_k \left(\chi_{i,\{2\}}^{k-1|k-1} \right), \quad 1 \leq i \leq N_{\{2\}}; \\
\hat{x}_{k|k-1,\{2\}} &= \sum_{i=1}^{N_{\{2\}}} w_{i,\{2\}} \chi_{i,\{2\}}^{k|k-1}; \\
\hat{P}_{xx,\{2\}}^{k|k-1} &= \sum_{i=1}^{N_{\{2\}}} w_{i,\{2\}} \left(\chi_{i,\{2\}}^{k|k-1} - \hat{x}_{k|k-1,\{2\}} \right) (\diamond)^T + Q_k. \quad (2.23)
\end{aligned}$$

3. *Measurements prediction.*

$$\begin{aligned}
\gamma_{i,\{2\}}^{k|k-1} &= h_k \left(\chi_{i,\{2\}}^{k-1|k} \right), \quad 1 \leq i \leq N_{\{2\}}; \\
\hat{y}_{k|k-1,\{2\}} &= \sum_{i=1}^{N_{\{2\}}} w_{i,\{2\}} \chi_{i,\{2\}}^{k|k-1}; \\
\hat{P}_{yy,\{2\}}^{k|k-1} &= \sum_{i=1}^{N_{\{2\}}} w_{i,\{2\}} \left(\chi_{i,\{2\}}^{k|k-1} - \hat{y}_{k|k-1,\{2\}} \right) (\diamond)^T + R_k; \\
\hat{P}_{xy,\{2\}}^{k|k-1} &= \sum_{i=1}^{N_{\{2\}}} w_{i,\{2\}} \left(\chi_{i,\{2\}}^{k|k-1} - \hat{x}_{k|k-1,\{2\}} \right) \left(\chi_{i,\{2\}}^{k|k-1} - \hat{y}_{k|k-1,\{2\}} \right)^T.
\end{aligned}$$

4. State's correction.

$$\begin{aligned}
G_{k,\{2\}} &= \hat{P}_{xy,\{2\}}^{k|k-1} \left(\hat{P}_{yy,\{2\}}^{k|k-1} \right)^{-1}; \\
\hat{x}_{k|k,\{2\}} &= \hat{x}_{k|k-1,\{2\}} + G_{k,\{2\}} \left(y_k - \hat{y}_{k|k-1,\{2\}} \right); \\
\hat{P}_{xx,\{2\}}^{k|k} &= \hat{P}_{xx,\{2\}}^{k|k-1} - G_{k,\{2\}} \hat{P}_{yy,\{2\}}^{k|k-1} G_{k,\{2\}}^T.
\end{aligned}$$

Algorithm 4 (*AdUKF 3* (in [67])). Perform the following steps:

1. $\hat{x}_{k-1|k-1,\{3\}}$, $\hat{P}_{xx,\{3\}}^{k-1|k-1}$, Q_k , R_k and a measurement y_k are given.

2. Prediction of the state.

(a) Predicted statistics.

$$\begin{aligned}
\left\{ \chi_{i,\{3\}}^{k-1|k-1}, w_{i,\{3\}}^* \right\}_{i=1}^{N_{\{3\}}^*} &= SS \left(\hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right); \\
\chi_{i,*,\{3\}}^{k|k-1} &= f_k \left(\chi_{i,\{3\}}^{k-1|k-1} \right), \quad 1 \leq i \leq N_{\{3\}}^*; \\
\hat{x}_{k|k-1,\{3\}} &= \sum_{i=1}^{N_{\{3\}}^*} w_{i,\{3\}}^* \chi_{i,*,\{3\}}^{k|k-1}; \\
\hat{P}_{xx,\{3\}}^{k|k-1} &= \sum_{i=1}^{N_{\{3\}}^*} w_{i,\{3\}}^* \left(\chi_{i,*,\{3\}}^{k|k-1} - \hat{x}_{k|k-1,\{3\}} \right) (\diamond)^T + Q_k. \quad (2.24)
\end{aligned}$$

(b) Regeneration of predicted state sigma points.

$$\chi_{i,\{3\}}^{k|k-1} = \chi_{i,*,\{3\}}^{k|k-1}, \quad 1 \leq i \leq N_{\{3\}}^*; \quad (2.25)$$

$$\left\{ \chi_{i,\{3\}}^{k|k-1}, w_{i,\{3\}} \right\}_{l=N_{\{3\}}^*+1}^{N_3} = SS \left(\hat{x}_{k|k-1}, Q_k \right); \quad (2.26)$$

$$w_{j,\{3\}} = \frac{w_{j,\{3\}}^*}{2}, \quad 1 \leq j \leq N_3. \quad (2.27)$$

3. Measurements prediction.

$$\begin{aligned}
\gamma_{i,\{3\}}^{k|k-1} &= h_k \left(\chi_{i,\{3\}}^{k-1|k} \right), \quad 1 \leq i \leq N_{\{3\}}; \\
\hat{y}_{k|k-1,\{3\}} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \chi_{i,\{3\}}^{k|k-1}; \\
\hat{P}_{yy,\{3\}}^{k|k-1} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \left(\chi_{i,\{3\}}^{k|k-1} - \hat{y}_{k|k-1,\{3\}} \right) (\diamond)^T + R_k; \\
\hat{P}_{xy,\{3\}}^{k|k-1} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \left(\chi_{i,\{3\}}^{k|k-1} - \hat{x}_{k|k-1,\{3\}} \right) \left(\chi_{i,\{3\}}^{k|k-1} - \hat{y}_{k|k-1,\{3\}} \right)^T.
\end{aligned}$$

4. State's correction.

$$\begin{aligned} G_{k,\{3\}} &= \hat{P}_{xy,\{3\}}^{k|k-1} \left(\hat{P}_{yy,\{3\}}^{k|k-1} \right)^{-1}; \\ \hat{x}_{k|k,\{3\}} &= \hat{x}_{k|k-1,\{3\}} + G_{k,\{3\}} \left(y_k - \hat{y}_{k|k-1,\{3\}} \right); \\ \hat{P}_{xx,\{3\}}^{k|k} &= \hat{P}_{xx,\{3\}}^{k|k-1} - G_{k,\{3\}} \hat{P}_{yy,\{3\}}^{k|k-1} G_{k,\{3\}}^T. \end{aligned}$$

Algorithm 5 (*AdUKF 4* (in [2])). Perform the following steps:

1. $\hat{x}_{k-1|k-1,\{4\}}$, $\hat{P}_{xx,\{4\}}^{k-1|k-1}$, Q_k , R_k and a measurement y_k are given.

2. Prediction of the state.

(a) Predicted statistics.

$$\begin{aligned} \left\{ \chi_{i,\{4\}}^{k-1|k-1}, w_{i,\{4\}} \right\}_{i=1}^{N_{\{4\}}} &= SS \left(\hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} + Q_k \right); \\ \chi_{i,\{4\}}^{k|k-1} &= f_k \left(\chi_{i,\{4\}}^{k-1|k-1} \right), \quad 1 \leq i \leq N_{\{4\}}; \\ \hat{x}_{k|k-1,\{4\}} &= \sum_{i=1}^{N_{\{4\}}} w_{i,\{4\}} \chi_{i,\{4\}}^{k|k-1}; \\ \hat{P}_{xx,\{4\}}^{k|k-1} &= \sum_{i=1}^{N_{\{4\}}} w_{i,\{4\}} \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1,\{4\}} \right) (\diamond)^T. \end{aligned} \quad (2.28)$$

3. Measurements prediction.

$$\begin{aligned} \gamma_{i,\{3\}}^{k|k-1} &= h_k \left(\chi_{i,\{3\}}^{k-1|k} \right), \quad 1 \leq i \leq N_{\{3\}}; \\ \hat{y}_{k|k-1,\{3\}} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \chi_{i,\{3\}}^{k|k-1}; \\ \hat{P}_{yy,\{3\}}^{k|k-1} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \left(\chi_{i,\{3\}}^{k|k-1} - \hat{y}_{k|k-1,\{3\}} \right) (\diamond)^T + R_k; \\ \hat{P}_{xy,\{3\}}^{k|k-1} &= \sum_{i=1}^{N_{\{3\}}} w_{i,\{3\}} \left(\chi_{i,\{3\}}^{k|k-1} - \hat{x}_{k|k-1,\{3\}} \right) \left(\chi_{i,\{3\}}^{k|k-1} - \hat{y}_{k|k-1,\{3\}} \right)^T. \end{aligned}$$

4. State's correction.

$$\begin{aligned} G_{k,\{3\}} &= \hat{P}_{xy,\{3\}}^{k|k-1} \left(\hat{P}_{yy,\{3\}}^{k|k-1} \right)^{-1}; \\ \hat{x}_{k|k,\{3\}} &= \hat{x}_{k|k-1,\{3\}} + G_{k,\{3\}} \left(y_k - \hat{y}_{k|k-1,\{3\}} \right); \\ \hat{P}_{xx,\{3\}}^{k|k} &= \hat{P}_{xx,\{3\}}^{k|k-1} - G_{k,\{3\}} \hat{P}_{yy,\{3\}}^{k|k-1} G_{k,\{3\}}^T. \end{aligned}$$

Note that there is no essential difference among i) the measurement's prediction

steps of each AdUKF (steps 3.), and ii) the state's correction steps of each AdUKF (steps 3.). Thus, the differences rely in the state's prediction steps (steps 2.).

The four classes of AdUKF's are divided according to the criteria 1, 2 and 3 described in the beginning of Section 2.8. Considering these criteria in each AdUKF we have that:

- in the AdUKF 1 (Algorithm 2), the covariance Q_k is considered in (2.21), and the predicted sigma set $\{\chi_{i,\{1\}}^{k|k-1}, w_{i,\{1\}}\}$ is regenerated in (2.22);
- in the AdUKF 2 (Algorithm 3), the covariance Q_k is considered in (2.23), and the predicted sigma set $\{\chi_{i,\{2\}}^{k|k-1}, w_{i,\{2\}}\}$ is *not* regenerated;
- in the AdUKF 3 (Algorithm 4), the covariance Q_k is considered in (2.24), and the predicted sigma set $\{\chi_{i,\{3\}}^{k|k-1}, w_{i,\{3\}}\}$ is regenerated in equations (2.25), (2.26), and (2.27);
- in the AdUKF 4 (Algorithm 5), the covariance Q_k is considered in (2.28), and the predicted sigma set $\{\chi_{i,\{4\}}^{k|k-1}, w_{i,\{4\}}\}$ is *not* regenerated.

The AdUKF 4 is the *only* filter to *not* consider Q_k in the equation of the predicted covariance $\hat{P}_{xx,\{j\}}^{k|k-1}$. Moreover, the AdUKF 1 and AdUKF 3 regenerate the predicted sigma set $\{\chi_{i,\{j\}}^{k|k-1}, w_{i,\{j\}}\}$ —AdUKF 2 and AdUKF 4 do not—, but in different ways.

Let us now investigate whether these differences result in differences in the final estimates $\hat{x}_{k|k,\{j\}}$ and $\hat{P}_{xx,\{j\}}^{k|k}$; and, if it is the case, which class of AdUKF provide the best estimates $\hat{x}_{k|k,\{j\}}$.

2.8.2 Numerical Example

In this section, we compare the AdUKF's of Section 2.8.1 in a numerical example. Suppose that, on time k , we have $y_k = 1000$, $Q_k = \text{diag}([100, 50]^T)$, $R_k = 100$, and

$$f(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}, \quad h(x) = x^T x,$$

$$\hat{x}_{k-1|k-1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \hat{P}_{xx,\{1\}}^{k-1|k-1} = \begin{bmatrix} 3 & 0 \\ 0 & 12 \end{bmatrix}.$$

For the sigma set of [2] with $\kappa = 2$, the posterior estimates provided by i) the AdUKF 1 are

$$\hat{x}_{k|k,\{1\}} = \begin{bmatrix} 120.0967 \\ 49.3841 \end{bmatrix},$$

$$\hat{P}_{xx,\{1\}}^{k|k} = 10^4 \begin{bmatrix} 2.99 & -2.24 \\ -2.24 & 17.41 \end{bmatrix};$$

by ii) the AdUKF 2 are

$$\hat{x}_{k|k,\{2\}} = \begin{bmatrix} 125.17 \\ -25.12 \end{bmatrix},$$

$$\hat{P}_{xx,\{2\}}^{k|k} = 10^4 \begin{bmatrix} 2.91 & -1.04 \\ -1.04 & 0.80 \end{bmatrix};$$

by iii) the AdUKF 3 are

$$\hat{x}_{k|k,\{3\}} = \begin{bmatrix} 88.9681 \\ -17.7611 \end{bmatrix},$$

$$\hat{P}_{xx,\{3\}}^{k|k} = 10^4 \begin{bmatrix} 3.00 & -4.34 \\ -4.34 & -8.40 \end{bmatrix}; \quad (2.29)$$

by iv) the AdUKF 4 are

$$\hat{x}_{k|k,\{4\}} = \begin{bmatrix} 202.56 \\ -94.53 \end{bmatrix},$$

$$\hat{P}_{xx,\{4\}}^{k|k} = 10^4 \begin{bmatrix} 12.08 & -6.89 \\ -6.89 & 4.44 \end{bmatrix};$$

and v) by a Monte Carlo simulation using 10^6 samples considering $x_{k-1|k-1}$ to be Gaussian are

$$\hat{x}_{k|k,\{MC\}} = \begin{bmatrix} 92.2690 \\ 54.5245 \end{bmatrix},$$

$$\hat{P}_{xx,\{MC\}}^{k|k} = 10^4 \begin{bmatrix} 2.10 & -0.39 \\ -0.39 & 9.39 \end{bmatrix}.$$

The relative deviation $e_{k|k,\{j\}}$ of each $\hat{x}_{k|k,\{j\}}$ of the AdUKF's from $\hat{x}_{k|k,MC}$ defined by

$$e_{k|k,\{j\}} := \frac{\|\hat{x}_{k|k,\{1\}} - \hat{x}_{k|k,\{MC\}}\|}{\|\hat{x}_{k|k,\{MC\}}\|}, \quad j = 1, 2, 3, 4; \quad (2.30)$$

is $e_{k|k,\{1\}} = 0.26$, $e_{k|k,\{2\}} = 0.80$, $e_{k|k,\{3\}} = 0.68$ and $e_{k|k,\{4\}} = 1.73$.

All four classes of AdUKF's present *distinct* estimates $\hat{x}_{k|k,\{j\}}$ and $\hat{P}_{xx,\{j\}}^{k|k}$ for this numerical example. Therefore, the three criteria distinguishing these classes (criteria 1, 2, and 3 in the beginning of Section 2.8) indeed influence the final estimates of the

AdUKF's.

The AdUKF 1 provides the smallest relative error ($e_{k|k,\{1\}} = 0.26$) in the sense of (2.30). The examples show that the error differences can be significant since the second best AdUKF, AdUKF 3, provides an error ($e_{k|k,\{3\}} = 0.68$) greater than 2.6 times the error of the AdUKF 1 ($e_{k|k,\{1\}} = 0.26$); and it also provides a non-positive definite covariance [cf. (2.29)].

An analytical example can give us further information about which of these filters is endowed with the best *mathematical* properties. Considering (2.1) with linear functions and Gaussian state random vectors is particularly interesting because we already know the best solution for the filter problem of such case, namely the (linear) KF.

2.8.3 Linear System

By the facts that i) the UKF's are extensions of the (linear) Kalman Filter (KF) to nonlinear system and ii) that the KF provides the minimum variance estimate of the state of a linear system with Gaussian noise and initial state [24, 26], it is expected [from i)] and desirable [from ii)] that the estimates of the AdUKF's are equal to the ones of the KF when linear systems are considered.

Suppose i) that, for $F_k \in \mathbb{R}^{n_x \times n_x}$ and $H_k \in \mathbb{R}^{n_y \times n_x}$, the system (2.1) can be written in the following form

$$x_k = F_k x_{k-1} + \varpi_k, \quad (2.31)$$

$$y_k = H_k x_k + \vartheta_k; \quad (2.32)$$

and that ii), at time step k , $\hat{x}_{k-1|k-1}$, $\hat{P}_{xx}^{k-1|k-1}$, Q_k , R_k and a measurement y_k are given. Then the KF's algorithm is given by [98]:

$$\hat{x}_{k|k-1,\{KF\}} = F_k \hat{x}_{k-1|k-1}, \quad (2.33)$$

$$P_{xx,\{KF\}}^{k|k-1} = F_k \hat{P}_{xx}^{k-1|k-1} F_k^T + Q_k, \quad (2.34)$$

$$\hat{y}_{k|k-1,\{KF\}} = H_k \hat{x}_{k|k-1,\{KF\}}, \quad (2.35)$$

$$P_{yy,\{KF\}}^{k|k-1} = H_k P_{xx,\{KF\}}^{k|k-1} H_k^T + R_k, \quad (2.36)$$

$$P_{xy,\{KF\}}^{k|k-1} = P_{xx,\{KF\}}^{k|k-1} H_k^T, \quad (2.37)$$

$$G_{k,\{KF\}} = P_{xy,\{KF\}}^{k|k-1} \left(P_{yy,\{KF\}}^{k|k-1} \right)^{-1}, \quad (2.38)$$

$$\hat{x}_{k|k,\{KF\}} = \hat{x}_{k|k-1,\{KF\}} + G_{k,\{KF\}} \left(y_k - \hat{y}_{k|k-1,\{KF\}} \right), \quad (2.39)$$

$$\hat{P}_{xx,\{KF\}}^{k|k} = P_{xx,\{KF\}}^{k|k-1} - G_{k,\{KF\}} P_{yy,\{KF\}}^{k|k-1} G_{k,\{KF\}}^T. \quad (2.40)$$

From a simple example, it can be seen that AdUKF's 2, 3, and 4 do not provide the same estimates as the KF for the posterior mean and covariance. Suppose that $Q_k = \hat{P}_{xx}^{k-1|k-1} = \hat{x}_{k-1|k-1} = 1$ and $y_k = F_k = R_k = H_k = 2$, then the KF's posterior estimates are

$$\hat{x}_{k|k,\{KF\}} = \frac{12}{11} \text{ and } \hat{P}_{xx,\{KF\}}^{k|k} = \frac{5}{11};$$

and, for the sigma of [2] with $\kappa = 0$, the posterior estimates for the AdUKF 1 are

$$\begin{aligned} \hat{x}_{k|k,\{1\}} &= \frac{12}{11} = \hat{x}_{k|k,\{KF\}} \text{ and} \\ \hat{P}_{xx,\{1\}}^{k|k} &= \frac{5}{11} = \hat{P}_{xx,\{KF\}}^{k|k}; \end{aligned}$$

for the AdUKF 2 are

$$\begin{aligned} \hat{x}_{k|k,\{2\}} &= \frac{1}{9} \neq \hat{x}_{k|k,\{KF\}} \text{ and} \\ \hat{P}_{xx,\{2\}}^{k|k} &= \frac{13}{9} \neq \hat{P}_{xx,\{KF\}}^{k|k}; \end{aligned}$$

for the AdUKF 3 are

$$\begin{aligned} \hat{x}_{k|k,\{3\}} &= \frac{7}{6} \neq \hat{x}_{k|k,\{KF\}} \text{ and} \\ \hat{P}_{xx,\{3\}}^{k|k} &= \frac{15}{4} \neq \hat{P}_{xx,\{KF\}}^{k|k}; \end{aligned}$$

for the AdUKF 4 are

$$\begin{aligned} \hat{x}_{k|k,\{4\}} &= \frac{1}{17} \neq \hat{x}_{k|k,\{KF\}} \text{ and} \\ \hat{P}_{xx,\{4\}}^{k|k} &= \frac{8}{17} \neq \hat{P}_{xx,\{KF\}}^{k|k}. \end{aligned}$$

Therefore, *the AdUKF's 2, 3 and 4 do not provide the same estimates as the KF for a linear system.*

For a general *linear* system, the estimates of the AdUKF 1 are given by:

$$\left\{ \chi_{i,\{1\}}^{k-1|k-1}, w_{i,\{1\}} \right\}_{i=1}^N = \text{SS} \left(\hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right); \quad (2.41)$$

$$\chi_{i,*,\{1\}}^{k|k-1} = F_k \chi_{i,\{1\}}^{k-1|k-1}, \quad 1 \leq i \leq N; \quad (2.42)$$

$$\begin{aligned} \hat{x}_{k|k-1,\{1\}} &= \sum_{i=1}^N w_{i,\{1\}} F_k \chi_{i,\{1\}}^{k-1|k-1} \\ &= F_k \hat{x}_{k-1|k-1} \end{aligned} \quad (2.43)$$

$$= \hat{x}_{k|k-1,\{KF\}}; \quad (2.44)$$

$$\begin{aligned}
\hat{P}_{xx,\{1\}}^{k|k-1} &= F_k \sum_{i=1}^N w_{i,\{1\}} \left(\chi_{i,\{1\}}^{k-1|k-1} - \hat{x}_{k-1|k-1} \right) (\diamond)^T + Q_k \\
&= F_k \hat{P}_{xx}^{k-1|k-1} F_k^T + Q_k \\
&= \hat{P}_{xx,\{KF\}}^{k|k-1};
\end{aligned} \tag{2.45}$$

$$\left\{ \chi_{i,\{1\}}^{k|k-1}, w_{i,\{1\}} \right\}_{i=1}^N = \text{SS} \left(\hat{x}_{k|k-1,\{1\}}, \hat{P}_{xx,\{1\}}^{k|k-1} \right); \tag{2.46}$$

$$\gamma_{i,\{1\}}^{k|k-1} = H_k \chi_{i,\{1\}}^{k|k-1}, \quad 1 \leq i \leq N; \tag{2.47}$$

$$\begin{aligned}
\hat{y}_{k|k-1,\{1\}} &= \sum_{i=1}^N w_i H_k \chi_{i,\{1\}}^{k|k-1} \\
&= H_k \hat{x}_{k|k-1,\{1\}} \\
&= \hat{y}_{k|k-1,\{KF\}};
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
\hat{P}_{yy,\{1\}}^{k|k-1} &= H_k \sum_{i=1}^N w_{i,\{1\}} \left(\chi_{i,\{1\}}^{k|k-1} - \hat{x}_{k|k-1,\{1\}} \right) (\diamond)^T + R_k \\
&= H_k \hat{P}_{xx,\{1\}}^{k|k-1} H_k^T + R_k \\
&= \hat{P}_{yy,\{KF\}}^{k|k-1};
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\hat{P}_{xy,\{1\}}^{k|k-1} &= \sum_{i=1}^N w_{i,\{1\}} \left(\chi_{i,\{1\}}^{k|k-1} - \hat{x}_{k|k-1,\{1\}} \right) \left(H_k \chi_{i,\{1\}}^{k|k-1} - H_k \hat{x}_{k|k-1,\{1\}} \right)^T \\
&= \hat{P}_{xx,\{1\}}^{k|k-1} H_k^T \\
&= \hat{P}_{xy,\{KF\}}^{k|k-1};
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
G_{k,\{1\}} &= \hat{P}_{xy,\{KF\}}^{k|k-1} \left(\hat{P}_{yy,\{KF\}}^{k|k-1} \right)^{-1} \\
&= G_{k,\{KF\}};
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
\hat{x}_{k|k,\{1\}} &= \hat{x}_{k|k-1,\{KF\}} + G_{k,\{KF\}} \left(y_k - \hat{y}_{k|k-1,\{KF\}} \right) \\
&= \hat{x}_{k|k,\{KF\}};
\end{aligned}$$

$$\begin{aligned}
\hat{P}_{xx,\{1\}}^{k|k} &= \hat{P}_{xx,\{KF\}}^{k|k-1} - G_{k,\{KF\}} \hat{P}_{yy,\{KF\}}^{k|k-1} G_{k,\{KF\}}^T \\
&= \hat{P}_{xx,\{KF\}}^{k|k}.
\end{aligned}$$

Hence, for a linear system, *the AdUKF 1 provides the same estimates for the mean and covariance of $x_k|y_{1:k}$ as the KF*. Note that this is also true for $\hat{x}_{k|k-1}$, $\hat{P}_{xx}^{k|k-1}$, $\hat{y}_{k|k-1}$, $\hat{P}_{yy}^{k|k-1}$ and $\hat{P}_{xy}^{k|k-1}$.

Summing up, there are, at least, two superior results of the AdUKF 1 comparative with the other AdUKF classes, namely: the AdUKF 1 a) is the only one to have this *linear* property—of providing the same estimates as the KF when the system is *linear*—, and b) was the best in the *nonlinear* numerical example of Section 2.8.2.

Together, these two superior results *indicate* that there might be a formal reason endowing the AdUKF 1 with *better mathematical properties* comparative with the other

AdUKF's for any *nonlinear* system (2.1). Later, in Chapter 5.1, by using results developed in Chapters 3 and 4, we will be able to develop stronger conclusions respective to this topic.

2.9 CONCLUSIONS REGARDING THE LITERATURE REVIEW ON EUCLIDEAN MANIFOLDS

In this chapter, we provided an extensive review of the Unscented Kalman filter theory in the literature. We were able to observe several problems concerning the following aspects of this theory:

1. multiple UKF definitions (Section 2.3.1);
2. the matching order of the transformed covariance (Sections 2.4.1 and 2.6.2) and the transformed cross-covariance (Sections 2.4.2 and 2.6.3) of both the Unscented Transformation (UT) and of the Scaled Unscented Transformation (SUT);
3. definitions of the reduced sigma sets of [45], [46] and [83] (Section 2.5);
4. the conservativeness of the SUT (Section 2.6.1);
5. the scaling effect of the SUT on both the transformed covariance and cross-covariance (Sections 2.6.2 and 2.6.3);
6. possibly ill-conditioned results in the square-root Unscented Kalman Filters (Section 2.7.1);
7. definitions for the Additive Unscented Kalman Filters (Section 2.8).

These problems, along with the difficulty in gathering all results related to the Unscented theory, reveal the existence of i) gaps in the fundamental mathematical concepts of this theory, and of ii) mathematical solutions generalizing the sigma sets, UT's and UKF's of the literature.

In order to fill these gaps and provide these mathematical solutions, we propose a systematization of this theory that treats the construction of UKF's by parts. We first consider the problem of estimating the mean of a non-linear transformation, which will lead us to the definition of a σ -representation.

3. SIGMA-REPRESENTATIONS

In this chapter, we propose the first results of our systematization of the Unscented Kalman filtering theory. We begin by considering diverse forms of estimating the expected value of a transformed random vector (Section 3.1). One interesting way of doing it is by creating an weighted set approximating the previous random vector; this provides us with the necessary intuition to define the σ -representations (Section 3.2). Broadly, σ -representations are weighted sets whose sample moments, up to a certain order, are equal to the ones of a given random vector.

We develop some results related to this new concept that facilitates finding closed forms for σ -representations. We present closed forms for the minimum symmetric σ -representation in Section 3.3, and one closed form for the minimum (non-symmetric) σ -representation in Section 3.4. We are able to show that i) one of these closed forms for the minimum symmetric σ -representations is equivalent to the classic sigma set of [2] (cf. Corollary 3.4), and ii) the closed form for the minimum σ -representation (cf. Theorem 3.2) is actually the only consistent of this class in the literature.

3.1 ESTIMATING A POSTERIOR EXPECTED VALUE

Given a random vector $X \in \Phi^n$ with probability density function $\text{pdf}_X(x)$, many problems, such as calculating the moments of a random variable, can be reduced to the problem of finding the posterior expectation

$$\mathcal{E}\{f(X)\} = \int_{\mathbb{R}^n} f(z)\text{pdf}_X(z)dz, \quad (3.1)$$

for an appropriate function $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n_y}$. As a first attempt to solve this problem, we could consider using numerical integration techniques. In the scalar case ($n = n_y = 1$) and if the function f is well approximated by a polynomial of order $2N - 1$ for a $N \in \mathbb{N}$, Gaussian quadrature methods give approximate solutions for (3.1) of the form (see [78, 99–102])

$$\mathcal{E}\{f(X)\} = \int_{-\infty}^{\infty} f(z)\text{pdf}_X(z)dz \approx \sum_{i=1}^N w_i f(x_i), \quad (3.2)$$

where $x_1, \dots, x_N \in \mathbb{R}^n$ are samples of X , and w_1, \dots, w_N their associated (scalar)

weights. For X being a standard scalar normal random variable, the solution is obtained by the Gauss-Hermite Quadrature (GHQ) [67, 78, 99–101, 103]. The multivariate case can be obtained by first using a stochastic decoupling technique

$$X' = \sqrt{P_{XX}}^{-1} (X - \bar{X}),$$

where X' is a multivariate standard Gaussian random variable. Then, for

$$\tilde{f}(X') = f\left(\sqrt{P_{XX}}^T X + \bar{X}\right),$$

the GHQ is applied on the form [78]

$$\begin{aligned} \mathcal{E}_{X'}\{\tilde{f}(X')\} &= \int_{\mathbb{R}^n} \tilde{f}(\xi) \text{pdf}_{X'}(\xi) d\xi \\ &\approx \left(w_1 \times w_1 \times \cdots \times w_1 \tilde{f}(x_1, \dots, x_1)\right) \\ &\quad + \left(w_2 \times w_1 \cdots \times w_1 \tilde{f}(x_2, x_1, \dots, x_1)\right) \\ &\quad + \cdots + \left(w_2 \times w_2 \times \cdots \times w_2 \tilde{f}(x_2, \dots, x_2)\right) \\ &\quad + \left(w_3 \times w_2 \cdots \times w_2 \tilde{f}(x_3, x_2, \dots, x_2)\right) \\ &\quad + \cdots + \left(w_{N-1} \times w_{N-1} \times \cdots \times w_{N-1} \tilde{f}(x_{N-1}, \dots, x_{N-1})\right) \\ &\quad + \left(w_N \times w_{N-1} \times \cdots \times w_{N-1} \tilde{f}(x_N, \dots, x_N, x_{N-1})\right) \\ &\quad + \cdots + \left(w_N \times \cdots \times w_N \tilde{f}(x_N, \dots, x_N)\right) \\ &= \sum_{i_1, \dots, i_n=1}^N w_{i_1} \times \cdots \times w_{i_n} \tilde{f}(x_{i_1}, x_{i_2}, \dots, x_{i_n}) \end{aligned}$$

and $\mathcal{E}_X\{f(X)\}$ is obtained from $\mathcal{E}_{X'}\{\tilde{f}(X')\}$. An alternative to solving the multivariate Gaussian case is to use the spherical curvature rule along with the Gaussian Quadrature after performing a Cartesian-to-spherical coordinate transformation. In fact, consider the Gaussian case $\text{pdf}_Z(z) = \exp(-zz^T)$ and let $z = by$, with $y^T y = 1$, $b \in [0, \infty)$. In this case, (3.1) becomes

$$\mathcal{E}\{f(X)\} = \int_0^\infty S(b) b^{n-1} \exp(-r^2) db, \quad (3.3)$$

$$S(\rho) := \int_{U_n} f(by) d\phi(y), \quad (3.4)$$

where $U_n := \{u \in \mathbb{R}^n | u^T u = 1\}$ and $\phi(\bullet)$ is the spherical surface measure of U_n [80]; equation (3.3) is called radial integral, and is solved by a Gaussian Quadrature rule [80]; and (3.4) is called spherical integral, and is solved by the spherical cubature rule.

Instead of using a quadrature solution, one can obtain a suboptimal solution by

approximating the function f . For instance, one can use linearization or higher-order polynomial approximations of the kind [104]

$$f(x) \approx \sum_i a_i x^i.$$

In this case, (3.1) would be approximated by

$$\mathcal{E}\{f(X)\} = \sum_i a_i \int_{\mathbb{R}^n} z^i \text{pdf}_X(z) dz.$$

Well-known methods are the trapezoidal rule, Simpson’s Rule, the Newton-Cotes Formulas, the Clenshaw-Curtis Integration, among others [104].

Another alternative for obtaining (3.1) is by approximating $\text{pdf}_X(x)$. We can classify this type of suboptimal approximation into two categories, namely Monte Carlo methods [73–77, 105] and sigma-point methods [67, 103]. Monte Carlo (MC) methods consist of taking a very large quantity of samples x_i of X (the method gets more accurate as the number of samples $N \rightarrow +\infty$) randomly [73, 74, 76, 77]. Sigma point methods, on the other hand, consist of analytically choosing finite N samples x_i and weights w_i [67]. These approaches can be viewed as generalized—negative weights are admitted—discrete approximations of $\text{pdf}_X(x)$. Figure 3.1 illustrates these different methods of obtaining the posterior expected value.

There is some overlap in this type of classification, as well as other interpretations. Some sigma-point formulas can be obtained from integration approaches [68, 87, 106]. For instance, [80] derives a particular case of the symmetric sigma-point set of [1] (Tab 2.1 [1,2]) using the spherical cubature quadrature; and [47] and [85] derive the fifth-order sigma-point set (Tab 2.1 [4,2]) also by this quadrature rule [68]. It is worthwhile to mention that the symmetric sigma-point set of [1] (Tab 2.1 [1,2]) can also be viewed as a statistical linear regression technique [82].

In order to estimate the state of dynamical systems such as (2.1) and (2.2), these techniques for expected value calculation can be used in recursive filters. For instance, GHQ yields the GHF [78] when applied in a KF framework; the cubature spherical rule yields the CKF [80,81]; the Central Difference technique, the CDF [78]; the linearization and the second order approximation of the functions yield the EKF and the SOEKF, respectively; different UT’s yield different forms of the UKF; Stirling’s interpolation formula yields the Divided Difference Filter (DDF) [79]; and the Monte Carlo methods yield SMCF’s (e.g. PF’s [73–76]) or MCMCF’s [77].

The DDF and the CDF are considered to be “essentially identical” [67]. The CKF is a particular case of the derivations in [47] and [85], where the CKF is also showed to be equivalent to the UKF of [2] (Tab 2.3 [1,*]) by making the central weight equal

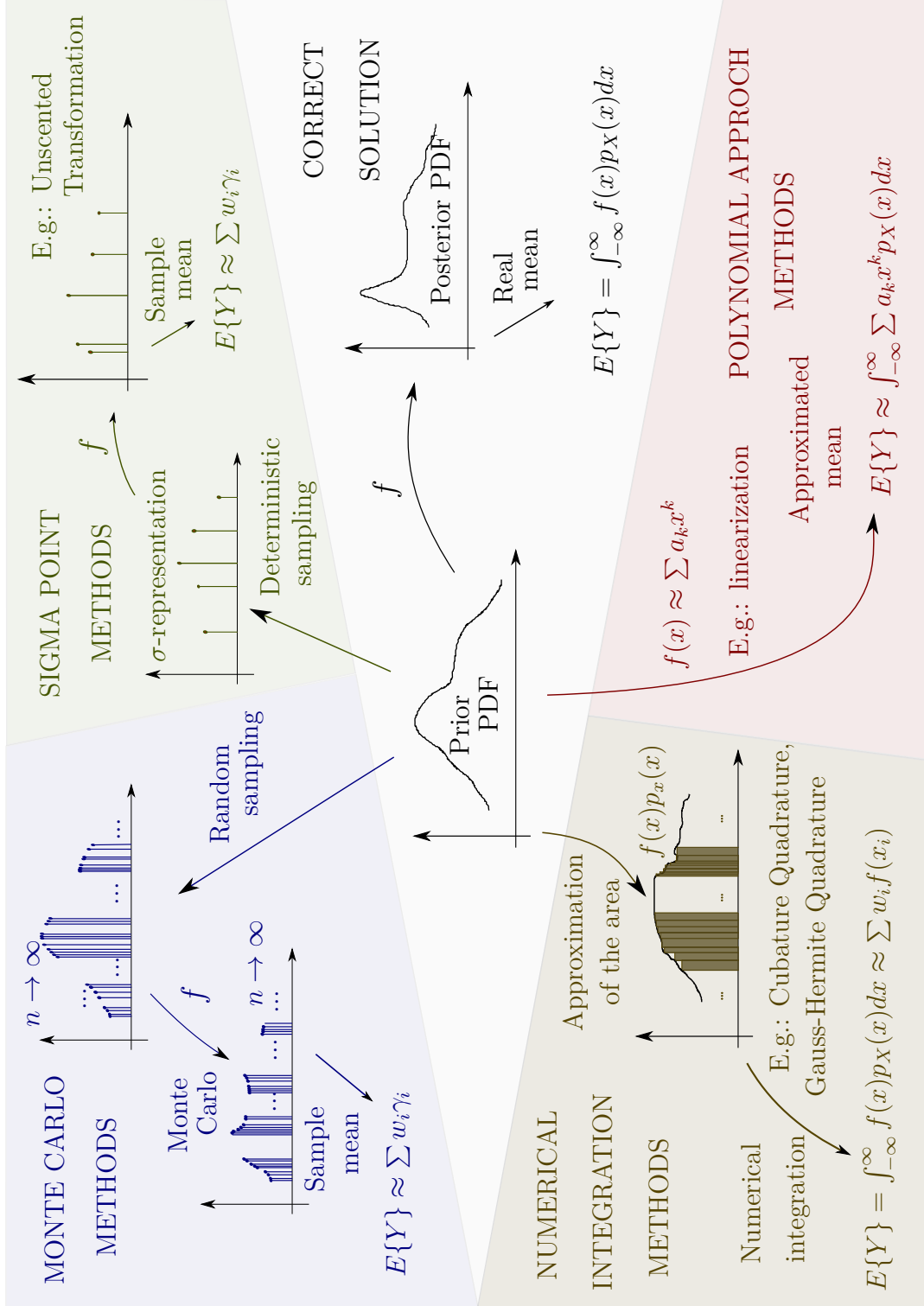


Figure 3.1: Different approaches to approximate the conditional mean.

to zero [59, 60].

The UKF of [2] is showed to be a particular case of the GHF in the scalar case ($n = n_y = 1$) [78]. In fact, consider a scalar standard normal random variable $X \sim N(0, 1)$. Both a GHQ approach of order $N = 3$ and a sigma set of [2] with $k = 2$ and $n = 1$ would yield the set with sigma points (cf. [78])

$$[\chi_1, \chi_2, \chi_3] = [-\sqrt{3}, 0, \sqrt{3}],$$

and weights

$$w_1 = w_3 = \frac{1}{6}, \quad w_2 = \frac{2}{3}.$$

However, for larger lengths of the state vector, this equivalence does not hold. The GHF is $\mathcal{O}(N^n)$, while the UKF of [2] is $\mathcal{O}(n^3)$ [78, 85, 103]. In fact, for $X \sim ([0]_{2 \times 1}, I_2)$, the Gauss-Hermite set would be composed by the sigma points

$$[\chi_1, \dots, \chi_4] = -[\chi_9, \dots, \chi_6] = \begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & 0 \end{bmatrix},$$

$$\chi_5 = [0]_{2 \times 1};$$

and weights

$$w_i = \begin{cases} \frac{1}{36} & , i = 1, 3, 7, 9; \\ \frac{1}{9} & , i = 2, 4, 6, 8; \\ \frac{4}{9} & , i = 5; \end{cases}$$

while the sigma set of [2] (Tab 2.1 [1,1]) for $k = 2$ and $n = 2$ would be composed by the points and weights

$$[\chi_0, \dots, \chi_4] = \begin{bmatrix} 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & -2 \end{bmatrix};$$

and weights

$$w_0 = \frac{1}{2}, \quad w_1 = \dots = w_4 = \frac{1}{8}.$$

In order to properly construct the systematization of the UKF filtering theory, we propose definitions of three fundamental mathematical elements: (i) the sigma (σ)-representation; (ii) the Unscented Transformation; and (iii) the recursive filters. The first is an approximation of a pdf by a set of weighted points. The second is an approximation of the joint pdf of two random variables by two sets of weighted points, where one is a function of the other. The third consists of solutions to the stochastic

filtering problems applying the UT in a recursive manner.

3.2 SIGMA-REPRESENTATION

The σ -representations (σR 's) are approximations of a random variable's pdf by a set of weighted points via moment matching. We say that a set is an l th order σR if the central moments of its samples are equal to the central moments of the chosen random variable up to, and including, order l .

The notation M_X^j stands for the j th central moment of $X \in \Phi^n$, and is defined as

$$M_X^j := \begin{cases} \mathcal{E} \left\{ \left[(X - \bar{X}) (X - \bar{X})^T \right]^{\otimes \frac{j}{2}} \right\} & \text{for even } j, \\ \mathcal{E} \left\{ \left[(X - \bar{X}) (X - \bar{X})^T \right]^{\otimes \frac{j-1}{2}} \otimes (X - \bar{X}) \right\} & \text{for odd } j. \end{cases}$$

Definition 3.1 (σ -Representation). Let

$$\mathbb{M}_\chi^j := \begin{cases} \sum_{i=1}^N w_i^{(j)} \left[(\chi_i - \mu_\chi) (\chi_i - \mu_\chi)^T \right]^{\otimes \frac{j}{2}} & \text{for even } j, \text{ and} \\ \sum_{i=1}^N w_i^{(j)} \left[(\chi_i - \mu_\chi) (\chi_i - \mu_\chi)^T \right]^{\otimes \frac{j-1}{2}} \otimes (\chi_i - \mu_\chi) & \text{for odd } j, \end{cases} \quad (3.5)$$

be the j th sample central moment of

$$\chi := \{\chi_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \chi_i \in \mathbb{R}^n; w_i^{(1)}, \dots, w_i^{(l)} \in \mathbb{R}\}_{i=1}^N;$$

let the sample mean of χ be

$$\mu_\chi := \sum_{i=1}^N w_i^{(1)} \chi_i,$$

and consider the random variable $X \sim (\bar{X}, M_X^2, \dots, M_X^l)^n$. Then χ is an l th order N points σ -representation (l th $N\sigma R$) of X if

$$w_i^{(j)} \neq 0, \quad i = 1, \dots, N \text{ and } j = 1, \dots, l; \quad (3.6)$$

$$\mu_\chi = \bar{X}; \quad (3.7)$$

$$\mathbb{M}_\chi^j = M_X^j, \quad j = 2, 3, \dots, l. \quad (3.8)$$

We define also the function

$$\sigma R(\bar{X}, M_X^2, \dots, M_X^l) := \chi \quad (3.9)$$

mapping the statistics $(\bar{X}, M_X^2, \dots, M_X^l)$ of X into an l th $N\sigma R$ χ of X .

Moreover, assume χ is an l th $N\sigma R$ of X , then:

- χ is *normalized* if

$$\sum_{i=1}^N w_i^{(j)} = 1, \quad j = 1, 2, \dots, l. \quad (3.10)$$

- χ is *homogeneous* if:

$$w_1^{(j)} = w_i^{(j)}, \quad 1 \leq i \leq N-1, \quad \text{for odd } N; \quad \text{or} \quad (3.11)$$

$$w_1^{(j)} = w_i^{(j)}, \quad 1 \leq i \leq N, \quad \text{for even } N. \quad (3.12)$$

- χ is *symmetric* (respectively to χ_N)—if χ is *symmetric* relative to other χ_i , we can rearrange the indices of the sigma points and weights—if:

$$\chi_i - \chi_N = -\left(\chi_{i+\frac{N-1}{2}} - \chi_N\right) \quad \text{and} \quad w_i^{(j)} = w_{i+\frac{N-1}{2}}^{(j)}, \quad 1 \leq i \leq \frac{N-1}{2}, \quad \text{for odd } N; \quad \text{or} \quad (3.13)$$

$$\chi_i - \chi_N = -\left(\chi_{i+\frac{N}{2}} - \chi_N\right) \quad \text{and} \quad w_i^{(j)} = w_{i+\frac{N}{2}}^{(j)}, \quad 1 \leq i \leq \frac{N}{2}, \quad \text{for even } N. \quad (3.14)$$

The case $l = 2$ is of particular interest, since the majority of works in Unscented literature focus on second order moment matching [1, 2, 7, 16, 21, 39, 40, 42, 44–46, 67, 107]. This is mainly motivated by three facts. First, these are usually the estimated statistics within a stochastic filter. Second, they fully describe a Gaussian distribution [103]. Third, the mean is the point estimate with the least mean squared error. Thus, when calling an l th $N\sigma R$ of X , the reference to the l th order can be omitted if $l = 2$. Also, the reference to N point and/or to X can be omitted in case they are obvious from the context or irrelevant for a given statement. Note that the Reduced set of [45], the Spherical simplex set of [46] and the Minimum set of [83] are not σ -R's (cf. Section 2.5).

The next theorem provides conditions for a given weighted set to be an l th $N\sigma R$ in a matrix form; this matrix result states the ground to develop some new results in the Unscented research field.

Theorem 3.1. *A random vector $X \sim (\bar{X}, M_X^2, \dots, M_X^l)^n$ admits a normalized l th $N\sigma R$ if and only if there exists a matrix $E \in \mathbb{R}^{n \times N}$ and the matrices $W^{(j)} := \text{diag}(w^{(j)})$, for $j = 2, 3, \dots, l$, where $w^{(j)} := [w_1^{(j)}, \dots, w_N^{(j)}]^T$, satisfying:*

- for even l , the following equations:

$$\left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right] W^{(j)} \left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right]^T = M_X^j, \quad j = 2, 4, \dots, l; \quad (3.15)$$

$$\left[E_{*1}^{\otimes \frac{j+1}{2}}, \dots, E_{*N}^{\otimes \frac{j+1}{2}} \right] W^{(j)} \left[E_{*1}^{\otimes \frac{j-1}{2}}, \dots, E_{*N}^{\otimes \frac{j-1}{2}} \right]^T = M_X^j, \quad j = 1, 3, \dots, l-1; \quad (3.16)$$

$$Ew^{(1)} = 0; \quad (3.17)$$

$$[1]_{1 \times N} w^{(j)} = 1, \quad j = 1, 2, \dots, l. \quad (3.18)$$

• for odd l , the following equations:

$$\left[E_{*1}^{\otimes \frac{j+1}{2}}, \dots, E_{*N}^{\otimes \frac{j+1}{2}} \right] W^{(j)} \left[E_{*1}^{\otimes \frac{j-1}{2}}, \dots, E_{*N}^{\otimes \frac{j-1}{2}} \right]^T = M_X^j, \quad j = 1, 3, \dots, l; \quad (3.19)$$

$$\left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right] W^{(j)} \left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right]^T = M_X^j, \quad j = 2, 4, \dots, l-1; \quad (3.20)$$

$$Ew^{(1)} = 0; \quad (3.21)$$

$$[1]_{1 \times N} w^{(j)} = 1, \quad j = 1, 2, \dots, l. \quad (3.22)$$

If (3.15)-(3.18) or (3.19)-(3.22) admits a solution $(E, w^{(1)}, W^{(2)}, \dots, W^{(l)})$, then a normalized l th $N\sigma R$ of X is $\{\chi_i, w_i^{(1)}, \dots, w_i^{(l)}\}_{i=1}^N$ such that

$$[\chi_1, \dots, \chi_N] := E + [\bar{X}]_{1 \times N}.$$

Proof. Define

$$E := [\chi_1 - \mu_\chi, \dots, \chi_N - \mu_\chi].$$

So, from (3.5), for even j and l , we have that

$$\mathbb{M}_X^j = \sum_{i=1}^N \left[E_{*i}^{\otimes \frac{j}{2}} \right] w_i^{(j)} \left[E_{*i}^{\otimes \frac{j}{2}} \right]^T = \left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right] W^{(j)} \left[E_{*1}^{\otimes \frac{j}{2}}, \dots, E_{*N}^{\otimes \frac{j}{2}} \right]^T,$$

which proves (3.15). Equations (3.16), (3.19) and (3.20) can be proven similarly; and the proofs of (3.17), (3.18), (3.21) and (3.22) are trivial. \square

The next corollary uses Theorem 3.1 to obtain two novel results: the minimum amount of σ -points for both the symmetric and the non-symmetric case when $P_{XX} \geq 0$. Note that the literature's result stating that the minimum number is $n+1$ for $P_{XX} > 0$ [1, 45, 46] is a particular case of Corollary 3.1 [rank(A) stands for the rank of a matrix A].

Corollary 3.1 (Minimum number of sigma points). *Let $\chi := \{\chi_i, w_i^{(1)}, \dots, w_i^{(l)}\}_{i=1}^N$ be an l th $N\sigma R$ of $X \in \Phi^n$, and X have covariance P_{XX} with $\text{rank}\{P_{XX}\} = r \leq n$. Then*

1. $N \geq r + 1$. If $N = r + 1$, then χ is called a minimum l th $N\sigma R$ of X .

2. If χ is symmetric, then $N \geq 2r$. In this case, if $N = 2r$, then χ is called a minimum symmetric lthN σ R of X .

Proof. To prove assertion 1, consider, first, $E \in \mathbb{R}^{n \times N}$ and the singular value decomposition of P_{XX} ,

$$P_{XX} = USV^T,$$

where

$$S := \text{diag} \left([a_1, \dots, a_r, [0]_{(n-r) \times 1}]^T \right),$$

and a_1, \dots, a_r are the singular values of P_{XX} . Assume, for contradiction,

$$\text{rank}\{E\} < r.$$

Then there exists

$$v := [v_1^T, [0]_{1 \times (n-r)}^T]^T$$

$v_1 \in \mathbb{R}^r$, $v_1 \neq 0$, such that $v^T E = 0$. Then, from (3.15),

$$EW^{(2)}E^T = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} SV^T \Leftrightarrow v_1^T U_1 S_1 = 0 \Leftrightarrow v_1^T = 0,$$

which is a contradiction. Therefore, $\text{rank}\{E\} \geq r$.

Second, suppose $N = \text{rank}\{E\}$. Then, E is full column rank and, from (3.17), $w^{(1)} = 0$, which is a contradiction for, from Definition 3.1, $w^{(1)} \neq 0$. So

$$N \geq \text{rank}\{E\} + 1 \geq r + 1.$$

To prove assertion 2, let χ be symmetric. Then

$$E = [E_2, -E_2]$$

where $E_2 \in \mathbb{R}^{n \times \frac{N}{2}}$. So ($\min\{a, b\}$ stands for the minimum between a and b)

$$r \leq \text{rank}\{E\} = \text{rank}\{[E_2, -E_2]\} = \min\left\{n, \frac{N}{2}\right\} \Leftrightarrow N \geq 2r.$$

□

Corollary 3.2. Let $\chi = \{\chi_i, w_i^{(1)}, \dots, w_i^{(l)}\}_{i=1}^N$ be a normalized lthN σ R of $X \sim (\bar{X}, M_{\bar{X}}^2, \dots, M_{\bar{X}}^l)^n$ and consider the random variable

$$Z = AX + b,$$

$A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$. Then, the set

$$\zeta := \left\{ \zeta_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \zeta_i = A\chi_i + b \right\}_{i=1}^N$$

is a normalized l th $N\sigma\mathbb{R}$ of Z . In particular, we have

$$\mu_\zeta = A\bar{X} + b = \bar{Z},$$

and

$$\Sigma_{\zeta\zeta} = AP_{XX}A^T = P_{ZZ}.$$

Proof. The sample mean of ζ is

$$\mu_\zeta := \sum_{i=1}^N w_i^{(1)} \zeta_i = \sum_{i=1}^N w_i^{(1)} (A\chi_i + b) = \bar{Z}.$$

The j th sample central moment of ζ , for $j = 2, 4, \dots, l$ (l even) is, from (3.5),

$$\begin{aligned} \mathbb{M}_\zeta^j &:= \sum_{i=1}^N w_i^{(j)} \left[(\zeta_i - \mu_\zeta) (\zeta_i - \mu_\zeta)^T \right]^{\otimes \frac{j}{2}} \\ &= \sum_{i=1}^N w_i^{(j)} \left[(A\chi_i + b - A\bar{X} - b) (\diamond)^T \right]^{\otimes \frac{j}{2}} \\ &= A^{\otimes \frac{j}{2}} \left[\sum_{i=1}^N w_i^{(j)} \left[(\chi_i - \mu_\chi) \otimes (\chi_i - \mu_\chi)^T \right]^{\otimes \frac{j}{2}} \right] (A^T)^{\otimes \frac{j}{2}} \\ &= A^{\otimes \frac{j}{2}} M_X^j (A^T)^{\otimes \frac{j}{2}} \\ &= M_Z^j. \end{aligned}$$

The odd j case can be similarly proven. □

The result used by [1, 37, 40] and others that a sigma set $\chi = \{\chi_i, w_i\}_{i=1}^N$ approximating a random variable $X \sim (\bar{X}, P_{XX})^n$ can be obtained by the transformation

$$\zeta_i = \sqrt{P_{XX}} \chi_i + \bar{X},$$

where $\zeta = \{\zeta_i, w_i\}_{i=1}^N$ is a sigma set of a random variable with mean $[0]_{n \times 1}$ and covariance I_n , is a particular case of Corollary 3.2.

With Theorem 3.1, and Corollaries 3.1 and 3.2, new results regarding characteristics of general $\sigma\mathbb{R}$'s were developed. In the next two sections, particular $\sigma\mathbb{R}$'s are focused on; following Corollary 3.1, closed forms of both a minimum symmetric l th $N\sigma\mathbb{R}$ and a minimum l th $N\sigma\mathbb{R}$ are introduced—minimum symmetric one is presented first because it will result in the classical sigma set of [2] (cf. Corollary 3.4).

3.3 MINIMUM SYMMETRIC SIGMA-REPRESENTATION

Let $\chi = \{\chi_i, w_i^m, w_i^c\}_{i=1}^{2n}$ be an σR of $X \sim (\bar{X}, P_{XX})^n$, $P_{XX} > 0$. Considering the equations of Theorem 3.1, suppose χ is minimum symmetric. Then, we have

$$E = [\bar{E}, -\bar{E}],$$

where $\bar{E} \in \mathbb{R}^{n \times n}$. Define

$$W := \text{diag}(w_1^c, \dots, w_n^c) > 0.$$

Then, from (3.15), it follows that

$$[\bar{E}\sqrt{W}, -\bar{E}\sqrt{W}] [\diamond]^T = \left[\sqrt{\frac{1}{2}P_{XX}}, -\sqrt{\frac{1}{2}P_{XX}} \right] [\diamond]^T.$$

Clearly, a sufficient condition is

$$\bar{E} = (\sqrt{2W})^{-1} \sqrt{P_{XX}}.$$

Since (3.21) is satisfied for all symmetric σR 's, a closed form for this case is obtained. The next corollary formalizes it.

Corollary 3.3 (Minimum Symmetric σR). *Consider a symmetric random vector $X \sim (\bar{X}, P_{XX})^n$ with $P_{XX} > 0$. For*

$$W := \text{diag}(w_1^c, \dots, w_n^c) > 0, \quad w_i^m \neq 0,$$

a minimum symmetric σR of X is the set $\chi = \{\chi_i, w_i^m, w_i^c\}_{i=1}^{2n}$ with the sigma points given by

$$[\chi_1 \cdots \chi_{2n+1}] = \left[(\sqrt{2W})^{-1} \sqrt{P_{XX}}, -(\sqrt{2W})^{-1} \sqrt{P_{XX}} \right].$$

In addition, if

$$\sum_{i=1}^{2n} w_i^m = \sum_{i=1}^{2n} w_i^c = 1,$$

then χ is a normalized. Moreover, if

$$W = \frac{1}{2n} I_n,$$

and

$$w_i := w_i^m = w_i^c, \quad i = 1, \dots, 2n;$$

then $\chi = \{\chi_i, w_i\}_{i=1}^{2n}$ is a (normalized) homogeneous minimum symmetric σ -representation of X .

If an extra point located on \bar{X} is added to this $\sigma\mathbf{R}$, then neither the sample mean, nor the sample covariance will be modified; and this extra point's *weight* can act as a *tuning parameter*; an user can choose the value of this weight to achieve a desired property for the $\sigma\mathbf{R}$, e.g. a specific value for the sample moment of degree 3.

Corollary 3.4 ((Odd) Minimum Symmetric σ -representation). *Consider a symmetric random vector $X \sim (\bar{X}, P_{XX})^n$ with $P_{XX} > 0$. For*

$$W := \text{diag}(w_1^c, \dots, w_n^c) > 0, \quad w_i^m \neq 0,$$

a minimum symmetric $\sigma\mathbf{R}$ of X is the set $\chi = \{\chi_i, w_i^m, w_i^c\}_{i=1}^{2n+1}$ with the sigma points given by

$$[\chi_1 \cdots \chi_{2n+1}] = \left[(\sqrt{2W})^{-1} \sqrt{P_{XX}}, -(\sqrt{2W})^{-1} \sqrt{P_{XX}}, [0]_{n \times 1} \right] + [\bar{X}]_{1 \times (2n+1)}.$$

In addition, if

$$\sum_{i=1}^{2n+1} w_i^m = \sum_{i=1}^{2n+1} w_i^c = 1,$$

then χ is a normalized (MiSy $\sigma\mathbf{R}$). Moreover, if

$$W = \frac{1 - w_{2n+1}}{2n} I_n$$

and

$$w_i := w_i^m = w_i^c, \quad i = 1, \dots, 2n + 1$$

then $\chi = \{\chi_i, w_i\}_{i=1}^{2n+1}$ is a (normalized) homogeneous (odd) minimum symmetric σ -representation (HoMiSy $\sigma\mathbf{R}$) of X ; the HoMiSy $\sigma\mathbf{R}$ is equivalent to the symmetric sigma set of [1] (Tab 2.1 [1,2]).

Corollaries 3.3 and 3.4 present the even and odd $\sigma\mathbf{R}$'s with the least amount of symmetric sigma points. The classical symmetric sigma sets of [1,2] (Table 2.1), which have been presented in the literature without formal justification, are rewritten forms of the homogeneous cases of these corollaries. In fact, heretofore, it was not known that these sigma sets are composed by the smallest amount of symmetric sigma points.

Regarding the choice of the tuning parameter, a couple of results have already been proposed in the literature. The authors in [108] provide an off-line way of computing it by maximizing the likelihood function with a training set of data. In [59], an on-line method of computing the tuning parameter by means of maximizing a Gaussian approximation of the likelihood function is proposed.

3.4 MINIMUM SIGMA-REPRESENTATION

As shown in [57], the $n + 1$ sigma sets found in the literature, [45] and [46], present some problems; the set of [45] has a numerical instability problem (see [46]), and both the sets of [45] and [46] do not have the properties of matching the mean and the covariance of the prior random variable (see Section 2.5). In response, we proposed in [57] a new sigma set composed by the minor quantity of points that proved to hold these two properties—therefore it is a 2nd order σR (cf. Definition 3.1). For easy reference, this set is presented in the following.

Consider the random variable $X \in \mathbb{R}^n$ with mean \bar{X} and covariance $P_{XX} > 0$ and the non-linear mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^{n_y}$ differentiable up, at least, to the second order defining the random variable Y by $Y := f(X)$. Then the σR of [57] $\chi = \{\chi_i, w_i\}_{i=1}^{n+1}$ is given by the following equations, for $0 < w_{n+1} < 1$ (Tab 2.1 [3,2]):

$$\rho := \sqrt{\frac{1 - w_{n+1}}{n}}; \quad (3.23)$$

$$C := \sqrt{I_n - \rho^2 [1]_{n \times n}}; \quad (3.24)$$

$$W := \text{diag}(w_1, \dots, w_n); \quad (3.25)$$

$$w_i = \left(w_{n+1} \rho^2 C^{-1} [1]_{n \times n} (C^T)^{-1} \right)_{ii}, \quad \forall i = 1, \dots, n; \quad (3.26)$$

$$[\chi_0, \dots, \chi_n] = \left[\sqrt{P_{XX}} C (\sqrt{W})^{-1}, -\sqrt{P_{XX}} \frac{[\rho]_{n \times 1}}{\sqrt{w_{n+1}}} \right] + [\bar{X}]_{1:n+1}. \quad (3.27)$$

This set will be called the *Rho Minimum σR (RhoMi σR)*. If X is non symmetrical, then the RhoMi σR requires less computational effort than the symmetric sets of [2], [1] and [41] while keeping the same estimate quality.

The Rho Minimum σR has, nevertheless, the limitation that its tuning parameter, w_{n+1} , has only one degree of freedom which can be limiting if one wants to achieve some additional properties. For instance, consider a random variable

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \begin{bmatrix} N(0, 1) \\ \chi^2(1) \end{bmatrix}$$

where $\chi^2(a)$ is the chi-square distribution with distribution parameter a . The mean and the covariance of X are

$$\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} := \mathcal{E}\{X\} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (3.28)$$

and

$$P_{XX} := \mathcal{E} \left\{ (X - \bar{X})(X - \bar{X})^T \right\} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad (3.29)$$

and the main third central moments are

$$M_{x_1}^3 := \mathcal{E} \left\{ (x_1 - \bar{x}_1)^3 \right\} = 0, \quad (3.30)$$

and

$$M_{x_2}^3 := \mathcal{E} \left\{ (x_2 - \bar{x}_2)^3 \right\} = 8. \quad (3.31)$$

If χ is the σ R of [57], then, from (3.23)-(3.27), the sample mean and covariance of χ are

$$\mu_\chi = \begin{bmatrix} \mu_{\chi,1} \\ \mu_{\chi,2} \end{bmatrix} := \sum_{i=1}^{n+1} w_i \chi_i = \begin{bmatrix} \sum_{i=1}^{n+1} w_i \chi_{i,1} \\ \sum_{i=1}^{n+1} w_i \chi_{i,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \stackrel{(3.28)}{=} \bar{X},$$

and

$$\Sigma_{\chi\chi} := \sum_{i=1}^{n+1} w_i \chi_i = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \stackrel{(3.29)}{=} P_{XX};$$

and its main third central moments are given by

$$\mathbb{M}_{\chi,j}^3 := \sum_{i=0}^2 w_i (\chi_{i,j} - \mu_{\chi,j})^3, \quad j = 1, 2. \quad (3.32)$$

Now it is easy to see that w_{n+1} can not be chosen such that

$$\mathbb{M}_{\chi,1}^3 = M_{x_1}^3 \text{ and } \mathbb{M}_{\chi,2}^3 = M_{x_2}^3$$

are both satisfied since we have two equations and only one free parameter (w_{n+1}). However, Theorem 3.2 can lead us to a more general minimum σ R that is able to fulfill this kind of property that the Rho Minimum σ R fails to. We first present a heuristic for finding this σ R, followed by a formal and more general minimum σ R in Theorem 3.2. At the end of this section, Corollary 3.5 shows that this minimum σ R has the minimum set of [57] as a particular case.

For $P_{XX} > 0$, least amount of (non-symmetric) sigma points is $n + 1$ (cf Theorem 3.1). Let $\chi = \{\chi_i, w_i\}_{i=1}^{n+1}$ be a minimum σ R of $X \sim (\bar{X}, P_{XX})^n$, $P_{XX} > 0$. Considering the equations of Theorem 3.1, suppose χ is minimum. Then,

$$E = [\bar{E}, e],$$

where $\bar{E} \in \mathbb{R}^{n \times n}$ and $e \in \mathbb{R}^n$. Define

$$\bar{w} = [w_1, \dots, w_n]^T$$

and

$$\bar{W} := \text{diag}(w_1, \dots, w_n) > 0.$$

Then, from (3.15),

$$e = -w_{n+1}^{-1} \bar{E} \bar{w}.$$

Substituting it on (3.17), it follows that

$$\begin{aligned} P_{XX} &= \bar{E} \bar{W} \bar{E}^T + w_{n+1}^{-1} \bar{E} \bar{w} \bar{w}^T \bar{E}^T \\ &= \bar{E} \sqrt{\bar{W}} (I_n + vv^T) (\bar{E} \sqrt{\bar{W}})^T, \end{aligned} \quad (3.33)$$

where

$$v := w_{n+1}^{-\frac{1}{2}} \sqrt{\bar{W}} \bar{w}.$$

Then it is easy to see that

$$\bar{E} = \sqrt{P_{XX}} (I_n + vv^T)^{-\frac{1}{2}} \bar{W}^{-\frac{1}{2}}$$

is a sufficient condition for (3.33). Moreover, from (3.17), it follows that

$$w_{n+1} = \frac{1}{1 + \sum_{i=1}^n v_i^2}.$$

The σR of this heuristic approach is more general than then the $\text{RhoMi}\sigma\text{R}$ (cf. Corollary 3.5 further ahead). The following theorem formalizes this heuristic approach with an even more general closed form of the minimum σR (without the $w_i > 0$ restriction).

Theorem 3.2 (Minimum σR). *Consider a random vector $X \sim (\bar{X}, P_{XX})^n$ with $P_{XX} > 0$. Then, for*

$$\begin{aligned} v &:= [v_1, \dots, v_n]^T \in \mathbb{R}^n, \quad v_i \neq 0; \\ \bar{w} &:= [w_1, \dots, w_n]^T, \\ \bar{W} &:= \text{diag}(w_1, \dots, w_n) > 0, \end{aligned}$$

the set $\chi = \{\chi_i, w_i\}_{i=1}^{n+1}$ with

$$w_{n+1} = \frac{1}{\sum_{i=1}^n (|w_i| + v_i^2)}, \quad (3.34)$$

$$|\bar{W}|^{-\frac{1}{2}} \bar{w} = \sqrt{w_{n+1}} v, \quad (3.35)$$

$$\bar{E} := \sqrt{P_{XX}} (\text{sign}(\bar{W}) + vv^T)^{-\frac{1}{2}} |\bar{W}|^{-\frac{1}{2}}, \quad (3.36)$$

$$e := -\frac{1}{w_{n+1}}E\bar{w}, \quad (3.37)$$

$$[\chi_1, \dots, \chi_{n+1}] = [\bar{E}, e] + [\bar{X}]_{1 \times (n+1)}; \quad (3.38)$$

is an $\text{Mio}\sigma\text{R}$ of X . Besides, χ is normalized if

$$\sum_{i=1}^{n+1} w_i = 1.$$

Proof. Define

$$w := [w_1, \dots, w_{n+1}]^T;$$

then, from Theorem 3.1, the set $\chi = \{\chi_i, w_i\}_{i=1}^N$ is a $2\text{nd}N\sigma\text{R}$ of X if, and only if,

$$\begin{aligned} EW E^T &= P_{XX}, \\ Ew &= 0, \end{aligned}$$

which, for

$$\begin{aligned} E &:= [\bar{E} \quad e], \\ \bar{w} &:= [w_1, \dots, w_n]^T, \\ \bar{W} &:= \text{diag}(w_1, \dots, w_n); \end{aligned}$$

can be written as

$$\bar{E}\bar{W}\bar{E}^T + w_{n+1}ee^T = P_{XX}, \quad (3.39)$$

$$\bar{E}\bar{w} + w_{n+1}e = 0, \quad (3.40)$$

Note that

$$w_i \neq 0 \quad (3.41)$$

since, otherwise, $w_i = 0$ would imply a σR of $N = n$ sigma points. From (3.40), e can be written as

$$e = -\frac{1}{w_{n+1}}\bar{E}\bar{w}, \quad (3.42)$$

proving (3.37). Substituting (3.42) into (3.39), we have that

$$\begin{aligned} P_{XX} &= \bar{E}\bar{W}\bar{E}^T + \frac{1}{w_{n+1}}\bar{E}\bar{w}\bar{w}^T\bar{E}^T \\ &= \bar{E} \left(\bar{W} + \frac{1}{w_{n+1}}\bar{w}\bar{w}^T \right) \bar{E}^T. \end{aligned} \quad (3.43)$$

From (3.41), \bar{W} is invertible. As \bar{W} is symmetric, we can write it in the following way

$$\bar{W} = |\bar{W}|^{\frac{1}{2}} S |\bar{W}|^{\frac{1}{2}} \quad (3.44)$$

where

$$\begin{aligned} |\bar{W}|^{\frac{1}{2}} &:= \text{diag} \left(\sqrt{|w_1|}, \dots, \sqrt{|w_n|} \right) \in \mathbb{R}^{n \times n}, \text{ and} \\ S &:= \text{diag} (\text{sign} (w_1), \dots, \text{sign} (w_n)) \in \mathbb{R}^{n \times n}. \end{aligned}$$

In this case, (3.43) can be written as

$$\begin{aligned} P_{XX} &= \bar{E} \left(\bar{W} + \frac{1}{w_{n+1}} \bar{w} \bar{w}^T \right) \bar{E}^T \\ &\stackrel{(3.44)}{=} \bar{E} |\bar{W}|^{\frac{1}{2}} \left(S + \frac{1}{w_{n+1}} |\bar{W}|^{-\frac{1}{2}} \bar{w} \bar{w}^T |\bar{W}|^{-\frac{1}{2}} \right) |\bar{W}|^{\frac{1}{2}} \bar{E}^T \\ &= F (S + vv^T) F^T, \end{aligned} \quad (3.45)$$

where

$$F := \bar{E} |\bar{W}|^{\frac{1}{2}}, \quad (3.46)$$

$$v := \frac{|\bar{W}|^{-\frac{1}{2}} \bar{w}}{\sqrt{w_{n+1}}}. \quad (3.47)$$

proving (3.35). Note that F is invertible because \bar{E} and $|\bar{W}|^{\frac{1}{2}}$ are. From (3.45) and by the fact that, by assumption, P_{XX} is invertible ($P_{XX} > 0$), we can write

$$P_{XX}^{-\frac{1}{2}} F (S + vv^T) \left(P_{XX}^{-\frac{1}{2}} F \right)^T = I$$

and a sufficient condition is

$$F = P_{XX}^{\frac{1}{2}} (S + vv^T)^{-\frac{1}{2}}. \quad (3.48)$$

From (3.46),

$$\begin{aligned} \bar{E} |\bar{W}|^{\frac{1}{2}} &= F \stackrel{(3.48)}{=} P_{XX}^{\frac{1}{2}} (S + vv^T)^{-\frac{1}{2}} \\ \Leftrightarrow \bar{E} &= P_{XX}^{\frac{1}{2}} (S + vv^T)^{-\frac{1}{2}} |\bar{W}|^{-\frac{1}{2}}, \end{aligned}$$

proving (3.36). From (3.47)

$$v := \frac{|\bar{W}|^{-\frac{1}{2}} \bar{w}}{\sqrt{w_{n+1}}}$$

$$= \frac{1}{\sqrt{w_{n+1}}} \begin{bmatrix} \text{sign}(w_1) \sqrt{|w_1|} \\ \vdots \\ \text{sign}(w_n) \sqrt{|w_n|} \end{bmatrix}. \quad (3.49)$$

From (3.41), we must have

$$v_i \neq 0.$$

Therefore, by choosing $v = [v_1, \dots, v_n]^T \in \mathbb{R}^n$ with $v_i \neq 0$ for $i = 1, \dots, n$, then w_i is such that, from (3.49),

$$\begin{aligned} \frac{1}{\sqrt{w_{n+1}}} \text{sign}(w_i) \sqrt{|w_i|} &= v_i \\ \Rightarrow \frac{1}{w_{n+1}} |w_i| &= v_i^2 \\ \Leftrightarrow |w_i| &= w_{n+1} v_i^2. \end{aligned}$$

Summing from $i = 1$ to $i = n$:

$$w_{n+1} = \frac{1}{\sum_{i=1}^n (|w_i| + v_i^2)},$$

proving (3.34). From Theorem 3.1, we have that

$$\begin{aligned} [\chi_1, \dots, \chi_N] &= E + [\bar{X}]_{1 \times N} \\ &= [\bar{E}, e] + [\bar{X}]_{1 \times (n+1)} \end{aligned}$$

proving (3.38) and completing the prove. \square

Corollary 3.5 (Minimum σR with positive weights). *If $w_i > 0$, then the normalized $Mi\sigma R$ of Theorem 3.2 becomes*

$$w_{n+1} = \frac{1}{1 + \sum_{i=1}^n v_i^2}, \quad (3.50)$$

$$\bar{w} = w_{n+1} [v_1^2, \dots, v_n^2]^T, \quad (3.51)$$

$$\bar{E} := \sqrt{\frac{P_{XX}}{w_{n+1}}} (I + vv^T)^{-\frac{1}{2}} \text{diag}(v)^{-1}, \quad (3.52)$$

$$e := -\frac{1}{w_{n+1}} \bar{E} \bar{w}, \quad (3.53)$$

$$[\chi_1, \dots, \chi_{n+1}] = [\bar{E}, e] + [\bar{X}]_{1 \times (n+1)}.$$

Moreover, if

$$w_i > 0 \quad \text{and} \quad v = \alpha C^{-1} [1]_{n \times 1},$$

then χ is the minimum σR of [57] (Tab 2.1 [3,2]).

The parameter vector v is a tuning parameter with n degrees of freedom (v_i , $i = 1, \dots, n$) and has the only restriction of not containing a zero element and can, therefore, also have negative values. v can be chosen according to a specific design. For instance, consider the problem of choosing it in order to match the mean, the covariance and the main third central moments of a real random variable X with mean

$$\bar{X} = \mathcal{E} \{X\} = [0]_{n \times 1},$$

covariance

$$P_{XX} = \mathcal{E} \{XX^T\} = I_n.$$

and j th main third central moments

$$M_{x_j}^3 := \mathcal{E} \{x_j^3\} = b.$$

In other words, we want to find a value for $v \in \mathbb{R}^n$ for the minimum σ R of Theorem 3.2 such that

$$\begin{aligned} \mu_\chi &:= \sum_{i=0}^n w_i \chi_i = \bar{X}, \\ \Sigma_{\chi\chi} &:= \sum_{i=0}^n w_i \chi_i \chi_i^T = P_{XX}, \\ \mathbb{M}_{\chi,j}^3 &:= \sum_{i=0}^n w_i (\chi_{j,i})^3 = M_{x_j}^3, \quad j = 1, \dots, n. \end{aligned}$$

For simplicity, suppose $v = \beta [1]_{n \times 1}^1$, then, from (3.50)-(3.53), the weights are

$$\begin{aligned} w_0 &= \frac{1}{1 + n\beta^2}, \\ w &= \frac{\beta^2}{1 + n\beta^2} [1]_{n \times 1}, \end{aligned} \tag{3.54}$$

and, for

$$\eta := \sqrt{1 + n\beta^2}, \tag{3.55}$$

the sigma points are

$$\left[\chi_1, \quad \dots, \quad \chi_{n+1} \right] := \left[-\eta\beta \left(I + \beta^2 [1]_{n \times n} \right)^{-\frac{1}{2}} [1]_{n \times 1}, \quad \frac{\eta}{\beta} \left(I + \beta^2 [1]_{n \times n} \right)^{-\frac{1}{2}} \right]. \tag{3.56}$$

The properties of matching the mean and the covariance are already proved by Theorem 3.2. In order to achieve the property of matching the main third central moments,

¹The most general case of v having n degrees of freedom is given after this example.

Table 3.1: Values of β for which $\mu_\chi = [0]_{n \times 1}$, $\Sigma_{\chi\chi} = I_n$ and $\mathbb{M}_{\chi,j}^3 = 0$.

n	1	2	3	4	5
β	± 1	$\pm\sqrt{2}$	$\pm\sqrt{5}$	$\pm\frac{3+\sqrt{5}}{\sqrt{2}}$	$\pm(3+\sqrt{5})$

assume that $(I + \beta^2 [1]_{n \times 1})^{-\frac{1}{2}}$ is of the form $I_n + a [1]$, for some $\phi \in \mathbb{R}$. Then it can be shown that

$$a = \pm \frac{1 - \eta}{n\eta}$$

and, from (3.56), we have

$$\left[\chi_1, \dots, \chi_{n+1} \right] = \left[-\beta(1 + \eta \mp \eta) [1]_{n \times 1}, \frac{1}{n\beta} (n\eta I_n \pm (1 - \eta) [1]_{n \times n}) \right]. \quad (3.57)$$

The j th third central moment of the set of sigma points is, from (3.54), (3.56), and (3.57):

$$\mathbb{M}_{\chi,j}^3 := \sum_{i=1}^{n+1} w_i (\chi_{j,i})^3 = \frac{-n^3 \beta^4 (1 + \eta \mp \eta)^3 + (n\eta \pm (1 - \eta))^3 \pm (n - 1) (1 - \eta)^3}{\eta^2 \beta n^3}.$$

In order to satisfy $\mathbb{M}_{\chi,j}^3 = M_{x_j}^3 = b$, β should be a solution of

$$-n^3 \beta^4 (1 + \eta \mp \eta)^3 + (n\eta \pm (1 - \eta))^3 \pm (n - 1) (1 - \eta)^3 - b\beta n^3 \eta^2 = 0. \quad (3.58)$$

Therefore any real value of $\beta \neq 0$, including negative values, satisfying (3.58) will make the set of sigma points have the same principal third moments of X . Particularly, for $b = 0$ and using (3.55) it follows that we can choose

$$\beta = \pm \sqrt{\frac{\eta^2 - 1}{n}},$$

where η is any real solution of

$$n\eta^2 - \left[(n - 1)^3 - (n - 1) \right] \eta - 3(n - 1)^2 - 3(n - 1) - 2n = 0. \quad (3.59)$$

Table 3.1 shows some possible values of β calculated from (3.59) such that $\mu_\chi = [0]_{n \times 1}$, $\Sigma_{\chi\chi} = I_n$ and $\mathbb{M}_{\chi,j}^3 = 0$.

By using v with more than one degree of freedom, one can set the new minimum sigma set to achieve some properties in cases where each element of a random vector X has a different distribution. For instance, consider the case of matching the third central moments of a random variable

$$X \sim \left[N(0, 1), \chi^2(1) \right]^T.$$

The sigma set of [57] is unable to attain this property [see (3.32) and the comments following it] whilst the sigma set of Theorem 3.2 is able. Indeed, if $v = [v_1, v_2]^T$ where $v_1 = 1$, v_2 is a real root of the polynomial

$$f(v_1, v_2) = v_2^4 + 4v_2^3 + 8v_2 - 4$$

(e.g. $v_2 = 0.4494$), and $(I_n + vv^T)^{1/2}$ is the lower triangular Cholesky factor, then

$$\mathbb{M}_{\chi,1}^3 = 0 = M_{x_1}^3,$$

and

$$\mathbb{M}_{\chi,2}^3 = 8 = M_{X_2}^3.$$

This example shows one of the benefits comparative to the sigma set of [57], which is of having the tuning parameter v with n degrees of freedom while the one of the sigma set of [57], the scalar w_{n+1} , has only one. This fact gives the new sigma set the possibility of achieving some properties in cases where the sigma set of [57] fails to do so. Besides the sigma set of Theorem 3.2 is a generalization of this other set (cf. Corollary 3.5).

In comparison to the symmetric sigma sets, such as the ones of [2], [1] and [41], the new sigma set will be the preferred choice when the prior distribution is not symmetric because all of these sets offer the same estimate quality (moments are matched up to the second order), but the symmetric sets require more computational effort. For the symmetric prior distribution case, the designer has a trade-off choice involving computational effort and precision. The new sigma set requires less computational effort, but offers less precision.

Comparing with the sigma sets of [45], [46], and [83], the one of Theorem 3.2 bares the advantage of matching both the mean and the covariance of the prior random variable even for values of n greater than one (see Section 2.5). Besides, the sigma set of [45] has a numerical instability problem for high values of n [46], which is another disadvantage of this sigma set in comparison to the new sigma set. It should be noted that the new minimum sigma set is neither a particular case nor a generalization of the sigma sets of [45], [46], and [83].

Note that the $\sigma\mathbb{R}$ from Theorem 3.2 is currently the only consistent $\sigma\mathbb{R}$ constituted of less than $2n$ points, given that the set of [57] is a particular case (cf. Corollary 3.5) and, to the best of our knowledge, the other reduced sets do not fit Definition 3.1—they do not match the mean and/or the covariance of the prior distribution (see Section 2.5). Numerical simulations comparing all the sigma sets constituted of less than $2n$ points are given in Section 4.4 because these results are more illustrative when

analyzed along with the results of their UT's.

Finally, due to the restriction $v_i \neq 0$, χ cannot have a sigma point equal to \bar{X} and, therefore, from Theorem 2.1, the SUT of [44] cannot be applied to the $\text{Mi}\sigma\text{R}$. For a similar reason, one cannot also obtain a scaled version of the Rho Minimum σR with the SUT of [44] (See Section 2.6.1). We will show that our definition for the Scaled Unscented Transformation can be used for these σR 's (Section 4.2).

3.5 CONCLUSIONS REGARDING σ -REPRESENTATIONS

Motivated by the problem of estimating the expected value of a transformed random vector (Section 3.1), we proposed the *lth order N points σ -representation* ($l\text{th}N\sigma\text{R}$, Definition 3.1); essentially, a set χ is an $l\text{th}N\sigma\text{R}$ of a random vector X if the sample moments of χ (of order 1 to l) are equal to the moments of X —a $l\text{th}N\sigma\text{R}$ can also be seen as the mapping X (or its moments) to a set χ with these properties.

By proposing a matrix form of the $l\text{th}N\sigma\text{R}$'s (Theorem 3.1), we discovered some key properties of these representations, to know:

1. the *minimum number* of sigma points of an $l\text{th}N\sigma\text{R}$ (Corollary 3.1);
2. the *minimum number* of sigma points of an *symmetric* $l\text{th}N\sigma\text{R}$ (Corollary 3.1);
3. the form of the $l\text{th}N\sigma\text{R}$ of a the random vector $Z = aX + b$ when the $l\text{th}N\sigma\text{R}$ of X is known (Corollary 3.2); with this, the $l\text{th}N\sigma\text{R}$ of a random vector Z with mean \bar{Z} e moments M_1, \dots, M_l can be found by *first calculating the $l\text{th}N\sigma\text{R}$ of a (simpler) random vector X* ; e.g. X with mean equal to zero , and (even) moments equal to identity matrices.

Lead by the results 1. and 2., we found closed forms of some $l\text{th}N\sigma\text{R}$, namely i) closed forms for the minimum symmetric $2\text{th}N\sigma\text{R}$'s (Section 3.3), and ii) a closed form for the minimum $2\text{th}N\sigma\text{R}$ (Section 3.4).

One of the closed forms of the minimum symmetric $2\text{th}N\sigma\text{R}$'s (the Homogeneous Minimum Symmetric σR , Corollary 3.4) is equivalent to the classical symmetric sigma sets of [1, 2] (Table 2.1); therefore, with this we show the reasons behind these sigma sets which, until now, was based only on intuitive ideas. In fact, heretofore, it was not even known that these sigma sets are composed by the smallest amount of symmetric sigma points.

As for the closed form for the minimum $2\text{th}N\sigma\text{R}$ (Theorem 3.2), it turned out to be *the only existing consistent minimum $2\text{th}N\sigma\text{R}$* ; we showed that this $2\text{th}N\sigma\text{R}$ is a

general case of the only consistent minimum 2th $N\sigma$ R *of the literature* (Corollary 3.5).

The initial motivational problem of estimating the expected value of a *transformed* random vector still persists. A solution to this problem is actually given by the Unscented Transformations.

4. UNSCENTED TRANSFORMATIONS

The concept of an UT follows naturally from the one of σ -representations. The σ -representation's goal is to approximate a random vector, while the UT's goal is to approximate a *transformed* random vector.

There are many ways to approximate a transformed random vector. An UT, particularly, does it by using a σ -representation of the previous random vector. Therefore, we can say that the approximation of an UT is based on matching the moments of an random vector—recall that a σ -representation is defined as being a weighted set matching the moments, up to a certain order, of a given random vector.

Even though definitions for the UT already exist in the literature, in Chapter 2 we showed that they present some drawbacks. Therefore, in Chapter 4, we present a new *definition of the UT* (Definition 4.1). Among other advantages comparative with the UT's for the literature, our UT is more general; it is defined for any order l (the order of the used l th $N\sigma$ R), while as far as our knowledge goes, the higher UT's order of the literature is 5 (the UT of [47]).

Based on Taylor Series expansions, we provide *the estimation quality of an l th order UT* (Theorem 4.1)—recall, from Chapter 2, that there were some errors in the UT's estimation quality, and that some UT's elements' estimation accuracy, such as the cross-covariances', were not yet determined.

Further, we propose new definitions for i) the scaled UT variants in Section 4.2, and ii) for the square-root UT variants in Section 4.3—recall, from Chapter 2, that also all these UT variants need to be corrected in some way. We are able to show that our definitions of scaled UT's and square-root UT are particular cases of our UT definition in Section 4.1. With this result, the properties already developed for the UT are naturally extended to the scaled and square-root variants. Moreover, we present an analysis of the influence of the scaling parameter on the estimation quality of the scaled UT variants, and introduce a scaled square-root UT variant.

In Section 4.4, some properties of the UT's developed in this chapter are verified in numerical simulations.

4.1 UNSCENTED TRANSFORMATION

In this section, a new definition for the Unscented Transformation is proposed. In general terms, an UT consists of two sets of weighted points (the sigma points) approximating the joint pdf of two random vectors in the case where there is a functional dependence between them.

For the remaining of this chapter, consider, for a natural number $l \geq 2$, the random vectors

$$X \sim (\bar{X}, M_X^2, \dots, M_X^l)^n$$

and

$$Y := f(X) \in \Phi^{n_y}.$$

For i) the vectors λ^η such that

$$\lambda^\eta \in \{\chi_1, \dots, \chi_N, \gamma_1, \dots, \gamma_N\}, \quad \text{for each } \eta = 1, 2, \dots, l;$$

and ii) the sets

$$\chi := \left\{ \chi_i, w_i^{(\mathbf{m}_\chi)}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \mid \chi_i \in \mathbb{R}^n; \right. \\ \left. w_i^{(\mathbf{m}_\chi)}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \in \mathbb{R} \right\}_{i=1}^N,$$

and

$$\gamma := \left\{ \gamma_i, w_i^{(\mathbf{m}_\gamma)}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \mid \gamma_i = f(\chi_i) \right\}_{i=1}^N;$$

define a) the sample means by

$$\mu_\chi := \sum_{i=1}^N w_i^{(\mathbf{m}_\chi)} \chi_i, \\ \mu_\gamma := \sum_{i=1}^N w_i^{(\mathbf{m}_\gamma)} \gamma_i; \quad (4.1)$$

b) the sample central moments for even j by

$$\mathbb{M}_\chi^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\chi_1, \dots, \chi_j}^j)} [(\chi_i - \mu_\chi) (\diamond)^T]^{\otimes \frac{j}{2}}, \quad (4.2)$$

$$\mathbb{M}_\gamma^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\gamma_1, \dots, \gamma_j}^j)} [(\gamma_i - \mu_\gamma) (\diamond)^T]^{\otimes \frac{j}{2}}, \quad (4.3)$$

$$\mathbb{M}_{\lambda^1 \dots \lambda^j}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^j}^j)} \bigotimes_{q=1}^{j/2} \left[(\lambda_i^q - \mu_{\lambda^q}) (\lambda_i^{q+1} - \mu_{\lambda^{(q+1)}})^T \right]; \quad (4.4)$$

and c) the sample central moments for odd j by

$$\mathbb{M}_{\chi}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\chi^1 \dots \chi^j}^j)} \left[(\chi_i - \mu_{\chi}) (\diamond)^T \right]^{\otimes \frac{j-1}{2}} \otimes (\chi_i - \mu_{\chi}), \quad (4.5)$$

$$\mathbb{M}_{\gamma}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\gamma^1 \dots \gamma^j}^j)} \left[(\gamma_i - \mu_{\gamma}) (\diamond)^T \right]^{\otimes \frac{j-1}{2}} \otimes (\gamma_i - \mu_{\gamma}), \quad (4.6)$$

$$\mathbb{M}_{\lambda^1 \dots \lambda^j}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^j}^j)} \bigotimes_{q=1}^{(j-1)/2} \left[(\lambda_i^q - \mu_{\lambda^q}) (\lambda_i^{q+1} - \mu_{\lambda^{(q+1)}})^T \right] \otimes (\lambda_i^j - \mu_{\lambda^j}). \quad (4.7)$$

Definition 4.1 (Unscented Transformation). Consider equations (4.1)-(4.7). If

$$\mu_{\chi} = \bar{X},$$

and

$$\mathbb{M}_{\chi}^j = M_X^j, \quad j = 2, \dots, l;$$

then the l th order Unscented Transformation (lUT) is defined by

$$lUT \left(f, \bar{X}, M_X^2, \dots, M_X^l \right) := \left[\mu_{\gamma}, \mathbb{M}_{\gamma}^2, \dots, \mathbb{M}_{\gamma}^l, \mathbb{M}_{\lambda^1 \dots \lambda^2}^2, \dots, \mathbb{M}_{\lambda^1 \dots \lambda^l}^l \right].$$

Remark 4.1. Every l th $N\sigma R$ is a set χ of an lUT .

This form of defining the lUT as a function mapping (f, \bar{X}, P_{XX}) to the transformed sample mean and covariances can also be used in Monte Carlo and quadrature methods. Moreover, one should notice that negative weights can lead to negative values of the sample moments.

Broadly, an lUT can be viewed as a mapping from 2 random vectors X and Y with a functional dependence $[Y = f(X)]$ to 2 sets (composed of weighted points) χ and γ acting as a discrete approximation of the *joint* pdf of (X, Y) . For instance, a 2UT can be viewed as the following approximation (this interpretation is inspired on [52])

$$\begin{pmatrix} X \\ Y \end{pmatrix} \approx \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \sim \left(\begin{pmatrix} \mu_{\chi} \\ \mu_{\gamma} \end{pmatrix}, \begin{pmatrix} \Sigma_{\chi\chi} & \Sigma_{\chi\gamma} \\ \Sigma_{\chi\gamma}^T & \Sigma_{\gamma\gamma} \end{pmatrix} \right).$$

The next theorem states the approximation quality for the Y 's. The notation $Y^{[c,l]}$ stands for the Taylor Series of $Y = f(X)$ around $X = c$ truncated at the l th order.

Theorem 4.1 (Unscented Transformation's estimation quality). *Consider Definition 4.1 and let χ be a normalized l th $N\sigma R$ of X . If*

- i. $\mu_\chi = \bar{X}$;
- ii. $\mathbb{M}_\chi^j = M_X^j$ for $j = 2, \dots, l$;
- iii. and f is l th differentiable;

then:

1. $\mu_\gamma^{[\mu_\chi, l]} = \bar{Y}^{[\bar{X}, l]}$;
2. $\Sigma_{\gamma\gamma}^{[\mu_\chi, l/2]} = P_{YY}^{[\bar{X}, l/2]}$ if l is even,
 $\Sigma_{\gamma\gamma}^{[\mu_\chi, (l-1)/2]} = P_{YY}^{[\bar{X}, (l-1)/2]}$ if l is odd;
3. $\Sigma_{\chi\gamma}^{[\mu_\chi, l-1]} = P_{XY}^{[\bar{X}, l-1]}$.

Proof. Suppose $\mu_\chi = \bar{X}$ and $\mathbb{M}_\chi^p = M_X^p$, $\forall p = 2, \dots, l$ and that f is l th differentiable. The first assertion is proven by¹

$$\begin{aligned} \mu_\gamma^{[\mu_\chi, l]} &= f(\mu_\chi) + \frac{1}{2!} \sum_{i_1, i_2=1}^n \left(\mathbb{M}_\chi^2 \right)_{i_1, i_2} \frac{\partial^2 f(x)}{\partial x^{(i_1)} \partial x^{(i_2)}} \Big|_{x=\mu_\chi} + \dots + \\ &\quad \frac{1}{l!} \sum_{i_1, \dots, i_l=1}^n \left(\mathbb{M}_\chi^l \right)_{i_1, (i_2 * i_3 * \dots * i_l)} \frac{\partial^l f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi} =: \bar{Y}^{[\bar{X}, l]}. \end{aligned}$$

In order to prove the second assertion, note that

$$\Sigma_{\gamma\gamma}^{[l/2, \mu_\chi]} = \Theta_{\Sigma\gamma\gamma}^2 + \dots + \Theta_{\Sigma\gamma\gamma}^{l/2},$$

where

$$\begin{aligned} \Theta_{\Sigma\gamma\gamma}^q &= \\ &\sum_{j=1}^{q-1} \left[\frac{1}{j!q!} \sum_{i_1, \dots, i_{(q+j)}=1}^n \left(\left(\mathbb{M}_\chi^{q+j} \right)_{i_1, (i_2 * \dots * i_{(q+j)})} - \left(\mathbb{M}_\chi^q \right)_{i_1, (i_2 * \dots * i_q)} \left(\mathbb{M}_\chi^j \right)_{i_{(l/2+1)}, (i_{(q+2)} * \dots * i_{(q+j)})} \right) \right. \\ &\times \left(\frac{\partial^q f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_q)}} \Big|_{x=\mu_\chi} \frac{\partial^j f^T(x)}{\partial x^{(i_{(q+1)})} \dots \partial x^{(i_{(q+j)})}} \Big|_{x=\mu_\chi} \right. \\ &\left. \left. + \frac{\partial^j f(x)}{\partial x^{(i_{(q+1)})} \dots \partial x^{(i_{(q+j)})}} \Big|_{x=\mu_\chi} \frac{\partial^q f^T(x)}{\partial x^{(i_1)} \dots \partial x^{(i_q)}} \Big|_{x=\mu_\chi} \right) \Big] \\ &+ \frac{1}{q!q!} \sum_{\substack{i_1, \dots, \\ i_{2q}=1}}^n \left(\left(\mathbb{M}_\chi^{2q} \right)_{i_1, (i_2 * \dots * i_{2q})} - \left(\mathbb{M}_\chi^q \right)_{i_1, (i_2 * \dots * i_q)} \left(\mathbb{M}_\chi^q \right)_{i_{(q+1)}, (i_{(q+2)} * \dots * i_{2q})} \right) \end{aligned}$$

¹Recall from Chapter 1.2 that $(\mathbb{M}_\chi^2)_{i_1, i_2}$ stand for the i_1 th row and i_2 th column element of the matrix \mathbb{M}_χ^2 .

$$\times \frac{\partial^q f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_q)}} \Big|_{x=\mu_\chi} \frac{\partial^q f^T(x)}{\partial x^{(i_{q+1})} \dots \partial x^{(i_{2q})}} \Big|_{x=\mu_\chi}. \quad (4.8)$$

For the third assertion, note that

$$\Sigma_{\chi\gamma}^{[\mu_\chi, l-1]} = \Theta_{\Sigma_{\chi\gamma}}^1 + \dots + \Theta_{\Sigma_{\chi\gamma}}^{l-1},$$

where

$$\Theta_{\Sigma_{\chi\gamma}}^q = \frac{1}{q!} \sum_{i_1, \dots, i_q=1}^n \left[\left(\mathbb{M}_\chi^{q+1} \right)_{1, (i_1 * \dots * i_q)}, \dots, \left(\mathbb{M}_\chi^{q+1} \right)_{n, (i_1 * \dots * i_q)} \right]^T \frac{\partial^q f^T(x)}{\partial x^{(i_1)} \dots \partial x^{(i_q)}} \Big|_{x=\mu_\chi}.$$

The remaining steps can be proven similarly. \square

Note that the approximations of the posterior random variable of Theorem 4.1 are *not* guaranteed for any function f (unlike the literature state; cf. [1]), but only for the l th differentiable ones. This theorem is the *first* to provide the estimation quality of the cross-covariance, which is of the order $l - 1$ (item 3). Thus for $l = 2$, the transformed covariance is approximated up to order 1; this solves the problem in the Unscented's literature pointed out in Section 2.4.2.

Furthermore, according to Theorem 4.1, a sufficient condition for a second order approximation of the transformed covariance is

$$l = 4,$$

since, from item 2 for even l ,

$$\Sigma_{\gamma\gamma}^{[\mu_\chi, l/2]} = P_{YY}^{[\bar{X}, l/2]},$$

this solves the problem in the Unscented's literature pointed out in Section 2.4.1. In order to verify this result, suppose $\mu_\chi = \bar{X}$ and $\mathbb{M}_\chi^l = M_X^l, \forall i = 2, \dots, 4$; then, from (4.8),

$$\Sigma_{\gamma\gamma}^{[\mu_\chi, 2]} = \Theta_{\Sigma_{\gamma\gamma}}^1 + \dots + \Theta_{\Sigma_{\gamma\gamma}}^2 = 2n = P_{YY}.$$

Moreover, consider $X \sim N(0_{n \times 1}, I_n)$,

$$Y := f(X) = [x_1^3, \dots, x_n^3]^T,$$

and suppose that $\mu_\chi = \bar{X}$ and $\mathbb{M}_\chi^l = M_X^l, \forall i = 2, \dots, 6$. Then,

$$\Sigma_{\gamma\gamma}^{[\mu_\chi, 3]} = \Theta_{\Sigma_{\gamma\gamma}}^1 + \dots + \Theta_{\Sigma_{\gamma\gamma}}^3 = 15n = P_{YY}.$$

This result does not imply that the mean and covariance estimates of a 2UT are

equal to the ones obtained through linearization. We can point out two reasons. First, for a 2UT,

$$\mu_{\gamma}^{[\mu_X, 2]} = \mu_Y^{[\bar{X}, 2]};$$

whereas for linearization,

$$\mu_X^{[\mu_X, 1]} = \mu_Y^{[\bar{X}, 1]}$$

Second, even though both linearization and 2UT have

$$\Theta_{\Sigma_{\gamma\gamma}}^1 = \Theta_{P_{YY}}^1,$$

it happens that, from (4.8), $\Theta_{\Sigma_{\gamma\gamma}}^2$ and $\Theta_{P_{YY}}^2$ are partially equal for a 2UT, but not for linearization ($\Theta_{\Sigma_{\gamma\gamma}}^2 = 0$).

Finally, note that the estimation quality of the transformed statistics of the sigma sets of [45] and [46] are not given by Theorem 4.1 (this is illustrated numerically in Section 4.4). Since these sigma sets are not σR 's—they do not match the mean and/or the covariance of the prior distribution (see Section 2.5)—, they do not compose a 2UT in the sense of Definition 4.1. This elucidates the problems in the UT's literature pointed out in Section 2.5.

4.2 SCALED UNSCENTED TRANSFORMATION

In this section, the Scaled Unscented Transformation (ScUT) is refined—we use the acronym ScUT referring to our definition of the Scaled Unscented Transformation, and SUT to the Scaled Unscented Transformation of [44] (second column of Table 2.2)—. This new definition is based on the AuxUT of [44] (third column of Table 2.2), and is less conservative than the SUT of [44]; this SUT can not be applied to *any* previous sigma set (see Section 2.6), but the ScUT can.

Definitions similar to the SUT of [44] and the UT of [41] (Tab 2.1 [4,1]) are presented at the end of this section as particular cases of the ScUT.

Unless otherwise specified, the term Scaled Unscented Transformation will henceforth refer to the following definition.

Definition 4.2 (Scaled Unscented Transformation). Consider, for $\alpha \in (0, 1]$ and $\kappa \in (0, 1]$, the scaling function

$$g(f, X, b, \alpha, \kappa) := \frac{f(b + \alpha(X - b)) - f(b)}{\kappa} + f(b), \quad (4.9)$$

and the sets

$$\chi := \{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\gamma := \left\{ \gamma_i, w_i^m, w_i^c, w_i^{cc} \mid \gamma_i = g\left(f, \chi_i, \mu_\chi, \alpha, \alpha^2\right) \right\}_{i=1}^N$$

with

$$\begin{aligned} \Sigma_{\gamma\gamma}^\alpha &:= \alpha^2 \sum_{i=0}^N w_i^c (\gamma_i - \mu_\gamma) (\gamma_i - \mu_\gamma)^T, \\ \Sigma_{\chi\gamma}^\alpha &:= \alpha \sum_{i=0}^N w_i^{cc} (\chi_i - \mu_\chi) (\gamma_i - \mu_\gamma)^T, \end{aligned} \tag{4.10}$$

where $\Sigma_{\gamma\gamma}^\alpha$ is the *scaled sample covariance* of γ and $\Sigma_{\chi\gamma}^\alpha$ is the *scaled sample cross covariance* of χ and γ . If

$$\mu_\chi = \bar{X}$$

and

$$\Sigma_{\chi\chi} = P_{XX},$$

then the *Scaled Unscented Transformation (ScUT)* is defined by

$$\text{ScUT}(f, \bar{X}, P_{XX}, \alpha) := [\mu_\gamma, \Sigma_{\gamma\gamma}^\alpha, \Sigma_{\chi\gamma}^\alpha].$$

Remark 4.2. Every 2thN σ R is a set χ of a ScUT.

Such a definition for the (scaled) cross-covariance of the ScUT cannot be found for the scaled UT's of the literature; this solves part of the problem in the UT literature pointed out in Section 2.6.3. Crossing covariances are not treated in the SUT of [44] nor in the AuxUT of [44]. In the UT of [41], the cross-covariance is defined differently and restricted only to the symmetric set defined there (see Section 2.6.3).

In Sections 2.6.1 and 3.4, we showed that the SUT could not be used for the Mi σ R and for the RhoMi σ R, but here we provide an example showing that the ScUT can. In fact, for

$$\begin{aligned} X &\sim N([1]_{2 \times 1}, I_2), \\ f(X) &:= X^T X, \\ v &= [1, 1]^T, \\ \alpha &= 10^{-3}, \end{aligned}$$

and χ defined as in Corollary 3.5, it follows that

$$[\chi_1 \quad \chi_2 \quad \chi_3] = \begin{bmatrix} 2.37 & 0.63 & 0.00 \\ 0.63 & 2.37 & 0.00 \end{bmatrix},$$

and

$$w_1 = w_2 = w_3 = \frac{1}{3}.$$

The sample statistics for the set

$$\gamma = \{\gamma_i, w_i | \gamma_i = f(\chi_i)\}$$

for a (non-scaled) UT are

$$\begin{aligned} \mu_\gamma &= 4.00, \\ \Sigma_{\gamma\gamma} &= 8.00, \\ \Sigma_{\chi\gamma} &= 2.00; \end{aligned}$$

and for the set

$$\xi = \{\xi_i, w_i | \xi_i = g(\chi_i, \mu_\chi, \alpha, \alpha^2, f)\}$$

for the new SUT are

$$\begin{aligned} \mu_\xi &= 4.00, \\ \Sigma_{\xi\xi}^\alpha &= 8.00, \\ \Sigma_{\chi\xi}^\alpha &= 2.00. \end{aligned}$$

This shows that the ScUT is suited for more sets of sigma points than the SUT, and this solves the problem in the Unscented's literature pointed out in Section 2.6.1. As expected from Remark 4.2 and Theorem 4.1, the results of mean, covariance and cross-covariance are the same for both the UT and the ScUT for this case.

In Sections 2.6.2 and 2.6.3, we showed that α modifies the second order terms of both $\Sigma_{\gamma\gamma}^\alpha$ and $\Sigma_{\chi\gamma}^\alpha$. In order to check the influence of α in the covariances of the ScUT, define $\Theta_{\Sigma_{\gamma\gamma}^\alpha}^l$ and $\Theta_{\Sigma_{\chi\gamma}^\alpha}^l$ as the l th term of the Taylor Series of $\Sigma_{\gamma\gamma}^\alpha$ and $\Sigma_{\chi\gamma}^\alpha$, respectively. Then, we have

$$\begin{aligned} \mu_\gamma &= f(\mu_\chi) + \sum_{i_1, i_2=1}^n \left(\mathbb{M}_\chi^2 \right)_{i_1, i_2} \frac{\partial^2 f(x)}{\partial x^{(i_1)} \partial x^{(i_2)}} \Big|_{x=\mu_\chi} \\ &\quad + \dots + \alpha^{l-2} \sum_{i_1, \dots, i_l=1}^n \left(\mathbb{M}_\chi^l \right)_{i_1, i_2^* \dots^* i_l} \frac{\partial^l f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi} + \dots, \end{aligned} \quad (4.11)$$

$$\begin{aligned}
\Theta_{\Sigma_{\gamma\gamma}^\alpha}^l = & \sum_{j=1}^{l-1} \left[\frac{\alpha^{j+l-2}}{j!l!} \sum_{i_1, \dots, i_{l+j}=1}^n \left(\left(\mathbb{M}_\chi^{l+j} \right)_{i_1, (i_2^* \dots^* i_{l+j})} - \left(\mathbb{M}_\chi^l \right)_{i_1, (i_2^* \dots^* i_l)} \left(\mathbb{M}_\chi^j \right)_{i_{(l+1)}, (i_{(l+2)}^* \dots^* i_{(l+j)})} \right) \right. \\
& \times \left(\frac{\partial^l f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi} \frac{\partial^j f^T(x)}{\partial x^{(i_{l+1})} \dots \partial x^{(i_{l+j})}} \Big|_{x=\mu_\chi} \right. \\
& \left. \left. + \frac{\partial^j f(x)}{\partial x^{(i_{l+1})} \dots \partial x^{(i_{l+j})}} \Big|_{x=\mu_\chi} \frac{\partial^l f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi}^T \right) \right] \\
& + \frac{\alpha^{2l-2}}{l!l!} \sum_{i_1, \dots, i_{2l}=1}^n \left(\left(\mathbb{M}_\chi^{2l} \right)_{i_1, (i_2^* \dots^* i_{2l})} - \left(\mathbb{M}_\chi^l \right)_{i_1, (i_2^* \dots^* i_l)} \left(\mathbb{M}_\chi^l \right)_{i_{(l+1)}, (i_{(l+2)}^* \dots^* i_{2l})} \right) \\
& \times \frac{\partial^l f(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi} \frac{\partial^l f^T(x)}{\partial x^{(i_{(l+1)})} \dots \partial x^{(i_{2l})}} \Big|_{x=\mu_\chi}, \tag{4.12}
\end{aligned}$$

$$\Theta_{\Sigma_{\chi\gamma}^\alpha}^l = \alpha^{l-1} \frac{1}{l!} \sum_{i_1, \dots, i_l=1}^n \left[\left(\mathbb{M}_\chi^{l+1} \right)_{1, (i_1^* \dots^* i_l)} , \dots , \left(\mathbb{M}_\chi^{l+1} \right)_{n, (i_1^* \dots^* i_l)} \right]^T \frac{\partial^l f^T(x)}{\partial x^{(i_1)} \dots \partial x^{(i_l)}} \Big|_{x=\mu_\chi}. \tag{4.13}$$

The ScUT scales the terms of order 3 and higher for μ_γ and of order 2 and higher for $\Sigma_{\gamma\gamma}^\alpha$ and $\Sigma_{\chi\gamma}^\alpha$. However, if χ is symmetric, then

$$\mathbb{M}_\chi^3 = [0]_{n \times 2n} \Rightarrow \Theta_{\Sigma_{\chi\gamma}^*}^3 = [0]_{n \times 1}$$

and α does not modify the second order of $\Sigma_{\chi\gamma}^*$ (cf. Section 2.6.3). The next theorem gives the estimation quality of the ScUT.

Theorem 4.2 (ScUT's estimation quality). *Consider Definition 4.2 and let χ be a normalized σR of X . If:*

- i. $\mu_\chi = \bar{X}$;
- ii. $\Sigma_{\chi\chi} = P_{XX}$;
- iii. and f is 2nd order differentiable;

then:

1. $\mu_\gamma^{[\mu_\chi, 2]} = \bar{Y}^{[\bar{X}, 2]}$,
2. $\Sigma_{\gamma\gamma}^{\alpha, [\mu_\chi, 1]} = P_{YY}^{[\bar{X}, 1]}$,
3. $\Sigma_{\chi\gamma}^{\alpha, [\mu_\chi, 1]} = P_{XY}^{[\bar{X}, 1]}$.

Furthermore, if X and χ are symmetric, then

$$\Sigma_{\chi\gamma}^{\alpha, [\mu_\chi, 2]} = P_{XY}^{[\bar{X}, 2]}.$$

Proof. Suppose $\mu_\chi = \bar{X}$, $\Sigma_{\chi\chi} = P_{XX}$ and that f is a 2nd order differentiable function. For the first assertion, we have that

$$\mu_\gamma^{[\mu_\chi, 2]} = f(\mu_\chi) + \frac{1}{2!} \sum_{i_1, i_2=1}^n (\mathbb{M}_\chi^2)_{i_1, i_2} \left. \frac{\partial^2 f(x)}{\partial x^{(i_1)} \partial x^{(i_2)}} \right|_{x=\mu_\chi} = \bar{Y}^{[\bar{X}, 2]},$$

which proves the first assertion. For the second assertion, we have that

$$\Sigma_{\gamma\gamma}^{\alpha, [\mu_\chi, 1]} = \sum_{i, j=1}^n (\Sigma_{\chi\chi})_{i, j} \left. \frac{\partial f(x)}{\partial x^{(i)}} \right|_{x=\mu_\chi} \left. \frac{\partial f^T(x)}{\partial x^{(j)}} \right|_{x=\mu_\chi} = P_{YY}^{[\mu_\chi, 1]}.$$

For the third assertion, we have that

$$\Sigma_{\chi\gamma}^{\alpha, [\mu_\chi, 1]} = \sum_{i=1}^n [(\Sigma_{\chi\chi})_{1, i}, \dots, (\Sigma_{\chi\chi})_{n, i}]^T \left. \frac{\partial f^T(x)}{\partial x^{(i)}} \right|_{x=\mu_\chi} = P_{XY}^{[\bar{X}, 1]}.$$

For the last assertion, note that X symmetric implies

$$M_\chi^3 = [0]_{n \times 2n} \Rightarrow \Theta_{P_{XY}}^3 = [0]_{n \times 1},$$

and χ symmetric implies

$$\mathbb{M}_\chi^3 = [0]_{n \times 2n} \Rightarrow \Theta_{\Sigma_{\chi\gamma}^\alpha}^3 = [0]_{n \times 1}.$$

□

Similar to the 2UT, the covariance of the transformed random variable is estimated only up to first order. Theorem 4.2 gives, for the first time, the estimation quality of the sample cross-covariance. These results showing 1) the degree of influence of the scale parameter α [equations (4.11), (4.12) and (4.13)], and 2) the ScUT's estimation quality (Theorem 4.2) solve the problems in the UT's literature pointed out in Sections 2.6.2 and 2.6.3. The next corollary states a new result.

Corollary 4.1. *A ScUT with sets*

$$\{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\left\{ \gamma_i, w_i^m, w_i^c, w_i^{cc} \mid \gamma_i = g(f, \chi_i, \mu_\chi, \alpha, \alpha^2) \right\}_{i=1}^N$$

is a 2UT with sets

$$\{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\{\gamma_i, w_i^m, w_i^{\alpha,c}, w_i^{\alpha,cc} | \gamma_i = g(f, \chi_i, \mu_\chi, \alpha, \alpha^2)\}$$

where

$$w_i^{\alpha,c} = \alpha^2 w_i^c$$

and

$$w_i^{\alpha,cc} = \alpha w_i^{cc}$$

are the weights to calculate the sample covariance and cross-covariance, respectively.

Because of the way these transformations were defined, every ScUT is a 2UT and, therefore, every result obtained for the 2UT can also be applied to the ScUT. We proceed by redefining the SUT of [44] and the UT of [41].

Definition 4.3 (Simplex Scaled Unscented Transformation). Let $\chi := \{\chi_i, w_i\}_{i=1}^N$ be a normalized σ R of X with $\chi_N = \bar{X}$. Consider the sets, for $\alpha \in (0, 1]$,

$$\chi' := \{\chi'_i, w'_i | \chi'_i = \bar{X} + \alpha(\chi_i - \bar{X})\}_{i=1}^N$$

and

$$\gamma' := \{\gamma'_i, w'_i | \gamma'_i = f(\chi'_i)\}_{i=1}^N,$$

where

$$\begin{aligned} w'_N &:= \alpha^{-2} w_N + 1 - \alpha^{-2}; \\ w'_i &= \alpha^{-2} w_i, \quad i = 1, \dots, N-1. \end{aligned}$$

Define the modified sample covariance and the modified sample cross-covariance of γ' , respectively, as

$$\Sigma_{\gamma'\gamma'}^{\alpha\alpha} := \sum_{i=1}^N w'_i (\gamma'_i - \mu_{\gamma'}) (\diamond)^T + (1 - \alpha^2) (\gamma'_N - \mu_{\gamma'}) (\diamond)^T$$

and

$$\Sigma_{\chi'\gamma'} := \sum_{i=1}^N w'_i (\chi'_i - \mu_{\chi'}) (\gamma'_i - \mu_{\gamma'})^T.$$

Then the *Simplex Scaled Unscented Transformation (SiScUT)* is defined by

$$\text{SiScUT}(f, \bar{X}, P_{XX}, \alpha) := [\mu_{\gamma'}, \Sigma_{\gamma'\gamma'}^{\alpha\alpha}, \Sigma_{\chi'\gamma'}].$$

Definition 4.4 (Symmetric Intrinsically-Scaled Unscented Transformation). Choose $\alpha \in (0, 1]$ and $\kappa \in \mathbb{R}$ such that

$$\lambda := \alpha^2(n + \kappa) - n > -n.$$

Let $\chi := \{\chi_i, w_i\}_{i=1}^{2n+1}$ with $w_{2n+1} = \lambda/(n + \lambda)$ be a normalized HoMiSy σ R of X . Consider the sets

$$\tilde{\chi} := \{\tilde{\chi}_i, \tilde{w}_i^m, \tilde{w}_i^c, \tilde{w}_i^{cc} | \tilde{\chi}_i = \chi_i\}_{i=1}^{2n+1}$$

$$\tilde{\gamma} := \{\tilde{\gamma}_i, \tilde{w}_i^m, \tilde{w}_i^c, \tilde{w}_i^{cc} | \tilde{\gamma}_i = f(\chi_i)\}_{i=1}^{2n+1}$$

where

$$\begin{aligned} \tilde{w}_{2n+1}^m &= w_{2n+1}; \\ \tilde{w}_{2n+1}^c &= w_{2n+1} + (1 - \alpha^2); \\ \tilde{w}_{2n+1}^{cc} &= w_{2n+1} + (1 - \alpha); \\ \tilde{w}_i^m &= \tilde{w}_i^c = \tilde{w}_i^{cc} = w_i, \quad i = 1, \dots, 2n; \end{aligned}$$

Then the *Symmetric Intrinsically-Scaled Unscented Transformation* (SyInScUT) is defined by

$$\text{SyInScUT}(f, \bar{X}, P_{XX}, \alpha) := [\mu_{\tilde{\gamma}}, \Sigma_{\tilde{\gamma}\tilde{\gamma}}, \Sigma_{X\tilde{\gamma}}].$$

Corollary 4.2. If $\chi := \{\chi_i, w_i\}_{i=1}^N$ is a normalized σ R of X with $\chi_N = \bar{X}$, then

$$\text{SiScUT}(f, \bar{X}, P_{XX}, \alpha) = \text{ScUT}(f, \bar{X}, P_{XX}, \alpha);$$

and if $\chi := \{\chi_i, w_i\}_{i=1}^{2n+1}$ is a normalized HoMiSy σ R of X with $\chi_N = \bar{X}$, then

$$\text{SyInScUT}(f, \bar{X}, P_{XX}, \alpha) = \text{SiScUT}(f, \bar{X}, P_{XX}, \alpha) = \text{ScUT}(f, \bar{X}, P_{XX}, \alpha).$$

Proof. Let χ and γ be the sets of a ScUT (Definition 4.2) with $w_i^{cc} = w_i^c = w_i^m = w_i$. To prove first part, consider Definition 4.3. First, from (4.1),

$$\begin{aligned} \mu_{\tilde{\gamma}} &= (1 - w_N) \left(1 - \alpha^{-2}\right) \gamma'_N + \sum_{i=1}^{N-1} \left(\alpha^{-2} w_i \gamma'_i\right) + w_N \gamma'_N \\ &= \left[(1 - w_N) \left(1 - \alpha^{-2}\right) + w_N\right] \gamma'_N + \sum_{i=1}^{N-1} w_i \gamma'_i \\ &= \left[1 - \alpha^{-2} - w_N + w_N \alpha^{-2} + w_N\right] \gamma'_N + \sum_{i=1}^{N-1} w_i \gamma'_i \end{aligned}$$

$$\begin{aligned}
&= \left[1 - \alpha^{-2} + w_N \alpha^{-2}\right] \gamma'_N + \sum_{i=1}^{N-1} w'_i \gamma'_i \\
&= w'_N \gamma'_N + \sum_{i=1}^{N-1} w'_i \gamma'_i \\
&= \sum_{i=1}^N w'_i \gamma'_i \\
&=: \mu_{\gamma'}
\end{aligned}$$

Second, from (4.10),

$$\begin{aligned}
\Sigma_{\gamma\gamma}^\alpha &= \sum_{i=1}^{N-1} w'_i (\gamma'_i - \mu_{\gamma'}) (\diamond)^T + \alpha^{-2} (1 - \alpha^2) (\gamma'_N - \mu_{\gamma'}) (\diamond)^T \\
&\quad - \alpha^{-2} (\alpha^2 - 1)^2 (\mu_{\gamma'} - \gamma'_N) (\diamond)^T = \Sigma_{\gamma'\gamma'}^{\alpha\alpha}.
\end{aligned}$$

Third, from (4.10),

$$\Sigma_{\chi\gamma}^\alpha = \sum_{i=1}^N w'_i (\chi'_i - \mu_{\chi'}) (\gamma'_i - \mu_{\gamma'})^T = \Sigma_{\chi'\gamma'}.$$

The remaining steps of the first part are trivial.

To prove the second part, consider Definition 4.4 and define the set

$$\tilde{\zeta} := \{\tilde{\zeta}_i, \tilde{w}_i | \tilde{\zeta}_i = \gamma_i\}_{i=1}^{2n+1},$$

where

$$\begin{aligned}
\tilde{w}_{2n+1} &:= \alpha^{-2} w_{2n+1} + 1 - \alpha^{-2}, \\
\tilde{w}_i &:= \alpha^{-2} w_i, \quad i = 1, \dots, 2n,
\end{aligned}$$

and note that, from Definition 4.3, the function

$$\gamma(f, \bar{X}, P_{XX}, \alpha) := [\mu_{\tilde{\zeta}}, \Sigma_{\tilde{\zeta}\tilde{\zeta}}^\alpha, \Sigma_{\chi\tilde{\zeta}}]$$

is a SiScUT. Then it can easily be proven that $\mu_\gamma = \mu_{\tilde{\zeta}}$, $\Sigma_{\gamma\gamma} = \Sigma_{\tilde{\zeta}\tilde{\zeta}}^\alpha$ and $\Sigma_{\chi\gamma} = \Sigma_{\chi\tilde{\zeta}}$. \square

The SUT of [44] is incorporated in the SiScUT (Definition 4.3) with the difference that now it states the restriction of having a point located in the mean (cf. Section 2.6.1) and defines the sample cross-covariance (cf. Section 2.6.2). Besides, with Corollary 4.2, the SiScUT follows naturally as a particular case of the ScUT and, therefore, we also have the estimation quality of P_{YY} and P_{XY} and the influence of α on the estimate of P_{XY} (see Section 2.6). Definition 4.4 provides similar results for the UT

of [41] which we now define as SyInScUT. Summing up, we provide unified and consistent new definitions for all the scaled transformations. Besides, the results of this section solve the problems in the Unscented literature pointed out in Section 2.6.

4.3 SQUARE-ROOT UNSCENTED TRANSFORMATION

In this section, we state the results for the Square-Root Unscented Transformation (SRUT). As Section 2.7.3 pointed out, Definition 4.5 should be the first definition for an SRUT.

The key idea of an SRUT is to map the square-root matrix of the *previous* covariance *directly* (without squaring) to the square-root matrix of the *posterior* covariance. One way of doing it for

$$\Sigma_{\gamma\gamma}^{\gamma} := \Sigma_{\gamma\gamma} + \sqrt{\gamma}\sqrt{\gamma}^T$$

is by the function

$$\text{cu} : (S_{\gamma}^+, S_{\gamma}^-, \sqrt{\gamma}) \mapsto \sqrt{\Sigma_{\gamma\gamma}^{\gamma}}, \quad (4.14)$$

where, for a set $\gamma = \{\gamma_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$ with at least one positive weight w_i^c and one negative, the subsets γ_+ and γ_- are defined by

$$\begin{aligned} \gamma_+ &:= \left\{ \gamma_{(+,j_+)}, w_{(+,j_+)}^m, w_{(+,j_+)}^c, w_{(+,j_+)}^{cc} \right\}_{j_+=1}^{N_+} = \left\{ \gamma_i, w_i^m, w_i^c, w_i^{cc} \mid w_i^c \geq 0 \right\}_{i=1}^N, \\ \gamma_- &:= \left\{ \gamma_{(-,j_-)}, w_{(-,j_-)}^m, w_{(-,j_-)}^c, w_{(-,j_-)}^{cc} \right\}_{j_-=1}^{N_-} = \left\{ \gamma_i, w_i^m, w_i^c, w_i^{cc} \mid w_i^c < 0 \right\}_{i=1}^N, \end{aligned}$$

the matrices S_{γ}^+ and S_{γ}^- by

$$S_{\gamma}^+ := \left[\sqrt{w_{(+,1)}^c} \left(\gamma_{(+,1)} - \mu_{\gamma} \right), \dots, \sqrt{w_{(+,N_+)}^c} \left(\gamma_{(+,N_+)} - \mu_{\gamma} \right) \right], \quad (4.15)$$

$$S_{\gamma}^- := \left[\sqrt{\|w_{(-,1)}^c\|} \left(\gamma_{(-,1)} - \mu_{\gamma} \right), \dots, \sqrt{\|w_{(-,N_-)}^c\|} \left(\gamma_{(-,N_-)} - \mu_{\gamma} \right) \right]; \quad (4.16)$$

and $\sqrt{\Sigma_{\gamma\gamma}^{\gamma}}$ is calculated by the following algorithm:

1. $D = \text{tria} \left(\left[S_{\gamma}^+, \sqrt{\gamma} \right] \right)$;
2. If $N_- > 0$, $\sqrt{\Sigma_{\gamma\gamma}^{\gamma}} = \text{cdown} \{A, S_{\gamma}^-\}$; else, $\sqrt{\Sigma_{\gamma\gamma}^{\gamma}} = D$.

In this way, $\sqrt{\Sigma_{\gamma\gamma}^{\gamma}}$ can be obtained by first updating the Cholesky factor, and then downdating it. The former operation can be done by means of triangularization (e.g. the QR decomposition) $S = \text{tria}\{A\}$, $A \in \mathbb{R}^{n \times n}$, where S is *lower* triangular (see [80, 93]). The latter can be achieved through $S = \text{cdown}\{A, B\}$, for $B \in \mathbb{R}^{n \times n_y}$, representing the Cholesky downdating of A by B (it is the same as doing $\text{cholupdate}\{A, B_{*,i}, -1\}$ as in [42]).

If the set $\gamma = \{\gamma_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$ has no negative weights, then

$$\sqrt{\Sigma_{\gamma\gamma}} = \text{tria} \left([S_{\gamma}^+, \sqrt{\gamma}] \right) \quad (4.17)$$

and no Cholesky downdatings are performed. Since downdatings might lead to ill-conditioned matrices [92] (see Section 2.7.1), it should be avoided whenever possible—it is only necessary when the σR contains negative weights. With an abuse of notation, when we write $\text{cu}(S_{\gamma}^+, S_{\gamma}^-, \sqrt{\gamma})$ we also refer to the case when S_{γ}^- does not exist ($N_- = 0$); in this case we have $\text{cu}(S_{\gamma}^+, S_{\gamma}^-, \sqrt{\gamma}) = \text{tria} \left([S_{\gamma}^+, \sqrt{\gamma}] \right)$.

For now on, in this section, consider the random variable X characterized by the mean \bar{X} and square-root of the covariance $\sqrt{P_{XX}}$.

Definition 4.5 (Square-Root Unscented Transformation). Consider the sets

$$\chi = \{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\gamma = \{\gamma_i, w_i^m, w_i^c, w_i^{cc} | \gamma_i = f(\chi_i)\}_{i=1}^N$$

with

$$\mu_{\chi} = \bar{X} \text{ and } \Sigma_{\chi\chi} = \sqrt{P_{XX}} \sqrt{P_{XX}}^T .$$

Given a matrix $\sqrt{\gamma}$ and $S_{\chi}^+, S_{\chi}^-, S_{\gamma}^+, S_{\gamma}^-$ defined as in (4.15) and (4.16), the *Square-Root Unscented Transformation (SRUT)* is defined by

$$\text{SRUT} \left(f, \bar{X}, \sqrt{P_{XX}}, \sqrt{\gamma} \right) := \left[\mu_{\gamma}, \sqrt{\Sigma_{\gamma\gamma}}, S_{\chi}^+, S_{\chi}^-, S_{\gamma}^+, S_{\gamma}^-, \Sigma_{\chi\gamma} \right] .$$

Next, we introduce the Scaled Square-Root Unscented Transformation and some results concerning this transformation. This definition is necessary for the Scaled Square-Root Unscented Kalman Filters (see Table 5.3), the first one in the literature.

Definition 4.6 (Scaled Square-Root Unscented Transformation). Consider the sets χ and γ as in Definition 4.2 with

$$\mu_{\chi} = \bar{X} \text{ and } \Sigma_{\chi\chi} = \sqrt{P_{XX}} \sqrt{P_{XX}}^T .$$

Given a matrix $\sqrt{\gamma}$, define the matrix

$$\Sigma_{\gamma\gamma}^{\alpha\gamma} := \Sigma_{\gamma\gamma}^{\alpha} + \sqrt{\gamma} \sqrt{\gamma}^T ;$$

then the *Scaled Square-Root Unscented Transformation (ScSRUT)* is defined by

$$\text{ScSRUT} \left(f, \bar{X}, \sqrt{P_{XX}}, \sqrt{\gamma}, \alpha \right) := \left[\mu_\gamma, \sqrt{\Sigma^{\alpha\gamma}}, S_\chi^+, S_\chi^-, S_\gamma^+, S_\gamma^-, \Sigma_{\chi\gamma}^\alpha \right].$$

Corollary 4.3. *A ScSRUT with sets*

$$\chi = \{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\{\gamma_i, w_i^m, w_i^c, w_i^{cc} | \gamma_i = g(f, \chi_i, \mu_\chi, \alpha, \alpha^2)\}_{i=1}^N$$

is a SRUT with the sets

$$\chi = \{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$$

and

$$\{\gamma_i, w_i^m, \alpha^2 w_i^c, \alpha w_i^{cc}\}_{i=1}^N.$$

Remark 4.3. Every 2thNσR is a set χ of an SRUT.

Finally, we state new ScSRUT results similar to the ones in Section 4.2 for the particular scaled transformations.

Definition 4.7 (Simplex Scaled Square-Root Unscented Transformation). Consider the sets χ' and γ' as in Definition 4.3 with

$$\mu_{\chi'} = \bar{X} \text{ and } \Sigma_{\chi'\chi'} = \sqrt{P_{XX}} \sqrt{P_{XX}}^T.$$

Given a matrix $\sqrt{\gamma}$, define the matrix

$$\Sigma_{\gamma'\gamma'}^{\alpha\alpha\gamma} := \Sigma_{\gamma'\gamma'}^{\alpha\alpha} + \sqrt{\gamma} \sqrt{\gamma}^T;$$

then the *Simplex Scaled Square-Root Unscented Transformation (SiScSRUT)* is defined by

$$\text{SiScSRUT} \left(f, \bar{X}, \sqrt{P_{XX}}, \sqrt{\gamma}, \alpha \right) := \left[\mu_{\gamma'}, \sqrt{\Sigma_{\gamma'\gamma'}^{\alpha\alpha\gamma}}, S_{\chi'}^+, S_{\chi'}^-, S_{\gamma'}^+, S_{\gamma'}^-, \Sigma_{\chi'\gamma'} \right].$$

Definition 4.8 (Symmetric Intrinsically-Scaled Square-Root Unscented Transformation). Consider the sets $\tilde{\chi}$ and $\tilde{\gamma}$ as in Definition 4.4 with

$$\mu_{\tilde{\chi}} = \bar{X} \text{ and } \Sigma_{\tilde{\chi}\tilde{\chi}} = \sqrt{P_{XX}} \sqrt{P_{XX}}^T.$$

Given a matrix $\sqrt{\gamma}$, define the matrix

$$\Sigma_{\tilde{\gamma}\tilde{\gamma}}^\gamma := \Sigma_{\tilde{\gamma}\tilde{\gamma}} + \sqrt{\gamma} \sqrt{\gamma}^T;$$

then the *Symmetric Intrinsically-Scaled Square-Root Unscented Transformation (SyIn-ScSRUT)* is defined by

$$\text{SyInScSRUT} \left(f, \bar{X}, \sqrt{P_{XX}}, \sqrt{\gamma}, \alpha \right) := \left[\mu_{\tilde{\gamma}}, \sqrt{\Sigma_{\tilde{\gamma}\tilde{\gamma}}}, S_{\tilde{X}}^+, S_{\tilde{X}}^-, S_{\tilde{\gamma}}^+, S_{\tilde{\gamma}}^-, \Sigma_{\tilde{X}\tilde{\gamma}} \right].$$

Corollary 4.4. *Every SiScSRUT is an ScSRUT and every SyInScSRUT is an ScSRUT.*

4.4 COMPARISON OF SIGMA SETS WITH LESS THAN $2N$ SIGMA POINTS

In this section we compare the estimation quality of the main sigma sets² (SS's) composed by less than $2n$ sigma points, which are the (Normalized) Minimum σ -representation (Mi σ R) of Theorem 3.5, the Rho Minimum σ -representation of [57] (RhoMi σ R, Tab 2.1 [3,2]), the Reduced sigma set of [45] (ReSS, Tab 2.1 [2,1]) and the Spherical Simplex sigma set of [46] (SSSS, Tab 2.1 [2,2]). The Unscented Transformations of these sigma sets are also compared, they are: the Minimum Unscented Transformation (MiUT, a 2UT with the Mi σ R), the Rho Minimum Unscented Transformation (RhoMiUT, a 2UT with the RhoMi σ R), the Reduced Unscented Transformation (ReUT, a 2UT with the ReSS) and the Spherical Simplex Unscented Transformation (SSUT, a 2UT with the SSSS)³.

For the examples of this section we consider sigma sets of the random variable

$$X \sim N \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix} \right),$$

$v = [0.5, 0.5]^T$ is chosen as the tuning parameter of the new minimum sigma set, and $w_0 = 1/3$ for the other three sigma sets. Figure 4.1 shows the sigma points of each of these sigma sets; the compositions of these sigma sets are given below:

- the new minimum SS is

$$\left\{ \left(\begin{bmatrix} 8.04 \\ 5.93 \end{bmatrix}, 0.17 \right), \left(\begin{bmatrix} 0.29 \\ 9.63 \end{bmatrix}, 0.17 \right), \left(\begin{bmatrix} -0.58 \\ 3.61 \end{bmatrix}, 0.17 \right) \right\},$$

²Here we use the name of sigma sets and not σ -representation because two of these sets, the Reduced sigma set of [45] and the Spherical sigma set of [46], are not σ -representations (cf. Definition 3.1 and comments following it).

³In order to simplify the presentation of this section, we consider the UT's in the cases of the ReUT and the SSUT as relaxed variants of Definition 4.1 .

- the Rho Min. SS of [57] is

$$\left\{ \left(\begin{bmatrix} -2.16 \\ 2.22 \end{bmatrix}, 0.33 \right), \left(\begin{bmatrix} 7.32 \\ 4.12 \end{bmatrix}, 0.17 \right), \left(\begin{bmatrix} 1.00 \\ 7.14 \end{bmatrix}, 0.50 \right) \right\},$$

- the Reduced SS of [45] is

$$\left\{ \left(\begin{bmatrix} 1.00 \\ 5.00 \end{bmatrix}, 0.33 \right), \left(\begin{bmatrix} 0.09 \\ 4.20 \end{bmatrix}, 0.17 \right), \left(\begin{bmatrix} 1.00 \\ 5.62 \end{bmatrix}, 0.17 \right) \right\},$$

- and the Spherical SS of [46] is

$$\left\{ \left(\begin{bmatrix} 1.00 \\ 5.00 \end{bmatrix}, 0.33 \right), \left(\begin{bmatrix} -0.05 \\ 2.31 \end{bmatrix}, 0.22 \right), \left(\begin{bmatrix} 2.05 \\ 2.73 \end{bmatrix}, 0.22 \right), \left(\begin{bmatrix} 1.00 \\ 5.00 \end{bmatrix}, 0.22 \right) \right\}.$$

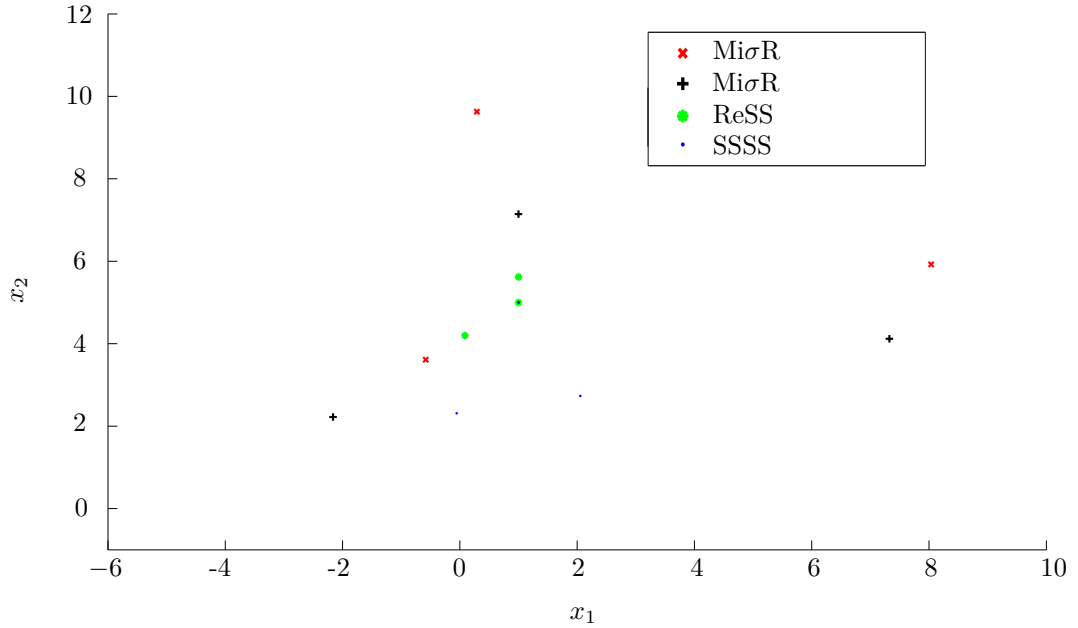


Figure 4.1: Geometry location in the \mathbb{R}^2 of the sigma points of the sigma sets composed by less than $2n$ sigma points.

Note that none of the sigma points of the $\text{Mi}\sigma\text{R}$ and the $\text{RhoMi}\sigma\text{R}$ are located on the mean $([1, 5]^T)$ while the other two sigma sets have sigma points located there. This verifies the fact that the SUT cannot be used with the $\text{Mi}\sigma\text{R}$ and the $\text{RhoMi}\sigma\text{R}$ because the SUT requires that the sigma set has a sigma point equal to \bar{X} (cf. Section 2.6.1 and the last paragraph of Section 3.4).

Table 4.1 shows the relative errors of the mean and the covariance generated by each of the aforementioned sigma sets in comparison to the mean and covariance of X .

Table 4.1: Relative errors of the sample mean and sample covariance for the main sigma sets composed by less than $2n$ sigma points in relation to the mean and covariance of X .

Mi σ R		RhoMi σ R		ReSS		SSSS	
mean	Cov. error	mean	Cov.	mean	Cov.	mean	Cov.
1.3×10^{-8}	2.3×10^{-8}	2.3×10^{-8}	1.9×10^{-8}	0.59	0.98	0.46	0.97

The main result relative to these data is that the relative errors of the previous means and covariances for the Mi σ R and the RhoMi σ R are almost zero (Tab 4.1 [1, 1-4]) while the ones for the ReSS and the SSSS are not (Tab 4.1 [1, 5-8]). The matching of the previous mean and covariance is important to assure that the sample mean and covariance of the posterior sigma points approximates well the mean and the covariance of the posterior random variable (cf. Theorem 4.1). In fact, in order to verify this, we compare the Unscented Transformations of the same four sigma sets by considering the transformation of a $X = [x_1, x_2]^T$ by the following functions:

$$\begin{aligned}
 f_1(x_1, x_2) &= x_1^2 + x_2^2, \\
 f_2(x_1, x_2) &= x_1^4 + x_2^4, \\
 f_3(x_1, x_2) &= e^{x_1} + e^{x_2}, \\
 f_4(x_1, x_2) &= x_1^{-1} + x_2^{-1}, \\
 f_5(x_1, x_2) &= \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ \arctan\left(\frac{x_2}{x_1}\right) \end{bmatrix}.
 \end{aligned}$$

Table 4.2 shows the errors concerning the transformed means and covariances of the sigma sets in comparison to the transformed mean and covariance of a 10^7 Monte Carlo simulation. The better performance of the MiUT and the RhoMiUT in comparison to the other two UT's is due to the property mentioned and verified above that their sigma sets match the first two moments of the previous random variable, whilst the sigma sets of the reduced UT of [45] and the spherical UT of [46] do not. Note that, as shown in Theorem 4.1, the matching of the mean and the covariance of X implies a second order approximation of the Taylor Series of the posterior mean which, for the case of f_1 (a second order polynomial), implied in a negligible error associated with the posterior mean of the MiUT (Tab 4.2 [2,2]) and of the RhoMiUT (Tab 4.2 [2,4]). Note also that, even though the tuning parameters were not set precisely for each function, the results for the the MiUT and the RhoMinUT are comparable. Nevertheless, for any case, there exists a suitable choice for v such that the MiUT provides the least errors since the RhoMi σ R is a particular case of the Mi σ R.

Table 4.2: Relative errors of the posterior sample mean and sample covariance for the main Unscented Transformations composed by less than $2n$ sigma points in relation to the mean and covariance of $f_i(X)$.

1	function	MiUT		RhoMiUT		ReUT		SSUT	
		mean	Cov.	mean	Cov.	mean	Cov.	mean	Cov.
2	f_1	9.5×10^{-8}	0.85	9.5×10^{-8}	0.63	0.76	0.96	0.75	0.95
3	f_2	0.59	0.76	0.14	0.87	0.87	1.00	0.89	0.99
4	f_3	0.66	0.97	0.77	1.00	1.00	1.00	1.00	1.00
5	f_4	1.3	1.00	0.65	1.00	2.20	1.00	2.90	1.00
6	f_5	0.22	0.80	0.18	0.54	0.66	0.80	0.55	0.86

4.5 CONCLUSIONS REGARDING UNSCENTED TRANSFORMATIONS

By looking at the new definition of the UT proposed in this chapter, we can say that it follows naturally from the definition of a σ -representation introduced; and by looking at the results derived from it, we can say that it provides an efficient tool to estimate a transformed random vector.

Among other advantages comparative with the UT's for the literature, our UT is more general. Based on Taylor Series expansions, we provide the estimation quality of the an l th order UT (Theorem 4.1). Moreover, we propose new definitions for i) the scaling UT's, and ii) for the square-root UT's. Overall, in this chapter, we corrected all the problems and filled all gaps presented in Chapter 2 regarding Unscented Transformations.

In the simulations of Section 4.4, the UT based on the minimum σ -representation introduced in Section 3.4 shows good results, comparative with the UT based on reduced sigma sets.

In the next chapter, our systematization of the Unscented Kalman filtering theory is further developed with the last of its three main concepts: the UKF.

5. UNSCENTED FILTERS FOR EUCLIDEAN MANIFOLDS

Previously, we i) introduced the concept of an σ -representation (Chapter 3), and ii) extended its idea to redefine the UT's in a more formal and consistent way (Chapter 4). With these results, we were able to correct all the problems and filled all gaps presented in Chapter 2 regarding these transformations. Now, we proceed by providing new consistent UKF definitions.

There are many UKF definitions in the literature. In order to investigate based on which of these UKF's we will construct our definitions, we first investigate the problems detected in Section 2.8 regarding the discrete-time Additive UKF's of the literature; this investigation is done in Section 5.1. We use the results of Chapter 4 regarding the UT's to study the possible *causes* of the misbehaviors of the Additive UKF's. We conclude that *only one definition of the AdUKF's is consistent* (in a way that will be established). Based in this consistent Additive UKF, we define our discrete-time Additive Unscented Kalman Filter (AdUKF, Section 5.2).

By extending our discrete-time AdUKF, we present new definitions for the general (non-additive) case (Section 5.2), and also for the square-root variants (Section 5.3).

Further, in Section 5.4, we provide a list of particular cases of these filters showing that all consistent UKF's of the literature are embodied by our systematization. Then, in Section 5.5, we provide comments relative to computational aspects of the proposed UKF filters; and, in Section 5.7, we present a discussion about higher order UKF's.

In Section 5.8, the UKF's for discrete-time systems developed in Sections 5.2 and 5.3 are extended to treat the cases of continuous-time and continuous-discrete-time dynamic systems. Even though these systems were not treated yet up to this points, the results of the discrete-time UKF's makes this transition suitable.

Since many UF's are proposed, in Section 5.9, we provide guidelines for practical users indicating some criteria for choosing the most suitable filter for a given practical problem.

Finally, in Section 5.6, we illustrate some properties of the UF's developed in this chapter with numerical examples.

5.1 CONSISTENCY OF THE ADDITIVE UNSCENTED FILTERS OF THE LITERATURE

In Section 2.8, we classified the AdUKF's of the literature according to the following three criteria:

1. *in which equation* the process noise's covariance Q_k is considered,
2. whether the predicted state sigma set $\{\chi_{i,\{j\}}^{k|k-1}, w_{i,\{j\}}\}$ is *re-generated or not*, and
3. *how* this regeneration is done if it is the case.

We found four distinctive classes, to know the AdUKF's 1, 2, 3 and 4 (cf. Section 2.8.1).

We showed two superior results of the AdUKF 1 comparative with the other AdUKF classes, namely: the AdUKF 1 a) is the only one to have the property of providing the same estimates as the KF when the system is linear, and b) was the best in the *nonlinear* numerical example of Section 2.8.2.

Together, these two superior results *indicate* that there might be a formal reason endowing the AdUKF 1 with *better mathematical properties* comparative with the other AdUKF's for any *nonlinear* system (2.1). In this section, we use results developed in Chapters 3 and 4, to develop stronger conclusions respective to this topic.

Particularly, by using the results regarding the UT definition (Definition 4.1), we get to the conclusion that classifying the AdUKF's of the literature respective to the criteria 1, 2, and 3 above is *equivalent to classifying respective to the input and output vectors of the UT's in these AdUKF's*. We show that each of these AdUKF's classes can be written by using two UT's; and each AdUKF class differentiate from each other from the considered input and output vectors of these UT's.

Depending on how an AdUKF uses the UT's to estimate the state of a given system, *we can say whether or not this AdUKF is consistent with this system*. Recall that, given two random vectors

$$X \text{ and } Y = F(X),$$

an UT provides an approximation of the statistics of Y . Suppose a stochastic filter defined for the following system

$$\begin{aligned} x_k &= F_1(x_{k-1}) \\ y_k &= F_2(x_k). \end{aligned} \tag{5.1}$$

In order to estimate this system, we can use, in a same AdUKF, i) one UT estimating

the statistics of x_k by making $X = x_{k-1}$ and $F = F_1(X)$, and ii) another UT estimating the statistics of y_k by making $X = x_k$ and $F = F_2(X)$. From Theorem 4.1, *we know these UT's provide good estimates* for the statistics of x_k and y_k ; and these estimates are required to the final estimates of every AdUKF (cf. the step 3 of the Algorithms 2, 3, 4, and 5). Therefore, *if an AdUKF uses the UT in this form, we can say that the AdUKF is consistent with system (5.1).*

Let us analyze whether the four AdUKF classes described above are consistent with system (2.1) or not.

5.1.1 Consistency analysis

Consider system (2.1), and define the following random variables

$$\begin{aligned} x_{k-1|k-1} &:= x_{k-1}|y_{1:k-1}, \\ x_{k|k-1} &:= x_k|y_{1:k-1}, \\ x_{k|k-1}^* &:= f_k(x_{k-1|k-1}), \\ x_{k|k} &:= x_k|y_{1:k}, \\ y_{k|k-1} &:= y_k|y_{1:k-1}, \\ y_{k|k-1}^* &:= h_k(x_{k|k-1}), \end{aligned}$$

where $x_{k-1|k-1}$ is the previous state, $x_{k|k-1}$ the predicted state, $x_{k|k}$ the posterior state, $x_{k|k-1}^*$ is the propagated state without the process noise and $y_{k|k-1}^*$ is the predicted measurement without the measurement noise.

In AdUKF's, the estimation quality of any estimate of $x_{k|k}$ and $P_{xx}^{k|k}$ depends on estimation quality of *the predicted estimates* $\hat{x}_{k|k-1}$, $\hat{y}_{k|k-1}$, $\hat{P}_{xx}^{k|k-1}$, $\hat{P}_{yy}^{k|k-1}$ and $\hat{P}_{xy}^{k|k-1}$ (cf. step 3. of Algorithms 2, 3, 4, and 5). Let us analyze each of these estimates of each AdUKF class based on the UT definition.

Since, from (2.1), $x_{k|k-1} = x_{k|k-1}^* + \varpi_k$ and $\varpi_k \sim ([0]_{n_x \times 1}, Q_k)$, then

$$\bar{x}_{k|k-1} = \bar{x}_{k|k-1}^* \text{ and } P_{xx}^{k|k-1} = P_{xx,*}^{k|k-1} + Q_k; \quad (5.2)$$

and, analogously, [recall that $\vartheta_k \sim ([0]_{n_y \times 1}, R_k)$]

$$\bar{y}_{k|k-1} = \bar{y}_{k|k-1}^* \text{ and } P_{yy}^{k|k-1} = P_{yy,*}^{k|k-1} + R_k. \quad (5.3)$$

Therefore, an AdUKF is said to be consistent with system (2.1) according to the following definition.

Definition 5.1. An AdUKF is consistent with system (2.1) if this filter's equations can be written in the form

$$\left[\hat{x}_{k|k-1}, \hat{P}_{xx,*}^{k|k-1} \right] = \text{UT} \left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right), \quad (5.4)$$

$$\hat{P}_{xx}^{k|k-1} = \hat{P}_{xx,*}^{k|k-1} + Q_k, \quad (5.5)$$

$$\left[\hat{y}_{k|k-1}, \hat{P}_{yy}^{k|k-1}, \hat{P}_{xy}^{k|k-1} \right] = \text{UT} \left(h_k, \hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1} \right), \quad (5.6)$$

$$\hat{P}_{yy}^{k|k-1} = \hat{P}_{yy,*}^{k|k-1} + R_k, \quad (5.7)$$

$$G_k := \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \quad (5.8)$$

$$\hat{x}_{k|k} := \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \quad (5.9)$$

$$\hat{P}_{xx}^{k|k} := \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \quad (5.10)$$

This consistency property is associated with the quality of the estimates in an AdUKF.

Suppose, ideally, that

$$\begin{aligned} \hat{x}_{k-1|k-1} &= \bar{x}_{k-1|k-1} \\ \hat{P}_{xx}^{k-1|k-1} &= P_{xx}^{k-1|k-1}. \end{aligned} \quad (5.11)$$

Then, from (5.2), (5.4), (5.5), and Theorem 4.1, it follows that

$$\hat{x}_{k|k-1}^{[\hat{x}_{k-1|k-1}, 2]} = \bar{x}_{k|k-1}^{*, [\bar{x}_{k-1|k-1}, 2]} = \bar{x}_{k|k-1}^{[\bar{x}_{k-1|k-1}, 2]}; \quad (5.12)$$

and

$$\begin{aligned} \hat{P}_{xx}^{k|k-1, [\hat{x}_{k-1|k-1}, 1]} &= \hat{P}_{xx,*}^{k|k-1, [\hat{x}_{k-1|k-1}, 1]} + Q_k \\ &= P_{xx,*}^{k|k-1, [\bar{x}_{k-1|k-1}, 1]} + Q_k \\ &= P_{xx}^{k|k-1, [\bar{x}_{k-1|k-1}, 1]}. \end{aligned} \quad (5.13)$$

Similarly, suppose, ideally, that

$$\begin{aligned} \hat{x}_{k|k-1} &= \bar{x}_{k|k-1} \\ \hat{P}_{xx}^{k|k-1} &= P_{xx}^{k|k-1}. \end{aligned} \quad (5.14)$$

then, from (5.3), (5.6), (5.7), and Theorem 4.1, it follows that

$$\hat{y}_{k|k-1}^{[\hat{x}_{k|k-1},2]} = \bar{y}_{k|k-1}^{*,[\bar{x}_{k-1|k-1},2]} = \bar{y}_{k|k-1}^{[\bar{x}_{k-1|k-1},2]}, \quad (5.15)$$

$$\begin{aligned} \hat{P}_{yy}^{k|k-1,[\hat{x}_{k|k-1},1]} &= \hat{P}_{yy,*,\{1\}}^{k|k-1,[\hat{x}_{k|k-1},1]} + R_k \\ &= P_{yy,*}^{k|k-1,[\bar{x}_{k|k-1},1]} + R_k \\ &= P_{yy}^{k|k-1,[\bar{x}_{k|k-1},1]}, \end{aligned} \quad (5.16)$$

$$\hat{P}_{xy}^{k|k-1,[\hat{x}_{k|k-1},1]} = P_{xy}^{k|k-1,[\bar{x}_{k|k-1},1]}. \quad (5.17)$$

Analogously, suppose that,

$$\begin{aligned} \hat{x}_{k|k-1} &= \bar{x}_{k|k-1}, \\ \hat{P}_{xx}^{k|k-1} &= P_{xx}^{k|k-1}, \\ \hat{y}_{k|k-1} &= \bar{y}_{k|k-1}, \\ \hat{P}_{yy}^{k|k-1} &= P_{yy}^{k|k-1}, \\ \hat{P}_{xy}^{k|k-1} &= P_{xy}^{k|k-1}, \end{aligned} \quad (5.18)$$

then, from (5.8), (5.9), and (5.10), we have that

$$\hat{x}_{k|k} = \bar{x}_{k|k} \text{ and} \quad (5.19)$$

$$\hat{P}_{xx}^{k|k} = P_{xx}^{k|k}. \quad (5.20)$$

Therefore, if an AdUKF is consistent with system (2.1), we are able to state the qualities of the estimates—naturally, based on the assumptions (5.11), (5.14), and (5.18). Moreover, these estimates are generally good since they are estimates provided by UT's.

Since the correction equations (step 3. of Algorithms 2, 3, 4, and 5) are equal for all AdUKF classes, the equations (5.19) and (5.20) will be true for every AdUKF. However, naturally, the equations (5.12), (5.13), (5.15), (5.16), and (5.17) will *not*; they will all be true only for the AdUKF's consistent with system (2.1).

The following propositions relates the consistency of an AdUKF and its performance when a linear system is considered

Theorem 5.1. *Consider an AdUKF estimating system (2.1). If the system functions f_k and h_k are linear, then each estimate $\hat{x}_{k|k-1}$, $\hat{y}_{k|k-1}$, $\hat{P}_{xx}^{k|k-1}$, $\hat{P}_{yy}^{k|k-1}$, $\hat{P}_{xy}^{k|k-1}$, $\hat{x}_{k|k}$,*

and $\hat{P}_{xx}^{k|k-1}$ of the AdUKF is equal to the corresponding one given by the linear Kalman Filter.

Proof. Suppose that, at a time $k \geq 1$, the estimates $\hat{x}_{k-1|k-1}$ and $\hat{P}_{xx}^{k-1|k-1}$ of the AdUKF are equal to the ones given by a linear KF. Since f_k is linear, from (5.12) and (5.13), we have that

$$\begin{aligned}\hat{x}_{k|k-1}^{[\hat{x}_{k-1|k-1}, 2]} &= \bar{x}_{k|k-1}^{[\bar{x}_{k-1|k-1}, 2]} = \bar{x}_{k|k-1}, \\ \hat{P}_{xx}^{k|k-1, [\hat{x}_{k-1|k-1}, 1]} &= P_{xx}^{k|k-1, [\bar{x}_{k-1|k-1}, 1]} = P_{xx}^{k|k-1},\end{aligned}$$

and the assumptions (5.11) hold. Thus, since h_k is also linear, from (5.15), (5.16), (5.17), we have that

$$\begin{aligned}\hat{y}_{k|k-1}^{[\hat{x}_{k|k-1}, 2]} &= \bar{y}_{k|k-1}^{[\bar{x}_{k-1|k-1}, 2]} = \bar{y}_{k|k-1}, \\ \hat{P}_{yy}^{k|k-1, [\hat{x}_{k|k-1}, 1]} &= P_{yy}^{k|k-1, [\bar{x}_{k|k-1}, 1]} = P_{yy}^{k|k-1}, \\ \hat{P}_{xy}^{k|k-1, [\hat{x}_{k|k-1}, 1]} &= P_{xy}^{k|k-1, [\bar{x}_{k|k-1}, 1]} = P_{xy}^{k|k-1},\end{aligned}$$

and assumptions (5.18) hold. Thus, from (5.19) and (5.20), we have that

$$\begin{aligned}\hat{x}_{k|k} &= \bar{x}_{k|k} \\ \hat{P}_{xx}^{k|k} &= P_{xx}^{k|k}.\end{aligned}$$

By *choosing* the initial estimates $\hat{x}_{0|0}$ and $\hat{P}_{xx}^{0|0}$ of the AdUKF *equal* to the ones of the KF, these equations will be true for all $k \geq 1$; hence, the theorem is proved. \square

Let us now analyze the consistency of each AdUKF. Again, below, some variables are written with a subscript $\{j\}$ as in $A_{\{j\}}$, for $j = 1, 2, 3$ and 4; this notation associates the element A to the AdUKF j . For example, $\hat{x}_{k|k-1, \{1\}}$ is an estimate of the AdUKF 1, $\hat{x}_{k|k-1, \{2\}}$ of the AdUKF 2, $\hat{x}_{k|k-1, \{3\}}$ of the AdUKF 3, and $\hat{x}_{k|k-1, \{4\}}$ of the AdUKF 4.

- The equations in the AdUKF 1 (Algorithm 2) can be rewritten in the following way:

$$\left[\hat{x}_{k|k-1, \{1\}}, \hat{P}_{xx, *, \{1\}}^{k|k-1} \right] = \text{UT} \left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right), \quad (5.21)$$

$$\hat{P}_{xx, \{1\}}^{k|k-1} = \hat{P}_{xx, *, \{1\}}^{k|k-1} + Q_k, \quad (5.22)$$

$$\left[\hat{y}_{k|k-1, \{1\}}, \hat{P}_{yy, *, \{1\}}^{k|k-1}, \hat{P}_{xy, \{1\}}^{k|k-1} \right] = \text{UT} \left(h_k, \hat{x}_{k|k-1, \{1\}}, \hat{P}_{xx, \{1\}}^{k|k-1} \right), \quad (5.23)$$

$$\hat{P}_{yy,\{1\}}^{k|k-1} = \hat{P}_{yy,*,\{1\}}^{k|k-1} + R_k. \quad (5.24)$$

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

Therefore, from Definition 5.1, the AdUKF 1 is consistent with system (2.1).

- The equations in the AdUKF 2 (Algorithm 3) can be rewritten in the following way:

$$\left[\hat{x}_{k|k-1,\{2\}}, \hat{P}_{xx,*,\{2\}}^{k|k-1} \right] = \text{UT} \left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right), \quad (5.25)$$

$$\hat{P}_{xx,\{2\}}^{k|k-1} = \hat{P}_{xx,*,\{2\}}^{k|k-1} + Q_k, \quad (5.26)$$

$$\left[\hat{y}_{k|k-1,\{2\}}, \hat{P}_{yy,*,\{2\}}^{k|k-1}, \hat{P}_{xy,\{2\}}^{k|k-1} \right] = \text{UT} \left(h_k, \hat{x}_{k|k-1,\{2\}}, \hat{P}_{xx,*,\{2\}}^{k|k-1} \right), \quad (5.27)$$

$$\hat{P}_{yy,\{2\}}^{k|k-1} = \hat{P}_{yy,*,\{2\}}^{k|k-1} + R_k. \quad (5.28)$$

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

Equation (5.27) is different from (5.6); thus, we can say that the AdUKF 2 is not consistent with system (2.1).

- The equations in the AdUKF 3 (Algorithm 4) for the the estimates $\hat{x}_{k|k-1,\{3\}}$ and $\hat{P}_{xx,\{3\}}^{k|k-1}$ can be rewritten in the following way:

$$\left[\hat{x}_{k|k-1,\{3\}}, \hat{P}_{xx,*,\{3\}}^{k|k-1} \right] = \text{UT} \left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} \right), \quad (5.29)$$

$$\hat{P}_{xx,\{3\}}^{k|k-1} = \hat{P}_{xx,*,\{3\}}^{k|k-1} + Q_k, \quad (5.30)$$

but the estimates $\hat{y}_{k|k-1,\{3\}}$, $\hat{P}_{yy,\{3\}}^{k|k-1}$ and $\hat{P}_{xy,\{3\}}^{k|k-1}$ can not be rewritten in a similar way. Note that they are not equivalent to

$$\begin{aligned} \left[\hat{y}_{k|k-1,\{3\}}, \hat{P}_{yy,*,\{3\}}^{k|k-1}, \hat{P}_{xy,\{3\}}^{k|k-1} \right] &= \text{UT} \left(h_k, \hat{x}_{k|k-1,\{3\}}, \hat{P}_{xx,*,\{3\}}^{k|k-1} + Q_k \right), \\ \hat{P}_{yy,\{3\}}^{k|k-1} &= \hat{P}_{yy,*,\{3\}}^{k|k-1} + R_k. \end{aligned}$$

Therefore, we can say that the AdUKF 3 is not consistent with system (2.1).

- The equations in the AdUKF 4 (Algorithm 5) can be rewritten in the following way:

$$\left[\hat{x}_{k|k-1, \{4\}}, \hat{P}_{xx, *, \{4\}}^{k|k-1} \right] = \text{UT} \left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} + Q_k \right), \quad (5.31)$$

$$\hat{P}_{xx, \{4\}}^{k|k-1} = \hat{P}_{xx, *, \{4\}}^{k|k-1} \quad (5.32)$$

$$\left[\hat{y}_{k|k-1, \{4\}}, \hat{P}_{yy, *, \{4\}}^{k|k-1}, \hat{P}_{xy, \{4\}}^{k|k-1} \right] = \text{UT} \left(h_k, \hat{x}_{k|k-1, \{4\}}, \hat{P}_{xx, \{4\}}^{k|k-1} \right),$$

$$\hat{P}_{yy, \{4\}}^{k|k-1} = \hat{P}_{yy, *, \{4\}}^{k|k-1} + R_k.$$

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

Equation 5.32 is different from (5.5); thus, we can say that the AdUKF 4 is not consistent with system (2.1).

Summarizing, among the studied AdUKF classes, *only the AdUKF 1 is consistent with system (2.1)*. Consequently, the following two statements can be made:

1. The reason behind the AdUKF 1 being the only AdUKF class providing the same estimates as the (linear) KF when (2.1) is linear (cf. Section 2.8.3) is given by *Theorem 5.1*.
2. The reason behind the AdUKF 1 outperforming the other AdUKF classes in the numerical example of Section 2.8.2 is, *probably, given by equations (5.12), (5.13), (5.15), (5.16), (5.17), (5.19) and (5.20)*.

Therefore, we shall define the Additive Unscented Kalman Filter in Section 5.2 based on the form of the AdUKF 1.

5.2 UNSCENTED KALMAN FILTERS

The analysis of Section 5.1 showed that, among the additive UKF, only the AdUKF 1 is consistent with system (2.1). Hence, we use the form of this filter to propose the

AdUKF of our systematization. Recall, from Algorithm 2, that this means considering Q_k in the equation of the predicted covariance $\hat{P}_{xx,\{j\}}^{k|k-1}$, and regenerating the predicted sigma set $\{\chi_{i,\{j\}}^{k|k-1}, w_{i,\{j\}}\}$ as in (2.22).

Definition 5.2. Consider the system

$$\begin{aligned}x_k &= f_k(x_{k-1}) + \varpi_k, \\y_k &= h_k(x_k) + \vartheta_k.\end{aligned}$$

Suppose that i) the noises ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state x_0 are characterized by

$$\begin{aligned}x_0 &\sim (\bar{x}_0, P_{xx}^0), \\ \varpi_k &\sim ([0]_{n_x \times 1}, Q_k), \\ \vartheta_k &\sim ([0]_{n_y \times 1}, R_k); \end{aligned}$$

and iii) the measurements y_1, y_2, \dots, y_{k_f} are given. Then the *Additive Unscented Kalman Filter (AdUKF)*—from now on, unless mentioned otherwise, AdUKF will refer to the following algorithm—is given by the following algorithm:

Algorithm 6 (Additive UKF (AdUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\hat{P}_{xx}^{0|0} := P_{xx}^0$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics by*

$$\begin{aligned}[\hat{x}_{k|k-1}, \hat{P}_{xx,*}^{k|k-1}] &:= \text{UT}_1\left(f_k, \hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1}\right), \\ \hat{P}_{xx}^{k|k-1} &:= \hat{P}_{xx,*}^{k|k-1} + Q_k.\end{aligned}\tag{5.33}$$

(b) *The measurement's predicted statistics by*

$$\begin{aligned}[\hat{y}_{k|k-1}, \hat{P}_{yy,*}^{k|k-1}, \hat{P}_{xy}^{k|k-1}] &:= \text{UT}_2\left(h_k, \hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1}\right), \\ \hat{P}_{yy}^{k|k-1} &:= \hat{P}_{yy,*}^{k|k-1} + R_k.\end{aligned}\tag{5.34}$$

(c) *The state's corrected statistics by*

$$\begin{aligned}G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1}\right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1}\right), \\ \hat{P}_{xx}^k &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T.\end{aligned}\tag{5.35}$$

Given that we only consider the second order UT in this subsection, we use the notation UT to refer to the 2UT (higher order UKF's are considered in Section 5.7). The notations UT₁ and UT₂ indicate that the transformations in the prediction and correction steps do not need to be the same. In fact, the number of sigma points can be different, and we could even use the ScUT. The output of UT₁ has only two terms meaning that only the first two elements of the output of Definition 4.1 are needed in the algorithm. If f_k is linear, then UT₁ can be substituted by the (linear) KF's *prediction* equations; likewise, If h_k is linear, then UT₂ can be substituted by the (linear) KF's *correction* equations. Comments analogous to these ones for the AdUKF can be made for the other filters of this chapter.

By definition, in the AdUKF, the posterior set of UT₁ in (5.33), $\chi^{k|k-1} = \{\chi_i^{k|k-1}, w_i\}$, is regenerated in (5.34), since it is the previous σ -representation of UT₂. One can consider to not regenerate $\chi^{k|k-1}$, but, in this case, i) the filter would not be consistent with system (2.1) (cf. Section 5.1), and ii) $\chi^{k|k-1}$ would not carry information about the process noise (cf. (2.3) and (2.4), and [49]).

By combining i) the proposed AdUKF with ii) the idea of extending the state vectors with the noise (cf. Section 2.2), we can propose an augmented UKF for the more general system (2.2). For this, define the augmented functions $f_k^a : \mathbb{R}^{n_x+n_\varpi} \rightarrow \mathbb{R}^{n_x}$ and $h_k^a : \mathbb{R}^{n_x+n_\vartheta} \rightarrow \mathbb{R}^{n_y}$ such that, for ,

$$\begin{aligned} f_k^a \left(\begin{bmatrix} x_{k-1} \\ \varpi_k \end{bmatrix} \right) &:= f_k(x_{k-1}, \varpi_k), \\ h_k^a \left(\begin{bmatrix} x_k \\ \vartheta_k \end{bmatrix} \right) &:= h_k(x_k, \vartheta_k). \end{aligned} \quad (5.36)$$

From now on, unless mentioned otherwise, AuUKF will refer to the following algorithm:

Definition 5.3. Consider the system

$$\begin{aligned} x_k &= f_k(x_{k-1}, \varpi_k), \\ y_k &= h_k(x_k, \vartheta_k); \end{aligned}$$

and the pair of equations (5.36). Suppose that i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state x_0 are characterized by

$$\begin{aligned} x_0 &\sim (\bar{x}_0, P_{xx}^0), \\ \varpi_k &\sim ([0]_{n_\varpi \times 1}, Q_k), \\ \vartheta_k &\sim ([0]_{n_\vartheta \times 1}, R_k); \end{aligned}$$

and iii) the measurements y_1, y_2, \dots, y_{k_f} are given. Then the *Augmented Unscented Kalman Filter* is given by the following algorithm:

Algorithm 7 (Augmented Unscented Kalman Filter (AuUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\hat{P}_{xx}^{0|0} := P_{xx}^0$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\begin{aligned}\hat{x}_{k-1|k-1}^a &:= \left[\hat{x}_{k-1|k-1}^T, [0]_{n_\varphi \times 1}^T \right]^T, \\ \hat{P}_{xx,a}^{k-1|k-1} &:= \text{diag} \left(\hat{P}_{xx}^{k-1|k-1}, Q_k \right).\end{aligned}$$

(b) *The predicted statistics of the state by*

$$\left[\hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1} \right] := \text{UT}_1 \left(f_k^a, \hat{x}_{k-1|k-1}^a, \hat{P}_{xx,a}^{k-1|k-1} \right). \quad (5.37)$$

(c) *The augmented predicted estimates by*

$$\begin{aligned}\hat{x}_{k|k-1}^a &:= \left[\hat{x}_{k|k-1}^T, [0]_{n_\vartheta \times 1}^T \right]^T, \\ \hat{P}_{xx,a}^{k|k-1} &:= \text{diag} \left(\hat{P}_{xx}^{k|k-1}, R_k \right).\end{aligned}$$

(d) *The predicted statistics of the measurement by*

$$\begin{aligned}\left[\hat{y}_{k|k-1}, \hat{P}_{yy}^{k|k-1}, \hat{P}_{xy,a}^{k|k-1} \right] &:= \text{UT}_2 \left(h_k^a, \hat{x}_{k|k-1}^a, \hat{P}_{xx,a}^{k|k-1} \right), \\ \hat{P}_{xy}^{k|k-1} &:= \left[\hat{P}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}.\end{aligned} \quad (5.38)$$

(e) *The corrected statistics of the state by*

$$\begin{aligned}G_k &:= \left(\hat{P}_{xy}^{k|k-1} \right) \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T.\end{aligned}$$

Unlike the AdUKF, we do not know if *not* regenerating $\chi^{k|k-1}$ in (5.38) makes the AuUKF inconsistent with system (2.2). Similar to the AdUKF, in the AuUKF, by definition, the posterior set of UT_1 in (5.37), $\chi^{k|k-1} = \{\chi_i^{k|k-1}, w_i\}$, is regenerated in (5.38), since it is the previous σ -representation of UT_2 . One can consider to not regenerate $\chi^{k|k-1}$. For the AdUKF, it would make the filters inconsistent with its associated

system, (2.2), but for the AuUKF we do not know. This analysis of consistency is yet to be done.

5.3 SQUARE-ROOT UNSCENTED KALMAN FILTERS

We now present the Square-Root Unscented Kalman Filter (SRUKF). The main difference between this filter and other types of UKF is the fact that the SRUKF propagate the square-root matrix of the covariance matrices directly, which is computationally more stable than squaring the propagated covariance matrix [93].

As pointed out in Section 2.7.1, the SRUKF's in the literature present three steps in which Cholesky factors are downdated: in the calculations of the square-root matrices of the covariance matrix for the predicted state; in the covariance matrix for the innovation; and in the covariance matrix for the corrected state. While, in the first two cases, downdating is only performed when negative weights exist, the last one is always performed. Due to the fact that downdating steps can be computationally unstable (see Section 2.7.1), we derive an alternative form—which is an extension of the results of [93] and [80]—that uses the downdating procedure only for the negative weight components.

According to (4.15), define $S_\chi^+ := S_{\chi^{k|k-1}}^+$ for $\chi^{k|k-1}$, and $S_\gamma^+ := S_{\gamma^{k|k-1}}^+$ for $\gamma^{k|k-1}$; and according to (4.16), define $S_\chi^- := S_{\chi^{k|k-1}}^-$ for $\chi^{k|k-1}$, and $S_\gamma^- := S_{\gamma^{k|k-1}}^-$ for $\gamma^{k|k-1}$. Note that

$$\begin{aligned}\hat{P}_{xx}^{k|k-1} &= S_\chi^+ S_\chi^{+T} - S_\chi^- S_\chi^{-T}, \\ \hat{P}_{yy}^{k|k-1} &= S_\gamma^+ S_\gamma^{+T} - S_\gamma^- S_\gamma^{-T},\end{aligned}$$

and

$$\hat{P}_{xy}^{k|k-1} = S_\chi^+ S_\gamma^{+T} - S_\chi^- S_\gamma^{-T} + R_k.$$

Therefore,

$$\hat{P}_{xx}^{k|k} = \left[S_\chi^+ - G_k S_\gamma^+, G_k R_k \right] \left[\diamond \right]^T - \left[S_\chi^- - G_k S_\gamma^-, G_k R_k \right] \left[\diamond \right]^T,$$

which shows that $\hat{P}_{xx}^{k|k}$ can be obtained through updating and downdating. The latter is only performed for the negative weight cases.

The SRUKF is presented below. It is more general than the algorithms currently in the literature, since these are restricted to the case where only the central weight, w_0 , can be negative, whereas our SRUKF does not restrict the quantity of negative weights. From now on, unless mentioned otherwise, AdSRUKF and AuSRUKF will

refer, respectively, to the following algorithms:

Definition 5.4. Consider the system

$$\begin{aligned}x_k &= f_k(x_{k-1}) + \varpi_k, \\y_k &= h_k(x_k) + \vartheta_k.\end{aligned}$$

Suppose that i) the noises ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state x_0 are characterized by

$$\begin{aligned}x_0 &\sim \left(\bar{x}_0, \sqrt{P_{xx}^0} \sqrt{P_{xx}^0}^T\right), \\ \varpi_k &\sim \left([0]_{n_x \times 1}, \sqrt{Q_k} \sqrt{Q_k}^T\right), \\ \vartheta_k &\sim \left([0]_{n_y \times 1}, \sqrt{R_k} \sqrt{R_k}^T\right);\end{aligned}$$

and iii) the measurements $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_{k_f}$ are given. Then the *Additive Square-Root Unscented Kalman Filter* is given by the following algorithm:

Algorithm 8 (*Additive Square-Root Unscented Kalman Filter (AdSRUKF)*). Perform the following steps:

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\sqrt{\hat{P}_{xx}^{0|0}} := \sqrt{P_{xx}^0}$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics by*

$$\left[\hat{x}_{k|k-1}, \sqrt{\hat{P}_{xx}^{k|k-1}}\right] := \text{SRUT}_1\left(f_k, \hat{x}_{k-1|k-1}, \sqrt{\hat{P}_{xx}^{k-1|k-1}}, \sqrt{Q_k}\right). \quad (5.39)$$

(b) *The measurement's predicted statistics by*

$$\begin{aligned}\left[\hat{y}_{k|k-1}, \sqrt{\hat{P}_{yy}^{k|k-1}}, S_\chi^+, S_\chi^-, S_\gamma^+, S_\gamma^-, \hat{P}_{xy}^{k|k-1}\right] := \\ \text{SRUT}_2\left(h_k, \hat{x}_{k|k-1}, \sqrt{\hat{P}_{xx}^{k|k-1}}, \sqrt{R_k}\right).\end{aligned} \quad (5.40)$$

(c) *The state's corrected statistics by*

$$\begin{aligned}G_k &:= \hat{P}_{xy}^{k|k-1} \left(\sqrt{\hat{P}_{yy}^{k|k-1}}\right)^{-T} \left(\sqrt{\hat{P}_{yy}^{k|k-1}}^{-1}\right), \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1}\right), \\ \sqrt{\hat{P}_{xx}^{k|k}} &:= \text{cu} \left(\left[S_\chi^+ - G_k S_\gamma^+\right], \left[S_\chi^- - G_k S_\gamma^-\right], G_k \sqrt{R_k}\right).\end{aligned} \quad (5.41)$$

Definition 5.5. Consider the system

$$\begin{aligned}x_k &= f_k(x_{k-1}, \varpi_k), \\y_k &= h_k(x_k, \vartheta_k);\end{aligned}$$

and the pair of equations

$$\begin{aligned}f_k^a\left(\begin{bmatrix}x_{k-1} \\ \varpi_k\end{bmatrix}\right) &:= f_k(x_{k-1}, \varpi_k), \\h_k^a\left(\begin{bmatrix}x_k \\ \vartheta_k\end{bmatrix}\right) &:= h_k(x_k, \vartheta_k).\end{aligned}$$

Suppose that i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state x_0 are characterized by

$$\begin{aligned}x_0 &\sim \left(\bar{x}_0, \sqrt{P_{xx}^0} \sqrt{P_{xx}^0}^T\right), \\ \varpi_k &\sim \left([0]_{n_x \times 1}, \sqrt{Q_k} \sqrt{Q_k}^T\right), \\ \vartheta_k &\sim \left([0]_{n_x \times 1}, \sqrt{R_k} \sqrt{R_k}^T\right);\end{aligned}$$

and iii) the measurements y_1, y_2, \dots, y_{k_f} are given. Then the *Augmented Square-Root Unscented Kalman Filter* is given by the following algorithm:

Algorithm 9 (Augmented Square-Root Unscented Kalman Filter (AuSRUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\sqrt{\hat{P}_{xx}^{0|0}} := \sqrt{P_{xx}^0}$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\begin{aligned}\hat{x}_{k-1|k-1}^a &:= \left[\hat{x}_{k-1|k-1}^T, [0]_{n_\varpi \times 1}^T\right]^T, \\ \sqrt{\hat{P}_{xx,a}^{k-1|k-1}} &:= \text{diag}\left(\sqrt{\hat{P}_{xx}^{k-1|k-1}}, \sqrt{Q_k}\right).\end{aligned}$$

(b) *The predicted statistics of the state by*

$$\left[\hat{x}_{k|k-1}, \sqrt{\hat{P}_{xx}^{k|k-1}}\right] := \text{SRUT}_1\left(f_k, \hat{x}_{k-1|k-1}^a, \sqrt{\hat{P}_{xx,a}^{k-1|k-1}}\right),$$

(c) *The augmented predicted estimates by*

$$\hat{x}_{k|k-1}^a := \left[\hat{x}_{k|k-1}^T, [0]_{n_\vartheta \times 1}^T\right]^T,$$

$$\sqrt{\hat{P}_{xx,a}^{k|k-1}} := \text{diag} \left(\sqrt{\hat{P}_{xx}^{k|k-1}}, \sqrt{R_k} \right).$$

(d) *The predicted statistics of the measurement by*

$$\begin{aligned} \left[\hat{y}_{k|k-1}, \sqrt{\hat{P}_{yy}^{k|k-1}}, \hat{P}_{xy,a}^{k|k-1} \right] &:= \text{SRUT}_2 \left(h_k, \hat{x}_{k|k-1}^a, \sqrt{\hat{P}_{xx,a}^{k|k-1}} \right), \\ \hat{P}_{xy}^{k|k-1} &:= \left[\hat{P}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}. \end{aligned}$$

(e) *The corrected statistics of the state by*

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\sqrt{\hat{P}_{yy}^{k|k-1}} \right)^{-T} \left(\sqrt{\hat{P}_{yy}^{k|k-1}} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \sqrt{\hat{P}_{xx}^{k|k}} &:= \text{cu} \left(\left[S_\chi^+ - G_k S_\gamma^+ \right], \left[S_\chi^- - G_k S_\gamma^- \right], G_k \sqrt{R_k} \right). \end{aligned}$$

5.4 CONSISTENT UNSCENTED FILTERS VARIANTS

Recall from Chapter 2, that some UKF's and SRUKF's are not consistent. In order to clarify which UKF's and SRUKF's in the literature are consistent, we put variants of the AdUKF and the AuUKF with $\text{UT}_1 = \text{UT}_2$, and of the AdSRUKF and AuSRUKF with $\text{SRUT}_1 = \text{SRUT}_2$ in Tables 5.1, 5.2, 5.3, and 5.4.

There are some abbreviations of words in these tables: Def. stands for Definition; Cor. for Corollary; Th. for Theorem; Ho. for Homogeneous; Intr. for Intrinsically; Mi. for Minimum; Sc. for Scaled; Si. for Simplex; and Sy. for Symmetric. Each final variant of the filters without a footnote comment is a new consistent version.

Table 5.1 contains the AdUKF and SRUKF variants using minimum σR 's developed in Section 3.4, and Table 5.2 the analogous variants for the AuUKF and SRUKF; Table 5.3 contains the AdUKF and SRUKF variants using the minimum symmetric σR 's developed in Section 3.3, and Table 5.4 the analogous variants for the AuUKF and SRUKF.

In each table, the particular filters are presented in all the columns, *except the first*; and in all the rows, *except the heading one*. In Table 5.1, each filter is the resulting variant of using the AdUKF or AdSRUKF (analogously for the other tables) with the corresponding i) UT or SRUT in the *first column* of its own row, and ii) with the corresponding σR in the *heading row* of its own column. For instance, the Minimum Scaled Additive Unscented Kalman Filter (Min. Sc. AdUKF in Tab 5.1 [2,2]) is the result of using the AdUKF with the ScUT (Tab 5.1 [2,1]) and the $\text{Min}\sigma R$ (heading of

the second column of Table 5.1).

Table 5.1: Some Consistent Minimum AdUKF and Riemannian Minimum AdSRUKF Variants.

	UT's	Mi σR ¹ (Th. 3.2)	RhoMi σR (Cor. 3.5)
1	UT (Def. 4.1)	Mi. AdUKF	Rho Mi. AdUKF ²
2	ScUT (Def. 4.2)	Min. Sc. AdUKF	Rho Mi. Sc. AdUKF
3	SRUT (Def. 4.5)	Mi. AdSRUKF	Rho Mi. AdSRUKF
4	ScSRUT (Def. 4.6)	Mi. Sc. AdSRUKF	Rho Mi. Sc. AdSRUKF

¹RhoMi σR (Rho Minimum σ -representation) stands for the σ -representation of [57];

²Equivalent to the filter in Tab 2.3 [8,*].

Table 5.2: Some Consistent Minimum AuUKF and Riemannian Minimum AuSRUKF Variants.

	UT's	Mi σR ¹ (Th. 3.2)	RhoMi σR (Cor. 3.5)
1	UT (Def. 4.1)	Mi. AuUKF	Rho Mi. AuUKF ²
2	ScUT (Def. 4.2)	Mi. Sc. AuUKF	Rho Mi. Sc. AuUKF
3	SRUT (Def. 4.5)	Mi. AuSRUKF	Rho Mi. AuSRUKF
4	ScSRUT (Def. 4.6)	Mi. Sc. AuSRUKF	Rho Mi. Sc. AuSRUKF

¹RhoMi σR (Rho Minimum σ -representation) stands for the σ -representation of [57];

²Equivalent to the filter in Tab 2.3 [8,*].

One should notice that consistent variants of the UKF (SRUKF) in the literature are particular cases of the proposed UKF (SRUKF) definitions in this work. Also, these definitions are able to provide new filter variants (e.g. the Scaled Square-Root Unscented Kalman Filters).

5.5 COMPUTATIONAL COMPLEXITY AND NUMERICAL IMPLEMENTATIONS

From the computational complexity point-of-view, the UKF's most expensive operations are the square-root matrix operation of $\hat{P}_{xx}^{k-1|k-1} + Q_k$ [$\mathcal{O}(n_x^3)$] and the matrix inversion of $\hat{P}_{yy}^{k|k-1}$ [$\mathcal{O}(n_y^3)$], where n_y is the dimension of the measurement vector]. Hence, for the case in which $n_y \leq n_x$, the computational complexity of the UKF is $\mathcal{O}(n_x^3)$; and for the case in which $n_y \geq n_x$, the computational complexity of the UKF is $\mathcal{O}(n_y^3)$, which is the same complexity as the EKF's [42]. From a numerical implementation standpoint, even though the Cholesky decomposition seems to be the most adopted method to compute the square-root matrix of the covariance matrix for the state, some studies indicate that other methods, such as SVD decomposition, provide

Table 5.3: Some Consistent Minimum Symmetric AdUKF and Minimum Symmetric AdSRUKF Variants.

UT's	MiSy σ R (Cor. 3.4)	HoMiSy σ R (Cor. 3.4)
UT (Def. 4.1)	Mi. Sy. AdUKF	Ho. Mi. Sy. AdUKF ¹
ScUT (Def. 4.2)	Mi. Sy. Sc. AdUKF	Ho. Mi. Sy. Sc. AdUKF ²
SySiScUT (Def. 4.3)	Mi. Sy. Si. Sc. AdUKF	Ho. Mi. Sy. Si. Sc. AdUKF ³
SyInScUT (Def. 4.4)	–	Sy. Intr.-Sc. AdUKF ⁴
SRUT (Def. 4.5)	Mi. Sy. AdSRUKF	Ho. Mi. Sy. AdSRUKF ⁵
ScSRUT (Def. 4.6)	Mi. Sy. Sc. AdSRUKF	Ho. Mi. Sy. Sc. AdSRUKF
SySiScSRUT (Def. 4.7)	Mi. Sy. Si. Sc. AdSRUKF	Ho. Mi. Sy. Si. Sc. AdSRUKF
SyInScSRUT (Def. 4.8)	–	Sy. Intr.-Sc. AdSRUKF ⁶

¹Equivalent to the filter in Tab 2.3 [1,*]. ²Corrected version of the filters in Tab 2.3 [9,*] with the set of Tab 2.1 [1,2]. ³Corrected version of the filter in Tab 2.3 [10,*] with the set of Tab 2.1 [1,2]. ⁴Equivalent to the filter in Tab 2.1 [5,*]. ⁵Equivalent to the SRUKF of [21]. ⁶Equivalent to the SRUKF of [67].

Table 5.4: Some Consistent Minimum Symmetric AuUKF and Minimum Symmetric AuSRUKF Variants.

UT's	MiSy σ R (Cor. 3.4)	HoMiSy σ R (Cor. 3.4)
UT (Def. 4.1)	Mi. Sy. AuUKF	Ho. Mi. Sy. AuUKF
ScUT (Def. 4.2)	Mi. Sy. Sc. AuUKF	Ho. Mi. Sy. Sc. AuUKF
SySiScUT (Def. 4.3)	Mi. Sy. Si. Sc. AuUKF	Ho. Mi. Sy. Si. Sc. AuUKF
SyInScUT (Def. 4.4)	–	Sy. Intr.-Sc. AuUKF
SRUT (Def. 4.5)	Mi. Sy. AuSRUKF	Ho. Mi. Sy. AuSRUKF
ScSRUT (Def. 4.6)	Mi. Sy. Sc. AuSRUKF	Ho. Mi. Sy. Sc. AuSRUKF
SySiScSRUT (Def. 4.7)	Mi. Sy. Si. Sc. AuSRUKF	Ho. Mi. Sy. Si. Sc. AuSRUKF
SyInScSRUT (Def. 4.8)	–	Sy. Intr.-Sc. AuSRUKF

better estimation quality (see [109] for more details). Some code implementations are available on-line (e.g. [110] and [111]).

For the SRUKF, the computational complexity is also $\mathcal{O}(n_x^3)$ due to the triangularization (tria{}), which is its most expensive operation. One example of triangularization is the QR decomposition, which has different implementations; for an $n \times n$ matrix, the Householder QR requires $n^3/3$ floating points operations (flops), the Givens QR $2n^3$ flops, and the modified Gram-Schmidt QR requires $2n^3$ flops [112]. Comparative with UKF's, from a computational perspective, SRUKF's are usually more expensive—demand more flops—, but tend to behave better when implemented in poor-precision machines [88].

5.6 SIMULATIONS

5.6.1 Comparison between sigma sets composed of less than $2n$ sigma points

In this section, we have the purpose of simulating the Minimum Additive Unscented Kalman Filters in order to verify its theoretical results and also to compare it with the Homogeneous Minimum Symmetric Additive Unscented Kalman Filter (Tab 5.3 [1,3]) (which is equivalent to the UKF of [1], Tab 2.3 [2, 1-5]). The scenario is a target tracking of civil aircraft with synthesized data; it is based on [98]. The state vector is $x = [p_x \ v_x \ p_y \ v_y]^T$ where p_x and p_y are, respectively, the Cartesian coordinates along the axes of the *abscissae* and the ordinates, and $v_x = \dot{p}_x$ and $v_y = \dot{p}_y$ are the associated velocities.

The discrete process and measurement equations are the ones of the Coordinated Turn model with measurements of range and azimuth:

$$x_k = \begin{bmatrix} 1 & \frac{\sin(\omega_k T)}{\omega_k} & 0 & -\frac{1-\cos(\omega_k T)}{\omega_k} \\ 0 & \cos(\omega_k T) & 0 & -\sin(\omega_k T) \\ 0 & \frac{1-\cos(\omega_k T)}{\omega_k} & 1 & \frac{\sin(\omega_k T)}{\omega_k} \\ 0 & \sin(\omega_k T) & 0 & \cos(\omega_k T) \end{bmatrix} x_{k-1} + \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \end{bmatrix} \varpi_k,$$

$$y_k = \begin{bmatrix} \sqrt{(p_x - p_{rx})^2 + (p_y - p_{ry})^2} \\ \arctan\left(\frac{p_y - p_{ry}}{p_x - p_{rx}}\right) \end{bmatrix} + \vartheta_k,$$

where $T = 5\text{s}$ is the sampling time, y_k the measurement vector on step time k ,

$\varpi_k \sim N([0]_{2 \times 1}, Q_k)$, $\vartheta_k \sim N([0]_{2 \times 1}, R_k)$ the process and measurement noise vectors, respectively, $p_{rx} = 6000\text{m}$ and $p_{ry} = -6000\text{m}$ the position coordinates of the radar, and ω_k the angular velocity; ω_k is supposed to be a known input. Standard deviations are supposed to be 1m/s^2 for the process error in both directions, 50m for the range measurement error and 1° for the azimuth measurement error. Therefore, $Q_k = I_2$ and

$$R_k = \begin{bmatrix} 2500 & 0 \\ 0 & \frac{(1\pi)^2}{180} \end{bmatrix}.$$

The initial values of the estimates of the state are chosen according to [98] (apparently, these choices are realistic):

$$\hat{x}_{0,0} = [2500, -120, 10000, 0]^T \text{ and } \hat{P}_{xx}^{0,0} = 100I_4.$$

The aircraft's trajectory is followed by the following sequence of movements: 120s with $\omega_k = 0\text{rad/s}$, 30s with $\omega_k = 5\text{rad/s}$, 120s with $\omega_k = 0\text{rad/s}$, 60s with $\omega_k = 1\text{rad/s}$, and 120s with $\omega_k = 0\text{rad/s}$.

The relative error at time k of the j th simulation is

$$\epsilon_{k,j} := \frac{(\hat{p}_x - p_x^c)^2}{(p_x^c)^2} + \frac{(\hat{p}_y - p_y^c)^2}{(p_y^c)^2},$$

where p_x^c and p_y^c are the correct position coordinates of the aircraft. We calculate the Root-Mean-Square Deviation (RMSD)

$$\text{RMSD} := \sqrt{\left(\frac{1}{N_{it}N_s} \sum_{j=1}^{N_s} \sum_{k=1}^{N_{it}} \epsilon_{k,j} \right)} \quad (5.42)$$

where N_{it} is the number of iterations and N_s the number of simulations. In these simulations, we perform $N_{it} = 2000$ iterations and $N_s = 10^5$ simulations.

We first investigate the different values of the tuning parameters for the Minimum Additive Unscented Kalman Filter (MiAdUKF, Tab 5.1 [1,2]), the Homogeneous Minimum Symmetric Additive Unscented Kalman Filter (HoMiSyAdUKF, Tab 5.3 [1,3]), and the Rho Minimum Additive Unscented Kalman Filter (RhoMiAdUKF, Tab 5.1 [1,3]). For the former, the tuning parameter is the vector $v \in \mathbb{R}^n$, and for the other two it is the weight w_0 , which is restricted to $0 < w_0 < 1$ for the RhoMiAdUKF. To simplify the analysis, we consider $v = \beta[1]_{n \times 1}$, $\beta \in \mathbb{R} - \{0\}$. Table 5.5 provides the mean errors μ_ϵ provided by these three filters for some different values of their tuning parameters. The best values were $\beta = 1$ for the MiAdUKF, $w_0 = 0.8$ for the HoMiSyAdUKF and for the RhoMiAdUKF.

Table 5.5: RMSD for different values of the tuning parameters.

MiAdUKF	β	0.1	0.5	1	$\sqrt{2}$	$\sqrt{5}$	5	10
	RMSD	0.563	0.541	0.478	0.501	0.649	73.572	1189.100
HoMiSyAdUKF	w_0	0.1	0.2	0.3	0.5	0.7	0.8	0.9
	RMSD	0.561	0.567	0.569	0.561	0.565	0.558	0.565
RhoMiAdUKF	w_0	0.1	0.2	0.3	0.5	0.7	0.8	0.9
	RMSD	0.563	0.568	0.570	0.562	0.566	0.559	0.566

Table 5.6: RMSD for different filters.

(a) In better conditions of flight and measurements.		(b) In worse conditions of flight and measurements.	
Filter	RMSD	Filter	RMSD
MiAdUKF	0.478	MiAdUKF	2.181
MiAdSRUKF	0.386	MiAdSRUKF	0.352
HoMiSyAdUKF	0.558	HoMiSyAdUKF	2.472
HoMiSyAdSRUKF	0.041	HoMiSyAdSRUKF	0.126
RhoMiAdUKF	0.559	RhoMiAdUKF	2.491
RhoMiAdSRUKF	0.041	RhoMiAdSRUKF	0.126

Now we use these values of tuning parameters to evaluate some filter's performances. Table 5.6a provides the mean errors μ_ϵ for the filters studied in Table 5.5 and also for their square-root forms, which are, the Minimum Additive Square-Root Unscented Kalman Filter (MiAdSRUKF, Tab 5.1 [3,2]), the Homogeneous Minimum Symmetric Additive Square-Root Unscented Kalman Filter (HoMiSyAdSRUKF, Tab 5.3 [5,3]), and the Rho Minimum Additive Square-Root Unscented Kalman Filter (RhoMiAdSRUKF, Tab 5.1 [3,3])

The minimum UKF's provided good estimates even in comparison to the HoMiSyAdSRUKF, which requires $2n+1$ sigma points, whilst the minimum ones require only $n+1$. The best performance was provided by both the RhoMiAdSRUKF and the HoMiSyAdSRUKF. However, one should note that as the Rho Minimum filters (RhoMiAdUKF and RhoMiAdSRUKF) are particular cases of the minimum filters (MiAdUKF and MiAdSRUKF, respectively; cf. Corollary 3.5) and, hence these ones can be tuned to provide the same results as the Rho Minimum filters according to Corollary 3.5. Overall, we can conclude that the minimum filters are able to provide good estimation quality to the problem in question.

In order to verify the performance of the filters in worse conditions, we simulate the same path with $Q_k = 10I_2$ and

$$R_k = \begin{bmatrix} 25000 & 0 \\ 0 & \frac{(5\pi)^2}{180} \end{bmatrix}.$$

Table 5.7: Mean of the CPU times.

Unscented Filter	McpuT(ms)
MiAdUKF	0.557
MiAdSRUKF	0.598
HoMiSyAdUKF	0.558
HoMiSyAdSRUKF	0.724
RhoMiAdUKF	0.481
RhoMiAdSRUKF	0.613

Table 5.6b shows the results. The performance of the filters is indeed worse, but the filters that presented the best results are the same as the ones of Table 5.5.

For each time step k and each simulation j , we measure the time spent ($\Delta t_{k,n}$) by the used CPU to run all the steps of each filter relative the time step k ; then we calculate the mean time consumed CPU in each filter as follows:

$$\text{McpuT} = \left(\frac{1}{N_{it}N_s} \sum_{j=1}^{N_s} \sum_{k=1}^{N_{it}} \Delta t_{k,n} \right).$$

Table 5.7 provides the McpuT's for each of the considered filters running in a machine with an Intel(R) Core (TM) i7 CPU. We can state the following conclusions:

1. the minimum *non-symmetric* UKF's (MiAdUKF and RhoMiAdUKF) were faster than the HoMiSyAdUKF; and the minimum *non-symmetric* SRUKF's (MiAdSRUKF and RhoMiAdSRUKF) were faster than the HoMiSyAdSRUKF. These behavior are consequences of the minimum *non-symmetric* UF's being composed of less sigma points than the minimum *symmetric* UF's.
2. each UKF was faster than its respective SRUKF; i.e., the MiAdUKF was faster than the MiAdSRUKF, the HoMiSyAdUKF was faster than the HoMiSyAdSRUKF, and the RhoMiAdUKF was faster than the RhoMiAdSRUKF. This was expected because there are some costly operations—such as QR decompositions—that are present in SRUKF's but not in UKF's (cf. Section 5.5).

5.6.2 Ill-conditioned measurement function

In comparison to the non-square-root filters, the square-root filters have better numerical properties and guarantee positive semi-definiteness of the state's covariance matrix. They are more convenient over the non-square-root filters specially when considering poor machine precision, since the square-root guarantee positive semi-definiteness of the state's covariance matrix even when round-off errors are considerable. Therefore, in this section, we provide an example with the objective of verifying this behavior.

We compare the new Homogeneous Minimum Symmetric Additive Square-Root Unscented Kalman Filter (HoMiSyAdSRUKF, Tab 5.3 [5,3]) with i) the Homogeneous Minimum Symmetric Additive Unscented Kalman Filter (HoMiSyAdUKF, Tab 5.3 [1,3]) (which is equivalent to the UKF of [1], Tab 2.3 [2, 1-5]), and ii) the SRUKF of [42] using the same method of Example 6.2 of [88]. The idea of this method is to test the influence of round-off errors in these filters by computing only their correction step with a ill-conditioned measurement function; it is considered the measurement function

$$h_k(x_k) := Hx_k$$

where

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 + \delta \end{bmatrix},$$

$$\delta = eps^{2/3}10^d,$$

d is an integer, and eps is the distance from 1.0 to the next largest double-precision number, which, in our case, is $eps = 2^{-52}$.

The SRUKF of [42] could not perform the simulations for $d \leq 10$ for presenting non-positive definite covariance-matrix. Figure 5.1 presents the relative errors of the (HoMiSyAdSRUKF) and the HoMiSyAdUKF for $d \in [-5, 8]$. The new HoMiSyAdSRUKF presented fewer errors than the HoMiSyAdUKF; thus, we can say that the new HoMiSyAdSRUKF is more robust to round-off errors than the SRUKF of [42] and the HoMiSyAdSRUKF.

5.7 HIGHER-ORDER UNSCENTED KALMAN FILTERS

In this work, the AdUKF and AdSRUKF were defined only with 2nd order UT's. Extensions to higher orders can be done in at least two ways. A first one is given by the following algorithm:

Definition 5.6. Consider the system

$$x_k = f_k(x_{k-1}) + \varpi_k,$$

$$y_k = h_k(x_k) + \vartheta_k.$$

Suppose that i) the noises ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state

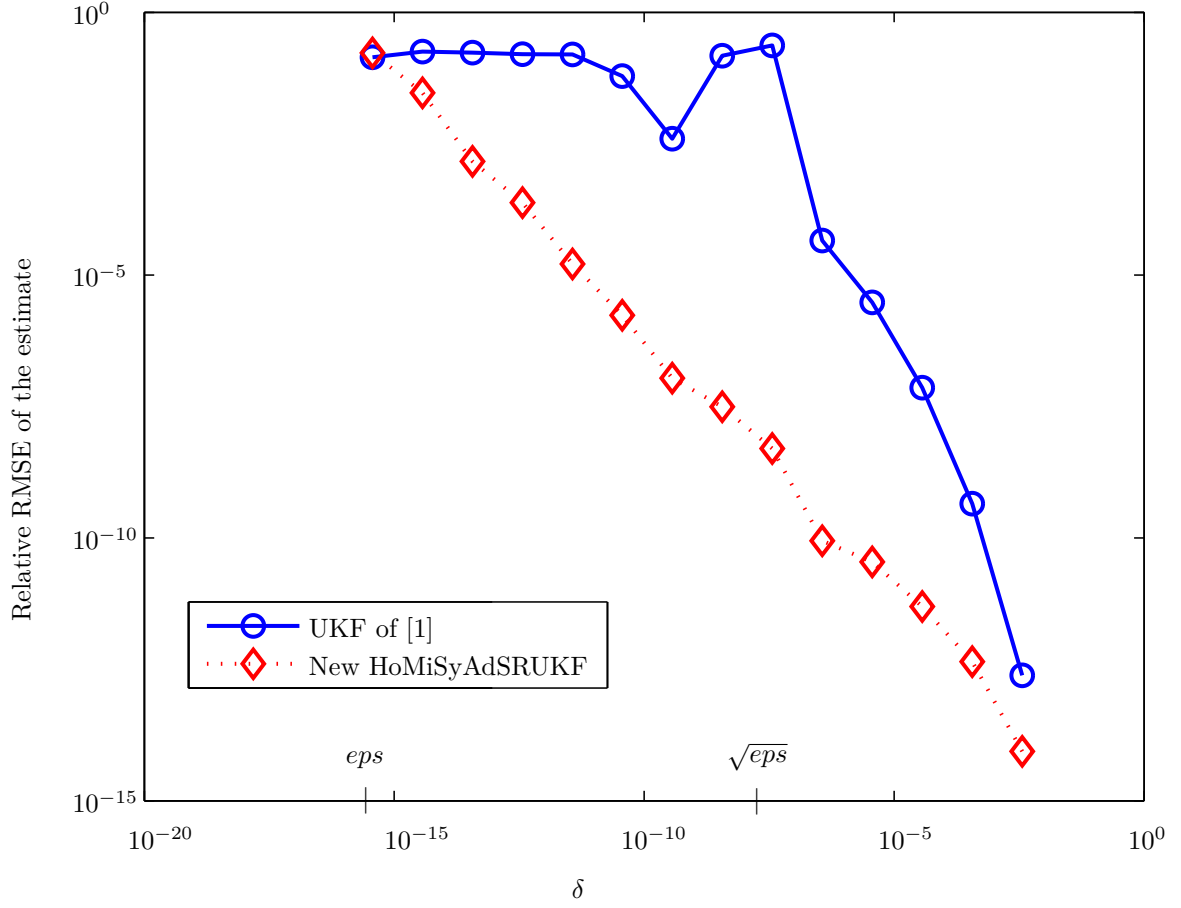


Figure 5.1: Comparison between filters.

x_0 are characterized by

$$\begin{aligned} x_0 &\sim (\bar{x}_0, P_{xx}^0), \\ \varpi_k &\sim ([0]_{n_x \times 1}, Q_k), \\ \vartheta_k &\sim ([0]_{n_y \times 1}, R_k); \end{aligned}$$

and iii) the measurements y_1, y_2, \dots, y_{k_f} are given. Then the l th order *Gaussian Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 10 (*l*th order *Gaussian Additive Unscented Kalman Filter*). Perform the following steps:

1. Initialization. Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\hat{P}_{xx}^{0|0} := P_{xx}^0$, and choose the order of the filter $l \in \mathbb{N}$, $l > 2$.
2. Filtering. For $k = 1, 2, \dots, k_f$; set the following elements:

- (a) The central moments

$$M_{x_{k-1|k-1}}^2, \dots, M_{x_{k-1|k-1}}^l$$

for

$$x_{k-1|k-1} \sim N\left(\hat{x}_{k-1|k-1}, \hat{P}_{xx}^{k-1|k-1} + Q_k\right).$$

(b) The predicted statistics of the state by

$$\left[\hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1}\right] = lUT_1\left(f_k, \hat{x}_{k-1|k-1}, M_{x_{k-1|k-1}}^2, \dots, M_{x_{k-1|k-1}}^l\right).$$

(c) The central moments

$$M_{x_{k|k-1}}^2, \dots, M_{x_{k|k-1}}^l$$

for

$$x_{k|k-1} \sim N\left(\hat{x}_{k|k-1}, \hat{P}_{xx}^{k|k-1} + R_k\right).$$

(d) The predicted statistics of the measurement by

$$\left[\hat{y}_{k|k-1}, \hat{P}_{yy}^{k|k-1}, \hat{P}_{xy}^{k|k-1}\right] = lUT_2\left(h_k, \hat{x}_{k|k-1}, M_{x_{k|k-1}}^2, \dots, M_{x_{k|k-1}}^l\right).$$

(e) The corrected statistics of the state by

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1}\right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1}\right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

This approach uses the Gaussian assumption of the Kalman Filter to obtain the previous first l moments of the state for each lUT . Generally, higher values of l result in a larger number of sigma-points and better state estimation (cf. Theorem 4.1). Note that the higher-order UKF of [91] is a particular case of this proposed filter for the scalar case.

A second way is to propagate, at every time step, not only the mean and the covariance matrix of the state, but also its higher-order moments up to a chosen l th order (a similar approach that does not use UT's is proposed by [113]). This method does not assume that the state follows a Gaussian distribution at every time step, and provides a better approximation when compared to the first one; but at the cost of increased effort in developing the recursive equations, and also of having a computationally more expensive algorithm.

5.8 CONTINUOUS-DISCRETE-TIME AND CONTINUOUS-TIME UNSCENTED KALMAN FILTERS

Instead of considering additive discrete-time systems as (2.1), we can consider the so called *continuous-discrete-time, stochastic, dynamic system* (for a vector x , dx stand for its differential) given by, for $t \geq t_0$,

$$\begin{aligned} dx(t) &= f_t(x(t)) + d\varpi(t), \\ y_k &= h_k(x_k, k) + \vartheta_k, \end{aligned} \tag{5.43}$$

where $\{\varpi(t), t \geq t_0\}$ is the process noise, and is supposed to be a vector of independent Brownian motions (see [24]); and the other elements are defined as in (2.1) and (2.2). The meaning of the first equation of (5.43) is given by its integral (when exists)

$$x(t) - x(t_0) = \int_{t_0}^t f_\tau(x_\tau) d\tau + \int_{t_0}^t \varpi_\tau;$$

the first integral can be defined as an Riemann integral and the second as an Itô integral (see [24]).

The work [52] derived a Unscented Filters for (5.43), namely the Continuous-discrete UKF (CdUKF) and the Square Continuous-discrete UKF (SRCdUKF). However the estimation's quality of these filters were not investigated yet. Because we know, from Theorem 4.1, the estimation quality of the UT, we can obtain the estimation quality of the CdUKF and the SRCdUKF (and also generalize the $\sigma\mathbb{R}$, since these filters were defined particularly for the InSy $\sigma\mathbb{R}$) by writing them in the form of our systematization developed so far. We also i) rename these filters following the reasoning used for the Unscented filters of this chapter, and ii) propose these filter's variants for the more general system

$$\begin{aligned} dx(t) &= f_t(x(t), \varpi(t)), \\ y_k &= h_k(x_k, \vartheta_k). \end{aligned} \tag{5.44}$$

For the augmented versions of these Unscented filters, define the augmented functions $f_t^a : \mathbb{R}^{n_x+n_\varpi} \rightarrow \mathbb{R}^{n_x}$ and $h_k^a : \mathbb{R}^{n_x+n_\vartheta} \rightarrow \mathbb{R}^{n_y}$ such that, for ,

$$f_t^a \left(\begin{bmatrix} x(t) \\ \varpi(t) \end{bmatrix} \right) := f_t(x(t), \varpi(t)), \tag{5.45}$$

$$h_k^a \left(\begin{bmatrix} x_k \\ \vartheta_k \end{bmatrix} \right) := h_k(x_k, \vartheta_k).$$

Definition 5.7. Consider the system (5.43). Suppose that i) the noises $\varpi(t)$ and ϑ_k are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, ϑ_k and the initial state x_0 are characterized by

$$\begin{aligned} x_0 &\sim (\bar{x}_0, P_{xx}^0), \\ \frac{d\varpi(t)}{dt} &\sim ([0]_{n_x \times 1}, Q(t)), \\ \vartheta_k &\sim ([0]_{n_y \times 1}, R_k); \end{aligned}$$

and iii) the measurements $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_{k_f}$ are given. Then the *Continuous-Discrete Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 11 (Continuous-discrete Additive UKF (CdUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\hat{P}_{xx}^{0|0} := P_{xx}^0$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics.* For the initial conditions

$$\begin{aligned} x^-(t_{k-1}) &:= \hat{x}_{k-1|k-1} \text{ and} \\ \hat{P}_{xx}^-(t_{k-1}) &:= \hat{P}_{xx}^{k-1|k-1}, \end{aligned}$$

solve i), for $\hat{x}^-(t_k)$, the differential equation

$$d\hat{x}^-(t) := \hat{m}^-(t);$$

and ii), for $\hat{P}_{xx}^-(t_k)$, the differential equation

$$d\hat{P}_{xx}^-(t) := \hat{P}_{xf(x)}^-(t) + \left(\hat{P}_{xf(x)}^-(t) \right)^T + Q(t);$$

where

$$\left[\hat{m}^-(t), \bullet, \hat{P}_{xf(x)}^-(t) \right] := \text{UT}_1 \left(f_t, \hat{x}^-(t), \hat{P}_{xx}^-(t) \right).$$

(b) *The measurement's predicted statistics by*

$$\begin{aligned} \left[\hat{y}_{k|k-1}, \hat{P}_{yy,*}^{k|k-1}, \hat{P}_{xy}^{k|k-1} \right] &:= \text{UT}_2 \left(h_k, \hat{x}^-(t_k), \hat{P}_{xx}^-(t_k) \right), \\ \hat{P}_{yy}^{k|k-1} &:= \hat{P}_{yy,*}^{k|k-1} + R_k. \end{aligned}$$

(c) The state's corrected statistics by

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

Definition 5.8. Consider the system (5.44) and the pair of equations (5.45). Suppose that i) the noises $\varpi(t)$ and ϑ_k are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, ϑ_k and the initial state $x(t_0)$ are characterized by

$$\begin{aligned} x(t_0) &\sim \left(\bar{x}_0, P_{xx}^0 \right), \\ \frac{d\varpi(t)}{dt} &\sim ([0]_{n_\varpi \times 1}, Q(t)), \\ \vartheta_k &\sim ([0]_{n_\vartheta \times 1}, R_k); \end{aligned}$$

and iii) the measurements y_1, y_2, \dots, y_{k_f} are given. Then the *Continuous-Discrete Augmented Unscented Kalman Filter* is given by the following algorithm.

Algorithm 12 (Continuous-discrete Augmented UKF (CdAuUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{x}_{0|0} := \bar{x}_0$ and $\hat{P}_{xx}^{0|0} := P_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics.* For the initial conditions

$$\begin{aligned} x^-(t_{k-1}) &:= \hat{x}_{k-1|k-1} \text{ and} \\ \hat{P}_{xx}^-(t_{k-1}) &:= \hat{P}_{xx}^{k-1|k-1}, \end{aligned}$$

solve i), for $\hat{x}^-(t_k)$, the differential equation

$$d\hat{x}^-(t) := \hat{m}^-(t);$$

and ii), for $\hat{P}_{xx}^-(t_k)$, the differential equation

$$d\hat{P}_{xx}^-(t) := \hat{P}_{xf(x)}^-(t) + \left(\hat{P}_{xf(x)}^-(t) \right)^T;$$

where

$$\begin{aligned} \hat{x}_a^-(t) &:= \left[\hat{x}^-(t)^T, [0]_{1 \times n_\varpi} \right]^T, \\ \hat{P}_{xx}^{-,a}(t) &:= \text{diag} \left(\hat{P}_{xx}^-(t), Q(t) \right), \end{aligned}$$

$$\begin{aligned} \left[\hat{m}^-(t), \bullet, \hat{P}_{xf(x)}^{-,a}(t) \right] &:= \text{UT}_1 \left(f_t^a, \hat{x}_a^-(t), \hat{P}_{xx}^{-,a}(t) \right), \\ \hat{P}_{xf(x)}^-(t) &:= \left[\hat{P}_{xf(x)}^{-,a}(t) \right]_{(1:n_x), (1:n_x)}. \end{aligned}$$

(b) *The measurement's predicted statistics by*

$$\begin{aligned} \hat{x}_{k|k-1}^a &:= \left[\left(\hat{x}^-(t_k) \right)^T, [0]_{n_\vartheta \times 1}^T \right]^T, \\ \hat{P}_{xx,a}^{k|k-1} &:= \text{diag} \left(\left(\hat{P}_{xx}^-(t_k) \right)^T, R_k \right), \\ \left[\hat{y}_{k|k-1}, \hat{P}_{yy}^{k|k-1}, \hat{P}_{xy,a}^{k|k-1} \right] &:= \text{UT}_2 \left(h_k^a, \hat{x}_{k|k-1}^a, \hat{P}_{xx,a}^{k|k-1} \right), \\ \hat{P}_{xy}^{k|k-1} &:= \left[\hat{P}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}. \end{aligned}$$

(c) *The state's corrected statistics by*

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned}$$

Note that, by writing the continuous-discrete Unscented filters in these forms we have, for each of these four filters, analog versions of all particular cases for the AdUKF in Table 5.3 (e.g., scaled variant, symmetric intrinsically-scaled variant, and so far).

There might be cases in which it would be more realistic to model a given system not only with the process equation being time continuous, but also the measurement equation. By doing so, we have the system, for $t \geq t_0$,

$$\begin{aligned} dx(t) &= f_t(x(t)) + d\varpi(t), \\ dy(t) &= h_t(x(t)) + d\vartheta(t), \end{aligned} \tag{5.46}$$

where $\{\varpi(t), t \geq t_0\}$ is the process noise, and $\{\vartheta(t), t \geq t_0\}$ the measurement noise, and are supposed to be vectors of independent Brownian motions (see [24]); and the other elements are defined as in (2.1) and (2.2).

Following the derivations of the Kalman-Bucy filter (this filter gives the minimum variance estimates for the linear case of the system (5.46), see [24]), [52] derived Unscented filters also for (5.46), namely the Unscented Kalman Bucy Filter (UKBF).

Similarly to the continuous-discrete-time we can obtain the estimation quality of the UKBF (and also generalize the σ R) by writing this filter in the form of our systematization developed so far. We also i) rename these filters following the reasoning used for the Unscented filters of this chapter, and ii) propose these filter's variants for

the general system

$$\begin{aligned} dx(t) &= f_t(x(t), \varpi(t)), \\ dy(t) &= h_t(x(t), \vartheta(t)), \end{aligned} \tag{5.47}$$

For the augmented versions of these Unscented filters, define the augmented functions $f_t^a : \mathbb{R}^{n_x+n_\varpi} \rightarrow \mathbb{R}^{n_x}$ and $h_t^a : \mathbb{R}^{n_x+n_\vartheta} \rightarrow \mathbb{R}^{n_y}$ such that, for ,

$$\begin{aligned} f_t^a \left(\begin{bmatrix} x(t) \\ \varpi(t) \end{bmatrix} \right) &:= f_t(x(t), \varpi(t)), \\ h_t^a \left(\begin{bmatrix} x(t) \\ \vartheta(t) \end{bmatrix} \right) &:= h_t(x(t), \vartheta(t)). \end{aligned} \tag{5.48}$$

Definition 5.9. Consider the system (5.46). Suppose that i) the noises $\varpi(t)$ and $\vartheta(t)$ are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, $\vartheta(t)$ and the initial state $x(t_0)$ are characterized by

$$\begin{aligned} x(t_0) &\sim (\bar{x}(t_0), P_{xx}(t_0)), \\ \frac{d\varpi(t)}{dt} &\sim ([0]_{n_x \times 1}, Q(t)), \\ \frac{d\vartheta(t)}{dt} &\sim ([0]_{n_y \times 1}, R(t)); \end{aligned}$$

and iii) the measurements $\{y(t), t \geq t_0\}$ are given. Then the *Continuous Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 13 (Continuous Additive UKF (CoAdUKF)). *For the initial conditions*

$$\begin{aligned} x(t_0) &:= \bar{x}(t_0) \text{ and} \\ \hat{P}_{xx}(t_0) &:= P_{xx}(t_0), \end{aligned}$$

solve i), for $\hat{x}(t_k)$, the differential equation

$$d\hat{x}(t) := \hat{m}(t) + G(t) \left(\underline{y}(t) - \hat{y}(t) \right);$$

and ii), for $\hat{P}_{xx}(t_k)$, the differential equation

$$d\hat{P}_{xx}(t) := \hat{P}_{xf(x)}(t) + \hat{P}_{xf(x)}^T(t) + Q(t) - G(t)R(t)G^T(t);$$

where

$$\begin{aligned} [\hat{m}(t), \bullet, \hat{P}_{xf(x)}(t)] &:= \text{UT}_1 \left(f_t, \hat{x}^-(t), \hat{P}_{xx}^-(t) \right), \\ [\hat{y}(t), \bullet, \hat{P}_{xh(x)}(t)] &:= \text{UT}_2 \left(h_t, \hat{x}^-(t), \hat{P}_{xx}^-(t) \right), \\ G(t) &:= \hat{P}_{xh(x)}(t) R^{-1}(t). \end{aligned}$$

Definition 5.10. Consider the system (5.47) and the pair of equations (5.48). Suppose that i) the noises $\varpi(t)$ and $\vartheta(t)$ are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, $\vartheta(t)$ and the initial state $x(t_0)$ are characterized by

$$\begin{aligned} x(t_0) &\sim (\bar{x}(t_0), P_{xx}(t_0)), \\ \frac{d\varpi(t)}{dt} &\sim ([0]_{n_x \times 1}, Q(t)), \\ \frac{d\vartheta(t)}{dt} &\sim ([0]_{n_x \times 1}, R(t)); \end{aligned}$$

and iii) the measurements $\{y(t), t \geq t_0\}$ are given. Then the *Continuous Augmented Unscented Kalman Filter* is given by the following algorithm:

Algorithm 14 (Continuous Augmented UKF (CoAuUKF)). *For the initial conditions*

$$\begin{aligned} x(t_0) &:= \bar{x}(t_0) \text{ and} \\ \hat{P}_{xx}(t_0) &:= P_{xx}(t_0), \end{aligned}$$

solve i), for $\hat{x}(t_k)$, the differential equation

$$d\hat{x}(t) := \hat{m}(t) + G(t) \left(\underline{y}(t) - \hat{y}(t) \right);$$

and ii), for $\hat{P}_{xx}(t_k)$, the differential equation

$$d\hat{P}_{xx}(t) := \hat{P}_{xf(x)}(t) + \hat{P}_{xf(x)}^T(t);$$

where

$$\begin{aligned} \hat{x}_a^-(t) &:= \left[\hat{x}^-(t)^T, [0]_{1 \times n_\varpi} \right]^T, \\ \hat{P}_{xx}^{-,a}(t) &:= \text{diag} \left(\hat{P}_{xx}(t), Q(t) \right), \\ \hat{x}_{a^*}^-(t) &:= \left[\hat{x}^-(t)^T, [0]_{1 \times n_\vartheta} \right]^T, \\ \hat{P}_{xx}^{-,a^*}(t) &:= \text{diag} \left(\hat{P}_{xx}(t), R(t) \right), \\ [\hat{m}(t), \bullet, \hat{P}_{xf(x)}^a(t)] &:= \text{UT}_1 \left(f_t^a, \hat{x}_a^-(t), \hat{P}_{xx}^{-,a}(t) \right), \end{aligned}$$

$$\begin{aligned}
[\hat{y}(t), \bullet, \hat{P}_{xh(x)}^a(t)] &:= \text{UT}_2 \left(h_t^a, \hat{x}_{a*}^-(t), \hat{P}_{xx}^{-,a*}(t) \right), \\
\hat{P}_{xf(x)}^-(t) &:= \left[\hat{P}_{xf(x)}^{-,a}(t) \right]_{(1:n_x), (1:n_x)}, \\
\hat{P}_{xh(x)}^-(t) &:= \left[\hat{P}_{xh(x)}^{-,a}(t) \right]_{(1:n_y), (1:n_y)}, \\
G(t) &:= \hat{P}_{xh(x)}^-(t) R^{-1}(t).
\end{aligned}$$

UKF's for the case in which the dynamics are time discrete (the process function) and the measurements are time continuous can easily be obtained by combined the filters of this section. However, this type of system is rare in practice; usually the measurements are modeled with discrete time because they are usually interpreted by digital machines. Yet, the measurements can be considered time continuous, but in this case, usually the dynamics are also considered time continuous.

Continuous UKF's are generally computationally more expensive than Continuous-discrete UKF's, and Continuous-discrete UKF's are generally more expensive than discrete-time UKF's. Computing integrals is costly and i) Continuous UKF's computes integrals in both the prediction and corrections steps, ii) Continuous-discrete UKF's computes integrals in the prediction step, and iii) discrete-time UKF's do not compute any integral.

5.9 GUIDELINES FOR USERS

In this chapter, we have proposed a collection of Unscented Filter's (UF's) and, in this section, we present some guidelines to a possible user for selecting one among all these filters.

In order to choose among all the presented UF's, let us recall some of their properties:

- Additive Unscented filters are computationally cheaper than augmented Unscented filters, but additive Unscented filters are not suitable to systems whose i) process noise is not additive relative to the process function and ii) measurement noise is not additive relative to the measurement function.
- Non Square-root Unscented filters are computationally cheaper than square-root Unscented filters, but square-root Unscented filters are computationally more stable than non Square-root Unscented filters.
- For an UF composed of the sigma-representations $l_1\text{th}N_1\sigma R_1$ and $l_2\text{th}N_2\sigma R_2$, the following statements are *generally* true:

- the estimation given by the UF is more accurate for bigger values of l_1 and l_2 ; recall, however, that in order to have the $\sigma\mathbf{R}$'s with either $l_1 > 2$ or $l_2 > 2$, central moments of order 3 or greater are needed, and we *rarely* have these moments at every instant of time.
- if a random vector is symmetric, than a symmetric $\sigma\mathbf{R}$ of this random vector will generally be a better approximation than a non-symmetric $\sigma\mathbf{R}$.
- the computational cost of the UF increases with the increase of N_1 and/or N_2 .

With these properties, we can choose an Unscented filter suitable to a given practical problem. An user should conjugate the properties above with the following characteristics of the problem:

1. *Form of the (mathematical) dynamic system.* The (mathematical) dynamic system modeling the practical problem can have one of the following forms:
 - (a) *Continuous-time or continuous-discrete-time.* When one or both the equations of a given dynamic system are time continuous, we *can* perform discretizations of these equations and estimate the resulting discrete-time system with a *discrete* UF. This technique may be advantageous in cases where the computational efforts of the non discrete filters are *high* because discrete UF's are computationally cheaper than their analogous continuous-discrete UF's and continuous UF's.
 - (b) *Discrete-time system with additive noise.* If the system is in the form of (2.1), then a discrete-time additive UF should be chosen, such as a particular AdUKF (Algorithm 6) or AdSRUKF (Algorithm 8)—e.g. the filters in Tables 5.3 and 5.1—; or even a particular l th order Gaussian Additive Unscented Kalman Filter.
 - (c) *Discrete-time system with non-additive noise.* If the system is in the form of (2.2)—and, naturally, not in the form of (2.1)—, then a discrete-time augmented UF should be chosen, such as a particular AuUKF (Algorithm 7) or the AuSRUKF (Algorithm 9)—e.g. the filters in Tables 5.2 and 5.4—; or even an *augmented variant* of the l th order Gaussian Additive Unscented Kalman Filter.
 - (d) *Continuous-discrete-time system with additive-noise.* If the system is in the form of (5.43), then a continuous-discrete-time additive UF should be chosen, such as a particular (CdAdUKF) (Algorithm 11), or a continuous-discrete-time variant of the l th order Gaussian Additive Unscented Kalman

Filter.

- (e) *Continuous-discrete-time system with non-additive noise.* If the system is in the form of (5.44)—and, naturally, not in the form of (5.43)—, then a continuous-discrete-time augmented UF should be chosen, such as a particular (CdAuUKF) (Algorithm 12), or a continuous-discrete-time augmented variant of the l th order Gaussian Additive Unscented Kalman Filter.
 - (f) *Continuous-time system with additive-noise.* If the system is in the form of (5.46), then a continuous-time additive UF should be chosen, such as a particular (CoAdUKF) (Algorithm 13), or a continuous-time variant of the l th order Gaussian Additive Unscented Kalman Filter.
 - (g) *Continuous-time system with non-additive noise.* If the system is in the form of (5.47)—and, naturally, not in the form of (5.46)—, then a continuous-time augmented UF should be chosen, such as a particular (CoAuUKF) (Algorithm 14), or a continuous-time augmented variant of the l th order Gaussian Additive Unscented Kalman Filter.
 - (h) *Continuous-time or continuous-discrete-time system with either additive-noise or non-additive noise.* This comment is a complement of the comments 1d, 1e, 1f, and 1g. Even when we have a continuous-time or a continuous-discrete-time system, we *can* perform discretizations of this system’s equations and estimate the resulting *discrete-time* system with a *discrete* UF. This technique may be advantageous in cases where the implementing machine’s computational power is insufficient to run properly the non discrete filters—recall that discrete UF’s are computationally cheaper than their analogous continuous-discrete UF’s and continuous UF’s.
2. *Computationally-ill conditions.* The choice between a square-root Unscented filter and an (non square-root) Unscented filter depends on the existence of computationally-ill conditions. If the filter will have to deal with computationally-ill conditions—e.g. almost non-positive covariances or poor machine precision—then we should choose an square-root Unscented filter (e.g. rows 5 to 8 of Tables 5.3 and 5.4, and rows 3 to 4 of Tables 5.1 and Tab 5.2); if not, then choose a non square-root Unscented filter (e.g. rows 1 to 4 of Tables 5.3 and 5.4, and rows 1 to 2 of Tables 5.1 and Tab 5.2).
 3. *Form of the state’s pdf.* The choice of the σ R’s depends on the approximate form of the state’s pdf at every instant of time. We should consider the following properties of this pdf:

- (a) *Normality.* If, at most of the instants of time t , the state's pdf is almost Normal, then an user should choose σR 's proper to Normal random vectors, such as the Fifth order set of [47] (Tab 2.1 [4,2]). In this case, a variant of the l th order Gaussian Additive Unscented Kalman Filter (Algorithm 10) would be a good choice; the value of l would depend on the capacity of the computer in which the filter would be implemented.
- (b) *Symmetry.* If, at most of the instants of time t , the state's pdf is *symmetric but not close to a Normal pdf*, then an user should choose minimum symmetric $2\sigma R$'s, such as the $MiSy\sigma R$ (Corollary 3.4) or the $HoMiSy\sigma R$ (Corollary 3.4). On the other hand, if, at most of the instants of time t , the state's pdf is *not symmetric*, then an user should choose a minimum (non-symmetric) σR such as the $Mi\sigma R$ (Theorem 3.2) or the $RhoMi\sigma R$ (Corollary 3.5).

5.10 CONCLUSIONS REGARDING UNSCENTED FILTERS

In this chapter, we showed that, among the AdUKF's of the literature, there is only one consistent with the UT and the system (2.1) (cf. Section 5.1). This is the reason behind the fact that, when (2.1) is linear, the estimates of most of the AdUKF's are not equivalent to the linear KF's one (cf. Section 2.8).

That consistent AdUKF of the literature was used as a basis to propose our AdUKF (Section 5.2). Our AdUKF is, nevertheless, more general and better principled because it is defined using the definitions of UT and σ -representation developed in the previous chapters. Besides, we extended our AdUKF and proposed i) a square-root variant (Section 5.3), ii) an UKF variant for the more general system (2.2) (Section 5.2), and iii) a square-root variant of this UKF for system (2.2) (Section 5.2). All the consistent UKF's and SRUKF's of the literature showed to be particular cases of our Unscented Filters. Numerical comments are provided in Section 5.5.

We extended even further our systematization of the Unscented Filter. In Section 5.7 we commented how higher order Unscented filters could be defined, and in Section 5.8 we proposed continuous-time and continuous-discrete-time variants of the proposed Unscented filters.

We also provided i) guidelines for choosing the most suitable Unscented Filter for a given practical problem (Section 5.9), and ii) numerical examples illustrating the results of this chapter are given (Section 5.6).

With this chapter we end the theoretical part of our systematization of the Unscented Kalman filtering theory for systems in the form of (2.1) and (2.2)—other forms

are considered in Part II. In the next chapter, we show the good properties of some UKF's proposed in this systematization in practical problem of estimating the position of an automotive electronic throttle valve.

6. APPLICATION: ESTIMATION OF AUTOMOTIVE ELECTRONIC THROTTLE VALVE'S POSITION

In the preceding chapters, the theory of Unscented Kalman Filters was systematized; in this systematization, new results were introduced, some problems were solved, and some scientific properties—such as formalism, and cohesion—were consolidated. Although some analytical and some numerical examples were presented to illustrate these new results, these contributions are theoretical and numerical. Completing the triad of scientific results—theory, simulation, and experiment—*this chapter presents an experimental/technological innovation* using some of the new UKF's developed in the preceding chapters; these filters are used to estimate the position of an automotive electronic throttle valve. Besides being a practical application of the UKF theory developed so far, this throttle valve's estimation is *also an innovation on its own*, from the technological point of view.

The electronic throttle valve of vehicles has been intensively improved by the automotive's industry in the last few years. Made up by a circular plate moving around a central axis, the throttle valve is a fundamental mechanism used in almost all modern spark-ignition combustion engines. The throttle's task is to regulate the power produced by the engine, and to do so, the throttle controls the amount of air entering into the combustion chambers. The rich literature has confirmed the importance of improving the throttle's functionality, see for instance [114–122] for a brief account.

The throttle is a single-input single-output process. When a voltage is applied in its input, the apparatus generates an angular movement of the throttle valve; and a sensor measures the angular position of the valve.

Even though reliable and vastly used by the automotive industry, the sensor of position is not free of failures at all. In case of failure, the throttle's functionality becomes deteriorated, a fact that increases the risks of damage—some specialists argue that the sudden acceleration in Toyota's vehicles are related to failures in the throttle [123, p. 478-479]. Also, failures in the throttle's functionality may appear due to tin whiskers [124, 125]. In summary, failures in the throttle's functionality are unacceptable.

Our main idea to overcome the effects of a failure in the sensor of position is to add in the circuitry a new sensor. This new sensor is detached from the throttle's body, but

it is positioned in series with the throttle’s input so as to measure the electrical current consumed by the throttle. The measure from the new sensor then feeds Unscented Kalman Filters, and so the filters estimate the position of the throttle—notice that the filters rely only on the measurements from this new sensor (Figure 6.1; a wattmeter was added in series with the throttle circuit to measure the electrical current consumed by it [variable i_k]; the real-time position of the throttle [see model in (6.1)] and its estimation from a Unscented Kalman Filter are denoted by θ_k and $\hat{\theta}_k$, respectively; the voltage input is denoted by u_k). Although simple, our idea is motivated by the fact that both the position and electrical current represent system states in the throttle’s model, an intricate nonlinear model [120,126,127]. Estimating the position of the throttle through Unscented Kalman Filters sets the main finding of this chapter.

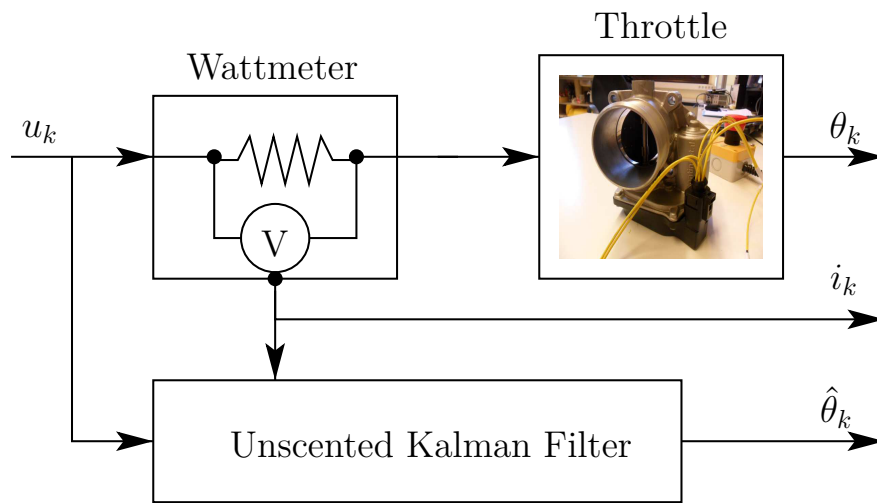


Figure 6.1: Diagram of the input-output relationship for an automotive electronic throttle device implemented in a laboratory testbed.

Unscented Kalman Filters are useful to processes with failures in sensors. For instance, in this chapter Unscented Kalman Filters are used to estimate the position of an automotive throttle valve with no sensor of position at all. The practical implications of the proposed approach is confirmed by accuracy the experimental results (see Section 6.4).

6.1 AUTOMOTIVE ELECTRONIC THROTTLE VALVE

The problem considered in this chapter can be modeled by the following additive stochastic discrete-time system

$$\begin{aligned}x_{k+1} &= f(x_k) + F\varpi_k, \\y_k &= h(x_k) + H\vartheta_k;\end{aligned}$$

where $x_k \in \Phi^{n_x}$ denotes the system's internal state, $y_k \in \Phi^{n_y}$ the measured output, $\varpi_k \in \Phi^{n_\varpi}$ the process noise, $\vartheta_k \in \Phi^{n_\vartheta}$ measurement noise. We suppose that the matrices $F \in \mathbb{R}^{n_x \times n_x}$ and $H \in \mathbb{R}^{n_y \times n_x}$, and the functions $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y}$ are given.

Even though successful for many instances, modeling the throttle remains a challenge since i) its assemblage is not unique, and ii) the throttle presents nonlinear dynamics due to the stick-slip, hysteresis, restoring springs, and limp-home constraints [115,120,126,128,129]. Our approach contributes towards the modeling and estimation of such nonlinear device, as detailed next.

The experiments presented in this section were conducted in a laboratory¹ testbed with the following equipments: a unity of Quanser Q4 Real-Time Control Board that allowed us to communicate real-time data with Matlab-Simulink software; a unity of Quanser UPM180-25-B-PWM Power Amplifier to supply the voltage and electrical current consumed by the equipments; and a unity of the automotive electronic throttle body made up by Continental Siemens VDO, Model A2C59511705, P.N. 06F133062J. The acquisition card of the Quanser Q4 Board was configured to work with data sampling fixed at 1 ms.

The throttle is assembled with an internal sensor of position, which maps the range of operation from zero to ninety degrees into zero to five Volts, in a linear relationship. The velocity of the valve can be computed by a numerical approximation of the derivative of the position. The electrical current (electric power) consumed by the throttle was measured by an ammeter (a wattmeter).

6.2 MODELING

According to [119] and [130], the throttle can be modeled as a piecewise linear system. An advantage of this piecewise setup is that it conveys the simplicity of linear systems to represent the throttle, a nonlinear device. A collateral effect is that of neglecting some significant nonlinear characteristics. Thus it seems reasonable to join these two setups into a single one, i.e., both piecewise linear dynamics [119,130] and nonlinear dynamics [115,120,126,128,129] into a single model.

The automotive electronic throttle body is usually represented by a three-dimensional system [120,126,127,131]; the three states of the system are (i) the angular position

¹In the Control, Dynamics and Applications Laboratory (CoDALab) at the Universitat Politècnica de Catalunya, in Barcelona, Spain. We would like to acknowledge the professors Leonardo Acho (with the CoDALab) and Alessandro Vargas (with the Universidade Tecnológica Federal do Paraná, in Paraná, Brazil) for collaborating on developing the results of this chapter.

of the throttle valve θ , (ii) the angular velocity of the throttle valve ϱ , and (iii) the electrical current consumed by the throttle i . The voltage applied in the terminals of the throttle represents the input of the model (i.e., u), recall the scheme shown in Figure 6.1.

The model used here is based on the physically driven, traditional continuous-time model (e.g. [126, Eq. (6)], [120, Eq. (6)], [127, Eq. (8)])

$$\frac{d}{dt} \begin{bmatrix} \theta_k \\ \varrho_k \\ i_k \end{bmatrix} = \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \theta_k \\ \varrho_k \\ i_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u_t + \begin{bmatrix} 0 \\ \varphi(\theta_k, \varrho_k) \\ 0 \end{bmatrix}, \quad (6.1)$$

where $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes a piecewise linear function. Each paper [120, 126, 127] proposes a distinct format for the function $\varphi(\cdot)$, so that there is no general consensus on $\varphi(\cdot)$.

Interestingly, experimental data indicated that the non-linearities of the throttle are more noticeable when the position of the throttle valve is near to the closed position; the effects of non-linearities decrease as long as the valve opens. This motivated us to split the region of operation of the throttle in three main regions, aiming for improving the throttle's nonlinear representation: $\Theta_1 = [0^\circ, 8^\circ]$, $\Theta_2 = (8^\circ, 16^\circ]$, and $\Theta_3 = (16^\circ, 90^\circ]$.

Under these three regions, we considered a discrete-time version of (6.1)—a discrete-time system was chosen to reduce the computational effort of the Unscented filters; in fact, discrete UF's are usually computationally cheaper than continuous UF's (cf. guideline 1h of Section 5.9)—; namely, with

$$x_k := [0.1 \times \theta_k \quad \varrho_k \quad i_k]^T \in \mathbb{R}^3,$$

the usual Euler discretization is applied in (6.1) to obtain

$$x_{k+1} = \begin{bmatrix} 1 & a_{12}^{(s)} & 0 \\ a_{21}^{(s)} & a_{22}^{(s)} & a_{23}^{(s)} \\ 0 & a_{32}^{(s)} & a_{33}^{(s)} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ b^{(s)} \end{bmatrix} u_k + F \varpi_k + \begin{bmatrix} 0 \\ c_1^{(s)} \operatorname{sgn}(\varrho_k) + c_2^{(s)} \operatorname{sgn}(\theta_k - 1) + c_3^{(s)} \\ 0 \end{bmatrix}; \quad (6.2)$$

$$\theta_k \in \Theta_s, \quad s = 1, 2, 3, \quad \forall k \geq 0;$$

where the values of $a_{12}^{(s)}, \dots, a_{33}^{(s)}, b^{(s)}, c_1^{(s)}, \dots, c_3^{(s)}$, $s = 1, 2, 3$, are available in Table 6.1; these values were identified according to a procedure described later. For the moment, notice in (6.2) that the s th mode is active at the k th stage when θ_k belongs to the set Θ_s .

Table 6.1: Parameters of the nonlinear stochastic model representing an automotive throttle body.

Parameter	$s = 1$	$s = 2$	$s = 3$
$a_{12}^{(s)}$	-0.003	0.0021	0.0442
$a_{21}^{(s)}$	0.148	-0.143	-0.0192
$a_{22}^{(s)}$	0.9625	0.9941	0.7981
$a_{23}^{(s)}$	-0.8673	1.8944	0.3538
$a_{32}^{(s)}$	0.0005	-0.0004	0.0349
$a_{33}^{(s)}$	0.944	0.9514	0.9043
$b^{(s)}$	0.0741	0.0346	0.0442
$c_1^{(s)}$	-0.0654	-0.1068	-0.0055
$c_2^{(s)}$	-0.007	0.0529	0.0615
$c_3^{(s)}$	0.2255	-0.3419	-0.0862

6.3 IDENTIFICATION

Persistent excitation signals were applied in u_k , and the corresponding real-time system state x_k was measured and stored. An amount of 3.8 million of points were used in u_k , and they were carefully chosen so as to excite all the possible input-output relations for the throttle. Indeed, the values of u_k were obtained by passing a train of pseudo-random rectangular pulses, with time-varying random amplitudes (from 0 to 10 Volts), through a fourth-order Butterworth low-pass filter with a cutoff frequency chosen randomly between 0.01 and 60 Hz.

The parameters of (6.2) were chosen so as to minimize the mean square error between part of the collected data and the simulated data from (6.2) (with $\varpi_k \equiv 0$). In this procedure, we used three blocks of data, and each block contained input-output data with ten thousand points generated via persistent excitation signals plus a DC offset.

After obtaining the parameters of (6.2) (cf. Table 6.1), we checked the statistical properties of the term ϖ_k , as follows. We calculated the error $e_k = x_k - \tilde{x}_k$, where x_k satisfies (6.2) with $\varpi_k \equiv 0$ and \tilde{x}_k represents the corresponding real-time measured point; in this evaluation, we used all the previously stored 3.8 million of points. Based on the calculated error, we made a statistical analysis (see Figure 6.2 for a pictorial illustration), which suggested that $\{\varpi_k\}$ is a Gaussian stationary process and F in (6.2) is

$$F = \text{diag}(\sqrt{0.35}, 0, \sqrt{0.18}).$$

A minor bias was detected in e_k with mean error of 1.5° for angular position and -0.12 A for electrical current (see Figure 6.2). Although the error bias was not repre-

sented in the model (6.2), it was accounted appropriately in the estimation procedure, the main experimental part of this chapter, to be detailed next.

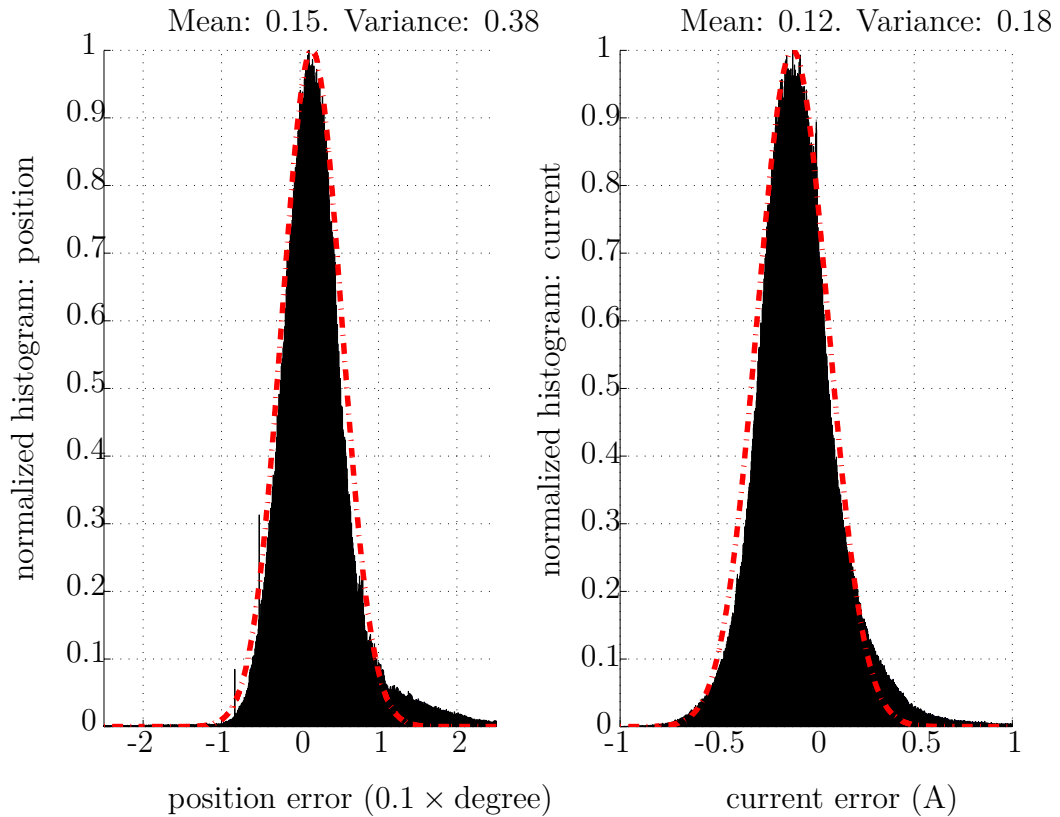


Figure 6.2: Automotive electronic throttle device: normalized histogram showing the error between the model and real-time data. The picture in the left (right) shows the error for the position (electrical current) of the throttle. The histograms tend to follow Gaussian functions with null mean and variance as indicated.

6.4 CASE STUDY: AUTOMOTIVE ELECTRONIC THROTTLE VALVE WITHOUT SENSOR OF POSITION

As previously discussed, a failure in the sensor of position is undesirable because it increases the risks of damage (e.g., [124]). To mitigate the effects of an eventual failure in the sensor of position, we suggest the use of Unscented Kalman Filters accompanied by measurements from an additional sensor, detached from the throttle’s structure but connected to it electronically, as shown in Figure 6.1. Showing the usefulness of this simple strategy represents the main contribution of this chapter.

To clarify our main contribution, we assume hereafter that the sensor of position is damaged. In this situation, we use a wattmeter in the circuitry of the throttle, as depicted in Figure 6.1.

Remark 6.1. Any instrument generating measurements that depend on the current i_k could be used in place of a wattmeter. For instance, the wattmeter reads the power consumption i_k^2 plus some imprecision ϑ_k , i.e.,

$$y_k = i_k^2 + \vartheta_k, \quad \forall k \geq 0, \quad (6.3)$$

where $\{\vartheta_k\}$ represents a standard Gaussian stationary noise. With $h(\cdot)$ being any continuous function, instruments giving measurements in the form $y_k = h(i_k) + \vartheta_k$ could be considered in place of (6.3). In our experiments, the wattmeter was the chosen sensor due to its low-cost.

The value of measurements y_k fed the Unscented Kalman Filters, which produce $\hat{\theta}_k$, an estimation of the position θ_k . Generating $\hat{\theta}_k$ in practice for the automotive throttle device reinforces the contribution of this chapter.

We use the following Additive Unscented Kalman Filters (AdUKF's):

1. Homogeneous Minimum Symmetric Additive Unscented Kalman Filter (HoMiSyAdUKF, Tab 5.3 [1,3]), which is equivalent to the UKF of [1] (second row of Table 2.3);
2. Rho Minimum Additive Unscented Kalman Filter (RhoMiAdUKF, Tab 5.1 [1,3]);
3. Minimum Additive Unscented Kalman Filter (MiAdUKF, Tab 5.1 [1,2]).

These three filters were evaluated in simulation and experiments with $n = 3$, $\hat{x}_{0|0} = [0 \ 0 \ 0]^T$, and $\hat{P}_{xx}^{0|0} = I$, as follows.

1. (Simulation). Two million points were considered in the input u_k . Then these points were used in (6.2) to compute both the statistical mean of (6.2), say \bar{x}_k , and the estimation value from the AdUKF's, say $\hat{x}_{k|k}$. The position error is obtained by extracting the first element from the computed vectors to obtain $e_k = \hat{\theta}_{k|k} - \bar{\theta}_k$.
2. (Experiment). The same input u_k used in the previous item [1. (Simulation)] was also used in the laboratory testbed to generate \tilde{y}_k , which denotes the value collected from the wattmeter in practice. Both u_k and \tilde{y}_k were applied in the AdUKF's to generate a estimations of the system state, say $\tilde{x}_{k|k}^{\text{est}}$. The first element of the vector $\tilde{x}_{k|k}^{\text{est}}$ is $\tilde{\theta}_{k|k}^{\text{est}}$, the estimated position. The sensor of position was used to generate $\tilde{\theta}_k$, the real value of the position of the throttle. Finally, the error produced by the estimation procedure was computed in $e_k = \tilde{\theta}_{k|k}^{\text{est}} - \tilde{\theta}_k$.

Table 6.2 presents the values of the mean and standard deviation of the error for the three filters for both cases, simulation and practice. As expected, the error in the simulation is smaller than the one observed in practice.

Table 6.2: Measure of the mean and standard deviation of the error produced by Unscented Kalman Filters when they were used to estimate the position of an automotive throttle body.

UKF Filters	Simulation		Experiment	
	Mean (°)	Std (°)	Mean (°)	Std (°)
HoMiSyAdUKF	-0.090	4.002	2.206	6.749
RhoMiAdUKF	-0.071	4.061	2.225	6.696
MiAdUKF	-0.078	3.983	2.219	6.560

From Table 6.2, it can be said that all filters produced a practical error of around $2.2^\circ \pm 13.4^\circ$ with a confidence interval of 95% (*c.f.* [132, Sec. D3, p. 553]). This signifies that the filters recovered the information of the position in practice with a precision close to $2.2^\circ \pm 13.4^\circ$. Subtracting the result by the bias error of 1.5° observed in the model (see Section 6.2), the estimation can be adjusted to the improved value $0.7^\circ \pm 13.4^\circ$. These findings reinforce the contribution of this chapter.

Concerning the individual performance of each filter, the UKF introduced in this work (the MiAdUKF) provided the smallest standard deviation in both the simulation (3.983, Tab 6.2 [4,3]) and experimental cases (6.560, Tab 6.2 [4,5]). The superior performance of the MiAdUKF over the HoMiSyAdUKF is further highlighted by the difference in their computational effort; the MiAdUKF (and the RhoMiUKF) is lighter—it uses $n_x + 1$ sigma points—than the HoMiSyAdUKF—it uses $2n_x + 1$ sigma points. Summarizing, the MiAdUKF was the best filter relative to the computational cost and the estimation quality.

For sake of illustration, part of the data is depicted in Figure 6.3. As can be seen, the estimated position recovered the real position within the prescribed accuracy (*i.e.* $0.7^\circ \pm 13.4^\circ$).

6.5 CONCLUSIONS REGARDING THE ESTIMATION OF THE THROTTLE VALVE

The findings of this chapter have practical implications, with special interest to automotive electronic throttle devices. Throttle device often have a unique sensor that measures the angular position of the throttle’s valve; thus, failures in this solitary sensor increase risks of damage in the whole system. Wishing to mitigate the impact

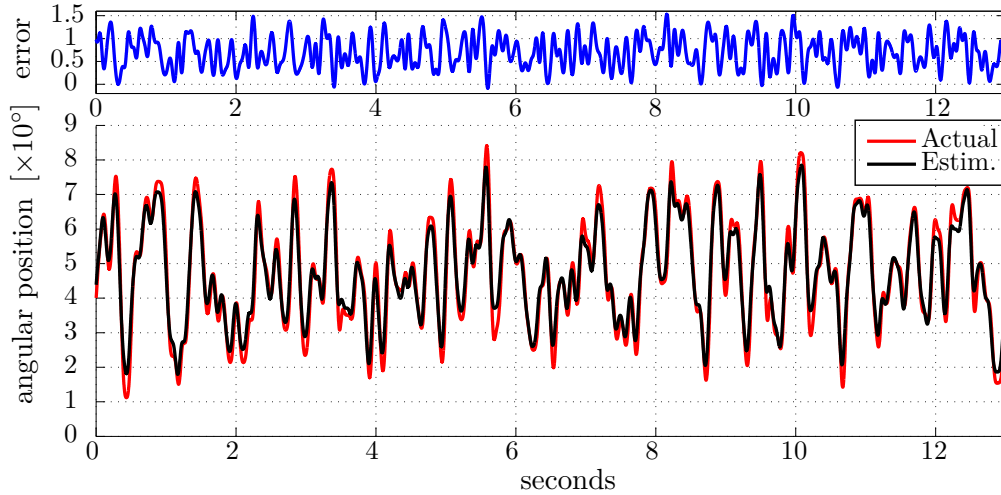


Figure 6.3: Real-time position (measured) and estimated position for an automotive throttle device. The estimated position was calculated by an Unscented Kalman Filter, which was fed only with measurements of the electrical power consumed by the throttle.

of a failure from the sensor of position, we suggest an approach that joins Unscented Kalman Filters with measurements produced by a wattmeter.

The novelty here relies on the use of a wattmeter to measure the electric power consumed by the throttle. As detailed in Remark 6.1, the wattmeter was preferred due to its low cost. However, any other kind of instruments could be used in place of a wattmeter without necessity to modify the proposed technique.

Measurements from the wattmeter feed UKF's, and these filters, in their turn, generate estimates for the position of the throttle. To the best of our knowledge, this work is the first to combine a filter with an external sensor aiming to improve a throttle's functionality.

Experiments that were carried out in laboratory showed promising results—the experimental data suggested an error of $0.7^\circ \pm 13.4^\circ$ (confidence level of 95%) for the estimated position. This finding was quite accurate, since the estimation was taken over a range from 0° to 90° . This evidence corroborates the novelty of this chapter's approach.

This chapter closes Part I. In this part, by reviewing the Unscented Kalman filtering theory's state-of-the-art, we showed some inconsistencies and gaps within this theory (Chapter 2). In consequence, in Chapters 3, 4 and 5 we proposed a systematization that is able to clear these inconsistencies and fill these gaps. Besides, new results were introduced with this systematization. Most of the results provided by this systematization were illustrated in numerical examples. Finally, in this chapter, a new experimental/technological technique was proposed using some of the new UKF's proposed with in the preceding chapter. Summing all the achievements of this part,

we can say that the developed theory so far is elegant, precise, strong; and have been verified in numerical simulations, and practical experiments.

Recall that all this theory developed so far is based on the concepts of stochastic dynamic systems—either in their discrete-time forms (2.1) and (2.2), or in their continuous-time form (5.43) and continuous-discrete-time forms (5.44). Note that, for all these systems, the variables—the state vector, measurement vector, and noises—take values in Euclidean spaces. Such Euclidean systems can be used to model numerous practical problems; yet, for certain practical problems, it might be better to use other classes of systems.

When we want to determine a dynamical model involving *rotations and/or orientations*, it may be advantageous to use *unit quaternions* rather than rotation matrices—these matrices are the natural way to model rotations in an three-dimensional Euclidean space. Hence, we can consider stochastic dynamic systems where at least some of their variables are unit quaternions; in this case, we could inquire whether the systematization developed so far can be extended to such unit quaternion systems or not.

Some fundamental concepts used to develop the theory of the preceding chapters—mainly the ones regarding the theories of probability and statistic—are not yet developed for unit quaternions, particularly. Nonetheless, there are some of these concepts developed for *Riemannian manifolds*, which is a general case of the set of unit quaternions.

Part II

Unscented Kalman Filtering on Riemannian manifolds

7. UNSCENTED KALMAN FILTERING FOR QUATERNION MODELS WITH ADDITIVE-NOISE

Euclidean state space models are adequate for problems whose dynamics can be considered as motion of dimensionless material points, that is, linear displacements and velocities. However, for extensive bodies, besides these linear extension characteristics, the body pointing direction and angular (rotational) movements are important [133–135]. In this chapter we consider filtering for *rotating systems*.

Within Euclidean spaces, rotations of 3-dimensional bodies are mathematically represented by an action (the usual matrix product) of the group of orthogonal 3×3 matrices with determinant equal to 1; these matrices are called *rotation matrices*, and this group is called the *Special Orthogonal Group* and denoted $SO(3)$.

Nevertheless, modeling rotations with *unit quaternions* may be advantageous comparative with rotation matrices. Performing calculations with the $SO(3)$ is often computationally expensive, but we can consider computationally-efficient parameterizations of this group such as Euler-angles, rotation vectors, and unit quaternions¹. Among other good properties, unit quaternions do not have singularities when representing rotations [33]. Unit quaternions form the set of points distanced by 1 (by the usual notion of distance in Euclidean spaces) from the origin of the \mathbb{R}^4 ; this set is called the 3-sphere and denote by S^3 .

We consider the following quaternionic pair of equations modeling a rotating system:

$$\begin{aligned}\mathbf{x}'_k &= f_k(\mathbf{x}'_{k-1}, \boldsymbol{\varpi}'_k), \\ \mathbf{y}'_k &= h_k(\mathbf{x}'_k, \boldsymbol{\vartheta}'_k);\end{aligned}$$

where

1. k is the time step, $\mathbf{x}'_k := (\mathbf{x}_k, x_k)$ the internal state, $\mathbf{y}'_k := (\mathbf{y}_k, y_k)$ the measured output, $\boldsymbol{\varpi}'_k := (\boldsymbol{\varpi}_k, \varpi_k)$ the process noise, and $\boldsymbol{\vartheta}'_k := (\boldsymbol{\vartheta}_k, \vartheta_k)$ the measurement noise;
2. $x_k \in \Phi^{n_x}$, $y_k \in \Phi^{n_y}$, $\varpi_k \in \Phi^{n_\varpi}$, and $\vartheta_k \in \Phi^{n_\vartheta}$; and

¹We say parameterizations of the *group* $SO(3)$ with an abuse of language, because we should rather refer to parameterizations of the *set* of rotation matrices.

3. \mathbf{x}_k , \mathbf{y}_k , $\boldsymbol{\varpi}_k$, and $\boldsymbol{\vartheta}_k$ take values on S^3 ; they are “random unit quaternions”—by “random unit quaternions”, we mean functions mapping from a set of events to S^3 ; in this chapter, we work only with this intuitive notion because the Unscented literature still has not presented a formal definition for these “random unit quaternions”; later, with the theory developed in the following chapters, we will introduce a consistent way of defining the “random unit quaternions”.

We suppose the distributions of $\boldsymbol{\varpi}_k$, $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by Gaussian, multidimensional-real-valued parameterizations². We can find (7.1) being used to model rotations or attitudes concerning spacecrafts [48, 138, 139], inertial navigation systems [48, 140], assisted surgeries [15, 141], pedestrian localization systems [142], and others.

In this chapter, we treat only additive-noise Unscented filters (UF’s)—UKF’s and SRUKF’s—for the system (7.1) because the majority of the UF’s for rotating systems with unit quaternions are additive-noise *filters* (cf. [48, 138, 139, 142–159] and [160])—meaning that, in these Unscented filters, i) the mean and covariance of $\boldsymbol{\varpi}'_k$ are *added*, respectively, to the ones of $f_k(\mathbf{x}_{k-1})$; and ii) the mean and covariance of $\boldsymbol{\vartheta}'_k$ are *added*, respectively, to the ones of $h_k(\mathbf{x}_k)$.

However, for now, we will not consider a *closed-formed* for a *additive-noise quaternion model* [additive-noise cases of (7.1)], because all the additive-noise quaternion *models* associated with the additive-noise *filters* of the literature present problems (see Remark 7.1). Instead, we will work with the following *additive-noise quaternion model*:

$$\begin{aligned}\mathbf{x}'_k &= f_k(\mathbf{x}'_{k-1}) \oplus \boldsymbol{\varpi}'_k, \\ \mathbf{y}'_k &= h_k(\mathbf{x}'_k) \oplus \boldsymbol{\vartheta}'_k;\end{aligned}\tag{7.1}$$

where, for the “random unit quaternions” \mathbf{q} with mean $\bar{\mathbf{q}}$ and covariance $P_{\mathbf{q}}$, and \mathbf{p} with mean $\bar{\mathbf{p}}$ and covariance $P_{\mathbf{p}}$; $\mathbf{q} \oplus \mathbf{p}$ is a well-defined operation (closed under the S^3) with mean

$$\overline{\mathbf{q} \oplus \mathbf{p}} := \bar{\mathbf{q}} \oplus \bar{\mathbf{p}}$$

and covariance

$$P_{\mathbf{q} \oplus \mathbf{p}} := P_{\mathbf{q}} \oplus P_{\mathbf{p}}.$$

We will work with system (7.1) *temporarily*; with the theory developed in the following

²We do not consider the Unscented filters of [136] and [137]; even though this is an Unscented filter for rotating systems with unit quaternions, the system in [136] is modeled with *the Bingham Distribution*, and in [137] with the *von Mises-Fisher Distributions*. These approaches deviate from the analysis of this chapter, which deals with *Gaussian Distributions*. See also the comments at the beginning of Chapter 8.

chapters, we will introduce a consistent way of representing additive-noise quaternion systems.

Extending the Unscented Kalman filtering theory developed in Part I to quaternion models is not trivial. All the UF's pertaining to our Euclidean systematization are composed of i) sums and ii) multiplications by scalars, but unit quaternions are not closed under these operations.

The Unscented literature already has some Unscented filters for quaternions systems. In this chapter, we analyze all the diverse additive UKF's and SRUKF's for quaternion systems proposed in the literature—considering essentially distinct algorithms, we can enumerate the following works [48, 55, 138–140, 142–161]. From this analysis, we show that i) a considerable amount of these filters do not guarantee the quaternion norm to be the unity; and ii) all UKF's preserving the quaternion norm are particular cases of a new algorithm, namely the *Quaternionic Additive Unscented Kalman Filter (QuAdUKF) for additive-noise quaternion models* (Section 7.3.1). Indeed, the QuAdUKF can result in any of additive quaternionic UKF's of the literature by particular choices of a σ -representation, weighted mean method, and vector parameterization of the S^3 (possible choices are provided).

We also introduce a square-root extension of the QuAdUKF, the *Quaternionic Additive Square-Root Unscented Kalman Filter (QuAdSRUKF)*, having better properties than all the SRUKF's for quaternion systems of the literature (Section 7.3.2). By simply choosing a particular σ -representation, a method for the weighted mean, and a vector-parameterization of the S^3 ; we obtain a list of new SRUKF's for quaternion systems having better properties than any SRUKF for quaternion models of the literature.

This superior performance of the QuAdSRUKF is illustrated in the numerical simulations of Section 7.4.2. In these simulations, we show that, in some computationally-unstable conditions, the QuAdSRUKF is able to provide good estimates in scenarios where even the most successful and/or new additive UKF's and SRUKF's for quaternion models fail to do so (Section 7.4.2.3). Furthermore, even in normal (computationally-stable) conditions the QuAdSRUKF outperforms the Unscented filters of the literature by presenting better estimates (Section 7.4.2.2; the second smallest mean error is 10, 56% higher than the error of the new SRUKF).

Remark 7.1. All additive-noise versions of (7.1) associated with the additive-noise *filters* of the literature— [48, 138, 139, 142–159]—present problems. These additive-noise Unscented filters are associated with three classes of models:

1. in [48, 138, 140, 150, 153–155] and [139], the quaternion models are written in the

following form:

$$\begin{aligned}\mathbf{x}'_k &= f_k(\mathbf{x}'_{k-1}) + \boldsymbol{\varpi}'_k, \\ \mathbf{y}'_k &= h_k(\mathbf{x}'_k) + \boldsymbol{\vartheta}'_k;\end{aligned}\tag{7.2}$$

which may result in the state variables \mathbf{x}_k and \mathbf{y}_k taking values outside of S^3 .

2. in [143] and [160], the quaternion part of the process equation is written in the following form:

$$\mathbf{x}_k = f'_k(\mathbf{x}_{k-1}) \otimes \boldsymbol{\varpi}_k\tag{7.3}$$

(\otimes represents the quaternion multiplication; see Section 7.1.1). However, in this case, their additive UF's—recall that, in additive UF's, the mean and covariance of the process noise are *added*, respectively, to the estimate's of the predicted state's mean and covariance—are not consistent with the associated quaternion model because generally, from (7.3), neither the mean of \mathbf{x}_k ($\bar{\mathbf{x}}_k$), is given by $\overline{f'_k(\mathbf{x}_{k-1})} + \bar{\boldsymbol{\varpi}}_k$; nor the covariance of \mathbf{x}_k ($P_{\mathbf{x}_k \mathbf{x}_k}$) is given by $P_{f'_k(\mathbf{x}_{k-1})f'_k(\mathbf{x}_{k-1})} + P_{\boldsymbol{\varpi}_k \boldsymbol{\varpi}_k}$.

3. [144, 145, 147–149, 151] and [152], the quaternion noises are *not written* in the considered system. Even though the UF's are with the means and covariances of the process and measurement noises, the equations of the quaternion models are presented *without* these noises. Naturally, in these cases, we can not determine what is the considered *noisy* model.

From the analysis of this Remark, we can say that writing a consistent additive-noise version of (7.1) is not trivial. With the theory developed in the following chapters, we will present a consistent way of representing additive-noise quaternion systems (see Section 9.3.1).

Before presenting the additive UF's for quaternion models of the literature, we present in the next section i) the main concepts of the quaternion algebra, ii) how unit quaternions relates with rotations, and iii) how to parameterize the set of unit quaternions with vector spaces; we will need these concepts when developing the Quaternionic UF's.

The Unscented filters for quaternion models of the literature are analyzed in Section 7.2. By investigating how each of these filters, we i) divide these filters under some

categories, one of them being the filters preserving the norm of the unit quaternions at every step; and ii) identify and classify, particularly, the solution given by each of these filters to each one of the steps in UF's for Euclidean systems that are difficult to extend to the case of UF's for quaternion models.

Afterwards, in Section 7.3, we i) unify all the UKF's preserving the norm of the unit quaternions at every step in one single algorithm, the QuAdUKF; and ii) introduce a square-root variant of this unifying algorithm, the QuAdSRUKF, having better computational properties comparative with all the SRUKF's of the literature.

Finally, in Section 7.4, we illustrate some of the results of the chapter in numerical simulations.

7.1 QUATERNIONS AND THEIR PARAMETERIZATIONS

Quaternions form a four-dimensional algebra over the real numbers and can be used to parameterize the $SO(3)$ [162]. By the fact that “globally nonsingular three-dimensional parametrization of the rotation group is topologically impossible”, they are a good choice to represent rotations in comparison to other three dimension parameterizations, such as the Euler angles; unit quaternions are singularity-free representations of rotations [34].

7.1.1 Quaternion Algebra

The algebra of quaternions, denoted by \mathbb{H} , is generated by its basis elements $1, \hat{i}, \hat{j}$ and \hat{k} , whose multiplication is defined pairwise as [162]:

$$-\hat{j}\hat{i} = \hat{i}\hat{j} = \hat{k}, \quad -\hat{k}\hat{j} = \hat{j}\hat{k} = \hat{i}, \quad -\hat{i}\hat{k} = \hat{k}\hat{i} = \hat{j},$$

$$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1;$$

an element of \mathbb{H} is of the form

$$\mathbf{q} := q_1 + \hat{i}q_2 + \hat{j}q_3 + \hat{k}q_4 = q_1 + \mathbf{v}_m q$$

where $q_1, q_2, q_3, q_4 \in \mathbb{R}$ are called the *Euler symmetric parameters* or the *Euler-Rodrigues parameters* [34]; $\mathbf{v}_m := [\hat{i}, \hat{j}, \hat{k}]$ the *imaginary vector unit*; and $q := [q_2, q_3, q_4]^T \in \mathbb{R}^3$ the *quaternion vector*. We call $\text{Re}(\mathbf{q}) := q_1 \in \mathbb{R}$ the *real part* or the *scalar part* of the quaternion, $\text{Im}(\mathbf{q}) := q$ the *imaginary part* of the quaternion (in analogy with

standard complex numbers). We should take care to the fact that some works define a quaternion by interchanging the order of the real and imaginary parts such that $\mathbf{q} := \hat{i}q_1 + \hat{j}q_2 + \hat{k}q_3 + q_4$ (cf. [48, 161]). The sum (subtraction) of two quaternions $\mathbf{a} = a_1 + \hat{i}a_2 + \hat{j}a_3 + \hat{k}a_4$ and $\mathbf{b} = b_1 + \hat{i}b_2 + \hat{j}b_3 + \hat{k}b_4$ is defined by

$$\mathbf{a} \pm \mathbf{b} := a_1 \pm b_1 + \hat{i}(a_2 \pm b_2) + \hat{j}(a_3 \pm b_3) + \hat{k}(a_4 \pm b_4),$$

and the multiplication by

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} &:= (a_1 + \hat{i}a_2 + \hat{j}a_3 + \hat{k}a_4) (b_1 + \hat{i}b_2 + \hat{j}b_3 + \hat{k}b_4) \\ &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) + \hat{i}(a_1b_2 + b_1a_2 + a_3b_4 - a_4b_3) \\ &\quad + \hat{j}(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2) + \hat{k}(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1). \end{aligned} \quad (7.4)$$

For a quaternion \mathbf{q} , $\mathbf{q}^{-1} \in \mathbb{H}$ is its inverse if

$$\mathbf{q} \otimes \mathbf{q}^{-1} = \mathbf{q}^{-1} \otimes \mathbf{q} = \mathbf{1}.$$

In analogy with complex numbers, the *norm* and the *conjugate* of \mathbf{q} are defined in order to calculate the inverse of an arbitrary quaternion. The conjugate of a quaternion \mathbf{q} , $\mathbf{q}^* \in \mathbb{H}$, is given by

$$\mathbf{q}^* := \text{Re}(\mathbf{q}) - \mathbf{i}_m \text{Im}(\mathbf{q});$$

and the norm by

$$\|\mathbf{q}\| := \sqrt{\text{Re}^2(\mathbf{q}) + \text{Im}^T(\mathbf{q}) \text{Im}(\mathbf{q})}.$$

If $\|\mathbf{q}\| \neq 0$, then

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2}.$$

If $\|\mathbf{q}\| = 1$, we call \mathbf{q} a *unit quaternion* or *quaternion of rotation*. The set of unit quaternions forms a group under the quaternion multiplication defined in (7.4), but *not* under the sum nor the scalar multiplication [33], hampering the creation of UKF's for quaternion systems (see Section 7.2). For a rotation of an angle θ around an unit vector n^* , there are two associated unit quaternions \mathbf{q} and \mathbf{q}' such that [162]

$$\mathbf{q} = \cos\left(\frac{\theta}{2}\right) + \mathbf{i}_m n^* \sin\left(\frac{\theta}{2}\right), \quad \mathbf{q}' = -\mathbf{q}.$$

Therefore the $SO(3)$ can be parameterized by unit quaternions, but the set of all unit quaternions covers the $SO(3)$ twice.

7.1.2 Vector Parameterizations of Unit Quaternions

Unit quaternions might be a good choice to model rotations. Sometimes, nevertheless, computations of unit quaternions may become problematic, and it may be convenient to use *vector parameterizations of the S^3* such as *rotation vectors (RoV's)*, *generalized Rodrigues vectors (GeRV's)* or *quaternion vectors (QuV's)*—if $v = [v_1, \dots, v_n]^T$ is one of these vector parameterizations, then the scalars v_1, \dots, v_n are the *parameters* of these parameterizations, e.g., if $v = [v_1, v_2, v_3]^T$ is a GeRV, then v_1, v_2 , and v_3 are the parameters; indeed they are known as the GeRV parameters (cf. [48, 163]).

For a unit quaternion $\mathbf{q} := q_1 + \mathbf{v}_m^T \mathbf{q}$ with $\|\mathbf{q}\| \neq 0$, the RoV q_{RoV}^v associated with \mathbf{q} is given by

$$\text{QtoRoV}(\mathbf{q}) := q_{RoV}^v \quad (7.5)$$

where [148]:

$$q_{RoV}^v := 2 \arccos(q_1) \frac{q}{\|\mathbf{q}\|}; \quad (7.6)$$

and the inverse transformation, for $\|q_{RoV}^v\| \neq 0$, is given by

$$\text{RoVtoQ}(q_{RoV}^v) := \mathbf{q} \quad (7.7)$$

where

$$\mathbf{q} := \cos\left(\frac{\|q_{RoV}^v\|}{2}\right) + \mathbf{v}_m \frac{q_{RoV}^v}{\|q_{RoV}^v\|} \sin\left(\frac{\|q_{RoV}^v\|}{2}\right). \quad (7.8)$$

A GeRV can be viewed as a stereographic projection of a unit quaternion. As the name generalized Rodrigues vector suggest, (standard) Rodrigues vectors are particular cases of GeRV's [163]. While Rodrigues vectors have singularities at $q_1 = 0$, GeRV's can modify the location of its singularities by changing a tuning parameter (a below).

For a unit quaternion $\mathbf{q} := q_1 + \mathbf{v}_m \mathbf{q}$, the GeRV $q_{GeRV}^v \in \mathbb{R}^3$ associated with \mathbf{q} is given by

$$\text{QtoGeRV}(\mathbf{q}) := q_{GeRV}^v$$

where

$$l := 2(a + 1), \quad q_{GeRV}^v := l \frac{q}{a + q_1}, \quad (7.9)$$

$a \neq -1$ is a chosen parameter and l is a scaling factor (see [48] for more details). The inverse transformation is, for $a \neq -1$, given by

$$\text{GeRVtoQ}(q_{GeRV}^v) := \mathbf{q}$$

where

$$\begin{aligned}
l &:= 2(a + 1), \\
q_1 &:= \frac{-a \|q_{GeRV}^v\|^2 + l\sqrt{l^2 + (1 - a^2) \|q_{GeRV}^v\|^2}}{l^2 + \|q_{GeRV}^v\|^2}, \\
q &:= l^{-1} [a + q_1] q_{GeRV}^v, \\
\mathbf{q} &:= q_1 + \mathbf{i}_m q.
\end{aligned}$$

For small rotations, the mapping of an unit quaternion to its own QuV can also be viewed as a parameterization of the S^3 . For $\mathbf{q} := q_1 + \mathbf{i}_m q$, its associated QuV q_{QuV}^v is given by

$$\text{QtoQuV}(\mathbf{q}) := q_{QuV}^v$$

where [138]

$$q_{QuV}^v := q \tag{7.10}$$

and the inverse transformation, for $\|q_{QuV}^v\| \leq 1$, by

$$\text{QuVtoQ}(q_{QuV}^v) := \mathbf{q}$$

where

$$\mathbf{q} := \sqrt{1 - \|q_{QuV}^v\|^2} + \mathbf{i}_m q_{QuV}^v. \tag{7.11}$$

However, all these vector parameterizations have limitations. The RoV parameterization has a singularity at the origin, the GeRV presents two singularities [163] and the QuV is only valid for small rotations. In fact, $\text{QtoRoV}(\bullet)$ is not defined for $q \neq 0$ [cf. (7.6)]—for $\|q\| \rightarrow 0$, the limit of $\text{QtoRoV}(\mathbf{q})$ is $2q$ —and the exponential of a unit quaternion is not defined at the origin [$\|q_{RoV}^v\| = 0$; cf. (7.6)]—for $\|q_{RoV}^v\| \rightarrow 0$, the limit of $\text{RoVtoQ}(q_{RoV}^v)$ is $1 + 0.5\mathbf{i}_m q_{RoV}^v$. GeRV's, on their turn, have singularities whose locations depend on the value of the chosen parameter a in the left equation of (7.9) (see [163] and [48] for more details). For instance, consider $a = 0$ (the case of the standard Rodrigues Vector), then, from the right equation of (7.9), the singularities would be the unit quaternions \mathbf{q} with $q_1 = 0$. As for QuV's, from (7.11), the transformation from a QuV q_{QuV}^v to a unit quaternion $\mathbf{q} := q_1 + \mathbf{i}_m q$ is not defined for $\|q_{QuV}^v\| > 1$, since it would result in a complex q_1 (remember that, by definition, $q_1 \in \mathbb{R}$). Besides, (7.10) is, in reality, an *approximating* parameterization of \mathbf{q} which is good only for small values of q_1 . Hence, this transformation cannot be viewed as a parameterization of the entire S^3 (and consequently of the entire rotation group), but only of the part of the S^3 associated with small rotations [in this work, QuV's are called parameterizations of the S^3 in this sense].

It might be convenient to use vector representations of the S^3 *when developing UKF's for quaternion models*. Representing rotations by unit quaternions is convenient in general. In some cases, nonetheless, we need to perform operations that are not well-defined in S^3 such as multiplications by scalars and additions. In such cases, representing unit quaternions by vectors parameterization might be convenient; and developing UKF's for quaternion models is one of these cases where multiplications by scalars and additions are required.

For easy reference, we define the function QtoV standing for any *consistent* vector parameterization of the type $S^3 \rightarrow \mathcal{V}$ (e.g. $\text{QtoV} \in \{\text{RoVtoQ}, \text{GeRVtoQ}, \text{QuVtoQ}\}$) where \mathcal{V} is a vector space; and VtoQ to the inverse of QtoV (e.g. $\text{VtoQ} \in \{\text{QtoRoV}, \text{QtoGeRV}, \text{QtoQuV}\}$).

7.2 UNSCENTED FILTERS FOR QUATERNION SYSTEMS

UKF's and SRUKF's were firstly defined for Euclidean dynamic systems and using them for quaternion models is not trivial. The UF's pertaining to our Euclidean systematization are composed of i) sums and ii) multiplications by scalars (cf. Algorithms 6, 7, 8, and 9), but unit quaternions are not closed under these operations.

For instance, the classical UKF of [2] is given by the following algorithm:

Algorithm 15 (UKF of [2]). *Perform the following steps:*

1. *Previous estimates at time step k .*

(a) $\hat{x}_{k-1|k-1}$, $\hat{P}_{xx}^{k-1|k-1}$ and a measurement y_k are given.

2. *Sigma points*

(a) *Previous sigma points: choose $\kappa > -n_x$ and set, for $1 \leq i \leq n_x$, the sigma points*

$$\chi_0^{k-1|k-1} = \hat{x}_{k-1|k-1} \tag{7.12}$$

$$\chi_i^{k-1|k-1} = \hat{x}_{k-1|k-1} + \left[\sqrt{(n_x + \kappa) \left(\hat{P}_{xx}^{k-1|k-1} + Q_k \right)} \right]_{*i} \tag{7.13}$$

$$\chi_{i+n_x}^{k-1|k-1} = \hat{x}_{k-1|k-1} - \left[\sqrt{(n_x + \kappa) \left(\hat{P}_{xx}^{k-1|k-1} + Q_k \right)} \right]_{*i} \tag{7.14}$$

and set the weights

$$w_0 = \frac{\kappa}{n_x + \kappa}, \quad w_i = w_{i+n_x} = \frac{1}{2(n_x + \kappa)}. \tag{7.15}$$

(b) *Predicted sigma points: for $0 \leq i \leq 2n_x$, set*

$$\chi_i^{k|k-1} = f_k \left(\chi_i^{k-1|k-1} \right), \quad \gamma_i^{k|k-1} = h_k \left(\chi_i^{k-1|k} \right). \quad (7.16)$$

3. *Statistics. Set*

$$\hat{x}_{k|k-1} = \sum_{i=0}^{2n_x} w_i \chi_i^{k|k-1} \quad (7.17)$$

$$\hat{P}_{xx}^{k|k-1} = \sum_{i=0}^{2n_x} w_i \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1} \right) (\diamond)^T \quad (7.18)$$

$$\hat{y}_{k|k-1} = \sum_{i=0}^{2n_x} w_i \gamma_i^{k|k-1} \quad (7.19)$$

$$\hat{P}_{yy}^{k|k-1} = \sum_{i=0}^{2n_x} w_i \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right) (\diamond)^T + R_k \quad (7.20)$$

$$\hat{P}_{xy}^{k|k-1} = \sum_{i=0}^{2n_x} w_i \left(\chi_i^{k|k-1} - \hat{x}_{k|k-1} \right) \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right)^T. \quad (7.21)$$

4. *Posterior estimates. Set*

$$G_k = \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1} \quad (7.22)$$

$$\nu_k = y_k - \hat{y}_{k|k-1} \quad (7.23)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + G_k \nu_k \quad (7.24)$$

$$\hat{P}_{xx}^{k|k} = \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \quad (7.25)$$

Some equations within this algorithm are composed of sums of unit quaternions, and/or multiplications of unit quaternions by scalars. Naturally, these operations most often result in non unit quaternions. They happen on the computation of the [we will refer to the following items as *problematic operations* (po's)]:

1. previous state's σ R: (7.13) and (7.14);
2. predicted state's estimate: (7.17);
3. predicted state's covariance: (7.18);
4. predicted measurement's estimate: (7.19);
5. predicted measurement's covariance: (7.20);
6. predicted cross-covariance: (7.21);
7. innovation term: (7.23);

8. posterior state’s estimate: (7.24).

Similar analyses can be developed for each of the particular version of the UF’s for chapter 5. In order to develop consistent UKF’s for quaternion models, we must give proper solutions to the problems concerning each of these equations when quaternion algebra is considered.

Within the literature, more than one solution has been given to the problem of creating additive UKF’s and SRUKF’s for quaternion systems—e.g. [48, 55, 138–140, 142–161]. Some works use the *same* algorithms of the UF’s for real systems in quaternions systems (these works are not considered in the comparative study that follows), that is, they do not take into consideration the norm restriction (e.g. [15] and [141]); others do take it into consideration, and can be divided in three groups according to *how* they treat this constraint:

1. a first group treats the norm constraint of the unit quaternions, but *do not* preserve them in any po (first row of Table 7.1);
2. a second group preserves the norm of the unit quaternions norms *only in some* (but not all) po’s (second row of Table 7.1);
3. and a third group preserves the norm of the unit quaternions norms in *all* the po’s (third row of Table 7.1).

Table 7.1: Classification of additive UF’s for quaternion models of the literature according to how these filters treat the norm constraint of the unit quaternions.

Unscented Filters	Algebraic characteristics
[55, 142, 147, 149, 156–158]	treat the norm constraint, but do not preserve them in any po
[143–146, 151, 153, 159, 160]	preserve the norm constraint <i>only in some</i> po’s
[48, 138–140, 148, 150, 152, 154, 155, 161]	preserve the norm constraint <i>in all</i> po’s

Among the group 1), essentially two approaches can be found. First, in [55], three UKF’s for systems subjected to a constraint equation are presented, to name the Equality-Constrained Unscented Kalman Filter (ECUKF), Projected Unscented Kalman Filter (PrUKF) and Measurement-Augmentation Unscented Kalman Filter (MAUKF), also used by [142, 147, 149, 157] and [158]. “These approaches do not guarantee that the non-linear equality constraint... is exactly satisfied, but they provide approximate solutions” [55]; a combination of the PrUKF with the MAUKF is shown to increase their performances [156]. Second, in [145, 151, 153], a normalization is performed after the posterior estimate of the state is calculated (see Section 7.2.2.1

for some restrictions concerning this technique) in the following way: suppose that a *quaternion* corrected estimate of the state $\hat{\mathbf{x}}'_{k|k} \in \mathbb{H}$ ($\hat{\mathbf{x}}'_{k|k} \notin S^3$) is given, then the *unit quaternion* corrected estimate is

$$\hat{\mathbf{x}}_{k|k} = \frac{\hat{\mathbf{x}}'_{k|k}}{\|\hat{\mathbf{x}}'_{k|k}\|}.$$

In the UKF's of the group 2), the quaternion norm is guaranteed to be the unity in some steps, but in others not: in the po 8) in [143]; 2) in [160]; in po's 1), 3), and 6) in [144]; 1), 3), 6) and 8) in [145, 146] and [159]; 1), 2), 3) and 6) in [151] and [153].

All the filters in the group 3) use vector parameterizations of the S^3 (see Section 7.1.2) in order to treat the po's; they are studied in the following subsections.

7.2.1 Previous State's Sigma Representation

Table 7.2 presents the σR 's used in each UF of the literature. All the σR 's require operations of additions and/or scalings. As a result, in order to obtain σR 's for quaternion state variables, all the UF's preserving the norm of the unit quaternion use some vector parameterizations of S^3 (see Section 7.1.2).

Table 7.2: Sigma-representations used by each of the UKF's and SRUKF's for quaternion systems.

Unscented Filters	σR or sigma set (SS)
[48, 143, 147, 152, 155, 161, 164]	HoMiSy σR (Corollary 3.4)
[55, 138, 139, 145, 148, 154]	σR of [41] (Table 2.1 [4,1])
[142, 144, 151, 153, 157, 158, 160]	
[140, 146, 150]	SS of [46] (Table 2.1 [2,2]) ^a

^aThe set of [46] is, generally, not a σR because it matches the mean and the covariance of the previous random vector *only* for the scalar case (cf. [23]). We keep it in this classification in order to simplify the exposition.

In the following, for a quaternion $\mathbf{q} \in S^3$, q^v stands for any vector parameterization of \mathbf{q} . Since the vector parameterizations of the S^3 present limitations (cf. Section 7.1.2), some UKF's for quaternion systems of the literature use *deviation quaternions* (or *error quaternions* as in [48]), which are *intended* to represent small rotations (we have not seen any proof in this sense), and hence can possibly overcome these limitations. A deviation quaternion is represented and defined with $\tilde{\mathbf{q}}$; and its vector parameterization, called *deviation vector*, by \tilde{q}^v .

Consider the system (7.2), and suppose that at time step k , $\hat{\mathbf{x}}_{k-1|k-1}$ (the previous state's estimated) and $\hat{P}_{xx}^{v,k|k-1}$ (the covariance of a vector parameterization of the previous state) are given. Consider also the function $\sigma R(\cdot)$ defined in (3.9). Then the

previous deviation vector σ -representation is obtained by

$$\tilde{\chi}^{v,k-1|k-1} := \left\{ \tilde{\chi}_i^{v,k-1|k-1}, w_i^m, w_i^c, w_i^{cc} \right\}_{i=1}^N = \sigma R \left([0]_{1 \times n_x}, \hat{P}_{xx}^{v,k-1|k-1} \right) \quad (7.26)$$

in [142, 144, 145, 148, 150, 152, 160]; or by

$$\tilde{\chi}^{v,k-1|k-1} := \left\{ \tilde{\chi}_i^{v,k-1|k-1}, w_i^m, w_i^c, w_i^{cc} \right\}_{i=1}^N = \sigma R \left([0]_{1 \times n_x}, \hat{P}_{xx}^{v,k-1|k-1} + Q_k^v \right) \quad (7.27)$$

in [48, 138, 143, 149, 155, 159] where $Q_k^v \in \mathbb{R}^{3 \times 3}$ is the covariance of a vector parameterization of ϖ_k (the quaternion part of the process noise). In the UKF's where $\tilde{\chi}^{v,k-1|k-1}$ is defined as in (7.26), $\hat{P}_{xx}^{v,k|k-1}$ (Section 7.2.3) should be calculated by (7.36); likewise where $\tilde{\chi}^{v,k-1|k-1}$ is defined as in (7.27), $\hat{P}_{xx}^{v,k|k-1}$ should be calculated by (7.37). The influence upon the UKF's of choosing between the pairs (7.26),(7.36) and (7.27),(7.37) was not considered in the literature yet.

The sigma points $\tilde{\chi}_i^{v,k-1|k-1}$ are supposed to be *deviation vector parameterizations* sigma points; the *deviation quaternion* sigma points $\tilde{\chi}_i^{k-1|k-1}$ are calculated by

$$\tilde{\chi}_i^{k-1|k-1} = \text{VtoQ} \left(\tilde{\chi}_i^{v,k-1|k-1} \right), \quad i = 1, \dots, N;$$

where $\text{VtoQ} = \text{RoVtoQ}$ in [138, 143, 148, 152, 160]; $\text{VtoQ} = \text{GeRVtoQ}$ in [48, 139, 154, 155]; and $\text{VtoQ} = \text{QuVtoQ}$ in [138, 140, 144, 150, 161] (in [138] the two possibilities $\text{VtoQ} = \text{RoVtoQ}$ and $\text{VtoQ} = \text{QuVtoQ}$ are considered, but only in this po, whilst in the others, only QuV 's are considered); Table 7.3 summarizes these relations. The *quaternion* sigma points $\chi_i^{k-1|k-1}$ are then calculated by

$$\chi_i^{k-1|k-1} = \tilde{\chi}_i^{k-1|k-1} \otimes \hat{\mathbf{x}}_{k-1|k-1}, \quad i = 1, \dots, N.$$

Table 7.3: Vector parameterization of the S^3 used by the additive UF's of the literature.

Unscented Filters	Vector parameterization of the S^3
[138, 143, 148, 152, 160]	Rotation Vector
[48, 139, 154, 155]	Generalized Rodrigues Vector
[138, 140, 144, 150, 161]	Quaternion Vector ^a

^aThe set of quaternion vectors parameterize the S^3 only for small rotations (cf. Section 7.1.2).

7.2.2 Predicted State Estimate

The calculation of the predicted state estimate (either in the form of a unit quaternion $\hat{\mathbf{x}}_{k|k-1}$ or a deviation vector parameterization $\tilde{\hat{x}}_{k|k-1}^v$) is also difficult for UKF's for quaternion systems; in this section, the solutions given by the literature to this

problem (Table 7.4) are described. For this, consider that a set of weighted predicted quaternion sigma points

$$\boldsymbol{\chi}^{k|k-1} := \left\{ \boldsymbol{\chi}_i^{k|k-1}, w_i^m, w_i^c, w_i^{cc} \mid \boldsymbol{\chi}_i^{k|k-1} = f_k(\boldsymbol{\chi}_i^{k-1|k-1}) \right\}_{i=1}^N$$

is given.

Table 7.4: Methods to calculate the sample weighted means in the additive UF's of the literature.

Unscented Filters	Method for the weighted mean
[138, 140, 144, 146, 152]	FN (Section 7.2.2.1)
[48, 139, 150, 154]	DPPSE(Section 7.2.2.2)
[143, 146, 148]	GDA (Section 7.2.2.3)
[161]	MQVCF (Section 7.2.2.4)
[155]	MAMCF (Section 7.2.2.5)

7.2.2.1 Forced Normalization (FN)

The works of [138, 140, 144, 146, 152] performs a *forced normalization* (FN), which consists of computing the weighted mean as in the real case (7.17) and dividing this results by its own norm:

$$\hat{\boldsymbol{x}}_{k|k-1} = \frac{\sum_{i=1}^N w_i^m \boldsymbol{\chi}_i^{k|k-1}}{\left\| \sum_{i=1}^N w_i^m \boldsymbol{\chi}_i^{k|k-1} \right\|}. \quad (7.28)$$

Then $\tilde{\hat{\boldsymbol{x}}}_{k|k-1}^v$ is given by

$$\tilde{\boldsymbol{\chi}}_i^{k|k-1} = \boldsymbol{\chi}_i^{k|k-1} \otimes \left(\hat{\boldsymbol{x}}_{k|k-1} \right)^{-1}, \quad 1 \leq i \leq N \quad (7.29)$$

$$\tilde{\boldsymbol{\chi}}_i^{v,k|k-1} = \text{QtoV}(\tilde{\boldsymbol{\chi}}_i^{k|k-1}), \quad 1 \leq i \leq N \quad (7.30)$$

$$\tilde{\hat{\boldsymbol{x}}}_{k|k-1}^v = \sum_{i=1}^N w_i^m \tilde{\boldsymbol{\chi}}_i^{v,k|k-1}, \quad (7.31)$$

where QtoV = QtoRoV in [138, 152]; and QtoV = QtoQuV in [138, 140, 144].

For the distance function

$$\text{dist}(\boldsymbol{q}_1, \boldsymbol{q}) := 2 \arccos(\boldsymbol{q}_1, \boldsymbol{q})$$

(the S^3 geodesic from \boldsymbol{q}_1 to \boldsymbol{q}_2), we can consider its Taylor expansion round \boldsymbol{q}_1 :

$$\text{dist}(\boldsymbol{q}_1, \boldsymbol{q}) = \text{dist}(\boldsymbol{q}_1, \boldsymbol{q})^{[\boldsymbol{q}_1, 1]} + \text{HOT},$$

where $\text{dist}(\mathbf{q}_1, \mathbf{q}_2)^{[q_1, 1]}$ is the first order term, and HOT stand for the remaining of this expansion. Then, the work [138] showed that $\hat{\mathbf{x}}_{k|k-1}$ in (7.28) is also

$$\hat{\mathbf{x}}_{k|k-1} = \arg \min_{\mathbf{q} \in S^3} \sum_{i=1}^N w_i^m \left(\text{dist}(\mathbf{q}_1, \mathbf{q})^{[q_1, 1]} \right)^2.$$

The FN, however, is often a rough approximation since each one of the sums in (7.28) probably leads to a non-unit quaternion, and therefore to a value that has not the physical meaning of a rotation. For $R(\theta, n^*)$ standing for a rotation of an angle θ around the axis n^* with $\|n^*\| = 1$, consider the rotations $R(\theta_i, n^*)$, $1 \leq i \leq N_r$. The mean rotation is

$$R_{\text{mean}} := R\left(\frac{\sum_{i=1}^{N_r} \theta_i}{N_r}, n^*\right);$$

suppose $N_r = 3$, $\theta_1 = 10^\circ$, $\theta_2 = 30^\circ$, $\theta_3 = -7^\circ$, and $n^* = [3^{-1/2}]_{3 \times 1}$; then we have that

$$R_{\text{mean}} = R\left(11^\circ, \left[\frac{1}{\sqrt{3}}\right]_{3 \times 1}\right) \approx R(11^\circ, [0.58]_{3 \times 1}),$$

and the unit quaternions associated with R_{mean} are

$$q_{\text{mean}} = \pm (0.9816 + \mathbf{v}_m [0.1102]_{3 \times 1}).$$

Define the quaternion representation of each rotation $R(\theta_i, [3^{-1/2}]_{3 \times 1})$, $\theta_1 = 10^\circ$, $\theta_2 = 30^\circ$, $\theta_3 = -7^\circ$, by

$$\mathbf{q}_i = \cos(\theta_i) + \mathbf{v}_m \sin(\theta_i) [3^{-1/2}]_{3 \times 1};$$

then, from (7.28), the forced normalization quaternion is given by

$$\begin{aligned} \mathbf{q}_{\text{FN}} &:= \frac{\sum_{i=1}^3 \mathbf{q}_i}{\|\sum_{i=1}^3 \mathbf{q}_i\|} \\ &= 0.3389 + \mathbf{v}_m [0.0380]_{3 \times 1}. \end{aligned}$$

The unit quaternion \mathbf{q}_{FN} is quite different from q_{mean} . Moreover, the rotation associated with \mathbf{q}_{FN} , R_{FN} , is

$$R_{\text{FN}} = R(140^\circ, [0.04]_{3 \times 1}),$$

which is quite different from R_{mean} . Note that we are considering rotations around the *same* axis; probably, rotations around different axes would result in even more different rotations; nevertheless, when considering smaller rotations, the FN should give better results.

In the simulations of this work (Section 7.4) the filters based in this method were numerically unstable in some scenarios (cf. Table 7.6).

7.2.2.2 Direct Propagation of the Previous State's Estimate (DPPSE)

In [48,139,150,154], the predicted deviation vector state's estimate $\tilde{\hat{x}}_{k|k-1}^v$ is obtained by propagating $\hat{\mathbf{x}}_{k-1|k-1}$ through f . First $\tilde{\chi}_i^{k|k-1}$ is obtained by

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1} &= f_k(\hat{\mathbf{x}}_{k-1|k-1}) \\ \tilde{\chi}_i^{k|k-1} &= \chi_i^{k|k-1} \otimes (\hat{\mathbf{x}}_{k|k-1})^{-1}, \quad 1 \leq i \leq N,\end{aligned}\tag{7.32}$$

and afterwards $\tilde{\hat{x}}_{k|k-1}^v$ is obtained by (7.30) and (7.31) with QtoV = QtoGeRV in [48, 139, 150, 154] and QtoV = QtoGeRV in [150]. These works do not calculate the predicted quaternion state's estimate $\hat{\mathbf{x}}_{k|k-1} \in S^3$ because the image of the measurement function h in the system considered by them is Euclidean, and therefore the innovation term can be calculated just with $\tilde{\hat{x}}_{k|k-1}^v$. (cf. Section 7.2.3). It is worthy to note that there is no guarantee that this method will provide a good estimate since the choice of (7.32) is *ad hoc* (we could not find a formal justification for it). Nonetheless, in the simulations of this work (Section 7.4), the filters based in this method provided satisfactory results (cf. Table 7.6).

7.2.2.3 Gradient Descent Algorithm (GDA)

In order to obtain $\hat{\mathbf{x}}_{k|k-1} \in S^3$, the works of [143,146,148] use the intrinsic gradient descent algorithm described in [165]; this algorithm, the GDA, consists in the following:

Algorithm 16 (Gradient Descent Algorithm). *1. Choose a threshold $\epsilon \in \mathbb{R}$, $\epsilon > 0$; and set an initial candidate*

$$\mathbf{q} := f_k(\hat{\mathbf{x}}_{k-1|k-1}).$$

2. Define, for $1 \leq i \leq N$:

$$e_i^v := \sum_{i=1}^N w_i^m \text{QtoRoV}(\chi_i^{k|k-1} \otimes \mathbf{q}^{-1}).$$

3. While ($\|\hat{e}^v\| > \epsilon$):

(a) Define a new candidate

$$\mathbf{q} := \text{RoVtoQ}(e^v) \otimes \mathbf{q},$$

and repeat step 2.

4. Assign the state's predicted mean estimate

$$\hat{\mathbf{x}}_{k|k-1} := \mathbf{q}.$$

In [148] and in the simulations of this work (Section 7.4), this algorithm converges to a satisfactory estimate within 2 to 4 iterations, and the UKF's based on the GDA provided satisfactory results (cf. Table 7.6). Afterwards, $\tilde{\hat{\mathbf{x}}}_{k|k-1}^v$ is obtained by (7.29)-(7.31) with QtoV = QtoRoV.

7.2.2.4 Minimization of a Quaternion Vector Cost Function (MQVCF)

In [161], $\hat{\mathbf{x}}_{k|k-1} \in S^3$ is such that its quaternion vector is the argument that “minimizes the weighted sum of squared length of the error quaternion vector part” [161], that is,

$$\text{Im}(\hat{\mathbf{x}}_{k|k-1}) = \arg \min_{\phi_1 \in S^3} \sum_{i=1}^N \text{Im}(\boldsymbol{\chi}_i^{k|k-1} \otimes \phi_1^{-1})^T \times W_i \text{Im}(\boldsymbol{\chi}_i^{k|k-1} \otimes \phi_1^{-1}) \quad (7.33)$$

where each $W_i \in \mathbb{R}^{3 \times 3}$ is a positive definite weighting matrix. The work [161] shows that, in this case, $\hat{\mathbf{x}}_{k|k-1} = v_{\lambda_{\min}(\Theta^x)}$ where $v_{\lambda_{\min}(\Theta)}$ is the eigenvector associated with the smallest eigenvalue of

$$\Theta^x := w_0 \left(\Psi(\boldsymbol{\chi}_0^{k|k-1}) \right) (\diamond)^T + \sum_{i=1}^N w_i \left(\Psi(\boldsymbol{\chi}_i^{k|k-1}) \right) (\diamond)^T \quad (7.34)$$

where

$$\Psi(\mathbf{q}) := \begin{bmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & q_4 & -q_3 \\ -q_4 & q_1 & q_2 \\ q_3 & -q_2 & q_1 \end{bmatrix} \quad (7.35)$$

is the attitude-matrix of a quaternion $\mathbf{q} := q_1 + \hat{i}q_2 + \hat{j}q_3 + \hat{k}q_4$.

Afterwards $\tilde{\hat{\mathbf{x}}}_{k|k-1}^v$, is obtained by (7.29)-(7.31) with VtoQ = QuVtoQ. This approach does not require the explicit manipulation of (7.33), but only the calculations of the eigenvectors and eigenvalues of Θ^x in (7.34), a 4×4 matrix. Nevertheless, since quaternion vectors represents rotations only for small angles (cf. Section 7.1.2), this approach should provide an accurate estimate of $\mathbf{x}_{k|k-1}$ only for the case when $\boldsymbol{\chi}_i^{k|k-1} \otimes \hat{\mathbf{x}}_{k|k-1}^{-1}$ [from (7.33)] results in quaternions associated with small rotations for each $i = 1, \dots, N$. In the simulations of this work (Section 7.4) the UKF's based in this method were numerically unstable in some scenarios and provided worse results in comparison to the filters based on the other weighted mean methods (cf. Table 7.6).

7.2.2.5 Minimization of an Attitude-Matrix Cost Function (MAMCF)

In [155], $\hat{\mathbf{x}}_{k|k-1} \in S^3$ is the quaternion minimizing the weighted sum of the squared Frobenius norms of the attitude-matrices of each quaternion sigma point, i.e.,

$$\hat{\mathbf{x}}_{k|k-1} = \arg \min_{\mathbf{q} \in S^3} \sum_{i=1}^N w_i^m \left\| \Psi(\mathbf{q}) - \Psi(\chi_i^{k|k-1}) \right\|_F^2$$

where $\Psi(\bullet)$ is defined as in (7.35); and, for a matrix $A \in \mathbb{R}^{p \times q}$ and $\text{Tr}(A)$ being its trace,

$$\|A\|_F^2 := \text{Tr}(A^T A)$$

is its Frobenius matrix norm. It is shown that, in this case, $\hat{\mathbf{x}}_{k|k-1}$ is the eigenvector associated with the maximum eigenvalue of $\Psi(\chi_i^{k|k-1})$ [155, 166]. Afterwards, $\tilde{\hat{x}}_{k|k-1}^v$ is obtained by (7.29)-(7.31) with QtoV = QtoGeRV.

7.2.3 Remaining Problematic Operations

Vector parameterizations of the S^3 are also used to calculate $\hat{P}_{xx}^{v,k|k-1} \in \mathbb{R}^{3 \times 3}$. Suppose that

$$\tilde{\chi}_i^{v,k|k-1} := \left\{ \tilde{\chi}_i^{v,k|k-1}, w_i^m, w_i^c, w_i^{cc} | \tilde{\chi}_i^{v,k|k-1} \in \right\}_{i=1}^N$$

(a set of weighted predicted deviation vector parameterization sigma points), $\hat{x}_{k|k-1}^v$ (the predicted deviation vector parameterization state's estimate) and $Q_k^v \in \mathbb{R}^{3 \times 3}$ (the covariance of a vector parameterization of the process noise ϖ_k) are given; then the predicted state's covariance $\hat{P}_{xx}^{v,k|k-1} \in \mathbb{R}^{3 \times 3}$ is obtained by

$$\hat{P}_{xx}^{v,k|k-1} = \sum_{i=1}^N w_i^c \left(\tilde{\chi}_i^{v,k|k-1} - \hat{x}_{k|k-1}^v \right) (\diamond)^T + Q_k^v \quad (7.36)$$

in [142, 144, 145, 148, 150, 152, 160]; or by

$$\hat{P}_{xx}^{v,k|k-1} = \sum_{i=1}^N w_i^c \left(\tilde{\chi}_i^{v,k|k-1} - \hat{x}_{k|k-1}^v \right) (\diamond)^T \quad (7.37)$$

in [48, 138, 143, 155, 159, 164]. Recall that in the UKF's where $\hat{P}_{xx}^{v,k|k-1}$ is calculated by (7.36), $\tilde{\chi}^{v,k-1|k-1}$ (Section 7.2.1) should be defined as in (7.26); likewise where $\hat{P}_{xx}^{v,k|k-1}$ is calculated by (7.37), $\tilde{\chi}^{v,k-1|k-1}$ should be as in (7.27). This ends the prediction steps and starts the correction ones.

Predicted measurement sigma points are calculated by transforming the predicted quaternion sigma points $\tilde{\chi}_i^{k|k-1}$ through the measurement function h_k . At this time, $\tilde{\chi}^{k|k-1}$ can be regenerated (as in [148, 152, 161]) or not (as in [48, 55, 138–140, 142–146,

149, 150, 155, 158–160]); this regeneration is done by, for $i = 1, \dots, N$,

$$\begin{aligned} \{w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N &= \sigma\text{R} \left([0]_{1 \times n_x}, \hat{P}_{xx}^{v,k|k-1} \right) \\ \boldsymbol{\chi}_i^{k|k-1} &= \text{VtoQ} \left(\tilde{\chi}_i^{v,k|k-1} \right) \otimes \hat{\boldsymbol{x}}_{k|k-1}. \end{aligned}$$

For (7.2), some works— [48, 55, 138, 140, 143–146, 148, 150–152, 154, 155, 160, 164]— consider the measurements belonging only to the Euclidean space ($\boldsymbol{y}'_k = \mathbb{R}^{n_y}$), and [161] to both the unit quaternion set and the Euclidean space [$\boldsymbol{y}'_k = (\boldsymbol{y}_k, y_k)$, $\boldsymbol{y}_k \in S^3, y_k \in \mathbb{R}^{n_y}$]. For the measurements belonging to the Euclidean space— y_k in both cases—the standard UKF equations (7.20), (7.21) and (7.23) are used to calculate the measurement's predicted estimate $\hat{y}_{k|k-1} \in \mathbb{R}^{n_y}$, the covariance $\hat{P}_{yy}^{v,k|k-1} \in \mathbb{R}^{n_y \times n_y}$ and the innovation term $\nu_k^v \in \mathbb{R}^{n_y}$ respectively. $\hat{P}_{xy}^{v,k|k-1} \in \mathbb{R}^{3 \times n_y}$ is calculated by

$$\hat{P}_{xy}^{v,k|k-1} = \sum_{i=1}^N w_i^c \left(\tilde{\chi}_i^{v,k|k-1} - \hat{x}_{k|k-1}^v \right) \left(\gamma_i^{k|k-1} - \hat{y}_{k|k-1} \right)^T.$$

In the case of the measurement being a unit quaternion (\boldsymbol{y}_k), $\hat{\boldsymbol{y}}_{k|k-1}$ is obtained similarly to $\hat{\boldsymbol{x}}_{k|k-1} \in S^3$ (Section 7.2.2);

$$\begin{aligned} \hat{P}_{xy}^{v,k|k-1} &= \sum_{i=1}^N w_i^c \tilde{\chi}_i^{v,k|k-1} (\tilde{\gamma}_i^{v,k|k-1})^T; \\ \hat{P}_{yy}^{v,k|k-1} &= w_i^c \tilde{\gamma}_i^{v,k|k-1} (\tilde{\gamma}_i^{v,k|k-1})^T + R_k^v \end{aligned}$$

where R_k^v is the covariance of the a vector parameterization of the measurement noise $\boldsymbol{\vartheta}'_k$; and $\nu_k^v \in \mathbb{R}^{n_y}$ is given by

$$\nu_k^v = \text{QtoQuV} \left(\boldsymbol{y}_k \otimes \left(\hat{\boldsymbol{y}}_{k|k-1} \right)^{-1} \right).$$

The Kalman Gain G_k is given by (7.22), $\hat{P}_{xx}^{v,k|k}$ by (7.25) and $\hat{\boldsymbol{x}}_{k|k} \in S^3$ by

$$\hat{\boldsymbol{x}}_{k|k} = \text{VtoQ} \left(\hat{\boldsymbol{x}}_{k|k-1}^v + G_k \nu_k^v \right) \otimes \hat{\boldsymbol{x}}_{k|k-1}$$

where $\text{VtoQ} = \text{RoVtoQ}$ in [140, 143, 148, 152, 160]; $\text{VtoQ} = \text{GeRVtoQ}$ in [48, 139, 154, 155] and $\text{VtoQ} = \text{QuVtoQ}$ in [138, 144, 150, 161].

In [138, 148, 161], $\hat{P}_{xx}^{v,k|k-1}$, $\hat{P}_{xy}^{v,k|k-1}$ and $\hat{\boldsymbol{x}}_{k|k}$ are calculated considering $\hat{x}_{k|k-1}^v = 0$. However, in general, $\hat{x}_{k|k-1}^v$ is not zero and, therefore, this does not represent the dispersion of the points around the mean, but only around the origin, at least in the sense of covariances for real valued random variables. In the simulations of this work (Section 7.4), the filters considering $\hat{x}_{k|k-1}^v = 0$ to calculate $\hat{P}_{xx}^{v,k|k-1}$, $\hat{P}_{xy}^{v,k|k-1}$ and $\hat{\boldsymbol{x}}_{k|k}$

provided slightly worse results comparative with the ones not considering so.

7.3 QUATERNIONIC ADDITIVE UNSCENTED FILTERS

This section introduces an algorithm able to gather all the additive UKF's for quaternion models of the literature and also provide new ones (Section 7.3.1). It is based on a new Unscented Transformation defined for this kind of systems; square-root extensions of these UKF's and this UT are also proposed.

7.3.1 Quaternionic Additive Unscented Kalman Filter

After analyzing the literature, we conclude that the additive UKF's of the literature preserving the norm of the unit quaternions in all steps (third row of Table 7.1) can be distinguished from each other by only three elements: i) the σ R, ii) the vector parameterization of the S^3 , and iii) the method for obtaining the weighted mean of the unit quaternion sigma points. This, along with the following definition, enables the construction of a general algorithm gathering all these filters.

For a given weighted set of unit quaternion points χ , the function

$$\mu_\chi := \text{QuatWeightedMean}(\chi)$$

maps χ to the weighted mean μ_χ by one weighted mean method, for example the methods in Table 7.4.

Definition 7.1. Consider the additive-noise quaternion model

$$\begin{aligned}\mathbf{x}'_k &= f_k(\mathbf{x}'_{k-1}) \oplus \boldsymbol{\varpi}'_k, \\ \mathbf{y}'_k &= h_k(\mathbf{x}'_k) \oplus \boldsymbol{\vartheta}'_k;\end{aligned}$$

where

1. $\mathbf{x}'_k := (\mathbf{x}_k, x_k)$, $\mathbf{y}'_k := (\mathbf{y}_k, y_k)$, $\boldsymbol{\varpi}'_k := (\boldsymbol{\varpi}_k, \varpi_k)$, and $\boldsymbol{\vartheta}'_k := (\boldsymbol{\vartheta}_k, \vartheta_k)$;
2. $x_k \in \Phi^{n_x}$, $y_k \in \Phi^{n_y}$, $\varpi_k \in \Phi^{n_\varpi}$, and $\vartheta_k \in \Phi^{n_\vartheta}$; and
3. \mathbf{x}_k , \mathbf{y}_k , $\boldsymbol{\varpi}_k$, and $\boldsymbol{\vartheta}_k$ take values in the S^3 ;

Suppose that i) the distributions of $\boldsymbol{\varpi}_k$, $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by Gaussian, multidimensional-real-valued parameterizations $\varpi_k^v \in \Phi^{n_{\varpi^v}}$, $\vartheta_k^v \in \Phi^{n_{\vartheta^v}}$

and x_0^v , respectively, and ii) the distributions of $\varpi'_k := (\varpi_k^v, \varpi_k)$ and $\vartheta'_k := (\vartheta_k^v, \vartheta_k)$ are given by

$$\begin{aligned}\varpi'_k &\sim ([0]_{n_x \times 1}, Q'_k), \\ \vartheta'_k &\sim ([0]_{n_y \times 1}, R'_k); \end{aligned}$$

iii) the mean of \mathbf{x}'_k is $\bar{\mathbf{x}}'_0$, and the covariance of $\mathbf{x}'_0 := (x_0^v, x_0)$ is $P_{x'x'}^0$; iv) the measurements $\mathbf{y}'_1, \mathbf{y}'_2, \dots, \mathbf{y}'_{k_f}$ are given, where $\mathbf{y}'_k = (\mathbf{y}_k, y_k)$ with $\mathbf{y}_k \in S^3$ and $y_k \in \mathbb{R}^{n_y}$. Then the *Quaternionic Additive Unscented Kalman Filter (for quaternion models)* is composed by the following algorithm:

Algorithm 17 (Quaternionic Additive Unscented Kalman Filter (QuAdUKF)). *Perform the following steps:*

1. *Initialization. Set the initial estimates $\hat{\mathbf{x}}'_{0|0} := \bar{\mathbf{x}}'_0$ and $\hat{P}_{x'x'}^{0|0} := P_{x'x'}^0$; and choose*
 - (a) *two σ -representations, and set the functions $2\sigma R_1$ and $2\sigma R_2$ accordingly;*
 - (b) *a vector parameterization and set the functions $QtoV$ and $VtoQ$ accordingly;*
 - (c) *a method for means of weighted sets composed of unit quaternions and set the function $QuatWeightedMean$ accordingly.*
2. *Filtering. For $k = 1, 2, \dots, k_f$; set the following elements:*

- (a) *Obtain the state sigma points by*

$$\left\{ \left[\begin{array}{c} \tilde{\chi}_i^{v, k-1|k-1} \\ \chi_i^{k-1|k-1} \end{array} \right], w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := 2\sigma R_1 \left([0]_{(n_x v + n_x) \times 1}, \hat{P}_{x'x'}^{k-1|k-1} \right);$$

and

$$\begin{aligned}\boldsymbol{\chi}_i^{k-1|k-1} &:= VtoQ \left(\tilde{\chi}_i^{v, k-1|k-1} \right) \otimes \hat{\mathbf{x}}'_{k-1|k-1}, \quad i = 1, \dots, N_1; \\ \boldsymbol{\chi}'_i^{k-1|k-1} &:= \left(\boldsymbol{\chi}_i^{k-1|k-1}, \chi_i^{k-1|k-1} \right), \quad 1 \leq i \leq N_1; \end{aligned} \quad (7.38)$$

where $\tilde{\chi}_i^{v, k-1|k-1} \in \mathbb{R}^{n_x v}$ and $\chi_i^{k-1|k-1} \in \mathbb{R}^{n_x}$.

- (b) *Obtain the predicted state's estimates by*

$$\begin{aligned}\left(\boldsymbol{\chi}_{*,i}^{k|k-1}, \chi_{*,i}^{k|k-1} \right) &:= f_k \left(\boldsymbol{\chi}'_i^{k-1|k-1} \right), \quad 1 \leq i \leq N_1; \\ \hat{\boldsymbol{\chi}}_{k|k-1} &:= QuatWeightedMean \left(\left\{ \boldsymbol{\chi}_{*,i}^{k|k-1}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc} \right\}_{i=1}^{N_1} \right); \\ \tilde{\chi}_{*,i}^{v, k|k-1} &:= QtoV \left(\boldsymbol{\chi}_{*,i}^{k|k-1} \otimes \hat{\boldsymbol{\chi}}_{k|k-1}^{-1} \right), \quad 1 \leq i \leq N_1; \end{aligned}$$

$$\begin{aligned}
\tilde{\hat{x}}_{k|k-1}^v &:= \sum_{i=1}^{N_1} w_i^{1,m} \tilde{\chi}_{*,i}^{v,k|k-1}; \\
\hat{x}_{k|k-1} &:= \sum_{i=1}^{N_1} w_i^{1,m} \chi_{*,i}^{k|k-1}; \\
\chi_{*,i}'^{k|k-1} &:= \left[\tilde{\chi}_{*,i}^{v,k|k-1}, \chi_{*,i}^{k|k-1} \right]^T; \\
\hat{x}'_{k|k-1} &:= \left[\tilde{\hat{x}}_{k|k-1}^v, \hat{x}_{k|k-1} \right]^T; \\
\hat{P}_{x'x'}^{k|k-1} &:= \sum_{i=1}^{N_1} w_i^{1,c} \left(\tilde{\chi}_{*,i}'^{k|k-1} - \tilde{\hat{x}}'_{k|k-1} \right) (\diamond)^T + Q'_k;
\end{aligned}$$

where $\chi_{*,i}^{k|k-1} \in S^3$ and $\chi_{*,i}^{k|k-1} \in \mathbb{R}^{n_x}$.

(c) Obtain the predicted measurement's estimates by

$$\left\{ \left[\begin{array}{c} \tilde{\chi}_i^{v,k|k-1} \\ \chi_i^{k|k-1} \end{array} \right], w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := 2\sigma R_2 \left([0]_{n_{x'} \times 1}, \hat{P}_{x'x'}^{k|k-1} \right);$$

and

$$\begin{aligned}
\chi_i^{k|k-1} &:= \text{VtoQ} \left(\tilde{\chi}_i^{v,k|k-1} \right) \otimes \hat{x}_{k|k-1}, \quad i = 1, \dots, N_2; \\
\chi_i'^{k|k-1} &:= \left(\chi_i^{k|k-1}, \chi_i^{k|k-1} \right), \quad 1 \leq i \leq N_2; \\
\left(\gamma_i^{k|k-1}, \gamma_i^{k|k-1} \right) &:= h_k \left(\chi_i'^{k|k-1} \right), \quad 1 \leq i \leq N_2; \\
\hat{\mathbf{y}}_{k|k-1} &:= \text{QuatWeightedMean} \left(\left\{ \gamma_i^{k|k-1}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} \right); \\
\tilde{\gamma}_i^{v,k|k-1} &:= \text{QtoV} \left(\gamma_i^{k|k-1} \otimes \hat{\mathbf{y}}_{k|k-1}^{-1} \right), \quad 1 \leq i \leq N_2; \\
\tilde{\hat{\mathbf{y}}}_{k|k-1}^v &:= \sum_{i=1}^{N_2} w_i^{2,m} \tilde{\gamma}_i^{v,k|k-1}; \\
\hat{\mathbf{y}}_{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,m} \gamma_i^{k|k-1}; \\
\gamma_i'^{k|k-1} &:= \left[\tilde{\gamma}_i^{v,k|k-1}, \gamma_i^{k|k-1} \right]^T; \\
\hat{\mathbf{y}}'_{k|k-1} &:= \left[\tilde{\hat{\mathbf{y}}}_{k|k-1}^v, \hat{\mathbf{y}}_{k|k-1} \right]^T; \\
\hat{P}_{y'y'}^{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,c} \left(\gamma_i'^{k|k-1} - \hat{\mathbf{y}}'_{k|k-1} \right) (\diamond)^T + R'_k;
\end{aligned}$$

where $\tilde{\chi}_i^{v,k|k-1} \in \mathbb{R}^{n_{x^v}}$, $\chi_i^{k|k-1} \in \mathbb{R}^{n_x}$, $\gamma_i^{k|k-1} \in S^3$, and $\gamma_i^{k|k-1} \in \mathbb{R}^{n_x}$.

(d) Obtain the corrected state's estimates by

$$\begin{aligned}
\chi_i'^{k|k-1} &:= \left[\tilde{\chi}_i^{v,k|k-1}, \chi_i^{k|k-1} \right]^T, \\
\hat{P}_{x'y'}^{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,cc} \left(\tilde{\chi}_i'^{k|k-1} - \tilde{\hat{x}}'_{k|k-1} \right) \left(\gamma_i'^{k|k-1} - \hat{\mathbf{y}}'_{k|k-1} \right),
\end{aligned}$$

$$\begin{aligned}
G_k &:= \hat{P}_{x'y'}^{k|k-1} \left(\hat{P}_{y'y'}^{k|k-1} \right)^{-1}, \\
\nu_k^v &:= \text{Qtov} \left(\mathbf{y}_k \otimes \left(\hat{\mathbf{y}}_{k|k-1} \right)^{-1} \right), \\
\nu_k &:= \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}, \\
\nu_k' &:= [\nu_k^v, \nu_k]^T, \\
\begin{bmatrix} \tilde{\hat{x}}_{k|k}^v \\ \hat{x}_{k|k} \end{bmatrix} &:= \tilde{\hat{x}}_{k|k-1}' + G_k \nu_k', \\
\hat{\mathbf{x}}_{k|k} &:= \text{VtoQ} \left(\tilde{\hat{x}}_{k|k}^v \right) \otimes \hat{\mathbf{x}}_{k|k-1}, \\
\hat{\mathbf{x}}_{k|k}' &:= \left(\hat{\mathbf{x}}_{k|k}, \hat{x}_{k|k} \right), \\
\hat{P}_{x'x'}^{k|k} &:= \hat{P}_{x'x'}^{k|k-1} - G_k \hat{P}_{y'y'}^{k|k-1} (G_k)^T;
\end{aligned} \tag{7.39}$$

where $\tilde{\hat{x}}_{k|k}^v \in \mathbb{R}^{n_{xv}}$ and $\hat{x}_{k|k} \in \mathbb{R}^{n_x}$.

In order to get the form of a particular QuAdUKF, only three choices have to be made: i) the σR 's, ii) the vector parameterization of the S^3 , and iii) the quaternion weighted mean method. All the filters guaranteeing to be the unity the quaternion norms in all steps (third row of Table 7.1) follow as particular cases of the QuAdUKF (see Table 7.5 and Figure 7.1). For instance, the UKF of [48] is the QuAdUKF with the Homogeneous Minimum Symmetric σR (HoMiSy σR , Corollary 3.4, which is equivalent to the σR of [2]), the GeRV (vector parameterization) and DPPSE (weighted mean method).

New filters are also obtained with the QuAdUKF. For instance, a UKF with the HoMiSy σR , RoV (Section 7.1.2) and DPPSE (Section 7.2.2.2); or any QuAdUKF using other σR 's, such as the Mi σR (Theorem 3.2) and RhoMi σR (Corollary 3.5), or the fifth-order one of [47] (Tab 2.1 [4,2]); or the QuAdUKF using the GeRV with the weighted mean method being any other than the DPPSE or the MAMCF; among others. Note also that it is straightforward to develop other variants of the QuAdUKF's, such as scaled and augmented ones, using the results of Chapters 3, 4, and 5.

7.3.2 Quaternionic Additive Square-Root Unscented Kalman Filter

The two SRUKF's for quaternion systems of the literature, SRUKF of [139] and [148], are based on the square-root techniques of the SRUKF of [42] for Euclidean systems. We could also propose a square-root version of the QuAdUKF adapting this filter with steps of the SRUKF of [42]. This would require simple changes, and we could show that the two SRUKF's for quaternion systems of the literature would be particular cases of this square-root version of the QuAdUKF. For instance, the SRUKF of [139] would be this square-root version of the QuAdUKF using the sigma set of [46]

Table 7.5: QuAdUKF's of the literature.

Particular QuAdUKF ^a	σ R or sigma set (SS)	vector par.	weighted mean method
UKF of [138]	σ R of [41]	QuV ^c	FN ^d
UKF of [48]	HoMiSy σ R	GeRV	DPPSE ^e
UKF of [140]	SS of [46] ^b	QuV ^c	FN ^d
UKF of [161]	HoMiSy σ R	QuV ^c	MQVCF ^f
UKF of [148]	σ R of [41]	RoV	GDA ^g
UKF of [150]	SS of [46] ^b	QuV ^c	DPPSE ^e
UKF of [152]	HoMiSy σ R	RoV	FN ^d
UKF of [154]	σ R of [41]	GeRV	DPPSE ^e
UKF of [155]	HoMiSy σ R	GeRV	MAMCF ^h

^aIn each line, an UKF in the first column is the QuAdUKF with the choices in the other three columns. ^bThe set of [46] is not a σ R because it matches the mean and the covariance of the previous random variable only for the scalar case (cf. Section 2.5); it is presented in this column in order to simplify the exposition. ^cThe set of QuV's parameterize the S^3 only for small rotations (cf. Section 7.1.2). ^dSection 7.2.2.1. ^eSection 7.2.2.2. ^fSection 7.2.2.4. ^gSection 7.2.2.3. ^hSection 7.2.2.5.

(Tab 2.1 [2,2]) with the GeRV and DPPSE; and the SRUKF of [148], this square-root version of the QuAdUKF using the σ R of [41] (Tab 2.1 [4,1]) with the RoV and GDA.

However, instead of defining a square-root version of the QuAdUKF using the square-root techniques of the SRUKF of [42], we introduce a square-root version of the QuAdUKF *using the square-root techniques of our AdSRUKF for Euclidean systems (Algorithm 8)*. Although this version does not generalize the SRUKF's of [148] and [139], it takes advantage of the better properties that our AdSRUKF has over the SRUKF of [42]. Recall that, essentially, the SRUKF of Section 5.3 is computationally more stable than the SRUKF of [42] when round-off errors are relevant (e.g. poor machine precision) or computationally ill-conditioned computations are present (e.g. inversions of quasi-singular matrices); this superior performance of our AdSRUKF is explained by the fewer number (or even the absence) of Cholesky factor downdatings in this algorithm (cf. Section 5.3).

For a set $\chi = \{\chi_i, w_i^m, w_i^c, w_i^{cc}\}_{i=1}^N$, define the subsets

$$\begin{aligned} \{\chi_j^+, w_j^{m,+}, w_j^{c,+}, w_j^{cc,+}\}_{j=1}^{N_+} &= \{\chi_i, w_i^m, w_i^c, w_i^{cc} | w_i^c \geq 0\}_{i=1}^N \\ \{\chi_j^-, w_j^{m,-}, w_j^{c,-}, w_j^{cc,-}\}_{j=1}^{N_-} &= \{\chi_i, w_i^m, w_i^c, w_i^{cc} | w_i^c < 0\}_{i=1}^N \end{aligned}$$

and the matrices

$$S_\chi^+ := \left[\sqrt{w_1^{c,+}} (\chi_1^+ - \mu_\chi), \dots, \sqrt{w_{N_+}^{c,+}} (\chi_{N_+}^+ - \mu_\chi) \right],$$

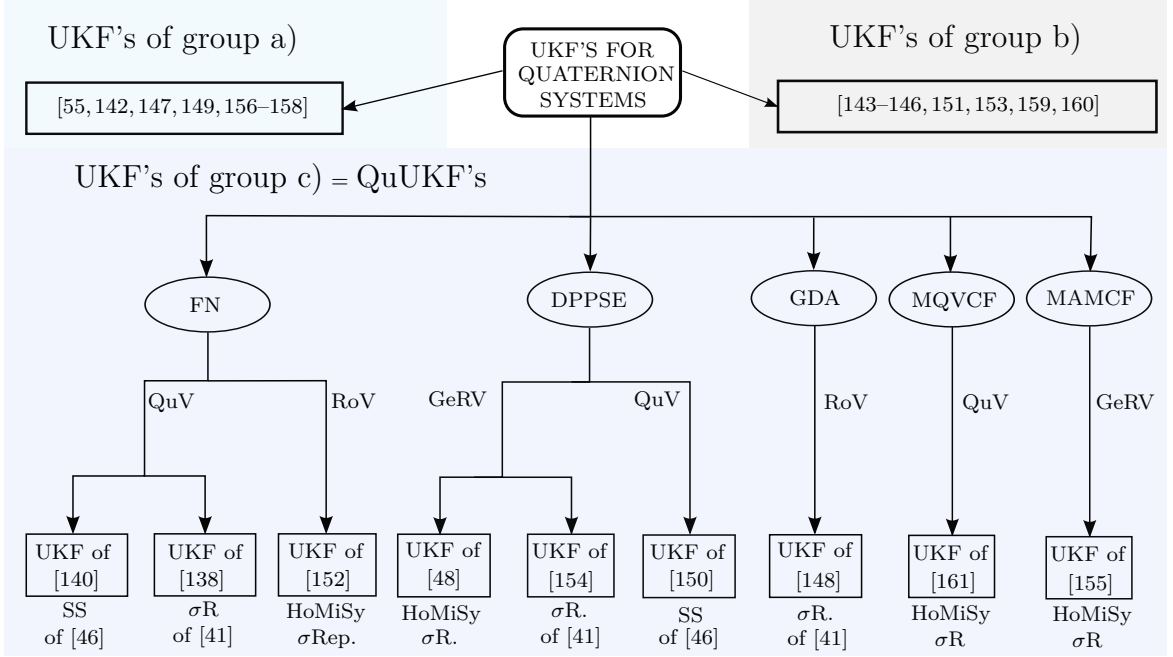


Figure 7.1: Taxonomy of the UKF's for quaternion models of the literature.

$$S_{\chi}^{-} := \left[\sqrt{\|w_1^{c,-}\|} (\chi_1^{-} - \mu_{\chi}), \dots, \sqrt{\|w_{N-}^{c,-}\|} (\chi_{N-}^{-} - \mu_{\chi}) \right].$$

Definition 7.2. Consider the additive-noise quaternion model

$$\begin{aligned} \mathbf{x}'_k &= f_k(\mathbf{x}'_{k-1}) \oplus \boldsymbol{\varpi}'_k, \\ \mathbf{y}'_k &= h_k(\mathbf{x}'_k) \oplus \boldsymbol{\vartheta}'_k; \end{aligned}$$

where

1. $\mathbf{x}'_k := (\mathbf{x}_k, x_k)$, $\mathbf{y}'_k := (\mathbf{y}_k, y_k)$, $\boldsymbol{\varpi}'_k := (\boldsymbol{\varpi}_k, \varpi_k)$, and $\boldsymbol{\vartheta}'_k := (\boldsymbol{\vartheta}_k, \vartheta_k)$;
2. $x_k \in \Phi^{n_x}$, $y_k \in \Phi^{n_y}$, $\varpi_k \in \Phi^{n_{\varpi}}$, and $\vartheta_k \in \Phi^{n_{\vartheta}}$; and
3. \mathbf{x}_k , \mathbf{y}_k , $\boldsymbol{\varpi}_k$, and $\boldsymbol{\vartheta}_k$ take values in the S^3 ;

Suppose that i) the distributions of $\boldsymbol{\varpi}_k$, $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by Gaussian, multidimensional-real-valued parameterizations $\varpi_k^v \in \Phi^{n_{\varpi^v}}$, $\vartheta_k^v \in \Phi^{n_{\vartheta^v}}$ and x_0^v , respectively, and ii) the distributions of $\boldsymbol{\varpi}'_k := (\boldsymbol{\varpi}_k^v, \varpi_k)$ and $\boldsymbol{\vartheta}'_k := (\boldsymbol{\vartheta}_k^v, \vartheta_k)$ are given by

$$\begin{aligned} \boldsymbol{\varpi}'_k &\sim \left([0]_{n_x \times 1}, \sqrt{Q'_k} \sqrt{Q'_k}^T \right), \\ \boldsymbol{\vartheta}'_k &\sim \left([0]_{n_y \times 1}, \sqrt{R'_k} \sqrt{R'_k}^T \right); \end{aligned}$$

iii) the mean of \mathbf{x}'_k is $\bar{\mathbf{x}}'_0$, and the covariance of $x'_0 := (x_0^v, x_0)$ is $\sqrt{P_{x'x'}^0} \sqrt{P_{x'x'}^0}^T$; iv) the measurements $\mathbf{y}'_1, \mathbf{y}'_2, \dots, \mathbf{y}'_{k_f}$ are given, where $\mathbf{y}'_k = (\mathbf{y}_k, y_k)$ with $\mathbf{y}_k \in S^3$ and $y_k \in \mathbb{R}^{n_y}$. Then the *Quaternionic Additive Square-Root Unscented Kalman Filter (for quaternion models)* is composed by the following algorithm:

Algorithm 18 (Quaternionic Additive Square-Root Unscented Kalman Filter (QuAdSRUKF)). *Perform the following steps:*

1. *Initialization. Set the initial estimates $\hat{\mathbf{x}}'_{0|0} := \bar{\mathbf{x}}'_0$ and $\sqrt{\hat{P}'_{x'x'}^{0|0}} := \sqrt{P_{x'x'}^0}$; and choose*

- (a) *two σ -representations, and set the functions $2\sigma R_1$ and $2\sigma R_2$ accordingly;*
- (b) *a vector parameterization, and set the functions $QtoV$ and $VtoQ$ accordingly;*
- (c) *a method for means of weighted sets composed of unit quaternions, and set the function $QuatWeightedMean$ accordingly.*

2. *Filtering. For $k = 1, 2, \dots, k_f$; set the following elements:*

(a) *Obtain the state sigma points by*

$$\left\{ \left[\begin{array}{c} \tilde{\chi}_i^{v, k-1|k-1} \\ \chi_i^{k-1|k-1} \end{array} \right], w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := 2\sigma R_1 \left([0]_{(n_{x^v} + n_x) \times 1}, \sqrt{\hat{P}'_{x'x'}^{k-1|k-1}} \sqrt{\hat{P}'_{x'x'}^{k-1|k-1}}^T \right);$$

and

$$\begin{aligned} \boldsymbol{\chi}_i^{k-1|k-1} &:= QtoV \left(\tilde{\chi}_i^{v, k-1|k-1} \right) \otimes \hat{\mathbf{x}}'_{k-1|k-1}, \quad i = 1, \dots, N_1; \\ \boldsymbol{\chi}'_i^{k-1|k-1} &:= \left(\boldsymbol{\chi}_i^{k-1|k-1}, \chi_i^{k-1|k-1} \right), \quad 1 \leq i \leq N_1; \end{aligned} \quad (7.40)$$

where $\tilde{\chi}_i^{v, k-1|k-1} \in \mathbb{R}^{n_{x^v}}$ and $\chi_i^{k-1|k-1} \in \mathbb{R}^{n_x}$.

i. *Obtain the predicted state's estimates by*

$$\begin{aligned} (\boldsymbol{\chi}_{*,i}^{k|k-1}, \chi_{*,i}^{k|k-1}) &:= f_k \left(\boldsymbol{\chi}'_i^{k-1|k-1} \right), \quad 1 \leq i \leq N_1; \\ \boldsymbol{\chi}_*^{k|k-1} &:= \left\{ \boldsymbol{\chi}_{*,i}^{k|k-1}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc} \right\}_{i=1}^{N_1}; \\ \hat{\mathbf{x}}_{k|k-1} &:= QuatWeightedMean \left(\boldsymbol{\chi}_*^{k|k-1} \right); \\ \tilde{\chi}_{*,i}^{v, k|k-1} &:= QtoV \left(\boldsymbol{\chi}_{*,i}^{k|k-1} \otimes \hat{\mathbf{x}}_{k|k-1}^{-1} \right), \quad 1 \leq i \leq N_1; \\ \tilde{\hat{\mathbf{x}}}_{k|k-1}^v &:= \sum_{i=1}^{N_1} w_i^{1,m} \tilde{\chi}_{*,i}^{v, k|k-1}; \end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{x}}_{k|k-1} &:= \sum_{i=1}^{N_1} w_i^{1,m} \boldsymbol{\chi}_{*,i}^{k|k-1}; \\
\boldsymbol{\chi}_{*,i}'^{k|k-1} &:= \left[\tilde{\chi}_{*,i}^{v,k|k-1}, \boldsymbol{\chi}_{*,i}^{k|k-1} \right]^T; \\
\hat{\boldsymbol{x}}_{k|k-1}' &:= \left[\tilde{\hat{\boldsymbol{x}}}_{k|k-1}^v, \hat{\boldsymbol{x}}_{k|k-1} \right]^T; \\
\left\{ \tilde{\chi}_{*,j}'^{k|k-1,+}, w_j^{1,c,+} \right\}_{j=1}^{N_{1+}} &:= \left\{ \boldsymbol{\chi}_{*,i}'^{k|k-1} - \hat{\boldsymbol{x}}_{k|k-1}', w_i^{1,c} | w_i^{1,c} > 0 \right\}_{i=1}^{N_1}; \\
\left\{ \tilde{\chi}_{*,j}'^{k|k-1,-}, w_j^{1,c,-} \right\}_{j=1}^{N_{1-}} &:= \left\{ \boldsymbol{\chi}_{*,i}'^{k|k-1} - \hat{\boldsymbol{x}}_{k|k-1}', w_i^{1,c} | w_i^{1,c} < 0 \right\}_{i=1}^{N_1}; \\
S_{\boldsymbol{\chi}_*}^{+,k|k-1} &:= \left[\sqrt{w_1^{1,c,+}} \tilde{\chi}_{*,1}'^{k|k-1,+}, \dots, \sqrt{w_{N_+}^{1,c,+}} \tilde{\chi}_{*,N_+}'^{k|k-1,+} \right]; \\
S_{\boldsymbol{\chi}_*}^{-,k|k-1} &:= \left[\sqrt{|w_1^{1,c,-}|} \tilde{\chi}_{*,1}'^{k|k-1,-}, \dots, \sqrt{|w_{N_-}^{1,c,-}|} \tilde{\chi}_{*,N_-}'^{k|k-1,-} \right]; \\
\sqrt{\hat{P}_{x'x'}^{k|k-1}} &:= \text{cu} \left(S_{\boldsymbol{\chi}_*}^{+,k|k-1}, S_{\boldsymbol{\chi}_*}^{-,k|k-1}, \sqrt{Q_k} \right);
\end{aligned}$$

where $\boldsymbol{\chi}_{*,i}^{k|k-1} \in S^3$ and $\boldsymbol{\chi}_{*,i}^{k|k-1} \in \mathbb{R}^{n_x}$.

(b) Obtain the predicted measurement's estimates by

$$\left\{ \left[\begin{array}{c} \tilde{\chi}_i^{v,k|k-1} \\ \boldsymbol{\chi}_i^{k|k-1} \end{array} \right], w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := 2\sigma R_2 \left([0]_{n_x' \times 1}, \sqrt{\hat{P}_{x'x'}^{k|k-1}} \sqrt{\hat{P}_{x'x'}^{k|k-1}{}^T} \right);$$

and

$$\begin{aligned}
\boldsymbol{\chi}_i^{k|k-1} &:= \text{VtoQ} \left(\tilde{\chi}_i^{v,k|k-1} \right) \otimes \hat{\boldsymbol{x}}_{k|k-1}, \quad i = 1, \dots, N_2; \\
\boldsymbol{\chi}_i'^{k|k-1} &:= \left(\boldsymbol{\chi}_i^{k|k-1}, \chi_i^{k|k-1} \right), \quad 1 \leq i \leq N_2; \\
\left(\boldsymbol{\gamma}_i^{k|k-1}, \gamma_i^{k|k-1} \right) &:= h_k \left(\boldsymbol{\chi}_i'^{k|k-1} \right), \quad 1 \leq i \leq N_2; \\
\boldsymbol{\gamma}^{k|k-1} &:= \left\{ \boldsymbol{\gamma}_i^{k|k-1}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2}; \\
\hat{\boldsymbol{y}}_{k|k-1} &:= \text{QuatWeightedMean} \left(\boldsymbol{\gamma}^{k|k-1} \right); \\
\tilde{\boldsymbol{\gamma}}_i^{v,k|k-1} &:= \text{QtoV} \left(\boldsymbol{\gamma}_i^{k|k-1} \otimes \hat{\boldsymbol{y}}_{k|k-1}^{-1} \right), \quad 1 \leq i \leq N_2; \\
\tilde{\hat{\boldsymbol{y}}}_{k|k-1}^v &:= \sum_{i=1}^{N_2} w_i^{2,m} \tilde{\boldsymbol{\gamma}}_i^{v,k|k-1}; \\
\hat{\boldsymbol{y}}_{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,m} \boldsymbol{\gamma}_i^{k|k-1}; \\
\boldsymbol{\gamma}_i'^{k|k-1} &:= \left[\tilde{\boldsymbol{\gamma}}_i^{v,k|k-1}, \boldsymbol{\gamma}_i^{k|k-1} \right]^T; \\
\hat{\boldsymbol{y}}_{k|k-1}' &:= \left[\tilde{\hat{\boldsymbol{y}}}_{k|k-1}^v, \hat{\boldsymbol{y}}_{k|k-1} \right]^T; \\
\left\{ \tilde{\boldsymbol{\gamma}}_j'^{k|k-1,+}, w_j^{2,c,+} \right\}_{j=1}^{N_{1+}} &:= \left\{ \boldsymbol{\gamma}_i'^{k|k-1} - \hat{\boldsymbol{y}}_{k|k-1}', w_i^{2,c} | w_i^{2,c} > 0 \right\}_{i=1}^{N_1};
\end{aligned}$$

$$\begin{aligned}
\{\tilde{\gamma}'_j{}^{v,k|k-1,-}, w_j^{2,c,-}\}_{j=1}^{N_1-} &:= \{\gamma'_i{}^{v,k|k-1} - \hat{y}'_{k|k-1}, w_i^{2,c}|w_i^{2,c} < 0\}_{i=1}^{N_1}; \\
S_{\gamma'{}^{v,k|k-1}}^+ &:= \left[\sqrt{w_1^{2,c,+}|\tilde{\gamma}'_1{}^{v,k|k-1,+}}, \dots, \sqrt{w_{N_+}^{2,c,+}|\tilde{\gamma}'_{N_+}{}^{v,k|k-1,+}} \right]; \\
S_{\gamma'{}^{v,k|k-1}}^- &:= \left[\sqrt{|w_1^{2,c,-}|\tilde{\gamma}'_1{}^{v,k|k-1,-}}, \dots, \sqrt{|w_{N_-}^{2,c,-}|\tilde{\gamma}'_{N_-}{}^{v,k|k-1,-}} \right]; \\
\sqrt{\hat{P}'_{y' y'}{}^{k|k-1}} &:= \text{cu} \left(S_{\gamma'{}^{v,k|k-1}}^+, S_{\gamma'{}^{v,k|k-1}}^-, \sqrt{R'_k} \right);
\end{aligned}$$

where $\tilde{\chi}_i^{v,k|k-1} \in \mathbb{R}^{n_x v}$, $\chi_i^{k|k-1} \in \mathbb{R}^{n_x}$, $\gamma_i^{k|k-1} \in S^3$, and $\gamma_i^{k|k-1} \in \mathbb{R}^{n_x}$.

(c) Obtain the corrected state's estimates by

$$\begin{aligned}
\chi_i'{}^{v,k|k-1} &:= \left[\tilde{\chi}_i^{v,k|k-1}, \chi_i^{k|k-1} \right]^T, \\
\hat{P}'_{x' y'}{}^{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,cc} \left(\tilde{\chi}_i'{}^{v,k|k-1} - \tilde{x}'_{k|k-1} \right) \left(\gamma_i'{}^{v,k|k-1} - \hat{y}'_{k|k-1} \right), \\
G_k &:= \hat{P}'_{x' y'}{}^{k|k-1} \left(\sqrt{\hat{P}'_{y' y'}{}^{k|k-1}} \right)^{-1} \left(\sqrt{\hat{P}'_{y' y'}{}^{k|k-1}} \right)^{-1}, \\
\nu_k^v &:= \text{QtoV} \left(\mathbf{y}_k \otimes \left(\hat{\mathbf{y}}_{k|k-1} \right)^{-1} \right), \\
\nu_k &:= y_k - \hat{y}_{k|k-1}, \\
\nu_k' &:= [\nu_k^v, \nu_k]^T, \\
\begin{bmatrix} \tilde{\hat{x}}_{k|k}^v \\ \hat{x}_{k|k} \end{bmatrix} &:= \tilde{x}'_{k|k-1} + G_k \nu_k', \\
\hat{\mathbf{x}}_{k|k} &:= \text{VtoQ} \left(\tilde{\hat{x}}_{k|k}^v \right) \otimes \hat{\mathbf{x}}_{k|k-1}, \\
\hat{\mathbf{x}}'_{k|k} &:= \left(\hat{\mathbf{x}}_{k|k}, \hat{x}_{k|k} \right), \\
\{\tilde{\chi}'_j{}^{v,k|k-1,+}, w_j^{2,c,+}\}_{j=1}^{N_1+} &:= \{\chi_i'{}^{v,k|k-1} - \hat{x}'_{k|k-1}, w_i^{2,c}|w_i^{2,c} > 0\}_{i=1}^{N_1}; \\
\{\tilde{\chi}'_j{}^{v,k|k-1,-}, w_j^{2,c,-}\}_{j=1}^{N_1-} &:= \{\chi_i'{}^{v,k|k-1} - \hat{x}'_{k|k-1}, w_i^{2,c}|w_i^{2,c} < 0\}_{i=1}^{N_1}; \\
S_{\chi'{}^{v,k|k-1}}^+ &:= \left[\sqrt{w_1^{2,c,+}|\tilde{\chi}'_1{}^{v,k|k-1,+}}, \dots, \sqrt{w_{N_+}^{2,c,+}|\tilde{\chi}'_{N_+}{}^{v,k|k-1,+}} \right]; \\
S_{\chi'{}^{v,k|k-1}}^- &:= \left[\sqrt{|w_1^{2,c,-}|\tilde{\chi}'_1{}^{v,k|k-1,-}}, \dots, \sqrt{|w_{N_-}^{2,c,-}|\tilde{\chi}'_{N_-}{}^{v,k|k-1,-}} \right];
\end{aligned}$$

and

$$\sqrt{\hat{P}'_{x' x'}{}^{k|k}} = \left(\text{cu} \left[S_{\chi'{}^{v,k|k-1}}^+ - G_k S_{\gamma'{}^{v,k|k-1}}^+ \right], \left[S_{\chi'{}^{v,k|k-1}}^- - G_k S_{\gamma'{}^{v,k|k-1}}^- \right], G_k \sqrt{R_k^v} \right);$$

where $\tilde{\hat{x}}_{k|k}^v \in \mathbb{R}^{n_x v}$ and $\hat{x}_{k|k} \in \mathbb{R}^{n_x}$.

Since the QuAdSRUKF inherits properties from our AdSRUKF (Algorithm 8), it outperforms all the SRUKF quaternion systems of the literature, which are based on the

SRUKF of the [42]. Note that, similarly to the case of the QuAdUKF, a great number of particular cases of the QuAdSRUKF can be obtained by simply choosing different σ_R 's (e.g. the ones presented in Sections 3.3 and 3.4), vector parameterizations of S^3 (e.g. the ones presented in Section 7.1.2) and methods for weighted means of unit quaternion sets (e.g. the ones presented in Section 7.2.2).

7.4 SIMULATIONS OF QUATERNION UNSCENTED FILTERS

In this section, numerical simulations are performed comparing UKF's and SRUKF's for quaternion systems. The scenario is of a satellite attitude estimation based on [48]; it is supposed that measurements from a three-axis magnetometer (TAM) and from gyroscopic rate sensors are acquired. Data is generated by a fourth order Runge-Kutta integration of the function [48, 55]

$$\dot{\mathbf{e}}(t) = \frac{1}{2} \mathbf{l}_m \boldsymbol{\omega}(t) \otimes \mathbf{e}(t) \quad (7.41)$$

where

$$\boldsymbol{\omega}(t) := \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} := \begin{bmatrix} 0.03 \sin\left(\left[\frac{\pi}{600}t\right]^\circ\right) \\ 0.03 \sin\left(\left[\frac{\pi}{600}t\right]^\circ - 300^\circ\right) \\ 0.03 \sin\left(\left[\frac{\pi}{600}t\right]^\circ - 600^\circ\right) \end{bmatrix};$$

is the angular velocity acting as an input and $\mathbf{e} \in S^3$ is the attitude of the satellite. The initial condition was chosen according to [55]:

$$\mathbf{e}(0) = 0.9603 + \mathbf{l}_m \begin{bmatrix} 0.1387 \\ 0.1981 \\ \sqrt{1 - 0.9603^2 - 0.1387^2 - 0.1981^2} \end{bmatrix}.$$

For the filtering process, it is assumed that corrupted measurements $\tilde{\boldsymbol{\omega}}(k)$ of the angular velocity $\boldsymbol{\omega}(k)$ are provided by biased gyros

$$\tilde{\boldsymbol{\omega}}(k) = \boldsymbol{\omega}(k) + \beta_k + \boldsymbol{\varpi}_k^\omega$$

where $\boldsymbol{\varpi}_k^\omega \sim N\left([0, 0, 0]^T, \sigma_\omega^2 I_3\right)$ is a zero mean Gaussian noise, σ_ω is the standard deviation of the gyro measurements, and β_k is drift error with $\dot{\beta}_k = \boldsymbol{\varpi}_k^\beta \sim \left([0, 0, 0]^T, \sigma_\beta^2 I_3\right)$. The sample time is $T = 0.1$ s and the filter's state at time step k is

$$\mathbf{x}_k := (\mathbf{e}(k), \beta_k).$$

The process function is

$$\begin{bmatrix} \text{vec}(\mathbf{e}(k)) \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_k \text{vec}(\mathbf{e}(k-1)) \\ \beta_{k-1} \end{bmatrix} + \varpi_k,$$

where $\text{vec}(\mathbf{e}) := [e_1, e_2, e_3, e_4]^T \in \mathbb{R}^4$,

$$s_k := \tilde{\omega}(k) - \beta_k, \quad \psi_k := \sin\left(\frac{T}{2} \|s_k\|\right) \frac{s_k^T}{\|s_k\|},$$

$$A_k := \begin{bmatrix} \cos\left(\frac{T}{2} \|s_k\|\right) & -\psi_k^T \\ \psi_k & \cos\left(\frac{T}{2} \|s_k\|\right) I_3 - (s_k)^x \end{bmatrix},$$

and $\omega_k \sim N([0]_{6 \times 1}, Q_k)$ is the process noise with

$$Q_k = \begin{bmatrix} (\sigma_\omega^2 T + \frac{1}{3} \sigma_\beta^2 T^3) I_3 & \frac{1}{2} \sigma_\beta^2 T^2 I_3 \\ \frac{1}{2} \sigma_\beta^2 T^2 I_3 & \sigma_\beta^2 T^2 I_3 \end{bmatrix};$$

The equation

$$\text{vec}(\mathbf{e}(k)) = A_k \text{vec}(\mathbf{e}(k-1))$$

is obtained by performing a trapezoidal discretization (relative to time) of (7.41) (cf. [48]).

The measurement function is, for $i = 1, 2, 3$,

$$y_k^{[i]} = h_k(x_k) d^{[i]} + \vartheta_k^{[i]},$$

where, for $x_k = [x_{1,k}, \dots, x_{7,k}]^T$,

$$h_k(x_k) = \begin{bmatrix} x_{1,k}^2 + x_{2,k}^2 - x_{3,k}^2 - x_{4,k}^2 & 2(x_{2,k}x_{3,k} + x_{1,k}x_{4,k}) & 2(x_{2,k}x_{4,k} - x_{1,k}x_{3,k}) \\ 2(x_{2,k}x_{3,k} - x_{1,k}x_{4,k}) & x_{1,k}^2 - x_{2,k}^2 + x_{3,k}^2 - x_{4,k}^2 & 2(x_{3,k}x_{4,k} + x_{1,k}x_{2,k}) \\ 2(x_{2,k}x_{4,k} + x_{1,k}x_{3,k}) & 2(x_{3,k}x_{4,k} - x_{1,k}x_{2,k}) & x_{1,k}^2 - x_{2,k}^2 - x_{3,k}^2 + x_{4,k}^2 \end{bmatrix}, \quad (7.42)$$

$d^{[i]}$ is a reference direction vector to a known point and $\vartheta_k^{[i]}$ the measurement noise [48, 167]. In this case, $d^{[i]}$ is given by the TAM:

$$d^{[1]} = [1, 0, 0]^T, d^{[2]} = [0, 1, 0]^T \text{ and } d^{[3]} = [0, 0, 1]^T;$$

and $\vartheta_k^{[1]} = \vartheta_k^{[2]} = \vartheta_k^{[3]} \sim N([0]_{3 \times 1}, \sigma_v^2 I_3)$, where σ_v is the standard deviation of the

TAM's. The initial conditions for the filter are $\hat{e}(0) = 1$, $\hat{\beta}_0 = [0]_{3 \times 1}$ and

$$\hat{P}_{xx}^{v,0|0} = \begin{bmatrix} 3.0462 \times 10^{-6} I_3 & [0]_{3 \times 3} \\ [0]_{3 \times 3} & 9.4018 \times 10^{-13} I_3 \end{bmatrix}.$$

The deviations are $\sigma_\beta = 3.1623 \times 10^{-4} \mu\text{rad} \times \text{s}^{-3/2}$, $\sigma_\omega = 3.1623 \mu\text{rad} \times \text{s}^{-1/2}$, $\sigma_v = 50 \text{ nT}$, and the bias $\beta(t) = [0.001]_{3 \times 1} \text{ rad} \times \text{s}^{-1}$ [48].

A quantitative comparison is also provided. We calculate i) for each time step k at each simulation j , an relative error

$$\epsilon_{k,j} := \sum_{i=1}^4 \frac{(\hat{e}_i(k,j) - e_i(k,j))^2}{e_i(k,j)} \quad (7.43)$$

for each filter; and ii) for $N_{it} = 2000$ iterations and $N_s = 1000$ simulations, the RMSD [defined in (5.42)] of (7.43) and the Root-Mean-Square Trace (RMST)

$$\text{RMST} := \sqrt{\sum_{j=1}^{N_s} \sum_{k=1}^{N_i} \text{tr}(\hat{P}_{xx}^{v,k|k})} \quad (7.44)$$

—the RMST quantifies the uncertainty of the estimate $\hat{\mathbf{x}}_{k|k}$.

7.4.1 Simulations of Quaternion Unscented Kalman Filters

In this subsection, UKF's for additive-noise quaternion models are considered. First, the three different vector representations of Section 7.1.2 are compared, and second the different methods for obtaining the weighted mean.

Recall that $\hat{P}_{xx}^{v,k|k-1}$, $\hat{P}_{xy}^{v,k|k-1}$ and $\hat{\mathbf{x}}_{k|k}$ are calculated considering $\hat{x}_{k|k-1}^v = 0$ in [138, 148, 161] (cf. Section 7.2.3). In the simulations here, making $\hat{x}_{k|k-1}^v = 0$ slightly decreased the performance of the filters, e.g., some UKF's turned numerically unstable.

Numerical problems occurred. Some methods (all QuAdUKF's with FN and MQVCF, some QuAdUKF's with RoV's, some with MAMCF, and some UKF's for equality-constrained systems) provided complex numbers within the elements of the unit quaternions or within the covariance of the state. This was treated here by considering only the real part of these complex numbers. Furthermore, simulations of some filters with the Rho Minimum σR of [57] and the Minimum σR of [23] were interrupted when performing the Cholesky factorization of the state's covariance because this matrix became non-positive definite. A well known trick of forcing to be zero the negative eigenvalues was used, as suggested in [88]. The square-root matrix is, then, computed using the singular value decomposition.

Figure 7.2 contains plots with the correct values of e_1, e_2, e_3 and e_4 and also their estimates given by a multiplicative EKF (the one presented in Table 7.1 of [167]) and QuAdUKF's with the σ R of [2], the DPPSE and different vector parameterizations. As expected, the UKF's behaved better than the EKF (the EKF's estimates are given by the line that is a bit far from all the others). The QuAdUKF's did not present problems with singularities and provided very close estimates in comparison with the correct value. Nonetheless, we can not distinguish the performances of the QuAdUKF's with this visual evaluation.

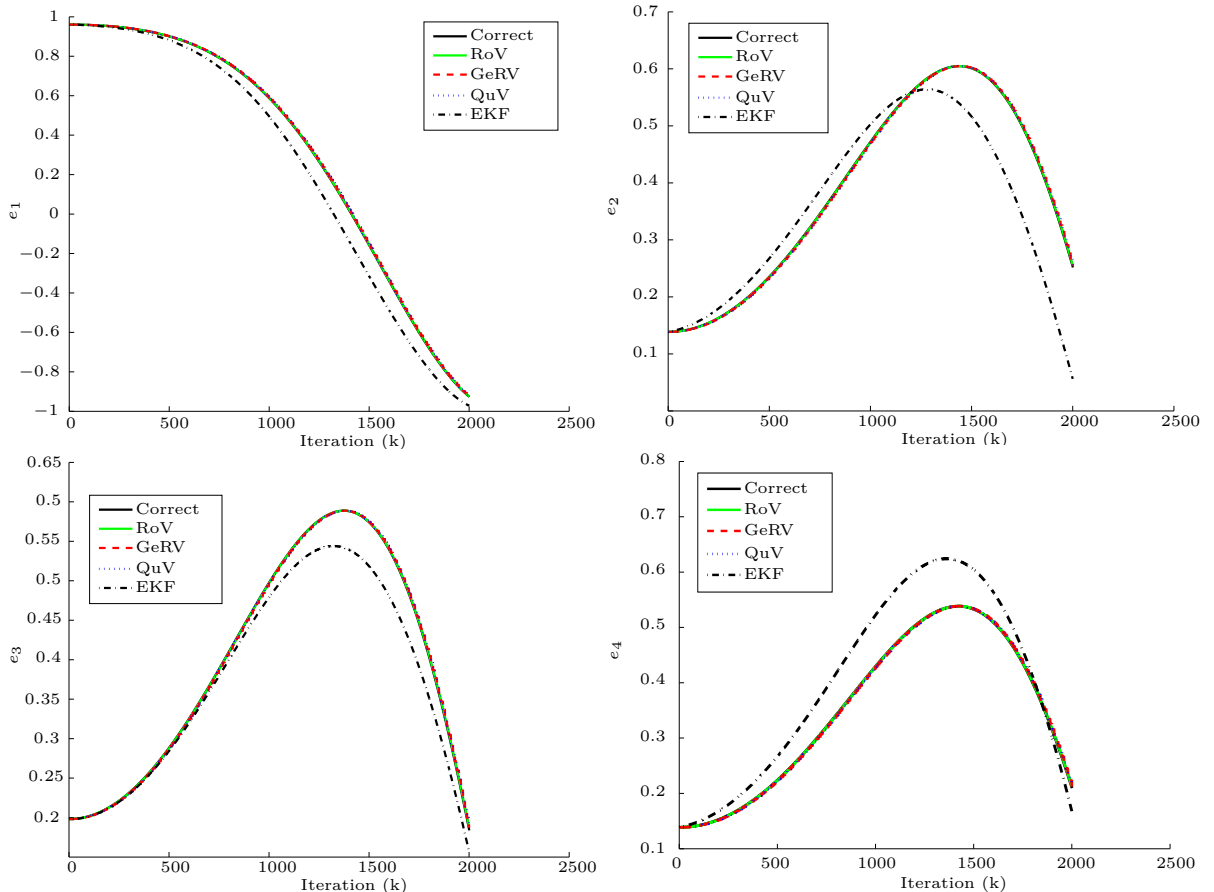


Figure 7.2: Values of e_1, e_2, e_3 and e_4 for the new QuAdUKF's with different parameterizations.

Table 7.6 shows RMSD's [equation (5.42)] and RMST's [equation (7.44)] for the each of the QuAdUKF's for unit quaternions considering each of the weighted mean methods. Among these, the DPPSE (second row of Table 7.6) and the GDA (third row of Table 7.6) provided the smallest sum of RMSD with RMST. The MAMCF provided the best μ_ϵ with the RoV, but the highest RMST; also it was numerically unstable with the GeRV and the QuV. Changing the vector representation results in changes in the estimation quality when using the FN (row 1), the MQVCF (row 4) or the MAMCF (row 5), but not when using the DPPSE (row 2) or the GDA (row 3); some filters were numerically unstable (NU) (row 2, columns 4-5; row 4, column 5; and

row 5, columns 4-5).

Table 7.6: RMSD and RMST (10^{-5}) for different weighted mean methods ($T = 0.1$ s).

Weighted Mean Method		RoV	GeRV	QuV
1	FN	RMSD	2.52	NU
	(Section 7.2.2.1)	RMST	3.05	
2	DPPSE	RMSD	1.48	1.48
	(Section 7.2.2.2)	RMST	6.76	
3	GDA	RMSD	1.48	1.48
	(Section 7.2.2.3)	RMST	6.76	
4	MQVCF	RMSD	1.56	2.58
	(Section 7.2.2.4)	RMST	1.23	
5	MAMCF	RMSD	1.29	NU
	(Section 7.2.2.5)	RMST	1.27×10^5	

Tables 7.7 (for a sampling time of $T = 0.1$ s) and 7.8 (for a sampling time of $T = 10$ s) show the errors of the UKF's implemented with the following σ R's (see [23] for their expressions): Homogeneous Minimum Symmetric σ R of [2] (HoMi σ R), Rho Minimum σ R of [57] (RhoMi σ R), Minimum σ R of [23] (Mi σ R), and 5th order σ R [47] (5th σ R). These last three σ R's are used here for first time in the literature in UKF's for quaternion systems; the DPPSE (Section 7.2.2.2) was used as the weighted mean method.

Table 7.7: RMSD and RMST (10^{-5}) for different σ R's ($T = 0.1$ s).

UKF's		HoMi σ R	RhoMi σ R	Mi σ R	5th σ R
1	with	RMSD	1.48	1.48	1.48
	RoV	RMST	6.76	6.76	6.76
2	with	RMSD	1.48	1.48	1.48
	GeRV	RMST	6.76	6.76	6.76
3	with	RMSD	1.48	1.48	1.48
	QuV	RMST	6.76	6.76	6.76
4	ECUKF	RMSD	NU	1.49	NU
		RMST		5.55	
5	PrUKF	RMSD	1.48	1.48	1.48
		RMST	6.76	6.76	6.76
6	MAUKF	RMSD	1.48	1.48	1.48
		RMST	6.76	6.76	6.76

In general, the values of RMSD and RMST for $T = 0.1$ s (Table 7.7) were smaller than the ones for $T = 10$ s (Table 7.8); this was expected since the discrete-time approximation is better for smaller values of T .

For $T = 0.1$ s, changing the σ R did not result in any substantial change in the quality of the estimations (cf. Table 7.7), but for $T = 10$ s the HoMi σ R (column 3 Table 7.8) and the 5th σ R (column 6 of Table 7.8) provided the best estimation qualities.

Table 7.8: RMSD and RMST(10^{-5}) for UKF's with different σ R's ($T = 10$ s).

	UKF		HoMi σ R	RhoMi σ R	Mi σ R	5th σ R
1	with	RMSD	2.00	2.12	2.04	2.00
	RoV	RMST	6.98	6.98	6.98	6.98
2	with	RMSD	2.05	2.49	2.42	2.05
	GeRV	RMST	6.98	6.98	6.98	6.98
3	with	RMSD	2.02	2.29	2.28	2.02
	QuV	RMST	6.98	6.98	6.98	6.98
4	ECUKF	RMSD	NU	NU	NU	NU
		RMST				
5	PrUKF	RMSD	2.13	2.13	2.13	2.13
		RMST	6.98	6.98	6.98	6.98
6	MAUKF	RMSD	2.12	2.12	2.12	2.12
		RMST	6.98	6.98	6.98	6.98

The 5th σ R (73 sigma points) and the HoMi σ R (13 sigma points), nevertheless, are composed by more sigma points than the RhoMi σ R [column 4 of Tables 7.7 and 7.8] and the Mi σ R [column 5 of Tables 7.7 and 7.8] (both with 7 sigma points).

Concerning the diverse vector parameterizations (rows 1, 2 and 3), there was no difference in the errors for the case of $T = 0.1$ s. However, for $T = 10$ s, the QuAdUKF with RoV was the best. The QuAdUKF with GeRV was a bit slower than the other two UKF's, but it was more robust to changes in the parameters of the filter (this fact is not shown in the tables nor in the graphics), such as in the noise covariances and tuning parameters of the σ R's (κ in the HoMi σ R, ρ in the RhoMi σ R and v in the Mi σ R as defined in [23]).

The QuAdUKF's provided better results comparative with the Projected UKF and the Measurement Augmented UKF for $T = 0.1$ s and for $T = 10$ s, although the differences for $T = 0.1$ s were very small. The Equality-Constrained UKF (row 4) was numerically unstable, except in the case using the RhoMi σ R for $T = 0.1$ s (Table 7.7, row 4, column 4).

7.4.2 Simulations of Quaternion Square-Root Unscented Kalman Filters

Recall from Section 7.3.2 that, when computationally ill-conditioned computations are present (e.g. inversions of quasi-singular matrices), the QuAdSRUKF should perform better than the SRUKF for quaternion systems of the literature; it should also perform better, in this cases, than UKF's in general because of its square-root properties. In order to assess this outperformance of the QuAdSRUKF, this filter is compared with the following filters: the SRUKF's of [139] and [148], the UKF of [138], the USQUE

of [48], the MUKF of [150], the QBUKF of [143], the UUF of [161], the UKF of [154], and the three UKF for non-linear equality-constrained systems of [55]—the ECUKF, the PUKF, and the MAUKF. The simulations were ran using a Matlab 2011b, and the tuning parameters were chosen as suggested in each respective work:

- the SRUKF of [139] with $a = 1$, $\alpha = 10^{-3}$, $\beta = 2$, and $\kappa = 0$
- the SRUKF of [148] with $\alpha = 10^{-3}$, $\beta = 2$, $\kappa = 1$, and 10^{-3} for the threshold of the gradient descent algorithm (smaller values of this threshold were making the simulation time extremely high);
- the UKF of [138] with $\alpha = 10^{-3}$, $\beta = 2$, and $\kappa = 0$;
- the USQUE of [48] with $a = 1$ and $\lambda = 1$;
- the MUKF of [150] with $\alpha = 10^{-3}$ and $w_0 = 1/3$;
- the QBUKF of [143] with 10^{-3} for the threshold of the gradient descent algorithm (smaller values of this threshold were making the simulation time extremely high);
- the UUF of [161] with $\kappa = 0$;
- the UKF of [154] with $\kappa = 0$;
- the UKF's of [55] with $\alpha = 1$, $\beta = 2$, and $\kappa = 0$.

The proposed QuAdSRUKF (using the HoMi σ R, the GeRV and the DPPSE) was ran with $a = 1$ using the Homogeneous Minimum Symmetric σ R—the standard symmetric σ R of [1] (cf. [23])—with the central weight (its tuning parameter) equal to $1/7$; this value provided good estimates in the examples considered here.

7.4.2.1 Ill-conditioned measurement function

The objective of this first example is to analyze the filters in a situation of poor machine precision. The new SRUKF is compared with the other aforementioned filters in a simple problem considering only the correction step of these filters. It is considered the measurement function

$$h_k(x_k) := Hx_k$$

where

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 + \delta \end{bmatrix},$$

$$\delta = eps^{2/3}10^d,$$

d is an integer, and eps is the distance from 1.0 to the next largest double-precision number, which, in our case, is $eps = 2^{-52}$. Even though this measurement function $h_k(x_k) := Hx_k$ is different from the original h in (7.42), the simulations of this subsection are still able to study the behavior the filters in a situation where round-off errors are able to deteriorate their performance.

In the simulations of the SRUKF's of [139] and [148], some covariances lost the semi-positiveness for $d \leq 10$ and the simulations could not be completed. Figure 7.3 presents the relative errors of the new SRUKF, the UKF of [138], the USQUE of [48], the MUKF of [150], the QBUKF of [143], the UUF of [161], and the UKF of [154], considering $d \in [-5, 8]$ —note that, since only correction steps are considered, some filters provided the same estimates. The new SRUKF presented fewer errors than the other filters; these simulations corroborates the exception that the new SRUKF should outperform the other Unscented filters in a situation with poor machine precision.

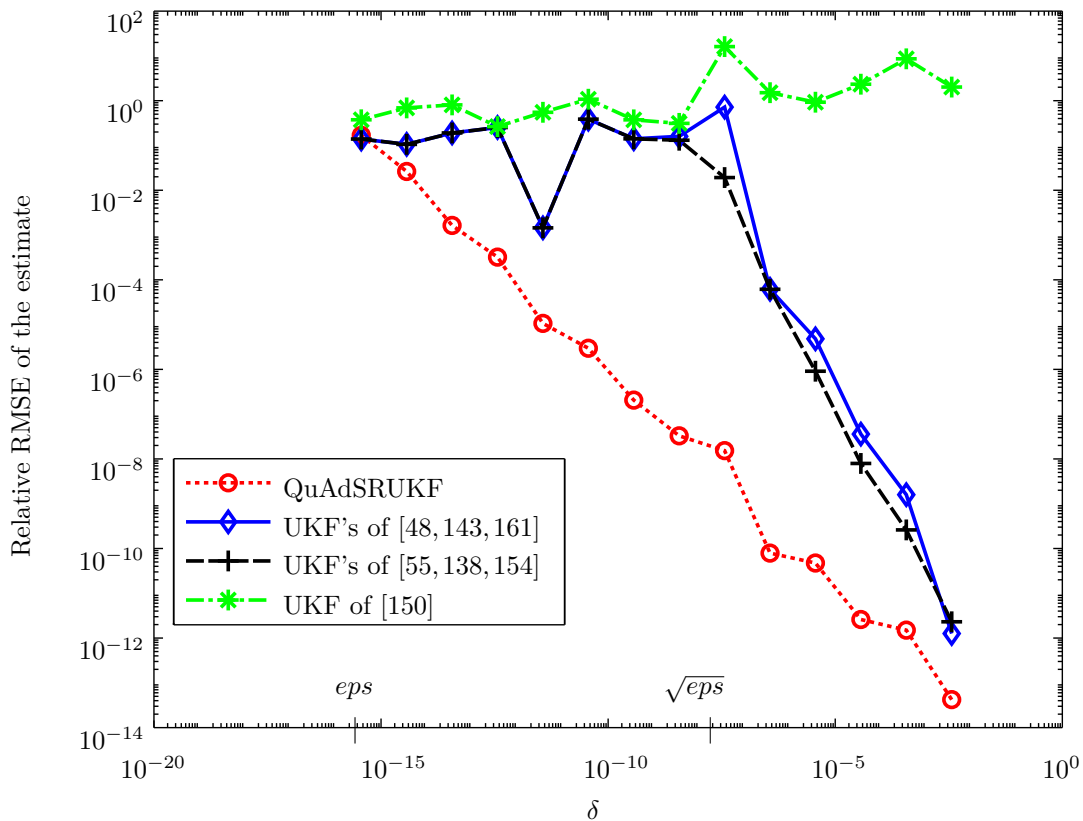


Figure 7.3: Relative RMSD Unscented filters for attitude estimation in a problem with an ill-conditioned measurement function.

Table 7.9: μ_e 's and MT's of Unscented filters for a problem of satellite attitude estimation in normal conditions.

Unscented Filter	RMSD ($\times 10^{-3}$)	RMST ($\times 10^{-5}$)
New SRUKF	3.41	1.07
USQUE of [48]	3.77	1.07
QBUKF of [143]	3.92	1.07
UKF of [154]	3.77	1.07

7.4.2.2 Satellite attitude estimation: normal conditions

In this example, the scenario is configured according to [48]: $T = 10$ s (measurements of both the TAM and the gyros are available at every 10 s), $\sigma_\omega = 0.31623 \mu\text{rad} \times \text{s}^{-1/2}$, $\sigma_\beta = 3.1623 \times 10^{-4} \mu\text{rad} \times \text{s}^{-3/2}$, $\beta_0 = [0.1]_{3 \times 1} \text{ deg/hr}$, $\sigma_v = 50 \text{ nT}$,

$$\hat{e}_0 = 0.85 + \hat{i}0.1387 + \hat{j}0.1981 + \hat{k}\sqrt{1 - 0.85^2 - 0.1387^2 - 0.1981^2},$$

$\hat{\beta}_0 = \beta_0 + [0, 20, 0]^T \text{ deg/hr}$, and

$$\hat{P}_{xx}^{\rho,0|0} = \begin{bmatrix} (\sigma_{xx}^{0|0,e})^2 I_{3 \times 3} & [0]_{3 \times 3} \\ [0]_{3 \times 3} & (\sigma_{xx}^{0|0,\beta})^2 I_{3 \times 3} \end{bmatrix}$$

with $\sigma_{xx}^{0|0,e} = 5 \text{ deg}$ and $\sigma_{xx}^{0|0,\beta} = 20 \text{ deg/hr}$.

The SRUKF's of [139] and [148], the UKF of [138], the MUKF of [150], the UUF of [161], and the UKF's of [55] failed to complete all the 1000 simulations for losing the positiveness of the state's covariance. The mean errors RMSD and the RMST of the other filters are shown in Table 7.9. The values of RMST were all equal (1.07×10^{-5}), indicating that the uncertainties in their estimates are the same; but the SRUKF presented the smallest RMSD (3.41×10^{-3}). Comparative to this value, the second smallest RMSD (3.77×10^{-3} , presented by both the USQUE of [48] and the UKF of [154]) was 10, 56% higher.

7.4.2.3 Satellite attitude estimation: computationally unstable conditions

In this example, some parameters of the simulations are changed from the values of Section 7.4.2.2 in order to create an computationally unstable situation: σ_ω is changed to $0.31623 \times 10^{-8} \mu\text{rad} \times \text{s}^{-1/2}$, σ_β to $3.1623 \times 10^{-12} \mu\text{rad} \times \text{s}^{-3/2}$, and σ_v to $50 \times 10^{-8} \text{ nT}$.

While all the Unscented filters of the literature failed complete all the 1000 simulations for presenting non-positive definite covariances, the new SRUKF finished with good estimate (RMSD = 3.45×10^{-3} and RMST = 1.07×10^{-5}). These errors and the plots for one simulation of e_1 , e_2 , e_3 , and e_4 of the new SRUKF comparative with the

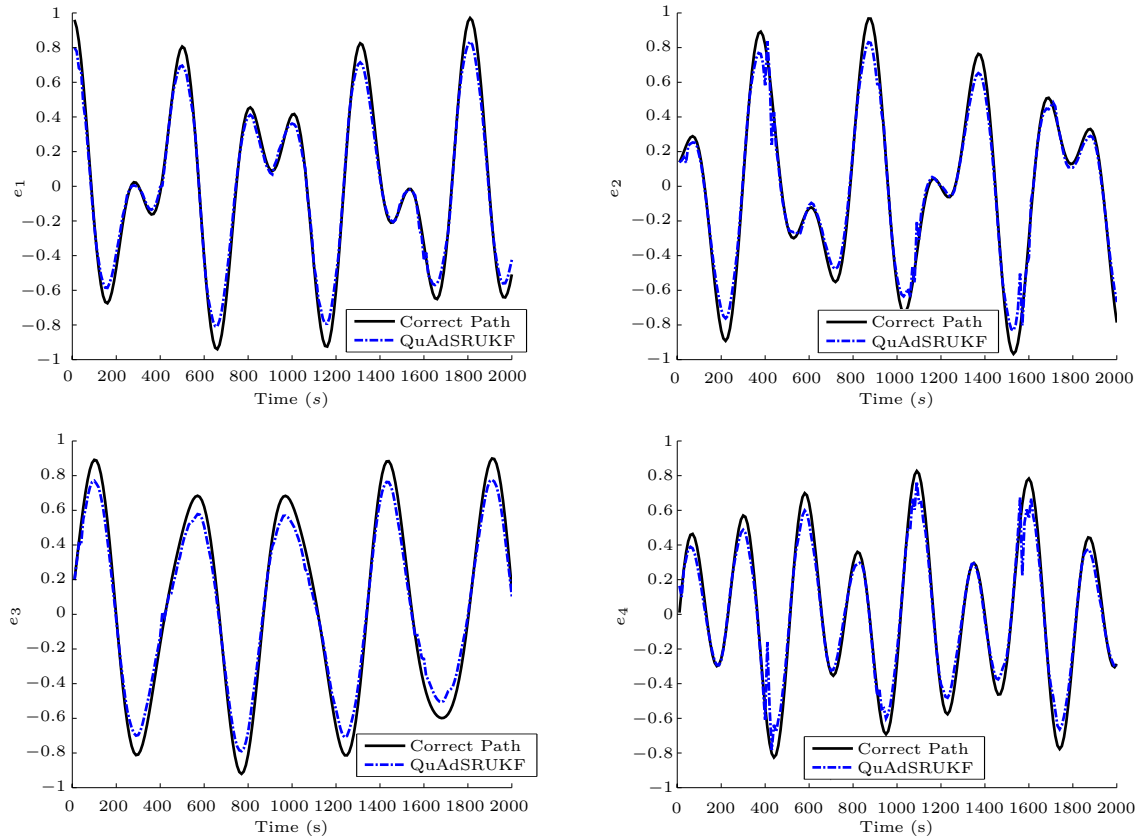


Figure 7.4: Values of e_1 , e_2 , e_3 and e_4 for the new SRUKF for a problem of satellite attitude estimation in heavy conditions.

correct ones (Figure 7.4), indicates that the estimates of the QuAdSRUKF are reliable. This shows that the QuAdSRUKF outperforms the additive UF's for quaternion models of the literature in a computationally unstable situation. This outperformance can be explained, at least in part, by the following characteristics:

1. the square-root properties of the QuAdSRUKF comparative with the UKF's of the literature. Square-root filters tend to perform better than non square-root filters in computationally ill-conditioned situations [88].
2. the lower (or even none) number of Cholesky factor downdatings of the QuAdSRUKF comparative with the SRUKF's of the literature. Recall that downdating a Cholesky factor A by a matrix B means finding a matrix C such that $CC^T = AA^T - BB^T$; the direct downdating of a Cholesky factor is "inherently more ill-conditioned than if Q (the usual triangular matrix Q of a QR decomposition) is also available" [92] (the comment in the parenthesis is ours).

7.5 CONCLUSIONS REGARDING UNSCENTED FILTERS FOR QUATERNION SYSTEMS

In this chapter, we show that constructing Unscented filters for quaternion systems is not trivial because there are steps in these algorithms composed of sums and scalings of unit quaternions. These operations, in general, result in non-unit quaternions.

A detailed analysis on the topic is provided. By comparing the properties of each UKF for quaternion systems, this analysis shows that, in a considerable amount of cases, the unit norm constraint of the unit quaternions is not completely respected (Section 7.2). We were able to gather all the algorithms that completely preserve this constraint in a single filter algorithm, the Quaternionic Unscented Kalman Filter (QuAdUKF, Section 7.3.1). By choosing only three elements of these filters—the sigma-representation, the vector parameterization of the S^3 and the method for calculating the weighted mean of a set of quaternion points—the QuAdUKF can result in every one of these filters and also to new ones. Numerical simulations of spacecraft attitude filtering illustrates these results (Section 7.4.1).

A square-root variant of the QuAdUKF is also proposed, the *Quaternionic Additive Square-Root Unscented Kalman Filter* (QuAdSRUKF); this filter has better computational properties than the other SRUKF's and Unscented Kalman Filter's (UKF's) for attitude systems of the literature (Section 7.3.2). Comparative with the UKF's of the literature, the QuAdSRUKF is computationally more stable in ill-conditioned situations because of its square-root properties; and comparative with the SRUKF's of the literature, the QuAdSRUKF is always computationally more stable because it has less (or even none) Cholesky factor downdatings (Section 7.3.2). These superior properties of the QuAdSRUKF were verified in numerical simulations considering the Unscented filters (UKF's and SRUKF's) for attitude systems in two problems (Section 7.4.2): 1) a theoretical problem with the performance of the filters being deteriorated by round-off errors; and 2) a satellite attitude estimation problem in two different situations considering i) normal conditions, ii) and computationally ill-conditioned conditions. In two of all these three situations [the only situation of the problem 1), and the situations ii) of the problem 2)], the QuAdSRUKF provided reliable estimates, but all the Unscented filters for attitude systems of the literature did not. Besides, even in normal conditions [situation i) of problem 2)], the QuAdSRUKF outperformed the Unscented filters of the literature by presenting better estimates (the second smallest mean error was 10,56% higher than the error of the QuAdSRUKF).

Our initial goal in this chapter was to extend the systematization of Part I to treat

quaternion systems. However, from the analysis developed in this chapter, we can conclude that the additive UKF's for quaternions systems of the literature were built upon some intuitive, but not mathematically-sound concepts; indeed, we can cite the following conclusions regarding this analysis.

1. The additive quaternion models are not consistent (cf. Remark 7.1).
2. Some of the probability and statistic concepts for the quaternion space *need further study*. For instance, it is not clear what are the definitions and properties of i) quaternionic random variables, their distributions, and their statistics; ii) the statistics of quaternionic weighted sets (such as quaternionic σ -representations); iii) the statistics of a transformed quaternionic random variable.
3. The form of the filters are extended from the Euclidean filters *without enough explanation*. For instance, what is the reason behind the correction equations of these UKF's [e.g. step (2d) of the QuAdUKF]? What kind of approximation does it provide?

In the next chapters, we will present a theory able to cover these—and possibly other—gaps in the current Unscented Kalman filtering theory for quaternion models. We will work with manifolds because i) the set of unit quaternions is a Riemannian manifold, and ii) there are some probability and statistic results for Riemannian manifolds in the literature (specially in [66]).

8. INTRINSIC STATISTICS ON RIEMANNIAN MANIFOLDS

In the first chapter of this part, Part II, we focused our attention upon extending the theory of Part I to rotating systems whose state spaces are in the space of unit quaternions. However, in Chapter 7, after analyzing the Additive UF's for these systems, we came to the conclusion that more attention have to be given to the mathematical concepts supporting these filters. For instance, we pointed out that probability and statistic concepts for the quaternion space need further study.

In this chapter, we will present statistical results on Riemannian manifolds. The main reasons for this choice are:

1. The set of unit quaternions is a Riemannian manifold. Therefore, the additive UF's for quaternion systems are particular cases of UF's for systems whose state variables belong to Riemannian manifolds.
2. There are some probability and statistic results for Riemannian manifolds in the literature, specially in [66].
3. Riemannian manifolds can model wider range of real systems than unit quaternions. Recall, from Section 1.1, that while unit quaternions can model rotations of 3-dimensional rigid bodies, Riemannian manifolds can model more complex real problems, such as the ones treated by the general theory of relativity. Besides, in Section 9.6, we present an extension of our UF's for Riemannian manifolds to the case of unit dual quaternions; these dual quaternions are adequate to model 3-dimensional rigid bodies displacements (rotations along with translations).

We highlight that other approaches could be taken for extending the theory of Part I to rotating systems, such as using the theory known as *Directional Statistics* (for more information on this topic, see [168]). Recently, some works have proposed consistent Unscented-based filters for quaternion models using directional distributions (distributions from Directional Statistics), such as the *Bingham Distribution* and *von Mises-Fisher Distribution* [136, 137]. However, working on Riemannian manifolds may be more appropriate for us; the following two arguments can be considered to defend this choice:

1. Riemannian manifolds are more general than the manifolds considered by Directional Statistics. The manifolds considered by Directional Statistics are spheres

and projective spaces, which are particular Riemannian manifolds.

2. the statistics developed in [66] for Riemannian manifolds might present less difficulties comparative with Directional Statistics. The results based on Directional Statistics are mostly *extrinsic*, i.e., they rely on the embedding space \mathbb{R}^n of the sphere S^{n-1} [66]. This means working with operations that are not well-defined in the S^{n-1} , such as usual Euclidean sums. Consequently, developing consistent results might turn troublesome at some point. On the other hand, the statistics developed in [66] are *intrinsic* to the manifolds; consequently, we work only with operations that are well-defined on the manifolds.

In this chapter, we present the results of probability and statistic for Riemannian manifolds which are required for the development of the UF's for these manifolds. Some of these probability and statistic results were introduced by [66]; and other results, by us. In Appendix A we provide the background on Riemannian manifolds needed upon which the results of this chapter and of Chapter 9 are built.

In Section 8.1, we present Riemannian random points; they are extensions of random vectors for Riemannian manifolds. In Section 8.2, we present the definition of the Riemannian mean; naturally, this concept is more complex than its analogous of the Euclidean case, the expected value. In Section 8.3, we present definitions of Riemannian moments; recall that UF's are based on means and covariances, which are moments. In Section 8.4, we present the concepts regarding jointly distributed Riemannian random points. In Section 8.6, we define statistics for weighted sets. Finally, in Section 8.7, we present the conclusions of this chapter.

8.1 RANDOM POINTS ON A RIEMANNIAN MANIFOLD

The work [66] introduces concepts of probability and statics defined, intrinsically, in Riemannian manifolds—that is, the concepts do not use results of embedding ambient spaces—that are necessary for our development. We now present i) some of these concepts of [66], ii) make some extensions in some of them (e.g. our definitions of moments are extended), and iii) propose other related results (e.g. all the results concerning joint Riemannian random points, all moments of order higher than 2). These novelties will be necessary in the development of the Riemannian Unscented filters of Chapter 9.

We are interested in measurements of elements of a Riemannian manifold that

depend on the outcome of a random experiment. Particular cases are given by random transformation and random feature for the particular case of transformation groups and homogeneous manifolds [66].

Definition 8.1 (Random point on a Riemannian Manifold). Let $(\Omega, \mathcal{B}(\Omega), \Pr)$ be a probability space, $\mathcal{B}(\Omega)$ being the Borel σ -algebra of Ω (i.e. the smallest σ -algebra containing all the open subsets of Ω) and \Pr a measure on $\mathcal{B}(\Omega)$ such that $\Pr(\Omega) = 1$. A (Riemannian) random point (or random variable) in the Riemannian manifold \mathcal{M} is a Borel measurable function \mathbf{X} from Ω to \mathcal{M} . The set of all Riemannian random points taking value on a Riemannian manifold \mathcal{M} will be denoted by $\Phi_{\mathcal{M}}$.

As in the real or vector case, we can now make abstraction of the original space Ω and directly work with the induced probability measure on \mathcal{M} . In a vector space with basis $\mathcal{A} = (a_1, \dots, a_n)$, the local representation of the metric is given by $G = A^T A$ where $A := [a_1, \dots, a_n]$ is the matrix of coordinates change from \mathcal{A} to an orthonormal basis. Similarly, the measure (or the infinitesimal volume element) is given by the volume of the parallelepipedon spanned by the basis vectors:

$$d\mathcal{V} = \|\det(A)\| dx = \sqrt{\|\det(G)\|} dx.$$

Assuming now a Riemannian manifold \mathcal{M} , we can see that the Riemannian metric $G(x)$ induces an infinitesimal volume element on each tangent space, and thus a measure on the manifold [66, p.131]:

$$d\mathcal{M}(\mathbf{p}) = \sqrt{\|\det(G(x))\|} dx. \quad (8.1)$$

One can show that the cut locus has a null measure. This means that we can integrate real functions indifferently in \mathcal{M} or in any exponential chart. If f is an integrable function of the manifold and

$$f_q(\vec{q\mathbf{p}}) := f(\exp_q(\vec{q\mathbf{p}}))$$

is its image in the exponential chart at \mathbf{q} , we have:

$$\int_{\mathcal{M}} f(\mathbf{q}) d\mathcal{M}(\mathbf{q}) = \int_{D(\mathbf{q})} f_q(z) \sqrt{G(z)} dz, \quad (8.2)$$

where $D(\mathbf{q})$ is the maximal definition domain for the exponential chart at a point $\mathbf{p} \in \mathcal{M}$.

Definition 8.2. Let $\mathcal{B}(\mathcal{M})$ be the Borel σ -algebra of \mathcal{M} . The random point \mathbf{X} has a (Riemannian) probability density function $\text{pdf}_{\mathbf{X}}$ (real, positive and integrable function)

if [66, p.132]:

$$\begin{aligned} \forall \mathcal{Y} \in \mathcal{B}(\mathcal{M}), \quad \Pr(\mathbf{X} \in \mathcal{Y}) &= \int_{\mathcal{Y}} \text{pdf}_{\mathbf{X}}(\mathbf{y}) d\mathcal{M}(\mathbf{y}); \\ \text{and} \quad \Pr(\mathcal{M}) &= \int_{\mathcal{M}} \text{pdf}_{\mathbf{X}}(\mathbf{y}) d\mathcal{M}(\mathbf{y}) = 1. \end{aligned}$$

A simple example of a pdf is the *uniform pdf* in a bounded set \mathcal{Y} :

$$\text{pdf}_{\mathbf{X}}(\mathbf{y}) = \frac{1}{\int_{\mathcal{Y}} d\mathcal{M}} 1_{\mathcal{Y}(\mathbf{y})} = \frac{1_{\mathcal{Y}(\mathbf{y})}}{\text{Vol}(\mathcal{Y})},$$

where $\text{Vol}(\mathcal{Y})$ stands for the volume of \mathcal{Y} . One must be careful that this pdf is uniform with respect to the measure $d\mathcal{M}$ and is not uniform for another measure on the manifold. This problem is the basis of the Bertrand paradox for geometrical probabilities and raise the problem of the measure to choose on the manifold. In our case, the measure is induced by the Riemannian metric, but the problem is only lifted: which Riemannian metric do we have to choose? For transformation groups and homogeneous manifolds, an invariant metric is a good geometric choice, even if such a metric does not always exist for homogeneous manifolds or if it leads in general to a partial consistency only between the geometric and statistical operations in non compact transformation groups [66, p.132].

Working with pdf's and integrals in a Riemannian manifolds may become hard. We can work in an Euclidean space instead. Let \mathbf{X} be a Riemannian random point with $\text{pdf}_{\mathbf{X}}$ taking values on a Riemannian manifold \mathcal{M} , and let $\varphi : U \subset \mathbb{R}^n \rightarrow \mathcal{M}$ be a chart of \mathcal{M} such that $\mathbf{X}(\omega) \in \mathcal{M}$ for some events ω . Then $X = \varphi^{-1}(\mathbf{X}(\omega))$ is an (Euclidean) random *vector* defined in U whose pdf_X is defined with respect to the Lebesgue measure dx in \mathbb{R}^n instead of $d\mathcal{M}$ in \mathcal{M} . Using the expression of the Riemannian measure, the two pdf's are related by [66, p.132]

$$\text{pdf}_X(u) = \text{pdf}_{\mathbf{X}}(\mathbf{q}) \sqrt{\|\det(G(x))\|}, \quad \mathbf{q} \in \mathcal{Y} \in \mathcal{B}(\mathcal{M}) \text{ and } u \in Z \in \mathcal{B}(\mathbb{R}^n). \quad (8.3)$$

Note that the density pdf_X *depends* on the chart used whereas the $\text{pdf}_{\mathbf{X}}$ does *not*—it is intrinsic to the manifold [66, p.132].

Let $f(\mathbf{X})$ be a Borelian real valued function defined on \mathcal{M} and \mathbf{X} a Riemannian random point with $\text{pdf}_{\mathbf{X}}$. Then $f(\mathbf{X})$ is a real random variable and we can compute its expectation [66, p.132]:

$$\mathcal{E}_X \{ \varphi(\mathbf{q}) \} := \int_{\mathcal{M}} f(\mathbf{p}) \text{pdf}_{\mathbf{X}}(\mathbf{p}) d\mathcal{M}(\mathbf{p}) \quad (8.4)$$

$$\begin{aligned}
&= \int_{\mathbb{R}^n} y \text{pdf}_{f(\mathbf{X})}(x) dx \\
&= \mathcal{E}_{f(\mathbf{X})} \{f(\mathbf{q})\}.
\end{aligned}$$

This notion of expectation corresponds to the one we defined on real random variables and vectors. However, we cannot directly extend it to the case where $f(\mathbf{X})$ take values in manifold because we do not have defined the integral in (8.4) for such cases. We need other notions for mean values.

8.2 EXPECTATION OR MEAN OF A RANDOM POINT

In this section we focus in the notion of central value of a distribution. We will preferably use the denomination *mean value* or *mean point* than expected point to stress the difference between this notion and the expectation of a real function [66, p.132].

8.2.1 Fréchet Expectation or Mean Value

Let X be a random vector on \mathbb{R}^n . Fréchet observed that the variance

$$\sigma_{\mathbf{X}}^2(c) := \mathcal{E}_X \{ \text{dist}^2(X, c) \}$$

is minimized for the mean vector $\bar{X} = \mathcal{E}_X \{X\}$. The major point for the generalization is that the expectation of a real valued function is well defined for our connected and geodesically complete Riemannian manifold \mathcal{M} .

Definition 8.3 (Variance of a random point [66]). Let $\mathbf{X} \in \Phi_{\mathcal{M}}$ be a Riemannian random point. The (Riemannian) variance $\sigma_{\mathbf{X}}^2(\mathbf{c})$ is the expectation of the squared distance between the random point and the fixed point $\mathbf{c} \in \mathcal{M}$:

$$\sigma_{\mathbf{X}}^2(\mathbf{c}) := \mathcal{E}_{\mathbf{X}} \{ \text{dist}^2(\mathbf{c}, \mathbf{q}) \} = \int_{\mathcal{M}} \text{dist}^2(\mathbf{c}, \mathbf{u}) \text{pdf}_{\mathbf{X}}(\mathbf{u}) d\mathcal{M}(\mathbf{u}). \quad (8.5)$$

Definition 8.4 (Fréchet expectation of a random point [66]). Consider a Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$. If the variance $\sigma_{\mathbf{X}}^2(\mathbf{c})$ is finite for all point $\mathbf{c} \in \mathcal{M}$ (which is in particular true for a density with a compact support), then every point $\bar{\mathbf{X}}$ minimizing this $\sigma_{\mathbf{X}}^2(\mathbf{c})$ is called an *expected* or (*Riemannian*) *mean (point)*. Thus, the Riemannian mean of χ is defined by:

$$\bar{\mathbf{X}} := \arg \min_{\mathbf{c} \in \mathcal{M}} \left(\mathcal{E}_{\mathbf{X}} \{ \text{dist}^2(\mathbf{c}, \mathbf{X}) \} \right), \quad (8.6)$$

and the set of all means of \mathbf{X} is represented by $\mathbb{E}(\mathbf{X})$ —it is possible to exist more than one point satisfying (8.6). If there exists at least one mean point $\bar{\mathbf{X}}$, we call *variance* the minimal value $\sigma_{\mathbf{X}}^2 := \sigma_{\mathbf{X}}^2(\bar{\mathbf{X}})$ and *standard deviation* ($\sigma_{\mathbf{X}} \equiv \sigma_{\mathbf{X}}(\bar{\mathbf{X}})$) its square-root.

8.2.2 Existence and Uniqueness: Riemannian Center of Mass

As a mean point is the result of a minimization, its existence is not ensured (the global minimum could be unreachable) and anyway the result is a set and no longer a single element. This has to be compared with some central values in vector spaces, for instance the modes. However, the Fréchet expectation does not define all the modes even in vector spaces: one only keeps the modes of maximal intensity [66, p.133].

To get rid of this constraint, [169] proposed to consider the local minima of the variance $\sigma_{\mathbf{X}}^2(\mathbf{c})$ defined in (8.5) instead of the global ones. We call this new set of means *Riemannian centers of mass*. As global minima are local minima, the Fréchet expected points are a subset of the Riemannian centers of mass. However, the use of local minima allows to characterize the Riemannian centers of mass using only local derivatives of order two.

Using this extended definition, [169] and [170] established conditions on the manifold and on the distribution to ensure the existence and uniqueness of the mean. We just recall here the results without the proofs.

Definition 8.5 (Regular geodesic balls [66]). The ball $\mathbb{B}_c(r)$ is said *geodesic* if it does not meet the cut locus of its center. This means that there exists a unique minimizing geodesic from the center to any point of a geodesic ball. The ball is said *regular* if its radius verifies $2r\sqrt{\kappa} < \pi$, where κ is the maximum of the Riemannian curvature in this ball.

8.3 RIEMANNIAN CENTRAL MOMENTS

Euclidean KF's are built up with covariances; these matrices provide a measure of the error of the estimate that a KF is providing. We then define the covariance of a Riemannian random point in order to, ultimately, define UKF's for Riemannian manifolds.

Definition 8.6 (Riemannian covariance (extended from [66])). Let $\mathbf{X} \in \Phi_{\mathcal{M}}$ be a Riemannian random point with a mean $\bar{\mathbf{X}} \in \mathbb{E}(\mathbf{X})$. Consider a point $\mathbf{q} \in \mathcal{M}$ with *cut locus* $\mathcal{C}(\mathbf{q})$ and let $D(\mathbf{q})$ be the maximal definition domain for the exponential chart at

\mathbf{q} . If $\bar{\mathbf{X}} \in \mathcal{M} - \mathcal{C}(\mathbf{q})$, then the *covariance* of \mathbf{X} relative to $\bar{\mathbf{X}}$ at \mathbf{q} is defined by

$$\begin{aligned} \mathbf{P}_{\mathbf{X}\mathbf{X},\bar{\mathbf{X}}}^{\mathbf{q}} &:= \mathcal{E}_{\mathbf{X}} \left\{ \left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \right\} \\ &= \int_{\mathcal{M} - \mathcal{C}(\mathbf{q})} \left(\log_{\mathbf{q}}(\mathbf{x}) - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \text{pdf}_{\mathbf{X}}(\mathbf{x}) d\mathcal{M}(\mathbf{x}); \end{aligned} \quad (8.7)$$

we can omit the reference to the point \mathbf{q} when $\mathbf{q} = \bar{\mathbf{X}}$ for simplicity in some cases. If $\mathbb{E}(\mathbf{X}) = \{\bar{\mathbf{X}}\}$ —that is, $\bar{\mathbf{X}}$ is the unique mean according to (8.6)—, we can write $\mathbf{P}_{\mathbf{X}\mathbf{X}}^{\mathbf{q}} := \mathbf{P}_{\mathbf{X}\mathbf{X},\bar{\mathbf{X}}}^{\mathbf{q}}$ or even $\mathbf{P}_{\mathbf{X}\mathbf{X}} := \mathbf{P}_{\mathbf{X}\mathbf{X},\bar{\mathbf{X}}}^{\bar{\mathbf{X}}}$.

Definition (8.6) is more general than Definition 6 of [66] in two characteristics: i) it is defined when more than one Riemannian mean exists, whereas [66] it is assumed that $\bar{\mathbf{X}} \in \mathbb{E}(\mathbf{X})$ is unique; and ii) it is defined for any point $\mathbf{q} \in \mathcal{M}$, whereas in [66] it is defined only for $\mathbf{q} = \bar{\mathbf{X}}$. Particularly, this second extension will be necessary when developing UKF's for Riemannian because we will need to calculate covariance in points $\mathbf{q} \neq \bar{\mathbf{X}}$.

The covariance depends on the basis used for the exponential chart if we see it as a matrix— that is, expressing it with coordinates—, but it does not depend on it if we consider it as a bilinear form over the tangent plane [66].

The covariance $\mathbf{P}_{\mathbf{X}\mathbf{X}}$ is related to the variance just as in the vector case [66]:

$$\begin{aligned} \text{Tr}(\mathbf{P}_{\mathbf{X}\mathbf{X}}) &:= \text{Tr} \mathbf{M} \left(\mathcal{E}_{\mathbf{X}} \left\{ \left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \right\} \right) \\ &= \mathcal{E}_{\mathbf{X}} \left\{ \text{Tr} \left(\left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \right) \right\} \\ &= \mathcal{E}_{\mathbf{X}} \left\{ \text{dist}^2 \left(\bar{\mathbf{X}}, \mathbf{X} \right) \right\} \\ &=: \sigma_{\bar{\mathbf{X}}}^2. \end{aligned} \quad (8.8)$$

Recall from Chapters 3 and 4 that higher order central moments are necessary in order to develop higher order sigma representations and Unscented Transformations. Hence, we define higher order central moments for Riemannian random variables.

Definition 8.7 (Riemannian central moments). Let $\mathbf{X} \in \Phi_{\mathcal{M}}$ be a Riemannian random point with a mean $\bar{\mathbf{X}} \in \mathbb{E}(\mathbf{X})$. Consider a point $\mathbf{q} \in \mathcal{M}$ with *cut locus* $\mathcal{C}(\mathbf{q})$ and let $D(\mathbf{q})$ be the maximal definition domain for the exponential chart at \mathbf{q} . If $\bar{\mathbf{X}} \in \mathcal{M} - \mathcal{C}(\mathbf{q})$, then the j th (central) moment of \mathbf{X} relative to $\bar{\mathbf{X}}$ at \mathbf{q} is defined by

$$\mathbf{M}_{\mathbf{X},\bar{\mathbf{X}}}^{\mathbf{q},j} := \begin{cases} \mathcal{E} \left\{ \left[\left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \right]^{\otimes \frac{j}{2}} \right\} & \text{for even } j, \\ \mathcal{E} \left\{ \left[\left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) (\diamond)^T \right]^{\otimes \frac{j-1}{2}} \otimes \left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) \right\} & \text{for odd } j; \end{cases} \quad (8.9)$$

we can omit the reference to the point \mathbf{q} when $\mathbf{q} = \bar{\mathbf{X}}$ for simplicity in some cases. If $\mathbb{E}(\mathbf{X}) = \{\bar{\mathbf{X}}\}$, we can write $\mathbf{M}_{\bar{\mathbf{X}}}^{q,j} := \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,j}$ or even $\mathbf{M}_{\bar{\mathbf{X}}}^j := \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{\bar{\mathbf{X}},j}$.

Note that $\mathbf{P}_{\mathbf{X}\mathbf{X},\bar{\mathbf{X}}}^q = \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,2}$. The notations $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,1}, \dots, \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,l})_{\mathcal{M}}$ and $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,1}, \dots, \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,l})$ will stand for a Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with $\mathbf{x} \in \mathbb{E}(\mathbf{X})$ being one Riemannian mean and $\mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,1}, \dots, \mathbf{M}_{\bar{\mathbf{X}},\bar{\mathbf{X}}}^{q,l}$ its moments respective to $\bar{\mathbf{X}}$.

8.4 JOINT PROBABILITY AND STATISTICS

Definition 8.8 (Joint probability density function). Let $\mathcal{B}(\mathcal{M})$ be the Borel σ -algebra of \mathcal{M} . The Riemannian random points $\mathbf{X} \in \Phi_{\mathcal{M}}$ and $\mathbf{Y} \in \Phi_N$ have a (*Riemannian*) *joint probability density function* $\text{pdf}_{\mathbf{X},\mathbf{Y}}$ (real, positive and integrable function) if:

$$\begin{aligned} \forall A \in \mathcal{B}(\mathcal{M} \times N), \quad \Pr((\mathbf{X}, \mathbf{Y}) \in A) &= \int_A \text{pdf}_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) d\mathcal{M}(\mathbf{x}) d\mathcal{M}(\mathbf{y}); \\ \text{and} \quad \Pr(\mathcal{M}) &= \int_{\mathcal{M} \times N} \text{pdf}_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) d\mathcal{M}(\mathbf{x}) d\mathcal{M}(\mathbf{y}) = 1. \end{aligned}$$

Definition 8.9 (Joint Expected moment). Let $\mathbf{X} \in \Phi_{\mathcal{M}}$ and $\mathbf{Y} \in \Phi_N$ be Riemannian random points with joint pdf $\text{pdf}_{\mathbf{X},\mathbf{Y}}$, and f be a function from a subset $\mathcal{U} \subset \mathcal{M} \times N$ to \mathbb{R}^n . Then the (*Riemannian*) *joint expected moment* of f respective to \mathbf{X} and \mathbf{Y} —or to (\mathbf{X}, \mathbf{Y}) —is defined by

$$\mathcal{E}_{\mathbf{X},\mathbf{Y}} \{f(\mathbf{x})\} := \int_{\mathcal{U}} f(\mathbf{x}, \mathbf{y}) \text{pdf}_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) d\mathcal{M}(\mathbf{x}) d\mathcal{M}(\mathbf{y}).$$

Definition 8.10 (Cross-covariance). Let $\mathbf{X} \in \Phi_{\mathcal{M}}$ and $\mathbf{Y} \in \Phi_N$ be Riemannian random points with a mean $\bar{\mathbf{X}} \in \mathbb{E}(\mathbf{X})$ and $\bar{\mathbf{Y}} \in \mathbb{E}(\mathbf{Y})$, respectively. Consider two points of $\mathbf{q} \in \mathcal{M}$ and $\mathbf{b} \in N$ with *cut loci* $\mathcal{C}(\mathbf{q})$ and $\mathcal{C}(\mathbf{b})$, respectively, and let $D(\mathbf{q})$ and $D(\mathbf{b})$ be the maximal definition domains for the exponential charts at \mathbf{q} and \mathbf{b} , respectively. If $\bar{\mathbf{X}} \in \mathcal{M} - \mathcal{C}(\mathbf{q})$ and $\bar{\mathbf{Y}} \in N - \mathcal{C}(\mathbf{b})$, then the (*Riemannian*) *cross-covariance* of \mathbf{X} and \mathbf{Y} respective to $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$ —or the (*Riemannian*) *cross-covariance* of (\mathbf{X}, \mathbf{Y}) respective to $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ —at (\mathbf{q}, \mathbf{b}) is defined by

$$\begin{aligned} \mathbf{P}_{\mathbf{X},\mathbf{Y},(\bar{\mathbf{X}},\bar{\mathbf{Y}})}^{qb} &:= \mathcal{E}_{\mathbf{X},\mathbf{Y}} \left\{ \left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) \left(\log_{\mathbf{b}} \mathbf{Y} - \log_{\mathbf{b}} \bar{\mathbf{Y}} \right)^T \right\} \\ &= \int_{\mathcal{U}} \left(\log_{\mathbf{q}} \mathbf{X} - \log_{\mathbf{q}} \bar{\mathbf{X}} \right) \left(\log_{\mathbf{b}} \mathbf{Y} - \log_{\mathbf{b}} \bar{\mathbf{Y}} \right)^T \text{pdf}_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) d\mathcal{M}(\mathbf{x}) d\mathcal{M}(\mathbf{y}), \end{aligned} \tag{8.10}$$

where $\mathcal{U} := (\mathcal{M} - \mathcal{C}(\mathbf{q})) \times (N - \mathcal{C}(\mathbf{b}))$; we can omit the reference to (\mathbf{q}, \mathbf{b}) when $(\mathbf{q}, \mathbf{b}) =$

(\bar{X}, \bar{Y}) for simplicity in some cases. If $\mathbb{E}\{\mathbf{X}\} = \{\bar{X}\}$ and $\mathbb{E}\{\mathbf{Y}\} = \{\bar{Y}\}$, we can write $P_{XY}^{qb} := P_{XY,(\bar{X},\bar{Y})}^{qb}$ or even $P_{XY} := P_{XY,(\bar{X},\bar{Y})}^{\bar{X}\bar{Y}}$.

8.5 SOME TRANSFORMATIONS OF RIEMANNIAN RANDOM VARIABLES

In this section, we provide two propositions concerning transformations of Riemannian random points that will be important further in this work.

Proposition 8.1. *Consider the Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}^n}$, for $\mathbf{q} \in \mathcal{M}^n$,*

$$\log_{\mathbf{q}} \mathbf{X} \sim (\bar{X}, P_{XX});$$

and, for the point $\mathbf{p} \in \mathcal{M}$ and linear mappings $A : T_{\mathbf{q}}\mathcal{M}^n \rightarrow T_{\mathbf{q}}\mathcal{M}^n, B \in T_{\mathbf{q}}\mathcal{M}^n \rightarrow T_{\mathbf{q}}\mathcal{M}^n$, the Riemannian random point

$$\mathbf{Z} := \exp_{\mathbf{q}} \left(A \log_{\mathbf{q}} \mathbf{X} + B \log_{\mathbf{q}} \mathbf{p} \right).$$

Then the Riemannian mean \bar{Z} of \mathbf{Z} , and its covariance $P_{ZZ}^{\mathbf{q}} := P_{ZZ,\bar{Z}}^{\mathbf{q}}$ (relative to \bar{Z} at \mathbf{q}) are, respectively,

$$\bar{Z} = \exp_{\mathbf{q}} \left(A\bar{X} + B \log_{\mathbf{q}} \mathbf{p} \right) \tag{8.11}$$

$$P_{ZZ}^{\mathbf{q}} = AP_{XX}A^T + B \log_{\mathbf{q}} \mathbf{p} \log_{\mathbf{q}}^T \mathbf{p} B^T. \tag{8.12}$$

In particular, for $\mathbf{q} = \bar{X}$, we have that

$$\mathbf{Z} \sim \left(\exp_{\bar{X}} (B \log_{\bar{X}} \mathbf{p}), AP_{XX}A^T + B \log_{\bar{X}} \mathbf{p} \log_{\bar{X}}^T \mathbf{p} B^T \right). \tag{8.13}$$

Proof. From (8.6), a Riemannian mean \bar{Z} of \mathbf{Z} is such that it solves the following optimization problem

$$\begin{aligned} & \text{minimize } g(\mathbf{c}) := \mathcal{E}_{\mathbf{Z}} \left\{ \text{dist}^2(\mathbf{c}, \mathbf{Z}) \right\} \\ & \text{subject to } \mathbf{c} \in \mathcal{M}; \end{aligned} \tag{8.14}$$

now consider the function

$$\tilde{g}(\tilde{c}) := g \left(\log_{\mathbf{q}} \mathbf{Z} \right) = \mathcal{E}_{\log_{\mathbf{q}} \mathbf{Z}} \left\{ \text{dist}^2(\tilde{c}, x) \right\},$$

and the following optimization problem

$$\begin{aligned} & \text{minimize } \tilde{g}(\tilde{c}) := \mathcal{E}_{\log_q \mathbf{Z}} \left\{ \text{dist}^2(\tilde{c}, x) \right\} \\ & \text{subject to } \tilde{c} \in T_q \mathcal{M}; \end{aligned} \tag{8.15}$$

Because i) the function \log_q is one-to-one, and ii) $\tilde{c} = \mathcal{E}\{\log_q \mathbf{X} + \log_q \mathbf{p}\}$ minimizes (8.15), then $\log_q^{-1} \bar{X} = \exp_q \bar{X}$ minimizes (8.14), and consequently $\bar{\mathbf{Z}} = \exp_q \mathcal{E}\{A \log_q \mathbf{X} + B \log_q \mathbf{p}\}$. Now we have that, since $\log_q \mathbf{p}$ is constant to the integral of the expected value, and using

$$\begin{aligned} \mathcal{E}\{A \log_q \mathbf{X} + B \log_q \mathbf{p}\} &:= \mathcal{E}\{A \log_q \mathbf{X}\} + B \log_q \mathbf{p} \\ &= A \bar{X} + B \log_q \mathbf{p}; \end{aligned}$$

this proves (8.11).

For the part relative to the covariance, we have that, from (8.7), \mathbf{P}_{ZZ}^q is given by

$$\mathbf{P}_{ZZ}^q := \int_{\mathcal{M}-\mathcal{C}(\bar{\mathbf{Z}})} (\log_q z - \log_q \bar{\mathbf{Z}}) (\diamond)^T \text{pdf}_{\mathbf{Z}}(z) d\mathcal{M}(z).$$

By making the change of variables $Az = \log_q z$ (cf. (8.3) and (8.2)) and using (8.11), it follows that

$$\begin{aligned} \mathbf{P}_{ZZ}^q &= \int_{D(\bar{\mathbf{Z}})} (Az - \log_q \bar{\mathbf{Z}}) (\diamond)^T \text{pdf}_{(\log_q \mathbf{Z})}(z) \sqrt{\|\det G(z)\|} dz \\ &= \int_{D(\bar{\mathbf{Z}})} (Az - \log_q (\exp_p (A \bar{X} + B \log_q \mathbf{p}))) (\diamond)^T \text{pdf}_{(\log_q \mathbf{Z})}(z) \sqrt{\|\det G(z)\|} dz \\ &= \int_{D(\bar{\mathbf{Z}})} (Az - A \bar{X} - B \log_q \mathbf{p}) (\diamond)^T \text{pdf}_{(\log_q \mathbf{Z})}(z) \sqrt{\|\det G(z)\|} dz \\ &= \mathcal{E} \left\{ (Az - A \bar{X} - B \log_q \mathbf{p}) (\diamond)^T \right\} \\ &= \mathcal{E} \left\{ (Az - A \bar{X}) (z - B \bar{X})^T \right\} + \mathcal{E} \left\{ (-B \log_q \mathbf{p}) (-B \log_q \mathbf{p})^T \right\} \\ &+ \mathcal{E} \left\{ (Az - A \bar{X}) (-B \log_q \mathbf{p})^T \right\} + \mathcal{E} \left\{ (-B \log_q \mathbf{p}) (Az - A \bar{X})^T \right\} \\ &= AP_{XX} A^T + B \log_q \mathbf{p} \log_q^T \mathbf{p} B^T \\ &+ A (\bar{X} - \bar{X}) (-B \log_q \mathbf{p})^T + (-B \log_q \mathbf{p}) (\bar{X} - \bar{X})^T A^T \\ &= AP_{XX} A^T + B \log_q \mathbf{p} \log_q^T \mathbf{p} B^T; \end{aligned}$$

this proves (8.12). The equation (8.13) follows directly from (8.11) and (8.12) (notice that, in this case, $\bar{X} = \log_{\bar{X}} \mathbf{X} = [0]_{n \times 1}$). \square

Proposition 8.2. *For a Riemannian point $\mathbf{q} \sim (\bar{\mathbf{q}}, \mathbf{P}_{q\mathbf{q}})_{\mathcal{M}^n}$ and a random vector*

$p \sim (\bar{p}, P_{pp})^n$, it follows that

$$\exp_{\bar{q}} [\log_{\bar{q}}(\mathbf{q}) + p] \sim (\exp_{\bar{q}} \bar{p}, \mathbf{P}_q + P_p)_{\mathcal{M}_x}. \quad (8.16)$$

Proof. From (8.6), a Riemannian mean $\bar{\mathbf{X}}$ of

$$\mathbf{X} := \exp_{\bar{q}} [\log_{\bar{q}}(\mathbf{q}) + p]$$

is such that it solves the following optimization problem

$$\begin{aligned} & \text{minimize } g(\mathbf{c}) := \mathcal{E}_{\mathbf{X}} \left\{ \text{dist}^2(\mathbf{c}, \mathbf{x}) \right\} \\ & \text{subject to } \mathbf{c} \in \mathcal{M}; \end{aligned} \quad (8.17)$$

now consider the function

$$\tilde{g}(\tilde{c}) := g(\log_{\bar{q}} \mathbf{c}) = \mathcal{E}_{\log_{\bar{q}} \mathbf{X}} \left\{ \text{dist}^2(\tilde{c}, x) \right\}, \quad (8.18)$$

and the following optimization problem

$$\begin{aligned} & \text{minimize } \tilde{g}(\tilde{c}) := \mathcal{E}_{\log_{\bar{q}} \mathbf{X}} \left\{ \text{dist}^2(\tilde{c}, x) \right\} \\ & \text{subject to } \mathbf{c} \in \mathcal{M}; \end{aligned} \quad (8.19)$$

Since the function $\log_{\bar{q}}$ is one-to-one, it follows that if \tilde{c} minimizes (8.19), then $\log_{\bar{q}}^{-1} \tilde{c} = \exp_{\bar{q}} \tilde{c}$ minimizes (8.17), and consequently $\bar{\mathbf{X}} = \exp_{\bar{q}} \tilde{c}$.

We now show that \bar{p} minimizes (8.19). From (8.18), we have that

$$\begin{aligned} \tilde{g}(\mathbf{c}) &:= \mathcal{E}_{\log_{\bar{q}} \mathbf{X}} \left\{ \text{dist}^2(\tilde{c}, x) \right\} \\ &= \mathcal{E}_{\log_{\bar{q}}(\mathbf{q})+p} \left\{ \text{dist}^2(\tilde{c}, x) \right\} \\ &= \sigma_{\log_{\bar{q}}(\mathbf{q})+p}^2(\tilde{c}), \end{aligned}$$

since $\sigma_{\log_{\bar{q}}(\mathbf{q})+p}^2(\tilde{c})$ is the variance of $\log_{\bar{q}}(\mathbf{q}) + p$ it follows that the $\tilde{g}(\mathbf{c})$ is minimized by

$$\mathcal{E}_{\log_{\bar{q}}(\mathbf{q})+p} \left\{ \log_{\bar{q}}(\mathbf{q}) + p \right\} = [0]_{n \times 1} + \bar{p} = \bar{p}.$$

Thus $\bar{\mathbf{X}} := \exp_{\bar{q}} \bar{p}$, proving the part relative to the mean of $\exp_{\bar{q}} p$. For the part relative to the covariance, we have that, from (8.7), $\mathbf{P}_{\mathbf{X}\mathbf{X}}$ (relative to $\bar{\mathbf{X}}$ at $\bar{\mathbf{X}}$) of \mathbf{X} is

$$\begin{aligned} \mathbf{P}_{\mathbf{X}\mathbf{X}} &:= \int_{\mathcal{M}-\mathcal{C}(\bar{\mathbf{X}})} \left(\log_{\bar{\mathbf{X}}}(\mathbf{x}) - \log_{\bar{\mathbf{X}}}(\bar{\mathbf{X}}) \right) (\diamond)^T \text{pdf}_{\exp_{\bar{q}} p}(\mathbf{x}) d\mathcal{M}(\mathbf{x}) \\ &= \int_{\mathcal{M}-\mathcal{C}(\bar{\mathbf{X}})} \log_{\bar{\mathbf{X}}}(\mathbf{x}) \log_{\bar{\mathbf{X}}}(\mathbf{x})^T \text{pdf}_{\mathbf{X}}(\mathbf{x}) d\mathcal{M}(\mathbf{x}). \end{aligned}$$

Using (8.3) and (8.2),

$$\begin{aligned}
\mathbf{P}_{\mathbf{X}\mathbf{X}} &= \int_{D(\bar{\mathbf{X}})} \log_{\bar{\mathbf{X}}}(\mathbf{x}) \log_{\bar{\mathbf{X}}}(\mathbf{x})^T \text{pdf}_{(\log_{\bar{\mathbf{q}}}(\mathbf{q})+p)}(x) \sqrt{G(x)} d\mathcal{M}(x) \\
&= \int_{D(\bar{\mathbf{X}})} \log_{\bar{\mathbf{X}}}(\mathbf{x}) \log_{\bar{\mathbf{X}}}(\mathbf{x})^T \left(\text{pdf}_{(\log_{\bar{\mathbf{q}}}(\mathbf{q}))}(x) + \text{pdf}_{(p)}(x) \right) \sqrt{G(x)} d\mathcal{M}(x) \\
&= \mathbf{P}_{\mathbf{q}\mathbf{q}} + P_{pp}.
\end{aligned}$$

□

8.6 STATISTICS OF WEIGHTED SETS

For our intentions in this work, we also need definitions of statistics of a set of weighted points in a Riemannian manifold. Consider a (geodesically complete) Riemannian manifold \mathcal{M} and the weighted set

$$\boldsymbol{\chi} := \left\{ \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \boldsymbol{\chi}_i \in \mathcal{M}; w_i^{(1)}, \dots, w_i^{(l)} \in \mathbb{R} \right\}_{i=1}^N$$

—note that we do not restrict these definition to the case $w_i > 0$, nor to $\sum_i^N w_i = 1$. The (*Riemannian*) *sample (empirical) variance* of $\boldsymbol{\chi}$ respective to a point $\mathbf{c} \in \mathcal{M}$ is defined by

$$s_{\boldsymbol{\chi}}^2(\mathbf{c}) := \sum_{i=1}^N w_i^{(1)} \text{dist}^2(\mathbf{c}, \boldsymbol{\chi}_i).$$

If the variance $s_{\boldsymbol{\chi}}^2(\mathbf{c})$ is finite for all point $\mathbf{c} \in \mathcal{M}$, then every point $\boldsymbol{\mu}_{\boldsymbol{\chi}}$ minimizing this $s_{\boldsymbol{\chi}}^2(\mathbf{c})$ is called an *sample expected* or *sample mean point*. Thus, a sample mean point of $\boldsymbol{\chi}$ is defined by

$$\boldsymbol{\mu}_{\boldsymbol{\chi}} := \arg \min_{\mathbf{c} \in \mathcal{M}} \left(\sum_{i=1}^N w_i^{(1)} \text{dist}^2(\mathbf{c}, \boldsymbol{\chi}_i) \right), \quad (8.20)$$

and the set of all sample means of $\boldsymbol{\chi}$ is represented by $\mathcal{E}(\boldsymbol{\chi})$ —it is possible to exist more than one point satisfying (8.20).

If there exists a least one sample mean point $\boldsymbol{\mu}_{\boldsymbol{\chi}}$, we call *sample variance* the minimal value $s_{\boldsymbol{\chi}}^2 := s_{\boldsymbol{\chi}}^2(\boldsymbol{\mu}_{\boldsymbol{\chi}})$ and *standard deviation* ($s_{\boldsymbol{\chi}} \equiv s_{\boldsymbol{\chi}}(\boldsymbol{\mu}_{\boldsymbol{\chi}})$) its square-root. Besides, consider a point $\mathbf{q} \in \mathcal{M}$ with *cut locus* $\mathcal{C}(\mathbf{q})$; if $\boldsymbol{\mu}_{\boldsymbol{\chi}}, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \dots, \boldsymbol{\chi}_N \in \mathcal{M} - \mathcal{C}(\mathbf{q})$, then the (*Riemannian*) *j*th (*central*) *sample (or empirical) moment* of $\boldsymbol{\chi}$ respective to

$\bar{\mathbf{X}}$ at \mathbf{q} is defined by

$$\mathbb{M}_{\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^{q,j} := \begin{cases} \sum_{i=1}^N w_i^{(j)} \left[(\log_q \boldsymbol{\chi}_i - \log_q \boldsymbol{\mu}_{\mathbf{X}}) (\diamond)^T \right]^{\otimes \frac{j}{2}} & \text{for even } j, \\ \sum_{i=1}^N w_i^{(j)} \left[(\log_q \boldsymbol{\chi}_i - \log_q \boldsymbol{\mu}_{\mathbf{X}}) (\diamond)^T \right]^{\otimes \frac{j-1}{2}} \otimes (\log_q \boldsymbol{\chi}_i - \log_q \boldsymbol{\mu}_{\mathbf{X}}) & \text{for odd } j; \end{cases} \quad (8.21)$$

A second sample moment is called a (*Riemannian*) *sample covariance* and represented by $\Sigma_{\mathbf{X}\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^q := \mathbb{M}_{\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^{q,2}$. We can omit the references to the point \mathbf{q} when $\mathbf{q} = \boldsymbol{\mu}_{\mathbf{X}}$ for simplicity in some cases. If $\mathcal{E}(\boldsymbol{\chi}) \equiv \{\boldsymbol{\mu}_{\mathbf{X}}\}$, we can write $\mathbb{M}_{\mathbf{X}}^{q,j} := \mathbb{M}_{\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^{q,j}$, or $\mathbb{M}_{\mathbf{X}}^j := \mathbb{M}_{\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^{q,j}$; and $\Sigma_{\mathbf{X}\mathbf{X}}^q := \Sigma_{\mathbf{X}\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^q$ or $\Sigma_{\mathbf{X}\mathbf{X}} := \Sigma_{\mathbf{X}\mathbf{X}, \boldsymbol{\mu}_{\mathbf{X}}}^q$.

Moreover, for the Riemannian manifolds \mathcal{M} and N , consider the weighted sets $\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i | \boldsymbol{\gamma}_i \in \mathcal{M}, w_i \in \mathbb{R}\}_{i=1}^N$ with a mean $\boldsymbol{\mu}_{\boldsymbol{\chi}} \in \mathcal{E}(\boldsymbol{\chi})$ and $\boldsymbol{\gamma} = \{\boldsymbol{\gamma}_i, w_i | \boldsymbol{\gamma}_i \in N, w_i \in \mathbb{R}\}_{i=1}^N$ with sample mean $\boldsymbol{\mu}_{\boldsymbol{\gamma}} \in \mathcal{E}(\boldsymbol{\gamma})$; and two points of $\mathbf{q} \in \mathcal{M}$ and $\mathbf{b} \in N$ with *cut loci* $\mathcal{C}(\mathbf{q})$ and $\mathcal{C}(\mathbf{b})$, respectively. If $\boldsymbol{\mu}_{\boldsymbol{\chi}}, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \dots, \boldsymbol{\chi}_N \in \mathcal{M} - \mathcal{C}(\mathbf{q})$ and $\boldsymbol{\mu}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \dots, \boldsymbol{\gamma}_N \in N - \mathcal{C}(\mathbf{b})$ then the (*Riemannian*) *sample cross-covariance* of $(\boldsymbol{\chi}, \boldsymbol{\gamma})$ respective to $(\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ at (\mathbf{q}, \mathbf{b}) is defined by

$$\Sigma_{\boldsymbol{\chi}\boldsymbol{\gamma}, (\bar{\mathbf{X}}, \bar{\mathbf{Y}})}^q := \sum_{i=1}^N w_i (\log_q \boldsymbol{\chi}_i - \log_q \boldsymbol{\mu}_{\boldsymbol{\chi}}) (\log_b \boldsymbol{\gamma}_i - \log_b \boldsymbol{\mu}_{\boldsymbol{\gamma}})^T;$$

we can omit the references to the point (\mathbf{q}, \mathbf{b}) when $(\mathbf{q}, \mathbf{b}) = (\bar{\mathbf{X}}, \bar{\mathbf{Y}})$ for simplicity in some cases. If $\mathcal{E}(\boldsymbol{\chi}) \equiv \{\boldsymbol{\mu}_{\boldsymbol{\chi}}\}$ and $\mathcal{E}(\boldsymbol{\gamma}) \equiv \{\boldsymbol{\mu}_{\boldsymbol{\gamma}}\}$, then can write $\Sigma_{\boldsymbol{\chi}\boldsymbol{\gamma}}^{qb} := \Sigma_{\boldsymbol{\chi}\boldsymbol{\gamma}, (\bar{\mathbf{X}}, \bar{\mathbf{Y}})}^{qb}$ or even $\Sigma_{\boldsymbol{\chi}\boldsymbol{\gamma}} := \Sigma_{\boldsymbol{\chi}\boldsymbol{\gamma}, (\bar{\mathbf{X}}, \bar{\mathbf{Y}})}^{\bar{\mathbf{X}}\bar{\mathbf{Y}}}$.

We will also need to treat a more general situation; for a Riemannian manifold \mathcal{M} , consider the weighted set

$$\boldsymbol{\chi} := \left\{ \boldsymbol{\chi}_i, w_i^{(\mathbf{m})}, w_i^{\binom{\mathbf{m}^2}{\lambda^1 \lambda^2}}, \dots, w_i^{\binom{\mathbf{m}^l}{\lambda^1 \dots \lambda^l}} \middle| \boldsymbol{\chi}_i \in \mathcal{M}; \right. \\ \left. w_i^{\binom{\mathbf{m}^2}{\lambda^1 \lambda^2}}, \dots, w_i^{\binom{\mathbf{m}^l}{\lambda^1 \dots \lambda^l}} > 0 \right\}_{i=1}^N,$$

and the vectors

$$\lambda^\eta \in \{\chi_1, \dots, \chi_N, \gamma_1, \dots, \gamma_N\}, \quad \eta = 2, 3, \dots;$$

and a point $\mathbf{q} \in \mathcal{M}$ with *cut locus* $\mathcal{C}(\mathbf{q})$. The sample mean point $\boldsymbol{\mu}_{\boldsymbol{\chi}}$ is also given by (8.20). For $\boldsymbol{\mu}_{\boldsymbol{\chi}}, \boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \dots, \boldsymbol{\chi}_N \in \mathcal{M} - \mathcal{C}(\mathbf{q})$, the (*Riemannian*) *jth (central) sample*

moment of χ respective to $\bar{\mathbf{X}}$ at \mathbf{q} is defined by

$$\mathbb{M}_{\chi, \mu_\chi}^{\mathbf{q}, j} := \begin{cases} \sum_{i=1}^N w_i^{(\mathbf{m}_{\chi_1, \dots, \chi_j}^j)} \left[(\log_{\mathbf{q}} \chi_i - \log_{\mathbf{q}} \mu_\chi) (\diamond)^T \right]^{\otimes \frac{j}{2}} & \text{for even } j, \\ \sum_{i=1}^N w_i^{(\mathbf{m}_{\chi_1, \dots, \chi_j}^j)} \left[(\log_{\mathbf{q}} \chi_i - \log_{\mathbf{q}} \mu_\chi) (\diamond)^T \right]^{\otimes \frac{j-1}{2}} \\ \quad \otimes (\log_{\mathbf{q}} \chi_i - \log_{\mathbf{q}} \mu_\chi) & \text{for odd } j. \end{cases}$$

Additionally, for another Riemannian manifold N , another weighted set

$$\gamma := \left\{ \gamma_i, w_i^{(\mathbf{m})}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \mid \gamma_i \in \mathcal{N} \right\}_{i=1}^N,$$

with a mean μ_γ , and the points \mathbf{q}_l , $l = 1, 2, \dots, j$ define the cross-central moments $\mathbb{M}_{\lambda^1 \dots \lambda^j}^j$ according to the following—supposing that all the log functions are well defined in the points considered: for even j ,

$$\mathbb{M}_{\lambda^1 \dots \lambda^j}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^j}^j)} \bigotimes_{q=1}^{j/2} \left[(\log_{\mathbf{q}_q} \lambda_i^q - \log_{\mathbf{q}_q} \mu_{\lambda^q}) \right. \\ \left. \times (\log_{\mathbf{q}_{(q+1)}} \lambda_i^{q+1} - \log_{\mathbf{q}_{(q+1)}} \mu_{\lambda^{(q+1)}}) \right]^T;$$

and for odd j ,

$$\mathbb{M}_{\lambda^1 \dots \lambda^j}^j := \sum_{i=1}^N w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^j}^j)} \bigotimes_{q=1}^{(j-1)/2} \left[(\log_{\mathbf{q}_q} \lambda_i^q - \log_{\mathbf{q}_q} \mu_{\lambda^q}) \right. \\ \left. \times (\log_{\mathbf{q}_{(q+1)}} \lambda_i^{(q+1)} - \log_{\mathbf{q}_{(q+1)}} \mu_{\lambda^{(q+1)}}) \right]^T \otimes (\log_{\mathbf{q}_j} \lambda_i^j - \log_{\mathbf{q}_j} \mu_{\lambda^j});$$

8.7 CONCLUSIONS REGARDING STATISTICS IN RIEMANNIAN MANIFOLDS

In this chapter, we i) presented some *results of [66]* regarding statistics intrinsically developed in Riemannian manifolds, ii) made some *extensions* these results of [66]—e.g., among others, definitions of moments are extensions—, and iii) propose *other results* regarding statistics in Riemannian manifolds—e.g., among others, moments and sample moments of order higher than 2 (Section 8.3 and 8.6), propositions concerning transformations of Riemannian random points (Section 8.5), and results concerning joint Riemannian random points (Section 8.4).

Using the theory presented in this chapter, we will extend the Unscented Kalman filtering systematization developed in Part I to the case of Riemannian manifolds.

9. UNSCENTED FILTERS FOR RIEMANNIAN MANIFOLDS

In chapter 7, the problem of developing UKF's for quaternion models in the form of (7.2) was addressed using \mathbb{R}^3 parameterizations of S^3 (e.g. Rotation vectors, Rodrigues vectors, and Quaternion vectors). In this chapter, this problem is addressed from another perspective; the theory of Riemannian manifolds is used to develop UKF's for dynamic systems belonging to these manifolds. These UKF's are general cases of UKF's for dynamic systems belonging to S^3 .

The systematization of Part 1 was developed upon the concepts of σ -representation, Unscented Transformation and Unscented Kalman Filter. We want to extend this systematization to the Riemannian case, and hence the first concept that needs to be extended is the σ -representation.

We make the following assumptions in this chapter:

1. all Riemannian manifolds are *geodesically complete* (cf. Section A.4);
2. all Riemannian exponential mappings are defined with their domain allowing them to *realize diffeomorphisms* (cf. Section A.5, this means their inverse mappings, the Riemannian logarithms mappings, always exist and are differentiable);
3. every time a Riemannian exponential of the form $\exp_q v$ is considered, we assume v belonging to the domain the maximal definition domain $D(\mathbf{p})$ of \exp_q (cf. Section A.5);
4. every time a Riemannian logarithm of the form $\log_q \mathbf{p}$ is considered, we assume \mathbf{p} belonging to the domain of \log_q ;
5. every Riemannian random point admits *one, and only one, Riemannian mean* (cf. Definition 8.4);
6. every set of weighted points belonging to a Riemannian manifold admits *one, and only one, Riemannian sample mean* [cf. equation (8.20)].

9.1 RIEMANNIAN σ -REPRESENTATIONS

Riemannian random points are analogous, for Riemannian manifolds, to random vectors for Euclidean spaces. Recall from chapter 3 that random vectors can be approximated by weighted sets called σ -representations (σ R's). Similarly, Riemannian random points can be approximated by weighted sets called *Riemannian σ -representations*.

Definition 9.1. Consider, for a point $\mathbf{q} \in \mathcal{M}$, i) a Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with mean $\bar{\mathbf{X}} \in \mathbb{E}(\mathbf{X})$ (Definition 8.4) and Riemannian central moments $\mathbf{M}_{\mathbf{X}, \bar{\mathbf{X}}}^{\mathbf{q}, j}$, $j = 1, 2, \dots, l$ (cf. Definition 8.6); and ii) a weighted set

$$\boldsymbol{\chi} := \left\{ \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \boldsymbol{\chi}_i \in \mathcal{M} \right\}_{i=1}^N$$

with Riemannian sample mean $\boldsymbol{\mu}_{\boldsymbol{\chi}} \in \mathcal{E}(\boldsymbol{\chi})$ [cf. equation (8.20)] and Riemannian sample moments $\mathbb{M}_{\boldsymbol{\chi}}^j$, $j = 1, 2, \dots, l$ [cf. equation 8.21]. If

$$w_i^{(j)} > 0, \quad \forall i = 1, \dots, N \text{ and } j = 1, \dots, l; \quad (9.1)$$

$$\boldsymbol{\mu}_{\boldsymbol{\chi}} = \bar{\mathbf{X}}; \quad (9.2)$$

$$\mathbb{M}_{\boldsymbol{\chi}}^j = \mathbf{M}_{\mathbf{X}}^j, \quad j = 2, 3, \dots, l; \quad (9.3)$$

then $\boldsymbol{\chi}$ is a *Riemannian l th order N points σ -representation (Rilth $N\sigma$ R) of \mathbf{X} .*

Moreover, assume $\boldsymbol{\chi}$ is an Rilth $N\sigma$ R of X , then:

- $\boldsymbol{\chi}$ is *normalized* if

$$\sum_{i=1}^N w_i^{(j)} = 1, \quad j = 1, 2, \dots, l.$$

- $\boldsymbol{\chi}$ is *homogeneous* if:

$$w_1^{(j)} = w_i^{(j)}, \quad 1 \leq i \leq N-1, \text{ for odd } N; \text{ or} \quad (9.4)$$

$$w_1^{(j)} = w_i^{(j)}, \quad 1 \leq i \leq N, \text{ for even } N. \quad (9.5)$$

- $\boldsymbol{\chi}$ is *symmetric (relative to $\boldsymbol{\chi}_N$)* if—in the case where $\boldsymbol{\chi}$ is *symmetric* relative to other $\boldsymbol{\chi}_i$, we can rearrange the indices of the sigma points and weights—:

$$\begin{aligned} \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_i) - \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_N) &= - \left(\log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_{i+\frac{N-1}{2}}) - \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_N) \right), \\ \text{and } w_i^{(j)} &= w_{i+\frac{N-1}{2}}^{(j)}, \quad 1 \leq i \leq \frac{N-1}{2}, \text{ for odd } N; \text{ or} \end{aligned} \quad (9.6)$$

$$\log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_i) - \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_N) = - \left(\log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_{i+\frac{N}{2}}) - \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_N) \right),$$

$$\text{and } w_i^{(j)} = w_{i+\frac{N}{2}}^{(j)}, 1 \leq i \leq \frac{N}{2}, \text{ for even } N. \quad (9.7)$$

When calling an $\text{Rilth}N\sigma\text{R}$ of \mathbf{X} , the reference to the l th order can be omitted if $l = 2$. Also, the reference to N point and/or to \mathbf{X} can be omitted in case they are obvious from the context or irrelevant for a given statement.

Note that the $\text{Rilth}N\sigma\text{R}$'s are restricted to positive weights $w_i^{(j)}$ [cf. (9.1)]; this will facilitate some results ahead (specially Theorem 9.1).

Definition 9.1 provides concepts analogous to the σR for an Euclidean random variable. However, finding closed forms for $\text{Ri}\sigma\text{R}$'s may be troublesome; the next theorem provides a way to extend the expression of a particular σR to a $\text{Ri}\sigma\text{R}$.

Theorem 9.1. *Consider a Riemannian manifold \mathcal{M}^n , a point $\mathbf{q} \in \mathcal{M} - \mathcal{C}(\bar{\mathbf{X}})$, and a Riemannian random point*

$$\mathbf{X} \sim \left(\bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, l} \right)_{\mathcal{M}^n}.$$

Then the set

$$\boldsymbol{\chi} := \left\{ \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \boldsymbol{\chi}_i \in \mathcal{M} \right\}_{i=1}^N$$

is a normalized $\text{Rilth}N\sigma\text{R}$ of \mathbf{X} if, and only if, the set

$$\chi := \left\{ \log_{\mathbf{q}} \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \right\}_{i=1}^N$$

is a normalized $l\text{th}N\sigma\text{R}$ of the random vector

$$X \sim \left(\log_{\mathbf{q}} \bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, l} \right)^n \in \Phi_{T_{\mathbf{q}}\mathcal{M}}.$$

Moreover, the following statements are true:

1. $\boldsymbol{\chi}$ is homogeneous if, and only if, χ is homogeneous;
2. $\boldsymbol{\chi}$ is symmetric if, and only if, χ is symmetric.

Proof. Suppose the set

$$\boldsymbol{\chi} := \left\{ \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \boldsymbol{\chi}_i \in \mathcal{M} \right\}_{i=1}^N$$

is a $\text{Rilth}N\sigma\text{R}$ of $\mathbf{X} \sim \left(\bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{\mathbf{q}, l} \right)_{\mathcal{M}^n}$. Define the set

$$\chi := \left\{ \log_{\mathbf{q}} \boldsymbol{\chi}_i, w_i^{(1)}, \dots, w_i^{(l)} \right\}_{i=1}^N,$$

and consider $X := \log_q \mathbf{X} \sim (\log_q \bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,l})^n \in \Phi_{T_q \mathcal{M}}$. Then from (9.1), (3.6) is satisfied. We want to show that $\log_q \bar{\mathbf{X}}$ is a sample mean μ_χ of χ . Because χ is a l th $N\sigma$ R of \mathbf{X} , from of (9.2), $\bar{\mathbf{X}}$ is a Riemannian sample mean of χ and, therefore, from (8.20), $\bar{\mathbf{X}}$ minimizes the function

$$g(\mathbf{x}) := \sum_{i=1}^N w_i^{(1)} \text{dist}^2(\mathbf{x}, \exp_q \chi_i). \quad (9.8)$$

The function $g \circ \exp_q : D(\mathbf{q}) \subset T_q \mathcal{M} \rightarrow [0, \infty)$ is a real valued function defined in a subset of the vector space $T_q \mathcal{M}$. We can, therefore, use results of optimization for such cases. Since $g \circ \exp_q$ is a quadratic function, it is clear that it has a minimum $x^* \in D(\mathbf{q})$ and that the derivative of $g \circ \exp_q$ on x^* is zero, that is,

$$\begin{aligned} [0]_{n \times 1} &= \left. \frac{d(g \circ \exp_q)(x)}{dx} \right|_{x=x^*} \\ &= 2 \sum_{i=1}^N w_i^{(1)} (x^* - \chi_i) \\ \Leftrightarrow [0]_{n \times 1} &= x^* \sum_{i=1}^N w_i^{(1)} - \sum_{i=1}^N w_i^{(1)} \chi_i \\ \Leftrightarrow x^* &= \sum_{i=1}^N w_i^{(1)} \chi_i. \end{aligned} \quad (9.9)$$

Since $\bar{\mathbf{X}}$ minimizes g , then $\log_q \bar{\mathbf{X}}$ minimizes $g \circ \exp_q$ —we are assuming that \exp_q is one-to-one—; hence

$$\log_q \bar{\mathbf{X}} = x^* = \sum_{i=1}^N w_i^{(1)} \chi_i =: \mu_\chi, \quad (9.10)$$

and (3.7) is satisfied.

Now let us prove the reverse for the mean. Consider the Riemannian random point $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,l})_{\mathcal{M}^n}$ and the set

$$\chi := \left\{ \chi_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \chi_i \in T_{\bar{\mathbf{X}}} \mathcal{M}, w_i^{(1)}, \dots, w_i^{(l)} > 0 \right\}_{i=1}^N;$$

suppose that all the points χ_i 's belong to the domain of \exp_q , and that χ is an l th $N\sigma$ R of $X := \log_q \mathbf{X} \sim (\log_q \bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,l})^n \in \Phi_{T_q \mathcal{M}}$. Define the set

$$\chi := \left\{ \exp_q(\chi_i), w_i^{(1)}, \dots, w_i^{(l)} \mid \chi_i \in \mathcal{M} \right\}_{i=1}^N$$

[recall that, if $\chi_i = \exp_q(\chi_i)$, then $\chi_i = \log_q(\chi_i)$]. Then, from (3.6) and $w_i^{(1)}, \dots, w_i^{(l)} > 0$, (9.1) is satisfied. We want to show that $\bar{\mathbf{X}}$ is a Riemannian sample mean of χ ; from (9.9), we have that $\mu_\chi := \sum_{i=1}^N w_i^{(1)} \chi_i$ minimizes the function $g \circ \exp_q$, therefore, from

(9.10),

$$\exp_q(\mu_\chi) = \exp_q(\log_q \bar{\mathbf{X}}) = \bar{\mathbf{X}}$$

minimizes g . Then $\bar{\mathbf{X}}$ is a Riemannian sample mean of χ and (9.2) is satisfied.

Now we want to show that (3.8) and (9.3) are equivalent. For even j , we have that, from (3.7) and (8.21),

$$\begin{aligned} \mathbb{M}_{\chi, \bar{\mathbf{X}}}^{q,j} &:= \sum_{i=1}^N w_i^{(j)} \left[(\log_q \chi_i - \log_q \bar{\mathbf{X}}) (\diamond)^T \right]^{\otimes \frac{j}{2}} \\ &= \sum_{i=1}^N w_i^{(j)} \left[(\chi_i - \mu_\chi) (\diamond)^T \right]^{\otimes \frac{j}{2}} \\ &=: \mathbb{M}_\chi^j; \end{aligned}$$

and from (3.8), it follows that

$$\mathbb{M}_{\chi, \bar{\mathbf{X}}}^{q,j} = \mathbb{M}_\chi^j = M_{\bar{\mathbf{X}}}^j.$$

Likewise, for odd j , we have

$$\begin{aligned} \mathbb{M}_{\chi, \bar{\mathbf{X}}}^{q,j} &:= \sum_{i=1}^N w_i^{(j)} \left[(\log_q \chi_i - \log_q \bar{\mathbf{X}}) (\diamond)^T \right]^{\otimes \frac{j}{2}} \otimes (\log_q \chi_i - \log_q \bar{\mathbf{X}}) \\ &= \sum_{i=1}^N w_i^{(j)} \left[(\chi_i - \mu_\chi) (\diamond)^T \right]^{\otimes \frac{j}{2}} \otimes (\chi_i - \mu_\chi) \\ &=: \mathbb{M}_\chi^j; \end{aligned}$$

and from (3.8), it follows that

$$\mathbb{M}_{\chi, \bar{\mathbf{X}}}^{q,j} = \mathbb{M}_\chi^j = M_{\bar{\mathbf{X}}}^j;$$

then (3.8) and (9.3) are equivalent.

It remains to prove statements 1. and 2. Note that statement 2. follows directly from the equivalence between (3.11) and (9.4), and between (3.12) and (9.5).

Now we prove that (3.13) is equivalent to (9.6), and (3.14) to 9.7. From (3.13), for odd N , we have that

$$w_i^{(j)} = w_{i+\frac{N-1}{2}}^{(j)}, \quad 1 \leq i \leq \frac{N-1}{2};$$

and that

$$\log_q(\chi_i) - \log_q(\chi_N) = \chi_i - \chi_N$$

$$\begin{aligned}
&= -\left(\chi_{i+\frac{N-1}{2}} - \chi_N\right) \\
&= -\left(\log_q\left(\chi_{i+\frac{N-1}{2}}\right) - \log_q\left(\chi_N\right)\right);
\end{aligned}$$

therefore, (3.13) is equivalent to (9.6). From (3.14), for even N , we have that

$$w_i^{(j)} = w_{i+\frac{N}{2}}^{(j)}, \quad 1 \leq i \leq \frac{N}{2};$$

and that

$$\begin{aligned}
\log_q(\chi_i) - \log_q(\chi_N) &= \chi_i - \chi_N \\
&= -\left(\chi_{i+\frac{N}{2}} - \chi_N\right) \\
&= -\left(\log_{\mu_{\chi}}\left(\chi_{i+\frac{N}{2}}\right) - \log_{\mu_{\chi}}\left(\chi_N\right)\right);
\end{aligned}$$

therefore(3.14) is equivalent to (9.7), and statement 2. is proved. \square

Of particular importance is the case where $\mathbf{q} = \bar{\mathbf{X}}$: the set χ is a normalized RilthN σ R of

$$\mathbf{X} \sim \left(\bar{\mathbf{X}}, \mathbf{M}_{\mathbf{X}}^2, \dots, \mathbf{M}_{\mathbf{X}}^l\right)_{\mathcal{M}}$$

if, and only if, the set

$$\chi := \left\{\log_{\bar{\mathbf{X}}}(\chi_i), w_i^{(1)}, \dots, w_i^{(l)}\right\}_{i=1}^N$$

is a normalized l thN σ R of

$$X \sim \left([0]_{n \times 1}, \mathbf{M}_{\mathbf{X}}^2, \dots, \mathbf{M}_{\mathbf{X}}^l\right)^n \in \Phi_{T_{\bar{\mathbf{X}}}\mathcal{M}}.$$

With Theorem 9.1 we can extend some results from l thN σ R's to RilthN σ R's, such as the minimum number of sigma points of a RilthN σ R, among others.

Corollary 9.1. *Let $\chi := \{\chi_i, w_i^{(1)}, \dots, w_i^{(l)} \mid \chi_i \in \mathcal{M}\}_{i=1}^N$ be a normalized RilthN σ R of a Riemannian random point $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{P}_{\mathbf{X}\mathbf{X}})_{\mathcal{M}^n}$. Let the rank of the covariance $\mathbf{P}_{\mathbf{X}\mathbf{X}}$ be $r \leq n$. Then the following statements are true:*

1. $N \geq r + 1$. If $N = r + 1$, then χ is called a minimum RilthN σ R of \mathbf{X} .
2. If χ is symmetric, then $N \geq 2r$. If χ is symmetric and $N = 2r$, then χ is called a minimum symmetric RilthN σ R of \mathbf{X} .

Moreover, consider the set

$$\chi := \left\{ \log_q \chi_i, w_i^{(1)}, \dots, w_i^{(l)} \right\}_{i=1}^N$$

and the random vector

$$X \sim \left(\log_q \bar{\mathbf{X}}, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,j}, \dots, \mathbf{M}_{\bar{\mathbf{X}}, \bar{\mathbf{X}}}^{q,l} \right)^n \in \Phi_{T_q \mathcal{M}}.$$

Then the following statements are true:

- If N is even and χ is a (normalized) homogeneous minimum symmetric σ - representation of X (Corollary 3.3), then $\boldsymbol{\chi}$ is also minimum and symmetric and is called a Riemannian (even) (normalized) homogeneous minimum symmetric σ -representation of \mathbf{X} .
- If χ is a HoMiSy σ R of X (Corollary 3.4), then $\boldsymbol{\chi}$ is also minimum and symmetric, and is called a Riemannian (odd) (normalized) homogeneous minimum symmetric σ -representation (RiHoMiSy σ R) of \mathbf{X} .
- If χ is a RhoMi σ R of X (Tab 2.1 [3,2]), then $\boldsymbol{\chi}$ is also minimum, and is called a Riemannian Rho Minimum σ -representation (RiRhoMi σ R) of \mathbf{X} .
- If χ is a Mi σ R of X (Theorem 3.2), then $\boldsymbol{\chi}$ is also minimum, and is called a Riemannian Minimum σ -representation (RiMi σ R) of \mathbf{X} .

Proof. From Theorem 9.1, the set

$$\chi := \left\{ \log_q (\chi_i), w_i^{(1)}, \dots, w_i^{(l)} \right\}_{i=1}^N$$

is a normalized l th $N\sigma$ R of $(\log_q(\chi_i), \mathbf{P}_{\chi \chi}^q)^n$. From the item 1. of Corollary 3.1, it follows that $N \geq r + 1$. Suppose that $\boldsymbol{\chi}$ is symmetric, then, from statement 2. of Theorem 9.1, χ is symmetric; and from statement of 2. of Corollary 3.1 it follows that $N \geq r + 1$. \square

9.2 RIEMANNIAN UNSCENTED TRANSFORMATIONS

The concept of a σ -representation is a requisite for the definition of the UT in Chapter 4. Essentially, an UT is an approximation of the joint pdf of two functionally-related random vectors X and $Y = f(X)$ by two weighted sets χ with points χ_i and γ with points $\gamma_i = f(\chi_i)$, where χ is a σ -representation of X . For a Riemannian extension of the UT, we develop likewise.

Definition 9.2 (Riemannian Unscented Transformation). Consider two Riemannian manifolds \mathcal{M} and \mathcal{N} , the function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$, the Riemannian random point $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{M}_{\mathbf{X}}^2, \dots, \mathbf{M}_{\mathbf{X}}^l)_{\mathcal{M}}$ taking values on \mathcal{M} , and the sets

$$\chi := \left\{ \chi_i, w_i^{(\mathbf{m})}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \middle| \chi_i \in \mathcal{M}; \right. \\ \left. w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} > 0 \right\}_{i=1}^N,$$

and

$$\gamma := \left\{ \gamma_i, w_i^{(\mathbf{m})}, w_i^{(\mathbf{m}_{\lambda^1 \lambda^2}^2)}, \dots, w_i^{(\mathbf{m}_{\lambda^1 \dots \lambda^l}^l)} \middle| \gamma_i = f(\chi_i) \right\}_{i=1}^N.$$

If χ is an $\text{Rilth}N\sigma\text{R}$ of \mathbf{X} , then the l th order Riemannian Unscented Transformation (*RilUT*) is defined by

$$\text{RilUT}(f, \bar{\mathbf{X}}, \mathbf{M}_{\mathbf{X}}^2, \dots, \mathbf{M}_{\mathbf{X}}^l) := [\boldsymbol{\mu}_{\gamma}, \mathbb{M}_{\gamma}^2, \dots, \mathbb{M}_{\gamma}^l, \mathbb{M}_{\lambda^1 \lambda^2}^2, \dots, \mathbb{M}_{\lambda^1 \dots \lambda^l}^l].$$

If $l = 2$ or l is irrelevant for a given discussion, we can omit the reference to l and write $\text{RilUT} := \text{RI2UT}$.

Following the order of the results in Chapter 4, we proceed towards defining Riemannian scaled and square-root Unscented Transformations.

9.2.1 Scaled Riemannian Unscented Transformations

For defining Riemannian scaled transformations, we need a Riemannian “scaling” function g similar to the one in (4.9) for the Scaled Unscented Transformation. From (4.9), we need i) g to perform operations of sums and multiplications by scalars, and ii) the domain \mathcal{U} of $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$ to be convex—this means that the domain \mathcal{U} of f must contain every line segment between any two points in \mathcal{U} .

Since i) convexity is a conservative assumption (few manifolds are convex), and ii) operations of sums and multiplications by scalars are not always defined in Riemannian manifolds, we define g using tangent spaces of the domain of f —tangent spaces have the required operations because they are vector spaces. For this, consider a mapping $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$ between two Riemannian manifolds \mathcal{M} and \mathcal{N} ; and, for $\alpha, \kappa \in (0, 1]$, $\mathbf{q} \in \mathcal{U}, \mathbf{X} \in \mathcal{U}$, and $\mathbf{b} \in \mathcal{N}$, define the function

$$g(f, \mathbf{X}, \mathbf{q}, \mathbf{b}, \alpha, \kappa) := \exp_{f(\mathbf{q})} \left(\kappa^{-1} \log_{f(\mathbf{q})} \left[f \left(\exp_{\mathbf{q}} \left[(1 - \alpha) \log_{\mathbf{q}} \mathbf{X} \right] \right) \right] \right). \quad (9.11)$$

Definition 9.3 (Riemannian Scaled Unscented Transformation). Consider two Riemannian manifolds \mathcal{M} and \mathcal{N} , the mapping $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$, the Riemannian random point $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{P}_{\mathbf{X}\mathbf{X}})_{\mathcal{M}}$ taking values on \mathcal{M} and the sets

$$\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i^m, w_i^c, w_i^{cc} | \boldsymbol{\chi}_i \in \mathcal{M}\}_{i=1}^N. \quad (9.12)$$

Define g as in (9.11), the set

$$\boldsymbol{\gamma} := \left\{ \boldsymbol{\gamma}_i, w_i^m, w_i^c, w_i^{cc} | \boldsymbol{\gamma}_i = g\left(f, \boldsymbol{\chi}_i, \boldsymbol{\mu}_{\boldsymbol{\chi}}, f\left(\alpha^{-2} \log\left[f\left(\boldsymbol{\mu}_{\boldsymbol{\chi}}\right)\right]\right), \alpha, \alpha^2\right) \right\}_{i=1}^N, \quad (9.13)$$

and the Riemannian scaled sample moments

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^\alpha &:= \alpha^2 \sum_{i=0}^N w_i^c \left(\log_{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}_i)\right) (\diamond)^T, \\ \boldsymbol{\Sigma}_{\boldsymbol{\chi}\boldsymbol{\gamma}}^\alpha &:= \alpha \sum_{i=0}^N w_i^{cc} \left(\log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}}(\boldsymbol{\chi}_i)\right) \left(\log_{\boldsymbol{\mu}_{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}_i)\right)^T. \end{aligned}$$

If $\boldsymbol{\chi}$ is a Ri σ R of \mathbf{X} , then the *Riemannian Scaled Unscented Transformation (RiScUT)* is defined by

$$\text{RiScUT}\left(f, \bar{\mathbf{X}}, \mathbf{P}_{\mathbf{X}\mathbf{X}}, \alpha\right) := \left[\boldsymbol{\mu}_{\boldsymbol{\gamma}}, \boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^\alpha, \boldsymbol{\Sigma}_{\boldsymbol{\chi}\boldsymbol{\gamma}}^\alpha\right].$$

Note that, similarly to the Euclidean case, every RiScUT with sets $\boldsymbol{\chi}$ in (9.12) and $\boldsymbol{\gamma}$ in (9.12) is a 2RiUT with sets $\boldsymbol{\chi}$ and

$$\{\boldsymbol{\gamma}_i, w_i^m, w_i^{\alpha,c}, w_i^{\alpha,cc} | \boldsymbol{\gamma}_i\}$$

where $w_i^{\alpha,c} = \alpha^2 w_i^c$ and $w_i^{\alpha,cc} = \alpha w_i^{cc}$.

Next, we extend the particular scaled UT's of Section 4.2 to the case of Riemannian manifolds.

Definition 9.4 (Riemannian Simplex Scaled Unscented Transformation). Let the weighted set $\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i\}_{i=1}^N$ with be a Ri σ R of $\mathbf{X} \sim (\bar{\mathbf{X}}, \mathbf{P}_{\mathbf{X}\mathbf{X}})$ with $\boldsymbol{\chi}_N = \bar{\mathbf{X}}$. Choose $\alpha \in (0, 1]$ and define i) the set

$$\boldsymbol{\chi}' := \{\boldsymbol{\chi}'_i, w'_i | \boldsymbol{\chi}'_i = \exp_{\bar{\mathbf{X}}}\left((1 - \alpha) \log_{\bar{\mathbf{X}}}\boldsymbol{\chi}_i\right)\}_{i=1}^N \quad (9.14)$$

where

$$\begin{aligned} w'_N &:= \alpha^{-2} w_N + 1 - \alpha^{-2}, \\ w'_i &= \alpha^{-2} w_i, \quad i = 1, \dots, N - 1; \end{aligned}$$

ii) for a function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$ on an open set \mathcal{U} of \mathcal{M} , the set

$$\boldsymbol{\gamma}' := \left\{ \boldsymbol{\gamma}'_i, w'_i \mid \boldsymbol{\gamma}'_i = n_{T\mathcal{M}_y} f(\boldsymbol{\chi}_i) \right\}_{i=1}^N; \quad (9.15)$$

and iii) the modified sample covariance of $\boldsymbol{\gamma}'$ as

$$\boldsymbol{\Sigma}_{\boldsymbol{\gamma}'\boldsymbol{\gamma}'}^{\alpha\alpha} := \sum_{i=1}^N w'_i \left(\log_{\mu_{\boldsymbol{\gamma}'}}(\boldsymbol{\gamma}'_i) \right) (\diamond)^T + (1 - \alpha^2) \left(\log_{\mu_{\boldsymbol{\gamma}'}}(\boldsymbol{\gamma}'_N) \right) (\diamond)^T.$$

Then the *Riemannian Simplex Scaled Unscented Transformation (RISiScUT)* is defined by

$$\text{RiSiScUT}(f, \bar{\boldsymbol{X}}, \boldsymbol{P}_{\boldsymbol{X}\boldsymbol{X}}, \alpha) := \left[\boldsymbol{\mu}_{\boldsymbol{\gamma}'}, \boldsymbol{\Sigma}_{\boldsymbol{\gamma}'\boldsymbol{\gamma}'}^{\alpha\alpha}, \boldsymbol{\Sigma}_{\boldsymbol{\chi}'\boldsymbol{\gamma}'} \right].$$

Definition 9.5 (Riemannian Symmetric Intrinsically-Scaled Unscented Transformation). Choose $\alpha \in (0, 1]$ and $\kappa \in \mathbb{R}$ such that

$$\lambda := \alpha^2(n + \kappa) - n > -n;$$

and let $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$, \mathcal{M} with dimension n , be a function mapping an open set \mathcal{U} of \mathcal{M} to \mathcal{N} , and the weighted set $\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i\}_{i=1}^{2n+1}$ with $w_{2n+1} = \lambda/(n + \lambda)$ be a RiHoMiSy σ R of $\boldsymbol{X} \sim (\bar{\boldsymbol{X}}, \boldsymbol{P}_{\boldsymbol{X}\boldsymbol{X}})$. Define the sets

$$\tilde{\boldsymbol{\chi}} := \{\tilde{\boldsymbol{\chi}}_i, \tilde{w}_i^m, \tilde{w}_i^c, \tilde{w}_i^{cc} \mid \tilde{\boldsymbol{\chi}}_i = \boldsymbol{\chi}_i\}_{i=1}^{2n+1} \quad (9.16)$$

and

$$\tilde{\boldsymbol{\gamma}} := \{\tilde{\boldsymbol{\gamma}}_i, \tilde{w}_i^m, \tilde{w}_i^c, \tilde{w}_i^{cc} \mid \tilde{\boldsymbol{\gamma}}_i = f(\boldsymbol{\chi}_i)\}_{i=1}^{2n+1} \quad (9.17)$$

where

$$\begin{aligned} \tilde{w}_{2n+1}^m &= w_{2n+1}; \\ \tilde{w}_{2n+1}^c &= w_{2n+1} + (1 - \alpha^2); \\ \tilde{w}_{2n+1}^{cc} &= w_{2n+1} + (1 - \alpha); \\ \tilde{w}_i^m &= \tilde{w}_i^c = \tilde{w}_i^{cc} = w_i, \quad i = 1, \dots, 2n; \end{aligned}$$

Then the *Riemannian Symmetric Intrinsically-Scaled Unscented Transformation (RiSyInScUT)* is defined by

$$\text{RiSyInScUT}(f, \bar{\boldsymbol{X}}, \boldsymbol{P}_{\boldsymbol{X}\boldsymbol{X}}, \alpha) := \left[\boldsymbol{\mu}_{\tilde{\boldsymbol{\gamma}}}, \boldsymbol{\Sigma}_{\tilde{\boldsymbol{\gamma}}\tilde{\boldsymbol{\gamma}}}, \boldsymbol{\Sigma}_{\boldsymbol{\chi}\tilde{\boldsymbol{\gamma}}} \right].$$

9.2.2 Riemannian Square-Root Unscented Transformation

In this section, we extend results related to the SRUT (Section 4.3) to the case of Riemannian manifolds.

Consider the Riemannian random point \mathbf{X} with mean $\bar{\mathbf{X}}$ and square-root of the covariance $\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}$. For a set

$$\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i^m, w_i^c, w_i^{cc} | \boldsymbol{\chi}_i \in \mathcal{M}\}_{i=1}^N,$$

and a point $\mathbf{q} \in \mathcal{M}$, define the matrix $S_{\boldsymbol{\chi}}^{\mathbf{q}}$ by

$$S_{\boldsymbol{\chi}}^{\mathbf{q}} := \left[\sqrt{w_1^c} (\log_{\mathbf{q}} \boldsymbol{\chi}_1 - \log_{\mathbf{q}} \boldsymbol{\mu}_{\boldsymbol{\chi}}), \dots, \sqrt{w_N^c} (\log_{\mathbf{q}} \boldsymbol{\chi}_N - \log_{\mathbf{q}} \boldsymbol{\mu}_{\boldsymbol{\chi}}) \right];$$

and, for $\mathbf{q} = \boldsymbol{\mu}_{\boldsymbol{\chi}}$, the matrix

$$S_{\boldsymbol{\chi}} := S_{\boldsymbol{\chi}}^{\boldsymbol{\mu}_{\boldsymbol{\chi}}} := \left[\sqrt{w_1^c} \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}} \boldsymbol{\chi}_1, \dots, \sqrt{w_N^c} \log_{\boldsymbol{\mu}_{\boldsymbol{\chi}}} \boldsymbol{\chi}_N \right]. \quad (9.18)$$

To clear notations, in the definitions below, we will restrict to the case of $\mathbf{q} = \boldsymbol{\mu}_{\boldsymbol{\chi}}$ ($S_{\boldsymbol{\chi}} = S_{\boldsymbol{\chi}}^{\boldsymbol{\mu}_{\boldsymbol{\chi}}}$), but they are easily extended to the case of $\mathbf{q} \neq \boldsymbol{\mu}_{\boldsymbol{\chi}}$ ($S_{\boldsymbol{\chi}} \neq S_{\boldsymbol{\chi}}^{\boldsymbol{\mu}_{\boldsymbol{\chi}}}$).

Definition 9.6 (Riemannian Square-Root Unscented Transformation). Consider two Riemannian manifolds \mathcal{M} and \mathcal{N} ; the function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$; the Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with mean $\bar{\mathbf{X}}$ and its covariance's square-root $\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}$; and the sets

$$\boldsymbol{\chi} := \{\boldsymbol{\chi}_i, w_i^m, w_i^c, w_i^{cc} | \boldsymbol{\chi}_i \in \mathcal{M}\}_{i=1}^N,$$

and

$$\boldsymbol{\gamma} := \{\boldsymbol{\gamma}_i, w_i^m, w_i^c, w_i^{cc} | \boldsymbol{\gamma}_i := f(\boldsymbol{\chi}_i)\}_{i=1}^N.$$

Given a matrix $\sqrt{\boldsymbol{\Gamma}}$, define $S_{\boldsymbol{\chi}}, S_{\boldsymbol{\gamma}}$ as in (9.18), and the matrix

$$\sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}} := \sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} + \sqrt{\boldsymbol{\Gamma}}\sqrt{\boldsymbol{\Gamma}}^T}.$$

If $\boldsymbol{\chi}$ is a Ri σ R of $\mathbf{X} \sim (\bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}^T})$, then the *Riemannian Square-Root Unscented Transformation (RiSRUT)* is defined by

$$\text{RiSRUT}\left(f, \bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}, \sqrt{\boldsymbol{\Gamma}}\right) := \left[\boldsymbol{\mu}_{\boldsymbol{\gamma}}, \sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}}, S_{\boldsymbol{\chi}}, S_{\boldsymbol{\gamma}}, \boldsymbol{\Sigma}_{\boldsymbol{\chi}\boldsymbol{\gamma}} \right].$$

We could think of calculating $\sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}}$ by $\sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}} = \text{cu}(S_{\boldsymbol{\gamma}}^+, S_{\boldsymbol{\gamma}}^-, \sqrt{\boldsymbol{\Gamma}})$ as for the SRUT's [cf. equation (4.14)]. However, since the Ri σ R's are defined only for positive weights, the matrices $S_{\boldsymbol{\chi}}^-, S_{\boldsymbol{\gamma}}^-$ defined in (4.15) are not defined for Ri σ R's. Therefore, $\sqrt{\boldsymbol{\Sigma}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}}$ can

be calculated as in (4.17):

$$\sqrt{\Sigma_{\gamma\gamma}^{\Gamma}} = \text{tria} \left([S_{\gamma}, \sqrt{\Gamma}] \right).$$

Definition 9.7 (Riemannian Scaled Square-Root Unscented Transformation). Consider two Riemannian manifolds \mathcal{M} and \mathcal{N} ; the function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$; the Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with mean $\bar{\mathbf{X}}$ and its covariance's square-root $\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}$; and the sets χ' in (9.14) and γ' in (9.14). Given a matrix $\sqrt{\Gamma}$, define S_{χ}, S_{γ} as in (9.18), and the matrix

$$\sqrt{\Sigma_{\gamma\gamma}^{\alpha\Gamma}} := \sqrt{\Sigma_{\gamma\gamma}^{\alpha} + \sqrt{\Gamma}\sqrt{\Gamma}^T}.$$

If χ is a Ri σ R of $\mathbf{X} \sim (\bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}^T})$, then the *Riemannian Scaled Square-Root Unscented Transformation (RiScSRUT)* is defined by

$$\text{RiScSRUT} \left(f, \bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}, \sqrt{\Gamma}, \alpha \right) := \left[\boldsymbol{\mu}_{\gamma}, \sqrt{\Sigma_{\gamma\gamma}^{\alpha\Gamma}}, S_{\chi}, S_{\gamma}, \Sigma_{\chi\gamma}^{\alpha} \right].$$

Note that, similarly to the Euclidean case, RiScSRUT with sets χ in (9.12) and γ in (9.12) is a RiSRUT with sets χ and

$$\{\gamma_i, w_i^m, w_i^{\alpha,c}, w_i^{\alpha,cc} | \gamma_i\}$$

where $w_i^{\alpha,c} = \alpha^2 w_i^c$ and $w_i^{\alpha,cc} = \alpha w_i^{cc}$.

Definition 9.8 (Riemannian Simplex Scaled Square-Root Unscented Transformation). Consider two Riemannian manifolds \mathcal{M} and \mathcal{N} ; the function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$; the Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with mean $\bar{\mathbf{X}}$ and its covariance's square-root $\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}$; and the sets χ' in (9.12) and γ' in (9.13). Given a matrix $\sqrt{\Gamma}$, define $S_{\chi'}, S_{\gamma'}$ as in (9.18), and the matrix

$$\sqrt{\Sigma_{\gamma'\gamma'}^{\alpha\alpha\Gamma}} := \sqrt{\Sigma_{\gamma'\gamma'}^{\alpha\alpha} + \sqrt{\Gamma}\sqrt{\Gamma}^T}.$$

If χ is a Ri σ R of $\mathbf{X} \sim (\bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}^T})$, then the *Riemannian Simplex Scaled Square-Root Unscented Transformation (RiSiScSRUT)* is defined by

$$\text{RiSiScSRUT} \left(f, \bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}, \sqrt{\Gamma}, \alpha \right) := \left[\boldsymbol{\mu}_{\gamma'}, \sqrt{\Sigma_{\gamma'\gamma'}^{\alpha\alpha\Gamma}}, S_{\chi'}, S_{\gamma'}, \Sigma_{\chi'\gamma'} \right].$$

Definition 9.9 (Riemannian *Symmetric Intrinsically-Scaled* Square-Root Unscented Transformation). Consider two (geodesically complete) Riemannian manifolds \mathcal{M} and \mathcal{N} ; the function $f : \mathcal{U} \subset \mathcal{M} \rightarrow \mathcal{N}$; the Riemannian random point $\mathbf{X} \in \Phi_{\mathcal{M}}$ with mean $\bar{\mathbf{X}}$ and its covariance's square-root $\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}$; and the sets $\tilde{\chi}$ in (9.16) and $\tilde{\gamma}$ in (9.17).

Given a matrix $\sqrt{\Gamma}$, define $S_{\bar{\chi}}, S_{\bar{\gamma}}$ as in (9.18), and the matrix

$$\sqrt{\Sigma_{\bar{\gamma}\bar{\gamma}}^{\Gamma}} := \sqrt{\Sigma_{\bar{\gamma}\bar{\gamma}} + \sqrt{\Gamma}\sqrt{\Gamma}^T}.$$

If χ is a RiσR of $\mathbf{X} \sim (\bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}\sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}^T})$, then the *Riemannian Symmetric Intrinsically-Scaled Square-Root Unscented Transformation (RiSyInSRUT)* is defined by

$$\text{RiSyInScSRUT}\left(f, \bar{\mathbf{X}}, \sqrt{\mathbf{P}_{\mathbf{X}\mathbf{X}}}, \sqrt{\Gamma}, \alpha\right) := \left[\boldsymbol{\mu}_{\bar{\gamma}}, \sqrt{\Sigma_{\bar{\gamma}\bar{\gamma}}^{\Gamma}}, S_{\bar{\chi}}, S_{\bar{\gamma}}, \Sigma_{\bar{\chi}\bar{\gamma}}\right].$$

We have Riemannian extensions of all the UT's for the Euclidean defined in chapter 4. Now we can define the Riemannian Unscented Filters.

9.3 RIEMANNIAN UNSCENTED FILTERS

UKF's are solutions to the problem of estimating the state of stochastic dynamic systems. In order to define Riemannian UKF's—the extension of the UKF's for the Riemannian case—we need to extend the systems (2.1) and (2.2) to the Riemannian case.

9.3.1 Riemannian Dynamics Systems

Consider the following system:

$$\begin{aligned}\mathbf{x}_k &= f_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_k), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k)\end{aligned}\tag{9.19}$$

where k is the time step; $\mathbf{x}_k \in \Phi_{\mathcal{M}_{\mathbf{x}}^{n_x}}$ the internal state; $\mathbf{y}_k \in \Phi_{\mathcal{M}_{\mathbf{y}}^{n_y}}$ is the measured output; $\boldsymbol{\varpi}_k \in \Phi_{\mathcal{M}_{\boldsymbol{\varpi}}^{n_{\boldsymbol{\varpi}}}}$ the process noise; and $\boldsymbol{\vartheta}_k \in \Phi_{\mathcal{M}_{\boldsymbol{\vartheta}}^{n_{\boldsymbol{\vartheta}}}}$ the measurement noise. The noise $\boldsymbol{\varpi}_k$ is assumed to have mean $\bar{\boldsymbol{\varpi}}_k$ and covariance Q_k ; and $\boldsymbol{\vartheta}_k$, mean $\bar{\boldsymbol{\vartheta}}_k$ and covariance R_k . We call the pair of equations (9.19) the *Riemannian (stochastic, discrete-time, dynamic) system*.

We also want to consider an *additive variant* of (9.19). Filters for these systems are computationally cheaper. Moreover, we want additive UKF's for Riemannian manifolds to be solutions to the problems encountered with the additive UKF's for quaternions models (cf. Chapter 7).

Nonetheless, defining these additive variants of (9.19) is not straightforward; sums are not defined for all Riemannian manifolds. For instance, recall, from Remark 7.1,

that all the additive-noise quaternions models in the literature present problems. The problem of defining an additive-noise quaternion system still persists since, in Chapter 7, we did not provide a definition for these systems. Now, we solve this problem by introducing an additive variant of (9.19).

Essentially, we want of an additive variant of (9.19) for i) ϖ_k to act on $f_k(\mathbf{x}_{k-1})$ by “adding” a) its mean to the mean of $f_k(\mathbf{x}_{k-1})$, and b) its covariance to the covariance of $f_k(\mathbf{x}_{k-1})$; and ii) for ϑ_k to act on $h_k(\mathbf{x}_k)$ by “adding” a) its mean to the mean of $h_k(\mathbf{x}_k)$ and b) its covariance to the covariance of $h_k(\mathbf{x}_k)$. Since the tangent spaces are vector spaces, we can work with sums in these spaces using Proposition 8.2.

Consider Proposition 8.2 two times: one for the process function with $\mathbf{q} = f_k(\mathbf{x}_{k-1})$ and $p = \varpi_k$, and another for the measurement function with $\mathbf{q} = h_k(\mathbf{x}_k)$ and $p = \vartheta_k$. Then we define the *additive Riemannian (stochastic, discrete-time, dynamic) system* as follows:

$$\begin{aligned}\mathbf{x}_k &= \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \varpi_k \right], \\ \mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \vartheta_k \right];\end{aligned}\tag{9.20}$$

where $\mathbf{x}_k \in \Phi_{\mathcal{M}_x^{n_x}}$, $\mathbf{y}_k \in \Phi_{\mathcal{M}_y^{n_y}}$, $\varpi_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_x^{n_x}$, and $\vartheta_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_y^{n_y}$. The noise ϖ_k is assumed to have mean $\bar{\varpi}_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_x^{n_x}$ and covariance $Q_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_x^{n_x} \times T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_x^{n_x}$, and ϑ_k mean $\bar{\vartheta}_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_y^{n_y}$ and covariance $R_k \in T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_y^{n_y} \times T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_y^{n_y}$. We highlight that the noise ϖ_k is defined in the tangent space $T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_x^{n_x}$ and ϑ_k , in $T_{f_k(\mathbf{x}_{k-1})}\mathcal{M}_y^{n_y}$. An alternative definition in which these noises belong to Riemannian manifolds is discussed in Remark 9.1.

As far as our knowledge goes, system (9.20) is the *first consistent additive-noise Riemannian stochastic discrete-time dynamic system*; and also, for the particular case of unit quaternions, the first consistent additive-noise *unit-quaternion* stochastic discrete-time dynamic system.

Remark 9.1. System (9.20) is defined with the process and measurement noises belonging to tangent spaces. An alternative definition in which these noises belong to Riemannian manifolds is the following:

$$\begin{aligned}\mathbf{x}_k &= \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \log_{f_k(\mathbf{x}_{k-1})} \varpi_k \right], \\ \mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \log_{h_k(\mathbf{x}_k)} \vartheta_k \right];\end{aligned}$$

where $\mathbf{x}_k \in \Phi_{\mathcal{M}_x^{n_x}}$, $\mathbf{y}_k \in \Phi_{\mathcal{M}_y^{n_y}}$, $\varpi_k \in \Phi_{\mathcal{M}_x^{n_x}}$, and $\vartheta_k \in \Phi_{\mathcal{M}_y^{n_y}}$. In this case, it would be interesting to assume one of the following two cases:

1. That were known i) the means of ϖ_k and ϑ_k —e.g. $\bar{\varpi}_k \in \mathcal{M}_x^{n_x} - \mathcal{C}(\overline{f_k(\mathbf{x}_{k-1})})$

and $\bar{\boldsymbol{\vartheta}}_k \in \mathcal{M}_y^{n_y} - \mathcal{C}(\overline{h_k(\mathbf{x}_k)})$ —, b) the covariance of $\boldsymbol{\varpi}_k$ respective to $\bar{\boldsymbol{\omega}}_k$ at $\overline{f_k(\mathbf{x}_{k-1})}$ —see the definition of Riemannian covariance in (8.7)—, and iii) the covariance of $\boldsymbol{\vartheta}_k$ respective to $\bar{\boldsymbol{\vartheta}}_k$ at $\overline{h_k(\mathbf{x}_k)}$.

2. That the means and covariances of $\log_{\overline{f_k(\mathbf{x}_{k-1})}} \boldsymbol{\varpi}_k$ and $\log_{\overline{h_k(\mathbf{x}_k)}} \boldsymbol{\vartheta}_k$ were known— e.g. the means $\bar{\boldsymbol{\omega}}_k \in T_{\overline{f_k(\mathbf{x}_{k-1})}} \Phi_{\mathcal{M}_x^{n_x}}$ and $\bar{r}_k \in T_{\overline{h_k(\mathbf{x}_k)}} \Phi_{\mathcal{M}_y^{n_y}}$; and the covariances $Q_k \in T_{\overline{f_k(\mathbf{x}_{k-1})}} \Phi_{\mathcal{M}_x^{n_x}} \times T_{\overline{f_k(\mathbf{x}_{k-1})}} \Phi_{\mathcal{M}_x^{n_x}}$ and $R_k \in T_{\overline{h_k(\mathbf{x}_k)}} \Phi_{\mathcal{M}_x^{n_x}} \times T_{\overline{h_k(\mathbf{x}_k)}} \Phi_{\mathcal{M}_y^{n_y}}$.

9.3.2 Correction equations

Essentially, Unscented filters are composed of two UT's along with the KF correction equations. We already have the analogous of the UT's for the Riemannian case, but not the analogous of the correction equations. Let us consider these equations for the additive systems.

From the Algorithm 6, the corrections equations of the AdUKF are

$$\begin{aligned} G_k &:= \hat{P}_{xy}^{k|k-1} \left(\hat{P}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{x}_{k|k} &:= \hat{x}_{k|k-1} + G_k \left(y_k - \hat{y}_{k|k-1} \right), \\ \hat{P}_{xx}^{k|k} &:= \hat{P}_{xx}^{k|k-1} - G_k \hat{P}_{yy}^{k|k-1} G_k^T. \end{aligned} \quad (9.21)$$

Again, we have operations of sums, which are not defined for all Riemannian manifolds. We can try to work on the tangent space of a given point, but here we have another problem. Equation (9.21) have sums involving the estimates of the state ($\hat{x}_{k|k-1}$) and of the measurement ($\hat{y}_{k|k-1}$), but in the Riemannian case, the state and the measurement do not belong, necessarily, to the same Riemannian manifold— $\mathbf{x}_k \in \Phi_{\mathcal{M}_x^{n_x}}$ and $\mathbf{y}_k \in \Phi_{\mathcal{M}_y^{n_y}}$. Choosing a tangent space as a set to perform operations similar to (9.21) when $\Phi_{\mathcal{M}_x^{n_x}} \neq \Phi_{\mathcal{M}_y^{n_y}}$; so let us treat, first, the simpler case of $\Phi_{\mathcal{M}_x^{n_x}} = \Phi_{\mathcal{M}_y^{n_y}}$.

Remark 9.2. Recall that the intrinsic statistics for Riemannian manifolds presented in Chapter 8 are an extension of the results presented in [66]. In the present section, we will need specially two of these extensions: i) the definition of a *covariance* of a given Riemannian random point \mathbf{X} at a point \mathbf{q} different from $\bar{\mathbf{X}}$, and ii) the definition of the cross-covariance of two Riemannian random points.

9.3.2.1 State and measurement in the same manifold

Suppose that

$$\mathcal{M}_x^{n_x} = \mathcal{M}_y^{n_y},$$

and that a measurement \mathbf{y}_k is acquired. Define the Riemannian random points

$$\mathbf{x}_{k|k-1} := \mathbf{x}_k | \mathbf{y}_{1:k-1},$$

$$\mathbf{x}_{k|k} := \mathbf{x}_k | \mathbf{y}_{1:k},$$

$$\mathbf{y}_{k|k-1} := \mathbf{y}_k | \mathbf{y}_{1:k-1};$$

and the projections on the tangent space of $\mathbf{x}_{k|k-1}$

$$x_{k|k-1}^{TM} := \log_{\bar{\mathbf{x}}_{k|k-1}} \mathbf{x}_{k|k-1}, \quad (9.22)$$

$$x_{k|k}^{TM} := \log_{\bar{\mathbf{x}}_{k|k-1}} \mathbf{x}_{k|k},$$

$$y_{k|k-1}^{TM} := \log_{\bar{\mathbf{x}}_{k|k-1}} \mathbf{y}_{k|k-1}, \quad (9.23)$$

$$\underset{\sim}{y}_k^{TM} := \log_{\bar{\mathbf{x}}_{k|k-1}} \underset{\sim}{\mathbf{y}}_k. \quad (9.24)$$

Let i) $\mathbf{x}_{k|k-1}$ and $\mathbf{y}_{k|k-1}$ be characterized by their projection on the tangent space of $\mathbf{x}_{k|k-1}$ according to the following equation:

$$\begin{bmatrix} x_{k|k-1}^{TM} \\ y_{k|k-1}^{TM} \end{bmatrix} \sim N \left(\begin{bmatrix} [0]_{n_x,1} \\ y_{k|k-1}^{TM} \end{bmatrix}, \begin{bmatrix} \mathbf{P}_{xx}^{k|k-1} & \mathbf{P}_{xy}^{k|k-1} \\ (\mathbf{P}_{xy}^{k|k-1})^T & \mathbf{P}_{yy}^{k|k-1} \end{bmatrix} \right)^{n_x}; \quad (9.25)$$

and ii) the projection $x_{k|k}^{TM}$ be given by the following linear correction of $x_{k|k-1}^{TM}$

$$x_{k|k}^{TM} = x_{k|k-1}^{TM} + \mathbf{G}_k \left(\underset{\sim}{y}_k^{TM} - y_{k|k-1}^{TM} \right), \quad (9.26)$$

where $\mathbf{G}_k \in \mathbb{R}^{n_x \times n_x}$ is a gain matrix. From known results of the Kalman filtering theory (cf. [25]), we have that

$$\mathbf{G}_k := \mathbf{P}_{xy}^{k|k-1} \left(\mathbf{P}_{yy}^{k|k-1} \right)^{-1}, \quad (9.27)$$

and

$$x_{k|k}^{TM} \sim N \left(\bar{x}_{k|k}^{TM}, \mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k-1}} \right)$$

where

$$\bar{x}_{k|k}^{TM} := \mathbf{G}_k \left(\underset{\sim}{y}_k^{TM} - \bar{y}_{k|k-1}^{TM} \right), \quad (9.28)$$

$$\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k-1}} := \mathbf{P}_{xx}^{k|k-1} - (\mathbf{G}_k) \mathbf{P}_{yy}^{k|k-1} (\mathbf{G}_k)^T. \quad (9.29)$$

From (9.22),

$$\bar{\mathbf{x}}_{k|k} = \exp_{\bar{\mathbf{x}}_{k|k-1}} x_{k|k}^{TM}, \quad (9.30)$$

and $\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k-1}}$ is the covariance of $\mathbf{x}_{k|k}$ relative to $\bar{\mathbf{x}}_{k|k}$ at $\bar{\mathbf{x}}_{k|k-1}$. We want the covariance $\mathbf{P}_{xx}^{k|k-1} := \mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k}}$ of $\mathbf{x}_{k|k}$ at $\bar{\mathbf{x}}_{k|k}$, and the following theorem from [171] provides the mechanism to obtain $\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k}}$ from $\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k-1}}$.

Theorem 9.2 (Parallel Transport of a Bilinear Mapping [171]). *Let P be a symmetric bilinear mapping on the tangent space $T_{\mathbf{q}}\mathcal{M}$ of the Riemannian manifold \mathcal{M} at $\mathbf{q} \in \mathcal{M}$, and $\gamma : [0, 1] \rightarrow \mathcal{M}$ a differentiable curve on \mathcal{M} with $\gamma(0) = \mathbf{q}$. Since P is symmetric, it can be written as*

$$P = \sum_{i=1}^n \lambda_i v_i v_i^T$$

where (v_1, \dots, v_n) is an orthonormal basis of $T_{\mathbf{q}}\mathcal{M}$, and each λ_i is the eigenvalue of P associated with the eigenvector v_i . Let $v_i(t)$ be the parallel transport of v_i along $\gamma(t)$ (see Section A.3). With this,

$$P_t := \sum_{i=1}^n \lambda_i v_i(t) v_i(t)^T \quad (9.31)$$

is the parallel transport of P along $\gamma(t)$.

Note, from (9.31), that parallel transport of tangent vectors $[v_i(t)]$ are needed in the definition of a parallel transport of a symmetric bilinear mapping $[P_t]$. When closed forms of parallel transport of tangent vectors are not known for manifolds in question, algorithms such as the *Schild's Ladder* can be used (cf. [171]; see [172] for other implementations and algorithms of parallel transports).

For a Riemannian manifold \mathcal{M} , and the points $\mathbf{q} \in \mathcal{M}$ and $\mathbf{p} \in \mathcal{M}$; we define the function

$$\begin{aligned} \text{PT} : T_{\mathbf{q}}\mathcal{M} \times T_{\mathbf{q}}\mathcal{M} \times \mathcal{M} \times \mathcal{M} &\rightarrow T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{M} \\ (P^{\mathbf{q}}, \mathbf{q}, \mathbf{p}) &\mapsto P^{\mathbf{p}} \end{aligned}$$

mapping the symmetric matrix $P^{\mathbf{q}} \in T_{\mathbf{q}}\mathcal{M} \times T_{\mathbf{q}}\mathcal{M}$ to the symmetric matrix $P^{\mathbf{p}} \in T_{\mathbf{p}}\mathcal{M} \times T_{\mathbf{p}}\mathcal{M}$ according to (9.31).

Because we want the covariance $\mathbf{P}_{xx}^{k|k-1} := \mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k}}$ of $\mathbf{x}_{k|k}$ at $\bar{\mathbf{x}}_{k|k}$, we obtain $\mathbf{P}_{xx}^{k|k-1}$ by performing the parallel transport of $\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k}}$ —which we already calculated—from $\bar{\mathbf{x}}_{k|k-1}$ to $\bar{\mathbf{x}}_{k|k}$. Thus, the covariance of $\mathbf{P}_{xx}^{k|k-1}$ at $\bar{\mathbf{x}}_{k|k}$ is given by

$$\mathbf{P}_{xx}^{k|k-1} = \text{PT} \left(\mathbf{P}_{xx}^{k|k-1, \bar{\mathbf{x}}_{k|k}}, \bar{\mathbf{x}}_{k|k-1}, \bar{\mathbf{x}}_{k|k} \right). \quad (9.32)$$

With this, we can compute all the estimates required by an UKF for Riemannian systems. Nevertheless, these estimates are calculated only for the particular case treated in this section ($\mathcal{M}_{\mathbf{x}}^{n_x} = \mathcal{M}_{\mathbf{y}}^{n_y}$); next, we extend the approach of this section towards finding correction equations for the case where $\mathcal{M}_{\mathbf{x}}^{n_x}$ can be different from

$\mathcal{M}_y^{n_y}$.

9.3.2.2 State and measurement in different manifolds

Equations (9.27), (9.28), (9.29), (9.30), and (9.32) are the correction equations for the Riemannian Additive UKF considering the state and the measurement in the same manifold. In this section, we consider the state and the measurement belonging to different manifolds.

If \mathbf{x}_k belongs to a manifold $\Phi_{\mathcal{M}_x^{n_x}}$ and \mathbf{y}_k to another manifold $\Phi_{\mathcal{M}_y^{n_y}}$, then $y_{k|k-1}^{TM}$ and \underline{y}_k^{TM} can not be defined, respectively, as in (9.23) and (9.24); consequently, $x_{k|k}^{TM}$ can not be defined as in (9.26).

Since we know the equations for the case of \mathbf{x}_k and the \mathbf{y}_k belonging to the same manifold, we can look for a manifold of which both \mathcal{M}_x and \mathcal{M}_y are subsets. The simplest of such a class of sets is, of course, the Riemannian manifold $\mathcal{M}_x \times \mathcal{M}_y$ —the Cartesian product of two Riemannian manifolds is a Riemannian manifold (cf. Section A.2).

Suppose that $x_{k|k-1}^{TM}$ and $y_{k|k-1}^{TM}$ are jointly Gaussian random vectors according to (9.25). Define i) the Riemannian Manifold $\mathcal{M}_{x,y} := \mathcal{M}_x \times \mathcal{M}_y$; ii) the points $\mathbf{c} := (\mathbf{c}_x, \mathbf{c}_y) \in \mathcal{M}_{x,y}$, $\mathbf{b}_x \in \mathcal{M}_x$, and $\mathbf{b}_y \in \mathcal{M}_y$ (these points are chosen); and the following random vector belonging to $T_c \mathcal{M}_{x,y}$:

$$x_{k|k,**}^{T_c \mathcal{M}_{x,y}} := \log_c \begin{bmatrix} \mathbf{x}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} + \mathbf{G}_{k,**} \left(\log_c \begin{bmatrix} \mathbf{b}_x \\ \mathbf{y}_k \end{bmatrix} - \log_c \begin{bmatrix} \mathbf{b}_x \\ \mathbf{y}_{k|k-1} \end{bmatrix} \right)$$

where $\mathbf{G}_{k,**} \in \mathbb{R}^{(n_x+n_y) \times (n_x+n_y)}$ is a gain matrix. The tangent vector $x_{k|k,**}^{T_c \mathcal{M}_{x,y}}$ is clearly related with $x_{k|k}^{TM}$ by

$$x_{k|k}^{TM} := [\hat{x}_{k|k,**}^{T_c \mathcal{M}_{x,y}}]_{1:n_x,1}. \quad (9.33)$$

Therefore, by finding the mean and the covariance of $x_{k|k,**}^{T_c \mathcal{M}_{x,y}}$, we find the mean and covariance of $x_{k|k}^{TM}$.

Since $x_{k|k-1}^{TM}$ and $y_{k|k-1}^{TM}$ are jointly Gaussian random vectors, it follows that—we use the same reasoning used to obtain (9.27), (9.28), (9.29), (9.30), and (9.32), for the following covariances—

$$\begin{aligned} \mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k-1} &:= \mathcal{E}_{\mathbf{x}_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \mathbf{x} \\ \mathbf{b}_y \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} \right) (\diamond)^T \right\}, \\ \mathbf{P}_{\mathbf{y}\mathbf{y},**}^{k|k-1} &:= \mathcal{E}_{\mathbf{y}_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \mathbf{b}_x \\ \mathbf{y} \end{bmatrix} - \log_c \begin{bmatrix} \mathbf{b}_x \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} \right) (\diamond)^T \right\}, \end{aligned}$$

$$\mathbf{P}_{xy,**}^{k|k-1} := \boldsymbol{\mathcal{E}}_{x_{k|k-1}, y_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \mathbf{x} \\ \mathbf{b}_y \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} \right) \times \left(\log_c \begin{bmatrix} \mathbf{b}_x \\ \mathbf{y} \end{bmatrix} - \log_c \begin{bmatrix} \mathbf{b}_x \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} \right)^T \right\};$$

and the mean and covariance of $x_{k|k}^{T_c \mathcal{M}_{x,y}}$ are, therefore, given by

$$\mathbf{G}_{k,**} := \mathbf{P}_{xy,**}^{k|k-1} \left(\mathbf{P}_{yy,**}^{k|k-1} \right)^{-1}, \quad (9.34)$$

$$\bar{x}_{k|k}^{T_c \mathcal{M}_{x,y}} := \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} + \mathbf{G}_{k,**} \log_c \begin{bmatrix} \mathbf{b}_x \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix}, \quad (9.35)$$

$$\mathbf{P}_{xx,**}^{k|k, T_c M} := \mathbf{P}_{xx,**}^{k|k-1} - (\mathbf{G}_{k,**}) \mathbf{P}_{yy,**}^{k|k-1} (\mathbf{G}_{k,**})^T; \quad (9.36)$$

From (9.33), we can find the mean and covariance of $x_{k|k}^{TM}$.

The points \mathbf{c} , \mathbf{b}_x and \mathbf{b}_y can be chosen arbitrary within the limits required by the operations above, specially by the Riemannian logarithm maps. However, there is an special case.

Theorem 9.3. *Given (9.33), (9.34), (9.35), and (9.36); if $\mathbf{c}_x = \mathbf{b}_x = \hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{c}_y = \mathbf{b}_y = \hat{\mathbf{y}}_{k|k-1}$, then*

$$x_{k|k}^{TM} = \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k) \quad (9.37)$$

and

$$\mathbf{P}_{xx}^{k|k, \bar{\mathbf{x}}_{k|k-1}} = \hat{\mathbf{P}}_{xx}^{k|k-1} - \mathbf{G}_k \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1} (\mathbf{G}_k)^T. \quad (9.38)$$

Proof. Substituting $\mathbf{c}_x = \mathbf{b}_x = \bar{\mathbf{x}}_{k|k-1}$ and $\mathbf{c}_y = \mathbf{b}_y = \bar{\mathbf{y}}_{k|k-1}$ on the definitions of $\mathbf{P}_{xx,**}^{k|k-1}$, $\mathbf{P}_{yy,**}^{k|k-1}$, and $\mathbf{P}_{xy,**}^{k|k-1}$, we have that

$$\begin{aligned} \mathbf{P}_{xx,**}^{k|k-1} &:= \boldsymbol{\mathcal{E}}_{x_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \mathbf{x} \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} \right) (\diamond)^T \right\} \\ &= \begin{bmatrix} \mathbf{P}_{xx}^{k|k-1} & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix}, \end{aligned} \quad (9.39)$$

$$\begin{aligned} \mathbf{P}_{yy,**}^{k|k-1} &:= \boldsymbol{\mathcal{E}}_{y_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{y} \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} \right) (\diamond)^T \right\} \\ &= \begin{bmatrix} [0]_{n_x \times n_x} & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & \mathbf{P}_{yy}^{k|k-1} \end{bmatrix}, \end{aligned} \quad (9.40)$$

and

$$\begin{aligned}
\mathbf{P}_{xy,**}^{k|k-1} &:= \boldsymbol{\varepsilon}_{x_{k|k-1}; y_{k|k-1}} \left\{ \left(\log_c \begin{bmatrix} \mathbf{x} \\ \mathbf{b}_y \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} \right); \right. \\
&\quad \left. \times \left(\log_c \begin{bmatrix} \mathbf{b}_x \\ \mathbf{y} \end{bmatrix} - \log_c \begin{bmatrix} \mathbf{b}_x \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} \right)^T \right\} = \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{P}_{xy}^{k|k-1} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix}. \quad (9.41)
\end{aligned}$$

Substituting (9.40) and (9.41) into (9.34) gives

$$\begin{aligned}
\mathbf{G}_{k,**} &:= \mathbf{P}_{xy,**}^{k|k-1} \left(\mathbf{P}_{yy,**}^{k|k-1} \right)^{-1} \\
&= \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{P}_{xy}^{k|k-1} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} [0]_{n_x \times n_x} & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & \mathbf{P}_{yy}^{k|k-1} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{P}_{xy}^{k|k-1} \left(\mathbf{P}_{yy}^{k|k-1} \right)^{-1} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \\
&= \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{G}_k \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \quad (9.42)
\end{aligned}$$

substituting $\mathbf{c}_x = \mathbf{b}_x = \hat{\mathbf{x}}_{k|k-1}$, $\mathbf{c}_y = \mathbf{b}_y = \hat{\mathbf{y}}_{k|k-1}$, and (9.42) into (9.35) gives

$$\begin{aligned}
\bar{x}_{k|k,**}^{T_c \mathcal{M}_{x,y}} &:= \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{b}_y \end{bmatrix} + \mathbf{G}_{k,**} \log_c \left(\begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{y}_k \end{bmatrix} - \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \mathbf{y}_{k|k-1} \end{bmatrix} \right) \\
&= \log_c \begin{bmatrix} \bar{\mathbf{x}}_{k|k-1} \\ \bar{\mathbf{y}}_{k|k-1} \end{bmatrix} + \mathbf{G}_{k,**} \begin{bmatrix} [0]_{n_x \times 1} \\ \log_{\bar{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k) \end{bmatrix} \\
&= \begin{bmatrix} \log_{\bar{\mathbf{x}}_{k|k-1}}(\bar{\mathbf{x}}_{k|k-1}) \\ \log_{\bar{\mathbf{y}}_{k|k-1}}(\bar{\mathbf{y}}_{k|k-1}) \end{bmatrix} + \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{G}_k \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} [0]_{n_x \times 1} \\ \log_{\bar{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k) \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{G}_k \log_{\bar{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k) \\ [0]_{n_y \times 1} \end{bmatrix};
\end{aligned}$$

consequently, from (9.33)

$$\begin{aligned}
x_{k|k}^{TM} &:= \left[x_{k|k,**}^{T_c \mathcal{M}_{x,y}} \right]_{1:n_x,1} \\
&= \begin{bmatrix} \mathbf{G}_k \log_{\bar{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k) \\ [0]_{n_y \times 1} \end{bmatrix} \\
&= \mathbf{G}_k \log_{\bar{\mathbf{y}}_{k|k-1}}(\mathbf{y}_k);
\end{aligned}$$

which proves the first equality of the theorem. The proof of the second equation comes from substituting $\mathbf{c}_x = \mathbf{b}_x = \hat{\mathbf{x}}_{k|k-1}$, $\mathbf{c}_y = \mathbf{b}_y = \hat{\mathbf{y}}_{k|k-1}$, (9.39), (9.40), and (9.42) into

(9.36)

$$\begin{aligned}
\mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k,T_cM} &:= \mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k-1} - (\mathbf{G}_{k,**}) \mathbf{P}_{\mathbf{y}\mathbf{y},**}^{k|k-1} (\mathbf{G}_{k,**})^T; \\
&= \mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k-1} - \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{G}_k \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} [0]_{n_x \times n_x} & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & \mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1} \end{bmatrix}^{-1} (\mathbf{G}_{k,**})^T \\
&= \mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k-1} - \begin{bmatrix} [0]_{n_x \times n_x} & \mathbf{G}_k (\mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1})^{-1} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \begin{bmatrix} [0]_{n_x \times n_x} & [0]_{n_x \times n_y} \\ (\mathbf{G}_k)^T & [0]_{n_y \times n_y} \end{bmatrix}^T \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{x}\mathbf{x}}^{k|k-1} & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} - \begin{bmatrix} \mathbf{G}_k (\mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1})^{-1} (\mathbf{G}_k)^T & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{x}\mathbf{x}}^{k|k-1} - \mathbf{G}_k (\mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1})^{-1} (\mathbf{G}_k)^T & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix};
\end{aligned}$$

finally, from 9.33, it follows that

$$\begin{aligned}
\mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k,\hat{\mathbf{x}}_k|k-1} &:= \left[\mathbf{P}_{\mathbf{x}\mathbf{x},**}^{k|k,T_cM} \right]_{1:n_x, 1:n_x} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{x}\mathbf{x}}^{k|k-1} - \mathbf{G}_k (\mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1})^{-1} (\mathbf{G}_k)^T & [0]_{n_x \times n_y} \\ [0]_{n_y \times n_x} & [0]_{n_y \times n_y} \end{bmatrix}_{1:n_x, 1:n_x} \\
&= \mathbf{P}_{\mathbf{x}\mathbf{x}}^{k|k-1} - \mathbf{G}_k (\mathbf{P}_{\mathbf{y}\mathbf{y}}^{k|k-1})^{-1} (\mathbf{G}_k)^T;
\end{aligned}$$

which proves the second equality. \square

According to Theorem 9.3, the correction equations—(9.27), (9.28), (9.29), (9.30), and (9.32)—are correct even when the state and the measurement belong to different manifolds. Initially in this section, we considered a manifold with dimension bigger than the ones of the state and of the measurement; yet, Theorem 9.3 shows that, at the end, we do not have to perform calculations on this bigger manifold. Instead, we can work with the manifolds of the state and the measurement separately with (9.37) and (9.38).

9.3.3 New Riemannian Unscented Filters

We already have all the elements needed to extended the UKF's to the Riemannian case. By analogy with the Unscented Filters of Chapter 5, we introduce the following four filters. For the augmented ones, define the augmented functions $f_k^a : \mathcal{M}_x^{n_x} \times \mathcal{M}_{\varpi}^{n_{\varpi}} \rightarrow \mathbb{R}^{n_x}$ and $h_k^a : \mathcal{M}_x^{n_x} \times \mathcal{M}_{\vartheta}^{n_{\vartheta}} \rightarrow \mathbb{R}^{n_y}$

$$\begin{aligned}
f_k^a \left(\begin{bmatrix} \mathbf{x}_{k-1} \\ \boldsymbol{\varpi}_k \end{bmatrix} \right) &:= f_k(\mathbf{x}_{k-1}, \mathbf{q}_k), \\
h_k^a \left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{r}_k \end{bmatrix} \right) &:= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k).
\end{aligned} \tag{9.43}$$

Definition 9.10. Consider the system

$$\begin{aligned}
\mathbf{x}_k &= f_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_k), \\
\mathbf{y}_k &= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k);
\end{aligned}$$

the pair of equations (9.43); and the functions RiUT in Definition 9.2, and PT in the equation (9.32). Suppose that i) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ are independent; ii) $\boldsymbol{\varpi}_k$, $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned}
\mathbf{x}_0 &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x}, \\
\boldsymbol{\varpi}_k &\sim (\bar{\boldsymbol{\varpi}}_k, \mathbf{Q}_k)_{\mathcal{M}_{\boldsymbol{\varpi}}}, \\
\boldsymbol{\vartheta}_k &\sim (\bar{\boldsymbol{\vartheta}}_k, \mathbf{R}_k)_{\mathcal{M}_{\boldsymbol{\vartheta}}};
\end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Augmented Unscented Kalman Filter* is given by the following algorithm:

Algorithm 19 (Riemannian Augmented Unscented Kalman Filter (RiAuUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\begin{aligned}
\hat{\mathbf{x}}_{k-1|k-1}^a &:= \left[\hat{\mathbf{x}}_{k-1|k-1}^T, \bar{\boldsymbol{\varpi}}_k^T \right]^T, \\
\hat{\mathbf{P}}_{xx,a}^{k-1|k-1} &:= \text{diag} \left(\hat{\mathbf{P}}_{xx}^{k-1|k-1}, \mathbf{Q}_k \right).
\end{aligned}$$

(b) *The predicted statistics of the state by*

$$\left[\hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{P}}_{xx}^{k|k-1} \right] := \text{RiUT}_1 \left(f_k^a, \hat{\mathbf{x}}_{k-1|k-1}^a, \hat{\mathbf{P}}_{xx,a}^{k-1|k-1} \right). \tag{9.44}$$

(c) *The augmented predicted estimates by*

$$\hat{\mathbf{x}}_{k|k-1}^a := \left[\hat{\mathbf{x}}_{k|k-1}^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T,$$

$$\hat{\mathbf{P}}_{xx,a}^{k|k-1} := \text{diag} \left(\hat{\mathbf{P}}_{xx}^{k|k-1}, \mathbf{R}_k \right).$$

(d) The predicted statistics of the measurement by

$$\begin{aligned} \left[\hat{\mathbf{y}}_{k|k-1}, \hat{\mathbf{P}}_{yy}^{k|k-1}, \hat{\mathbf{P}}_{xy,a}^{k|k-1} \right] &:= \text{RiUT}_2 \left(h_k^a, \hat{\mathbf{x}}_{k|k-1}^a, \hat{\mathbf{P}}_{xx,a}^{k|k-1} \right), \\ \hat{\mathbf{P}}_{xy}^{k|k-1} &:= \left[\hat{\mathbf{P}}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}. \end{aligned} \quad (9.45)$$

(e) The corrected statistics of the state by

$$\begin{aligned} \mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\mathcal{S}_{\hat{\mathbf{y}}_{k|k-1}}} \left(\mathbf{y}_k \right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\ \hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} &:= \hat{\mathbf{P}}_{xx}^{k|k-1} - (\mathbf{G}_k) \hat{\mathbf{P}}_{yy}^{k|k-1} (\mathbf{G}_k)^T, \\ \hat{\mathbf{P}}_{xx}^{k|k} &:= \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \end{aligned} \quad (9.46)$$

Definition 9.11. Consider the system

$$\begin{aligned} \mathbf{x}_k &= f_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_k), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k); \end{aligned}$$

the pair of equations (9.43); and the functions RiSRUT in Definition 9.6, and PT in the equation (9.32). Suppose that i) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ are independent; ii) $\boldsymbol{\varpi}_k$, $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned} \mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{xx}^0} \sqrt{\mathbf{P}_{xx}^0}^T \right)_{\mathcal{M}_x}, \\ \boldsymbol{\varpi}_k &\sim \left(\bar{\boldsymbol{\varpi}}_k, \sqrt{\mathbf{Q}_k} \sqrt{\mathbf{Q}_k}^T \right)_{\mathcal{M}_{\boldsymbol{\varpi}}}, \\ \boldsymbol{\vartheta}_k &\sim \left(\bar{\boldsymbol{\vartheta}}_k, \sqrt{\mathbf{R}_k} \sqrt{\mathbf{R}_k}^T \right)_{\mathcal{M}_{\boldsymbol{\vartheta}}}; \end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Augmented Square-Root Unscented Kalman Filter* is given by the following algorithm:

Algorithm 20 (Riemannian Augmented Square-Root Unscented Kalman Filter (Ri-AuSRUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) The augmented previous estimates by

$$\begin{aligned}\hat{\mathbf{x}}_{k-1|k-1}^a &:= \left[\hat{\mathbf{x}}_{k-1|k-1}^T, \bar{\boldsymbol{\omega}}_k^T \right]^T, \\ \sqrt{\hat{\mathbf{P}}_{xx,a}^{k-1|k-1}} &:= \text{diag} \left(\sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1}}, \sqrt{\mathbf{Q}_k} \right).\end{aligned}$$

(b) The predicted statistics of the state:

$$\left[\hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}} \right] = \text{RiSRUT}_1 \left(f_k^a, \hat{\mathbf{x}}_{k-1|k-1}, \sqrt{\hat{\mathbf{P}}_{xx,a}^{k-1|k-1}}, [0]_{n_{\boldsymbol{\omega}} \times n_{\boldsymbol{\omega}}} \right). \quad (9.47)$$

(c) The augmented predicted estimates by

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1}^a &:= \left[\hat{\mathbf{x}}_{k|k-1}^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T, \\ \sqrt{\hat{\mathbf{P}}_{xx,a}^{k|k-1}} &:= \text{diag} \left(\sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}, \sqrt{\mathbf{R}_k} \right).\end{aligned}$$

(d) The predicted statistics of the measurement:

$$\left[\hat{\mathbf{y}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}, S_{\chi}, S_{\gamma}, \hat{\mathbf{P}}_{xy,a}^{k|k-1} \right] = \text{RiSRUT}_2 \left(h_k^a, \hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx,a}^{k|k-1}}, [0]_{n_{\boldsymbol{\theta}} \times n_{\boldsymbol{\theta}}} \right), \quad (9.48)$$

$$\hat{\mathbf{P}}_{xy}^{k|k-1} := \left[\hat{\mathbf{P}}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}.$$

(e) The corrected statistics of the state:

$$\mathbf{G}_k := \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-T} \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-1}, \quad (9.49)$$

$$\hat{x}_{k|k}^{TM} := \hat{x}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{y}_{k|k-1}} \left(\mathbf{y}_k \right),$$

$$\hat{\mathbf{x}}_{k|k} := \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{x}_{k|k}^{TM} \right),$$

$$\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} := \text{triag} \left([S_{\chi} - \mathbf{G}_k S_{\gamma}] \right), \quad (9.50)$$

$$\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} := \text{PT} \left(\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \quad (9.51)$$

Definition 9.12. Consider the system

$$\mathbf{x}_k = \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \boldsymbol{\omega}_k \right],$$

$$\mathbf{y}_k = \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \boldsymbol{\vartheta}_k \right];$$

and the functions RiUT in Definition 9.2, and PT in the equation (9.32). Suppose that
i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned}\mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0\right)_{\mathcal{M}_x}, \\ \varpi_k &\sim \left(\bar{\varpi}_k, Q_k\right)_{T_{f(\mathbf{x}_{k-1}, k)} \mathcal{M}_x}, \\ \vartheta_k &\sim \left(\bar{\vartheta}_k, R_k\right)_{T_{h(\mathbf{x}_k, k)} \mathcal{M}_y};\end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 21 (Riemannian Additive Unscented Kalman Filter (RiAdUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The predicted statistics of the state by*

$$\left[\hat{\mathbf{x}}_{k|k-1}^*, \hat{\mathbf{P}}_{xx,*}^{k|k-1}\right] := \text{RiUT}_1\left(f_k, \hat{\mathbf{x}}_{k-1|k-1}, \hat{\mathbf{P}}_{xx}^{k-1|k-1}\right), \quad (9.52)$$

$$\hat{\mathbf{x}}_{k|k-1} := \exp_{\hat{\mathbf{x}}_{k|k-1}^*} \bar{\varpi}_k, \quad (9.53)$$

$$\hat{\mathbf{P}}_{xx}^{k|k-1} := \hat{\mathbf{P}}_{xx,*}^{k|k-1} + Q_k. \quad (9.54)$$

(b) *The predicted statistics of the measurement by*

$$\left[\hat{\mathbf{y}}_{k|k-1}^*, \hat{\mathbf{P}}_{yy,*}^{k|k-1}, \hat{\mathbf{P}}_{xy}^{k|k-1}\right] := \text{RiUT}_2\left(h_k, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{P}}_{xx}^{k|k-1}\right), \quad (9.55)$$

$$\hat{\mathbf{y}}_{k|k-1} := \exp_{\hat{\mathbf{y}}_{k|k-1}^*} \bar{\vartheta}_k, \quad (9.56)$$

$$\hat{\mathbf{P}}_{yy}^{k|k-1} := \hat{\mathbf{P}}_{yy,*}^{k|k-1} + R_k. \quad (9.57)$$

(c) *The corrected statistics of the state by*

$$\mathbf{G}_k := \left(\hat{\mathbf{P}}_{xy}^{k|k-1}\right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1}\right)^{-1}, \quad (9.58)$$

$$\hat{\mathbf{x}}_{k|k}^{TM} := \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}} \left(\mathbf{y}_k\right), \quad (9.59)$$

$$\hat{\mathbf{x}}_{k|k} := \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM}\right), \quad (9.60)$$

$$\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} := \hat{\mathbf{P}}_{xx}^{k|k-1} - \mathbf{G}_k \hat{\mathbf{P}}_{yy}^{k|k-1} \mathbf{G}_k^T,$$

$$\hat{\mathbf{P}}_{xx}^{k|k} := \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k}\right). \quad (9.61)$$

Definition 9.13. Consider the system

$$\begin{aligned}\mathbf{x}_k &= \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \varpi_k \right], \\ \mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \vartheta_k \right];\end{aligned}$$

and the functions RiSRUT in Definition 9.6, and PT in the equation (9.32). Suppose that i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned}\mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{xx}^0} \sqrt{\mathbf{P}_{xx}^0}^T \right)_{\mathcal{M}_x}, \\ \varpi_k &\sim \left(\bar{\varpi}_k, \sqrt{Q_k} \sqrt{Q_k}^T \right)_{T_{f(\mathbf{x}_{k-1}, k)} \mathcal{M}_x}, \\ \vartheta_k &\sim \left(\bar{\vartheta}_k, \sqrt{R_k} \sqrt{R_k}^T \right)_{T_{h(\mathbf{x}_k, k)} \mathcal{M}_y};\end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Additive Square-Root Unscented Kalman Filter* is given by the following algorithm:

Algorithm 22 (Riemannian Additive Square-Root Unscented Kalman Filter (RiAdSRUKF)). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The predicted statistics of the state by*

$$\left[\hat{\mathbf{x}}_{k|k-1}^*, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}} \right] := \text{RiSRUT}_1 \left(f_k, \hat{\mathbf{x}}_{k-1|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1}}, \sqrt{Q_k} \right), \quad (9.62)$$

$$\hat{\mathbf{x}}_{k|k-1} := \exp_{\hat{\mathbf{x}}_{k|k-1}^*} \bar{\varpi}_k. \quad (9.63)$$

(b) *The predicted statistics of the measurement by*

$$\left[\hat{\mathbf{y}}_{k|k-1}^*, \sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}, S_\chi, S_\gamma, \hat{\mathbf{P}}_{xy}^{k|k-1} \right] := \text{RiSRUT}_2 \left(h_k, \hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}, \sqrt{R_k} \right), \quad (9.64)$$

$$\hat{\mathbf{y}}_{k|k-1} := \exp_{\hat{\mathbf{y}}_{k|k-1}^*} \bar{\vartheta}_k. \quad (9.65)$$

(c) *The corrected statistics of the state by*

$$\mathbf{G}_k := \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-T} \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-1}, \quad (9.66)$$

$$\hat{\boldsymbol{x}}_{k|k}^{TM} := \hat{\boldsymbol{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\boldsymbol{y}}_{k|k-1}} \left(\tilde{\boldsymbol{y}}_k \right), \quad (9.67)$$

$$\hat{\boldsymbol{x}}_{k|k} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}} \left(\hat{\boldsymbol{x}}_{k|k}^{TM} \right), \quad (9.68)$$

$$\sqrt{\hat{\mathbf{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}}} := \text{triag} \left(\left[S_{\boldsymbol{\chi}} - \mathbf{G}_k S_{\boldsymbol{\gamma}}, \mathbf{G}_k \sqrt{R_k} \right] \right), \quad (9.69)$$

$$\sqrt{\hat{\mathbf{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}}} := \text{PT} \left(\sqrt{\hat{\mathbf{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}}}, \hat{\boldsymbol{x}}_{k|k-1}, \hat{\boldsymbol{x}}_{k|k} \right). \quad (9.70)$$

The notations RiUT₁ and RiUT₂ [in (9.44), (9.45), (9.52), and (9.55)] indicate that the transformations in the prediction and correction steps do not need to be the same^a. In fact, the number of sigma points can be different, and we could use the RiScUT (recall that the RiScUT is a particular case of the RiUT). The output of RiUT₁ has only two terms meaning that only the first two elements of the output of Definition 9.2 are needed.

By definition, in the RiAdUKF, the set posterior set $\boldsymbol{\chi}_*^{k|k-1} = \{\boldsymbol{\chi}_{*,i}^{k|k-1}, w_i\}$ of RiUT₁ [in (9.52)] is regenerated in (9.55) because it is the previous σ -representation of RiUT₂. One can consider not regenerating $\boldsymbol{\chi}_*^{k|k-1}$ by making $\boldsymbol{\chi}^{k|k-1} = \{\boldsymbol{\chi}_i^{k|k-1}, w_i\} = \boldsymbol{\chi}^{k|k-1}$, but the filter could have the same consistency problems that the Euclidean AdUKF's without re-sampling have (cf. Section 5.1).

The functions RiUT₁ and RiUT₂ require the calculation of Ri σ R's. It can be difficult to find these Ri σ R's by making the calculations in the manifolds; fortunately, there is an easier way.

From Theorem 9.1, we can find a Ri σ R by first finding a normalized σ R in the tangent space of the considered manifold; each one of the normalized σ R's of Chapter 3 have their associated Ri σ R's (cf. Corollary 9.1). For instance, suppose we want to calculate (9.52) with the normalized RiMi σ R (Theorem 3.2); that is, we want

$$\boldsymbol{\chi} = \{\boldsymbol{\chi}_i, w_i | \boldsymbol{\chi}_i \in \mathcal{M}\}_{i=1}^{n_x+1} := \text{RiMi}\sigma\text{R} \left(\hat{\boldsymbol{x}}_{k-1|k-1}, \hat{\mathbf{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k-1|k-1} \right).$$

We can compute the (Euclidean) Mi σ R (Corollary 3.4)

$$\boldsymbol{\chi} = \{\boldsymbol{\chi}_i, w_i | \boldsymbol{\chi}_i \in T_{\hat{\boldsymbol{x}}_{k-1|k-1}} \mathcal{M}\}_{i=1}^{n_x+1} := \text{Mi}\sigma\text{R} \left([0]_{n_x}, \hat{\mathbf{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k-1|k-1} \right),$$

and then, from Theorem 9.1, we would have

$$\boldsymbol{\chi} = \left\{ \exp_{\hat{\boldsymbol{x}}_{k-1|k-1}} \boldsymbol{\chi}_i, w_i | \boldsymbol{\chi}_i \in \mathcal{M} \right\}_{i=1}^{n_x+1}.$$

^aFor simplicity, we will make comments only to the non square-root filters. Nonetheless, the comments in the remaining of this section can be applied analogously to the Riemannian *square-root* Unscented filters.

The Kalman gain \mathbf{G}_k in (9.46) and (9.58) could be defined in a more general way, as done in (9.34). However, it would imply in more computational effort—the dimension of the sigma points and matrices would be higher—at the exchange of no advantages, at least at the present time; perhaps benefits can be drawn from (9.34) in future works.

Each Riemannian Unscented filter is a *general case* of the respective Euclidean Unscented filter of Chapter 5. In fact, it is easy to see that, if \mathcal{M}_x and \mathcal{M}_y are Euclidean spaces, then the i) RiAuUKF is the AuUKF (Algorithm 7), ii) RiAuSRUKF the AuSRUKF (Algorithm 9), iii) RiAdUKF the AdUKF (Algorithm 6), iv) RiAdSRUKF the AdSRUKF (Algorithm 8).

Since *Cartesian products* of Riemannian manifolds are also Riemannian manifolds (cf. Section A.2), systems composed of Cartesian-product manifolds—e.g. the manifold $S^3 \times \mathbb{R}^n$ in the satellite attitude estimation of Section 7.4.1—can be estimated by the Riemannian Unscented filters of this section.

For each RiUT in the filters above, we need to calculate the Riemannian *sample mean* of the posterior Riemannian weighted sets. Since these means are defined by optimization problems (Section 8.6), obtaining closed forms for them is generally challenging; usually, optimization algorithms are used, such as the Gauss-Newton Gradient Descent Algorithm of [173], or the Newton algorithms and the trust region algorithms of [174].

In the square-root Unscented filters of this section (Algorithms 20 and 22), the triang operations in (9.50) and (9.69) *must return symmetric square-root matrices* $\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}}$, because the PT function in (9.51) and (9.70) is defined only for this class of matrices.

In each of the Riemannian Unscented filters above, Riemannian exponentials and logarithms are used. These functions, as well as other elements in these filters such as covariances, have different expressions *depending on the parameterization* chosen for the manifolds (cf. Section A.5). Therefore, after choosing a parameterization for each of the manifolds in a considered filter, *all* the expressions for the Riemannian exponentials, logarithms, covariances, etc, should be *coherent* with these parameterizations.

We can define *particular cases* of all the Unscented filters above by choosing particular forms of the Ri σ R's and RiUT's; some are shown in Tables 9.1, 9.2, 9.3, and 9.4. In all these tables Def. stands for Definition; Cor. for Corollary; Ho. for Homogeneous; Intr. for Intrinsically; Mi. for Minimum; Sc. for Scaled; Si. for Simplex; Sy. for Symmetric; and Ri. for Riemannian.

The variants of the RiAuUKF with RiUT₁ = RiUT₂, and of the RiAuSRUKF with RiSRUT₁ = RiSRUT₂ are shown in Tables 9.1 and 9.3; particularly, Table 9.1 contains these variants with the *minimum* (non-symmetric) Riemannian σ -representations of

Corollary 3.4, and Table 9.3 with the *minimum symmetric* Riemannian σ -representations of Corollary 3.4. Table 9.2 contains the *additive* analogous of the filters in Table 9.1, and Table 9.4 the *additive* analogous of the filters in Table 9.3.

In each table, the particular filters are presented in all the columns, *except the first*; and in all the rows, *except the heading one*. In Table 9.1, each filter is the resulting variant of using the RiAuUKF or the RiAuSRUKF (analogously for the other tables) with the corresponding i) RiUT or RiSRUT written in the *first column* of its own row, and ii) Ri σ R written in the *heading row* of its own column. For instance, the Riemannian Minimum Scaled Augmented Unscented Kalman Filter (Ri. Mi. Sc. AuUKF in Tab 9.1 [2,2]), is the result of the RiAuUKF with the RiScUT (Tab 9.1 [2,1]) and the RiMi σ R (heading of the second column of Table 9.1). It is worthy to mention that all the filters in Tables 9.1, 9.2, 9.3, and 9.4 are *new*.

Table 9.1: Some Consistent Riemannian Minimum AuUKF and Riemannian Minimum AuSRUKF Variants.

	RiUT's	RiMi σ R (Cor. 9.1)	RiRhoMi σ R (Cor. 9.1)
1	RiUT (Def. 9.2)	Ri. Mi. AuUKF	Ri. Rho Mi. AuUKF
2	RiScUT (Def. 9.3)	Ri. Mi. Sc. AuUKF	Ri. Rho Mi. Sc. AuUKF
3	RiSRUT (Def. 9.6)	Ri. Mi. AuUKF	Ri. Rho Mi. AuUKF
4	RiScSRUT (Def. 9.7)	Ri. Mi. Sc. AuUKF	Ri. Rho Mi. Sc. AuUKF

Table 9.2: Some Consistent Riemannian Minimum AdUKF and Riemannian Minimum AdSRUKF Variants.

	RiUT's	RiMi σ R (Cor. 9.1)	RiRhoMi σ R (Cor. 9.1)
1	RiUT (Def. 9.2)	Ri. Mi. AdUKF	Ri. Rho Mi. AdUKF
2	RiScUT (Def. 9.3)	Ri. Mi. Sc. AdUKF	Ri. Rho Mi. Sc. AdUKF
3	RiSRUT (Def. 9.6)	Ri. Mi. AdSRUKF	Ri. Rho Mi. AdSRUKF
4	RiScSRUT (Def. 9.7)	Ri. Mi. Sc. AdSRUKF	Ri. Rho Mi. Sc. AdSRUKF

9.4 RELATION WITH THE LITERATURE

The unique Unscented filter for Riemannian systems in the literature was proposed by [171].

Definition 9.14. Consider the system

$$\mathbf{x}_k = f_k(\mathbf{x}_{k-1}) := f'_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_{k-1}), \quad (9.71)$$

Table 9.3: Some Consistent Riemannian Minimum Symmetric AuUKF and Riemannian Minimum Symmetric AuSRUKF Variants.

RiUT's	RiMiSy σ R (Cor. 9.1)	RiHoMiSy σ R (Cor. 9.1)
RiUT (Def. 9.2)	Ri. Mi. Sy. AuUKF	Ri. Ho. Mi. Sy. AuUKF
RiScUT (Def. 9.3)	Ri. Mi. Sy. Sc. AuUKF	Ri. Ho. Mi. Sy. Sc. AuUKF
RiSySiScUT (Def. 9.4)	Ri. Mi. Sy. Si. Sc. AuUKF	Ri. Ho. Mi. Sy. Si. Sc. AuUKF
RiSyInScUT (Def. 9.5)	–	Ri. Sy. Intr. Sc AuUKF
RiSRUT (Def. 9.6)	Ri. Mi. Sy. AuSRUKF	Ri. Ho. Mi. Sy. AuSRUKF
RiScSRUT (Def. 9.7)	Ri. Mi. Sy. Sc. AuSRUKF	Ri. Ho. Mi. Sy. Sc. AuSRUKF
RiSySiScSRUT (Def. 9.8)	Ri. Mi. Sy. Si. Sc. AuSRUKF	Ri. Ho. Mi. Sy. Si. Sc. AuSRUKF
RiSyInScSRUT (Def. 9.9)	–	Ri. Sy. Intr. Sc. AuSRUKF

Table 9.4: Some Consistent Riemannian Minimum Symmetric AdUKF and Riemannian Minimum Symmetric AdSRUKF Variants.

RiUT's	RiMiSy σ R (Cor. 9.1)	RiHoMiSy σ R (Cor. 9.1)
RiUT (Def. 9.2)	Ri. Mi. Sy. AdUKF	Ri. Ho. Mi. Sy. AdUKF
RiScUT (Def. 9.3)	Ri. Mi. Sy. Sc. AdUKF	Ri. Ho. Mi. Sy. Sc. AdUKF
RiSySiScUT (Def. 9.4)	Ri. Mi. Sy. Si. Sc. AdUKF	Ri. Ho. Mi. Sy. Si. Sc. AdUKF
RiSyInScUT (Def. 9.5)	–	Ri. Sy. Intr. Sc AdUKF
RiSRUT (Def. 9.6)	Ri. Mi. Sy. AdSRUKF	Ri. Ho. Mi. Sy. AdSRUKF
RiScSRUT (Def. 9.7)	Ri. Mi. Sy. Sc. AdSRUKF	Ri. Ho. Mi. Sy. Sc. AdSRUKF
RiSySiScSRUT (Def. 9.8)	Ri. Mi. Sy. Si. Sc. AdSRUKF	Ri. Ho. Mi. Sy. Si. Sc. AdSRUKF
RiSyInScSRUT (Def. 9.9)	–	Ri. Sy. Intr. Sc. AdSRUKF

$$\mathbf{y}_k = h_k(\mathbf{x}_k) := h'_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k);$$

and let n_x be the dimension of \mathcal{M}_x , and n_y of \mathcal{M}_y . Suppose that i) the initial state \mathbf{x}_0 is characterized by

$$\mathbf{x}_0 \sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x},$$

and ii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the Unscented Kalman Filter for Riemannian manifolds of [171] is given by the following algorithm.

Algorithm 23 (Unscented Kalman Filter for Riemannian manifolds (UKFRM) of [171]). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's tangent previous sigma points by*

$$\left\{ \chi_{i, k-1|k-1}^{TM}, w_i \right\}_{i=1}^{2n_x+1} := \text{HoMiSy}\sigma\text{R} \left([0]_{n_x}, \hat{\mathbf{P}}_{xx}^{k-1|k-1} \right); \quad (9.72)$$

(b) *The state's previous sigma points by*

$$\boldsymbol{\chi}_i^{k-1|k-1} := \exp_{\hat{\mathbf{x}}_{k-1|k-1}} \left(\chi_{i, k-1|k-1}^{TM} \right), \quad i = 1, \dots, 2n_x + 1; \quad (9.73)$$

(c) *The state's predicted sigma points by*

$$\boldsymbol{\chi}_{i,*}^{k|k-1} := f_k \left(\boldsymbol{\chi}_i^{k-1|k-1} \right), \quad i = 1, \dots, 2n_x + 1;$$

(d) *The state's predicted estimate by*

$$\hat{\mathbf{x}}_{k|k-1} := \arg \min_{\mathbf{a} \in \mathcal{M}_x} \sum_{i=1}^{2n_x+1} w_i \text{dist}^2 \left(\boldsymbol{\chi}_{i,*}^{k|k-1}, \mathbf{a} \right); \quad (9.74)$$

(e) *The state's predicted covariance by*

$$\hat{\mathbf{P}}_{xx}^{k|k-1} := \sum_{i=1}^{2n_x+1} w_i \left(\log_{\hat{\mathbf{x}}_{k|k-1}} \left(\boldsymbol{\chi}_{i,*}^{k|k-1} \right) \right) \left(\diamond \right)^T; \quad (9.75)$$

(f) *The state's tangent predicted sigma points by*

$$\left\{ \chi_{i, k|k-1}^{TM}, w_i \right\}_{i=1}^{2n_x+1} := \text{HoMiSy}\sigma\text{R} \left([0]_{n_x}, \hat{\mathbf{P}}_{xx}^{k-1|k-1} \right); \quad (9.76)$$

(g) The state's new predicted sigma points by

$$\boldsymbol{\chi}_i^{k|k-1} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}} \left(\boldsymbol{\chi}_{i,k|k-1}^{TM} \right), \quad i = 1, \dots, 2n_x + 1; \quad (9.77)$$

(h) The measurement's predicted sigma points by

$$\boldsymbol{\gamma}_i^{k|k-1} := h_k \left(\boldsymbol{\chi}_i^{k|k-1} \right), \quad i = 1, \dots, 2n_x + 1;$$

(i) The measurement's predicted estimate by

$$\hat{\boldsymbol{y}}_{k|k-1} := \arg \min_{\boldsymbol{b} \in \mathcal{M}_y} \sum_{i=1}^{2n_x+1} w_i \text{dist}^2 \left(\boldsymbol{\gamma}_i^{k|k-1}, \boldsymbol{b} \right); \quad (9.78)$$

(j) The measurement's predicted covariance by

$$\hat{\boldsymbol{P}}_{yy}^{k|k-1} := \sum_{i=1}^{2n_x+1} w_i \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right) \left(\diamond \right)^T;$$

(k) The predicted cross-covariance by

$$\hat{\boldsymbol{P}}_{xy}^{k|k-1} := \sum_{i=1}^{2n_x+1} w_i \left(\log_{\hat{\boldsymbol{x}}_{k|k-1}} \left(\boldsymbol{\chi}_i^{k|k-1} \right) \right) \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right)^T; \quad (9.79)$$

(l) The Kalman Gain by

$$\boldsymbol{G}_k := \left(\hat{\boldsymbol{P}}_{xy}^{k|k-1} \right) \left(\hat{\boldsymbol{P}}_{yy}^{k|k-1} \right)^{-1}; \quad (9.80)$$

(m) The state's tangent corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k}^{TM} := \hat{\boldsymbol{x}}_{k|k-1}^{TM} + \boldsymbol{G}_k \log_{\hat{\boldsymbol{y}}_{k|k-1}} \left(\boldsymbol{y}_k \right) \quad (9.81)$$

(n) The state's corrected covariance estimate at $\hat{\boldsymbol{x}}_{k|k-1}$ by

$$\hat{\boldsymbol{P}}_{xx}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}} = \hat{\boldsymbol{P}}_{xx}^{k|k-1} - (\boldsymbol{G}_k) \hat{\boldsymbol{P}}_{yy}^{k|k-1} (\boldsymbol{G}_k)^T; \quad (9.82)$$

(o) The state's corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}} \left(\hat{\boldsymbol{x}}_{k|k}^{TM} \right); \quad (9.83)$$

(p) The state's corrected covariance estimate at $\hat{\boldsymbol{x}}_{k|k}$ by

$$\hat{\boldsymbol{P}}_{xx}^{k|k} := \text{PT} \left(\hat{\boldsymbol{P}}_{xx}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}}, \hat{\boldsymbol{x}}_{k|k-1}, \hat{\boldsymbol{x}}_{k|k} \right); \quad (9.84)$$

Among the Riemannian Unscented filters presented in Section 9.3, we compare the UKFRM of [171] with the Riemannian Homogeneous Minimum Symmetric AdUKF (RiHoMiSyAdUKF, Table 9.4 [1,1]) because i) it does not augment the state vectors with the noise vectors (as the augmented filters do); and ii) it is composed of the RiHoMiSy σ R (Cor. 9.1). Therefore, we can say that *all* the *other* filters of Table 9.4, and *all* e filters of Tables 9.1, 9.2, and 9.3 are novelties of our present work. By comparing the UKFRM of [171] with the RiHoMiSyAdUKF, we want to show that also the RiHoMiSyAdUKF is a novelty.

There are some inconsistencies in the UKFRM of [171]; we can cite the following ones:

1. Although the UKFRM of [171] (Algorithm 23) considers the system (9.71)—cf. equations (1) and (2) of [171]—, the noises ϖ_k and ϑ_k do *not* influence any estimate within the UKFRM; these noises' statistics *do not appear at any step* of Algorithm 23; not even the covariances \mathbf{Q}_k and \mathbf{R}_k , which usually appear in Unscented filters. We believe this inconsistency may lead Algorithm 23 to poor estimates, and sometimes, to diverge. For our Riemmanian *augmented* filters such as the RiHoMiSyAuUKF, the influence of the noises in the estimate is given by realizing the augmented sigma points in the process and measurement functions [cf. (9.44), (9.45), (9.47), and (9.45)]; and for our Riemmanian *additive* filters, the influence of the noises in the estimate is given by “adding” (in the tangent space) their means and covariances [cf. (9.53), (9.54), (9.56), (9.57), (9.62), (9.63), (9.64), and (9.65)].
2. The term $\hat{x}_{k|k-1}^{TM}$ appears in (9.81), but it should not, since it is *always* equal to zero; it is the origin of $T_{\hat{x}_{k|k-1}}\mathcal{M}_x$ ^b. In the RiHoMiSyAdUKF, this problem does not appear.

Moreover, we could not find *formal justifications* in [171] for some equations of the UKFRM; namely the following ones:

1. *Equations (9.72), (9.73), (9.76), and (9.77)*. These equations perform the generation of new Ri σ R's, and in these equations, these Ri σ R's are generated by first generating σ R's in tangent spaces, and then the associated Ri σ R's are obtained using (Riemannian) exponential mappings. In (9.72), the state's previous σ R, $\chi_{k-1|k-1}^{TM} := \{\chi_{i,k-1|k-1}^{TM}, w_i\}$, is generated in the tangent space $T_{\hat{x}_{k-1|k-1}}\mathcal{M}_x$, and the associated Ri σ R, $\chi^{k-1|k-1} := \{\chi_i^{k-1|k-1}, w_i\}$, is obtained from $\chi_{k-1|k-1}^{TM}$ in (9.73); similarly in (9.76), the (second) predicted state's σ R, $\chi_{k|k-1}^{TM} := \{\chi_{i,k|k-1}^{TM}, w_i\}$,

^bIn a personal email, Soren Hauberg himself, the main author of [171], acknowledged us that this argument is true.

is generated in the tangent space $T_{\hat{x}_{k|k-1}}\mathcal{M}_x$, and the associated $\text{Ri}\sigma\text{R}$, $\chi^{k|k-1} := \{\chi_i^{k|k-1}, w_i\}$, is obtained from $\chi_{k|k-1}^{TM}$ in (9.77). However we could not find results in [171] proving that this equations result in proper forms for $\chi^{k-1|k-1}$ and $\chi^{k|k-1}$.

2. *Equation (9.80)*. This equation defines the Kalman Gain \mathbf{G}_k ; this gain is stated by [171] as being a “linear transformation between the two tangent spaces $[\mathcal{M}_y$ and $\mathcal{M}_x]$ ” (the comment among brackets is ours). For example, in the case of \mathcal{M}_x and \mathcal{M}_y being very different Riemannian manifolds, \mathbf{G}_k would be a linear transformation from \mathcal{M}_y to \mathcal{M}_x ; such a transformation is counterintuitive, at least. Thus, we can say that it is not intuitive nor straightforward to assume that a filter with transformation provide *a consistent final estimate of the state*; it is natural to ask for a justification of (9.80).
3. *Equations (9.81), (9.82), (9.83), and (9.84)*. These equations correct the predicted state estimate. However, in [171], we could not find results showing whether these equations do or do not provide consistent estimates $\hat{\mathbf{x}}_{k|k}$ and $\hat{\mathbf{P}}_{xx}^{k|k}$.

On the other hand, for the RiHoMiSyAdUKF, we presented i) these equations as *natural results* within the Riemannian Unscented Kalman filtering theory presented in this chapter, and ii) *formal justifications* for all these equations. These justifications are the following ones:

1. *Equations (9.72), (9.73), (9.76), and (9.77) are justified by Theorem 9.1*. Indeed, it is straightforward to see that the relations i) between $\chi_{k-1|k-1}^{TM}$ and $\chi^{k-1|k-1}$, and ii) between $\chi_{k|k-1}^{TM}$ and $\chi^{k|k-1}$ are given by this theorem.
2. *Equation (9.80) is justified in Section 9.3.2.2*. This form of the Kalman gain \mathbf{G}_k in (9.80) followed as a particular case of the Kalman gain $\mathbf{G}_{k,**}$ of a more general system where the state and the measurement belong to the product of $\mathcal{M}_x \times \mathcal{M}_y$.
3. *Equations (9.81), (9.82), (9.83), and (9.84) are justified in Section 9.3.2*. We showed that they follow from considering i) $x_{k|k-1}^{TM}$ and $y_{k|k-1}^{TM}$ normally-joint distributed [equation (9.25)], and ii) $x_{k|k}^{TM}$ given by a linear correction of $x_{k|k-1}^{TM}$ by $(y_k^{TM} - y_{k|k-1}^{TM})$ [equation (9.26)].

Finally, we can say that the RiHoMiSyAdUKF is *more general* than the UKFRM. In the UKFRM, $\chi^{k-1|k-1}$ and $\chi^{k|k-1}$ are *necessarily* calculated by (9.72), (9.73), (9.76), and (9.77), but in the RiHoMiSyAdUKF they *can be calculated by other equations*. In the UKFRM, these $\text{Ri}\sigma\text{R}$'s are *defined* by (9.73), and (9.77); then they *are* calculated by (9.72), (9.73), (9.76), and (9.77). On the other hand, in the RiHoMiSyAdUKF, $\chi^{k-1|k-1}$ and $\chi^{k|k-1}$ are *defined according to Definition 9.1*. Therefore, in RiHoMiSyAdUKF,

the tangent $\sigma\mathbb{R}$'s $\chi_{k-1|k-1}^{TM}$ and $\chi_{k|k-1}^{TM}$ are not *required*; there may exist other forms of calculating $\chi^{k-1|k-1}$ and $\chi^{k|k-1}$. Nevertheless, calculating $\chi^{k-1|k-1}$ and $\chi^{k|k-1}$ by (9.72), (9.73), (9.76), and (9.77) should be, *in general*, easier.

Altogether, we can say that our RiUF's have novelties comparative with the UKF for Riemannian manifolds of the literature.

9.5 RIEMANNIAN UNSCENTED FILTERING FOR STATE VARIABLES IN UNIT SPHERES

In most of the times, if not always, Unscented filters are implemented in computers, but for *Riemannian* Unscented filters this task might not be trivial. Concepts of the Riemannian manifold theory can be very abstract, but usually computer languages are not designed to work with such level of abstraction. Instead, often we have to work either with particular closed forms or with numerical approximations. For instance, computing geodesics is not easy; in the cases we can compute them, either we restrict the manifold to a few particular cases whose closed forms are known, or we compute these geodesics numerically [174].

In this section, we show that our RiUF's are elegant solutions to the problem of finding *consistent computationally-implementable* UKF's for systems whose state variables belong to the S^3 , the set of unit quaternions. Recall that, initially, this problem has been the motivation to move towards Riemannian manifolds (cf. Chapter 7).

In order to develop these filters, we need computationally-implementable bases to express the elements of S^{n-1} . Because the S^{n-1} is a Riemannian manifold embedded in the \mathbb{R}^n (cf. Section A.2), we can write its elements and mappings (e.g. exponential and logarithm mappings) in the same coordinate system as the \mathbb{R}^n —for the remaining of this work, we will represent the canonical basis of the \mathbb{R}^n by $e := \{e_1, \dots, e_n\}$, $e_i = [0, \dots, 0, 1, 0, \dots, 0]^T$. Besides, we already provided closed forms of some results relative to S^{n-1} in e (cf. Examples A.6, A.7, A.8, A.10, and A.12).

However, representing the *tangent sigma points* of an Unscented filter in a $(n-1)$ -basis (a basis composed of $n-1$ elements) results in a *computationally-cheaper filter* comparative with representing them in e (e is an n -basis). Because the dimension of the tangent spaces of S^{n-1} is $n-1$, we *can* represent the tangent vectors of S^{n-1} in a basis with $n-1$ elements. From Theorem 9.1, a Ri $\sigma\mathbb{R}$ $\chi = \{\chi_i, w_i\}_{i=1}^N$ with sample mean μ_χ on a Riemannian manifold S^{n-1} can be calculated from a $\sigma\mathbb{R}$ $\chi = \{\chi_i, w_i\}_{i=1}^N$ on $T_{\mu_\chi} S^{n-1}$. In this case, the computational effort of an Unscented filter increases with

the following numbers:

1. the *number of sigma points* N ,
2. the *number of rows* (or columns) of the sample covariance $\Sigma_{\chi\chi}$ of χ (recall that the most expensive operations of the Unscented filters are the square-rooting and inversion of *covariances*);

and these two numbers increase with the length of the tangent sigma points χ_i 's. Then, the smaller is the number of the elements in the basis representing χ_i , the smaller will be the computational effort of the filter. For instance, in the UKFRM of [171] for $\mathcal{M} = S^{n-1}$, the tangent sigma points χ_i 's are expressed in the basis e (cf. Section 4.1 of [171]); thus i) its Ri σ R's (HoMiSy σ R's) are composed of $N = 2n + 1$ sigma points, and ii) $\Sigma_{\chi\chi}$ is composed of n columns. On the other hand, if χ_i 's were expressed in a $(n - 1)$ -basis, then i) N would be $2n - 1$, and ii) $\Sigma_{\chi\chi}$ would be composed of $n - 1$ columns.

For any differentiable manifold of dimension $n - 1$, an orthogonal $(n - 1)$ -basis for a tangent space is naturally induced by a chosen parameterization of the manifold. Consider a point \mathbf{q} of a differentiable manifold \mathcal{M} , and let $\varphi : U \subset \mathbb{R}^{n-1} \rightarrow \mathcal{M}$ be a parameterization from the open set U to \mathcal{M} such that $\mathbf{q} = \varphi(u_1, \dots, u_{n-1})$. Then the set—for a function $f(x, y, \dots)$, the notation ∂f_x stands for $\partial f_x := \partial f / \partial x$ —

$$\left\{ \partial\varphi_{u_1}, \dots, \partial\varphi_{u_{n-1}} \right\}$$

is an basis of $T_{\mathbf{q}}S^{n-1}$ (cf. Section A.1).

For a parameterization $\varphi \in \{\phi^i\}_{i=1}^{2n}$, we define i) the function

$$\begin{aligned} \text{TBtoCB} : T_{\mathbf{q}}S^{n-1} &\rightarrow T_{\mathbf{q}}S^{n-1} \\ v_{TB} &\mapsto v_e \end{aligned} \tag{9.85}$$

mapping the coordinates of the vector $v_{TB} \in T_{\mathbf{q}}S^{n-1}$ in the basis $\{\partial\phi^i_{u_1}, \dots, \partial\phi^i_{u_{n-1}}\}$ to the canonical basis e according to (A.7) and (A.9); and ii) the function

$$\begin{aligned} \text{CBtoTB} : T_{\mathbf{q}}S^{n-1} &\rightarrow T_{\mathbf{q}}S^{n-1} \\ v_e &\mapsto v_{TB} \end{aligned} \tag{9.86}$$

as the inverse mapping of TBtoCB according to (A.8) and (A.10).

Summing up, we use the following closed forms:

1. (A.22) for the exponential mapping expressed on e , $\exp_{\mathbf{q}}^{e\mathbf{x}}$;

2. (A.23) for the logarithm mapping expressed on e , $\log_q^{e_x}$;
3. (9.85) for the transformation from the basis $\{\partial\varphi_{u_1}, \dots, \partial\varphi_{u_{n-1}}\}$ to e , TBtoCB;
4. (9.86) for the transformation from the basis e to $\{\partial\varphi_{u_1}, \dots, \partial\varphi_{u_{n-1}}\}$, CBtoTB;
and
5. (A.13) for the parallel transport of *tangent vectors* expressed on the basis e [these operations are used in the parallel transport of the *covariances* in (9.89), (9.92), (9.104), and (9.92)].

The *Riemannian sample means* of the Ri σ R's still have to be calculated by *approximations or numerical solutions*—e.g. the weighted mean methods in Section 7.2.2, or the algorithms of [174]. Unfortunately, to the best of our knowledge, there is no closed form for the *Riemannian sample means* of weighted sets composed of Riemannian points, such as Ri σ R's, belonging to S^{n-1} .

Definition 9.15. Consider the system

$$\begin{aligned}\mathbf{x}_k &= f_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_k), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k); \end{aligned}$$

and the pair of equations

$$\begin{aligned}f_k^a\left(\begin{bmatrix} \mathbf{x}_{k-1} \\ \boldsymbol{\varpi}_k \end{bmatrix}\right) &:= f_k(\mathbf{x}_{k-1}, \mathbf{q}_k), \\ h_k^a\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{r}_k \end{bmatrix}\right) &:= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k); \end{aligned}$$

with $\mathcal{M}_{\mathbf{x}}^{n_x} = S^{n_x}$, $\mathcal{M}_{\boldsymbol{\varpi}}^{n_{\boldsymbol{\varpi}}} = S^{n_{\boldsymbol{\varpi}}}$, $\mathcal{M}_{\boldsymbol{\vartheta}}^{n_{\boldsymbol{\vartheta}}} = S^{n_{\boldsymbol{\vartheta}}}$, and $\mathcal{M}_{\mathbf{y}}^{n_y} = S^{n_y}$; and let e_x be the canonical basis of \mathbb{R}^{n_x+1} and e_y of \mathbb{R}^{n_y+1} . Suppose that i) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ are independent; ii) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned}\mathbf{x}_0 &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{S^{n_x}}, \\ \boldsymbol{\varpi}_k &\sim (\bar{\boldsymbol{\varpi}}_k, \mathbf{Q}_k)_{S^{n_{\boldsymbol{\varpi}}}}, \\ \boldsymbol{\vartheta}_k &\sim (\bar{\boldsymbol{\vartheta}}_k, \mathbf{R}_k)_{S^{n_{\boldsymbol{\vartheta}}}}; \end{aligned}$$

with \mathbf{P}_{xx}^0 , \mathbf{Q}_k , and \mathbf{R}_k expressed in the differentiable structure in (A.1), and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian-Spheric Augmented Unscented Kalman Filter (RiSAuUKF)* is given by the following algorithm:

Algorithm 24 (Riemannian-Spheric Augmented Unscented Kalman Filter). *Perform the following steps:*

1. Initialization. Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.

2. Filtering. For $k = 1, 2, \dots, k_f$; set the following elements:

(a) The augmented previous estimates by

$$\begin{aligned}\hat{\mathbf{x}}_{k-1|k-1}^a &:= \left[\hat{\mathbf{x}}_{k-1|k-1}^T, \bar{\boldsymbol{\varpi}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{xx,a}^{k-1|k-1} &:= \text{diag} \left(\hat{\mathbf{P}}_{xx}^{k-1|k-1}, \mathbf{Q}_k \right).\end{aligned}$$

(b) The state's tangent previous sigma representation by

$$\left\{ \chi_{i,k-1|k-1}^{TM,a}, w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := \sigma \mathbf{R}_1 \left([0]_{(n_x+n_\varpi) \times 1}, \hat{\mathbf{P}}_{xx,a}^{k-1|k-1} \right).$$

(c) The state's previous sigma points, for $i = 1, \dots, N_1$, by

$$\begin{aligned}\boldsymbol{\chi}_i^{k-1|k-1} &:= \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k-1|k-1}^{TM,a} \right]_{1:n_x,1} \right) \right), \\ \boldsymbol{\chi}_{i,\varpi}^{k-1|k-1} &:= \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k-1|k-1}^{TM,a} \right]_{(n_x+1):(n_x+n_\varpi),1} \right) \right).\end{aligned}$$

(d) The state's predicted sigma points by

$$\boldsymbol{\chi}_{i,*}^{k|k-1} := f_k \left(\boldsymbol{\chi}_i^{k-1|k-1}, \boldsymbol{\chi}_{i,\varpi}^{k-1|k-1} \right), \quad i = 1, \dots, N_1.$$

(e) The state's predicted estimate by

$$\hat{\mathbf{x}}_{k|k-1} := \arg \min_{\mathbf{a} \in \mathcal{M}_x} \sum_{i=1}^{N_1} w_i^{1,m} \text{dist}^2 \left(\boldsymbol{\chi}_{i,*}^{k|k-1}, \mathbf{a} \right). \quad (9.87)$$

(f) The state's predicted covariance estimate by

$$\hat{\mathbf{P}}_{xx}^{k|k-1} := \sum_{i=1}^{N_1} w_i^{1,c} \left(\log_{\hat{\mathbf{x}}_{k|k-1}}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_{i,*}^{k|k-1} \right) \right) \right) \left(\diamond \right)^T.$$

(g) The augmented predicted estimates by

$$\begin{aligned}\hat{\mathbf{x}}_{k|k-1}^a &:= \left[\hat{\mathbf{x}}_{k|k-1}^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{xx,a}^{k|k-1} &:= \text{diag} \left(\hat{\mathbf{P}}_{xx,a}^{k|k-1}, \mathbf{R}_k \right).\end{aligned}$$

(h) The regenerated state's predicted sigma points by

$$\left\{ \chi_{i,k|k-1}^{TM}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := \sigma \mathbf{R}_2 \left([0]_{(n_x+n_\vartheta) \times 1}, \hat{\mathbf{P}}_{xx,a}^{k|k-1} \right);$$

and, for $i = 1, \dots, N_2$, by

$$\begin{aligned}\boldsymbol{\chi}_i^{k|k-1} &:= \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k|k-1}^{TM,a} \right]_{1:n_x,1} \right) \right), \\ \boldsymbol{\chi}_{i,\boldsymbol{\vartheta}}^{k|k-1} &:= \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k|k-1}^{TM,a} \right]_{(n_x+1):(n_x+n_\vartheta),1} \right) \right).\end{aligned}$$

(i) The measurement's predicted sigma points by

$$\boldsymbol{\gamma}_i^{k|k-1} := h_k \left(\boldsymbol{\chi}_i^{k|k-1}, \boldsymbol{\chi}_{i,\boldsymbol{\vartheta}}^{k|k-1} \right), \quad i = 1, \dots, N_2.$$

(j) The measurement's predicted estimate by

$$\hat{\boldsymbol{y}}_{k|k-1} := \arg \min_{\boldsymbol{b} \in \mathcal{M}_y} \sum_{i=1}^{N_2} w_i^{2,m} \text{dist}^2 \left(\boldsymbol{\gamma}_i^{k|k-1}, \boldsymbol{b} \right). \quad (9.88)$$

(k) The measurement's predicted covariance estimate by

$$\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1} := \sum_{i=1}^{N_2} w_i^{2,c} \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right) \right) \left(\diamond \right)^T.$$

(l) The predicted cross-covariance estimate by

$$\begin{aligned}\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,cc} \left(\log_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_i^{k|k-1} \right) \right) \right) \\ &\quad \times \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right) \right)^T.\end{aligned}$$

(m) The Kalman Gain by

$$\boldsymbol{G}_k := \left(\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} \right) \left(\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1} \right)^{-1}.$$

(n) The state's tangent corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k}^{TM} := \hat{\boldsymbol{x}}_{k|k-1}^{TM} + \boldsymbol{G}_k \log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\underline{\boldsymbol{y}}_k \right) \right).$$

(o) The state's corrected covariance estimate at $\hat{\boldsymbol{x}}_{k|k-1}$ by

$$\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k, \hat{\boldsymbol{x}}_{k|k-1}} = \hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k-1} - (\boldsymbol{G}_k) \hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1} (\boldsymbol{G}_k)^T.$$

(p) The state's corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\hat{\boldsymbol{x}}_{k|k}^{TM} \right) \right).$$

(q) The state's corrected covariance estimate at $\hat{\mathbf{x}}_{k|k}$ by

$$\hat{\mathbf{P}}_{xx}^{k|k} := \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \quad (9.89)$$

Definition 9.16. Consider the system

$$\begin{aligned} \mathbf{x}_k &= f_k(\mathbf{x}_{k-1}, \boldsymbol{\varpi}_k), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k); \end{aligned}$$

and the pair of equations

$$\begin{aligned} f_k^a \left(\begin{bmatrix} \mathbf{x}_{k-1} \\ \boldsymbol{\varpi}_k \end{bmatrix} \right) &:= f_k(\mathbf{x}_{k-1}, \mathbf{q}_k), \\ h_k^a \left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{r}_k \end{bmatrix} \right) &:= h_k(\mathbf{x}_k, \boldsymbol{\vartheta}_k); \end{aligned}$$

with $\mathcal{M}_{\mathbf{x}}^{n_x} = S^{n_x}$, $\mathcal{M}_{\boldsymbol{\varpi}}^{n_{\boldsymbol{\varpi}}} = S^{n_{\boldsymbol{\varpi}}}$, $\mathcal{M}_{\boldsymbol{\vartheta}}^{n_{\boldsymbol{\vartheta}}} = S^{n_{\boldsymbol{\vartheta}}}$, and $\mathcal{M}_{\mathbf{y}}^{n_y} = S^{n_y}$; and let $e_{\mathbf{x}}$ be the canonical basis of \mathbb{R}^{n_x+1} and $e_{\mathbf{y}}$ of \mathbb{R}^{n_y+1} . Suppose that i) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ are independent; ii) $\boldsymbol{\varpi}_k$ and $\boldsymbol{\vartheta}_k$ and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned} \mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{xx}^0} \sqrt{\mathbf{P}_{xx}^0}^T \right)_{S^{n_x}}, \\ \boldsymbol{\varpi}_k &\sim \left(\bar{\boldsymbol{\varpi}}_k, \sqrt{\mathbf{Q}_k} \sqrt{\mathbf{Q}_k}^T \right)_{S^{n_{\boldsymbol{\varpi}}}}, \\ \boldsymbol{\vartheta}_k &\sim \left(\bar{\boldsymbol{\vartheta}}_k, \sqrt{\mathbf{R}_k} \sqrt{\mathbf{R}_k}^T \right)_{S^{n_{\boldsymbol{\vartheta}}}}; \end{aligned}$$

with $\sqrt{\mathbf{P}_{xx}^0}$, $\sqrt{\mathbf{Q}_k}$, and $\sqrt{\mathbf{R}_k}$ expressed in the differentiable structure in (A.1), and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian-Spheric Augmented Square-Root Unscented Kalman Filter (RiSAuSRUKF)* is given by the following algorithm:

Algorithm 25 (Riemannian-Spheric Augmented Square-Root Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\hat{\mathbf{x}}_{k-1|k-1}^a := \left[\hat{\mathbf{x}}_{k-1|k-1}^T, \bar{\boldsymbol{\varpi}}_k^T \right]^T,$$

$$\sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k-1|k-1}} := \text{diag} \left(\sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x}}^{k-1|k-1}}, \sqrt{\mathbf{Q}_k} \right).$$

(b) The state's tangent previous sigma representation by

$$\left\{ \chi_{i,k-1|k-1}^{TM,a}, w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := \sigma \mathbf{R}_1 \left([0]_{(n_x+n_\varpi) \times 1}, \sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k-1|k-1}} \sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k-1|k-1}T} \right).$$

(c) The state's previous sigma points, for $i = 1, \dots, N$, by

$$\begin{aligned} \boldsymbol{\chi}_i^{k-1|k-1} &:= \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_{\mathbf{x}}} \left(\text{TBtoCB} \left(\left[\chi_{i,k-1|k-1}^{TM,a} \right]_{1:n_x,1} \right) \right), \\ \boldsymbol{\chi}_{i,\varpi}^{k-1|k-1} &:= \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_{\mathbf{x}}} \left(\text{TBtoCB} \left(\left[\chi_{i,k-1|k-1}^{TM,a} \right]_{(n_x+1):(n_x+n_\varpi),1} \right) \right). \end{aligned}$$

(d) The state's predicted sigma points by

$$\boldsymbol{\chi}_{i,*}^{k|k-1} := f_k \left(\boldsymbol{\chi}_i^{k-1|k-1}, \boldsymbol{\chi}_{i,\varpi}^{k-1|k-1} \right), \quad i = 1, \dots, N_1.$$

(e) The state's predicted estimate by

$$\hat{\mathbf{x}}_{k|k-1} := \arg \min_{\mathbf{a} \in \mathcal{M}_{\mathbf{x}}} \sum_{i=1}^N w_i^{1,m} \text{dist}^2 \left(\boldsymbol{\chi}_{i,*}^{k|k-1}, \mathbf{a} \right). \quad (9.90)$$

(f) The state's predicted square-root covariance estimate, for $i = 1, \dots, N_1$, by

$$\begin{aligned} \tilde{\boldsymbol{\chi}}_{i,*}^{k|k-1} &:= \log_{\hat{\mathbf{x}}_{k|k-1}}^{e_{\mathbf{x}}} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_{i,*}^{k|k-1} \right) \right), \\ S_{\boldsymbol{\chi}^{k|k-1}}^* &:= \left[\sqrt{w_1^{1,c}} \tilde{\boldsymbol{\chi}}_{1,*}^{k|k-1}, \dots, \sqrt{w_N^{1,c}} \tilde{\boldsymbol{\chi}}_{N,*}^{k|k-1} \right], \\ \sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x}}^{k|k-1}} &:= \text{tria} \left(\left[S_{\boldsymbol{\chi}^{k|k-1}}^*, \sqrt{\mathbf{Q}_k} \right] \right). \end{aligned}$$

(g) The augmented predicted estimates by

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1}^a &:= \left[\hat{\mathbf{x}}_{k|k-1}^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k|k-1} &:= \text{diag} \left(\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k|k-1}, \mathbf{R}_k \right). \end{aligned}$$

(h) The regenerated state's predicted sigma points by

$$\left\{ \chi_{i,k|k-1}^{TM}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := \sigma \mathbf{R}_2 \left([0]_{(n_x+n_\vartheta) \times 1}, \hat{\mathbf{P}}_{\mathbf{x}\mathbf{x},a}^{k|k-1} \right);$$

and, for $i = 1, \dots, N_2$, by

$$\begin{aligned}\boldsymbol{\chi}_i^{k|k-1} &:= \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k|k-1}^{TM,a} \right]_{1:n_x,1} \right) \right), \\ \boldsymbol{\chi}_{i,\vartheta}^{k|k-1} &:= \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\left[\chi_{i,k|k-1}^{TM,a} \right]_{(n_x+1):(n_x+n_\vartheta),1} \right) \right).\end{aligned}$$

(i) The measurement's predicted sigma points by

$$\boldsymbol{\gamma}_i^{k|k-1} := h_k \left(\boldsymbol{\chi}_i^{k|k-1}, \boldsymbol{\chi}_{i,\vartheta}^{k|k-1} \right), \quad i = 1, \dots, N_2.$$

(j) The measurement's predicted estimate by

$$\hat{\boldsymbol{y}}_{k|k-1} := \arg \min_{\boldsymbol{b} \in \mathcal{M}_y} \sum_{i=1}^{N_2} w_i^{2,m} \text{dist}^2 \left(\boldsymbol{\gamma}_i^{k|k-1}, \boldsymbol{b} \right). \quad (9.91)$$

(k) The measurement's predicted square-root covariance estimate, for $i = 1, \dots, N_2$, by

$$\begin{aligned}\tilde{\boldsymbol{\gamma}}_i^{k|k-1} &:= \log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right), \\ S_{\boldsymbol{\gamma}^{k|k-1}} &:= \left[\sqrt{w_1^{2,c}} \tilde{\boldsymbol{\gamma}}_1^{k|k-1}, \dots, \sqrt{w_N^{2,c}} \tilde{\boldsymbol{\gamma}}_N^{k|k-1} \right], \\ \sqrt{\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1}} &:= \text{tria} \left(\left[S_{\boldsymbol{\gamma}^{k|k-1}}, \sqrt{\boldsymbol{R}_k} \right] \right).\end{aligned}$$

(l) The predicted cross-covariance estimate by

$$\begin{aligned}\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} &:= \sum_{i=1}^{N_2} w_i^{2,cc} \left(\log_{\hat{\boldsymbol{x}}_{k|k-1}}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_i^{k|k-1} \right) \right) \right) \\ &\quad \times \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right) \right)^T.\end{aligned}$$

(m) The Kalman Gain by

$$\boldsymbol{G}_k := \left(\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} \right) \left(\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1} \right)^{-1}.$$

(n) The state's tangent corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k}^{TM} := \hat{\boldsymbol{x}}_{k|k-1}^{TM} + \boldsymbol{G}_k \log_{\hat{\boldsymbol{y}}_{k|k-1}}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{y}_k \right) \right).$$

(o) The state's corrected square-root covariance estimate at $\hat{\boldsymbol{x}}_{k|k-1}$ by

$$S_{\boldsymbol{\chi}^{k|k-1}} := \left[\sqrt{w_1^{2,c}} \tilde{\boldsymbol{\chi}}_1^{k|k-1}, \dots, \sqrt{w_N^{2,c}} \tilde{\boldsymbol{\chi}}_N^{k|k-1} \right];$$

$$\sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}} := \text{tria} \left(\left[S_{\chi^{k|k-1}} - \mathbf{G}_k S_{\gamma^{k|k-1}}, \mathbf{G}_k \sqrt{\mathbf{R}_k} \right] \right).$$

(p) The state's corrected estimate by

$$\hat{\mathbf{x}}_{k|k} := \exp_{\hat{\mathbf{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right) \right).$$

(q) The state's corrected square-root covariance estimate at $\hat{\mathbf{x}}_{k|k}$ by

$$\hat{\mathbf{P}}_{xx}^{k|k} := \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \quad (9.92)$$

Definition 9.17. Consider the system

$$\begin{aligned} \mathbf{x}_k &= \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \varpi_k \right], \\ \mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \vartheta_k \right]; \end{aligned}$$

with $\mathcal{M}_x^{n_x} = S^{n_x}$ and $\mathcal{M}_y = S^{n_y}$; and let e_x be the canonical basis of \mathbb{R}^{n_x+1} and e_y of \mathbb{R}^{n_y+1} . Suppose that i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned} \mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0 \right)_{S^{n_x}}, \\ \varpi_k &\sim \left(\bar{\varpi}_k, Q_k \right)_{S^{n_x}}, \\ \vartheta_k &\sim \left(\bar{\vartheta}_k, R_k \right)_{S^{n_y}}, \end{aligned}$$

with \mathbf{P}_{xx}^0 , Q_k , and R_k expressed in the differentiable structure in (A.1), and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian-Spheric Additive Unscented Kalman Filter (RiSAdUKF)* is given by the following algorithm:

Algorithm 26 (Riemannian-Spheric Additive Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) The state's tangent previous sigma representation by

$$\left\{ \chi_{i,k-1|k-1}^{TM}, w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := \sigma \mathbf{R}_1 \left([0]_{n_x \times 1}, \hat{\mathbf{P}}_{xx}^{k-1|k-1} \right).$$

(b) The state's previous sigma points by

$$\chi_i^{k-1|k-1} := \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_x} \left(\text{TBtoCB} \left(\chi_{i,k-1|k-1}^{TM} \right) \right), \quad i = 1, \dots, N_1. \quad (9.93)$$

(c) The state's predicted sigma points by

$$\boldsymbol{\chi}_{i,*}^{k|k-1} := f_k \left(\boldsymbol{\chi}_i^{k-1|k-1} \right), \quad i = 1, \dots, N_1.$$

(d) The state's predicted estimate by

$$\hat{\boldsymbol{x}}_{k|k-1}^* := \arg \min_{\boldsymbol{a} \in \mathcal{M}_x} \sum_{i=1}^{N_1} w_i^{1,m} \text{dist}^2 \left(\boldsymbol{\chi}_{i,*}^{k|k-1}, \boldsymbol{a} \right), \quad (9.94)$$

$$\hat{\boldsymbol{x}}_{k|k-1} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}^*} \bar{\boldsymbol{\omega}}_k. \quad (9.95)$$

(e) The state's predicted covariance estimate by

$$\hat{\boldsymbol{P}}_{xx}^{k|k-1} := \sum_{i=1}^{N_1} w_i^{1,c} \left(\log_{\hat{\boldsymbol{x}}_{k|k-1}^*}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_{i,*}^{k|k-1} \right) \right) \right) \left(\diamond \right)^T + Q_k. \quad (9.96)$$

(f) The regenerated state's predicted sigma points by

$$\left\{ \chi_{i,k|k-1}^{TM}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := \sigma R_2 \left([0]_{n_x \times 1}, \hat{\boldsymbol{P}}_{xx}^{k|k-1} \right);$$

and, for $i = 1, \dots, N_2$, by

$$\boldsymbol{\chi}_i^{k|k-1} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}^*}^{e_x} \left(\text{TBtoCB} \left(\chi_{i,k|k-1}^{TM} \right) \right). \quad (9.97)$$

(g) The measurement's predicted sigma points by

$$\boldsymbol{\gamma}_i^{k|k-1} := h_k \left(\boldsymbol{\chi}_i^{k|k-1} \right), \quad i = 1, \dots, N_2.$$

(h) The measurement's predicted estimate by

$$\hat{\boldsymbol{y}}_{k|k-1}^* := \arg \min_{\boldsymbol{b} \in \mathcal{M}_y} \sum_{i=1}^{N_2} w_i^{2,m} \text{dist}^2 \left(\boldsymbol{\gamma}_i^{k|k-1}, \boldsymbol{b} \right), \quad (9.98)$$

$$\hat{\boldsymbol{y}}_{k|k-1} := \exp_{\hat{\boldsymbol{y}}_{k|k-1}^*} \bar{\boldsymbol{\vartheta}}_k. \quad (9.99)$$

(i) The measurement's predicted covariance estimate by

$$\hat{\boldsymbol{P}}_{yy}^{k|k-1} := \sum_{i=1}^{N_2} w_i^{2,c} \left(\log_{\hat{\boldsymbol{y}}_{k|k-1}^*}^{e_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right) \right) \left(\diamond \right)^T + R_k. \quad (9.100)$$

(j) The predicted cross-covariance estimate by

$$\hat{\boldsymbol{P}}_{xy}^{k|k-1} := \sum_{i=1}^{N_2} w_i^{2,cc} \left(\log_{\hat{\boldsymbol{x}}_{k|k-1}^*}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_i^{k|k-1} \right) \right) \right)$$

$$\times \left(\log_{\underline{\mathbf{y}}_{k|k-1}}^{e_{\mathbf{y}}} \left(\text{CBtoTB} \left(\gamma_i^{k|k-1} \right) \right) \right)^T. \quad (9.101)$$

(k) The Kalman Gain by

$$\mathbf{G}_k := \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1}.$$

(l) The state's tangent corrected estimate by

$$\hat{\mathbf{x}}_{k|k}^{TM} := \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\underline{\mathbf{y}}_{k|k-1}}^{e_{\mathbf{y}}} \left(\text{CBtoTB} \left(\underline{\mathbf{y}}_k \right) \right). \quad (9.102)$$

(m) The state's corrected covariance estimate at $\hat{\mathbf{x}}_{k|k-1}$ by

$$\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} = \hat{\mathbf{P}}_{xx}^{k|k-1} - (\mathbf{G}_k) \hat{\mathbf{P}}_{yy}^{k|k-1} (\mathbf{G}_k)^T.$$

(n) The state's corrected estimate by

$$\hat{\mathbf{x}}_{k|k} := \exp_{\hat{\mathbf{x}}_{k|k-1}}^{e_{\mathbf{x}}} \left(\text{TBtoCB} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right) \right). \quad (9.103)$$

(o) The state's corrected covariance estimate at $\hat{\mathbf{x}}_{k|k}$ by

$$\hat{\mathbf{P}}_{xx}^{k|k} := \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \quad (9.104)$$

Definition 9.18. Consider the system

$$\begin{aligned} \mathbf{x}_k &= \exp_{f_k(\mathbf{x}_{k-1})} \left[\log_{f_k(\mathbf{x}_{k-1})} f_k(\mathbf{x}_{k-1}) + \varpi_k \right], \\ \mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \vartheta_k \right]; \end{aligned}$$

with $\mathcal{M}_{\mathbf{x}}^{n_x} = S^{n_x}$ and $\mathcal{M}_{\mathbf{y}}^{n_y} = S^{n_y}$; and let $e_{\mathbf{x}}$ be the canonical basis of \mathbb{R}^{n_x+1} and $e_{\mathbf{y}}$ of \mathbb{R}^{n_y+1} . Suppose that i) ϖ_k and ϑ_k are independent; ii) ϖ_k , ϑ_k and the initial state \mathbf{x}_0 are characterized by

$$\begin{aligned} \mathbf{x}_0 &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{xx}^0} \sqrt{\mathbf{P}_{xx}^0}^T \right)_{S^{n_x}}, \\ \varpi_k &\sim \left(\bar{\varpi}_k, \sqrt{Q_k} \sqrt{Q_k}^T \right)_{S^{n_x}}, \\ \vartheta_k &\sim \left(\bar{\vartheta}_k, \sqrt{R_k} \sqrt{R_k}^T \right)_{S^{n_y}}, \end{aligned}$$

with $\sqrt{\mathbf{P}_{xx}^0}$, $\sqrt{Q_k}$, and $\sqrt{R_k}$ expressed in the differentiable structure in (A.1), and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Then the *Riemannian-Spheric Additive Square-Root Unscented Kalman Filter (RiSAdSRUKF)* is given by the following algorithm:

Algorithm 27 (Riemannian-Spheric Additive Square-Root Unscented Kalman Filter).

Perform the following steps:

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's tangent previous sigma representation by*

$$\left\{ \chi_{i,k-1|k-1}^{TM}, w_i^{1,m}, w_i^{1,c}, \bullet \right\}_{i=1}^{N_1} := \sigma R_1 \left([0]_{n_x \times 1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1}} \sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1} T} \right).$$

(b) *The state's previous sigma points by*

$$\boldsymbol{\chi}_i^{k-1|k-1} := \exp_{\hat{\mathbf{x}}_{k-1|k-1}}^{e_x} \left(\text{TBtoCB} \left(\chi_{i,k-1|k-1}^{TM} \right) \right), \quad i = 1, \dots, N_1. \quad (9.105)$$

(c) *The state's predicted sigma points by*

$$\boldsymbol{\chi}_{i,*}^{k|k-1} := f_k \left(\boldsymbol{\chi}_i^{k-1|k-1} \right), \quad i = 1, \dots, N_1.$$

(d) *The state's predicted estimate by*

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1}^* &:= \arg \min_{\mathbf{a} \in \mathcal{M}_x} \sum_{i=1}^{N_1} w_i^{1,m} \text{dist}^2 \left(\boldsymbol{\chi}_{i,*}^{k|k-1}, \mathbf{a} \right), \quad (9.106) \\ \hat{\mathbf{x}}_{k|k-1} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}^*} \bar{\boldsymbol{\omega}}_k. \end{aligned}$$

(e) *The state's predicted square-root covariance estimate, for $i = 1, \dots, N_1$, by*

$$\begin{aligned} \tilde{\boldsymbol{\chi}}_{i,*}^{k|k-1} &:= \log_{\hat{\mathbf{x}}_{k|k-1}^*}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_{i,*}^{k|k-1} \right) \right), \\ S_{\boldsymbol{\chi}^{k|k-1}}^* &:= \left[\sqrt{w_1^{1,c}} \tilde{\boldsymbol{\chi}}_{1,*}^{k|k-1}, \dots, \sqrt{w_N^{1,c}} \tilde{\boldsymbol{\chi}}_{N,*}^{k|k-1} \right], \\ \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}} &:= \text{tria} \left(\left[S_{\boldsymbol{\chi}^{k|k-1}}^*, \sqrt{Q_k} \right] \right). \end{aligned}$$

(f) *The regenerated state's previous sigma points by*

$$\left\{ \chi_{i,k|k-1}^{TM}, w_i^{2,m}, w_i^{2,c}, w_i^{2,cc} \right\}_{i=1}^{N_2} := \sigma R_2 \left([0]_{n_x \times 1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}} \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1} T} \right);$$

and, for $i = 1, \dots, N_2$, by

$$\begin{aligned} \boldsymbol{\chi}_i^{k-1|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}^*}^{e_x} \left(\text{TBtoCB} \left(\chi_{i,k|k-1}^{TM} \right) \right), \\ \tilde{\boldsymbol{\chi}}_i^{k|k-1} &:= \log_{\hat{\mathbf{x}}_{k|k-1}^*}^{e_x} \left(\text{CBtoTB} \left(\boldsymbol{\chi}_i^{k-1|k} \right) \right), \end{aligned}$$

$$S_{\boldsymbol{\chi}^{k|k-1}} := \left[\sqrt{w_1^{2,c} \tilde{\boldsymbol{\chi}}_1^{k|k-1}}, \dots, \sqrt{w_N^{2,c} \tilde{\boldsymbol{\chi}}_N^{k|k-1}} \right].$$

(g) The measurement's predicted sigma points by

$$\boldsymbol{\gamma}_i^{k|k-1} := h_k \left(\boldsymbol{\chi}_i^{k|k-1} \right), \quad i = 1, \dots, N_2.$$

(h) The measurement's predicted estimate by

$$\begin{aligned} \hat{\boldsymbol{y}}_{k|k-1}^* &:= \arg \min_{\boldsymbol{b} \in \mathcal{M}_y} \sum_{i=1}^{N_2} w_i^{2,m} \text{dist}^2 \left(\boldsymbol{\gamma}_i^{k|k-1}, \boldsymbol{b} \right), \\ \hat{\boldsymbol{y}}_{k|k-1} &:= \exp_{\hat{\boldsymbol{y}}_{k|k-1}^*} \bar{\boldsymbol{v}}_k. \end{aligned} \quad (9.107)$$

(i) The measurement's predicted square-root covariance estimate, for $i = 1, \dots, N_2$, by

$$\begin{aligned} \tilde{\boldsymbol{\gamma}}_i^{k|k-1} &:= \log_{\hat{\boldsymbol{y}}_{k|k-1}^*}^{\boldsymbol{e}_y} \left(\text{CBtoTB} \left(\boldsymbol{\gamma}_i^{k|k-1} \right) \right), \\ S_{\boldsymbol{\gamma}^{k|k-1}} &:= \left[\sqrt{w_1^{2,c} \tilde{\boldsymbol{\gamma}}_1^{k|k-1}}, \dots, \sqrt{w_{N_2}^{2,c} \tilde{\boldsymbol{\gamma}}_{N_2}^{k|k-1}} \right], \\ \sqrt{\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1}} &:= \text{tria} \left(\left[S_{\boldsymbol{\gamma}^{k|k-1}}, \sqrt{R_k} \right] \right). \end{aligned}$$

(j) The predicted cross-covariance by

$$\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} := \sum_{i=1}^{N_2} w_i^{cc} \left(\tilde{\boldsymbol{\chi}}_i^{k|k-1} \right) \left(\tilde{\boldsymbol{\gamma}}_i^{k|k-1} \right)^T.$$

(k) The Kalman Gain by

$$\boldsymbol{G}_k := \left(\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{y}}^{k|k-1} \right) \left(\hat{\boldsymbol{P}}_{\boldsymbol{y}\boldsymbol{y}}^{k|k-1} \right)^{-1}.$$

(l) The state's tangent corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k}^{TM} := \hat{\boldsymbol{x}}_{k|k-1}^{TM} + \boldsymbol{G}_k \log_{\hat{\boldsymbol{y}}_{k|k-1}^*}^{\boldsymbol{e}_y} \left(\text{CBtoTB} \left(\boldsymbol{y}_k \right) \right).$$

(m) The state's corrected square-root covariance estimate at $\hat{\boldsymbol{x}}_{k|k-1}$ by

$$\sqrt{\hat{\boldsymbol{P}}_{\boldsymbol{x}\boldsymbol{x}}^{k|k-1}} := \text{tria} \left(\left[S_{\boldsymbol{\chi}^{k|k-1}} - \boldsymbol{G}_k S_{\boldsymbol{\gamma}^{k|k-1}}, \boldsymbol{G}_k \sqrt{R_k} \right] \right).$$

(n) The state's corrected estimate by

$$\hat{\boldsymbol{x}}_{k|k} := \exp_{\hat{\boldsymbol{x}}_{k|k-1}}^{\boldsymbol{e}_x} \left(\text{TBtoCB} \left(\hat{\boldsymbol{x}}_{k|k}^{TM} \right) \right).$$

(o) The state's corrected square-root covariance estimate at $\hat{\mathbf{x}}_{k|k}$ by

$$\sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x}}^{k|k-1}} := \text{PT} \left(\sqrt{\hat{\mathbf{P}}_{\mathbf{x}\mathbf{x}}^{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \quad (9.108)$$

The Riemannian sample means are the only elements in these algorithms that still have to be calculated by approximations or algorithms. The following Riemannian sample means are required:

- $\hat{\mathbf{x}}_{k|k-1}$ in (9.87) and $\hat{\mathbf{y}}_{k|k-1}$ in (9.88) for the RiAuUKF;
- $\hat{\mathbf{x}}_{k|k-1}$ in (9.90) and $\hat{\mathbf{y}}_{k|k-1}$ in (9.91) for the RiAuSRUKF;
- $\hat{\mathbf{x}}_{k|k-1}$ in (9.94) and $\hat{\mathbf{y}}_{k|k-1}$ in (9.98) for the RiAdUKF;
- $\hat{\mathbf{x}}_{k|k-1}$ in (9.106) and $\hat{\mathbf{y}}_{k|k-1}$ in (9.107) for the RiAdSRUKF;

Examples of numeric techniques for computing these means are the five methods for weighted means of unit quaternions presented in Section 7.2.2 (the FN, DPPSE, GDA, MQVCF, and MAMCF; cf. Table 7.4), or the optimization algorithms presented in [174] (e.g. their Newton's method, or trust-region methods).

Other cases such as Riemannian Unscented filters for products of Euclidean spaces and spheres can be obtained from these Riemannian-Spheric UF's by making simple extensions.

9.5.1 Riemannian-Spheric Unscented filters and Quaternionic Unscented filters

In Section 7.3, we introduced the Quaternionic Unscented filters (QuAdUF's), namely the Quaternionic Additive Unscented Kalman Filter (QuAdUKF, Algorithm 17) and the Quaternionic Additive Square-Root Unscented Kalman Filter (QuAdSRUKF, Algorithm 18)—recall that QuAdUF's are defined for systems where the state variables belong to $S^3 \times \mathbb{R}^n$, but for simplicity we will restrict the discussion of this section to state variables belonging only to the S^3 ; this does not imply any loss of generality for this discussion. It is natural to ask for the relation between these filters and the Riemannian-Spheric Additive Unscented filters (RiSAdUF's)—the RiSAdUKF and RiSAdSRUKF. We can point out at least six advantages of RiSAdUF's comparative with QuAdUKF's:

1. RiSAdUF's *preserve distances and angles*, but the QuAdUKF's *with i) generalized Rodrigues vectors*, and *ii) quaternion vectors* do not. While all operations in the

RiSAdUF's *are isometries* (functions preserving distances and angles; cf. Section A.2), in the QuAdUF's the Quaternionic functions QtoGeRV and QtoQuV (and their inverses) *are not isometries*; indeed the distance from the quaternion $\mathbf{1} = (1, 0, 0, 0)$ to the quaternion $-\mathbf{1} = (-1, 0, 0, 0)$ is

$$\text{dist}(\mathbf{1}, -\mathbf{1}) = \pi,$$

but the distance from the transforms of these quaternions by the QtoQuV is

$$\text{dist}(\text{QtoQuV}(\mathbf{1}), \text{QtoQuV}(-\mathbf{1})) = \|[0]_{3 \times 1} - [0]_{3 \times 1}\| = 0,$$

and by the QtoGeRV is

$$\text{dist}(\text{QtoGeRV}(\mathbf{1}), \text{QtoGeRV}[-\mathbf{1}]) = \|[0]_{3 \times 1} - [0]_{3 \times 1}\| = 0.$$

2. RiSAdUF's are *more robust to miss-defined operations* than the QuAdUF's *with rotation vectors*. For the tangent vector $v = [v_1, v_2, v_3, v_4]^T \in T_{\mathbf{1}}S^{n-1}$ we have that—using the canonical basis of the embedding space \mathbb{R}^4 —

$$0 = \langle \mathbf{1}, v \rangle = v_1;$$

hence

$$\begin{aligned} \exp_{\mathbf{1}}(v) &:= \cos(\|v\|) \mathbf{1} + \sin(\|v\|) \frac{v}{\|v\|} \\ &= \begin{bmatrix} \cos(\|v\|) \\ \sin(\|v\|) \frac{v_2}{\|v\|} \\ \sin(\|v\|) \frac{v_3}{\|v\|} \\ \sin(\|v\|) \frac{v_4}{\|v\|} \end{bmatrix}. \end{aligned}$$

For the function e defined by

$$e : \mathbb{B}_{[0]_{(n-1) \times 1}}(\pi) \rightarrow S^{n-1} - \{-\mathbf{1}\} : x \mapsto \exp_{\mathbf{1}} \begin{bmatrix} 0 \\ \frac{1}{2}x \end{bmatrix} := \begin{bmatrix} \cos\left(\frac{\|x\|}{2}\right) \\ \sin\left(\frac{\|x\|}{2}\right) \frac{x}{\|x\|} \end{bmatrix}, \quad (9.109)$$

we have that

$$e(x) := \text{RoVtoQ}(x).$$

Therefore, the RoVtoQ can be viewed as the Riemannian exponential mapping on S^{n-1} at $\mathbf{1}$ (thus is a isometry); likewise, the function QtoRoV defined in (7.5) is equal $e^{-1}(x)$ and can be viewed as Riemannian logarithm mapping on S^{n-1} at $\mathbf{1}$.

Both the RiSAdUF's and the QuAdUF's *may present bad-definition problems* when mappings $\exp_{\mathbf{q}}$ and $\log_{\mathbf{q}}$ are realized at points where the distance from \mathbf{q} is equal or greater than π (we can consider more than one turn on the sphere). Indeed, the logarithm map $\log_{\mathbf{q}}(\mathbf{p})$ is not defined at the *antipodal point* $\mathbf{p} = -\mathbf{q}$ [cf. (A.23)], and the exponential map $\exp_{\mathbf{q}}(v)$ is not defined for vectors $v \in T_{\mathbf{q}}S^{n-1}$ with $\|v\| \geq \pi$ [cf. (A.22)].

Nonetheless, *the RiSAdUF's are more robust to these problems than the QuAdUF's with rotation vectors.*

In the QuAdUF's with rotation vectors, the exponential and logarithm mappings are always calculated *at the point* $\mathbf{q} = \mathbf{1}$ [cf. (9.109)]. Therefore, for the QuAdUF's, at any time in the *history of the system*, whenever i) the state or the measurement are calculated at $-\mathbf{1}$ the logarithm mapping $\log_{\mathbf{1}}$ *will be undefined*, and ii) the system perform a complete turn on the sphere, the exponential mapping $\exp_{\mathbf{1}}$ *will be undefined* (the distance from $\mathbf{1}$ will be equal or greater than π). On the other hand, in the RiSAdUF's, the exponential and logarithm mappings are calculated *at different points* (not always at $\mathbf{q} = \mathbf{1}$). For instance, in the RiSAdUKF, these mappings are calculated at $\hat{\mathbf{x}}_{k-1|k-1}$ in (9.93); at $\hat{\mathbf{x}}_{k|k-1}$ in (9.95), (9.96), (9.97), (9.101) and (9.103); at $\hat{\mathbf{y}}_{k-1|k}$ in (9.99), (9.100), (9.101) and (9.102); and at $\hat{\mathbf{x}}_{k|k}$ in (9.104).

Because these mappings are realized at different points in RiSAdUF's, the bad-definition problems of these mappings are *less likely* to happen in the RiSAdUF's than in the QuAdUF's with rotation vectors. Let us compare, for instance, the equation (9.103) of the RiSAdUKF with (7.39) of the QuAdUKF. In (9.103), if

$$\left\| \hat{\mathbf{x}}_{k|k}^{TM} \right\| \geq \pi, \quad (9.110)$$

then

$$\exp_{\hat{\mathbf{x}}_{k|k-1}}^{e_x} \left(\text{TBtoCB} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right) \right)$$

will be undefined (TBtoCB is just a change of basis and do not change the value of norm); and in (7.39), if

$$\left\| \tilde{\hat{\mathbf{x}}}_{k|k}^v \right\| \geq \pi, \quad (9.111)$$

than

$$\text{VtoQ} \left(\tilde{\hat{\mathbf{x}}}_{k|k}^v \right)$$

will be undefined (for the QuAdUKF, $\text{VtoQ} = \text{RoVtoQ}$). For (9.110) to be true, the corrected estimate $\hat{\mathbf{x}}_{k|k}$ would have to be at least a complete turn away *from* $\hat{\mathbf{x}}_{k|k-1}$; for (9.111) to be true, $\hat{\mathbf{x}}_{k|k}$ would have to be at least a complete turn away *from* $\mathbf{1}$. Naturally, it is more likely for (9.111) to be true, than for (9.110) to be, when *all the history* of the system is considered.

3. The (corrected) *state's covariance estimates* $\hat{\mathbf{P}}_{xx}^{k|k}$ in the RiSAdUF's *are more significant* than the (corrected) state's covariance estimates $\hat{P}_{xx}^{v,k|k}$ in the QuAdUF's. The matrix $\hat{\mathbf{P}}_{xx}^{k|k}$ is an estimate of the covariance $\mathbf{P}_{xx}^{k|k}$ of \mathbf{x}_k at $\hat{\mathbf{x}}_{k|k}$, and from (8.8), we can see that $\text{Tr}(\hat{\mathbf{P}}_{xx}^{k|k})$ provides an estimate of the error of the estimate $\hat{\mathbf{x}}_{k|k}$.

However, for $\hat{P}_{xx}^{v,k|k}$, we can not find a similar relation between $\hat{P}_{xx}^{v,k|k}$ and the error of the estimate $\hat{\mathbf{x}}_{k|k}$ because $\hat{P}_{xx}^{v,k|k}$ are calculated on a parameterization *at* $\hat{\mathbf{x}}_{k|k-1}$; a covariance calculated *at* $\hat{\mathbf{x}}_{k|k}$ is required for this relation to be made.

We highlight that, in the RiSAdUF, $\hat{\mathbf{P}}_{xx}^{k|k}$ is calculated by performing the parallel transport of the state's covariance estimate $\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}$ from $T_{\hat{\mathbf{x}}_{k|k-1}} \mathcal{M}_x \times T_{\hat{\mathbf{x}}_{k-1}} \mathcal{M}_x$ to $T_{\hat{\mathbf{x}}_{k|k}} \mathcal{M}_x \times T_{\hat{\mathbf{x}}_{k|k}} \mathcal{M}_x$ [cf. equation (9.104)]; and, in the RiSAdSRUKF, $\sqrt{\hat{\mathbf{P}}_{xx}^{k|k}}$ is calculated by performing the parallel transport of $\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}}$ from $T_{\hat{\mathbf{x}}_{k|k-1}} \mathcal{M}_x \times T_{\hat{\mathbf{x}}_{k-1}} \mathcal{M}_x$ to $T_{\hat{\mathbf{x}}_{k|k}} \mathcal{M}_x \times T_{\hat{\mathbf{x}}_{k|k}} \mathcal{M}_x$ [cf. equation (9.108)].

4. The previous state's $\sigma\mathbf{R}$'s $\boldsymbol{\chi}^{k-1|k-1} := \{\chi_i^{k-1|k-1}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc}\}$ are *better defined* in the RiSAdUF's than the previous state's quaternion sets $\boldsymbol{\chi}^{k-1|k-1} := \{\chi_i^{k-1|k-1}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc}\}$ in the QuAdUF's.

In the RiSAdUF's, $\boldsymbol{\chi}^{k-1|k-1}$ is obtained from $\chi_{k-1|k-1}^{TM} := \{\chi_{i,k-1|k-1}^{TM}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc}\}$ [cf. equations (9.93) and (9.105)], and $\chi_{k-1|k-1}^{TM}$ is a $\sigma\mathbf{R}$ of $x_k \sim ([0]_{n_x \times 1}, \hat{\mathbf{P}}_{xx}^{k-1|k-1})$. From Theorem 9.1, we know that $\boldsymbol{\chi}^{k-1|k-1}$ is a *Ri $\sigma\mathbf{R}$* of $\mathbf{x}_{k-1} \sim (\hat{\mathbf{x}}_{k-1,k-1}, \hat{\mathbf{P}}_{xx}^{k-1|k-1})$.

Even though a similar relation between $\boldsymbol{\chi}^{k-1|k-1}$ and \mathbf{x}_{k-1} can be established for the QuAdUF's with *rotations vectors*, this relation is not *consistent* with the filter. In the QuAdUF's, $\boldsymbol{\chi}^{k-1|k-1}$ is obtained from $\tilde{\chi}^{v,k-1|k-1} := \{\tilde{\chi}_i^{v,k-1|k-1}, w_i^{1,m}, w_i^{1,c}, w_i^{1,cc}\}$ [cf. equations 7.38 and 7.40], and $\tilde{\chi}^{v,k-1|k-1}$ is a $\sigma\mathbf{R}$ of $x_k \sim ([0]_{n_x \times 1}, \hat{P}_{xx}^{v,k|k})$. Since the mapping RoVtoQ can be viewed as the Riemannian exponential mapping on S^{n-1} at $\mathbf{1}$ [cf. equation 9.109], each $\tilde{\chi}_i^{v,k-1|k-1}$ belongs to the tangent space $T_1 S^{n-1}$. Again, from Theorem 9.1, we know that $\boldsymbol{\chi}^{k-1|k-1}$ is a *Ri $\sigma\mathbf{R}$* of $\mathbf{x}_{k-1} \sim (\hat{\mathbf{x}}_{k-1,k-1}, \hat{P}_{xx}^{v,k|k})$ with $\hat{P}_{xx}^{v,k|k} = \hat{\mathbf{P}}_{xx}^{k-1|k-1, \hat{\mathbf{x}}_{k|k-1}}$, i.e., with $\hat{P}_{xx}^{v,k|k}$ being the covariance of \mathbf{x}_{k-1} *at* $\hat{\mathbf{x}}_{k|k-1}$ (cf. item 3). We believe that this may lead the QuAdUF's with rotation vectors to provide poor estimates.

In this case of QuAdUF's with generalized Rodrigues vectors or Quaternion vectors, it is difficult to state a similar relation between $\boldsymbol{\chi}^{k-1|k-1}$ and the state \mathbf{x}_{k-1} because i) QtoGeRV and QtoQuV are not Riemannian exponentials, and ii) $\hat{P}_{xx}^{v,k|k}$ is calculated on a parameterization *at* $\hat{\mathbf{x}}_{k|k-1}$ (cf. item 3).

5. In the RiSAdUF's the *concepts of probability and statistic theories* are well defined, whereas in the QuAdUF's, some of them *are not*. The RiSAdUF's are built upon the well defined Riemannian random points and statistics of Chapter 8, but

the QuAdUF's were not build upon any kind of definition of the random variables and their statistics in the S^{n-1} . Theorem 9.1 provides the relation between a $\sigma\mathbb{R}$ on the tangent space of S^{n-1} and its associated $\text{Ri}\sigma\mathbb{R}$ on the S^{n-1} , but we do not have a similar result for any of the QuAdUF's.

6. The *additive* Riemannian system is *well defined* for RiSAdUF's, while the additive quaternion model for the QuAdUF's *is not*. The additive system for the RiSAdUF's is defined in (9.20) and is built upon the intrinsic statistics for Riemannian manifolds presented in Chapter 8. On the other hand, all additive-noise quaternion models associated QuAdUF's *present problems* (cf. Remark 7.1).

Altogether, we can say that RiSAdUF's are better principled than QuAdUF's. We illustrate this statement in numerical simulations.

We now perform simulations comparing a RiSAdUKF with the USQUE of [48] (which is a QuAdUKF) in the same satellite scenario of Section 7.4.

In this example, the scenario is configured according to [48]: $T = 10\text{s}$, $\sigma_\omega = 0.31623 \mu\text{rad} \times \text{s}^{-1/2}$, $\sigma_\beta = 3.1623 \times 10^{-4} \mu\text{rad} \times \text{s}^{-3/2}$, $\beta_0 = [0.1]_{3 \times 1} \text{ deg/hr}$, $\sigma_v = 50 \text{ nT}$, $\hat{e}_0 = 1$, $\hat{\beta}_0 = \beta_0 + [0, 20, 0]^T \text{ deg/hr}$, and

$$\hat{P}_{xx}^{\rho,0|0} = \begin{bmatrix} (\sigma_{xx}^{0|0,e})^2 I_{3 \times 3} & [0]_{3 \times 3} \\ [0]_{3 \times 3} & (\sigma_{xx}^{0|0,\beta})^2 I_{3 \times 3} \end{bmatrix}$$

with $\sigma_{xx}^{0|0,e} = 5 \text{ deg}$ and $\sigma_{xx}^{0|0,\beta} = 20 \text{ deg/hr}$.

The setting parameters of the filters were:

1. the USQUE of [48] with $a = 1$ and $\lambda = 1$;
2. the RiSAdUKF with the RhoRi $\sigma\mathbb{R}$ and tuning parameter $\rho = 0.5$; and using the Direct Propagation of the Previous State's Estimate (Section 7.2.2.2);

The RiSAdUKF outperformed the USQUE of [48]. This outperformance can be verified both qualitatively and quantitatively.

Qualitatively, this outperformance can be seen in Figure 9.1. This figure presents, for one simulation of each e_1 , e_2 , e_3 and e_4 , the plots of the i) correct path (in black, solid line), ii) estimates of the RiSAdUKF (in red, dot line), and iii) estimates of the USQUE of [48] (in blue, dash-dot line). The plot of the RiSAdUKF's estimates is almost *indistinguishable* from the correct path's plot (they are overlapped), but the plot of the USQUE's estimates is clearly displaced from the correct path's plot.

Quantitatively, this outperformance can be seen from the RMSD [defined in (5.42)] of (7.43) and the RMST [defined in (7.44)] of the estimates of these two filters: for $N_{it} =$

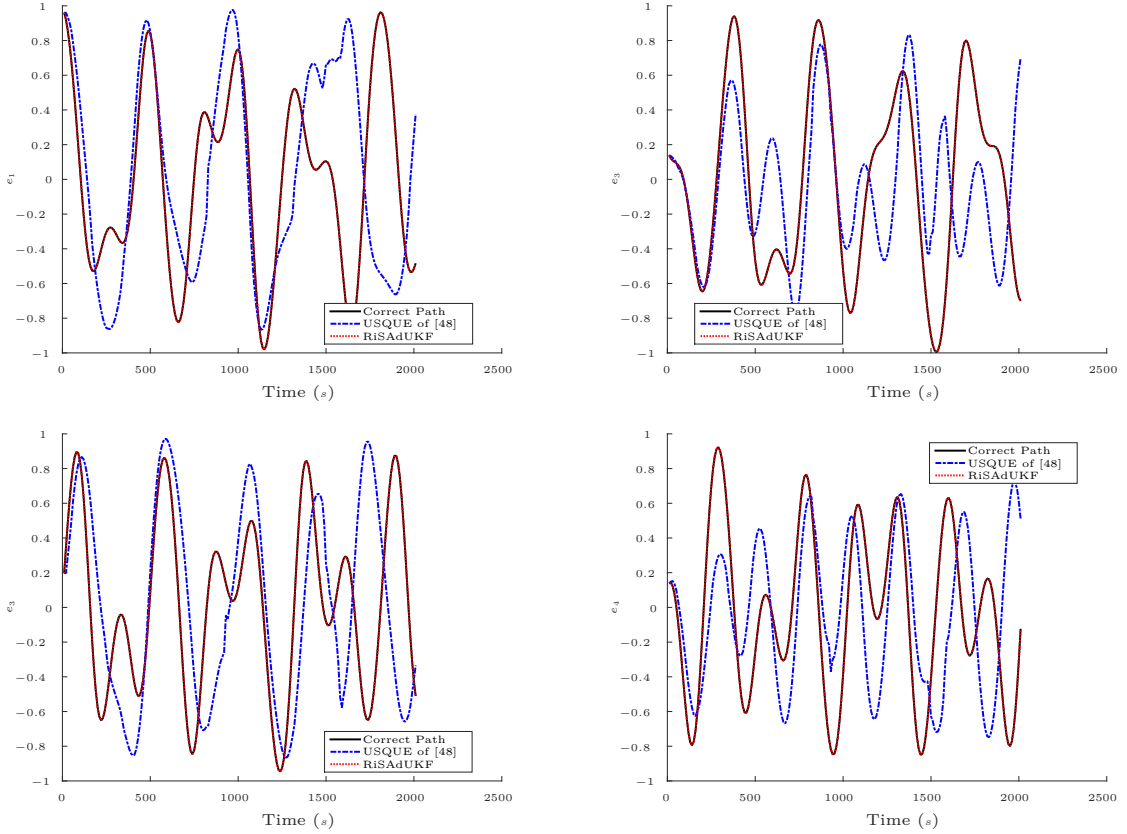


Figure 9.1: Values of e_1 , e_2 , e_3 and e_4 for the new RiSAdUKF and the USQUE for a problem of satellite attitude estimation.

201 iterations and $N_s = 1000$ simulations, i) the RiSAdUKF's RMSD is, approximately, 1.541×10^{-3} , and the USQUE's RMSD is 1.522×10^{-1} ; and ii) the RiSAdUKF's RMST is 3.37×10^{-6} , and the USQUE's RMST is, approximately, 4.23×10^{-6} . The USQUE's RMSD is, approximately, 100 times the RiSAdUKF's RMSD!

9.6 RIEMANNIAN UNSCENTED FILTERING FOR STATE VARIABLES BEING UNIT DUAL QUATERNIONS

Throughout Chapter 8 and in Section 9.5, we studied Unscented filters for quaternions system. Representing rotations of 3-dimensional bodies with unit quaternions may have some advantages comparative with other representations of rotations (cf. Chapter 8). The good properties of the unit quaternions for representing rotations can be extended to full movement of rigid bodies—a translation and a rotation of a 3-dimensional rigid body—by the so called *unit dual quaternions*.

A dual quaternion $\underline{\mathbf{q}}$ can be written in the form

$$\underline{\mathbf{q}} = \mathcal{P}(\underline{\mathbf{q}}) + \varepsilon \mathcal{D}(\underline{\mathbf{q}}),$$

where $\mathcal{P}(\underline{\mathbf{q}}) \in \mathbb{H}$ is called the *primary part* of $\underline{\mathbf{q}}$, the $\mathcal{D}(\underline{\mathbf{q}}) \in \mathbb{H}$ the *dual part* of $\underline{\mathbf{q}}$, and ε the *dual unit*; ε is a generalized complex number such that $\varepsilon^2 = 0$ [35]. The dual quaternion algebra will be denoted as \mathcal{H} .

The addition and multiplication between two dual quaternions are defined as follows:

$$\begin{aligned} \underline{\mathbf{q}} \pm \underline{\mathbf{p}} &:= \mathcal{P}(\underline{\mathbf{q}}) \pm \mathcal{P}(\underline{\mathbf{p}}) + \varepsilon [\mathcal{D}(\underline{\mathbf{q}}) \pm \mathcal{D}(\underline{\mathbf{p}})]; \\ \underline{\mathbf{q}} \underline{\mathbf{p}} &:= [\mathcal{P}(\underline{\mathbf{q}}) + \varepsilon \mathcal{D}(\underline{\mathbf{q}})] [\mathcal{P}(\underline{\mathbf{p}}) + \varepsilon \mathcal{D}(\underline{\mathbf{p}})]. \end{aligned}$$

The conjugate $\underline{\mathbf{q}}^*$ of a dual quaternion $\underline{\mathbf{q}}$ is defined using the conjugate of quaternions in the following way:

$$\underline{\mathbf{q}}^* := \mathcal{P}(\underline{\mathbf{q}})^* + \varepsilon \mathcal{D}(\underline{\mathbf{q}})^*;$$

with the conjugate, we can define the following function

$$\begin{aligned} \|\underline{\mathbf{q}}\| &:= \underline{\mathbf{q}} \underline{\mathbf{q}}^* \\ &= \underline{\mathbf{q}}^* \underline{\mathbf{q}}. \end{aligned}$$

$\|\underline{\mathbf{q}}\|$ is also a dual scalar number of the form

$$\|\underline{\mathbf{q}}\| = a + \varepsilon b, \quad a, b \in [0, \infty);$$

even though this function $\|\|\|$ is *not* a norm (positive function with the triangle inequality), it is generally named as the *pseudo-norm* of $\underline{\mathbf{q}}$ or, for simplicity, just as the *norm* of $\underline{\mathbf{q}}$.

If the norm of a dual quaternion $\underline{\mathbf{q}}$ is equal to $1 + \varepsilon 0 = 1$, then we call $\underline{\mathbf{q}}$ an *unit dual quaternion*. The set of all unit dual quaternions forms a quadric—essentially, a quadric is a set comprising the zeros of a polynomial of degree 2—and is called the Study Quadric [175]; we will denote the Study Quadric by $\mathcal{H}^{\|1\|}$.

While unit quaternions can represent rotations of 3-dimensional rigid bodies, *unit dual quaternions can represent full movements*; full movements of rigid bodies are compositions of rotations and translations. In fact, any unit dual quaternion can be written in the form

$$\underline{\mathbf{q}} = \mathbf{q} + \varepsilon \frac{1}{2} \mathbf{q} \mathbf{q}, \quad \mathbf{q} \in S^3, \mathbf{q} \in \mathbb{R}^3;$$

where $\underline{\mathbf{q}}$ represents the rigid body displacement composed of the rotation \mathbf{q} and translation q [35].

Naturally, rigid body displacements can be represented by other elements, such as homogeneous transformation matrices. These matrices are the natural way of describing rigid body displacements in homogeneous coordinates. The group formed by these matrices along with the usual matrix product is called the Special Euclidean Group and usually denoted by $SE(3)$.

Nevertheless, unit dual quaternions present benefits comparative with other representations of rigid body displacements. For instance, comparative with homogeneous transformation matrices, unit dual quaternions present *computational advantages*; among other advantages, the cost of some important operations, such as calculating Jacobians and forward kinematics, is smaller for unit dual quaternions than for homogeneous transformation matrices [35].

Because of the good properties of the unit dual quaternions when representing rigid body motions, we develop Unscented filters for systems composed by them.

9.6.1 Riemannian UKF for dual quaternions

In order to define a UKF for dual quaternions, we must describe a stochastic dynamic system with the random variables belonging to the $\mathcal{H}^{\parallel 1\parallel}$. We are unaware of any *probability and statistic theory* for unit dual quaternions, but we can use the one developed for Riemannian manifolds.

For a unit dual quaternion $\underline{\mathbf{q}} = \mathbf{q} + \varepsilon \frac{1}{2} \mathbf{q} \mathbf{q}$, $\mathbf{q} \in S^3$, $q \in \mathbb{R}^3$, the function

$$\begin{aligned} \psi : \mathcal{H}^{\parallel 1\parallel} &\rightarrow S^3 \times \mathbb{R}^3 \\ \underline{\mathbf{q}} &\mapsto [\mathbf{q}, q]^T \end{aligned} \tag{9.112}$$

is one-to-one, and its inverse is

$$\begin{aligned} \psi^{-1} : S^3 \times \mathbb{R}^3 &\rightarrow \mathcal{H}^{\parallel 1\parallel} \\ [\mathbf{q}, q]^T &\mapsto \underline{\mathbf{q}} := \mathbf{q} + \varepsilon \frac{1}{2} \mathbf{q} \mathbf{q}. \end{aligned}$$

Because ψ maps unit dual quaternion uniquely to a the Riemannian manifold $S^3 \times \mathbb{R}^3$, we can construct UKF's for the $\mathcal{H}^{\parallel 1\parallel}$ using the Riemannian-Spheric Unscented Filters of Section 9.5. Define the following system:

$$\begin{aligned}\underline{\mathbf{x}}_k &= f_k(\underline{\mathbf{x}}_{k-1}, \underline{\boldsymbol{\varpi}}_k), \\ \underline{\mathbf{y}}_k &= h_k(\underline{\mathbf{x}}_k, \underline{\boldsymbol{\vartheta}}_k),\end{aligned}\tag{9.113}$$

where k is the time step; $\underline{\mathbf{x}}_k$ the internal state characterized by the Riemannian random point $\psi(\underline{\mathbf{x}}_k) \in \Phi_{S^3 \times \mathbb{R}^3}$; $\underline{\mathbf{y}}_k$ is the measured output characterized by $\psi(\underline{\mathbf{y}}_k) \in \Phi_{S^3 \times \mathbb{R}^3}$; $\underline{\boldsymbol{\varpi}}_k \in \Phi_{\mathcal{M}_{\boldsymbol{\varpi}}}$ the process noise; and $\underline{\boldsymbol{\vartheta}}_k \in \Phi_{\mathcal{M}_{\boldsymbol{\vartheta}}}$ the measurement noise. The system (9.113) will be called *dual-quaternion (stochastic, discrete-time, dynamic) system*.

We can define an additive variant of system (9.113) in a way similar to the case of Riemannian manifolds. Define the following operation between a dual quaternion $\underline{\mathbf{q}}$ characterized by $\psi(\underline{\mathbf{q}}) \sim (\bar{\mathbf{q}}, \mathbf{P}_q)_{S^3 \times \mathbb{R}^3}$ and an Euclidean random vector $p \sim (\bar{p}, P_p)_{\mathbb{R}^6}$

$$\psi(\underline{\mathbf{q}} \boxplus p) \sim (\exp_{\bar{\mathbf{q}}} \bar{p}, \mathbf{P}_q + P_p).$$

Then the additive variant of (9.19) is a system in the form

$$\begin{aligned}\underline{\mathbf{x}}_k &= f_k(\underline{\mathbf{x}}_{k-1}) \boxplus \underline{\boldsymbol{\varpi}}_k, \\ \underline{\mathbf{y}}_k &= h_k(\underline{\mathbf{x}}_k) \boxplus \underline{\boldsymbol{\vartheta}}_k,\end{aligned}\tag{9.114}$$

with $\underline{\boldsymbol{\varpi}}_k = \boldsymbol{\varpi}_k$, $\mathcal{M}_{\underline{\boldsymbol{\varpi}}_k} = \mathbb{R}^6$, $\underline{\boldsymbol{\vartheta}}_k = \boldsymbol{\vartheta}_k$, and $\mathcal{M}_{\underline{\boldsymbol{\vartheta}}_k} = \mathbb{R}^6$.

Since $S^3 \times \mathbb{R}^3$ is a product of two Riemannian manifolds, it is also a Riemannian manifold. Let $\exp_{\mathbf{q}}^{S^3}$ be the Riemannian exponential application in the S^3 at $\mathbf{q} \in S^3$ —one expression is given in (A.22)—, and $\exp_{\mathbf{q}}^{\mathbb{R}^3}$ the Riemannian exponential of \mathbb{R}^3 at $\mathbf{q} \in \mathbb{R}^3$ —see (A.20)—; then the Riemannian exponential $\exp_{[\mathbf{q}, \mathbf{q}]^T}^{S^3 \times \mathbb{R}^3}$ of $S^3 \times \mathbb{R}^3$ at $[\mathbf{q}, \mathbf{q}]^T$ is

$$\begin{aligned}\exp_{[\mathbf{q}, \mathbf{q}]^T}^{S^3 \times \mathbb{R}^3} : T_{\mathbf{q}} S^3 \times T_{\mathbf{q}} \mathbb{R}^3 &\rightarrow S^3 \times \mathbb{R}^3 \\ [v, x]^T &\mapsto [\exp_{\mathbf{q}}^{S^3}(v), \exp_{\mathbf{q}}^{\mathbb{R}^3}(x)]^T;\end{aligned}$$

similarly, for $\log_{\mathbf{q}}^{S^3}$ being the Riemannian logarithm application in the S^3 at $\mathbf{q} \in S^3$ —one expression is given in (A.23)—, and $\log_{\mathbf{q}}^{\mathbb{R}^3}$ the Riemannian logarithm of \mathbb{R}^3 at $\mathbf{q} \in \mathbb{R}^3$ —see (A.21)—; then the Riemannian logarithm $\log_{[\mathbf{q}, \mathbf{q}]^T}^{S^3 \times \mathbb{R}^3}$ of $S^3 \times \mathbb{R}^3$ at $[\mathbf{q}, \mathbf{q}]^T$ is

$$\begin{aligned}\log_{[\mathbf{q}, \mathbf{q}]^T}^{S^3 \times \mathbb{R}^3} : S^3 \times \mathbb{R}^3 &\rightarrow T_{\mathbf{q}} S^3 \times T_{\mathbf{q}} \mathbb{R}^3 \\ [\mathbf{p}, p]^T &\mapsto [\log_{\mathbf{q}}^{S^3}(\mathbf{p}), \log_{\mathbf{q}}^{\mathbb{R}^3}(p)]^T.\end{aligned}$$

Then we can define the Unscented filters for the dual quaternions. For the augmented filters, define the augmented functions $f_k^a : \mathcal{H}^{\|1\|} \times \mathcal{M}_{\varpi} \rightarrow \mathcal{H}^{\|1\|}$ and $h_k^a : \mathcal{H}^{\|1\|} \times \mathcal{M}_{\vartheta} \rightarrow \mathcal{H}^{\|1\|}$ such that, for ,

$$\begin{aligned} f_k^a \left(\begin{bmatrix} \underline{\mathbf{x}}_{k-1} \\ \underline{\varpi}_k \end{bmatrix} \right) &:= f_k(\underline{\mathbf{x}}_{k-1}, \underline{\varpi}_k), \\ h_k^a \left(\begin{bmatrix} \underline{\mathbf{x}}_k \\ \underline{\vartheta}_k \end{bmatrix} \right) &:= h_k(\underline{\mathbf{x}}_k, \underline{\vartheta}_k). \end{aligned} \quad (9.115)$$

Definition 9.19. Consider the system (9.113)

$$\begin{aligned} \underline{\mathbf{x}}_k &= f_k(\underline{\mathbf{x}}_{k-1}, \underline{\varpi}_k), \\ \underline{\mathbf{y}}_k &= h_k(\underline{\mathbf{x}}_k, \underline{\vartheta}_k); \end{aligned}$$

the pair of equations (9.115), and the function ψ defined in (9.112). Suppose that i) $\underline{\varpi}_k$ and $\underline{\vartheta}_k$ are independent; ii) $\underline{\varpi}_k$, $\underline{\vartheta}_k$ and the initial state $\underline{\mathbf{x}}_0$ are characterized by

$$\begin{aligned} \psi(\underline{\mathbf{x}}_0) &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{S^3 \times \mathbb{R}^3}, \\ \psi(\underline{\varpi}_k) &\sim (\bar{\varpi}_k, \mathbf{Q}_k)_{S^3 \times \mathbb{R}^3}, \\ \psi(\underline{\vartheta}_k) &\sim (\bar{\vartheta}_k, \mathbf{R}_k)_{S^3 \times \mathbb{R}^3}, \end{aligned}$$

and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Then the *Dual-Quaternionic Riemannian Augmented Unscented Kalman Filter (DQRiAUUKF)* is given by the following algorithm:

Algorithm 28 (Dual-Quaternionic Riemannian Augmented Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\underline{\mathbf{x}}}_{0|0} := \psi^{-1}(\bar{\mathbf{x}}_0) \in$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\begin{aligned} \hat{\underline{\mathbf{x}}}_{k-1|k-1} &:= \psi(\hat{\underline{\mathbf{x}}}_{k-1|k-1}), \\ \hat{\underline{\mathbf{x}}}_{k-1|k-1}^a &:= [\hat{\underline{\mathbf{x}}}_{k-1|k-1}^T, \bar{\varpi}_k^T]^T, \\ \hat{\mathbf{P}}_{xx,a}^{k-1|k-1} &:= \text{diag}(\hat{\mathbf{P}}_{xx}^{k-1|k-1}, \mathbf{Q}_k). \end{aligned}$$

(b) The predicted statistics of the state by

$$\left[\hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{P}}_{xx}^{k|k-1} \right] := \text{RiUT}_1 \left(\psi^{-1} \circ f_k \circ \psi, \hat{\mathbf{x}}_{k-1|k-1}^a, \hat{\mathbf{P}}_{xx,a}^{k-1|k-1} \right).$$

(c) The augmented predicted estimates by

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1}^a &:= \left[\hat{\mathbf{x}}_{k|k-1}^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{xx,a}^{k|k-1} &:= \text{diag} \left(\hat{\mathbf{P}}_{xx}^{k|k-1}, \mathbf{R}_k \right). \end{aligned}$$

(d) The predicted statistics of the measurement by

$$\begin{aligned} \left[\hat{\mathbf{y}}_{k|k-1}, \hat{\mathbf{P}}_{yy}^{k|k-1}, \hat{\mathbf{P}}_{xy,a}^{k|k-1} \right] &:= \text{RiUT}_2 \left(\psi^{-1} \circ h_k \circ \psi, \hat{\mathbf{x}}_{k|k-1}^a, \hat{\mathbf{P}}_{xx,a}^{k|k-1} \right), \\ \hat{\mathbf{P}}_{xy}^{k|k-1} &:= \left[\hat{\mathbf{P}}_{xy,a}^{k|k-1} \right]_{(1:6),(1:6)}. \end{aligned}$$

(e) The corrected statistics of the state by

$$\begin{aligned} \mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1}, \\ \tilde{\mathbf{y}}_k &:= \psi \left(\tilde{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}} \left(\tilde{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\ \hat{\underline{\mathbf{x}}}_{k|k} &:= \psi^{-1} \left(\hat{\mathbf{x}}_{k|k} \right), \\ \hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} &:= \hat{\mathbf{P}}_{xx}^{k|k-1} - (\mathbf{G}_k) \hat{\mathbf{P}}_{yy}^{k|k-1} (\mathbf{G}_k)^T, \\ \hat{\mathbf{P}}_{xx}^{k|k} &:= \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \end{aligned}$$

Definition 9.20. Consider the system (9.113)

$$\begin{aligned} \underline{\mathbf{x}}_k &= f_k \left(\underline{\mathbf{x}}_{k-1}, \underline{\boldsymbol{\varpi}}_k \right), \\ \underline{\mathbf{y}}_k &= h_k \left(\underline{\mathbf{x}}_k, \underline{\boldsymbol{\vartheta}}_k \right); \end{aligned}$$

the pair of equations (9.115), and the function ψ defined in (9.112). Suppose that i) $\underline{\boldsymbol{\varpi}}_k$ and $\underline{\boldsymbol{\vartheta}}_k$ are independent; ii) $\underline{\boldsymbol{\varpi}}_k$, $\underline{\boldsymbol{\vartheta}}_k$ and the initial state $\underline{\mathbf{x}}_0$ are characterized by

$$\begin{aligned} \psi \left(\underline{\mathbf{x}}_0 \right) &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{xx}^0} \sqrt{\mathbf{P}_{xx}^0}^T \right)_{S^3 \times \mathbb{R}^3}, \\ \psi \left(\underline{\boldsymbol{\varpi}}_k \right) &\sim \left(\bar{\boldsymbol{\varpi}}_k, \sqrt{\mathbf{Q}_k} \sqrt{\mathbf{Q}_k}^T \right)_{S^3 \times \mathbb{R}^3}, \end{aligned}$$

$$\psi(\underline{\boldsymbol{\vartheta}}_k) \sim \left(\bar{\boldsymbol{\vartheta}}_k, \sqrt{\mathbf{R}_k} \sqrt{\mathbf{R}_k}^T \right)_{S^3 \times \mathbb{R}^3},$$

where $\sqrt{\mathbf{Q}_k}$ is $n_{\bar{\boldsymbol{\omega}}_k} \times n_{\bar{\boldsymbol{\omega}}_k}$ and $n_{\bar{\boldsymbol{\omega}}_k}$ and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Let $n_{\bar{\boldsymbol{\omega}}_k}$ be the number of columns of $\sqrt{\mathbf{Q}_k}$ and $n_{\bar{\boldsymbol{\vartheta}}_k}$ of $\sqrt{\mathbf{R}_k}$. Then the *Dual-Quaternionic Riemannian Augmented Square-Root Unscented Kalman Filter (DqRi-AuSRUKF)* is given by the following algorithm:

Algorithm 29 (Dual-Quaternionic Riemannian Augmented Square-Root Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \psi^{-1}(\bar{\mathbf{x}}_0)$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The augmented previous estimates by*

$$\begin{aligned} \hat{\mathbf{x}}_{k-1|k-1} &:= \psi\left(\hat{\mathbf{x}}_{k-1|k-1}\right), \\ \hat{\mathbf{x}}_{k-1|k-1}^a &:= \left[\hat{\mathbf{x}}_{k-1|k-1}, \bar{\boldsymbol{\omega}}_k\right]^T, \\ \sqrt{\hat{\mathbf{P}}_{xx,a}^{k-1|k-1}} &:= \text{diag}\left(\sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1}}, \sqrt{\mathbf{Q}_k}\right). \end{aligned}$$

(b) *The predicted statistics of the state by*

$$\begin{aligned} &\left[\hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}\right] \\ &= \text{RiSRUT}_1\left(\psi^{-1} \circ f_k \circ \psi, \hat{\mathbf{x}}_{k-1|k-1}, \sqrt{\hat{\mathbf{P}}_{xx,a}^{k-1|k-1}}, [0]_{n_{\bar{\boldsymbol{\omega}}_k} \times n_{\bar{\boldsymbol{\omega}}_k}}\right). \end{aligned}$$

(c) *The augmented predicted estimates by*

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1}^a &:= \left[\hat{\mathbf{x}}_{k|k-1}, \bar{\boldsymbol{\vartheta}}_k\right]^T, \\ \sqrt{\hat{\mathbf{P}}_{xx,a}^{k|k-1}} &:= \text{diag}\left(\sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}, \sqrt{\mathbf{R}_k}\right). \end{aligned}$$

(d) *The predicted statistics of the measurement by*

$$\begin{aligned} &\left[\hat{\mathbf{y}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}, S_{\chi}, S_{\gamma}, \hat{\mathbf{P}}_{xy,a}^{k|k-1}\right] \\ &= \text{RiSRUT}_2\left(\psi^{-1} \circ h_k \circ \psi, \hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx,a}^{k|k-1}}, [0]_{n_{\bar{\boldsymbol{\vartheta}}_k} \times n_{\bar{\boldsymbol{\vartheta}}_k}}\right), \end{aligned}$$

and

$$\hat{\mathbf{P}}_{xy}^{k|k-1} := \left[\hat{\mathbf{P}}_{xy,a}^{k|k-1} \right]_{(1:n_x),(1:n_y)}.$$

(e) The corrected statistics of the state by

$$\begin{aligned} \mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-T} \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}} \right)^{-1}, \\ \underline{\mathbf{y}}_k &:= \psi \left(\underline{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\underline{\mathbf{y}}_{k|k-1}} \left(\underline{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\ \underline{\hat{\mathbf{x}}}_{k|k} &:= \psi^{-1} \left(\hat{\mathbf{x}}_{k|k} \right), \\ \sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} &:= \text{triag} \left([S_{\mathcal{X}} - \mathbf{G}_k S_{\gamma}] \right), \\ \sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} &:= \text{PT} \left(\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \end{aligned}$$

Definition 9.21. Consider the system (9.113)

$$\begin{aligned} \underline{\mathbf{x}}_k &= f_k \left(\underline{\mathbf{x}}_{k-1} \right) \boxplus \underline{\varpi}_k, \\ \underline{\mathbf{y}}_k &= h_k \left(\underline{\mathbf{x}}_k \right) \boxplus \underline{\vartheta}_k, \end{aligned}$$

and the function ψ defined in (9.112). Suppose that i) $\underline{\varpi}_k$ and $\underline{\vartheta}_k$ are independent; ii) $\underline{\varpi}_k$, $\underline{\vartheta}_k$ and the initial state $\underline{\mathbf{x}}_0$ are characterized by

$$\begin{aligned} \psi \left(\underline{\mathbf{x}}_0 \right) &\sim \left(\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0 \right)_{S^3 \times \mathbb{R}^3}, \\ \underline{\varpi}_k &\sim \left(\bar{\varpi}_k, Q_k \right)_{S^3 \times \mathbb{R}^3}, \\ \underline{\vartheta}_k &\sim \left(\bar{\vartheta}_k, R_k \right)_{S^3 \times \mathbb{R}^3}, \end{aligned}$$

and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Then the *Dual-Quaternionic Riemannian Additive Unscented Kalman Filter (DqRiAdUKF)* is given by the following algorithm:

Algorithm 30 (Dual-Quaternionic Riemannian Additive Unscented Kalman Filter).
Perform the following steps:

1. Initialization. Set the initial estimates $\underline{\hat{\mathbf{x}}}_{0|0} := \psi^{-1} \left(\bar{\mathbf{x}}_0 \right)$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. Filtering. For $k = 1, 2, \dots, k_f$; set the following elements:

(a) The predicted statistics of the state by

$$\begin{aligned}\hat{\mathbf{x}}_{k-1|k-1} &:= \psi \left(\hat{\mathbf{x}}_{k-1|k-1} \right), \\ \left[\hat{\mathbf{x}}_{k|k-1}^*, \hat{\mathbf{P}}_{\mathbf{xx},*}^{k|k-1} \right] &:= \text{RiUT}_1 \left(\psi^{-1} \circ f_k \circ \psi, \hat{\mathbf{x}}_{k-1|k-1}, \hat{\mathbf{P}}_{\mathbf{xx}}^{k-1|k-1} \right), \\ \hat{\mathbf{x}}_{k|k-1} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}^*} \bar{\omega}_k, \\ \hat{\mathbf{P}}_{\mathbf{xx}}^{k|k-1} &:= \hat{\mathbf{P}}_{\mathbf{xx},*}^{k|k-1} + Q_k.\end{aligned}$$

(b) The predicted statistics of the measurement by

$$\begin{aligned}\left[\hat{\mathbf{y}}_{k|k-1}^*, \hat{\mathbf{P}}_{\mathbf{yy},*}^{k|k-1}, \hat{\mathbf{P}}_{\mathbf{xy}}^{k|k-1} \right] &:= \text{RiUT}_2 \left(\psi^{-1} \circ h_k \circ \psi, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{P}}_{\mathbf{xx}}^{k|k-1} \right), \\ \hat{\mathbf{y}}_{k|k-1} &:= \exp_{\hat{\mathbf{y}}_{k|k-1}^*} \bar{\vartheta}_k, \\ \hat{\mathbf{P}}_{\mathbf{yy}}^{k|k-1} &:= \hat{\mathbf{P}}_{\mathbf{yy},*}^{k|k-1} + R_k.\end{aligned}$$

(c) The corrected statistics of the state by

$$\begin{aligned}\mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{\mathbf{xy}}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{\mathbf{yy}}^{k|k-1} \right)^{-1}, \\ \underline{\mathbf{y}}_k &:= \psi \left(\underline{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}} \left(\underline{\mathbf{y}}_k \right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\ \hat{\underline{\mathbf{x}}}_{k|k} &:= \psi^{-1} \left(\hat{\mathbf{x}}_{k|k} \right), \\ \hat{\mathbf{P}}_{\mathbf{xx}}^{k|k, \hat{\mathbf{x}}_{k|k-1}} &:= \hat{\mathbf{P}}_{\mathbf{xx}}^{k|k-1} - (\mathbf{G}_k) \hat{\mathbf{P}}_{\mathbf{yy}}^{k|k-1} (\mathbf{G}_k)^T, \\ \hat{\mathbf{P}}_{\mathbf{xx}}^{k|k} &:= \text{PT} \left(\hat{\mathbf{P}}_{\mathbf{xx}}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right).\end{aligned}$$

Definition 9.22. Consider the system (9.113) and the function ψ defined in (9.112). Suppose that i) $\underline{\omega}_k$ and $\underline{\vartheta}_k$ are independent; ii) $\underline{\omega}_k$, $\underline{\vartheta}_k$ and the initial state $\underline{\mathbf{x}}_0$ are characterized by

$$\begin{aligned}\psi(\underline{\mathbf{x}}_0) &\sim \left(\bar{\mathbf{x}}_0, \sqrt{\mathbf{P}_{\mathbf{xx}}^0} \sqrt{\mathbf{P}_{\mathbf{xx}}^0}^T \right)_{S^3 \times \mathbb{R}^3}, \\ \psi(\underline{\omega}_k) &\sim \left(\bar{\omega}_k, \sqrt{Q_k} \sqrt{Q_k}^T \right)_{S^3 \times \mathbb{R}^3}, \\ \psi(\underline{\vartheta}_k) &\sim \left(\bar{\vartheta}_k, \sqrt{R_k} \sqrt{R_k}^T \right)_{S^3 \times \mathbb{R}^3};\end{aligned}$$

and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Then the *Dual-Quaternionic Riemannian Additive Square-Root Unscented Kalman Filter (DqRiAdSRUKF)* is given

by the following algorithm:

Algorithm 31 (Dual-Quaternionic Riemannian Additive Square-Root Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \psi^{-1}(\bar{\mathbf{x}}_0)$ and $\sqrt{\hat{\mathbf{P}}_{xx}^{0|0}} := \sqrt{\mathbf{P}_{xx}^0}$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The predicted statistics of the state by*

$$\begin{aligned}\hat{\mathbf{x}}_{k-1|k-1} &:= \psi\left(\hat{\mathbf{x}}_{k-1|k-1}\right), \\ \left[\hat{\mathbf{x}}_{k|k-1}^*, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}\right] &:= \text{RiSRUT}_1\left(f_k, \hat{\mathbf{x}}_{k-1|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k-1|k-1}}, \sqrt{Q_k}\right), \\ \hat{\mathbf{x}}_{k|k-1} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}^*} \bar{\omega}_k.\end{aligned}$$

(b) *The predicted statistics of the measurement by*

$$\begin{aligned}\left[\hat{\mathbf{y}}_{k|k-1}^*, \sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}, S_\chi, S_\gamma, \hat{\mathbf{P}}_{xy}^{k|k-1}\right] &:= \text{RiSRUT}_2\left(h_k, \hat{\mathbf{x}}_{k|k-1}, \sqrt{\hat{\mathbf{P}}_{xx}^{k|k-1}}, \sqrt{R_k}\right), \\ \hat{\mathbf{y}}_{k|k-1} &:= \exp_{\hat{\mathbf{y}}_{k|k-1}^*} \bar{\vartheta}_k.\end{aligned}$$

(c) *The corrected statistics of the state by*

$$\begin{aligned}\mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1}\right) \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}\right)^{-T} \left(\sqrt{\hat{\mathbf{P}}_{yy}^{k|k-1}}\right)^{-1}, \\ \underset{\sim}{\mathbf{y}}_k &:= \psi\left(\underset{\sim}{\mathbf{y}}_k\right), \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}^*} \left(\underset{\sim}{\mathbf{y}}_k\right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM}\right), \\ \hat{\mathbf{x}}_{k|k} &:= \psi^{-1}\left(\hat{\mathbf{x}}_{k|k}\right), \\ \sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} &:= \text{triag}\left(\left[S_\chi - \mathbf{G}_k S_\gamma, \mathbf{G}_k \sqrt{R_k}\right]\right), \\ \sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}} &:= \text{PT}\left(\sqrt{\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k}\right).\end{aligned}$$

As far as our knowledge goes, these dual quaternions UF's (the DqRiUKF, DqRiSRUKF, DqRiAd-UKF and DqRiAdSRUKF) are *the first UF's for unit dual quaternions of the literature*. We highlight the following properties of these filters:

1. they preserve norm of the unit dual quaternions *at every time step*;

2. their probability and statistic elements are well defined;
3. all operations within them are well defined (e.g. there are no sums of two elements for whom the sums would not be well defined, such as sums of unit dual quaternions);
4. the rotation part and the translation part of the dual quaternions—for a unit dual quaternion $\underline{\mathbf{q}} = \mathbf{q} + \varepsilon \frac{1}{2} q \mathbf{q}$, \mathbf{q} is the rotation part and q the translation one—are *not being treated separately* since these two parts are not supposed to be independent Riemannian random points. In fact, the cross-covariances between the rotation and translation parts can be different from zero.

9.7 CONTINUOUS-DISCRETE-TIME AND CONTINUOUS-TIME RIUKF'S

Similar to the Euclidean case, instead of considering the dynamics and the measurements time discrete, we can consider i) the dynamics being time continuous and the measurements being time discrete, ii) the dynamics being time continuous and the measurements being also time continuous, or iii) the dynamics being time discrete and the measurements being time continuous. In the first case [i)], we call the system continuous-discrete-time; and in the second case [ii)], we call the system continuous-time. We do not treat the third case [iii)] because it is usually not considered in practice (cf. the comments at the end of Section 5.8).

A *Riemannian continuous-discrete-time (stochastic, dynamic) system* can be written in the form given by, for $t \geq t_0$,

$$\begin{aligned} d\mathbf{x}(t) &= f_t(\mathbf{x}(t), \varpi(t)), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \vartheta_k); \end{aligned} \tag{9.116}$$

or in the additive form

$$\begin{aligned} d\mathbf{x}(t) &= \exp_{f_t(\mathbf{x}(t))} \left[\log_{f_t(\mathbf{x}(t))} f_t(\mathbf{x}(t)) + d\varpi(t) \right], \\ d\mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \vartheta_k \right]. \end{aligned} \tag{9.117}$$

In the systems above $d\varpi(t)$ and $d\vartheta(t)$ are well defined; they are differentials of Euclidean stochastic processes. However differentials of Riemannian stochastic processes such as $d\mathbf{x}(t)$ and $d\mathbf{y}(t)$ were not defined yet in this work. We are unaware of any work with results related to these differentials, and we do not know the conditions for their

existence; we shall *only suppose they exist*.

We can extend the continuous-discrete-time and continuous-time UKF's of Section 5.8 to the relative Riemannian cases by using the forms of the Riemannian filters.

For the augmented versions of these Unscented filters, define, for the Riemannian continuous-discrete-time system, the augmented functions $f_t^a : \mathcal{M}_x \times \mathcal{M}_\varpi \rightarrow \mathcal{M}_x$ and $h_k^a : \mathcal{M}_x \times \mathcal{M}_\vartheta \rightarrow \mathcal{M}_y$ such that, for ,

$$\begin{aligned} f_t^a \left(\begin{bmatrix} \mathbf{x}(t) \\ \varpi(t) \end{bmatrix} \right) &:= f_t(\mathbf{x}(t), \varpi(t)), \\ h_k^a \left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{r}_k \end{bmatrix} \right) &:= h_k(\mathbf{x}_k, \vartheta_k). \end{aligned} \quad (9.118)$$

Definition 9.23. Consider the system (9.116)

$$\begin{aligned} d\mathbf{x}(t) &= f_t(\mathbf{x}(t), \varpi(t)), \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \vartheta_k); \end{aligned}$$

and the pair of equations (9.118). Suppose that i) the noises $\varpi(t)$ and ϑ_k are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, ϑ_k and the initial state $\mathbf{x}(t_0)$ are characterized by

$$\begin{aligned} \mathbf{x}(t_0) &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x}, \\ \frac{d\varpi(t)}{dt} &\sim (\bar{\varpi}_k, \mathbf{Q}(t))_{\mathcal{M}_\varpi}, \\ \vartheta_k &\sim (\bar{\vartheta}_k, \mathbf{R}_k)_{\mathcal{M}_\vartheta}; \end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Continuous-Discrete Augmented Unscented Kalman Filter* is given by the following algorithm:

Algorithm 32 (Riemannian Continuous-discrete Augmented Unscented Kalman Filter). *Perform the following steps:*

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.
2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics. For the initial conditions*

$$\begin{aligned} \mathbf{x}^-(t_{k-1}) &:= \hat{\mathbf{x}}_{k-1|k-1}^- \text{ and} \\ \hat{\mathbf{P}}_{xx}^-(t_{k-1}) &:= \hat{\mathbf{P}}_{xx}^{k-1|k-1}^-, \end{aligned}$$

solve i), for $\hat{\mathbf{x}}^-(t_k)$, the differential equation

$$d\hat{\mathbf{x}}^-(t) := \hat{\mathbf{m}}^-(t);$$

and ii), for $\hat{\mathbf{P}}_{xx}^-(t_k)$, the differential equation

$$d\hat{\mathbf{P}}_{xx}^-(t) := \hat{\mathbf{P}}_{xf(x)}^-(t) + \left(\hat{\mathbf{P}}_{xf(x)}^-(t) \right)^T;$$

where

$$\begin{aligned} \hat{\mathbf{x}}_a^-(t) &:= \left[\hat{\mathbf{x}}^-(t)^T, \bar{\boldsymbol{\omega}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{xx}^{-,a}(t) &:= \text{diag} \left(\hat{\mathbf{P}}_{xx}(t), \mathbf{Q}(t) \right), \\ \left[\hat{\mathbf{m}}^-(t), \bullet, \hat{\mathbf{P}}_{xf(x)}^{-,a}(t) \right] &:= \text{RiUT}_1 \left(f_t^a, \hat{\mathbf{x}}_a^-(t), \hat{\mathbf{P}}_{xx}^{-,a}(t) \right), \\ \hat{\mathbf{P}}_{xf(x)}^-(t) &:= \left[\hat{\mathbf{P}}_{xf(x)}^{-,a}(t) \right]_{(1:n_x), (1:n_x)}. \end{aligned}$$

(b) The measurement's predicted statistics by

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1}^a &:= \left[\left(\hat{\mathbf{x}}^-(t_k) \right)^T, \bar{\boldsymbol{\vartheta}}_k^T \right]^T, \\ \hat{\mathbf{P}}_{xx,a}^{k|k-1} &:= \text{diag} \left(\left(\hat{\mathbf{P}}_{xx}^-(t_k) \right)^T, \mathbf{R}_k \right), \\ \left[\hat{\mathbf{y}}_{k|k-1}, \hat{\mathbf{P}}_{yy}^{k|k-1}, \hat{\mathbf{P}}_{xy,a}^{k|k-1} \right] &:= \text{RiUT}_2 \left(h_k, \hat{\mathbf{x}}_{k|k-1}^a, \hat{\mathbf{P}}_{xx,a}^{k|k-1} \right), \\ \hat{\mathbf{P}}_{xy}^{k|k-1} &:= \left[\hat{\mathbf{P}}_{xy,a}^{k|k-1} \right]_{(1:n_x), (1:n_y)}. \end{aligned}$$

(c) The corrected statistics of the state by

$$\begin{aligned} \mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1}, \\ \hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}} \left(\mathbf{y}_k \right), \\ \hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\ \hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} &:= \hat{\mathbf{P}}_{xx}^{k|k-1} - \left(\mathbf{G}_k \right) \hat{\mathbf{P}}_{yy}^{k|k-1} \left(\mathbf{G}_k \right)^T, \\ \hat{\mathbf{P}}_{xx}^{k|k} &:= \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right). \end{aligned}$$

Definition 9.24. Consider the system (9.117)

$$\begin{aligned} d\mathbf{x}(t) &= \exp_{f_t(\mathbf{x}(t))} \left[\log_{f_t(\mathbf{x}(t))} f_t(\mathbf{x}(t)) + d\boldsymbol{\omega}(t) \right], \\ d\mathbf{y}_k &= \exp_{h_k(\mathbf{x}_k)} \left[\log_{h_k(\mathbf{x}_k)} h_k(\mathbf{x}_k) + \boldsymbol{\vartheta}_k \right]. \end{aligned}$$

Suppose that i) the noises $\varpi(t)$ and ϑ_k are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, ϑ_k and the initial state $\mathbf{x}(t_0)$ are characterized by

$$\begin{aligned}\mathbf{x}(t_0) &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x}, \\ \frac{d\varpi(t)}{dt} &\sim (\bar{\varpi}_k, Q(t))^{n_x}, \\ \vartheta_k &\sim (\bar{\vartheta}_k, R_k)^{n_y};\end{aligned}$$

and iii) the measurements $\underline{\mathbf{y}}_1, \underline{\mathbf{y}}_2, \dots, \underline{\mathbf{y}}_{k_f}$ are given. Then the *Riemannian Continuous-Discrete Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 33 (Riemannian Continuous-discrete Additive Unscented Kalman Filter).
Perform the following steps:

1. *Initialization.* Set the initial estimates $\hat{\mathbf{x}}_{0|0} := \bar{\mathbf{x}}_0$ and $\hat{\mathbf{P}}_{xx}^{0|0} := \mathbf{P}_{xx}^0$.

2. *Filtering.* For $k = 1, 2, \dots, k_f$; set the following elements:

(a) *The state's predicted statistics.* For the initial conditions

$$\begin{aligned}\mathbf{x}^-(t_{k-1}) &:= \hat{\mathbf{x}}_{k-1|k-1} \text{ and} \\ \hat{\mathbf{P}}_{xx}^-(t_{k-1}) &:= \hat{\mathbf{P}}_{xx}^{k-1|k-1},\end{aligned}$$

solve i), for $\hat{\mathbf{x}}^-(t_k)$, the differential equation

$$d\hat{\mathbf{x}}^-(t) := \hat{\mathbf{m}}^-(t);$$

and ii), for $\hat{\mathbf{P}}_{xx}^-(t_k)$, the differential equation

$$d\hat{\mathbf{P}}_{xx}^-(t) := \hat{\mathbf{P}}_{xf(x)}^-(t) + \left(\hat{\mathbf{P}}_{xf(x)}^-(t)\right)^T + Q(t);$$

where

$$\begin{aligned}\left[\hat{\mathbf{m}}_*^-(t), \bullet, \hat{\mathbf{P}}_{xf(x)}^-(t)\right] &:= \text{RiUT}_1\left(f_t^a, \hat{\mathbf{x}}^-(t), \hat{\mathbf{P}}_{xx}^-(t)\right) \\ \hat{\mathbf{m}}^-(t) &:= \exp_{\hat{\mathbf{m}}_*^-(t)} \bar{\varpi}_k.\end{aligned}$$

(b) *The measurement's predicted statistics by*

$$\begin{aligned}\left[\hat{\mathbf{y}}_{k|k-1}^*, \hat{\mathbf{P}}_{yy,*}^{k|k-1}, \hat{\mathbf{P}}_{xy}^{k|k-1}\right] &:= \text{RiUT}_2\left(h_k, \hat{\mathbf{x}}^-(t_k), \hat{\mathbf{P}}_{xx}^-(t_k)\right), \\ \hat{\mathbf{y}}_{k|k-1} &:= \exp_{\hat{\mathbf{y}}_{k|k-1}^*} \bar{\vartheta}_k, \\ \hat{\mathbf{P}}_{yy}^{k|k-1} &:= \hat{\mathbf{P}}_{yy,*}^{k|k-1} + R_k.\end{aligned}$$

(c) The corrected statistics of the state by

$$\begin{aligned}
\mathbf{G}_k &:= \left(\hat{\mathbf{P}}_{xy}^{k|k-1} \right) \left(\hat{\mathbf{P}}_{yy}^{k|k-1} \right)^{-1}, \\
\hat{\mathbf{x}}_{k|k}^{TM} &:= \hat{\mathbf{x}}_{k|k-1}^{TM} + \mathbf{G}_k \log_{\hat{\mathbf{y}}_{k|k-1}} \left(\mathbf{y}_k \right), \\
\hat{\mathbf{x}}_{k|k} &:= \exp_{\hat{\mathbf{x}}_{k|k-1}} \left(\hat{\mathbf{x}}_{k|k}^{TM} \right), \\
\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}} &:= \hat{\mathbf{P}}_{xx}^{k|k-1} - (\mathbf{G}_k) \hat{\mathbf{P}}_{yy}^{k|k-1} (\mathbf{G}_k)^T, \\
\hat{\mathbf{P}}_{xx}^{k|k} &:= \text{PT} \left(\hat{\mathbf{P}}_{xx}^{k|k, \hat{\mathbf{x}}_{k|k-1}}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathbf{x}}_{k|k} \right).
\end{aligned}$$

Similarly, a *Riemannian continuous-time (stochastic, dynamic) system* can be written in the form (for a vector x , dx stand for its differential) given by, for $t \geq t_0$,

$$\begin{aligned}
d\mathbf{x}(t) &= f_t(\mathbf{x}(t), \boldsymbol{\varpi}(t)), \\
d\mathbf{y}(t) &= h_t(\mathbf{x}(t), \boldsymbol{\vartheta}(t)).
\end{aligned} \tag{9.119}$$

or in the additive form

$$\begin{aligned}
d\mathbf{x}(t) &= \exp_{f_t(\mathbf{x}(t))} \left[\log_{f_t(\mathbf{x}(t))} f_t(\mathbf{x}(t)) + d\boldsymbol{\varpi}(t) \right], \\
d\mathbf{y}(t) &= \exp_{h_t(\mathbf{x}(t))} \left[\log_{h_t(\mathbf{x}(t))} h_t(\mathbf{x}(t)) + d\boldsymbol{\vartheta}(t) \right].
\end{aligned} \tag{9.120}$$

For the augmented versions of these Unscented filters, define, for the Riemannian continuous-discrete-time system, the augmented functions $f_t^a : \mathcal{M}_x \times \mathcal{M}_{\boldsymbol{\varpi}} \rightarrow \mathcal{M}_x$ and $h_t^a : \mathcal{M}_x \times \mathcal{M}_{\boldsymbol{\vartheta}} \rightarrow \mathcal{M}_y$ such that, for ,

$$\begin{aligned}
f_t^a \left(\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\varpi}(t) \end{bmatrix} \right) &:= f_t(\mathbf{x}(t), \boldsymbol{\varpi}(t)), \\
h_t^a \left(\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\vartheta}(t) \end{bmatrix} \right) &:= h_t(\mathbf{x}(t), \boldsymbol{\vartheta}(t)).
\end{aligned} \tag{9.121}$$

Definition 9.25. Consider the system (9.119)

$$\begin{aligned}
d\mathbf{x}(t) &= f_t(\mathbf{x}(t), \boldsymbol{\varpi}(t)), \\
d\mathbf{y}(t) &= h_t(\mathbf{x}(t), \boldsymbol{\vartheta}(t));
\end{aligned}$$

and the pair of equations (9.121). Suppose that i) the noises $\boldsymbol{\varpi}(t)$ and $\boldsymbol{\vartheta}(t)$ are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\boldsymbol{\varpi}(t)$, $\boldsymbol{\vartheta}(t)$ and the initial state $\mathbf{x}(t_0)$ are

characterized by

$$\begin{aligned}\mathbf{x}(t_0) &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x}, \\ \frac{d\bar{\boldsymbol{\omega}}(t)}{dt} &\sim (\bar{\boldsymbol{\omega}}_k, \mathbf{Q}(t))_{\mathcal{M}_{\bar{\boldsymbol{\omega}}}}, \\ \frac{d\bar{\boldsymbol{\vartheta}}_k}{dt} &\sim (\bar{\boldsymbol{\vartheta}}_k, \mathbf{R}_k)_{\mathcal{M}_{\bar{\boldsymbol{\vartheta}}}};\end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Continuous Augmented Unscented Kalman Filter* is given by the following algorithm:

Algorithm 34 (Riemannian Continuous Augmented Unscented Kalman Filter). *For the initial conditions*

$$\begin{aligned}\hat{\mathbf{x}}(t)_0 &:= \bar{\mathbf{x}}(t_0) \text{ and} \\ \hat{\mathbf{P}}_{xx}(t_0) &:= P_{xx}(t_0),\end{aligned}$$

solve i), for $\hat{\mathbf{x}}(t_k)$, the differential equation

$$d\hat{\mathbf{x}}(t) := \hat{\mathbf{m}}(t) + G(t) (\mathbf{y}(t) - \hat{\mathbf{y}}(t));$$

and ii), for $\hat{P}_{xx}(t_k)$, the differential equation

$$d\hat{P}_{xx}(t) := \hat{\mathbf{P}}_{xf(x)}(t) + \hat{\mathbf{P}}_{xf(x)}^T(t);$$

where

$$\begin{aligned}\hat{\mathbf{x}}_a(t) &:= [\hat{\mathbf{x}}^-(t)^T, \bar{\boldsymbol{\omega}}_k^T]^T, \\ \hat{\mathbf{P}}_{xx}^a(t) &:= \text{diag}(\hat{\mathbf{P}}_{xx}(t), \mathbf{Q}(t)), \\ \hat{\mathbf{x}}_a^*(t) &:= [\hat{\mathbf{x}}^-(t)^T, \bar{\boldsymbol{\vartheta}}_k^T]^T, \\ \hat{\mathbf{P}}_{xx}^{a,*}(t) &:= \text{diag}(\hat{\mathbf{P}}_{xx}(t), \mathbf{R}(t)), \\ [\hat{\mathbf{m}}(t), \bullet, \hat{\mathbf{P}}_{xf(x)}^a(t)] &:= \text{RiUT}_1(f_t^a, \hat{\mathbf{x}}_a(t), \hat{\mathbf{P}}_{xx}^a(t)), \\ [\hat{\mathbf{y}}_{k|k-1}, \bullet, \hat{\mathbf{P}}_{xh(x)}^a(t)] &:= \text{RiUT}_2(h_k^a, \hat{\mathbf{x}}_a^*(t), \hat{\mathbf{P}}_{xx}^{a,*}(t)), \\ \hat{\mathbf{P}}_{xf(x)}(t) &:= [\hat{\mathbf{P}}_{xf(x)}^a(t)]_{(1:n_x), (1:n_x)}, \\ \hat{\mathbf{P}}_{xh(x)}(t) &:= [\hat{\mathbf{P}}_{xh(x)}^a(t)]_{(1:n_y), (1:n_y)}, \\ \mathbf{G}(t) &:= \hat{\mathbf{P}}_{xh(x)}(t) \mathbf{R}^{-1}(t).\end{aligned}$$

Definition 9.26. Consider the system (9.120)

$$\begin{aligned} d\mathbf{x}(t) &= \exp^{\overline{f_t(\mathbf{x}(t))}} \left[\log^{\overline{f_t(\mathbf{x}(t))}} f_t(\mathbf{x}(t)) + d\varpi(t) \right], \\ d\mathbf{y}(t) &= \exp^{\overline{h_t(\mathbf{x}(t))}} \left[\log^{\overline{h_t(\mathbf{x}(t))}} h_t(\mathbf{x}(t)) + d\vartheta(t) \right]. \end{aligned}$$

Suppose that i) the noises $\varpi(t)$ and $\vartheta(t)$ are independent for all $t \geq t_0$ and $k \geq t_0$; ii) $\varpi(t)$, $\vartheta(t)$ and the initial state $\mathbf{x}(t_0)$ are characterized by

$$\begin{aligned} \mathbf{x}(t_0) &\sim (\bar{\mathbf{x}}_0, \mathbf{P}_{xx}^0)_{\mathcal{M}_x}, \\ \frac{d\varpi(t)}{dt} &\sim (\bar{\varpi}_k, Q(t))^{n_x}, \\ \frac{d\vartheta_k}{dt} &\sim (\bar{\vartheta}_k, R_k)^{n_y}; \end{aligned}$$

and iii) the measurements $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k_f}$ are given. Then the *Riemannian Continuous Additive Unscented Kalman Filter* is given by the following algorithm:

Algorithm 35 (Riemannian Continuous-discrete Additive Unscented Kalman Filter).
For the initial conditions

$$\begin{aligned} \hat{\mathbf{x}}(t)_0 &:= \bar{\mathbf{x}}(t_0) \text{ and} \\ \hat{\mathbf{P}}_{xx}(t_0) &:= P_{xx}(t_0), \end{aligned}$$

solve i), for $\hat{\mathbf{x}}(t_k)$, the differential equation

$$d\hat{\mathbf{x}}(t) := \hat{\mathbf{m}}(t) + G(t) (\mathbf{y}(t) - \hat{\mathbf{y}}(t));$$

and ii), for $\hat{\mathbf{P}}_{xx}(t_k)$, the differential equation

$$d\hat{\mathbf{P}}_{xx}(t) := \hat{\mathbf{P}}_{xf(x)}(t) + \hat{\mathbf{P}}_{xf(x)}^T(t) + \mathbf{Q}(t) - \mathbf{G}(t)\mathbf{R}(t)\mathbf{G}^T(t);$$

where

$$\begin{aligned} [\hat{\mathbf{m}}(t), \bullet, \hat{\mathbf{P}}_{xf(x)}(t)] &:= \text{RiUT}_1 \left(f_t, \hat{\mathbf{x}}(t), \hat{\mathbf{P}}_{xx}(t) \right), \\ [\hat{\mathbf{y}}_{k|k-1}, \bullet, \hat{\mathbf{P}}_{xh(x)}(t)] &:= \text{RiUT}_2 \left(h_k, \hat{\mathbf{x}}(t), \hat{\mathbf{P}}_{xx}(t) \right), \\ \mathbf{G}(t) &:= \hat{\mathbf{P}}_{xh(x)}(t)\mathbf{R}^{-1}(t). \end{aligned}$$

9.8 CONCLUSIONS REGARDING UNSCENTED FILTERING ON RIEMANNIAN MANIFOLDS

We initiated the systematization of the theory of Unscented Kalman filters for Riemannian manifolds by introducing the Riemannian σ -representation (Ri σ R, in Section 9.1). In Theorem 9.1 we showed that closed forms of the σ -representations can be used to find closed forms for Ri σ R's; with this, in Corollary 9.1, we determined i) the minimum number sigma points of a Ri σ R, ii) the minimum number of a symmetric Ri σ R, iii) closed forms for a minimum Ri σ R, and iv) closed forms for a minimum symmetric Ri σ R.

Similarly to the systematization of Part I, we define the Riemannian Unscented Transformation (RiUT, Section 9.2) based on the the concept of a Ri σ R. Besides, we extended all the UT variants of Chapter 4 to the Riemannian case; among other, we propose the Scaled RiUT, and the Square-Root RiUT.

In Section 9.3, we treated the desired discrete-time Riemannian Unscented *filters*.

We introduced a definition of a Riemannian additive system (Section 9.3.1). These systems are necessary in order to define additive-noise Riemannian Unscented filters, but, generally, Riemannian manifolds are not endowed with sums.

Furthermore, we found consistent Kalman correction equations for the Riemannian Unscented filters (Section 9.3.2). To find these equations, we considered, first, a particular case where the state and the measurement belonged to the same manifold (Section 9.3.2.1); only then, by extending this result, we could get to the final form of the Kalman correction equations (Section 9.3.2.2).

In Section 9.3.3, we propose four new discrete-time Riemannian Unscented Filters. At the end of this section, we provide a list of numerous variants of these four Riemannian filters (Tables 9.3, 9.1, 9.4, and 9.2); all variants are new consistent Riemannian Unscented filters.

Further, in Section 9.4, we compared our Riemannian Unscented filters with the only Unscented Kalman filter of the literature, namely the Unscented Kalman Filter for Riemannian manifolds (UKFRM) of [171]. The UKFRM of [171] is *essentially* different from all the filters of Tables 9.1, 9.2, 9.3, ,and 9.4, except for one: the Riemannian Homogeneous Minimum Symmetric AdUKF (RiHoMiSyAdUKF, Table 9.4 [1,1]). Yet, even though there are similarities between the UKFRM of [171] and the RiHoMiSyAdUKF, the RiHoMiSyAdUKF is based on more solid concepts (cf. Section 9.4).

The initial intention of Part II of developing Unscented filters for quaternion systems

is materialized by the Riemannian-Spheric Additive Unscented Filters (RiSAdUF's) in Section 9.5. More than being just a particular form the Riemannian Unscented filters of Section 9.3, these Riemannian-Spheric filters are *computationally-implementable*.

Concepts of the Riemannian manifolds theory can be very abstract, but usually computer languages are not designed to work with such level of abstraction. Instead, often we have to work either with closed forms of particular cases or even with numerical approximations. We presented closed *forms of almost all the operations* performed in these filters—such as exponential mappings, logarithm mappings, and parallel transports—; *only sample means of Riemannian σ -representations still have to be found numerically*.

We showed that the RiSAdUF's are better than the Quaternionic Additive UF's of Section 7.3 (QuAdUF's). The RiSAdUF's have better mathematical properties than the QuAdUF's and, in a numerical example, one form of the RiSAdUF outperformed the USQUE of [48] (it is a well-established QuAdUF of the literature) by a great margin.

Unscented filters for dual quaternion systems are introduced in Section 9.6. Unit quaternions are computationally-efficient representations of rotations, and unit dual quaternions can be viewed as the extension of unit quaternions to representations of rigid body displacements—rotations along with translations. The filters of Section 9.6 are the *first consistent Unscented filters for dual quaternion systems*, and are based on the Riemannian Unscented filters of Section 9.3.

In Section 9.7, the continuous-time and continuous-discrete-time variants of the Riemannian filters of Section 9.3.3 were introduced for the *first* time in the literature.

10. CONCLUSIONS OF THIS THESIS

In Chapter 2, we provided an analysis of the literature of discrete-time Unscented Kalman filtering on Euclidean manifolds. We were able to observe several problems concerning

1. the matching order of the transformed covariance (Sections 2.4.1 and 2.6.2) and the transformed cross-covariance (Sections 2.4.2 and 2.6.3) of both the Unscented Transformation (UT) and of the Scaled Unscented Transformation (SUT);
2. multiple UKF definitions (Section 2.3.1);
3. issues with the reduced sets of [45], [46] and [83] (Section 2.5);
4. the conservativeness of the SUT (Section 2.6.1);
5. the scaling effect of the SUT on both the transformed covariance and cross-covariance (Sections 2.6.2 and 2.6.3);
6. possibly ill-conditioned results in the square-root Unscented Kalman Filters (Section 2.7.1);
7. definitions for the Additive Unscented Kalman Filters (Section 2.8).

These problems along with the difficulty in gathering all results related to the Unscented theory reveal a lack of foundation in terms of mathematical principles, and also the absence of mathematical solutions generalizing the sigma sets, UT's and UKF's of the literature. In order to address these needs, we propose *a systematization of the Unscented Kalman filter theory* that treats the construction of UKF's in parts.

We start the construction of this theory by considering diverse forms of estimating the expected value of a transformed random vector (Section 3.1). Motivated by this problem, we propose a key concept of our systematization: the *lth order N points σ -representation* ($lthN\sigma R$, Definition 3.1); essentially, σ -representations are weighted sets whose sample moments up to a certain order are equal to the ones of a given random vector.

By proposing a matrix form of the $lthN\sigma R$'s (Theorem 3.1), we discovered some key properties of these representations, to know i) the minimum number of sigma points of an $lthN\sigma R$ (Corollary 3.1), ii) the minimum number of sigma points of an symmetric $lthN\sigma R$ (Corollary 3.1), and iii) the form of the $lthN\sigma R$ of a the random

vector $Z = aX + b$ when the l th $N\sigma$ R of X is known (Corollary 3.2). With this third result, the l th $N\sigma$ R of a random vector Z can be found by first calculating the l th $N\sigma$ R of an associated random vector X with mean equal to zero vector, and covariance equal to the identity matrix.

Lead by these other two results, results i) and ii), we found a) closed forms for the minimum symmetric σ R's (Section 3.3)—when the order of the l th $N\sigma$ R is 2, we can omit the reference to it (to l); we can also omit the reference to the number of sigma points (N)—, and b) a closed form for the minimum σ R's (Section 3.4).

One of the closed forms of the minimum symmetric σ R's (the Homogeneous Minimum Symmetric σ R) is equivalent to the classical symmetric sigma sets of [1,2] (Table 2.1); therefore, with this we show the reasons behind these sigma sets which, until now, were based only on intuitive ideas. In fact, heretofore, it was not known that these sigma sets are composed by the smallest amount of symmetric sigma points.

As for the closed form for the minimum σ R's, it turned out to be *the only existing consistent minimum σ R*; we showed that this σ R is a general case of the only other consistent minimum σ R of the literature (Corollary 3.5).

The initial motivational problem of estimating the expected value of a *transformed* random vector is not completely solved by the σ R's. A solution to this problem is actually given by the *Unscented Transformations* (UT's).

The concept of an UT follows naturally from the one of σ -representations. A σ -representation's goal is to approximate *a random vector*, and an UT's goal is to approximate *a transformed random vector*.

There are many ways to approximate a transformed random vector. An UT, particularly, does it by using a σ -representation of the previous random vector. Therefore, we can say that the approximation of an UT is based on matching the moments of an random vector—recall that a σ -representation is defined as being a weighted set matching the moments, up to a certain order, of a given random vector.

Even though definitions for the UT already exist in the literature, in Chapter 2 we showed that they present some drawbacks. Therefore, in Chapter 4, we present a *definition of the UT* (Definition 4.1). This new definition is more general than the ones of the literature; our UT is defined for any order l (the order of the used l th $N\sigma$ R), while as far as our knowledge goes, the higher UT's order of the literature is 5 (the UT of [47]). Besides, based on Taylor Series expansions, we provide *the estimation quality of the an l th order UT* (Theorem 4.1)—recall, from Chapter 2, that there were some errors in the UT's estimation quality, and that some UT's elements' estimation accuracy, such as the cross-covariances', were not yet determined.

Further in Chapter 4, we propose new definitions for i) *the scaled UT variants* in Section 4.2, and ii) for *the square-root UT variants* in Section 4.3—recall, from Chapter 2, that also all these UT variants need to be corrected in some way. We are able to show that our definitions of scaled UT's and square-root UT are particular cases of our UT definition of Section 4.1. With this result, the properties already developed for the UT are naturally extended to the scaled and square-root variants. Moreover, we present *an analysis of the influence of the scaling parameter* on the estimation quality of the scaled UT variants, and introduce, for the first time in the literature, *a scaled square-root UT variant*. In Section 4.4, some properties of the UT's developed in this chapter are verified in numerical simulations.

With the defined σ R's and UT's, we are endowed with the necessary tools to study the Unscented Kalman Filters (UKF's) in more detail and to provide new consistent definitions.

There are many UKF definitions. In order to investigate from which of these we would construct the new definitions, we first tackle the problems presented in Section 2.8 regarding the Additive UKF's of the literature (Section 5.1)—for instance, when (2.1) is linear, the estimates of most of the AdUKF's are not equivalent to the linear KF's one (cf. Section 2.8). We use the results of Chapter 4 regarding the UT's to study the possible causes for the inconsistencies of these filters. This study reveals that *only one definition of the AdUKF's is consistent with the additive dynamic system*. Based in this consistent Additive UKF, we present the definition for the discrete-time Unscented Kalman Filters (Section 5.2).

By extending this new filter, we present new definitions for i) a square-root variant (Section 5.3), ii) an UKF variant for the more general system (2.2) (Section 5.2), and iii) a square-root variant of this UKF for system (2.2) (Section 5.2).

Further, in Section 5.4, we provide a list of particular cases of these filters *showing that all consistent UKF's of the literature are embodied by our systematization*. Then, in Section 5.5, we provide comments relative to computational aspects of the proposed UKF filters.

Afterwards, we extend even further our systematization of the Unscented Filter. In Section 5.7 we comment how *higher order Unscented filters could be defined*, and in Section 5.8 we *propose continuous-time and continuous-discrete-time variants of the proposed Unscented filters*. Numerical examples illustrating the results of this chapter are given (Section 5.6).

With Chapter 5, we end the theoretical part of our systematization of the Unscented Kalman filtering theory for systems in the form of (2.1) and (2.2). In this systematization, new results were introduced, some problems were solved, and some

scientific qualities—such as elegance, formalism, and cohesion—were achieved.

Up to this points, only analytical and numerical examples were presented to illustrate the new results. Completing the triad of scientific results—theory, simulation, and experiment—in Chapter 6, we present an *experimental/technological innovation* using some of the new UKF's developed in the preceding chapters; these filters are used to estimate the position of an automotive electronic throttle valve. Besides being a practical application of the UKF theory developed so far, this throttle valve's estimation is *also an innovation on its own*, from the technological point of view.

The findings of Chapter 6 have practical implications, with special interest to automotive electronic throttle devices. Throttle devices often have a unique sensor that measures the angular position of a throttle's valve; thus, failures in this solitary sensor increase risks of damage in the whole system. Wishing to mitigate the impact of a failure from the sensor of position, we suggest an approach that joins UKF's with measurements produced by a wattmeter.

The novelty here relies on the use of a wattmeter to measure the electric power consumed by the throttle. As detailed in Remark 6.1, the wattmeter was preferred due to its low cost. However, any other kind of instruments could be used in place of a wattmeter without necessity to modify the proposed technique.

Measurements from the wattmeter feed UKF's, and these filters, in their turn, generate estimates for the position of the throttle. To the best of our knowledge, this work is the first to combine a filter with an external sensor aiming to improve a throttle's functionality. Experiments that were carried out in laboratory showed promising results.

Chapter 6 closes Part I. In this part, by reviewing the Unscented Kalman filtering theory's state-of-the-art, we show some inconsistencies and gaps within this theory (Chapter 2). In consequence, in Chapters 3, 4 and 5 we propose a systematization that is able to clear these inconsistencies and fill these gaps. Besides, new results were introduced with this systematization. Most of the results provided by this systematization are illustrated in numerical examples. Finally, in Chapter 6, a new experimental/technological technique was proposed using some of the new UKF's proposed with in the preceding chapter.

Overall, in Part I, we developed a consistent Unscented Kalman filter theory which has been verified in numerical simulations and a practical experiment.

All the theory developed in Part I is *based in the concepts of stochastic dynamic systems*; either in their discrete-time forms (2.1) and (2.2), or in the their continuous-

time form (5.43) and continuous-discrete-time forms (5.44). Note that, for all these systems, the variables—the state vector, measurement vector, and noises—take values in *Euclidean spaces*. Such Euclidean systems can be used to model numerous practical problems; yet, for certain practical problems, it might be better to use other classes of systems.

When we want to determine a dynamical model involving *rotations and/or orientations*, it may be advantageous to use *unit quaternions*, rather than rotation matrices—these matrices are the natural way to model rotations in an 3-dimensional Euclidean space. Hence, we can consider stochastic dynamic systems where at least some of their variables are unit quaternions; in this case, we could question whether the systematization developed in Part I can be extended to such unit quaternion systems.

The Unscented literature already has some Unscented filters for quaternions systems. Hence, in Chapter 7, we analyze all the diverse UKF's and SRUKF's for quaternion systems proposed in the literature. From this analysis, we show that i) a considerable amount of these filters *do not preserve the norm of the unit quaternions*; and ii) *all UKF's preserving the norm of the unit quaternions are particular cases of a new algorithm*, namely of the *Quaternionic Additive Unscented Kalman Filter (QuAdUKF)*, Section 7.3.1). Indeed, the QuAdUKF can result in each of these filters of the literature by particular choices of i) the σ -representation, ii) weighted mean method of a unit quaternion set, and iii) vector parameterization of the set of unit quaternions (S^3 , possible choices are provided).

We also introduce a square-root extension of the QuAdUKF, the *Quaternionic Additive Square-Root Unscented Kalman Filter (QuAdSRUKF)*, *having better properties than all the SRUKF's for quaternion systems of the literature* (Section 7.3.2). Comparative with the UKF's of the literature, the QuAdSRUKF is computationally more stable in ill-conditioned situations because of its square-root properties; and comparative with the SRUKF's of the literature, the QuAdSRUKF is always computationally more stable because it has less (or even none) Cholesky factor downdatings (Section 7.3.2). These superior properties of the QuAdSRUKF were verified in numerical simulations considering the Unscented filters (UKF's and SRUKF's) for attitude systems in two problems (Section 7.4.2): 1) a theoretical problem with the performance of the filters being deteriorated by round-off errors; and 2) a satellite attitude estimation problem in two different situations considering i) normal conditions, ii) and computationally ill-conditioned conditions. In two of all these three situations [the only situation of the problem 1), and the situations ii) of the problem 2)], the QuAdSRUKF provided reliable estimates, but all the Unscented filters for attitude systems of the literature did not. Besides, even in normal conditions [situation i) of problem 2)], the QuAdSRUKF outperformed the Unscented filters of the literature by presenting better estimates (the

second smallest mean error was 10,56% higher than the error of the QuAdSRUKF).

The initial goal of Chapter 7 was to extend the systematization of Part I to quaternion systems. However, from the analysis developed in that chapter, we can conclude that the UKF's for quaternions systems of the literature were built upon some intuitive, but not mathematically-sound concepts; indeed, we can cite the following properties upon which these UKF's are built:

1. The additive quaternion models *are not consistent* (cf. Remark 7.1).
2. Some of the probability and statistic concepts for the quaternion space *need further study*. For instance, it is not clear what are the definitions and properties of i) quaternionic random variables, their distributions, and their statistics; ii) the statistics of quaternionic weighted sets (such as quaternionic σ -representations); iii) the statistics of a transformed quaternionic random variable.
3. The form of the filters are extended from the Euclidean filters *without enough explanation*. For instance, what is the reason behind the correction equations of these UKF's [e.g. step (2d) of the QuAdUKF]? What kind of approximation does it provide?

Our solution to extend the systematization of Part I to quaternion systems is based on *Riemannian manifolds*. We work with manifolds because i) *the set of unit quaternions is a Riemannian manifold*, and ii) *there are some probability and statistic results for Riemannian manifolds in the literature*.

In Chapter 8, we i) present some *results of the literature* regarding statistics intrinsically developed in Riemannian manifolds, ii) made some *extensions* these results of [66]—e.g., among others, definitions of moments are extended—, and iii) propose *other results* regarding statistics in Riemannian manifolds—e.g., among others, moments and sample moments of order higher than 2 (Section 8.3 and 8.6), propositions concerning transformations of Riemannian random points (Section 8.5), and results concerning joint Riemannian random points (Section 8.4).

Using the theory presented in Chapter 8, *we extend the Unscented Kalman filtering systematization developed in Part I to the case of Riemannian manifolds*; we do this constructively.

We initiate the systematization of the theory of Unscented Kalman filters for Riemannian manifolds by introducing the Riemannian σ -representation ($\text{Ri}\sigma\text{R}$, Section 9.1). In Theorem 9.1 *we show that closed forms of the σ -representations can be used to find closed forms for $\text{Ri}\sigma\text{R}$'s*; with this, in Corollary 9.1, we determine i) the minimum number sigma points of a $\text{Ri}\sigma\text{R}$, ii) the minimum number of a symmetric $\text{Ri}\sigma\text{R}$, iii)

closed forms for a minimum $\text{Ri}\sigma\text{R}$, and iv) closed forms for a minimum symmetric $\text{Ri}\sigma\text{R}$.

Similarly to the systematization of Part I, we define the Riemannian Unscented Transformation (RiUT, Section 9.2) based on the the concept of a $\text{Ri}\sigma\text{R}$. Besides, we extend all the UT variants of Chapter 4 to the Riemannian case; among other, we propose the Scaled RiUT, and the Square-Root RiUT.

In Section 9.3, we treat the desired discrete-time Riemannian Unscented *filters*.

We introduce a *definition of a Riemannian additive system* (Section 9.3.1). These systems are necessary in order to define additive-noise Riemannian Unscented filters, but, generally, Riemannian manifolds are not endowed with sums.

Furthermore, we found *consistent Kalman correction equations* for the Riemannian Unscented filters (Section 9.3.2). To find these equations, we consider, first, a particular case where the state and the measurement belonged to the same manifold (Section 9.3.2.1); only then, by extending this result, we can get to the final form of the Kalman correction equations (Section 9.3.2.2).

In Section 9.3.3, we propose *four new discrete-time Riemannian Unscented Filters*. At the end of this section, we provide a list of numerous variants of these four Riemannian filters (Tables 9.3, 9.1, 9.4, and 9.2); *all these variants are new consistent Riemannian Unscented filters*.

Further, in Section 9.4, we compared our Riemannian Unscented filters with the only Unscented Kalman filter of the literature, namely the Unscented Kalman Filter for Riemannian manifolds (UKFRM) of [171]. The UKFRM of [171] is *essentially* different from all the filters of Tables 9.3, 9.1, 9.4, and 9.2, except for one: the Riemannian Homogeneous Minimum Symmetric AdUKF (RiHoMiSyAdUKF, Table 9.4 [1,1]). Yet, even though there are similarities between the UKFRM of [171] and the RiHoMiSyAdUKF, the RiHoMiSyAdUKF is based on more solid concepts (cf. Section 9.4).

The initial intention of Part II of developing Unscented filters for quaternion systems is materialized by the Riemannian-Spheric Additive Unscented Filters (RiSAdUF's) in Section 9.5. More than being just a particular form the Riemannian Unscented filters of Section 9.3, these Riemannian-Spheric filters are *computationally-implementable*.

Concepts of the Riemannian manifolds theory can be very abstract, but usually computer languages are not designed to work with such level of abstraction. Instead, often we have to work either with closed forms of particular cases or even with numerical approximations. We present closed *forms of almost all the operations* performed in these filters—such as exponential mappings, logarithm mappings, and par-

allel transports—; *only sample means of Riemannian σ -representations still have to be found numerically.*

We show that the RiSAdUF's are better than the Quaternionic Additive UF's (QuAdUF's) of Section 7.3. The RiSAdUF's have better mathematical properties than the QuAdUF's and, in a numerical example, one form of the RiSAdUF outperforms the USQUE of [48] (it is a well-established QuAdUF of the literature) by a great margin.

Unscented filters for dual quaternion systems are introduced in Section 9.6. Unit quaternions are computationally-efficient representations of rotations, and unit dual quaternions can be viewed as the extension of unit quaternions to representations of rigid body displacements—rotations along with translations. The filters of Section 9.6 are the *first consistent Unscented filters for dual quaternion systems*, and are based on the Riemannian Unscented filters of Section 9.3.

In Section 9.7, the continuous-time and continuous-discrete-time variants of the Riemannian filters of Section 9.3.3 were introduced for the *first time* in the literature.

Overall, we can say that, in this work, *we developed a new, consistent Unscented Kalman filtering theory for Euclidean and Riemannian manifolds.*

10.1 FUTURE WORK

For future work, we suggest extending the present work by providing the following results:

1. an analysis of stability and convergence of all the Unscented Filters presented in this thesis. There are works in the literature treating this topic for some Unscented Filters (e.g. [50, 176]).
2. square-root continuous-time filters. The literature already have some continuous-time and continuous-discrete-time SRUKF's (e.g. [52]).
3. computationally-implementable Riemannian Unscented Filters to other Riemannian manifolds besides the S^3 , such as the projective spaces, special orthogonal groups, special Euclidean groups, among others.
4. applications of some of the proposed Unscented Filters.

10.2 SCIENTIFIC PUBLICATIONS

In the course of our research, some results of this thesis resulted in scientific publications.

- The works published in scientific journals are the following ones:
 1. A. N. Vargas, H. M. T. Menegaz, J. Y. Ishihara, and L. Acho, “Unscented Kalman Filters for Estimating the Position of an Automotive Electronic Throttle Valve,” *IEEE Transactions on Vehicular Technology.*, vol. 65, no. 6, pp. 4627–4632, Jun. 2016.
 2. H. M. T. Menegaz, J. Y. Ishihara, G. A. Borges, and A. N. Vargas, “A Systematization of the Unscented Kalman Filter Theory,” *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2583–2598, Oct. 2015.
 3. H. M. Menegaz, J. Y. Ishihara, and G. A. Borges, “New minimum sigma set for unscented filtering,” *International Journal of Robust and Nonlinear Control*, online preview;
- The works published in scientific conferences are the following on:
 1. H. M. T. Menegaz, J. Y. Ishihara, and P. P. M. Magro, “A Unscented Kalman Filter for Attitude Estimation of Satellites,” in *Proceedings of the Simpósio Brasileiro de Automação Inteligente (SBAI)*, 2015.
 2. C. Ochoa-Diaz, H. M. Menegaz, A. P. L. Bó, and G. A. Borges, “An EKF-based approach for estimating leg stiffness during walking,” in *Proceedings of the Annual International Conference. IEEE Eng. Medicine and Biology Society*, 2013, pp. 7226–7228.
 3. H. M. Menegaz, P. H. R. Q. A. Santana, J. Y. Ishihara, and G. A. Borges, “Scaled Minimum Unscented Multiple Hypotheses Mixing Filter,” in *Proceedings of the IEEE American Control Conference*, 2013, pp. 2466–2471.

We highlight that our work “A Systematization of the Unscented Kalman Filter Theory,” (item 2. of the published journals above) has been one of *the five most popular articles* of the *IEEE Transactions on Automatic Control* (Figure 10.1).

Moreover, in the following months, we intend proposing at least three works as scientific publications; one work for each of the following results of this thesis:

- the analysis of consistency of AdUKF’s presented in Sections 2.8 and 5.1;

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<p>Convergence of an Upwind Finite-Difference Scheme for Hamilton–Jacobi–Bellman Equation in Optimal Control</p> <p>Bing Sun ; Bao-Zhu Guo</p> <p>Out-26 2015</p>	<p>Nonlinear Complementary Filters on the Special Orthogonal Group</p> <p>Mahony, R. ; Hamel, T. ; Pfimlin, Jean-Michel</p> <p>Ago-26 2008</p>
<p>Second-Order Integral Sliding Mode Control for</p>	<p>A Systematization of the Unscented Kalman Filter Theory</p> <p>Menegaz, H.M.T. ; Ishihara, J.Y. ; Borges, G.A. ; Vargas, A.N.</p> <p>Set-23 2015</p>

Figure 10.1: Screenshot of the IEEE Transactions on Automatic Control’s webpage. This screenshot was taken at 10:18 a.m. (time of Brasília) on Friday, November the 20th, 2015.

- the analysis of the Unscented Filters for additive-noise quaternion models developed in Chapter 7.
- the Unscented filtering theory for Riemannian manifolds of Chapter 9.

Bibliography

- [1] JULIER, S. J.; UHLMANN, J. K. Unscented Filtering and Nonlinear Estimation. *Proc. of the IEEE*, v. 92, n. 3, p. 401–422, 2004. ISSN 0018-9219.
- [2] JULIER, S. J.; UHLMANN, J. K.; DURRANT-WHYTE, H. F. A new approach for filtering nonlinear systems. In: *Proc. IEEE American Control Conf.* Seattle, WA: [s.n.], 1995. p. 1628–1632.
- [3] MENG, J.; LUO, G.; GAO, F. Lithium Polymer Battery State-of-Charge Estimation Based on Adaptive Unscented Kalman Filter and Support Vector Machine. *IEEE Transactions on Power Electronics*, v. 31, n. 3, p. 2226–2238, 2016. ISSN 0885-8993.
- [4] AUNG, H.; LOW, K. S. Temperature dependent state-of-charge estimation of lithium ion battery using dual spherical unscented Kalman filter. *IET Power Electronics*, v. 8, n. 10, p. 2026–2033, 2015. ISSN 1755-4535.
- [5] AUNG, H.; LOW, K. S.; GOH, S. T. State-of-Charge Estimation of Lithium-Ion Battery Using Square Root Spherical Unscented Kalman Filter (Sqrt-UKFST) in Nanosatellite. *IEEE Transactions on Power Electronics*, v. 30, n. 9, p. 4774–4783, 2015. ISSN 0885-8993.
- [6] PARTOVIBAKHSH, M.; LIU, G. An Adaptive Unscented Kalman Filtering Approach for Online Estimation of Model Parameters and State-of-Charge of Lithium-Ion Batteries for Autonomous Mobile Robots. *IEEE Transactions on Control Systems Technology*, v. 23, n. 1, p. 357–363, jan 2015. ISSN 1063-6536.
- [7] ZHANG, J.; XIA, C. State-of-charge estimation of valve regulated lead acid battery based on multi-state Unscented Kalman Filter. *Int. J. Elect. Power Energy Syst.*, Elsevier Ltd, v. 33, n. 3, p. 472–476, mar 2011. ISSN 01420615. Disponível em: <<http://linkinghub.elsevier.com/retrieve/pii/S0142061510001973>>.
- [8] HUA, K.; MISHRA, Y.; LEDWICH, G. Fast Unscented Transformation-Based Transient Stability Margin Estimation Incorporating Uncertainty of Wind Generation. *IEEE Transactions on Sustainable Energy*, v. 6, n. 4, p. 1254–1262, 2015. ISSN 1949-3029.
- [9] EMAMI, K. et al. Application of Unscented Transform in Frequency Control of a Complex Power System Using Noisy PMU Data. *IEEE Transactions on Industrial Informatics*, v. 12, n. 2, p. 853–863, 2016. ISSN 1551-3203.

- [10] LORENZ, M. et al. Estimation of Non-Idealities in Sigma-Delta Modulators for Test and Correction Using Unscented Kalman Filters. *IEEE Transactions on Circuits and Systems I: Regular Papers*, v. 62, n. 5, p. 1240–1249, 2015. ISSN 1549-8328.
- [11] LORENZ, M.; BECKER, J.; ORTMANN, M. Hybrid of unscented Kalman filter and genetic algorithm for state and parameter estimation in sigma-delta modulators. *Electronics Letters*, v. 51, n. 17, p. 1318–1320, 2015. ISSN 0013-5194.
- [12] NARASIMHAPPA, M.; SABAT, S. L.; NAYAK, J. Adaptive sampling strong tracking scaled unscented Kalman filter for denoising the fibre optic gyroscope drift signal. *IET Science, Measurement Technology*, v. 9, n. 3, p. 241–249, 2015. ISSN 1751-8822.
- [13] RAHIMI, A.; KUMAR, K. D.; ALIGHANBARI, H. Enhanced adaptive unscented Kalman filter for reaction wheels. *IEEE Transactions on Aerospace and Electronic Systems*, v. 51, n. 2, p. 1568–1575, 2015. ISSN 0018-9251.
- [14] CHENG, G. et al. Tractography From HARDI Using an Intrinsic Unscented Kalman Filter. *IEEE Transactions on Medical Imaging*, v. 34, n. 1, p. 298–305, jan 2015. ISSN 0278-0062.
- [15] ENAYATI, N.; MOMI, E. D.; FERRIGNO, G. A Quaternion-Based Unscented Kalman Filter for Robust Optical/Inertial Motion Tracking in Computer-Assisted Surgery. *IEEE Trans. Instrum. Meas*, v. 64, n. 8, p. 2291–2301, jan 2015.
- [16] EBERLE, C.; AMENT, C. The Unscented Kalman Filter estimates the plasma insulin from glucose measurement. *BioSystems*, v. 103, n. 1, p. 67–72, 2011. ISSN 1872-8324.
- [17] GOH, S. T.; ZEKAVAT, S. A. ; PAHLAVAN, K. DOA-Based Endoscopy Capsule Localization and Orientation Estimation via Unscented Kalman Filter. *IEEE Sensors Journal*, v. 14, n. 11, p. 3819–3829, nov 2014. ISSN 1530-437X.
- [18] TIAN, Y.; CHEN, Z.; YIN, F. Distributed IMM-Unscented Kalman Filter for Speaker Tracking in Microphone Array Networks. *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, v. 23, n. 10, p. 1637–1647, 2015. ISSN 2329-9290.
- [19] KOLOURI, S.; AZIMI-SADJADI, M. R.; ZIEMANN, A. Acoustic Tomography of the Atmosphere Using Unscented Kalman Filter. *IEEE Transactions on Geoscience and Remote Sensing*, v. 52, n. 4, p. 2159–2171, 2014. ISSN 0196-2892.

- [20] HAVANGI, R. et al. A Square Root Unscented FastSLAM With Improved Proposal Distribution and Resampling. *IEEE Transactions on Industrial Electronics*, v. 61, n. 5, p. 2334–2345, 2014. ISSN 0278-0046.
- [21] HOLMES, S.; KLEIN, G.; MURRAY, D. W. An $O(N(2))$ Square Root Unscented Kalman Filter for Visual Simultaneous Localization and Mapping. *IEEE Trans. Pattern Anal. Mach. Intell.*, v. 31, n. 7, p. 1251–63, 2009. ISSN 0162-8828.
- [22] HOLMES, S.; KLEIN, G.; MURRAY, D. A square root unscented Kalman filter for visual monoSLAM. In: *Proc. Int. Conf. on Robotics and Automation*. [S.l.: s.n.], 2008. p. 3710–3716.
- [23] MENEGAZ, H. M. T. et al. A Systematization of the Unscented Kalman Filter Theory. *IEEE Trans. Autom. Control*, v. 60, n. 10, p. 2583–2598, oct 2015. ISSN 0018-9286.
- [24] JAZWINSKI, A. H. *Stochastic Processes and Filtering Theory*. [S.l.]: Academic Press, 1970. 373 p.
- [25] ANDERSON, B. D. O.; MOORE, J. B. *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979. 357 p. (Information and System Science Series, 2). ISSN 0-13-638122-7.
- [26] SIMON, D. *Optimal State Estimation. Kalman, H_∞ , and Nonlinear Approaches*. Hoboken, New Jersey: John Wiley & Sons, 2006. 526 p. ISBN 9780471708582.
- [27] CRASSIDIS, J. L.; MARKLEY, F. L.; CHENG, Y. Survey of Nonlinear Attitude Estimation Methods. *J. Guid. Control Dynam.*, v. 30, n. 1, p. 12–28, 2007. ISSN 0731-5090.
- [28] FILIPE, N.; KONTITSIS, M.; TSIOTRAS, P. Extended Kalman Filter for Spacecraft Pose Estimation Using Dual Quaternions. *Journal of Guidance, Control, and Dynamics*, p. 1–17, 2015. ISSN 0731-5090. Disponível em: <<http://arc.aiaa.org/doi/10.2514/1.G000977>>.
- [29] BONNABEL, S. Left-invariant extended Kalman filter and attitude estimation. In: *Proc. IEEE Conference on Decision and Control*. New Orleans, LA: [s.n.], 2007. p. 1027–1032. ISBN 978-1-4244-1497-0. Disponível em: <<http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=4434662>>.
- [30] SHENOI, B. A. *Introduction to Digital Signal Processing and Filter Design*. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2005. 440 p. ISBN 9780471656371.
- [31] MITSCHKE, F.; MÖLLER, M.; LANGE, W. Measuring filtered chaotic signals. *Phys. Rev. A*, v. 37, n. 11, p. 4518–4521, jun 1988. ISSN 0556-2791.

- [32] KALMAN, R. A new approach to linear filtering and prediction problems. *Trans. ASME J. Basic Eng.*, v. 82, p. 35–45, 1960.
- [33] ALTMANN, S. L. *Rotations, quaternions, and double groups*. New York, NY: Oxford University Press, 1986. 312 p. ISBN 0198553722.
- [34] MARKLEY, F. L. Attitude Error Representations for Kalman Filtering. *J. Guid. Control Dynam.*, v. 26, n. 2, p. 311–317, mar 2003.
- [35] ADORNO, B. V. *Two-arm Manipulation: From Manipulators to Enhanced Human-Robot Collaboration*. 163 p. Tese (Doutorado) — Laboratoire d’Informatique, de Robotique et de Microélectronique de Montpellier (LIRMM) Université Montpellier 2, 2011.
- [36] GODINHO, L.; NATÁRIO, J. *An Introduction to Riemannian Geometry: With Applications to Mechanics and Relativity*. [S.l.]: Springer International Publishing, 2014. 467 p. (Universitext). ISBN 978-3-319-08666-8.
- [37] JULIER, S. J.; UHLMANN, J. K. Consistent debiased method for converting between polar and Cartesian coordinate systems. In: *Proc. SPIE*. [S.l.: s.n.], 1997. v. 3086, p. 110–121. ISSN 0277786X.
- [38] JULIER, S. J.; UHLMANN, J. K. *A General Method for Approximating Nonlinear Transformations of Probability Distributions*. 1994.
- [39] JULIER, S. J.; UHLMANN, J. K. A New Extension of the Kalman Filter to Nonlinear Systems. In: *Proc. SPIE AeroSense*. [S.l.: s.n.], 1997. v. 3068, p. 182–193.
- [40] JULIER, S. J.; UHLMANN, J. K.; DURRANT-WHYTE, H. F. A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators. *IEEE Trans. Autom. Control*, v. 45, n. 3, p. 477–482, 2000.
- [41] WAN, E. A.; MERWE, R. van der. The unscented Kalman Filter for Nonlinear Estimation. In: *Proc. IEEE Adaptive Syst. Signal Process. Commun. Control Symp.* Lake Louise, Alta: [s.n.], 2000. p. 153–158. ISBN 0780358007.
- [42] MERWE, R. van der; WAN, E. A. The square-root unscented Kalman filter for state and parameter-estimation. In: *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.* Salt Lake City, UT: [s.n.], 2001. v. 6, p. 3461–3464. ISBN 0-7803-7041-4.
- [43] MERWE, R. van der et al. The Unscented Particle Filter. In: *Proc. Advances Neural Inform. Process. Syst. (NIPS’13)*. [S.l.: s.n.], 2000.
- [44] JULIER, S. J. The scaled unscented transformation. In: *Proc. IEEE American Control Conf.* Anchorage, AK: [s.n.], 2002. v. 6, p. 4555–4559. ISBN 0-7803-7298-0.

- [45] JULIER, S. J. Reduced sigma point filters for the propagation of means and covariances through nonlinear transformations. In: *Proc. IEEE American Control Conf.* Anchorage, AK: [s.n.], 2002. v. 2, p. 887–892.
- [46] JULIER, S. J. The spherical simplex unscented transformation. In: *Proc. IEEE American Control Conf.* Denver, CO: [s.n.], 2003. v. 3, p. 2430–2434. ISBN 0780378962.
- [47] LERNER, U. N. *Hybrid Bayesian networks for reasoning about complex systems.* 275 p. Tese (Ph. D.) — Stanford University, 2002.
- [48] CRASSIDIS, J. L.; MARKLEY, F. L. Unscented Filtering for Spacecraft Attitude Estimation. *J. Guid. Control Dynam.*, v. 26, n. 4, p. 536–542, 2003. ISSN 0731-5090.
- [49] WU, Y. et al. Unscented Kalman filtering for additive noise case: augmented versus nonaugmented. *IEEE Signal Processing Letters*, v. 12, n. 5, p. 357–360, may 2005. ISSN 1070-9908.
- [50] XIONG, K.; ZHANG, H. Y.; CHAN, C. W. Performance evaluation of UKF-based nonlinear filtering. *Automatica*, v. 42, n. 2, p. 261–270, feb 2006. ISSN 00051098.
- [51] XIONG, K.; ZHANG, H.; CHAN, C. Author's reply to "Comments on 'Performance evaluation of UKF-based nonlinear filtering'". *Automatica*, v. 43, n. 3, p. 569–570, mar 2007. ISSN 00051098.
- [52] SÄRKKÄ, S. On Unscented Kalman Filtering for State Estimation of Continuous-Time Nonlinear Systems. *IEEE Trans. Autom. Control*, v. 52, n. 9, p. 1631–1641, 2007. ISSN 0018-9286.
- [53] SÄRKKÄ, S. Unscented Rauch-Tung-Striebel Smoother. *IEEE Trans. Autom. Control*, v. 53, n. 3, p. 845–849, 2008.
- [54] TEIXEIRA, B. O. S. et al. Gain-Constrained Kalman Filtering for Linear and Nonlinear Systems. *IEEE Trans. Signal Process.*, v. 56, n. 9, p. 4113–4123, sep 2008. ISSN 1053-587X. Disponível em: <<http://ieeexplore.ieee.org/lpdocs/epic03/wrapper.htm?arnumber=4531183>>.
- [55] TEIXEIRA, B. O. et al. State estimation for linear and non-linear equality-constrained systems. *Int. J. Control*, v. 82, n. 5, p. 918–936, may 2009. ISSN 0020-7179.
- [56] TEIXEIRA, B. O. S. et al. On unscented Kalman filtering with state interval constraints. *Journal of Process Control*, v. 20, n. 1, p. 45–57, jan 2010. ISSN 0959-1524.

- [57] MENEGAZ, H. M.; ISHIHARA, J. Y.; BORGES, G. A. A new smallest sigma set for the Unscented Transform and its applications on SLAM. In: *Proc. IEEE Conf. Decis. Control and European Control Conf.* Orlando, FL: [s.n.], 2011. p. 3172–3177.
- [58] GARCÍA-FERNÁNDEZ, Á. F.; MORELANDE, M. R.; GRAJAL, J. Truncated Unscented Kalman Filtering. *IEEE Trans. Signal Process.*, v. 60, n. 7, p. 3372–3386, 2012. ISSN 1053-587X.
- [59] DUNÍK, J.; ŠIMANDL, M.; STRAKA, O. Unscented Kalman Filter: Aspects and Adaptive Setting of Scaling Parameter. *IEEE Trans. Autom. Control*, v. 57, n. 9, p. 2411–2416, sep 2012. ISSN 0018-9286.
- [60] GUSTAFSSON, F.; HENDEBY, G. Some Relations Between Extended and Unscented Kalman Filters. *IEEE Trans. Signal Process.*, v. 60, n. 2, p. 545–555, 2012. ISSN 1053-587X.
- [61] XINLONG, W.; BIN, W.; HENGNIAN, L. An autonomous navigation scheme based on geomagnetic and starlight for small satellites. *Acta Astron.*, v. 81, n. 1, p. 40–50, 2012. ISSN 00945765.
- [62] BARCZYK, M.; LYNCH, A. F. Invariant Extended Kalman Filter Design for a Magnetometer-plus-GPS Aided Inertial Navigation System. In: *Proc. IEEE Conf. Decision Control and European Control Conf.* Orlando, FL: [s.n.], 2011. p. 5389–5394. ISBN 9781612848013.
- [63] BARCZYK, M.; LYNCH, A. F. Invariant Observer Design for a Helicopter UAV Aided Inertial Navigation System. *IEEE Trans. Control Syst. Technol.*, v. 21, n. 3, p. 791–806, may 2013.
- [64] GUPTA, M. et al. A Robust Visual Human Detection Approach With UKF-Based Motion Tracking for a Mobile Robot. *IEEE Syst. J.*, v. 9, n. 4, p. 1363–1375, dec 2015. ISSN 1932-8184.
- [65] SWENSEN, J. P.; COWAN, N. J. An Almost Global Estimator on $SO(3)$ with Measurement on S^2 . In: *Proc. IEEE American Control Conf.* Montreal, Canada: [s.n.], 2012. p. 1780–1786. ISBN 9781457710964.
- [66] PENNEC, X. Intrinsic Statistics on Riemannian Manifolds: Basic Tools for Geometric Measurements. *J. Math. Imaging Vision*, v. 25, n. 1, p. 127–154, jul 2006. ISSN 0924-9907.
- [67] MERWE, R. van der. *Sigma-point Kalman filters for probabilistic inference in dynamic state-space models*. 378 p. Tese (Doutorado) — Oregon Health & Science University, 2004.

- [68] SÄRKKÄ, S. *Bayesian Filtering and Smoothing*. Cambridge, UK: Cambridge University Press, 2013.
- [69] DAUM, F. Nonlinear filters: beyond the Kalman filter. *IEEE Trans. Aerosp. Electron. Syst.*, v. 20, n. 8, p. 57–69, 2005.
- [70] ATHANS, M.; WISHNER, R.; BERTOLINI, A. Suboptimal state estimation for continuous-time nonlinear systems from discrete noisy measurements. *IEEE Trans. Autom. Control*, v. 13, n. 5, p. 504–514, 1968. ISSN 0018-9286.
- [71] ALSPACH, D.; SORENSON, H. Nonlinear Bayesian estimation using Gaussian sum approximations. *IEEE Trans. Autom. Control*, v. 17, n. 4, p. 439–448, 1972.
- [72] ŠIMANDL, M.; KRÁLOVEC, J.; SÖDERSTRÖM, T. Advanced point-mass method for nonlinear state estimation. *Automatica*, v. 42, n. 7, p. 1133–1145, 2006. ISSN 00051098.
- [73] DOUCET, A.; FREITAS, N. de; GORDON, N. *Sequential Monte Carlo Methods in Practice*. [S.l.]: Springer, 2001. 582 p.
- [74] CAPPE, O.; GODSILL, S. J.; MOULINES, E. An Overview of Existing Methods and Recent Advances in Sequential Monte Carlo. *Proc. of the IEEE*, v. 95, n. 5, p. 899–924, 2007. ISSN 0018-9219.
- [75] BUGALLO, M. F.; DJURIC, P. M. Joint Model Selection and Parameter Estimation by Population Monte Carlo Simulation. *IEEE J. Sel. Topics Signal Process.*, v. 4, n. 3, p. 526–539, 2010. ISSN 1932-4553.
- [76] CRISAN, D.; ROZOVSKII, B. *The Oxford Handbook of Nonlinear Filtering*. Oxford, UK: Oxford University Press, 2011.
- [77] SPALL, J. C. Estimation via Markov chain Monte Carlo. *IEEE Control Syst. Mag.*, v. 23, n. 2, p. 34–45, 2003.
- [78] ITO, K.; XIONG, K. Gaussian filters for nonlinear filtering problems. *IEEE Trans. Autom. Control*, v. 45, n. 5, p. 910–927, may 2000. ISSN 00189286.
- [79] NØRGAARD, M.; POULSEN, N.; RAVN, O. New developments in state estimation for nonlinear systems. *Automatica*, v. 36, n. 11, p. 1627–1638, 2000.
- [80] ARASARATNAM, I.; HAYKIN, S. Cubature Kalman Filters. *IEEE Trans. Autom. Control*, v. 54, n. 6, p. 1254–1269, 2009.
- [81] ARASARATNAM, I.; HAYKIN, S.; HURD, T. R. Cubature Kalman Filtering for Continuous-Discrete Systems: Theory and Simulations. *IEEE Trans. Signal Process.*, v. 58, n. 10, p. 4977–4993, 2010.

- [82] LEFEBVRE, T.; BRUYNINCKX, H.; SCHUTTER, J. D. Comment on "A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators". *IEEE Trans. Autom. Control*, v. 47, n. 8, p. 1406–1408, aug 2002.
- [83] WAN-CHUN, L.; PING, W.; XIAN-CI, X. A novel simplex unscented transform and filter. In: *Proc. IEEE Int. Symp. Comm. Inform. Technologies*. [S.l.: s.n.], 2007. p. 926–931. ISBN 978-1-4244-0976-1.
- [84] CHARALAMPIDIS, A. C.; PAPAVALASSILOPOULOS, G. P. Development and numerical investigation of new non-linear Kalman filter variants. *IET Control Theory & Applications*, v. 5, n. 10, p. 1155, aug 2011. ISSN 17518644.
- [85] WU, Y. et al. A Numerical-Integration Perspective on Gaussian Filters. *IEEE Trans. Signal Process.*, v. 54, n. 8, p. 2910–2921, aug 2006.
- [86] JULIER, S. J. A skewed approach to Filtering. In: *Proc. SPIE 3373, Conf. Signal Data Process. Small Targets*. Orlando, FL: [s.n.], 1998. p. 271–282.
- [87] CHARALAMPIDIS, A. Computationally Efficient Kalman Filtering For a Class of Nonlinear Systems. *IEEE Trans. Autom. Control*, v. 56, n. 3, p. 483–491, mar 2011.
- [88] GREWAL, M. S.; ANDREWS, A. P. *Kalman Filtering: Theory and Practice: Using Matlab*. New York, NY: John Wiley & Sons, 2001. 401 p. ISBN 0471392545.
- [89] TANG, X.; ZHAO, X.; ZHANG, X. The square-root spherical simplex unscented Kalman filter for state and parameter estimation. In: *Proc. IEEE Int. Conf. Signal Process.* [S.l.: s.n.], 2008. p. 260–263. ISBN 978-1-4244-2178-7.
- [90] LIU, Z. et al. An improved square root unscented Kalman filter for projectile's attitude determination. In: *Proc. IEEE Conf. Ind. Electron. Applicat.* [S.l.: s.n.], 2010. p. 1747–1751. ISBN 9781424450466.
- [91] TENNE, D.; SINGH, T. The higher order unscented filter. In: *Proc. IEEE American Control Conf.* Denver, CO: [s.n.], 2003. v. 3, p. 2441–2446. ISBN 0-7803-7896-2.
- [92] BJÖRCK, A. *Numerical methods for least squares problems*. [S.l.]: SIAM, 1996. 408 p. ISBN 0898713609.
- [93] ARASARATNAM, I.; HAYKIN, S. Square-Root Quadrature Kalman Filtering. *IEEE Trans. Signal Process.*, v. 56, n. 6, p. 2589–2593, 2008. ISSN 1053-587X.
- [94] CARNEIRO, M. L. et al. Doherty amplifier optimization using robust genetic algorithm and Unscented Transform. In: *Proc. IEEE Int. New Circuits Syst. Conf.* [S.l.: s.n.], 2011. p. 77–80. ISBN 9781612841373.

- [95] SÄRKKÄ, S. Continuous-time and continuous-discrete-time unscented Rauch-Tung-Striebel smoothers. *Signal Process.*, v. 90, n. 1, p. 225–235, 2010. ISSN 01651684.
- [96] MERWE, R. van der; WAN, E. Sigma-Point Kalman Filters for Probabilistic Inference in Dynamic State-Space Models. In: *Proc. Workshop on Advances in Machine Learning*. [S.l.: s.n.], 2003.
- [97] MENEGAZ, H. M.; ISHIHARA, J. Y.; BORGES, G. A. New minimum sigma set for unscented filtering. *Int. J. Robust Nonlinear Control*, p. n/a—n/a, oct 2014. ISSN 10498923.
- [98] BAR-SHALOM, Y.; LI, X.-R.; KIRUBARAJAN, T. *Estimation with applications to tracking and navigation*. New York: John Wiley & Sons, 2001.
- [99] NIA, V. P. Gauss-Hermite Quadrature: Numerical or Statistical Method? In: *Proc. Iranian Stat. Conf.* [S.l.: s.n.], 2006. p. 209–215.
- [100] GIESE, T. J. *Numerical quadrature*. [S.l.], 2006.
- [101] JÄCKEL, P. *A note on multivariate Gauss-Hermite quadrature*. [S.l.], 2005. 1–5 p.
- [102] HASCELIK, A. I. Gauss quadrature rules for a generalized Hermite weight function. *Applied Math. Comput.*, v. 180, n. 1, p. 86–96, 2006. ISSN 00963003.
- [103] ARASARATNAM, I.; HAYKIN, S.; ELLIOTT, R. J. Discrete-Time Nonlinear Filtering Algorithms Using Gauss-Hermite Quadrature. *Proc. of the IEEE*, v. 95, n. 5, p. 953–977, 2007. ISSN 0018-9219.
- [104] DAVIS, P. J.; RABINOWITZ, P. *Methods of Numerical Integration*. 2nd. ed. San Diego, CA: Academic Press, 1984. (Computer Science and Applied Mathematics).
- [105] KROESE, D. P.; TAIMRE, T.; BOTEV, Z. I. *Handbook of Monte Carlo Methods*. [S.l.]: John Wiley & Sons, 2011. 772 p.
- [106] CHARALAMPIDIS, A. C.; PAPAVALASSILOPOULOS, G. P. Kalman Filtering for a Generalized Class of Nonlinear Systems and a New Gaussian Quadrature Technique. *IEEE Trans. Autom. Control*, v. 57, n. 11, p. 2967–2973, nov 2012. ISSN 0018-9286.
- [107] WALTERS-WILLIAMS, J.; LI, Y. Comparison of extended and unscented Kalman Filters applied to EEG signals. In: *Proc. IEEE/ICME Int. Conf. Complex Medical Eng. (ICCME'10)*. Gold Coast, QLD: [s.n.], 2010. p. 45–51. ISBN 9781424468430.

- [108] SAKAI, A.; KURODA, Y. Discriminatively Trained Unscented Kalman Filter for Mobile Robot Localization. *J. Advanced Research Mech. Eng.*, v. 1, n. 3, p. 153–161, 2010.
- [109] STRAKA, O. et al. Aspects and Comparison of Matrix Decompositions in Unscented Kalman Filter. In: *Proc. IEEE American Control Conf.* Washington, DC: [s.n.], 2013. p. 3075–3080. ISBN 9781479901784.
- [110] CAO, Y. *ukf.m*. Disponível em: <<http://www.mathworks.com/matlabcentral/fileexchange/18217-learning-the-unscented-kalman-filter>>.
- [111] DYER, D. *ukf.cpp*. Disponível em: <<https://github.com/sfwa/ukf/blob/master/src/ukf.cpp>>.
- [112] GOLUB, G. H.; LOAN, C. F. V. *Matrix Computations*. 4th. ed. Baltimore, Maryland: John Hopkins University Press, 2013. ISBN 9781421407944.
- [113] BENAVALI, A. The Generalized Moment-Based Filter. *IEEE Trans. Autom. Control*, v. 58, n. 10, p. 2642–2647, oct 2013. ISSN 0018-9286.
- [114] CONATSER, R. et al. Diagnosis of automotive electronic throttle control systems. *Control Engineering Practice*, v. 12, n. 1, p. 23–30, 2004.
- [115] DEUR, J. et al. An electronic throttle control strategy including compensation of friction and limp-home effects. *IEEE Trans. Industry Appl.*, v. 40, n. 3, p. 821–834, 2004.
- [116] BERNARDO, M. di et al. Experimental validation of the discrete-time {MCS} adaptive strategy. *Control Engineering Practice*, v. 21, n. 6, p. 847–859, 2013.
- [117] BERNARDO, M. di et al. Synthesis and Experimental Validation of the Novel {LQ-NEMCSI} Adaptive Strategy on an Electronic Throttle Valve. *IEEE Trans. Control Syst. Technol.*, v. 18, n. 6, p. 1325–1337, 2010.
- [118] GREPL, R.; LEE, B. Modeling, parameter estimation and nonlinear control of automotive electronic throttle using a Rapid-Control Prototyping technique. *International Journal of Automotive Technology*, v. 11, n. 4, p. 601–610, 2010.
- [119] VASAK, M. et al. Hybrid Theory-Based Time-Optimal Control of an Electronic Throttle. *IEEE Trans. Industrial Electronics*, v. 54, n. 3, p. 1483–1494, 2007.
- [120] PAN, Y.; OZGUNER, U.; DAGCI, O. H. Variable-Structure Control of Electronic Throttle Valve. *IEEE Trans. Industrial Electronics*, v. 55, n. 11, p. 3899–3907, 2008.

- [121] MONTANARO, U.; GAETA, A. di; GIGLIO, V. Robust Discrete-Time {MRAC} With Minimal Controller Synthesis of an Electronic Throttle Body. *IEEE/ASME Transactions on Mechatronics*, v. 19, n. 2, p. 524–537, 2014.
- [122] REICHHARTINGER, M.; HORN, M. Application of Higher Order Sliding-Mode Concepts to a Throttle Actuator for Gasoline Engines. *IEEE Trans. Industrial Electronics*, v. 56, n. 9, p. 3322–3329, 2009.
- [123] FINCH, J. Toyota Sudden Acceleration: A Case Study of the National Highway Traffic Safety Administration—Recalls for Change. *Loyola Consumer Law Review*, v. 22, n. 4, p. 472–496, 2010. Disponível em: <<http://lawecommons.luc.edu/cgi/viewcontent.cgi?article=1055&context=lclr>>.
- [124] GEORGE, E.; PECHT, M. Tin whisker analysis of an automotive engine control unit. *Microelectronics Reliability*, v. 54, n. 1, p. 214–219, jan 2014. ISSN 00262714. Disponível em: <<http://linkinghub.elsevier.com/retrieve/pii/S0026271413003107>>.
- [125] SOOD, B.; OSTERMAN, M.; PECHT, M. Tin whisker analysis of Toyota’s electronic throttle controls. *Circuit World*, v. 37, n. 3, p. 4–9, aug 2011. ISSN 0305-6120. Disponível em: <<http://www.emeraldinsight.com/doi/abs/10.1108/03056121111155611>>.
- [126] YUAN, X.; WANG, Y.; WU, L. {SVM}-Based Approximate Model Control for Electronic Throttle Valve. *IEEE Trans. Vehicular Technology*, v. 57, n. 5, p. 2747–2756, 2008.
- [127] JIAO, X.; ZHANG, J.; SHEN, T. An Adaptive Servo Control Strategy for Automotive Electronic Throttle and Experimental Validation. *IEEE Transactions on Industrial Electronics*, v. 61, n. 11, p. 6275–6284, 2014.
- [128] ROSSI, C.; TILLI, A.; TONIELLI, A. Robust control of a throttle body for drive by wire operation of automotive engines. *IEEE Trans. Control Syst. Technol.*, v. 8, n. 6, p. 993–1002, 2000.
- [129] THOMASSON, A.; ERIKSSON, L. Model-Based Throttle Control using Static Compensators and Pole Placement. *Oil & Gas Science and Technology–Rev. IFP*, v. 66, n. 4, p. 717–727, 2011.
- [130] MORARI, M.; BAOTIC, M.; BORRELLI, F. Hybrid Systems Modeling and Control. *European Journal of Control*, v. 9, n. 2-3, p. 177–189, 2003.
- [131] YUAN, X. et al. Neural Network Based Self-Learning Control Strategy for Electronic Throttle Valve. *IEEE Trans. Vehicular Technology*, v. 59, n. 8, p. 3757–3765, 2010.

- [132] GRAFAREND, E. W. *Linear and nonlinear models: fixed effects, random effects, and mixed models*. Berlin, Germany: Walter de Gruyter GmbH, 2006.
- [133] SICILIANO, B. et al. *Robotics Modelling, Planning and Control*. [S.l.]: Springer, 2009. (Advanced Textbooks in Control and Signal Processing). ISBN 978-1-84628-641-4.
- [134] SPONG, M.; HUTCHINSON, S.; VIDYASAGAR, M. *Robot modeling and control*. John Wiley & Sons Hoboken, NJ, 2006. Disponível em: <<http://elib.tu-darmstadt.de/tocs/134922867.pdf>>.
- [135] SELIG, J. M. Lie Groups and Lie Algebras in Robotics. In: BYRNES, J. (Ed.). *Computational Noncommutative Algebra and Applications*. Springer Netherlands, 2005. p. 101–125. ISBN 978-1-4020-1982-1. Disponível em: <http://link.springer.com/chapter/10.1007/1-4020-2307-3_5>.
- [136] GILITSCHENSKI, I. et al. Unscented Orientation Estimation Based on the Bingham Distribution. *IEEE Trans. Autom. Control*, v. 61, n. 1, p. 172–177, jan 2016. ISSN 0018-9286.
- [137] KURZ, G.; GILITSCHENSKI, I.; HANEBECK, U. D. Unscented von Mises-Fisher Filtering. *IEEE Signal Processing Letters*, v. 23, n. 4, p. 463–467, 2016. ISSN 1070-9908.
- [138] CHEON, Y.-J.; KIM, J.-H. Unscented Filtering in a Unit Quaternion Space for Spacecraft Attitude Estimation. In: *Proc. IEEE Int. Symp. Ind. Elect.* Vigo: [s.n.], 2007. p. 66–71. ISBN 978-1-4244-0754-5.
- [139] TANG, X.; YAN, J.; ZHONG, D. Square-root sigma-point Kalman filtering for spacecraft relative navigation. *Acta Astron.*, v. 66, n. 5-6, p. 704–713, 2010. ISSN 00945765.
- [140] KHODER, W.; FASSINUT-MOMBOT, B.; BENJELLOUN, M. Inertial navigation attitude velocity and position algorithms using quaternion Scaled Unscented Kalman filtering. In: *Proc. IEEE Ind. Electron. Annu. Conf.* Orlando, FL: [s.n.], 2008. p. 1754–1759. ISBN 978-1-4244-1767-4.
- [141] VACCARELLA, A. et al. Unscented Kalman Filter Based Sensor Fusion for Robust Optical and Electromagnetic Tracking in Surgical Navigation. *IEEE Trans. Instrum. Meas.*, v. 62, n. 7, p. 2067–2081, jul 2013.
- [142] AMIRHOSSEINI, S. F. et al. Stochastic Cloning Unscented Kalman Filtering for Pedestrian Localization Applications. In: *Proc. IEEE Int. Pos. Indoor Navig.* Montbeliard-Belfort: [s.n.], 2013. p. 28–31. ISBN 9781479940431.

- [143] KRAFT, E. A Quaternion-based Unscented Kalman Filter for Orientation Tracking. In: *Proc. IEEE Int. Conf. Inform. Fusion*. Cairns, Queensland: [s.n.], 2003. v. 1, p. 47–54.
- [144] LARSEN, J. A.; VINTHER, K. Inexpensive CubeSat attitude estimation using COTS components and Unscented Kalman Filtering. In: *Proc. IEEE Int. Conf. Recent Advances Space Techn.* Istanbul: [s.n.], 2011. p. 341–346. ISBN 978-1-4244-9617-4.
- [145] Laviola Jr., J. J. A Comparison of Unscented and Extended Kalman Filtering for Estimating Quaternion Motion. In: *Proc. IEEE American Control Conf.* Denver, CO: [s.n.], 2003. v. 3, p. 2435–2440. ISBN 0780378962.
- [146] ROMANOVAS, M. et al. Efficient Orientation Estimation Algorithm for low cost inertial and magnetic sensor systems. In: *Proc. IEEE/SP Worksh. Stat. Signal Process.* Cardiff: [s.n.], 2009. p. 586–589. ISBN 9781424427109.
- [147] SEKHAVAT, P.; GONG, Q.; ROSS, I. M. NPSAT1 Parameter Estimation Using Unscented Kalman Filtering. In: *Proc. IEEE American Control Conf.* New York, NY: [s.n.], 2007. p. 4445–4451. ISBN 1-4244-0988-8. ISSN 0743-1619.
- [148] SIPOS, B. Application of the manifold-constrained unscented Kalman filter. In: *Proc. IEEE/ION Posit., Location Nav. Symp.* Monterey, CA: [s.n.], 2008. p. 30–43. ISBN 1424415373.
- [149] SOKEN, H. E.; SAKAI, S.-i.; SCIENCE, A. UKF Based On-Orbit Gyro and Magnetometer Bias Estimation as a part of the Attitude Determination Procedure for a Small Satellite. In: *Proc. Int. Conf. Control. Automat. Syst.* Gyeonggi-do: [s.n.], 2011. p. 1891–1896. ISBN 9788993215038.
- [150] XING, Y.; CAO, X.; ZHANG, S. Modified UKF for integrated orbit and attitude determination based on gyro and magnetometer. In: *Proc. IEEE Int. Symp. Syst. Control Aerosp. Astron.* Shenzhen: [s.n.], 2008. p. 1–4. ISBN 978-1-4244-2385-9.
- [151] XUEDONG, W. et al. An Application of Unscented Kalman Filter for Pose and Motion Estimation Based on Monocular Vision. In: *Proc. IEEE Int. Symp. Ind. Elect.* Montréal, Québec: [s.n.], 2006. v. 4, p. 2614–2619. ISBN 1424404975.
- [152] YANG, Y.; ZHOU, J.; LOFFELD, O. Quaternion-based Kalman Filtering on INS/GPS. In: *Proc. IEEE Int. Conf. Inform. Fusion*. Singapore: [s.n.], 2012. p. 511–518.

- [153] ZHANG, Z.-Q. et al. Adaptive Information Fusion for Human Upper Limb Movement Estimation. *IEEE Trans. Syst., Man, Cybern. {A}*, v. 42, n. 5, p. 1100–1108, 2012. ISSN 1083-4427.
- [154] ZHAO, L.; NIE, Q.; GUO, Q. Unscented Kalman Filtering for SINS Attitude Estimation. In: *Proc. IEEE Int. Conf. Control Automat.* Guangzhou: [s.n.], 2007. p. 228–232. ISBN 978-1-4244-0817-7.
- [155] CHANG, L.; HU, B.; CHANG, G. Modified Unscented Quaternion Estimator Based on Quaternion Averaging. *J. Guid. Control Dynam.*, v. 37, n. 1, p. 305–308, jan 2014. ISSN 0731-5090. Disponível em: <<http://arc.aiaa.org/doi/abs/10.2514/1.61723>>.
- [156] JULIER, S. J.; Laviola Jr., J. J. On Kalman Filtering with Nonlinear Equality Constraints. *IEEE Trans. Signal Process.*, v. 55, n. 6, p. 2774–2784, 2007.
- [157] KONDO, A.; DOKI, H.; HIROSE, K. An Estimation Method of 3D posture Using Quaternion-based Unscented Kalman filter. In: *IEEE Proc. SICE Annu. Conf.* Nagoya: [s.n.], 2013. p. 78–83.
- [158] LE, H. X.; MATUNAGA, S. A residual based adaptive unscented Kalman filter for fault recovery in attitude determination system of microsatellites. *Acta Astronautica*, Elsevier, v. 105, n. 1, p. 30–39, dec 2014. ISSN 00945765. Disponível em: <<http://linkinghub.elsevier.com/retrieve/pii/S0094576514003270>>.
- [159] SILVA, N. B. F. da; WILSON, D. B.; BRANCO, K. R. L. J. Performance Evaluation of the Extended Kalman Filter and Unscented Kalman Filter. In: *Proc. IEEE Conf. Unman. Aircr. Syst.* Denver, CO: [s.n.], 2015. p. 733 – 741. ISBN 9781479960095.
- [160] VARTIAINEN, P. et al. Nonlinear State-Space Modeling of Human Motion Using 2-D Marker Observations. *IEEE Trans. Biomed. Eng.*, v. 61, n. 7, p. 2167–2178, jul 2014. ISSN 15582531.
- [161] LEEGHIM, H.; CHOI, Y.; JAROUX, B. A. Uncorrelated unscented filtering for spacecraft attitude determination. *Acta Astron.*, v. 67, n. 1-2, p. 135–144, 2009. ISSN 0094-5765.
- [162] COHN, M. P. *Further Algebra and Applications*. London, UK: Springer-Verlag, 2003. 452 p.
- [163] SCHAUB, H.; JUNKINS, J. L. Stereographic Orientation Parameters For Attitude Dynamics: A Generalization of the Rodrigues Parameters. *J. Astronaut. Sci.*, v. 44, n. 1, p. 1–19, jan 1996.

- [164] SOKEN, H.; HAJIYEV, C. Adaptive Unscented Kalman Filter with multiple fading factors for pico satellite attitude estimation. In: *International Conference on Recent Advances in Space Technologies*. Istanbul, Turkey: [s.n.], 2009. p. 541–546. ISBN 9781424436286.
- [165] PENNEC, X. *Computing the Mean of Geometric Features Application to the Mean Rotation*. [S.l.], 1998.
- [166] MARKLEY, F. L. et al. Averaging quaternions. *J. Guid. Control Dynam.*, v. 30, n. 4, p. 1675–1682, jul 2007. ISSN 00653438.
- [167] CRASSIDIS, J. L.; JUNKINS, J. L. *Optimal Estimation of Dynamic Systems*. 2nd. ed. [S.l.]: CRC Press, 2012. 733 p. ISBN 9781439839867.
- [168] MARDIA, K. V.; JUPP, P. E. *Directional Statistics*. West Sussex, England: John Wiley & Sons, Inc., 2000. (Wiley Series in Probability and Statistics). ISBN 0471 953334.
- [169] KARCHER, H. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math.*, v. 30, n. 5, p. 509–541, 1977.
- [170] KENDALL, W. S. Probability, convexity, and harmonic maps with small image I: uniqueness and fine existence. *Proc. London Math. Soc.*, v. 61, n. 2, p. 371–406, sep 1990.
- [171] HAUBERG, S.; LAUZE, F.; PEDERSEN, K. S. Unscented Kalman Filtering on Riemannian Manifolds. *J. Math. Imaging Vis.*, v. 46, n. 1, p. 103–120, jan 2013.
- [172] LORENZI, M.; PENNEC, X. Efficient Parallel Transport of Deformations in Time Series of Images: From Schild’s to Pole Ladder. *J. Math. Imaging Vision*, Springer US, v. 50, n. 1-2, p. 5–17, sep 2014.
- [173] PENNEC, X. Probabilities and statistics on Riemannian manifolds: Basic tools for geometric measurements. In: *Int. Workshop Nonl. Signal Image Process*. Antalya: [s.n.], 1999. v. 1, p. 194–198.
- [174] ABSIL, P.-A.; MAHONY, R.; SEPULCHRE, R. *Optimization Algorithms on Matrix Manifolds*. [S.l.]: Princeton University Press, 2008. xvi+224 p. ISBN 978-0-691-13298-3.
- [175] SELIG, J. *Geometric Fundamentals of Robotics*. 2nd. ed. [S.l.]: Springer, 2005. 398 p. (Monographs in Computer Science). ISBN 0387208747.
- [176] WU, Y.; HU, D.; HU, X. Comments on "Performance evaluation of UKF-based nonlinear filtering". *Automatica*, v. 43, n. 3, p. 567–568, 2007. ISSN 00051098.

- [177] Do Carmo, M. P. *Riemannian Geometry*. Woodbine, NJ: Birkhäuser Boston, 1992. (Mathematics: Theory & Applications). ISBN 0817634908.
- [178] SPIVAK, M. *A Comprehensive Introduction to Differential Geometry*. 3rd. ed. Houston, TX: Publish or Perish, 1999. v. 1. ISBN 0914098721 9780914098720. Disponível em: <<http://www.maa.org/publications/maa-reviews/a-comprehensive-introduction-to-differential-geometry-vol-i>>.
- [179] PENNEC, X. *L'Incertitude dans les problemes de reconnaissance et de recalage: application en imagerie medicale et biologie moleculaire*. Tese (PhD Thesis) — Ecole Polytechnique, Palaiseau, dec 1996.
- [180] BERGER, M. *A Panoramic View of Riemannian Geometry*. Berlin: Springer-Verlag, 2003. v. 28. 73–74 p. ISSN 0343-6993. ISBN 978-3-642-18245-7.
- [181] CHAVEL, I. *Riemannian Geometry: A Modern Introduction*. 2nd. ed. [S.l.]: Cambridge University Press, 2006. (Cambridge Studies in Advanced Mathematics (No. 98)). ISBN 9780511616822.
- [182] JOST, J. *Riemannian Geometry and Geometric Analysis*. 6th. ed. [S.l.]: Springer, 2011. (Universitytext). ISBN 9783642212970.
- [183] Do Carmo, M. P. *Differential geometry of curves and surfaces*. Canada: Pearson Education, 1976. 503 p. ISBN 0132125897.
- [184] LEE, J. M. *Introduction to Smooth Manifolds*. 2nd. ed. [S.l.]: Springer, 2013. 218 p. (Graduate Texts in Mathematics). ISBN 9781441999818.
- [185] BANCHOFF, T.; LOVETT, S. *Differential Geometry of Curves and Surfaces*. Boca Raton, FL: A K Peter/CRC Press, 2010. ISBN 9781439894057.
- [186] LIMA, E. L. *Elementos de Topologia Geral*. [S.l.]: IMPA, 1970. (Elementos de matemática).

A. RIEMANNIAN MANIFOLDS

In this appendix, we provide the results respective to the theory of Riemannian manifolds that are required to develop our theory of Unscented Kalman filtering on these manifolds (Chapter 9). We present the concepts of Riemannian manifold, geodesic, exponential and logarithm mappings, parallel transport, among others.

This presentation is based, mainly, on [177] (our notation is also similar), and, in less degree, on [66, 174] and [178]. For more information on the topic, we suggest consulting, apart from [174, 177–179], [36, 180–182]; for the readers not familiar with the theory of differential geometry, it might be interesting to study, beforehand, works introductory to this field, such as [183, 184] and [185].

In this work, a *differentiable function (or mapping, or transformations)* will mean that it is of class C^∞ (differentiable for all degrees of differentiation)—this nomenclature is used in [177], our main source on Riemannian manifolds, and in other classical works on the topic such as [178].

A.1 DIFFERENTIABLE MANIFOLDS AND TANGENT SPACES

The notion of surface is intuitive from the real world where human beings live. In a mathematical sense, a (regular) surface can be defined as a set $S \subset \mathbb{R}^3$ whose subsets are identified with subsets of the \mathbb{R}^2 by *charts* (injective mappings) [183]. This notion can be extended to more abstract and general concepts giving birth, for instance, to the so called *differentiable (smooth) manifolds*.

Charts are fundamental concepts for defining smooth manifolds. Consider a set \mathcal{M} ; a chart is a pair (U, φ) where U is an open subset of \mathbb{R}^n , and $\varphi : U \mapsto \mathcal{U}$ is a bijection (a one-to-one correspondence) from U to a subset \mathcal{U} of \mathcal{M} . When there is no risk of confusion, we can simply write φ for (U, φ) , and call φ a chart. We point out that φ is defined as being $\varphi : \mathcal{U} \mapsto U$ by part of the differential geometry literature (cf. [174, 178]). Note that, generally, there exists more than one chart for each point $\mathbf{q} \in \mathcal{M}$.

Definition A.1 (Differentiable manifold [177, 178]). A *differentiable manifold* (or a C^∞ manifold or a smooth manifold) of dimension n is a pair $(\mathcal{M}, \mathcal{A})$ where \mathcal{M} is a set and $\mathcal{A} = \{(U_a, \varphi_a)\}$ a family of injective mappings (*charts*) $\varphi_a : U_a \subset \mathbb{R}^n \rightarrow \mathcal{M}$ of

open sets U_a of \mathbb{R}^n into \mathcal{M} such that:

1. $\bigcup_a \varphi_a(U_a) = \mathcal{M}$.

2. for any pair a, b , with

$$\varphi_a(U_a) \cap \varphi_b(U_b) =: W \neq \emptyset,$$

the sets $\varphi_a^{-1}(W)$ and $\varphi_b^{-1}(W)$ are open sets in \mathbb{R}^n and the mappings $\varphi_b^{-1} \circ \varphi_a$ are $\varphi_b^{-1} \circ \varphi_a$ differentiable (smooth).

3. The family $\mathcal{A} = \{(U_a, \varphi_a)\}$ is *maximal* relative to the conditions 1. and 2.—that is, if a pair (U_a, φ_a) satisfies conditions 1. and 2., then it is contained in \mathcal{A} .

The pair (U_a, φ_a) (or the mapping φ_a) with $\mathbf{q} \in \varphi_a(U_a)$ is called a *parametrization* (or *system of coordinates*) of \mathcal{M} at \mathbf{q} ; $\varphi_a(U_a)$ is then called a *coordinate neighborhood* at \mathbf{q}^a . A family $\{(U_a, \varphi_a)\}$ satisfying 1. and 2. is called an *atlas* of \mathcal{M} or a *differentiable structure* for \mathcal{M} . If an atlas (or a differentiable structure) satisfies 1, then it is called *maximal*. When indication of the dimension n of a differential manifold \mathcal{M} is required, we use the notation \mathcal{M}^n . For simplicity, we will use the name “differentiable manifold” to refer either for the pair $(\mathcal{M}, \mathcal{A})$, or to the set \mathcal{M} ; since \mathcal{A} is unique (because it is maximal), this practice does not introduce confusion.

Figure A.1 illustrates a differentiable manifold. The condition 3 is included for purely technical reasons. Indeed, given a differentiable structure on \mathcal{M} , we can easily complete it to a maximal one by taking the union of all the parameterizations that, together with any of the parameterizations of the given structure, satisfy condition 3. Therefore, with a certain abuse of language, we can say that a differentiable manifold is a set provided with a differentiable structure. In general, the extension to maximal structure will be done without further comment.

In this work, we will treat only the particular class of differentiable manifolds whose *induced topologies* satisfy the following two axioms:

1. **Hausdorff Axiom:** Given two distinct points of \mathcal{M} , there exist neighborhoods of these two points that do not intersect.
2. **Countable Basis Axiom:** \mathcal{M} can be covered by a countable number of coordinate neighborhoods. We, then, say that \mathcal{M} *has a countable basis*.

^aUntil now, a bold notation has been indicating that a given variable belongs to the S^3 (cf. Chapter 7), but it will be extended for now on; henceforth, a bold notation indicates the belonging of a given variable to a Riemannian manifold (e.g. $\mathbf{q} \in \mathcal{M}$); note that this extension does not cause confusion because the S^3 is a Riemannian manifold.

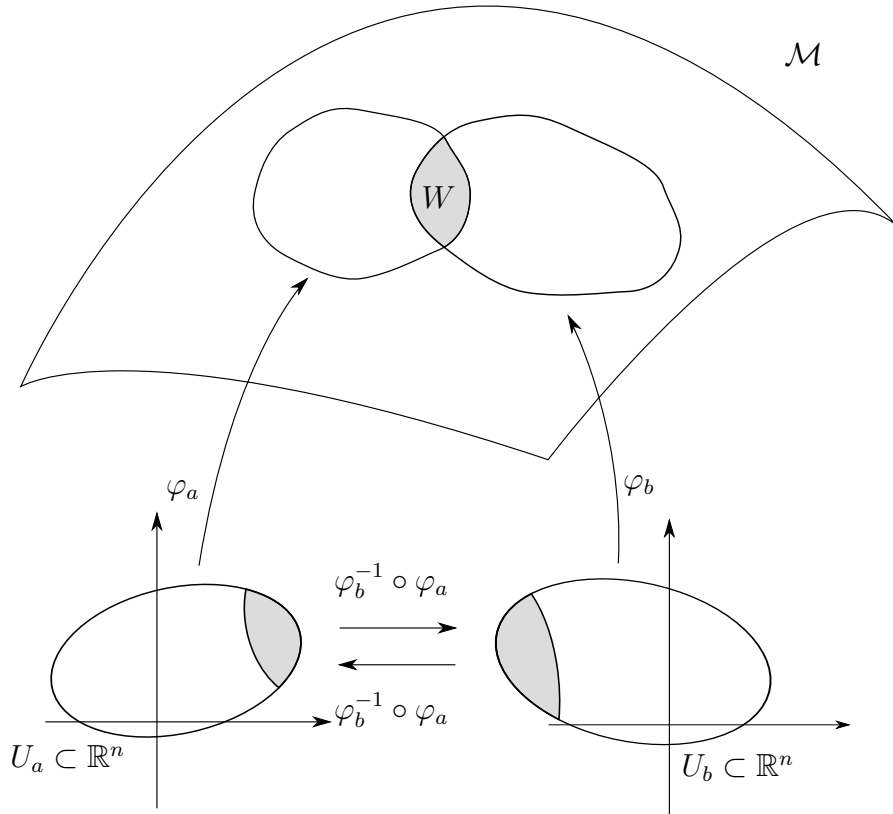


Figure A.1: A differentiable manifold.

Every atlas induces a topology—then we can say that every differentiable manifold induces a topology being the one induced by its maximal atlas; a topology is an abstraction of the notion of open sets in \mathbb{R}^n ; see [174, 186] for more information), but, generally, these topologies allow the existence of pathological behaviors such as convergent sequences having more than one limit point. By restricting the differentiable manifolds to the case of *Hausdorff* (satisfying axiom 1. above) and *second-countable* (satisfying the axiom 2. above) topologies, we exclude numerous strange behaviors. Thus, henceforth, we suppose that all differentiable manifolds are endowed with these classes of topologies.

Examples of differentiable manifolds are the Euclidean space \mathbb{R}^n , unit sphere S^{n-1} , set of $n \times m$ real matrices, set of $n \times m$ ($m \leq n$) real matrices whose columns are linearly independent (this manifold is called the noncompact Stiefel manifold).

Example A.1 (Vector Spaces [174]). The simplest example of a differentiable manifold is the \mathbb{R}^n endowed with the differentiable structure $\{(\varphi, \mathbb{R}^n)\}$ where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n : u \mapsto \mathbf{q}$. Indeed, any vector space is a differentiable manifold. Let \mathcal{V} be a d -dimensional vector space. Then, given the basis $\{e_1, \dots, e_d\}$ of \mathcal{V} , the function

$$\varphi : \mathbb{R}^d \rightarrow \mathcal{V} : [u_1, \dots, u_d]^T \mapsto \mathbf{q}$$

such that

$$\mathbf{q} = \sum_{i=1}^d u_i e_i$$

along with \mathbb{R}^d is a differentiable structure for \mathcal{V} .

Example A.2 (Unit sphere). The S^{n-1} can be viewed as a manifold embedded in the \mathbb{R}^n , meaning that it can be defined with a differentiable structure induced by the one of the \mathbb{R}^n .

A (non-maximal) atlas for the S^{n-1} is given by the family of pairs $\{(U, \varphi^i)\}_{i=1}^{2n}$ where $U = \{u \in \mathbb{R}^{n-1} | u^T u < 1\}$ and, for $u = [u_1, \dots, u_{n-1}]^T$,

$$\begin{aligned} \varphi^i : U &\rightarrow S^{n-1} \\ u &\mapsto \begin{cases} [u_1, \dots, u_{i-1}, \sqrt{1 - u^T u}, u_i, \dots, u_{n-1}]^T & \text{for } i = 1, 3, \dots, 2n - 1; \\ [u_1, \dots, u_{i-1}, -\sqrt{1 - u^T u}, u_i, \dots, u_{n-1}]^T & \text{for } i = 2, 4, \dots, 2n. \end{cases} \end{aligned} \quad (\text{A.1})$$

It can be shown that the conditions 1. and 2. of Definition A.1 are satisfied for $\{(U, \varphi^i)\}_{i=1}^{2n}$.

The *Cartesian product* of two differentiable manifolds is also a differentiable manifold. Given two differentiable manifolds $(\mathcal{M}_1^{n_1}, \{(U_a^1, \varphi_a^1)\})$ and $(\mathcal{M}_2^{n_2}, \{(U_a^2, \varphi_a^2)\})$, the pair $(\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}, \{(U_a^1 \times U_a^2, \varphi_a^1 \times \varphi_a^2)\})$ where $\varphi_a^1 \times \varphi_a^2 : U_a^1 \times U_a^2 \mapsto \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a differentiable manifold. The $S^{n-1} \times \mathbb{R}^m$ will be of particular importance for our purposes.

We will need the idea of differentiable mappings between manifolds. Let \mathcal{M}_1^n and \mathcal{M}_2^m be differentiable manifolds. A mapping $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is *differentiable at* $\mathbf{q} \in \mathcal{M}_1$ if, given a parametrization $\varphi_2 : V \subset \mathbb{R}^m \rightarrow \mathcal{M}_2$ at $f(\mathbf{q})$, there exists a parametrization $\varphi_1 : U \subset \mathbb{R}^n \rightarrow \mathcal{M}_1$ at \mathbf{q} such that $f(\varphi_1(U)) \subset \varphi_2(V)$ and the mapping

$$\tilde{f} := \varphi_2^{-1} \circ f \circ \varphi_1 : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (\text{A.2})$$

is differentiable at $\varphi_1^{-1}(\mathbf{q})$; \tilde{f} is called the *coordinate representation* of f (see Figure A.2). We say that f is differentiable on an open set of \mathcal{M}_1 if it is differentiable at all of the points of this open set. This definition is independent of the choice of the parameterizations. In all this work, *we suppose that all functions are differentiable unless otherwise stated.*

If $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a differentiable bijection and its inverse mapping f^{-1} is differentiable, then f is called a *diffeomorphism*. In this case, \mathcal{M}_1 and \mathcal{M}_2 are said to be *diffeomorphic*. The notion of diffeomorphism is a natural idea of equivalence between differentiable manifolds.

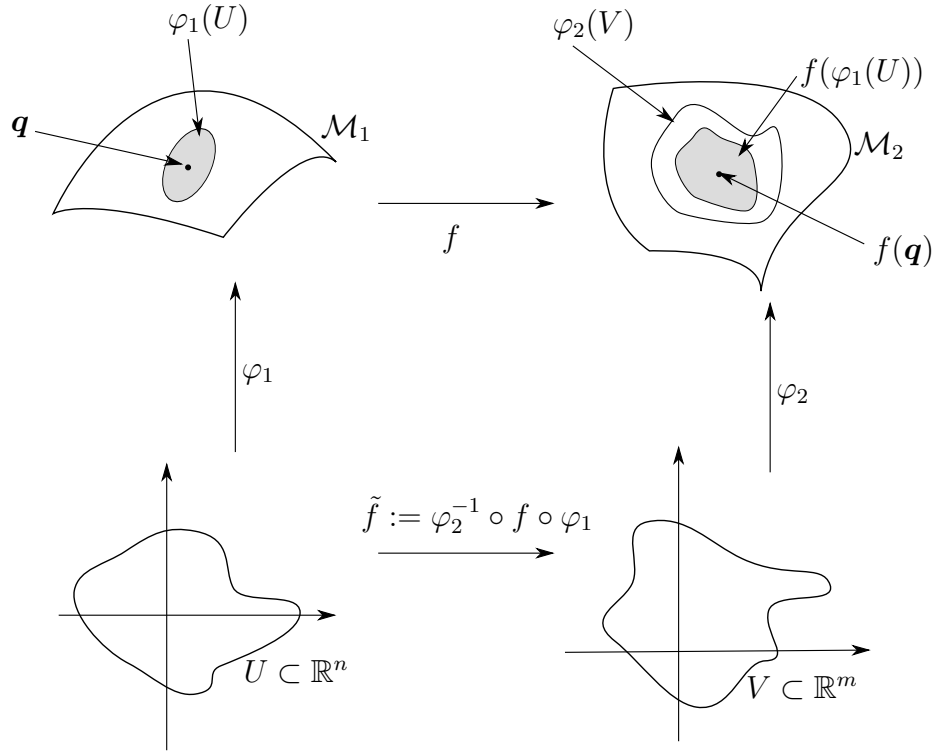


Figure A.2: Representation of a differentiable function.

We can apply differentiable mappings to the parameterizations; this will lead us to the definition of a tangent vector.

Definition A.2. Let \mathcal{M} be a differentiable manifold. A differentiable function $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ is called a (differentiable) *curve* in \mathcal{M} . Suppose that $\gamma(0) = \mathbf{q} \in \mathcal{M}$, and let $\mathcal{D}(\mathcal{M})$ be the set of all functions of the type $f : \mathcal{M} \rightarrow \mathbb{R}$ that are differentiable at \mathbf{q} . The *tangent vector to the curve* γ at $t = 0$ is a function $\gamma'(0) : \mathcal{D}(\mathcal{M}) \rightarrow \mathbb{R}$ given by

$$\gamma'(0)f = \left. \frac{d(f \circ \gamma)}{dt} \right|_{t=0}, \quad f \in \mathcal{D}(\mathcal{M}). \quad (\text{A.3})$$

A *tangent vector at* \mathbf{q} is the tangent vector at $t = 0$ of some curve $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}$ with $\gamma(0) = \mathbf{q}$. Note that $\gamma'(0)$ is an operator taking $f \in \mathcal{D}(\mathcal{M})$ to a scalar $\left. \frac{d(f \circ \gamma)}{dt} \right|_{t=0}$. The set of all tangent vectors to \mathcal{M} at \mathbf{q} will be indicated by $T_{\mathbf{q}}\mathcal{M}$.

If we choose a parametrization $\varphi : U \rightarrow \mathcal{M}^n$ at $\mathbf{q} = \varphi(p) = \gamma(0)$, we can express the function f and the curve γ in this parametrization by

$$f \circ \varphi(u) = \mathbf{f}(u_1, \dots, u_n), \quad u = (u_1, \dots, u_n) \in U,$$

and

$$\varphi^{-1} \circ \gamma(t) = u(t) = (u_1(t), \dots, u_n(t)),$$

respectively. Therefore, restricting \mathbf{f} to γ , we obtain

$$\begin{aligned} \gamma'(0)f &= \frac{d}{dt} (f \circ \gamma)|_{t=0} \\ &= \frac{d}{dt} (f \circ \varphi \circ \varphi^{-1} \circ \gamma)|_{t=0} \\ &= \frac{d}{dt} \mathbf{f}(u(t))|_{t=0} \\ &= \sum_{i=1}^n \frac{\partial \mathbf{f}(u)}{\partial u_i} \Big|_{u=p} \frac{d}{dt} u_i(t) \Big|_{t=0} \\ &= \sum_{i=1}^n \dot{u}_i(t)|_{t=0} \frac{\partial \mathbf{f}(u)}{\partial u_i} \Big|_{u=p} \\ &= \sum_{i=1}^n \dot{u}_i(0) \frac{\partial (f \circ \varphi(u))}{\partial u_i} \Big|_{u=p} \end{aligned} \tag{A.4}$$

For each u_i , the term

$$\frac{\partial (f \circ \varphi(u(t)))}{\partial u_i(t)} \Big|_{u=p}$$

can be interpreted as the tangent vector to the curve $\varphi(u)$ at $\varphi(u)|_{u=p}$; applying this notion on (A.4), we have that

$$\begin{aligned} \gamma'(0)f &= \sum_{i=1}^n \left(\dot{u}_i(0) \frac{\partial \varphi(u)}{\partial u_i} \Big|_{u=p} \right) f \\ &= \left(\sum_{i=1}^n \dot{u}_i(0) \frac{\partial \varphi(u)}{\partial u_i} \Big|_{u=p} \right) f \\ &= \left(\sum_{i=1}^n \dot{u}_i(0) \frac{\partial}{\partial u_i} \Big|_0 \right) f \end{aligned}$$

where

$$\left(\frac{\partial}{\partial u_i} \right)_0 := \frac{\partial \varphi(u(t))}{\partial u_i(t)} \Big|_{t=0}$$

is the tangent vector at \mathbf{q} of the “coordinate curve” (Figure A.3):

$$u_i \mapsto \varphi(0, \dots, 0, u_i, 0, \dots, 0).$$

In other words, the vector $\gamma'(0)$ can be expressed in the parametrization φ by

$$\gamma'(0) = \sum_{i=1}^n \dot{u}_i(0) \left(\frac{\partial}{\partial u_i} \right)_0. \tag{A.5}$$

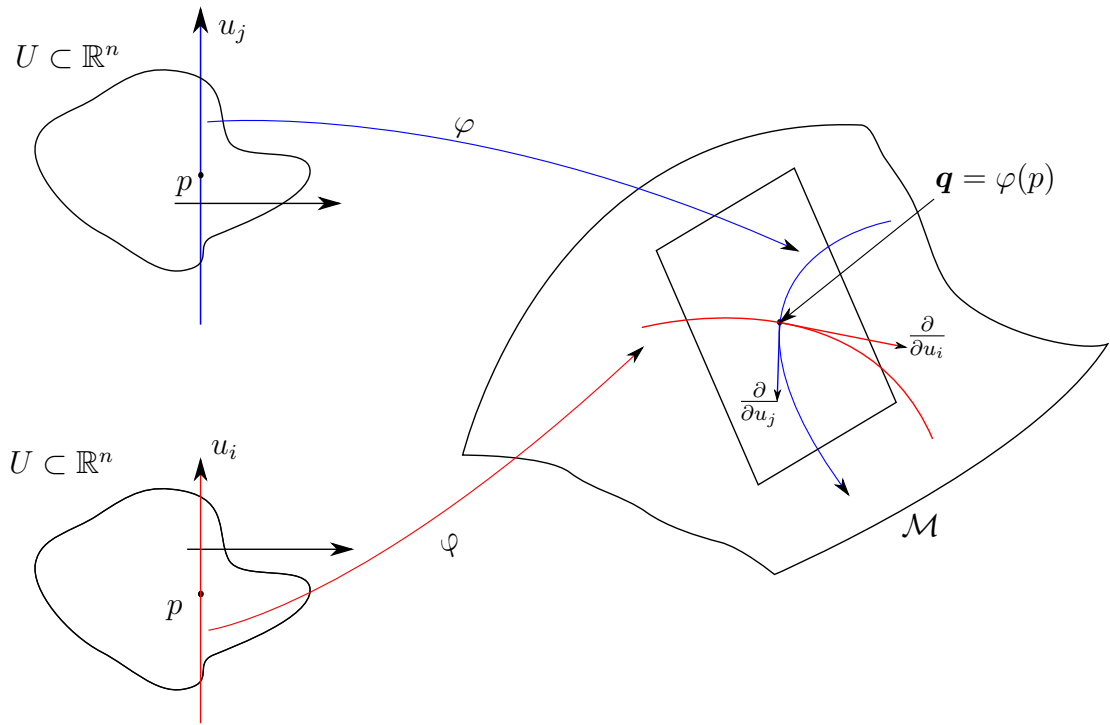


Figure A.3: Tangent vector of “coordinate curves”.

The expression (A.5) shows that the tangent vector to the curve γ at \mathbf{q} depends only on the derivative of γ in a coordinate system. It follows also from (A.5) that the set $T_{\mathbf{q}}\mathcal{M}$, with the usual operations of function, forms a vector space of dimension n , and that the choice of parametrization $\varphi : U \rightarrow \mathcal{M}$ determines an *associated basis*

$$\left\{ \left(\frac{\partial}{\partial u_1} \right)_0, \dots, \left(\frac{\partial}{\partial u_n} \right)_0 \right\}$$

where

$$\left(\frac{\partial}{\partial u_i} \right)_0 := \left. \frac{\partial \varphi(u(t))}{\partial u_i(t)} \right|_{t=0}.$$

in $T_{\mathbf{q}}\mathcal{M}$. It can be shown that this linear structure in $T_{\mathbf{q}}\mathcal{M}$ does not depend on the parametrization φ . The vector space $T_{\mathbf{q}}\mathcal{M}$ is called *tangent space of \mathcal{M} at \mathbf{q}* .

The tangent vector $\dot{\gamma}(0)$ should be distinguished from the *time derivative (vector)* (see Figure A.4)

$$\dot{\gamma}(t) := \lim_{\tau \rightarrow 0} \frac{\gamma(t + \tau) - \gamma(t)}{\tau}.$$

This definition of $\dot{\gamma}(t)$ requires \mathcal{M} to be endowed with a vector space structure, but the one of $\dot{\gamma}(0)$ does not; it is a mapping from $\mathcal{D}(\mathcal{M})$ to \mathbb{R} . If a differentiable manifold is not equipped with a vector space structure, then $\dot{\gamma}(t)$ is meaningless (it does not exist),

but if \mathcal{M} is (a submanifold of) a normed vector space \mathcal{V} , $\gamma'(0)$ and $\dot{\gamma}(t)$ are related: for all function f^* defined in a neighborhood \mathcal{U} of $\gamma(0)$ in \mathcal{V} , we have

$$\gamma'(0)f := Df^*(\gamma(0))[\dot{\gamma}(0)], \quad (\text{A.6})$$

where f denotes the restriction of f^* to $\mathcal{U} \cap \mathcal{M}$ [174], and Df_{xy} is the *directional derivative (function)* defined by

$$Df_{xy} := \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}.$$

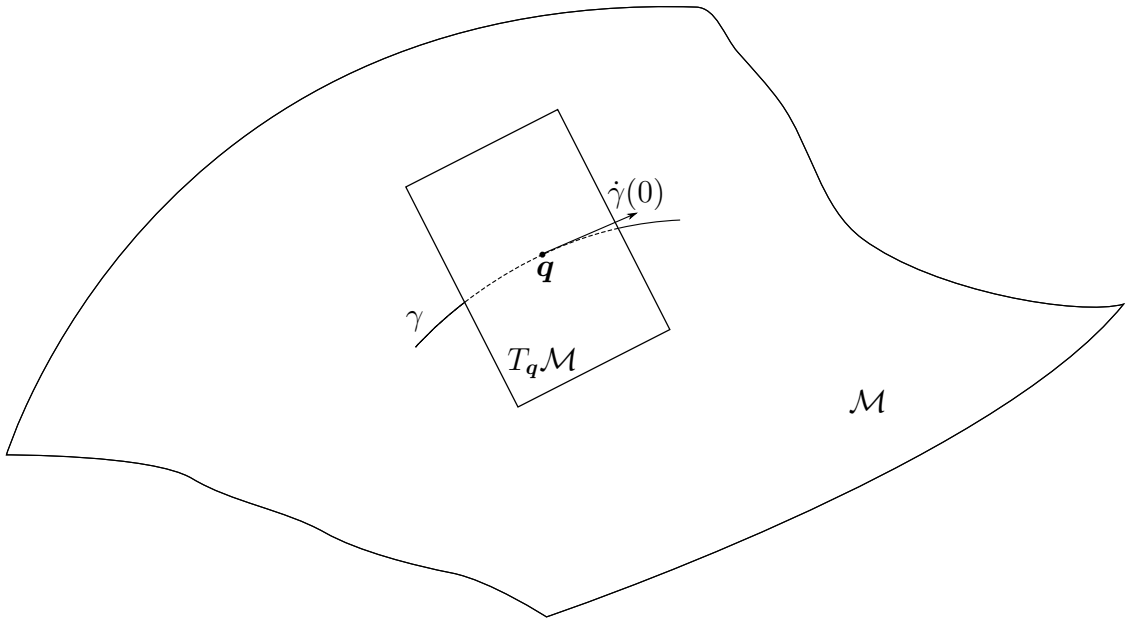


Figure A.4: Time derivative vector of a differentiable manifold.

Example A.3 (Tangent vectors of a tangent space). For a vector space \mathcal{V} , clearly the basis of $T_q \mathcal{V}$ associated with the differentiable structure of Example A.1 is the own canonical basis $\{e_1, \dots, e_n\}$ of \mathcal{V} . Thus, in an neighborhood of \mathbf{q} (tangent vectors are local objects), \mathcal{V} and $T_q \mathcal{V}$ are equivalent [174].

Example A.4 (Basis for $T_q S^{n-1}$). For the differentiable structure of the S^{n-1} given in Example A.2, for every point $\mathbf{q} \in S^{n-1}$, there is a chart φ^i such that $\varphi^i(u) = \mathbf{q} = [q_1, \dots, q_n]^T$ for a $u = [u_1, \dots, u_{n-1}]^T \in U$. Then we have that, for $i = 1, 3, \dots, 2n - 1$

$$\partial \varphi^i_{u_j} = \left[0, \dots, 0, 1, 0, \dots, 0, -\frac{u_j}{\sqrt{1 - u^T u}}, 0, \dots, 0 \right]^T$$

$$= \left[0, \dots, 0, 1, 0, \dots, 0, -\frac{q_j}{q_i}, 0, \dots, 0 \right]^T$$

and, for $i = 2, 4, \dots, 2n$,

$$\begin{aligned} \partial\varphi^i_{u_j} &= \left[0, \dots, 0, 1, 0, \dots, 0, \frac{u_j}{\sqrt{1-u^T u}}, 0, \dots, 0 \right]^T \\ &= \left[0, \dots, 0, 1, 0, \dots, 0, \frac{q_j}{q_i}, 0, \dots, 0 \right]^T \end{aligned}$$

Since $\{\partial\varphi^i_{u_1}, \dots, \partial\varphi^i_{u_{n-1}}\}$ is a basis of $T_q S^{n-1}$, and thus a vector $v \in T_q S^{n-1}$ can be written by, for $[v_1, \dots, v_{n-1}] \in \mathbb{R}^{n-1}$, $v = \sum_{j=1}^{n-1} v_j \partial\varphi^i_{u_j}$. Thus, for $i = 1, 3, \dots, 2n-1$,

$$\begin{aligned} v &= v_1 \partial\varphi^i_{u_1} + \dots + v_{n-1} \partial\varphi^i_{u_{n-1}} \\ &= v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{q_1}{q_i} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \dots + v_{n-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{q_{n-1}}{q_i} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ &= v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{q_j}{q_i} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \dots + v_{n-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{q_j}{q_i} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} v_1 \\ \vdots \\ v_{j-1} \\ 0 \\ v_j \\ \vdots \\ v_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ q_i^{-1} \sum_{j=1}^{n-1} v_j q_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} v_1 \\ \vdots \\ v_{j-1} \\ -q_i^{-1} \sum_{j=1}^{n-1} v_j q_j \\ v_j \\ \vdots \\ v_{n-1} \end{bmatrix} \\
&= v_1 e_1 + \cdots + v_{j-1} e_{i-1} + \left(-q_i^{-1} \sum_{j=1}^{n-1} v_j q_j \right) e_i + v_j e_{i+1} + \cdots + v_{n-1} e_n. \quad (\text{A.7})
\end{aligned}$$

Equation (A.7) provides the change of the coordinates of a vector $v \in T_q S^{n-1}$ from the basis $\{\partial\varphi^i_{u_1}, \dots, \partial\varphi^i_{u_{n-1}}\}$ to $\{e_1, \dots, e_n\}$ (the canonical basis of the \mathbb{R}^n). Conversely, the coordinates of $v \in T_q S^{n-1}$ in the basis $e := \{e_1, \dots, e_n\}$ are v'_1, \dots, v'_n ; then, from (A.7), the coordinates of v in $\{\partial\varphi^i_{u_1}, \dots, \partial\varphi^i_{u_{n-1}}\}$ are $[v'_1, \dots, v'_{i-1}, v'_{i+1}, v'_n]$, and hence

$$v = v'_1 \partial\varphi^i_{u_1} + \cdots + v'_{i-1} \partial\varphi^i_{u_{i-1}} + v'_{i+1} \partial\varphi^i_{u_i} + \cdots + v'_n \partial\varphi^i_{u_{n-1}}. \quad (\text{A.8})$$

Analogously, for $i = 2, 4, \dots, 2n$, we have that

$$v = v_1 e_1 + \cdots + v_{j-1} e_{i-1} + \left(q_i^{-1} \sum_{j=1}^{n-1} v_j q_j \right) e_i + v_j e_{i+1} + \cdots + v_{n-1} e_n \quad (\text{A.9})$$

$$= v'_1 \partial\varphi^i_{u_1} + \cdots + v'_{i-1} \partial\varphi^i_{u_{i-1}} + v'_{i+1} \partial\varphi^i_{u_i} + \cdots + v'_n \partial\varphi^i_{u_{n-1}}. \quad (\text{A.10})$$

Example A.5 (Tangent Bundle (based on [174])). Given a differentiable manifold \mathcal{M} , we define the set

$$T\mathcal{M} := \bigcup_q T_q \mathcal{M}.$$

For natural projection

$$\pi : T\mathcal{M} \rightarrow \mathcal{M} : T_q \mathcal{M} \ni v \mapsto \mathbf{q},$$

$\pi(v)$ is called the *foot* of v . The set $T\mathcal{M}$ admits a natural differentiable structure as

follows. Given a chart (U, φ) of \mathcal{M} , the mapping

$$T_{\mathbf{q}}\mathcal{M} \ni v \mapsto (\mathbf{q}, v)$$

is a chart of the set $T\mathcal{M}$ with domain $\pi^{-1}(\varphi(U))$. It can be shown that the collection of charts constructed in this way forms an atlas of the $T\mathcal{M}$. Then $T\mathcal{M}$ with this atlas is a differentiable manifold called the *tangent bundle* of \mathcal{M} . This is the natural space to work with when treating questions that involve positions and velocities, as in the case of mechanics. Besides, $T\mathcal{M}$ is also important for the concept of vector fields.

Definition A.3. A *vector field* \mathcal{X} on a differentiable manifold \mathcal{M} is a correspondence that associates to each point $\mathbf{q} \in \mathcal{M}$ a vector $\mathcal{X}(\mathbf{q}) \in T_{\mathbf{q}}\mathcal{M}$. In terms of mappings, \mathcal{X} is a mapping of \mathcal{M} into the tangent bundle $T\mathcal{M}$. The field is *differentiable* if the mapping

$$\mathcal{X} : \mathcal{M} \rightarrow T\mathcal{M}$$

is differentiable. On a submanifold of a vector space, a vector field can be pictured as a collection of arrows, one at each point of \mathcal{M} . Given a vector field \mathcal{X} on \mathcal{M} and a differentiable real-valued function $f : \mathcal{M} \rightarrow \mathbb{R}$ ($f \in \mathcal{D}(\mathcal{M})$), we let $\mathcal{X}f$ denote the real-valued function on \mathcal{M} defined by [recall that $v \in T_{\mathbf{q}}\mathcal{M}$ operates on real-valued function; cf. (A.3)]

$$\begin{aligned} (\mathcal{X}f) : \mathcal{M} &\rightarrow \mathbb{R} \\ \mathbf{q} &\mapsto v(f), \quad v \in T_{\mathbf{q}}\mathcal{M}. \end{aligned}$$

Let \mathcal{M}_1 and \mathcal{M}_2 be differentiable manifolds and $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ a differentiable mapping. For every $\mathbf{q} \in \mathcal{M}_1$ and for each $v \in T_{\mathbf{q}}\mathcal{M}_1$, choose a differentiable curve $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{M}_1$ with $\gamma(0) = \mathbf{q}$, $\gamma'(0) = v$. Take $\beta = f \circ \gamma$. Then it can be shown that the operator $df_{\mathbf{q}}(v)$ defined by

$$df_{\mathbf{q}}(v) := \beta'(0)$$

is a tangent vector of $T_{f(\mathbf{q})}\mathcal{M}_2$ (see Figure A.5). Moreover the mapping

$$df_{\mathbf{q}} : T_{\mathbf{q}}\mathcal{M}_1 \rightarrow T_{f(\mathbf{q})}\mathcal{M}_2 : v \mapsto \beta'(0)$$

is linear and does not depend on the choice of γ [177]. This linear mapping $df_{\mathbf{q}}$ is called the differential of f at \mathbf{q} .

The *rank* of a differentiable mapping $f : \mathcal{M}_1^{n_1} \rightarrow \mathcal{M}_2^{n_2}$ is the dimension of the range of $df_{\mathbf{q}}$. The mapping f is i) an *immersion* if its rank is equal to n_1 at every point of its domain (hence $n_1 \leq n_2$), and a *submersion* if its rank is equal n_2 at every point of

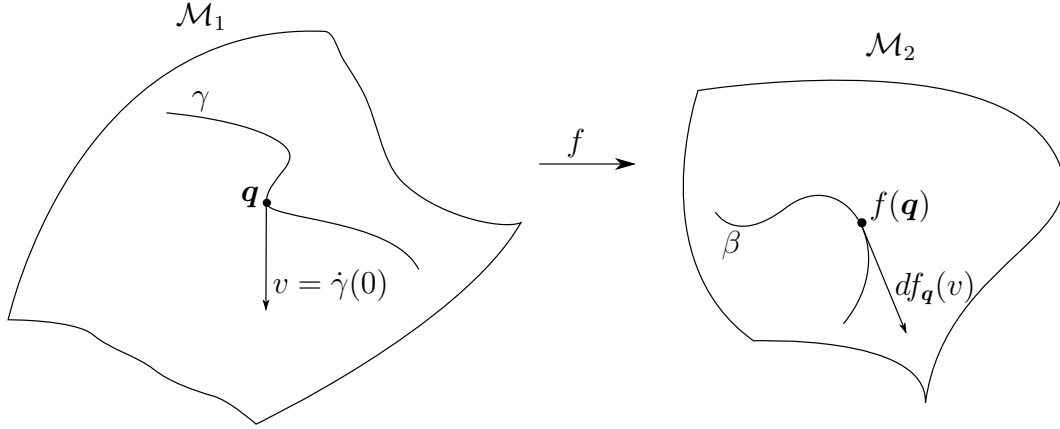


Figure A.5: Differential of a function.

its domain (hence $n_1 \geq n_2$) [174].

Consider two differentiable manifolds $(\mathcal{M}_1, \mathcal{A}_1)$ and $(\mathcal{M}_2, \mathcal{A}_2)$ such that $\mathcal{M}_2 \subset \mathcal{M}_1$. Then $(\mathcal{M}_2, \mathcal{A}_2)$ is called an *immersed submanifold* of $(\mathcal{M}_1, \mathcal{A}_1)$ if the inclusion map

$$i : \mathcal{M}_1 \rightarrow \mathcal{M}_2 : \mathbf{q} \mapsto \mathbf{q}$$

is an immersion. In this case, $(\mathcal{M}_1, \mathcal{A}_1)$ induces a topology on $(\mathcal{M}_2, \mathcal{A}_2)$, and $(\mathcal{M}_2, \mathcal{A}_2)$ is endowed with this topology induced from $(\mathcal{M}_1, \mathcal{A}_1)$ and with its original manifold topology (which is induced by \mathcal{A}_2). If these two topologies of $(\mathcal{M}_2, \mathcal{A}_2)$ are the same, $(\mathcal{M}_2, \mathcal{A}_2)$ is called an *embedded (sub)manifold* (or a regular manifold, or simply a submanifold). In this case, $(\mathcal{M}_1, \mathcal{A}_1)$ is called the *embedding manifold*, or *the ambient manifold*.

Embedded manifolds have interesting properties regarding their tangent vectors. Let \mathcal{M} be an embedded manifold of a normed vector space \mathcal{V} , and consider the curve in \mathcal{M} γ with $\gamma(0) = \mathbf{q}$. Let i be inclusion map of \mathcal{M} into \mathcal{V} and define directional derivative

$$\dot{\gamma}(0) := \lim_{t \rightarrow 0} \frac{i(\gamma(t)) - i(\gamma(0))}{t}.$$

Since γ is a curve, it also induces a tangent vector $\gamma'(0)$ according to (A.3). For f^* being defined in a neighborhood \mathcal{U} of $\gamma(0)$ in \mathcal{V} , and f being the restriction of f to $\mathcal{U} \cap \mathcal{M}$ we have that, from (A.6),

$$\gamma'(0)f := Df_{\gamma(0)}^*[\dot{\gamma}(0)]. \quad (\text{A.11})$$

Then, we can identify $T_{\mathbf{q}}\mathcal{M}$ with the set of all $\dot{\gamma}(0)$. Since $\dot{\gamma}(0)$ is can be graphically represented by an “arrow”, then $\gamma'(0)$ can also be picture as this arrow [174]. Note however, that this is meaningful only to the case of manifolds embedded in vector spaces. Furthermore, since i) the set of all vectors $\dot{\gamma}(0)$ is identified with $T_{\mathbf{q}}\mathcal{V}$, and ii) $T_{\mathbf{q}}\mathcal{V}$ is locally identified with \mathcal{V} itself, then *the vectors of $T_{\mathbf{q}}\mathcal{M}$ can be naturally represented*

by vectors of the embedding vector space \mathcal{V} . The following examples illustrates this point.

Example A.6 (Embedded S^{n-1}). The unit sphere S^{n-1} is an embedded manifold of the \mathbb{R}^n (the immersion is $i : S^{n-1} \rightarrow \mathbb{R}^n : \mathbf{q} \mapsto \mathbf{q}$). Since \mathbb{R}^n , we can use (A.11) to have a representation of $\gamma'(0)$ in \mathbb{R}^n . Let $\gamma : \mathbb{R} \rightarrow S^{n-1} : t \mapsto \gamma(t)$ be a curve in the unit sphere S^{n-1} with $\gamma(0) = \mathbf{q}$. Since $\gamma(t)$ belong to S^{n-1} for all t , we have that (considering the representation of $\gamma(t)$ in the ambient space \mathbb{R}^n)

$$\gamma(t)^T \gamma(t) = 1, \text{ for all } t.$$

Differentiating this equation (using the directional derivative in \mathbb{R}^n) with respect to t , we have that

$$\begin{aligned} \dot{\gamma}(t)^T \gamma(t) + \gamma(t)^T \dot{\gamma}(t) &= 0 \\ \Leftrightarrow 2\gamma(t)^T \dot{\gamma}(t) &= 0 \\ \Leftrightarrow \gamma(t)^T \dot{\gamma}(t) &= 0 \end{aligned}$$

for $t = 0$, this equation yields

$$\mathbf{q}^T \dot{\gamma}(0) = 0.$$

Therefore, the vectors $\dot{\gamma}(0)$ are the vectors orthogonal to \mathbf{q} [174]. These vectors $\dot{\gamma}(0)$ are associated with $\gamma'(0)$ in the basis \mathbb{R}^n . This representation is particularly useful when we want to implement $\gamma'(0)$ in computer programs because we know how to represent the elements of \mathbb{R}^n .

A.2 RIEMANNIAN METRICS

Riemannian geometry can be viewed as a further extension of the differential geometry of surfaces in the \mathbb{R}^3 . The length of a given curve in a surface $S \subset \mathbb{R}^3$ is defined by integrating the size (norm) of its velocity vector; on its turn, the length of a velocity vector is naturally defined by the inner product $\langle v, v \rangle$. Riemannian manifolds are differentiable manifolds with inner products induced in the tangent spaces.

With inner products, we can calculate the lengths of curves; then it is natural to ask for the shortest curves among two points of the manifold. These curves are called *geodesics*. This was the historical development of geodesics, but nowadays it is more usual among the scientific literature to define geodesics as the curves whose covariant derivative is zero at every point (cf. [177]).

Definition A.4 (Riemannian manifold). A *Riemannian metric* (or *Riemannian struc-*

ture) \langle, \rangle or g on a differentiable manifold \mathcal{M} is a correspondence which associates to each point \mathbf{q} of \mathcal{M} an inner product $g_{\mathbf{q}} := \langle, \rangle_{\mathbf{q}}$ (that is, a symmetric, bilinear, positive-definite form) on a tangent space $T_{\mathbf{q}}\mathcal{M}$, with $\langle, \rangle_{\mathbf{q}}$ varying differentially in the following sense: if $\varphi : U \subset \mathbb{R}^n \rightarrow \mathcal{M}$ is a system of coordinates (or chart) around \mathbf{q} , with

$$\varphi(u_1, u_2, \dots, u_n) = \mathbf{q} \in \varphi(U)$$

and

$$\frac{\partial}{\partial u_i}(\mathbf{q}) = d\varphi_{\mathbf{q}}(0, \dots, 0, 1, 0, \dots, 0),$$

then

$$\left\langle \frac{\partial}{\partial u_i}(\mathbf{q}), \frac{\partial}{\partial u_j}(\mathbf{q}) \right\rangle_{\mathbf{q}} = g_{\mathbf{q},ij}(u_1, u_2, \dots, u_n)$$

is a differentiable function on U [177]. The matrix

$$G_{\mathbf{q}}(\varphi) = [g_{\mathbf{q},ij}(\varphi)]$$

is called *local representation of the Riemannian metric in the chart* φ [66, p.129]. It is usual to delete the index \mathbf{q} in the function $\langle, \rangle_{\mathbf{q}}$ whenever there is no possibility of confusion. The function $g_{ij}(= g_{ji})$ is called *the local representation of the Riemannian metric* (or “the g_{ij} of the metric”) *in the coordinate system* $\varphi : U \subset \mathbb{R}^n \rightarrow \mathcal{M}$.

The pair (\mathcal{M}, g) is called a *Riemannian manifold* [174]. For simplicity, we will use the name “Riemannian manifold” to refer either for the pair (\mathcal{M}, g) , or to the set \mathcal{M} .

A diffeomorphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is called an *isometry* if

$$\langle v_a, v_b \rangle_{\mathbf{q}} = \langle df_{\mathbf{q}}(v_a), df_{\mathbf{q}}(v_b) \rangle_{f(\mathbf{q})}, \text{ for all } \mathbf{q} \in \mathcal{M}_1; \text{ and } v_a, v_b \in T_{\mathbf{q}}\mathcal{M}_1.$$

Similarly to vector spaces, we can associate norms with a Riemannian metrics. For $\mathbf{q} \in \mathcal{M}$, the (*Riemannian*) *norm* associated to \mathbf{q} is [66]

$$\|\mathbf{p}\|_{\mathbf{q}} := \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle_{\mathbf{q}}}, \quad \mathbf{p} \in \mathcal{M}.$$

The most trivial example of a Riemannian manifold is the Euclidean space. Indeed the \mathbb{R}^n with the metric given by $\langle e_i, e_j \rangle = \delta_{ij}$, where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ is a Riemannian manifold.

An embedded manifold \mathcal{M}_2 of a Riemannian manifold \mathcal{M}_1 can inherit the Riemannian metric of \mathcal{M}_1 ; in this case, \mathcal{M}_2 is called a *Riemannian submanifold* (see [174] for more information).

Example A.7 (Riemannian metric on the sphere). On the sphere S^{n-1} considered as

a Riemannian submanifold of \mathbb{R}^n , the inner product inherited from the standard inner product on \mathbb{R}^n is given by

$$\langle v_a, v_b \rangle_{\mathbf{q}} := v_a^T v_b, \quad v_a, v_b \in T_{\mathbf{q}}S^{n-1}. \quad (\text{A.12})$$

This inner product is called the *canonical Riemannian metric* of the S^{n-1} [177].

We saw that the Cartesian product of two differentiable manifolds is also a differentiable manifold (Section A.1); likewise, *the Cartesian product of two Riemannian manifolds is also a Riemannian manifold*. For two Riemannian manifolds \mathcal{M}_1 and \mathcal{M}_2 , consider the natural projections

$$\begin{aligned} \pi_1 : \mathcal{M}_1 \times \mathcal{M}_2 &\rightarrow \mathcal{M}_1 \\ (\mathbf{q}_1, \mathbf{q}_2) &\mapsto \mathbf{q}_1; \end{aligned}$$

and

$$\begin{aligned} \pi_2 : \mathcal{M}_1 \times \mathcal{M}_2 &\rightarrow \mathcal{M}_2 \\ (\mathbf{q}_1, \mathbf{q}_2) &\mapsto \mathbf{q}_2; \end{aligned}$$

Define, for every $\mathbf{q}_1 \in \mathcal{M}_1$, $\mathbf{q}_2 \in \mathcal{M}_2$, $v_a \in T_{\mathbf{q}_1}\mathcal{M}_1$, $v_b \in T_{\mathbf{q}_2}\mathcal{M}_2$, the following inner product on $T_{(\mathbf{q}_1, \mathbf{q}_2)}(\mathcal{M}_1 \times \mathcal{M}_2)$,

$$g_{(\mathbf{q}_1, \mathbf{q}_2)} := \langle v_a, v_b \rangle_{(\mathbf{q}_1, \mathbf{q}_2)} = \langle d\pi_1 \cdot v_a, d\pi_1 \cdot v_b \rangle_{\mathbf{q}_1} + \langle d\pi_2 \cdot v_a, d\pi_2 \cdot v_b \rangle_{\mathbf{q}_2};$$

then, for

$$g : (\mathbf{q}_1, \mathbf{q}_2) \mapsto g_{(\mathbf{q}_1, \mathbf{q}_2)},$$

the pair $(\mathcal{M}_1 \times \mathcal{M}_2, g)$ is a Riemannian manifold [177]. Further in Chapter 9, the Riemannian manifold $S^{n-1} \times \mathbb{R}^m$ will be particularly important.

It is worthy to consider a vector field *along a curve*. A differentiable mapping $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ of an open interval $\mathbf{I} \subset \mathbb{R}$ into a differentiable manifold \mathcal{M} is called a (parameterized) *curve*.

Definition A.5 (Vector field). A *vector field* V *along a curve* $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ is a differentiable mapping that associates to every $t \in \mathbf{I}$ a tangent vector $V(t) \in T_{\gamma(t)}\mathcal{M}$. To say that V is *differentiable* mean that for any differentiable function f on \mathcal{M} , the function $t \rightarrow V(t)f$ is a differentiable function on \mathbf{I} .

The vector field $d\gamma\left(\frac{d}{dt}\right)$, denoted by $\frac{d\gamma}{dt}$, is called the *velocity field* (or tangent vector field) of γ . Observe that a vector field along γ cannot necessarily be extended to a vector field on an open set of \mathcal{M} . The restriction of a curve γ to a closed interval

$[a, b] \subset \mathbf{I}$ is called *segment*.

A.3 AFFINE AND RIEMANNIAN CONNECTIONS

Let $S \subset \mathbb{R}^3$ be a manifold and let $c : \mathbf{I} \rightarrow S$ be a parameterized curve in S , with $V : \mathbf{I} \rightarrow \mathbb{R}^3$ a vector field along c tangent to S . The vector $dV(t)/dt$, $t \in \mathbf{I}$, does not, in general, belong to the tangent plane of $S, T_{c(t)}S$. The concept of differentiating a vector field is not, therefore, an “intrinsic” geometric notion on S . To remedy this state of affairs, we consider, instead of the usual derivative $dV(t)/dt$, the orthogonal projection of $dV(t)/dt$ on $T_{c(t)}S$. This orthogonally projected vector we call the covariant derivative, and denote it by $DV(t)/dt$. It is the derivative of V seen from the “viewpoint of S ”.

A basic point is that the covariant derivative depends only on the first fundamental form of S and is, therefore, a concept which can be considered within Riemannian geometry. In particular, the notion of covariant derivative permits us to take the derivative of the velocity vector of c , which gives the acceleration of the curves c in S . It is possible to show that curves with zero acceleration are precisely the geodesics of S and that the Gaussian curvature of S can be expressed in terms of the notion of the covariant derivative.

We say that a vector field V along c is parallel if $DV(t)/dt = 0$. Conversely, starting from the notion of parallelism it is possible to recover the notion of covariant derivative. These notions are then equivalent to each other.

We now present affine connections by the reason that a choice of a Riemannian metric on a manifold \mathcal{M} uniquely determines a certain affine connection on \mathcal{M} . We are then able, in this fashion, to differentiate vector fields on \mathcal{M} .

Let us indicate by $\mathcal{X}(\mathcal{M})$ the set of all vector fields of class C^∞ on \mathcal{M} and by $\mathcal{D}(\mathcal{M})$ the ring of real-valued functions of class C^∞ defined on \mathcal{M} .

Definition A.6 (Affine connection). An *affine connection* ∇ on a differentiable manifold \mathcal{M} is a mapping

$$\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \rightarrow \mathcal{X}(\mathcal{M})$$

which is denoted by $(\varphi, Y) \xrightarrow{\nabla} \nabla_\varphi Y$ and which satisfies the following properties, for $\varphi, Y, Z \in \mathcal{X}(\mathcal{M})$ and $f, g \in \mathcal{D}(\mathcal{M})$:

1. $\nabla_{f\varphi+gY}Z = f\nabla_\varphi Z + g\nabla_Y Z$,
- (a) $\nabla_\varphi(Y, Z) = \nabla_\varphi Y + \nabla_\varphi Z$,

$$(b) \nabla_{\varphi}(fY) = f\nabla_{\varphi}Y + \varphi(f)Y.$$

This definition is not as transparent as that of a Riemannian structure. The following theorem, nevertheless, should clarify the situation a little.

Theorem A.1 (Covariant derivative [177]). *Let \mathcal{M} be a differentiable manifold with an affine connection ∇ . There exists a unique correspondence which associates to a vector field V along the differentiable curve $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ another vector field $\frac{DV}{dt}$ along γ , called the covariant derivative of \mathcal{M} along γ , such that:*

$$1. \frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt};$$

$$(a) \frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}, \text{ where } W \text{ is a vector field along } \gamma \text{ and } f \text{ is a differentiable function on } \mathbf{I};$$

$$(b) \text{ if } V \text{ is induced by a vector field } Y \in \mathcal{X}(\mathcal{M}), \text{ i.e., } V(t) = Y(\gamma(t)), \text{ then } \frac{DV}{dt} = \nabla_{d\gamma/dt}Y.$$

Theorem A.1 shows that the choice of an affine connection on \mathcal{M} leads to a *bona fide* (i.e. satisfying 1 and 1a) derivative of vector fields along curves. The notion of connection furnishes, therefore, a manner of differentiating vector along curves; in particular, it will then be possible to speak of the acceleration of a curve in \mathcal{M} . The concept of parallelism now follows in a natural manner.

Let \mathcal{M} be a differentiable manifold with an affine connection ∇ . A vector field V along a curve $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ is called *parallel* when $\frac{DV}{dt} = 0$, for all $t \in \mathbf{I}$. Moreover, let γ be differentiable and v_0 a vector tangent to \mathcal{M} at $\gamma(t_0)$, $t_0 \in \mathbf{I}$ (i.e. $v_0 \in T_{\gamma(t_0)}\mathcal{M}$). Then there exists a unique parallel vector field V along γ , such that $V(t_0) = v_0$, ($V(t)$ is called the *parallel transport* of $V(t_0)$ along γ) [177].

Example A.8 (Parallel Transport for S^{n-1}). The parallel transport of a vector $\zeta \in T_{\mathbf{q}}S^{n-1}$ in a tangent vector to T_rS^{n-1} is used in the RiUKF in equation (9.104) (cf. Theorem 9.2). There is a closed form for this operation on S^{n-1} ; let $t \mapsto \gamma_{\mathbf{q},v}(t)$ be a geodesic, and then the parallel transport $\zeta(t)$ (expressed in the canonical basis e) of a vector $\zeta(0) = \zeta \in T_{\mathbf{q}}S^{n-1}$ along the geodesic $\gamma_{\mathbf{q},v}(t)$ is given by

$$\eta := \frac{\hat{\gamma}_{\mathbf{q},v}(0)}{\|\hat{\gamma}_{\mathbf{q},v}(0)\|}$$

$$\zeta(t) := -\gamma_{\mathbf{q},v}(0) \sin(\|\hat{\gamma}_{\mathbf{q},v}(0)\|t) \eta^T \zeta(0) + \eta \cos(\|\hat{\gamma}_{\mathbf{q},v}(0)\|t) \gamma_{\mathbf{q},v}(0)^T \zeta(0) + (I - \eta \eta^T) \zeta(0).$$

We have that:

$$\gamma_{\mathbf{q},v}(0) := \cos(0\|v\|) \mathbf{q} + \frac{v}{\|v\|} \sin(0\|v\|) = \mathbf{q};$$

$$\dot{\gamma}_{\mathbf{q},v}(0) = v$$

Then

$$\zeta(t) = -\mathbf{q} \sin(\|v\|t) \frac{v^T}{\|v\|} \zeta(0) + \frac{v}{\|v\|} \cos(\|v\|t) \mathbf{q}^T \zeta(0) + \left(I - \frac{v}{\|v\|} \frac{v^T}{\|v\|} \right) \zeta(0).$$

A vector $v = \exp_{\mathbf{q}} \mathbf{r}$ is such that $\mathbf{r} = \gamma_{\mathbf{q},v}(1)$; therefore, the parallel transport of $\zeta \in T_{\mathbf{q}}S^{n-1}$ to $T_{\mathbf{r}}S^{n-1}$ is given by

$$\zeta^{T_{\mathbf{r}}S^{n-1}} = \zeta(1) = -\mathbf{q} \sin(\|v\|) \frac{v^T}{\|v\|} \zeta + \frac{v}{\|v\|} \cos(\|v\|) \mathbf{q}^T \zeta + \left(I - \frac{v}{\|v\|} \frac{v^T}{\|v\|} \right) \zeta. \quad (\text{A.13})$$

Definition A.7. Let \mathcal{M} be a differentiable manifold with an affine connection ∇ and a Riemannian metric $\langle \cdot, \cdot \rangle$. A connection is said to be *compatible* with the metric $\langle \cdot, \cdot \rangle$, when for any smooth curve γ and any pair of parallel vector fields P and P' along γ , we have $\langle P, P' \rangle = \text{constant}$.

Definition A.7 is justified by the following fact. Let \mathcal{M} be a Riemannian manifold. A connection ∇ on \mathcal{M} is compatible with a metric if and only if, for any vector fields V and W along the differentiable curve $\gamma : \mathbf{I} \rightarrow \mathcal{M}$, we have (cf. [177])

$$\frac{d}{dt} \langle V, W \rangle = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle, \quad t \in \mathbf{I}. \quad (\text{A.14})$$

An affine connection ∇ on a smooth manifold \mathcal{M} is said to be *symmetric* when

$$\nabla_{\varphi} Y - \nabla_Y \varphi = [\varphi, Y] \text{ for all } \varphi, Y \in \mathcal{X}(\mathcal{M}), \quad (\text{A.15})$$

where, for two vectors fields X and Y , the bracket $[X, Y]$ is called the *Lie Bracket* of X and Y , and defined by

$$[X, Y] := XY - YX.$$

Theorem A.2 (Levi-Civita [177]). *Given a Riemannian manifold \mathcal{M} , there exists a unique affine connection ∇ on \mathcal{M} satisfying the conditions:*

1. ∇ is symmetric,
2. ∇ is compatible with the Riemannian metric.

The connection given by Theorem A.2 will be referred to, from now on, as the *Levi-Civita* (or *Riemannian*) *connection on \mathcal{M}* .

A.4 GEODESICS

In what follows, \mathcal{M} will be a Riemannian manifold, together with its Riemannian connection.

Given a curve $\gamma : [a, b] \rightarrow \mathcal{M}$ on a Riemannian manifold \mathcal{M} with $\gamma(a) = \mathbf{q}$ and $\gamma(b) = \mathbf{p}$, the arc length \mathbb{L} of γ is given by

$$\mathbb{L}(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt} \right\|_{\gamma(t)} dt = \int_a^b \sqrt{\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle_{\gamma(t)}} dt.$$

Generally, there is more than one curve connecting two points, and it is natural to ask for the which of these curves have the smallest arc length, and what is the value of this smallest arc length.

The smallest arc length gives the concept of the distance between two points; indeed, for two points \mathbf{q} and \mathbf{p} in \mathcal{M} connected by smooth curves $\gamma : [a, b] \rightarrow \mathcal{M}$, the *distance* between $\gamma(a) = \mathbf{q}$ and $\gamma(b) = \mathbf{p}$ is defined by

$$\text{dist}(\mathbf{q}, \mathbf{p}) := \min_{\gamma} \mathbb{L}(\gamma). \quad (\text{A.16})$$

The curves between two points whose lengths are the smallest are the so called *geodesics*. A geodesic can be seen as the analogous to an straight line in a Euclidean space in the sense that they are the shortest path between two points. Geodesics are defined as being the curves with zero covariant derivative at every point. It can be shown that this definition leads to the property of minimizing distances.

Definition A.8 (Geodesic). A parametrized curve $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ is a *geodesic* at $t_0 \in \mathbf{I}$ if $\frac{D}{dt} \left(\frac{d\gamma}{dt} \right) = 0$ at the point t_0 ; if γ is a geodesic at t , for all $t \in \mathbf{I}$, we say that γ is a *geodesic*. If $[a, b] \subset \mathbf{I}$ and $\gamma : \mathbf{I} \rightarrow \mathcal{M}$ is a geodesic, the restriction of γ to $[a, b]$ is called a *geodesic segment joining $\gamma(a)$ to $\gamma(b)$* [177]. If the definition domain of all geodesics of \mathcal{M} can be extended to \mathbb{R} , then \mathcal{M} is said to be *geodesically complete* [66, p.129].

From the Hopf-Rinow-De Rhom Theorem, it follows that there exists at least one geodesic connecting every two points of a geodesically complete manifold. In the Unscented theory developed in this work, *we suppose that all Riemannian manifolds are geodesically complete*, unless stated otherwise.

Example A.9 (Euclidean Space). For the Euclidean manifold \mathbb{R}^n , geodesics are given by

$$\gamma(t, x, v) = x + vt.$$

Example A.10 (Unit Sphere (from [177] and [174])). Let $\mathcal{M} = S^{n-1} \subset \mathbb{R}^n$ be the unit sphere of dimension n . The great circles of S^{n-1} parameterized by arc length

are geodesics (see Figure A.6). For S^{n-1} as a Riemannian submanifold of \mathbb{R}^n , the Riemannian metric is given by the inner product (A.12). Geodesics on the S^{n-1} are the curves $\gamma : \mathbb{R} \rightarrow S^{n-1} : t \mapsto \gamma(t)$ with $\gamma(0) = \mathbf{q}$ and $\gamma'(0) = v$ given by the following equation (the elements of the S^{n-1} are represented as vectors of the \mathbb{R}^n)

$$\gamma(t) = \mathbf{q} \cos(\|v\| t) + \frac{v}{\|v\|} \sin(\|v\| t). \quad (\text{A.17})$$

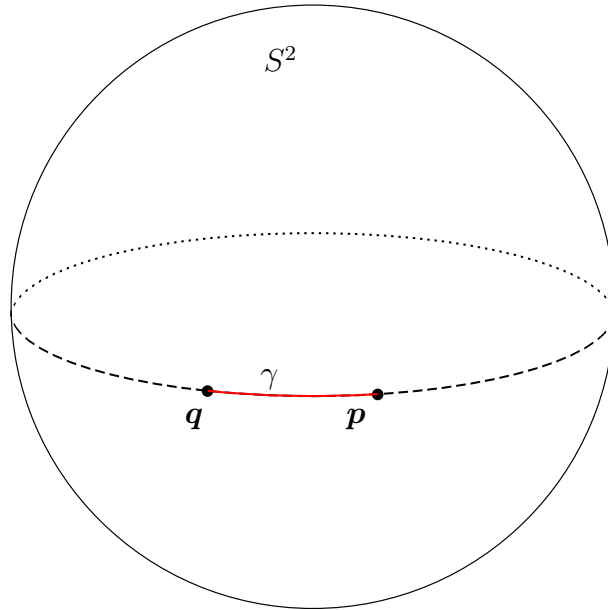


Figure A.6: A geodesic in S^2 .

A.5 EXPONENTIAL AND LOGARITHM MAPS

If it is known that a curve in a Riemannian manifold \mathcal{M} passes through a point $\mathbf{q} \in \mathcal{M}$ at time $t = 0$ with tangent vector $v \in T_{\mathbf{x}}\mathcal{M}$, then we can fully determine this curve. Particularly, if a curve determined by such a triad (t, \mathbf{q}, v) is a geodesic, then it is unique; i.e., this is the only geodesic passing through \mathbf{q} at time $t = 0$ with tangent vector $v \in T_{\mathbf{q}}\mathcal{M}$ [177]. Indeed, the coordinates of a geodesic follow a second order ordinary differential equation, and, from the theory of differential equations, given initial conditions of \mathbf{q} and its derivative, the solution is unique. Therefore, a geodesic γ can be expressed as a function of (t, \mathbf{q}, v) by $\gamma(t, \mathbf{q}, v)$.

For a geodesic $\gamma(t, \mathbf{q}, v)$, if we consider, for each $v \in T_{\mathbf{q}}\mathcal{M}$, the point $\gamma(1, \mathbf{q}, v) \in \mathcal{M}$, we have an interesting mapping from $T_{\mathbf{q}}\mathcal{M}$ to \mathcal{M} , the so called (Riemannian) exponential mapping [177].

Definition A.9 (Exponential map). Consider a point $\mathbf{q} \in \mathcal{M}$ and let $V \subset T_{\mathbf{q}}\mathcal{M}$ be

an open set of $T_q\mathcal{M}$. Then the map $\exp : \mathcal{U} \rightarrow \mathcal{M}$ given by

$$\exp_q(v) := \gamma(1, \mathbf{q}, v) = \gamma\left(\|v\|, \mathbf{q}, \frac{v}{\|v\|}\right), \quad v \in V; \quad (\text{A.18})$$

is called the (Riemannian) *exponential map* on \mathcal{U} .

Geometrically, $\exp_q(v)$ is a point of \mathcal{M} obtained by going out the length equal to $\|v\|$, starting from \mathbf{q} , along a geodesic which passes through \mathbf{q} with velocity equal to $v/\|v\|$. The map $\exp_{\mathcal{M}}$ is differentiable, and realizes a local diffeomorphism from a sufficiently small neighborhood around the origin of $T_q\mathcal{M}$, the so called (Riemannian) *logarithm map*. For \mathcal{U} being this neighborhood and $\mathbf{q}, \mathbf{p} \in \mathcal{U}$, $\mathbf{p} = \exp_q(v)$, then the inverse map $\log : \mathcal{U} \rightarrow T_q\mathcal{M}$ defined by

$$v := \log_q(\mathbf{p}),$$

or simply

$$\vec{\mathbf{q}\mathbf{p}} := \log_q(\mathbf{p}),$$

is called the (Riemannian) *logarithm map* [66, p.130].

The logarithm map of the geodesics going through $\mathbf{q} \in \mathcal{M}$ are represented by the lines going through the origin of $T_q\mathcal{M}$ [66]:

$$\log_q \gamma(t, \mathbf{q}, \vec{\mathbf{q}\mathbf{p}}) = t\vec{\mathbf{q}\mathbf{p}}.$$

Besides, the distance for paths passing through \mathbf{q} is preserved, that is,

$$\text{dist}(\mathbf{q}, \mathbf{p}) = \|\vec{\mathbf{q}\mathbf{p}}\|_q = \sqrt{\langle \vec{\mathbf{q}\mathbf{p}}, \vec{\mathbf{q}\mathbf{p}} \rangle_q}. \quad (\text{A.19})$$

From Example A.10, we see that, for some Riemannian manifolds, the exponential map for a given point $\mathbf{q} \in \mathcal{M}$ does not realize a diffeomorphism for all $T_q\mathcal{M}$, therefore, the logarithm is not defined for the whole $T_q\mathcal{M}$. We can reduce the domain of \exp_q to a certain subset such that \exp_q is a diffeomorphism; this subset is called the *tangential cut locus* $C(\mathbf{q}) \subset T_q\mathcal{M}$ of \exp_q , and the set $\mathcal{C}(\mathbf{p}) := \exp_q(C(\mathbf{q})) \in \mathcal{M}$ the *cut locus* of \exp_q .

Note that $\mathcal{C}(\mathbf{p}) = \exp_p(C(\mathbf{p}))$ and the *maximal definition domain* for the exponential map is the domain $D(\mathbf{p})$ containing the origin of $T_q\mathcal{M}$, and delimited by the tangential *cut locus* [66, p.130]. Therefore, the exponential map can be defined as covering all the manifold except the cut locus:

$$\mathcal{D}(\mathbf{p}) \in \mathbb{R}^n \longleftrightarrow \mathcal{M} - \mathcal{C}(\mathbf{p})$$

$$\overrightarrow{pq} = \log_p(\mathbf{q}) \longleftrightarrow \mathbf{q} = \exp_p(\overrightarrow{pq}).$$

For now on, we will consider that exponential maps are defined in this way, unless stated otherwise.

Example A.11 (Euclidean Space). Since the geodesics of the Euclidean manifold \mathbb{R}^n are given by (see Example A.9)

$$\gamma(t, x, v) = x + vt;$$

and consequently, the exponential map and the logarithm map (with usual inner product as the Riemannian metric) are given by

$$\begin{aligned} \exp_x^{\mathbb{R}^n} : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ v &\mapsto \gamma(1, x, v) = x + v; \end{aligned} \quad (\text{A.20})$$

and

$$\begin{aligned} \log_x^{\mathbb{R}^n} : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ y &\mapsto y - x. \end{aligned} \quad (\text{A.21})$$

Example A.12 (Exponential and logarithm mappings of the S^{n-1}). Applying the definition (A.18) on (A.17), we have that the exponential mapping on the sphere (in the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n), $\exp_{\mathbf{q}}^e(v)$, is given by

$$\begin{aligned} \exp_{\mathbf{q}}^e : \mathbb{B}_{[0]_{n \times 1}}(\pi) & \quad S^{n-1} - \{-\mathbf{q}\} \\ v & \quad \mapsto \gamma(1, \mathbf{q}, v) = \mathbf{q} \cos(\|v\|) + \frac{v}{\|v\|} \sin(\|v\|). \end{aligned} \quad (\text{A.22})$$

An illustration of this exponential mapping for the S^2 is shown in Figure A.7.

Let $\mathbf{p} = \exp_{\mathbf{q}}^e(v)$, then, for

$$\theta := \arccos(\langle \mathbf{q}, \mathbf{p} \rangle),$$

the logarithm mapping is

$$\begin{aligned} \log_{\mathbf{q}}^e : S^{n-1} - \{-\mathbf{q}\} &\rightarrow \mathbb{B}_{[0]_{n \times 1}}(\pi) \\ \mathbf{p} &\mapsto \ln_{\mathbf{q}}(\mathbf{p}) := \exp_{\mathbf{q}}^{-1}(v) = \frac{\theta}{\sin(\theta)} \mathbf{p} - \frac{\theta \cos \theta}{\sin(\theta)} \mathbf{q}. \end{aligned} \quad (\text{A.23})$$

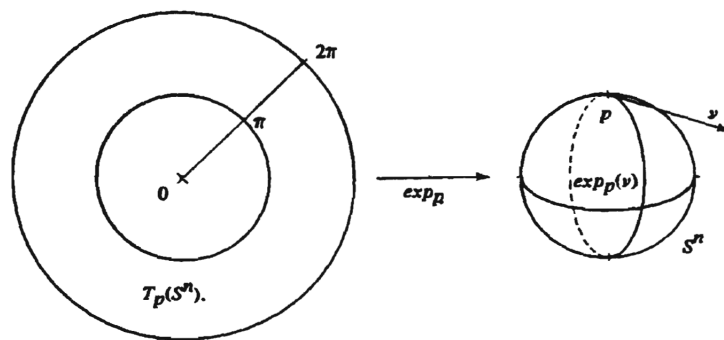


Figure A.7: Exponential map of the unit sphere of dimension 2.
Adapted from [177] with copyright.

B. RESUMO ESTENDIDO EM LÍNGUA PORTUGUESA

A filtragem de Kalman Unscented tornou-se extremamente popular na comunidade de controle. De acordo com o *IEEE Xplore Digital Library* (um sítio eletrônico do *Institute of Electrical and Electronics Engineers* [IEEE])^a, o trabalho [1] atingiu a impressionante marca de 8222 leituras; e 1279 citações no IEEE, 2735 no *Scopus* (<http://www.scopus.com>), e 1564 no *Web of Science* (<http://apps.webofknowledge.com>).

Desde o seu trabalho precursor [2], os Filtros de Kalman Unscented (FKUs) vêm sendo usado em diversas aplicações. Por exemplo, podemos encontrá-los sendo utilizados para estimar variáveis relativas a baterias [3–7], geradores eólicos [8], controle de frequência de sistemas de potências [9], circuitos integrados [10], moduladores sigma-delta [11], sistemas de navegação inerciais [12], satélites [13], imagens médicas [14], cirurgias assistidas por computador [15], insulinas plasmáticas [16], cápsulas endoscópicas [17], microfones [18], tomografias acústicas da atmosfera [19], robôs móveis [20–22], entre outros.

Algumas propriedades dos FKUs podem ser bem entendidas quando esses filtros são em relação com o conhecido Filtro de Kalman Estendido (FKE). Em muitas aplicações — por exemplo, em [7, 16, 21], e [22], entre outros — os FKUs comportaram-se melhor que o FKE. Esse comportamento superior pode ser explicado, pelo menos, pelas duas razões a seguir:

- as complexidades computacionais dos FKUs e do FKE são da mesma ordem — $\mathcal{O}(n_y)$ —, mas as estimativas dos FKU's tendem a ser melhores [23].
- o FKU é livre de derivadas (não precisa calcular matrizes jacobianas), enquanto o FKE requer que a dinâmica seja diferenciável. Portanto, diferente do FKE, os FKU's podem ser usados em sistemas em que matrizes jacobianas não existem, tais como sistemas com descontinuidades (cf. [1]).

Grande parte dos esforços dos pesquisadores da teoria Unscented tem sido direcionada a encontrar extensões do primeiro FKU. A direção dessas extensões são similares

^aEm [http://ieeexplore.ieee.org/xpl/abstractMetrics.jsp?arnumber=1271397&action=search&sortTyp e=&rowsPerPage=&searchField=Search_All&matchBoolean=true&queryText=\(julier%20unscented%20kalman%20filtering%20for%20nonlinear%20estimation\)](http://ieeexplore.ieee.org/xpl/abstractMetrics.jsp?arnumber=1271397&action=search&sortTyp e=&rowsPerPage=&searchField=Search_All&matchBoolean=true&queryText=(julier%20unscented%20kalman%20filtering%20for%20nonlinear%20estimation)), acessado às 21h00min, no dia 15 de fevereiro de 2016.

às tomadas para propor as extensões do FKE já propostas na literatura. Existem extensões do FKE em direção a diversas classes de filtros, espaços de estados, e sistemas dinâmicos; p. ex. nas seguintes direções:

1. de diferentes classes de espaços de estados com respeito às suas estruturas algébricas, tais como espaços de estado compostos por quatérnios unitários, quatérnios duais unitários, variedades riemannianas, álgebras de Lie, etc;
2. de diferentes classes de sistemas dinâmicos com respeito às formas dos seus conjuntos de tempo — os conjuntos compostos pelos parâmetros de tempo —, tais como sistemas tempo-discreto, sistemas tempo-contínuo, sistemas tempo-contínuo-discreto.

Neste trabalho, nós fazemos um estudo extenso da literatura de filtragem de Kalman Unscented considerando diferentes aspectos, tais como estruturas algébricas do espaços de estados e formas do conjuntos de tempo. Nós mostramos pontos fortes e fracos, fazemos comparações, propomos correções, e apresentamos uma tentativa de teoria sistemática.

B.1 FILTRAGEM DE KALMAN UNSCENTED EM VARIEDADES EUCLIDIANAS

Por meio de uma análise detalhada do estado-da-arte corrente da teoria de filtros de Kalman Unscented tempo-discreto para sistemas dinâmicos em variedades euclidianas, nós revelamos algumas *inconsistências* nessa teoria. Essas inconsistências estão relacionados aos seguintes aspectos dessa teoria:

1. a ordem de estimativa da covariância transformada (Seções 2.4.1 e 2.6.2) e da covariância-cruzada transformada (Seções 2.4.2 e 2.6.3) tanto da Transformação Unscented (TU) como da Transformação Unscented Escalada (TUE) .
2. múltiplas definições do FKU (Seção 2.5);
3. definição dos conjuntos sigma reduzidos de [45], [46] e [83] (Seção 2.5);
4. a conservadorismo da TUE (Seção 2.6.1);
5. o efeito de escalamento da TUE na covariância transformada e na covariância-cruzada transformada (Seções 2.6.2 e 2.6.3);
6. resultados possivelmente mal condicionados nos Filtros de Kalman Unscented Raiz Quadrada (FKURQ, Seção 2.7.1);

7. definições dos Filtros de Kalman Unscented Aditivos (FKUAds) (Seção 2.8).

Esses problemas em conjunto com a dificuldade de agrupar todos os resultados relacionados à teoria Unscented, revelam a existência de lacunas i) nos conceitos matemáticos basilares dessa teoria, e ii) de soluções matemáticas que generalizem os conjuntos sigma, as TUs, e os FKU's da literatura.

Para preencher essas lacunas, nós propomos uma *sistematização* da teoria de filtros de Kalman Unscented tempo-discreto para sistemas dinâmicos em variedades euclidianas. Essa sistematização é feita de forma construtiva, começando pelos conceitos mais simples da teoria.

Começamos a sistematização considerando diversas formas de estimar o valor esperado de um vetor aleatório transformado por uma dada função (Seção 3.1). Uma forma interessante de fazer isso é criar um conjunto de pontos ponderados que aproxime o vetor aleatório independente (não transformada, Seção 3.1). Isso nos fornece a intuição necessário para introduzir as σ -representações de N pontos de ordem l (σ RLN, Definição 3.1) de um vetor aleatório X . Essencialmente, dado um vetor aleatório X , um conjunto de pontos ponderado χ é uma σ RLN de X se seus momentos amostrais (de ordem 1 até l) são iguais aos de momentos de X — também podemos considerar uma σ RLN como sendo uma *transformação* que mapeia X (ou os seus momentos) para um conjunto χ com essas características.

Mediante a proposição de uma *forma matricial* das σ RLNs (Teorema 3.1), descobrimos algumas propriedades-chaves dessas representações, a saber:

1. o *menor número possível* de pontos sigma de uma σ RLN (Corolário 3.1);
2. o *menor número possível* de pontos sigma de uma σ RLN *simétrica* (Corolário 3.1);
3. a forma de uma σ RLN de um vetor aleatório $Z = aX + b$ no caso de uma σ RLN de X ser conhecida (Corolário 3.2); com isso, a σ RLN de um vetor Z com média \bar{Z} e momentos M_2, \dots, M_l pode ser encontrado como resultado, mediante a aplicação do Corolário 3.2, da obtenção prévia de uma σ RLN de um *caso mais simples*; p. ex. da σ RLN de X com média zero e momentos (pares) iguais a matrizes identidade.

Baseando-nos nos resultados 1. e 2., encontramos formas fechadas de algumas σ RLNs. Encontramos i) duas formas fechadas da σ RL2 *simétrica mínima* (Seção 3.3), e ii) uma forma fechada da σ RL2 *mínima* (Seção 3.4).

Uma das formas fechadas da σ RL2 *simétrica mínima* (a σ R Homogênea Simétrica Mínima, Corolário 3.4) é equivalente aos clássicos conjuntos sigma de $[1, 2]$ (Tabela

2.1); dessa forma, nós mostramos os fundamentos por trás desses conjuntos sigma que, até agora, eram baseados apenas em ideias intuitivas. De fato, até aqui, não se sabia nem mesmo que esses conjuntos sigma são compostos pelo menor número possível de pontos sigma.

Quanto à forma fechada da $\sigma Rl2$ mínima (Teorema 3.2), nós mostramos que ela é a *única $\sigma Rl2$ mínima consistente* existente; mostramos que essa $\sigma Rl2$ é um caso geral da *única $\sigma Rl2$ mínima consistente da literatura* (Corolário 3.5).

No entanto, as σRlN ainda não resolvem o problema inicial de estimar o valor esperado de um vetor aleatório *transformado*; uma solução para esse problema é dada pela *Transformação Unscented (TU)*.

O conceito de TU segue naturalmente o de uma σR — quando l ou N não forem importantes para uma discussão ou conhecidas pelo contexto, omitiremos a referência a ela e chamaremos uma σRlN simplesmente de σR . Uma σR pode ser vista como sendo uma transformação que mapeia uma variável aleatório X em um conjunto ponderado χ tal que χ é uma *aproximação de X* ; e já uma TU é uma transformação que mapeia dois vetores aleatórios X e $Y = f(X)$ em dois conjuntos $\chi := \{\chi_i, w_i\}$ e $\gamma := \{\gamma_i, w_i | \gamma_i = f(\chi_i)\}$ tal que $\{\chi, \gamma\}$ *aproxima o vetor aleatório conjunto $[X, Y]^T$* .

Há diversas maneiras de aproximar uma vetor aleatório dessa forma; particularmente, uma TU aproxima $[X, Y]^T$ com a condição de que χ seja uma σR de X . Portanto, podemos dizer que a aproximação de uma TU é baseada no casamento dos momentos de um vetor aleatório com a de uma conjunto ponderado.

Muito embora já existam definições de TU na literatura, no Capítulo 2 nós mostramos alguns problemas com essas definições. Dessa forma, no Capítulo 4, nós apresentamos uma nova definição da TU (Definição 4.1). Essa nova definição é mais geral que as da literatura; a nossa TU é definida para qualquer ordem l (a ordem da σRlN utilizada), ao passo que, até o limite do nosso conhecimento vai, a ordem mais alta da TUs da literatura é 5 (a TU de [47]).

Baseando-nos em Séries de Taylor, nós obtemos a qualidade da estimativa de uma TU de ordem l (Teorema 4.1) — no Capítulo 2, nós havíamos mostrados que i) havia alguns erros relacionados à qualidade de estimativa da TUs, e ii) a qualidade da estimativa de alguns momentos de uma UT não tinha sido ainda determinados, como a das covariâncias cruzadas.

Depois, propomos novas definições para i) a TU escalada (TUE) na Seção 4.2, e ii) a TU (TURQ) raiz-quadrada na seção Section 4.3 — antes, no Capítulo 2, nós havíamos mostrados também que as versões da literatura dessas TUs tinham problemas. Nós pudemos mostrar que nossas definições da TU escala e da TU raiz-quadrada são casos particulares da nossa definição de TU. Assim, todas as propriedades previamente

obtidas para TU são naturalmente herdadas pela TUE e pela TURQ.

Na Seção 4.4, algumas propriedades das TUs desenvolvidas no Capítulo 4 são verificadas em exemplos numéricos.

Com as definições de σR e de TU, nós já dispomos dos conceitos necessários para propor filtros Unscented (FUs) consistentes; isso é feito no Capítulo 5.

Há muitas definições de FKUs na literatura. Para saber em qual dessas definições nos apoiaremos para construir os nosso FKUs tempo-discreto, primeiro investigamos os problemas detectados na Seção 2.8 relativos aos FKUs Aditivos (FKUAds) tempo-discreto da literatura; essa investigação é feita na Seção 5.1. Nos utilizamos dos resultados desenvolvidos no Capítulo 4 concernentes as TUs para estudar as possíveis causas dos maus comportamentos dos FKUAds. Chegamos à conclusão de que *apenas um dos FKUAds da literatura é consistente* com i) o sistema aditivo e ii) a TU.

Baseado nesse FKUAd consistente da literatura, nós definimos o nosso FKUAd tempo-discreto (Seção 5.2). Esse nosso FKUAd é mais geral e baseado em princípios mais consistentes que esse FKUAd da literatura, porquanto é definido mediante a TU e a σR desenvolvidas neste trabalho.

Estendo esse nosso FKUAd tempo-discreto, apresentamos definições para o sistema mais geral (não aditivo) na Seção 5.2, e também para versões raiz-quadrada (Seção 5.3).

Na Seção 5.4, propomos uma lista de casos particulares de todos esses filtros; essa lista mostra que todos os filtros Unscented da literatura são englobados pela sistematização. Mais a frente, na Seção 5.5, apresentamos comentários relativos a aspectos computacionais dos filtros Unscented propostos; e , na Seção 5.7, apresentamos uma discussão sobre filtros Unscented de ordem maior que 2.

Na Seção 5.9, apresentamos critérios para escolher o filtro Unscented mais adequado a um dado problema prático, e na Seção 5.6, ilustramos alguns resultados relativos aos filtros Unscented em exemplos numéricos.

Nesse ponto, apenas exemplos analíticos e numéricos foram utilizados para ilustrar os novos resultados. Completando a tríade de resultados científicos — teoria, simulação, e experimento —, no Capítulo 6, apresentamos um inovação experimental/tecnológica utilizando alguns dos novos FKUs; esse filtros foram usados para estimar a posição de uma válvula eletrônica automática de aceleração. Além de ser uma aplicação prática da teoria de filtragem de Kalman Unscented desenvolvida até então neste trabalho, essa estimação da posição da válvula de aceleração *é uma inovação por si só* desde um ponto de vista prático/tecnológico.

Os resultados do Capítulo 6 têm implicações práticas, com interesse especial em dis-

positivos eletrônicos de aceleração. Esses dispositivos frequentemente possuem apenas um único sensor para medir a posição angular de uma válvula de aceleração; em razão disso, falhas nesse único sensor aumentam os riscos de dano em todo o sistema. Para mitigar o impacto dessas falhas, introduzimos uma técnica que conjuga estimativas de FKUs com medidas produzidas por um wattímetro.

A novidade reside no uso do wattímetro para medir a potência elétrica consumida pelo acelerador. O wattímetro foi preferido por causa do seu baixo custo, mas qualquer outro instrumento poderia ser usado no seu lugar.

Medidas do wattímetro alimentaram os FKUs, e esses filtros, por sua vez, geraram estimativas da posição do acelerador. Até limite do nosso conhecimento, este trabalho é o primeiro a combinar um filtros com um sensor externo para aprimorar a funcionalidade de um acelerador. Experimentos realizados em laboratório mostraram resultados promissores.

O Capítulo 6 encerra a Parte I. Nessa parte, por meio de uma revisão da teoria de filtragem Unscented, revelamos inconsistências e lacunas nessa teoria (Capítulo 2). Em consequência, nos Capítulos 3, 4 e 5, propusemos uma sistematização capaz de resolver essas inconsistências e preencher essas lacunas. Além disso, novos resultados foram introduzidos mediante dessa sistematização. A maior parte dos resultados dessa sistematização foram ilustrados em exemplo numéricos. Finalmente, no Capítulo 6, propusemos uma nova técnica experimental/tecnológica usando alguns dos novos FKUs.

Somando tudo, na Parte I, nós desenvolvemos uma sistematização da teoria de filtragem de Kalman Unscented que foi verificada em exemplos numéricos e em um experimento prático.

B.2 FILTRAGEM DE KALMAN UNSCENTED EM VARIEDADES RIEMANNIANAS

Toda a teoria desenvolvida na Parte I é baseada nos conceitos de *sistemas dinâmicos estocásticos*; tanto nas suas formas tempo-discreto em (2.1) e (2.2), quanto nas tempo-contínuo em (5.43), e tempo-contínuo-discreto em (5.44). Note que, para todos esses sistemas, as variáveis — os vetores de estado, medidas e ruídos — tomam valores em *espaço euclidianos*. Tais sistemas Euclidianas podem ser utilizados para modelar diversos problemas práticos; mesmo assim, para alguns problemas práticos, pode ser melhor utilizarmos outras classes de sistemas.

Quando queremos determinar um modelo dinâmico envolvendo *rotações e/ou ori-*

entações, pode ser vantajoso usarmos *quatérnios unitários*, ao invés de matrizes de rotação — essas matrizes são o modo natural de modelar rotações em um espaço euclidiano de tridimensional. Portanto, podemos considerar sistemas dinâmicos estocásticos em que ao menos algumas variáveis são quatérnios unitários; nesse caso, poderíamos nos perguntar se a sistematização desenvolvida na Parte I pode estendida para esses sistemas quaterniônicos.

A literatura Unscented já tem alguns filtros Unscented para sistemas quaterniônicos. Então, no Capítulo 7, nós analisamos todos os diferentes FKUs e FKUSRs para esses sistemas propostos na literatura. Dessa análise, mostramos que i) uma quantidade considerável desses filtros *não preservam a norma dos quatérnios unitários*; e ii) todos os FKUs aditivos que preservam a norma dos quatérnios unitários *são casos particulares de um novo algoritmo*, a saber do *Filtro Unscented de Kalman Aditivo Quaterniônico* (FKUAdQu, Section 7.3.1). De fato, os FKUAdQu pode resultar em qualquer um desses filtros da literatura por escolhas i) da $\sigma\mathbb{R}$, ii) do método para médias ponderadas de conjuntos de quatérnios unitários, e iii) da parametrização vetorial do conjuntos dos quatérnios unitários (S^3 , possíveis escolhas são apresentadas).

Também introduzimos uma extensão raiz-quadrada do FKUAdQu, o Filtro Unscented de Kalman Raiz-Quadrada Aditivo Quaterniônico (FKUSRAdQu), que tem propriedades melhores que os FKUSRs para sistemas quaterniônicos da literatura (Seção 7.3.2). Em comparação com os FKUs da literatura, o FKUSRAdQu é computacionalmente mais estável em situações (computacionalmente) mal condicionadas por causa das suas propriedades de filtro raiz-quadrada; e em comparação com os FKUSRs da literatura, o FKUSRAdQu é sempre computacionalmente mais estável porque tem menos (ou até nenhuma) *downdating* de fatores de Cholesky (Seção 7.3.2). Essas propriedades superiores do FKUSRAdQu foram verificadas computacionalmente considerando of filtros Unscented para sistemas de atitudes para dois problemas (Seção 7.4.2): 1) um sistema teórico com a performance dos filtros deterioradas por erros de arredondamento computacional; e 2) um problema de estimação de atitude de um satélite em duas situações diferentes: i) um considerando condições normais, e ii) outro considerando condições computacionalmente mal condicionadas. Em dois de todos esses três casos, [a única situação do problema 1), e as situações i) e ii) do problema 2)], o FKUSRAdQu proporcionou estimativas confiáveis, mas todos os filtros Unscented para sistemas quaterniônicos da literatura falharam. Além disso, até mesmo em condições normais [situações i) do problema 2)], o FKUSRAdQu superou os filtros Unscented da literatura, apresentando estimativas melhores (o segundo menor erro foi 10,56% maior que o erro do FKUSRAdQu).

O objetivo inicial do Capítulo 7 era estender a sistematização da Parte I para sistemas quaterniônicos. No entanto, pela análise desenvolvida nesse capítulo, concluímos

que os filtros Unscented para sistemas quaterniônicos da literatura foram construídos sobre alguns conceitos intuitivos, mas não tanto matemáticos; com efeito, podemos citar as seguintes propriedades que sobre as quais esses filtros Unscented foram construídos:

1. Os modelos quaterniônicos aditivos não são consistentes (cf. Nota 7.1).
2. Alguns dos conceitos de probabilidade e estatística o espaços quaterniônicos requerem de mais estudo. Por exemplo, não está claro quais são as definições e as propriedades das i) variáveis aleatórias quaterniônicas, suas distribuições, e suas estatísticas; ii) das estatísticas de conjuntos de quatérnios unitários (tais σR 's quaterniônicas); iii) as estatísticas de uma variável aleatória quaterniônica *transformada*.
3. A forma dos filtros quaterniônicos são estendidos dos euclidianos *sem explicações suficientes*. Por exemplo, qual é a explicação por trás das equações de correção dos filtros quaterniônicos [p. ex. o passo (2d) do FKUSRAdQu]? Qual tipo de aproximação ela dá?

A nossa solução para estender a sistematização da Parte I para sistemas quaterniônicos é baseada em *variedades riemannianas*. Trabalhamos com essas variedades porque i) *o conjuntos dos quatérnios unitários é uma variedades riemanniana*; e ii) *já existem alguns resultados de probabilidade e estatística para variedades riemannianas na literatura*.

No Capítulo 8, nós i) apresentamos *resultados da literatura* relativos a estatísticas desenvolvidas intrinsecamente para variedades riemannianas, ii) fazemos algumas *extensões* desses resultados da literatura — p. ex., entre outros resultados, definições de momentos são estendidas —, e iii) propomos *outros resultados* relativos a estatística em variedades riemannianas — p. ex., entre outros resultados, momentos e momentos amostrais de ordem maior do que 2 (Seção 8.3 e 8.6), resultados relativos a algumas transformações de pontos aleatórios riemannianos (Seção 8.5), e resultados relativos a conjuntos de pontos riemannianos (Seção 8.4).

Nós começamos essa sistematização da filtragem de Kalman Unscented para variedades riemannianas introduzindo a σ -representação riemanniana (σRRi , Seção 9.1). No Teorema 9.1, mostramos que fórmulas fechadas de σR s podem ser usados para encontrar σRRi s; com isso, no Corolário 9.1, determinamos i) o número mínimo de pontos sigma de uma σRRi , ii) o número mínimo de uma σRRi simétrica, e iii) formas fechadas para a mínima σRRi , e iv) formas fechadas para a σRRi simétrica mínima.

De modo parecido ao à sistematização da Parte I, definimos Transformação Unscented Riemanniana (TURi, Seção 9.2), baseando-nos na no conceito de σRRi . Além

disso, estendemos todas as versões da TU do Capítulo 4 para o caso riemanniano; entre outros, propusemos a TURi Escalada e a TURi Raiz-Quadrada.

Na Seção 9.3, nós tratamos dos *filtros* Unscented riemannianos desejados.

Nós introduzimos *uma definição de sistemas riemannianos aditivos* (Seção 9.3.1). Esses sistemas são necessários para definirmos filtros Unscented riemannianos com ruído aditivo, mas, em geral, variedades riemannianas não são equipadas com somas.

Ademais, encontramos equações de correção de Kalman consistentes para os filtros Unscented riemannianos (Seção 9.3.2). Para encontrar essas equações, consideremos, primeiro, um caso particular em que o estado e a medida pertencem à mesma variedade (Seção 9.3.2.1); só então, mediante a extensão desse resultado, conseguimos encontrar a forma final das equações de correção de Kalman (Seção 9.3.2.2).

Na Seção 9.3.3, introduzimos *quatro novos filtros Unscented riemannianos tempo-discreto*. No final dessa seção, providenciamos uma numerosa lista de versões desses quatro filtros riemannianos (Tabelas 9.1, 9.2, 9.3, e 9.4); *todas essas versões são novos filtros Unscented riemannianos*.

Depois, na Seção 9.4, comparamos, teoricamente, os nossos filtros Unscented riemannianos com o único filtro Unscented de Kalman riemanniano da literatura, a saber o Filtro de Kalman Unscented para Variedades riemannianas (FKURM) de [171]. O FKURM de [171] é *essencialmente* diferente de todos os filtros das Tabelas 9.3, 9.1, 9.4, e 9.2, exceto de um: o Filtro Unscented de Kalman Aditivo Simétrico Mínimo Homogêneo riemanniano (FKUAdSiMiRi, Tab 9.4 [1,1]). Mesmo assim, muito embora existam similaridades entre o FKURM de [171] e o FKUAdSiMiRi, o FKUAdSiMiRi apresenta vantagens (cf. Section 9.4).

A intenção inicial da Parte II de desenvolver filtros Unscented para sistemas quaterniônicos é materializada pelos *Filtros Unscented Esférico-riemannianos* (FUERis, Seção 9.5). Mais do que ser apenas uma forma particular dos filtros Unscented riemannianos da Seção 9.3, esses filtros esférico-riemannianos são *computacionalmente implementáveis*.

Conceitos da teoria de variedades riemannianas podem ser bem abstratos, mas geralmente linguagens computacionais não são desenvolvidas para trabalhar com esse nível de abstração. Em lugar disso, frequentemente temos que trabalhar com formas fechadas de casos particulares ou ainda com aproximações numéricas. Nós apresentamos *formas fechadas para quase todas as operações nesses filtros*—tais como mapeamentos exponenciais, mapeamentos logarítmicos, e transportes paralelo—; *apenas médias amostrais de σ RRis ainda precisam de ser encontradas numericamente*.

Mostramos que os FUERis são melhores que os Filtros Unscented Aditivos Qua-

terniônicos (QuAdUF's) da Seção 7.3. Os FUERis possuem melhores propriedades matemáticas que os QuAdUF's e, em um exemplo numérico, uma forma do FUERi superou o USQUE de [48] (este é um consagrado QuAdUF da literatura) por uma grande margem.

Filtros Unscented para *sistemas quaterniônicos duais* são introduzidos na Seção 9.6. Quatérnios unitários são computacionalmente eficientes para representar rotações, e os quatérnios unitários duais podem ser vistos como extensões dos quatérnios unitário para representar deslocamentos de corpos rígidos—rotações em conjunto com translações. Os filtros da Seção 9.6 são *os primeiros filtros Unscented consistentes Unscented para sistemas quaterniônicos duais*, e são baseados nos filtros Unscented Riemannianos

Na Seção 9.7, versões *tempo-contínuo e tempo-contínuo-discreto* dos filtros riemannianos da Seção 9.3.3 são introduzidos também pela primeira vez na literatura.

Somando tudo, podemos afirmar que, neste trabalho, *nós desenvolvemos uma nova e consistente teoria de filtragem de Kalman Unscented para variedades euclidianas e riemannianas*.