The Riemannian geometry of quantum surfaces - The AdS_2 case.

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A geometria Riemanniana das superfícies quânticas - O caso do AdS_2.

Resumo:

Esta tese tem como objetivo construir, de forma rigorosa, uma estrutura algébrica que possua as propriedades geométricas do espaço anti de Sitter bi-dimensional por meio da quantização das coordenadas do hiperboloide que o define. Inicialmente, são definidos os objetos matemáticos essenciais para essa construção, e é formalizado um análogo ao cálculo diferencial e integral sobre a álgebra criada. A partir dessas definições, são apresentados dois módulos importantes, um que representa o análogo ao espaço tangente e seu respectivo dual. Também é definida uma forma hermitiana, que funciona como a métrica do espaço, sendo construído explicitamente seu inverso. Com essas ferramentas, é desenvolvido um cálculo pseudo-Riemanniano para variedades algébricas, permitindo a obtenção dos símbolos de Christoffel, curvatura, Laplaciano, vetores de Killing e até mesmo a definição de autofunções algébricas, viabilizando a integração sobre a superfície quântica. O trabalho discute ainda a não unicidade de alguns elementos e a estrutura algébrica necessária para garantir que o ordenamento dos elementos não comutativos leve a resultados consistentes com a quantização pela deformação do parêntese de Poisson clássico. Além disso, é introduzido um programa que atua como uma ferramenta de cálculo para objetos não comutativos, facilitando os cálculos em superfícies não comutativas nesse formalismo.

Palavras-Chave: Geometria não comutativa, geometria algébrica de Poisson, Geometria Riemaniana não comutativa, espaço Anti-de Sitter bidimensional não comutativo, quantização geométrica, cálculo pseudo-Riemanniano.

Abstract:

The main objective of this thesis is to rigorously construct an algebraic structure that possesses the geometric properties of the two-dimensional anti-de Sitter space through the quantization of the coordinates of the hyperboloid that defines it. Initially, the essential mathematical objects for this endeavor are defined, and an analogue of differential and integral calculus on the constructed algebra is formalized. With these constructions, two important modules are defined: one serving as an analogue to the tangent space, and its respective dual. A Hermitian form is also defined to serve as the metric of the space, and its inverse is concretely constructed. With these tools, a pseudo-Riemannian calculus for algebraic varieties is developed, allowing for the determination of the Christoffel symbols, curvature, Laplacian, Killing vectors, and even the definition of algebraic eigenfunctions, enabling integration on the quantum surface. The article also discusses the non-uniqueness of some elements and the correct algebraic structure required for ensuring that the ordering of non-commutative elements leads to results consistent with quantization through the deformation of the classical Poisson bracket. Additionally, a program is introduced that functions as a calculator for non-commutative objects, facilitating computations involving non-commutative surfaces in this formalism.

Keywords: Non-commutative geometry, Poisson algebraic geometry, non-commutative Riemannian geometry, two-dimensional non-commutative anti-de Sitter space, geometrical quantization, pseudo-Riemannian calculus.

1 Introduction and Mathematical Preliminaries

The main goal of this thesis is to develop a coherent and rigorous prescription for constructing quantum surfaces that exhibit the same geometric objects found in commonly studied topological structures within Riemannian and Poisson geometries. Among these objects, I refer to the metric tensor, curvature, Ricci scalar, Christoffel symbols, and other geometric objects used in the theory of general relativity. We will apply this prescription for the study of AdS_2 as a quantum surface following [1], [2] and [3], as well as comparing all important results we obtained in our paper [7] with other works that do a similar analysis but in a different extent. As a new result we will derive all geometrical properties that allow us to analyse geometrically the $ncAdS_2$ as a quantum surface solving its field equations from a mainly geometrical formalism, in opposition to the usual deformation quantization procedure that is widely applied in the literature. The authors of [4] found exact solutions for the correspondence AdS_2/CFT_1 treating the AdS as a quantum surface, and some of the results obtained in my work can be demonstrated using a totally different approach, showing that these two formalisms could attain consistent results despite their foreseeable unrelatedness.

In this introductory chapter, we will establish the fundamental mathematical frameworks requisite for our endeavor. We will commence by delineating the algebraic structures suitable for the formalism of quantum surfaces, among which Weyl's algebra field of fractions and Heisenberg algebra stand out. Additionally, we will conduct a comprehensive review of the canonical formulation of Poisson geometry, an approach that is usually employed within non-commutative geometric schemes. This entails the promotion of the Poisson structure to the usual commutation relations, acting upon a given Hilbert space. Initially we present some important definitions, theorems and propositions regarding the Weyl and Heinsenberg algebras and we define over it a field of fractions, complex and differential structures in order to rigorously construct the "calculus" over these algebras. After this we study the main properties of the Poisson structure on manifolds and we will take as example the case of minimal surfaces. This example is chosen because our main references [1-3] use it as a starting point to build his theory. By the existence of some special features of these manifolds one can study some properties of the non-commutative minimal surfaces without worrying about unnecessary complications that may appear in a more general case.

In the second chapter we introduce some properties of the AdS space-time setting the coordinates and the embedding we will use in the following parts. The study of Anti-de Sitter (AdS) space is important in theoretical physics, particularly due to the AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence. This idea, proposed by Juan Maldacena in 1997, links gravity in AdS space to a conformal field theory on its boundary. AdS space provides a useful setting for exploring quantum gravity, helping to connect general relativity with quantum mechanics. The AdS/CFT correspondence suggests that a higher-dimensional gravitational theory in AdS space is equivalent to a lower-dimensional quantum field theory, offering insights into black hole physics and strongly coupled systems. Beyond fundamental physics, this correspondence has applications in condensed matter physics and quantum chromodynamics, linking gravitational theories with real-world systems. Following this we discuss some historical background for the non-commutative formulation of quantum mechanics and why one would construct a theory over a non-commutative space-time, presenting also a quantization scheme that was used in our previous paper. We finish this section with an outline of the AdS_2/CFT_1 correspondence for the commutative case and set the prescription we will use to verify it in a non-commutative background.

In the third chapter, we demonstrate how to construct the non-commutative analogue of the AdS_2/CFT_1 correspondence by quantizing the embedding coordinates and Killing vectors of the commutative theory. Through this quantization process, one should obtain a set of operators that retain the symmetries of the commutative case as the non-commutativity parameter approaches zero. Wepresent an alternative approach to quantizing the surface. We discuss how the main properties of Weyl algebras can be used as a tool for geometric quantization and gradually add features to this algebra to create an algebraic structure that enables the construction of the quantum AdS_2 using only the generators of the algebra as parameterization variables. To define a well-defined tangent space, we establish a module structure and discuss the existence of basis vectors and a metric within this framework, concluding this section with some examples. In the final part of this chapter, we construct the non-commutative analogues of the metric tensor, the Christoffel symbols, the curvature tensor, and the Ricci scalar for both ambient and local coordinates. We analyze the results obtained, explain why some ambiguities could arise, and how to address them. We conclude by obtaining the non-commutative Killing vector fields that are solutions to the Killing equation for the non-commutative metric and discuss non-commutative integration using the eigenfunctions of the non-commutative AdS_2 ($ncAdS_2$), which are found by solving the Laplace equation.

In the fourth chapter, we discuss potential developments that could arise from the results obtained in this thesis. The first proposal is to investigate how one could construct a well-defined non-commutative Einstein-Hilbert action that could yield the field equations. We raise a series of questions regarding these ideas and explore two main possibilities. The first approach involves a direct variation of the action using an integral form that must be rigorously defined. The second approach attempts to define the contracted Bianchi identity in the non-commutative setting and use it to derive the resulting field equations. The final proposal for future development is the construction of the spin connection and the Dirac operator using non-commutative vielbeins.

Lastly we present the tool developed to assist with calculations in this thesis in Appendix A. This tool is a non-commutative calculator that can verify the accuracy of some of the results obtained, helping to avoid small mistakes that may arise during lengthy non-commutative calculations. The tool was already functional for several calculations used in this thesis, and we intend to upgrade it to include new functionalities that could assist anyone interested in analyzing non-commutative surfaces.

1.1 Algebraic structures

We begin this subsection by defining the key concepts and objects that will be used throughout this thesis.

Definition 1.1. *Let F be a field and let V be a vector space over F. We define an associative non-commutative bilinear operation $\circ : V \times V \to V$ that will be denoted by the juxtaposition i.e. $v \circ w = vw \neq wv$ for v, w, vw and $wv \in V$. We call V equipped with $\circ$ an* ***F-Algebra****. If V has an element I such that for all $v \in V$ and $Iv = vI = v$ we say that the algebra is* ***unital****.*

Definition 1.2. *Let R be a ring and let 1 be the multiplicative identity*[1]*, we say that M is a* ***left-/right R-Module*** *if it has an Abelian group structure $(M, +)$ and a left/right scalar multiplication satisfying $1 \cdot x = x$,$\forall\ x \in M$ and obeying the distributive and associative rule.*

Definition 1.3. *Let $\mathcal{A}$ be an associative unital algebra, we say that $\mathcal{A}$ is* ***freely generated by*** *a sub-algebra B if there exists, for any map $f : B \longrightarrow C$ where C is any other algebraic object (Lie algebra, algebra, group, etc.), a unique homomorphism $h : \mathcal{A} \longrightarrow C$ such that $h|_B = f$.*[2]

Definition 1.4. *Let A be an algebra, a* ***right (left) ideal*** *of $\mathcal{A}$ is a linear subspace D of $\mathcal{A}$ such that $d \cdot a \in D$ $(a \cdot d \in D)$ $\forall a \in \mathcal{A}$ and $d \in D$. We say that an ideal D is two-sided if it is both a right and a left ideal.*

Definition 1.5. *Let $\mathcal{A}$ be an associative unital algebra and let $\mathcal{I}$ be a two-sided ideal in $\mathcal{A}$. The* ***quotient algebra*** *$\mathcal{A}/\mathcal{I}$ is the associative unital algebra of equivalence classes defined for $a \in \mathcal{A}$ as*

$$[a] = a + \mathcal{I} := \{a + b \mid b \in \mathcal{I}\} \ ,$$

for the equivalence relation $\sim$ defined as $a \sim b \implies (a - b) \in \mathcal{I}$ for $a,\ b \in \mathcal{A}$.

With these concepts in mind we define a Weyl algebra as

[1] In general the ring R doesn't need to be unital in order to define a Module properly.

[2] The sub-algebra B is a generalization of the notion of basis, related to the fact that a linear function $f : X \longrightarrow Y$ between two vector spaces is totally determined by its values on elements of the basis of Y.

Definition 1.6. *Let $\mathcal{A}$ be the free associative unital algebra over $\mathbb{C}$ generated by U, V. Moreover, let $\mathcal{I}$ denote the two-sided ideal generated by*

$$UV - VU - i\hbar\mathbb{1}.$$

The ***Weyl algebra*** *with the non-commutativity parameter $\hbar$ is denoted by $W_\hbar$ and it is defined by the expression below*

$$W_\hbar = A/\mathcal{I},$$

which states that the Weyl algebra is the quotient of the algebra $\mathcal{A}$ with the two-sided ideal $\mathcal{I}$.

As a simple example, if we consider the algebra $\mathcal{A} = \mathcal{A}(\hat{a}, \hat{a}^\dagger)$ to be the algebra generated by the ladder operators, the respective Weyl algebra is obtained by setting $\hbar = -i$, in other words $W_{-i} = \mathcal{A}(\hat{a}, \hat{a}^\dagger)$. Consider also the following example of the construction of $W_\hbar$ in a direct and formal way. Let $P(X^i) = P(X^1, ..., X^n)$ be a ring of polynomials over the set formal symbols $\{X^i\}$ with $p_\beta(X^i) \in P(X^i)$ being a polynomial of order $\beta \in \mathbb{N}$. The set

$$W^{(n)}(P) = \left\{ \sum_{\beta_k \in \mathbb{N}^n} p_\beta(X^i) \prod_{j=1}^{n} \partial_j^{\beta_k} \right\} , \tag{1.1}$$

is called the Weyl algebra of order n over $P(X^i)$. $W^{(n)}$ is a complex algebra with respect to the natural addition of $\mathbb{C}^n$. The product and the left multiplication are defined as

$$\partial_j \cdot f = f\partial_j + \partial_j(f) , \tag{1.2}$$

$$L_f g = fg , \tag{1.3}$$

for $f, g \in P(X^i)$ and $\partial_j, L_f \in \mathrm{End}(P(X^i))$. The space $\mathrm{End}(P(X^i))$ is a non-commutative ring for $n \geq 1$ with the product defined as the composition of endomorphisms satisfying

$$\partial_i \circ L_{X^j} = L_{X^j} + \delta_{ij} ,$$

$$\partial_i \circ \partial_j - \partial_j \circ \partial_i = L_{X^i} \circ L_{X^j} - L_{X^j} \circ L_{X^i} = 0 .$$

These relations lead us to an alternative definition of $W^{(n)}(P)$ as the sub-algebra of $\mathrm{End}(P(X^i))$ generated by $\{L_{X^i}, \partial_i\}$ for $i \in \overline{1,n}$. Using the notation $X^\sigma = \prod_{j=1}^n X_j^{\sigma_j}$ and $\partial^\sigma = \prod_{j=1}^n \partial_j^{\sigma_j}$ for $\sigma \in \mathbb{N}^n$ we can define the general multiplication rule on $W^{(n)}(P)$ as

$$X^\alpha \partial^\beta X^\gamma \partial^\omega = X^{\alpha+\gamma}\partial^{\beta+\omega} + \sum_{\sigma<\alpha,\beta \quad \sigma\neq 0} \frac{\beta!\,\gamma!}{\sigma!\,(\beta-\sigma)!\,(\gamma-\sigma)!} X^{\alpha+\gamma-\sigma}\partial^{\beta+\omega-\sigma} .$$

Now we continue defining some additional objects and structures that will be useful in our work.

Definition 1.7. *Let R be a commutative unital ring and V be a module over R equipped with a skew-symmetric bilinear form*

$$\omega : \ V \otimes_R V \longrightarrow \ R . \tag{1.4}$$

*The **Heisenberg Lie algebra** $\mathfrak{h}$ is the Lie algebra given by the R-module $V \oplus R$ with the unit element $\mathbb{1} = (0,1)$ and its left multiplication over an arbitrary element being $\mathbb{1}(V,r) := (0,r)$ and with the pair (V,r) related to the direct sum. We also define the Lie algebra bracket*

$$[(a,b),\ (a',b')] := (0, \omega(a,a')\mathbb{1}) \tag{1.5}$$

There is a relation between the Heisenberg Lie algebra and the Poisson algebra, which we will be introducing it in a more succinct manner.

Definition 1.8. *A **Poisson Algebra** is a module A over $\mathbb{K}$ (a field or a commutative ring) with two distinct products, namely $\circ$ and $\{\ ,\ \}$, where $\circ : A \otimes_{\mathbb{K}} A \to A$ is the usual product of an associative $\mathbb{K}$-algebra, and the second product $\{\ ,\ \} : A \otimes_{\mathbb{K}} A \to A$ is a Lie bracket, turning it into a Lie algebra such that $\forall a \in A$ the endomorphism $\{a, -\} : A \to A$ is a derivation, i.e. $\{a,-\} \in Der(A)$ satisfying the Leibnitz condition.*

The relation between the Poisson and Heisenberg algebras can be constructed starting by taking (V, ω) to be a symplectic vector space over $\mathbb{R}$. From the tangent bundle structure with the projection $\pi_{1,2} V \times V \to V$ we use the canonical isomorphism $\phi : TV \simeq V \times V$ to construct a differential 2-form $\tilde{\omega}$ from ω by the assignment

$$\tilde{\omega}(x,y) = \omega(\pi_2\phi(x), \pi_2\phi(y)) \ ,$$

$\forall x, y \in \Gamma(TV)$. A symplectic manifold X is $2n$-dimensional manifold equipped with a closed smooth non-degenerate[3] 2-form $\tilde{\omega} \in \Omega^2(X)$, from the assignment above it is clear that $X = (V, \tilde{\omega})$ is a symplectic manifold and the algebra of smooth functions $C^\infty(X, \mathbb{R})$ is indeed a Poisson algebra $\mathcal{P}(V, \tilde{\omega})$ when we define the Poisson bracket as

$$\{f, g\} := \tilde{\omega}(H_f, H_g) \ ,$$

where $f, g \in C^\infty(X, \mathbb{R})$ and $H_f \in \Gamma(TX)$ is the Hamiltonian vector field of X uniquely (from the non-degeneracy of ω) defined by

$$df = \tilde{\omega}(H_f, -)$$

The subspace of linear functions $\mathcal{L}(V, \mathbb{R}) \subset C^\infty(V)$ from V to $\mathbb{R}$ form the dual vector space V^* of V from the inclusion $V^* \hookrightarrow C^\infty(V)$. By the non-degeneracy of ω, $\forall f \in V^* \ \exists \ v_f \in V$ such that

$$f = \omega(v_f, -) \in C^\infty(V) \ ,$$

in this setting the extension H_f of $v_f \in \Gamma(TV)$ is a Hamiltonian vector field for f and from this follows that the Lie bracket of $f, g \in \mathcal{L}(V, \mathbb{R})$ is

$$\{f, g\} = \omega(v_f, v_g)$$

[3] This can be achieved by imposing that the form $\omega^{\wedge n} = \omega \wedge \omega \wedge \cdots \wedge \omega$ has the maximal rank at every point $x \in X$.

and as a last step we define the following inclusions $c_1 : \omega(v,-) \hookrightarrow C^\infty(V)$ for the Lie bracket and $c_2 : \mathbb{R} \hookrightarrow C^\infty(V)$ for the constant functions, then $c : V \oplus \mathbb{R} \xrightarrow{(c_1,c_2)} C^\infty(V)$ induces a Lie algebra homomorphism

$$c : \ \mathfrak{h}(V,\omega) \hookrightarrow \mathcal{P}(V,\tilde{\omega}) \ ,$$

showing that the Heisenberg Lie algebra is a sub-Lie algebra of the Poisson algebra $\mathcal{P}(V,\tilde{\omega})$. To explictly construct the structure written above consider first the Heisenberg group $\mathcal{H}(V) = \{V \times \mathbb{R}, \ \cdot \ \}$ on (V,ω) where the group law is defined as

$$(v,\alpha)\cdot(w,\beta) := \Big(v+w, \alpha+\beta+\frac{c}{2}\omega(v,w)\Big) \ ,$$

for $v, w \in V$, $\alpha, \beta \in \mathbb{R}$ and for some constant c that will be related to the Lie bracket on the Heisenberg algebra. Using the fact that every symplectic vector space has a Darboux basis $\{e_i, \tilde{e}^j\}_{i,j=1}^n$ and if we consider $\hat{r}$ as a basis for $\mathbb{R}$, we can define a new basis for $V \times \mathbb{R}$ as $\{e_i, \tilde{e}^j, \hat{r}\}_{i,j=1}^n$ where any vector from it can be written as $v = q^i e_i + p_j \tilde{e}^j + \alpha\hat{r}$. From it the group law becomes

$$(p,q,\alpha)\cdot(p',q',\beta) = \Big(p+p', q+q', \alpha+\beta+\frac{c}{2}(pq'-p'q)\Big) \ ,$$

and from the linearity of $\mathcal{H}$ we can indentify the vectors in the group with vectors of its Lie algebra $\mathfrak{h}$, which consequently gives as the Lie bracket of the group the usual commutation relation

$$[(v,\alpha),(w,\beta)] = c\omega(v,w) \ ,$$

which clearly implies that, for the Darboux basis $[e_i, \tilde{e}^j] = c\delta_i^j$ and all other commutators vanish. After the definition of the universal enveloping algebra, we will make this construction more suitable with the usual quantum mechanics setting.

Definition 1.9. *Let $\mathfrak{g}$ be a Lie algebra over a field F, we can construct a Tensor free algebra from it containing all possible tensor products of elements of $\mathfrak{g}$, or explicitly*

$$T(\mathfrak{g}) = F \bigoplus_{i\in\mathbb{N}} \otimes^i \mathfrak{g} \ ,$$

*the **universal enveloping algebra** is essentially the quotient $T(\mathfrak{g})/\mathcal{I}$ of the tensor algebra with the two-sided ideal $\mathcal{I}$ generated by the abstract Lie bracket on $\mathfrak{g}$ and for all a, b in the embedding of $\mathfrak{g}$ in $T(\mathfrak{g})$ this ideal can be explicitly constructed from*

$$a \otimes b - b \otimes a - [a,b] \ ,$$

where $[a,b]$ is the abstract Lie bracket on $\mathfrak{g} \in T(\mathfrak{g})$.

If we consider the universal enveloping algebra of the Heisenberg algebra with $2n$ generators obtained by identification of the center (elements that commutes with all other elements of the algebra) of the Heisenberg Lie algebra with multiples of the identity element, we obtain from it the Weyl algebra on 2n generators. For example, let $\mathfrak{h}$ be an $2n+1$ dimensional real Lie algebra with elements

$$\{p_1,\ ...,\ p_n,\ q_1,\ ...,\ q_n,\ c\}\ , \tag{1.6}$$

for $c \in \mathbb{C}$ the center of the algebra, it will become a Heisenberg algebra if we define on it the following Lie bracket

$$[p_i,\ p_j] = [q_i,\ q_j] = 0\ ,$$
$$[p_i,\ q_j] = c\delta_{ij}\ .$$

In this setting, the set of generators p, q acts as a Darboux basis, and the symplectic structure arises from the intrinsic definition of the commutation relation. Therefore, one can easily see that $\mathfrak{h}$ is constructed from $2n$ copies of the previously constructed Weyl algebra with $c = i\hbar$ and the center $\mathcal{C}$ being all complex-number-valued multiples of the identity. The structures of the Weyl and Heisenberg algebras arise naturally from the physical analysis of quantum phenomena. As will be discussed in the following chapters, these algebras are suited to introducing non-commutativity as an intrinsic feature of a given space-time when we promote the usual parametrization coordinates to operators that act on some Hilbert space. In a sense, this will allow us to define the geometry of the quantum algebras directly.

Now, we proceed with the next step in our enterprise. We turn our attention to $W_\hbar$ and aim to make it a field of fractions. For this purpose, we follow a similar path employed in [1] which is a brief review of the whole proccess done in [38], we omit some technicalities to make our text more legible. Since our main goal is to have a well defined field of fractions where elements[4] from $W_\hbar \times W_\hbar$ will be written as the ordered pair (A, B) in direct correspondence to the element AB^{-1}, we should first define the equivalence class underlying the so-called (right) Ore condition, namely, there exist a pair of elements $\alpha, \beta \in W_\hbar$ such that

$$A\alpha = B\beta\ , \tag{1.7}$$

this relation guarantees the existence of a common factor between two elements of the algebra. We also define the zero element, let A be an element of $W_\hbar$ that satisfies the following property $cA + B = B\ , \forall\ B \in W_\hbar$ and $c \in \mathbb{C}$, we denote A as 0 for the Weyl algebra and by definition A is not invertible. We can use this property to define an equivalence relation on $W_\hbar \times W_\hbar$, but first, for $B \neq 0$, we will denote a *fraction* as $\left(\frac{A}{B}\right)$ and we say that $\left(\frac{A}{B}\right) \sim \left(\frac{C}{D}\right)$ if there exist $\alpha,\ \beta \in W_\hbar$ such that

$$\begin{aligned} A\alpha &= C\beta\ , \\ B\alpha &= D\beta\ . \end{aligned} \tag{1.8}$$

and we will not verify that this indeed defines an equivalence relation since it is straightforward to show this. The relation above is equivalent to $\frac{a}{b} = \frac{x\cdot a}{x\cdot b}$ in a field of fractions. The quotient space

[4] Here we consider the Weyl algebra $W_\hbar$ generated by U and V as stated in the definition (1.6).

$W_\hbar \times W_\hbar/\sim$ is denoted by $\mathcal{F}_\hbar$ and will be called the field of fractions of the Weyl algebra $W_\hbar$. One can define the addition and multiplication on $\mathcal{F}_\hbar$ for $Y\alpha = W\beta = Z\gamma$ and $\alpha, \beta, \gamma,\ X,\ Y,\ Z,\ W \in W_\hbar$

$$\begin{aligned} \left(\frac{X}{Y}\right) + \left(\frac{Z}{W}\right) &= \left(\frac{X\alpha + Z\beta}{Y\alpha}\right) = \left(\frac{X\alpha + Z\beta}{W\beta}\right) \\ \left(\frac{X}{Y}\right) \cdot \left(\frac{Z}{W}\right) &= \left(\frac{X\alpha}{W\gamma}\right) \ . \end{aligned} \tag{1.9}$$

It is clear that one should verify if the operations above are well-defined (i.e., they respect the equivalence classes) and do not depend on the choice of α, β, γ, since this is done in [38] and [1], we will omit this verification here. Furthermore, we can represent the unit element by $\mathbb{1}_W = (\mathbb{1}/\mathbb{1})$ and the zero element $0_W = (0/\mathbb{1})$. For every non-zero element $B \in W_\hbar$ we identify B^{-1} with $(\mathbb{1}/B)$, and with this notation it is clear that $(A/B) = (A/\mathbb{1}) \cdot (\mathbb{1}/B) = AB^{-1}$, note that the way we defined the product of fractions make the element AB^{-1} not equal, in general, to $(\mathbb{1}/B) \cdot (A/\mathbb{1}) = B^{-1}A$ as one should expect (the equality holds if $[A, B] = 0$). To make $\mathcal{F}_\hbar$ a *-algebra we extend the involution operation to it. An involution is an anti-homomorphism that is its own inverse, we extend it to $\mathcal{F}_\hbar$ by setting $U^* = U$, $V^* = V$ and as consequence of the universal property of the fraction ring, we achieve our goal. Since now $\mathcal{F}_\hbar$ is a $*$-algebra we obtain an well defined structure that resembles the complex structure of $\mathbb{C}$ with the involution operation $*$ satisfying the following properties

$$\begin{aligned} &\left(\frac{A}{\mathbb{1}}\right)^* = \left(\frac{A^*}{\mathbb{1}}\right) \ , \\ &\left(\frac{\mathbb{1}}{A}\right)^* = \left(\frac{\mathbb{1}}{A^*}\right) \ , \\ &(AB^{-1})^* = \left(\left(\frac{A}{\mathbb{1}}\right) \cdot \left(\frac{\mathbb{1}}{B}\right)\right)^* = \left(\left(\frac{\mathbb{1}}{B^*}\right) \cdot \left(\frac{A^*}{\mathbb{1}}\right)\right) = (B^*)^{-1}A \ . \end{aligned} \tag{1.10}$$

From now on we will denote the fractions (A/B) on $\mathcal{F}_\hbar$ as AB^{-1} in this specific ordering. The real and imaginary elements of $\mathcal{F}_\hbar$ are

$$\begin{aligned} \Re(A) &= \frac{1}{2}(A + A^*) \ , \\ \Im(A) &= \frac{1}{2i}(A - A^*) \ . \end{aligned} \tag{1.11}$$

We can also define derivatives with respect to the generators in our algebra by noting that, for example, for some function $F(U)$ with a formal series expansion we have

$$[F(U), V] = i\hbar F'(U) \ ,$$

and the same is valid for some function of V with some change in sign. As expected we can define the derivatives as

$$\begin{aligned} \partial_U(A) &= \frac{1}{i\hbar}[A, V] \ , \\ \partial_V(A) &= -\frac{1}{i\hbar}[A, U] \ , \end{aligned} \tag{1.12}$$

and it is easy to verify that this definition obey the Leibniz rule and all important properties expected for a well-defined derivative (see [1]). We also extend these derivatives for $\mathcal{F}_\hbar$ by simply noting that $\partial_x(AA^{-1}) = (\partial_x A)A^{-1} + A(\partial_x A^{-1}) = 0$ which implies that $\partial_x A^{-1} = -A^{-1}(\partial_x A)A^{-1}$ for $x = U, V$ and $A \in \mathcal{F}_\hbar$. To keep the analogy with the complex numbers and their derivatives we introduce a new element $\Lambda \in \mathcal{F}_\hbar$ as $\Lambda = U + iV$. Clearly $[\Lambda, \Lambda^*] = 2\hbar\mathbb{1}$, in the following we define the derivatives with respect to these new elements

$$\begin{aligned} \partial(A) &= \frac{1}{2}(\partial_U - i\partial_V)[A] = \frac{1}{2\hbar}[A, \Lambda^*] \ , \\ \overline{\partial}(A) &= \frac{1}{2}(\partial_U + i\partial_V)[A] = -\frac{1}{2\hbar}[A, \Lambda] \ . \end{aligned} \tag{1.13}$$

We denote the sub-algebra $\mathbb{C}[\Lambda] \subset \mathcal{F}_\hbar$ as the algebra generated by Λ and $\mathbb{1}$.

Definition 1.10. *We say that an element $A \in \mathcal{F}_\hbar$ is said to be **r-holomorphic** if $\overline{\partial}A = 0$ and we call this element **holomorphic** if $A \in W_\hbar$ (equivalently if and only if $A \in \mathbb{C}[\Lambda]$).*

The r-holomorphic elements are analogues of meromorphic functions in the field of fractions, and generally they are written as fractions of polynomials in Λ. As the last step in this section we proceed to define the non-commutative Laplace operator, a notion of integrability in this setting and some additional objects that will be used in order to define a non-commutative pseudo-Riemannian geometry.

Definition 1.11. *We define the non-commutative **Laplace operator** Δ_0 as follows*

$$\Delta_0(A) = -\frac{1}{\hbar^2}\Big([[A, V], V] + [[A, U], U]\Big) \ ,$$

*An element A is called **harmonic** if $\Delta_0(A) = 0$.*[5]

Definition 1.12. *Let A and B be r-holomorphic elements, we say that B is a **primitive element** of A if $\partial B = A$ and we denote B for an arbitrary element A as*

$$B = \int A \, d\Lambda \ .$$

*If a r-holomorphic element A has at least one primitive element we say that A is **integrable**.*

Now we follow similar constructions done in [43], [44], [45] and [46] to introduce some concepts that we will need to construct the non-commutative pseudo-Riemannian geometry, we refer to these references for the proofs of any claim that may appear in the following. All the definitions and results will be taken for a right module and we refer that it is possible to extend to a left one by writing in parentheses, when a bimodule structure is implied we will say it explicitly.

[5] Two obvious results that follow from the last definition are

$$\begin{aligned} \Delta_0(A) = 4\partial\overline{\partial}(A) &= 4\overline{\partial}\partial(A) \ , \\ \partial_U \Re(A) = \partial_V \Im(A) \qquad \text{and} & \qquad \partial_V \Re(A) = -\partial_U \Im(A) \ , \end{aligned} \tag{1.14}$$

Definition 1.13. *Let $\mathcal{A}$ be a unital *-Algebra over $\mathbb{C}$. A left (right) $\mathcal{A}$-module M is said to have a* ***canonical structure*** *of a right (left) $\mathcal{A}$-module if we impose $m \cdot a = a^* \cdot m$ (respectively for the left one $a \cdot m = m \cdot a^*$) for $a \in \mathcal{A}$ and $m \in M$ and the operation $\cdot$ it the scalar multiplication for the module M.*

Definition 1.14. *For a right (left) $\mathcal{A}$-module M we denote as M^* its* ***dual module****. M^* will be canonically a left (right) module if we set $(a \cdot \omega)[m] = a\omega[m]$, for $\omega \in M^*$, $a \in \mathcal{A}$, the product $\cdot$ is the scalar multiplication in M^* and the action of ω on $m \in M$ is denoted by the square brackets. Using the same argument as in the Def 1.12. we can consider M^* as a right module by imposing a similar involution property.*

Definition 1.15. *Let $\mathcal{A}$ be a unital *-algebra and let M be a $\mathcal{A}$-module. A* ***hermitian form*** *on M is a right linear map $h : M \times M \to \mathcal{A}$ such that*

$$\begin{aligned} &(i)\ h(m,n) = h(n,m)^* \ , \\ &(ii)\ h(m, n \cdot a) = h(m,n)a \ , \\ &(iii)\ h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n) \ . \end{aligned} \tag{1.15}$$

for $m,\ m_1,\ m_2,\ n \in M$ and $a \in \mathcal{A}$ where the multiplication by juxtaposition is the natural $\mathcal{A}$ multiplication.

Definition 1.16. *Let $\mathcal{A}$ be a unital *- algebra and let M be a $\mathcal{A}$-module. The map $\hat{h} : M \to M^*$ is associated to a hermitian form as follows*

$$\begin{aligned} &(i)\ \hat{h}(m_1)[n] = h(m_1, n) \ , \\ &(ii)\ \hat{h}(m_1 + m_2) = \hat{h}(m_1) + \hat{h}(m_2) \ , \\ &(iii)\ \hat{h}(m_1 \cdot a) = a^* \cdot \hat{h}(m_1) \ , \end{aligned}$$

for $m_1,\ m_2,\ n \in M$ and $a \in \mathcal{A}$. If we consider M^* as a right (left) module, $\hat{h}$ is a homomorphism of right (left) modules. A hermitian form on the left and right $\mathcal{A}$-module M^* defined as $h^{-1} : M^* \times M^* \to \mathcal{A}$ with $h^{-1}(\omega_1, \omega_2) = \omega_1\Big(\hat{h}^{-1}(\omega_2)\Big)$ exists when $\hat{h}$ is a bijection, we call h^{-1} the ***inverse hermitian form*** *of h.*

Definition 1.17. *Let M be a right (left) $\mathcal{A}$-module and let h be a hermitian form on M. The pair (M,h) is called a* ***right (left) hermitian module****. Consider $\phi : (M,h) \to (M',h')$ satisfying $h(m,n) = h'(\phi(m), \phi(n))\ \forall\ m,n \in M$. If ϕ is also a module isomorphism then ϕ is called an* ***isometry*** *and (M,h) and (M,h) are said to be* ***isometric****.*

Definition 1.18. *A module M is* ***projective*** *if whenever M is a quotient of a free module N, there exist a module X such that N is isomorphic to $M \oplus X$.*

Definition 1.19. *If h is an invertible hermitian form on a finitely generated projective $\mathcal{A}$-module M, then (M,h) is called a* ***regular hermitian $\mathcal{A}$-module****.*

Definition 1.20. *Let (M,h) be a hermitian $\mathcal{A}$-module and let $\phi : M \to M$ be an endomorphism. If $h(\phi(m), n) = h(m, \phi(n))$ $\forall$ $m, n \in M$ then ϕ is said to be* ***orthogonal with respect to h****.*

Definition 1.21. *Let $\mathcal{A}$ be a unital $*$-algebra, a derivation in $\mathcal{A}$ is a linear map $\partial : \mathcal{A} \to \mathcal{A}$ that satisfies the Leibniz's rule, i.e., $\partial(AB) = A\partial(B) + \partial(A)B$. The collection of all derivations in A is called the* ***algebra of derivations*** *over $\mathcal{A}$ and it is denoted as $Der(\mathcal{A})$.*

Definition 1.22. *Let $\mathcal{A}$ be a unital *-algebra, the complex *-closed vector space $V \subset Der(\mathcal{A})$ is the complexification of the real vector space $V_{\mathbb{R}}$ with hermitian[6] basis vectors. A* ***Lie pair*** *is the pair $(\mathcal{A}, \mathfrak{g})$ where $\mathcal{A}$ is a unital *-algebra and $\mathfrak{g}$ is the Lie algebra over V. A Lie pair defines a* ***derivation based calculus*** *$\Omega(\mathcal{A})$ over the algebra $\mathcal{A}$.*

In [45] one can find some proven theorems about regular hermitian modules. One of the results we will use here claims that every regular hermitian module can be constructed as the image of an orthogonal projection on a free module if the hermitian form h is invertible. Now we will focus our attention on the geometry of Poisson manifolds in order to build a reasonable set of tools that will allow us to develop a structure analogous to the Poisson geometry using the previously defined algebras. These tools will help us to define surfaces in a non-commutative way and we will start analyzing, as a simple example, the quantum minimal surfaces and its properties.

1.2 Poisson Algebraic Geometry

In the following we recall some standard definitions and constructions regarding Poisson geometrical objects and minimal classic surfaces in order to finish this section analizing the example of non-commutative minimal surfaces. We follow an standard exposition of these topics and we refer to [34], [35] or any book on Riemannian geometry.

Using the definition 1.7 it is easy to see that the Poisson structure together with the pointwise multiplication makes the vector space of smooth functions on M a Poisson algebra. In this setup, derivations correspond to the Hamiltonian vector fields $X_f(g) = \{f, g\}$ which are related to the classical Hamiltonian function. It's well known that the geometry of surfaces can be expressed via Poisson brackets and all the information gathered by these means can be related to the non-commutative geometry and help to define these new objects (see [36] and [37] for a comprehensive exposition of these topics). Following this strategy we will show some geometrical properties of Poisson algebras. Let Σ be a 2-dimensional manifold with ambient coordinates u^i

[6] We say that a set $S \subset \mathrm{Der}(\mathcal{A})$ is *-closed if $\forall\ \partial \in S \iff \partial^* \in S$ where $\partial^*(f) = \Big(\partial(f^*)\Big)^*$ for $f \in C^\infty(\mathcal{A})$. A derivation is hermitian if $\partial = \partial^*$

embedded in $\mathbb{R}^n$ through the embedding coordinates $x^\mu(u^i)$, with Latin letters taking value in $1, 2$ and Greek letters taking value in $1, ..., n$. The induced metric takes the following form

$$g_{ab} = \sum_{\mu=1}^{n} (\partial_a x^\mu)(\partial_b x^\mu) \ . \tag{1.16}$$

We introduce a Poisson bracket $\{- \ , \ -\} \ : \ C^\infty(\Sigma) \times C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$ for an arbitrary density function ρ and $f, g \in C^\infty(\Sigma)$

$$\{f, g\} = \frac{\epsilon^{ab}}{\rho} (\partial_a f)(\partial_b g) \ , \tag{1.17}$$

where ϵ^{ab} is the usual $2D$ Levi-Civita symbol. We also define the Poisson bivector $\theta^{ab} = \frac{1}{\rho}\epsilon^{ab}$ and $\gamma = \sqrt{g}/\rho$, for g being the determinant of the induced metric. This gives us a way of expressing the inverse of the metric as $g^{ab} = (1/\gamma^2)\theta^{ac}\theta^{bd} g_{cd}$ since $g_{ac}\epsilon^{ab}\epsilon^{cd}$ is just the cofactor expansion of the inverse of the metric. The geometry of the sub-manifold $\Sigma \subset \mathbb{R}^n$ can be obtained from the definition of the projection operator $\mathcal{P} : T\mathbb{R}^n \longrightarrow T\Sigma$, for $X \in T\mathbb{R}^n$ and $T\Sigma \subset T\mathbb{R}^n$, which can be written as

$$\mathcal{P}^\mu(X) = \frac{1}{\gamma^2} \{x^\mu, x^\nu\}\{x^\alpha, x^\nu\} X^\alpha \ , \tag{1.18}$$

where the Einstein summation convention is implied. The projective property can be easily seen by calculating

$$\begin{aligned} \mathcal{P}^\mu(X) &= \frac{1}{\gamma^2}\theta^{ab}\theta^{cd}(\partial_a x^\mu)(\partial_b x^\nu)(\partial_c x^\alpha)(\partial_d x^\nu) X^\alpha = \frac{1}{\gamma^2}(\theta^{ab}\theta^{cd} g_{bd})(\partial_a x^\mu)(\partial_c x^\alpha) X^\alpha \ , \\ &= g^{ac}(\partial_a x^\mu)(\partial_c x^\alpha) X^\alpha \ , \end{aligned}$$

and from this expression we get to the following

$$\mathcal{P}^\rho(\mathcal{P}(X)) = g^{ef}(\partial_e x^\rho)(\partial_f x^\mu)\Big(g^{ac}(\partial_a x^\mu)(\partial_c x^\alpha) X^\alpha\Big) = g^{ef} g_{fa} g^{ac}(\partial_e x^\rho)(\partial_c x^\alpha) X^\alpha = \mathcal{P}^\rho(X) \ ,$$

showing that this is indeed a projector. In this setting the Laplace-Beltrami operator on Σ for some $f \in C^\infty(\Sigma)$ is

$$\Delta(f) = \frac{1}{\gamma}\left\{\frac{1}{\gamma}\{f, x^\mu\}, x^\mu\right\}, \tag{1.19}$$

and we can show that the definition above is equivalent to the usual $\Delta(f) = \frac{1}{\sqrt{g}}\partial_a\Big(\sqrt{g} g^{ab}\partial_b f\Big)$ by direct inspection

$$\begin{aligned} \Delta(f) &= \frac{1}{\gamma}\theta^{ab}\partial_a\Big(\frac{\theta^{cd}}{\gamma}(\partial_c f)(\partial_d x^\mu)(\partial_b x^\mu)\Big) = \frac{1}{\sqrt{g}}\partial_a\Big(\frac{\epsilon^{ab}}{\gamma}\theta^{cd} g_{bd}(\partial_c f)\Big) \ , \\ &= \frac{1}{\sqrt{g}}\partial_a\Big(\frac{\rho}{\gamma}\theta^{ab}\theta^{cd} g_{bd}(\partial_p f)\Big) = \frac{1}{\sqrt{g}}\partial_a\Big(\rho\gamma g^{ac}\partial_c f\Big) \ = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g} g^{ac}\partial_c f) \ . \end{aligned}$$

Now that we have defined in a more concrete manner how we can construct a surface from their Poisson structure we now define what is a minimal surface. Classically, surfaces that have zero mean curvature at every point are called minimal surfaces and one of its properties is that they minimize the area functional. In our setting, we can characterize minimal surfaces by the fact that their embedding coordinates, viewed as functions, belong to the Kernel of the Laplace operator Δ on Σ.

For some local conformal coordinates[7] u^1, u^2 satisfying $\{u^1, u^2\} = 1$ and for $x^\mu \in \ker(\Delta)\ \forall\ \mu = \overline{1, n}$ every set of embedding coordinates satisfy

$$\Delta(f) = \frac{1}{\eta(u^1,u^2)}\Big\{\{\{f, u^a\}\delta_{ab}, u^b\}\Big\} = \frac{1}{\eta(u^1,u^2)}\Big(\{\{f, u^1\}, u^1\} + \{\{f, u^2\}, u^2\}\Big) \ , \tag{1.20}$$

Another classical way to define minimal surfaces is using the fundamental forms. When evaluating the length of a element $w \in T\Sigma$ one can use the induced inner product from $\mathbb{R}^3$, then the first fundamental form I for $u^1 = u$ and $u^2 = v$ is given by

$$\begin{aligned} E &= \langle \partial_u X(u,v)\ , \partial_u X(u,v)\rangle \ , \\ F &= \langle \partial_v X(u,v)\ , \partial_u X(u,v)\rangle \ , \\ G &= \langle \partial_v X(u,v)\ , \partial_v X(u,v)\rangle \ , \end{aligned} \tag{1.21}$$

where $X(u,v)$ defines the parametric surface in $\mathbb{R}^3$. The coefficients of the second fundamental form $\mathrm{I\!I}$,

$$\begin{aligned} e &= \langle \partial_u^2 X(u,v)\ , N(u,v)\rangle \ , \\ f &= \langle \partial_u \partial_v X(u,v)\ , N(u,v)\rangle \ , \\ g &= \langle \partial_v^2 X(u,v)\ , N(u,v)\rangle \ , \end{aligned} \tag{1.22}$$

with $N(u,v) = \frac{\partial_u X(u,v)\ \wedge\ \partial_v X(u,v)}{|\partial_u X(u,v)\ \wedge\ \partial_v X(u,v)|}$ being the normal. Upon recalling the definition of mean curvature in terms of the first and second fundamental forms, which is $H = \frac{gE - 2fF + eG}{2(EG - F^2)}$, the equation $H = 0$ implies that the coefficients of the fundamental forms must satisfy

$$eG - 2fF + gE = 0 \ , \tag{1.23}$$

for all points in the surface. Now that we recalled some properties of minimal surfaces we intend to construct a non-commutative algebra containing the generators U, V in order to define an non-commutative minimal surface. The strategy we are employing here will be the following: i) We will define a Weyl algebra and its field of fractions for some generators U, V satisfying $[U, V] = i\hbar\mathbb{1}$ for some constant $\hbar$ that measures the non-commutativity of the surface; ii) we will introduce a module structure in order to replicate the main properties of a tangent space over this non-commutative manifold. Now we proceed to the next sections to apply this strategy.

1.3 Non-commutative Minimal surfaces

The classical theory of minimal surfaces can be developed, with some modifications, in the non-commutative set-up by relating all classical quantities to the new ones defined in a free[8] module

[7] We say a set of coordinates is conformal if the metric can be written as $g_{ab} = \eta(u^1, u^2)\delta_{ab}$ for some strictly positive function $\eta(u^1, u^2)$. One can always find such coordinates locally for any surface. From now on we will denote $\Delta_0 = \eta(u^1, u^2)\Delta = \{\{x^\mu, u^1\}, u^1\} + \{\{x^\mu, u^2\}, u^2\}$.

[8] Since $\mathcal{F}_\hbar$ was extended to a division ring, it suffices to consider a linearly independent generating set as a basis to promote the direct product of n copies of $\mathcal{F}_\hbar$ to a free module.

over $\mathcal{F}_\hbar$. Consider the canonical basis on $\mathcal{F}_\hbar^m$ as

$$E_n = (0,\ ...,\ 0,\ \mathbb{1},\ 0,\ ...,\ 0) \tag{1.24}$$

in which the m-th element is non-zero. Returning to the example where the Weyl algebra is constructed over a ring of polynomials in $W_\hbar$, we can consider the following set of monomials as a basis

$$\mathcal{S} = \{X^\alpha \partial^\beta | \alpha,\ \beta\ \in \mathbb{N}^m\}\ . \tag{1.25}$$

In this case, any non-zero element in the free module $\mathcal{F}_\hbar^m$ generated by m copies of $\mathcal{F}_\hbar$ can be written in a unique way as the finite sum

$$A = \sum_{\alpha,\ \beta} a_{\alpha\beta} X^\alpha \partial^\beta\ , \tag{1.26}$$

as a consequence, the left action of an element $A \in W_\hbar$ is

$$A(f) = \sum_{\alpha,\ \beta} a_{\alpha\beta} \prod_{i=1}^{m} X^{\alpha_i} \prod_{j=1}^{m} \partial^{\beta_j}(f)\ . \tag{1.27}$$

It is easy to see that any linear differential operator can be written uniquely using the basis $\mathcal{S}$ in the free module $\mathcal{F}_\hbar^m$. If we assume, for example, that $\Phi_{sol}(X)\ \in P(X^i)$ is the solution of a system of linear partial differential equations

$$\text{Sys} = \begin{cases} A_1(\Phi) & = 0 \\ \vdots & \vdots \\ A_n(\Phi) & = 0 \end{cases} \tag{1.28}$$

then one can assert that Φ_{sol} belongs to the left ideal generated by A_i.

Now that we have seen an concrete example of this formalism, we can continue introducing additional features to $\mathcal{F}_\hbar^m$. One can extend the action of the derivative operator as

$$\partial_U(K) = \partial_U(K^i)E_i\ , \tag{1.29}$$

for $K\ \in \mathcal{F}_\hbar^m$ and the same holding to ∂_V. We can also introduce a symmetric bi-linear form

$$\langle \overrightarrow{A}, \overrightarrow{B} \rangle = \frac{1}{2} \sum_{i=1}^{m} (A^i B^i + B^i A^i)\ , \tag{1.30}$$

for $\overrightarrow{A},\ \overrightarrow{B}\ \in \mathcal{F}_\hbar^m$ and we will use this arrow notation for any element of this space. One can easily show that

$$\partial_U \langle \overrightarrow{A}, \overrightarrow{B} \rangle = \frac{1}{i\hbar}[\langle \overrightarrow{A}, \overrightarrow{B} \rangle,\ V] = \frac{1}{i\hbar}\langle [A^i, V]E_i, \overrightarrow{B} \rangle + \frac{1}{i\hbar}\langle \overrightarrow{A}, [B^i, V]E_i \rangle\ , \tag{1.31}$$

which means that this bi-linear form obey the Leibnitz rule for the derivative operator. Keeping the analogy between the classical and the non-commutative setting, we shall introduce the non-commutative minimal surfaces.

Definition 1.23. *A hermitian element* $\overrightarrow{A} \in \mathcal{F}_\hbar^m$ *is called a non-commutative minimal surface if*

$$\begin{aligned} &\Delta_0(A^i) = 0 \qquad \textit{for } i = \overline{1,m} \\ &\langle \partial_U \overrightarrow{A},\ \partial_U \overrightarrow{A} \rangle = \langle \partial_V \overrightarrow{A},\ \partial_V \overrightarrow{A} \rangle \quad \textit{and } \langle \partial_U \overrightarrow{A},\ \partial_V \overrightarrow{A} \rangle = 0\ . \end{aligned} \tag{1.32}$$

If we denote $\mathcal{E} = \langle \partial_U \overrightarrow{A},\ \partial_U \overrightarrow{A} \rangle,\ \mathcal{F} = \langle \partial_U \overrightarrow{A},\ \partial_V \overrightarrow{A} \rangle$ and $\mathcal{G} = \langle \partial_V \overrightarrow{A},\ \partial_V \overrightarrow{A} \rangle$, the second relation in (1.28) is analogue to the definition of an isothermal surface[9] which means that it is the non-commutative analogue of a conformal parametrization. Sometimes the action of the free module $\mathcal{F}_\hbar^m$ over some vector space will not preserve this space, and the successive action of elements of $\mathcal{F}_\hbar^m$ will not be well defined, to avoid some trouble concerning this fact we define a new non-commutative vector field $\Phi \in \mathcal{F}_\hbar^m$ as

$$\Phi = \Phi^i E_i = 2\partial(A^i)E_i = \Big(\partial_U(A^i) - i\partial_V(A^i)\Big)E_i \tag{1.33}$$

which satisfies the following relations

$$\begin{aligned} &\langle \Phi,\ \Phi \rangle = \mathcal{E} - \mathcal{G} - 2i\mathcal{F}\ , \\ &\langle \Phi,\ \Phi \rangle = 0 \quad \text{if and only if } \mathcal{E} = \mathcal{G} \text{ and } \mathcal{F} = 0\ , \\ &\Phi \text{ is r-holomorphic } \longrightarrow \overrightarrow{A} \text{ is minimal.} \end{aligned} \tag{1.34}$$

These relations follow easily from the direct computation of the bi-linear form (1.30) using (1.33) and the last assertive follows from the fact that $\Delta_0(\Phi) = 4\overline{\partial}\Big(\partial(\Phi)\Big) = 2\overline{\partial}(\Phi) = 0$ since Φ is *r-holomorphic*. To finish this section we will apply this formalism to the case of the non-commutative catenoid.

1.3.1 The non-commutative Catenoid

Let V be the vector space of infinite sequences of complex numbers, with canonical basis vectors denoted by $|N\rangle$ with $N \in \mathbb{N}_0$. The element $X \in V$, with $A_K \in \mathbb{C}$, is written as

$$A = \sum_{K \in \mathbb{N}_0} A_K \cdot |K\rangle\ . \tag{1.35}$$

The subspace $V_0 \subset V$ of finite linear combinations of $|K\rangle$'s will be defined as the domain of the space of linear endomorphisms of V, denoted by $\mathcal{L}(V, V_0)$. Now we introduce two operators[10] that

[9] In classical differential geometry we say a surface is isothermal if $E = G = \alpha^2$ and $F = 0$ for some real smooth function $\alpha(u, v)$. In this case the parameters (u, v) are also called isothermal. The isothermal parametrization is also called conformal because it preserves angles. Another important result from classical differential geometry is that if a parametrized surface $\mathcal{S}$ is isothermal then $\Delta\mathcal{S} = 2\alpha^2(u, v)H|\overrightarrow{N}(u, v)|$, since in Def(1.8) we impose $\Delta_0(A^i) = 0$ it is straightforward to see that these two conditions imply that $H = 0$ and the surface is minimal.

[10] It is clear that they define the usual ladder operators if we divide them by $\sqrt{2\hbar}$.

leave the V_0 invariant under their left action

$$\begin{aligned} \Lambda N &= \sqrt{2\hbar N}|N-1\rangle \ , \\ \Lambda^\dagger N &= \sqrt{2\hbar(N+1)}|N+1\rangle \ , \qquad \text{with } \Lambda^\dagger = \Lambda^* \ . \end{aligned} \tag{1.36}$$

Using complex coordinates (z, z^*) the classical catenoid can be parametrized as

$$\begin{aligned} x^1 &= \Re(\cosh z) \ , \\ x^2 &= \Re(-i \sinh z) \ , \\ x^3 &= \Re(z) \ . \end{aligned} \tag{1.37}$$

Using the analogy between the commutative and the non-commutative setting we set

$$\begin{aligned} X^1 =& \frac{1}{4}\left(e^{\Lambda} + e^{-\Lambda} + e^{\Lambda^*} + e^{-\Lambda^*}\right) \ , \\ X^2 =& \frac{1}{4}\left(e^{\Lambda} - e^{-\Lambda} - e^{\Lambda^*} + e^{-\Lambda^*}\right) \ , \\ X^3 =& \Re(\Lambda) = \frac{1}{2}(\Lambda + \Lambda^*) \ , \end{aligned} \tag{1.38}$$

for $X^i \in \mathcal{L}(V, V_0)$ with $i = \overline{1,3}$. Now, it suffices to verify if this set of operators satisfy (1.36) to show that they define a minimal surface. To calculate $\Delta_0(\overrightarrow{X})$ we use the equations (1.13) and (1.14) to show that, for $\lambda \in \mathbb{C}$

$$\begin{aligned} \partial(e^{\lambda\Lambda})|N\rangle &= \frac{1}{2\hbar}[e^{\lambda\Lambda}, \Lambda^*]|N\rangle = \frac{1}{2\hbar}\sum_{i=0}^{\infty}\frac{\lambda^k}{k!}[\Lambda^k, \Lambda^*]|N\rangle \\ &= \frac{\lambda}{2\hbar}\sum_{i=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\Lambda^{k-1}[\Lambda,\ \Lambda^*]|N\rangle = \lambda e^{\lambda\Lambda}|N\rangle \ , \\ \overline{\partial}(e^{\lambda\Lambda})|N\rangle &= 0 \ . \end{aligned} \tag{1.39}$$

Similar calculations can be done for the derivatives ∂ and $\overline{\partial}$ applied to $e^{\lambda\Lambda^*}$ which gives $\partial(e^{\lambda\Lambda^*})|N\rangle = 0$ and $\overline{\partial}(e^{\lambda\Lambda^*})|N\rangle = \lambda e^{\lambda\Lambda^*}|N\rangle$. Since $\Delta_0(A) = 4\overline{\partial}\partial(A)$ we can calculate the Laplacian for each component X^i

$$\begin{aligned} \Delta_0(X^1) &= \overline{\partial}\partial\left(e^{\Lambda} + e^{-\Lambda} + e^{\Lambda^*} + e^{-\Lambda^*}\right) = 0 \ , \\ \Delta_0(X^2) &= -i\overline{\partial}\partial\left(e^{\Lambda} - e^{-\Lambda} - e^{\Lambda^*} + e^{-\Lambda^*}\right) = 0 \ , \\ \Delta_0(X^3) &= 2\overline{\partial}\partial\left(\Lambda + \Lambda^*\right) = 0 \ . \end{aligned} \tag{1.40}$$

The last result implies that $\Delta_0(X^i) \in \ker(\mathcal{L}(V, V_0))$. Now the last step is to show that $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$, but we must be careful since X^1 and X^2 don't leave V_0 invariant[11] so their composition is not well defined. In this case an equivalent approach is to show that $\langle\Phi,\ \Phi\rangle = 0$. First we calculate

$$\Phi|N\rangle = 2\partial(\overrightarrow{X})|N\rangle = \frac{1}{2}\Big(e^{\Lambda} - e^{-\Lambda},\ -i(e^{\Lambda} + e^{-\Lambda}),\ 2\cdot\mathbb{1}\Big)|N\rangle \ , \tag{1.41}$$

[11]This is a direct consequence of the action of e^{Λ^*} on $|N\rangle$ generating an infinite sequence of complex numbers

$$e^{\Lambda^*}|N\rangle = \sum_{i=0}^{\infty}\sqrt{\frac{(N+i)!}{N!\,(i!\,)^2}}|N+i\rangle \ ,$$

which clearly doesn't belong to V_0.

as expected $e^{\pm\Lambda} \in \text{End}(V_0)$ and the expression $\langle \Phi, \ \Phi \rangle = 0$ is well defined and yields that $\overrightarrow{X}$ defines a non-commutative minimal surface.

In the next chapter we will recall the most relevant properties of the AdS surface and introduce the AdS/CFT correspondence, exploring the commutative case and giving some results we found in recent papers regarding to the non-commutative setting. In chapter 3, we attempt to construct a non-commutative generalized surface in a similar way by defining a unique metric and a torsion-free connection that preserves the complex structure of this generalized space, by this time two natural questions arise from this context. Firstly, given the well known classical commutative limit, which can be achieved by taking $\hbar \to 0$, will the commutative differential geometric equations yield the same results? Secondly, what are the implications, from a physical perspective, of the non-commutative corrections that will follow from this new setting? We choose to start with the AdS_2 given the latest results achieved in [4], [6], [7], [18] and [41] aiming to find a suitable quantum metric for the non-comutative AdS_2. Lastly we will perform a detailed analysis of the full Riemannian Geometry of $ncAdS_2$ following the aforementioned setup, comparing them with the results obtained in [1-3].

2 The AdS_2 and its Quantization Schemes

In this chapter we follow a straightforward way to present the theoretical prerequisites to our work. We start by stating the most important properties of the AdS space and we briefly present the importance of the non-commutative geometry in the process of quantization. We will also discuss the underlying reasons that lead us to choose AdS as the targeted surface for applying the prescription we will use to define the geometrical objects of a surface. In this chapter, we will explore the existing attempts in the literature to construct non-commutative AdS using alternative strategies, and we will employ these outcomes as a point of comparison between established literature and the novel proposition introduced by my thesis.

2.1 AdS Spacetime

Anti-deSitter spacetime is a non-compact, maximally symmetric spacetime with constant negative curvature. By maximally symetric, we mean that it has the maximal number of symmetries for $d+1$ dimensions, from now on, we will call it AdS_{d+1}. The AdS_{d+1} has $\frac{1}{2}(d+1)(d+2)$ symmetries, that is the same number of the flat spacetime symmetries related to $(d+1)$ translations, d boosts and $\frac{1}{2}d(d-1)$ rotations. Usually we study $(d+1)$-dimensional AdS spaces because the CFT dual of AdS_{d+1} have d spacetime dimensions. In our work we will focus our attention in the $d=1$ case and the reason for this is that the geometry of AdS_2 is distinct because it has two disconnected time-like boundaries which brings more layers of complexity to the analysis, another fact is that it is possible to construct a quantum version of AdS_2 which preserves the isometry group $SO(2,1)$ (for instance see [26]) and retaining some useful notions from the commutative case as the notion of boundaries, vector fields and so on. We also have that the CFT_1 is realized as the de Alfaro-Fubini-Furlan model (dAFF)[12] or could be constructed as a matrix quantum mechanics making the CFT side of the correspondence more tractable.

In the general case, the AdS space-time is a solution to Einstein's equations with negative cosmological constant. There are a variety of coordinate systems for it and they satisfies the equation of the hyperboloid

$$X_A X^A = X_0^2 + X_{d+1}^2 - \sum_{n=1}^{d} X_n^2 = \ell^2 \ . \tag{2.1}$$

[12]For more information about this well known model see [22].

It can be embedded in a $(d+2)$-dimensional space as

$$
\begin{aligned}
X_0 &= \ell \frac{\cos(t)}{\cos(r)} \ , \\
X_{d+1} &= \ell \frac{\sin(t)}{\cos(r)} \ , \\
X_n &= \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega}_n \ ,
\end{aligned}
\tag{2.2}
$$

this embedding defines the Minkowskian AdS_{d+1} which has the following metric

$$
ds^2 = \frac{1}{\cos^2\left(\frac{r}{\ell}\right)} \left(dt^2 - dr^2 - \sin^2\left(\frac{r}{\ell}\right) d\Omega^2_{d-1} \right) \ . \tag{2.3}
$$

Here, ℓ is the length scale, which will be chosen in a convenient way in order to make the measurements of the energies be in the right scale, that is, unless specified diferently we are taking from now $\ell = 1$, r is the radial coordinate $r \in [0, \frac{\pi}{2})$, while $t \in (-\infty, \infty)$ and the angular coordinate Ω defines a $(d-1)$-dimensional sphere S^{d-1}.

The Euclidean AdS and the Euclidean conformal group which is $SO(d+1,1)$ can be better studied in this embedding space

$$
X_0^2 - \sum_{j=1}^{d+1} X_j^2 = \ell^2 \ . \tag{2.4}
$$

When we consider the global coordinates, the t term of the metric (2.3) changes the sign and it will just swap the trigonometric functions for the hyperbolic trigonometric ones in the global mapping (2.2), giving for $\tau = it$

$$
\begin{aligned}
X_0 &= \ell \frac{\cosh(\tau)}{\cos(r)} \ , \\
X_{d+1} &= \ell \frac{\sinh(\tau)}{\cos(r)} \ , \\
X_n &= \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega} \ .
\end{aligned}
\tag{2.5}
$$

This embedding defines the Euclidean AdS_{d+1}. There is a coordinate system that makes the d-dimensional Poincaré subgroup of the conformal group clear and manifest, we call it Poincaré Patch (PP). The relation between the Euclidean, Poicaré patch and global coordinates, respectively, is

$$
\begin{aligned}
X_0 &= \frac{z^2 + x^i x_i + \ell^2}{2z} = \ell \frac{\cosh(\tau)}{\cos(r)} \ , \\
X_{d+1} &= \frac{z^2 + x^i x_i - \ell^2}{2z} = \ell \frac{\sinh(\tau)}{\cos(r)} \ , \\
X_n &= \frac{\ell}{z} x_i = \ell \frac{\sin(r)}{\cos(r)} \hat{\Omega} \ ,
\end{aligned}
\tag{2.6}
$$

where x is a d-dimensional space vector, z runs from 0 to ∞ and τ is the global "time" coordinate, this fix the signal of X_0. From now on we will use the PP as our natural coordinate system and we will set $d = 1$. In our work we used extensively two set of coordinates charts for the lower hyperboloid of AdS_2, the canonical coordinates from [6] and the coordinalization defined in [5] by Fefferman and Graham and we will refer to it as FG coordinates from now on. First we consider the canonical coordinates (x, y) which satisfy the Poisson bracket $\{x, y\} = 1$ and can be explicitly defined as

$$\begin{aligned} X_0 &= -\, y, \\ X_1 &= -\, \frac{1}{2\ell} y^2 e^{-x} + \ell \sinh(x) \ , \\ X_2 &= -\, \frac{1}{2\ell} y^2 e^{-x} - \ell \cosh(x) \ , \end{aligned} \tag{2.7}$$

for $(x, y) \in \mathbb{R}^2$. The concrete construction for FG coordinates is

$$\begin{aligned} X_0 &= -\, \frac{\ell t}{z} \ , \\ X_1 &= -\, \frac{\ell}{2} \left(z + \frac{t^2 - 1}{z} \right) \ , \\ X_2 &= -\, \frac{\ell}{2} \left(z + \frac{t^2 - 1}{z} \right) \ , \end{aligned} \tag{2.8}$$

for $(z, t) \in \mathbb{R}_+ \times \mathbb{R}$ with the boundary in $z = 0$. The relation between these coordinates is given by

$$x = -\ln z \qquad \text{and} \qquad y = \ell \frac{t}{z} \ . \tag{2.9}$$

2.2 Non-Commutative Geometry and Quantization

The aim of non-commutative geometry is to reformulate geometrical structures of a manifold in terms of an algebra of functions defined on it, generalize this commutative algebra to a non-commutative one and then generalize the notion of the tangent and cotangent bundle in this non-commutative setup. Following [8], [9] one can understand that in the transition to the non-commutative setting the notion of a point is lost and this feature is manifested in non-relativistic quantum mechanics since the Heisenberg uncertainty principle that makes the whole geometry non-local. At length scales smaller than some fundamental length, the hypothesis that the geometry is based on a set of commuting variables is replaced by the rules of the non-commutative geometry since it is impossible to localize a point. At early stages of quantum mechanics the scientists considered the idea of replacing space-time by a lattice structure. This granular structure has an intrinsic point to point spacing that could be interpreted as a small scale cut-off and this could eliminate the UV divergences of quantum field theory. As proposed by Synder in 1947 [42], substituting the usual space-time coordinates by non-commutative ones makes it possible to transform the ordinary geometry in some similar structure that retains its main geometrical properties, but has the desirable features of a lattice structure while still remaining Lorentz invariant, which is a remarkable result (not shared by lattice models). As an example, one could replace the phase-space coordinates x^μ by generators of a non-commutative algebra which satisfy commutation relations of the form

$$[X^\mu, X^\nu] = i\beta J^{\mu\nu} \ , \tag{2.10}$$

for some parameter β that "measures" the non-commutativity of this algebra and $J^{\mu\nu}$ being some element of the algebra. Carrying the analogy with quantum mechanics now we impose that these generators could be represented as hermitian operators acting on some Hilbert space $\mathcal{H}$. Since (2.10) holds, one cannot simultaneously diagonalize the coordinates X^μ which makes the notion of a point ill-defined and following the quantum analogy, as the Bohr cells replace the classical phase-space points, the classical geometry of the manifold is replaced by some kind of a fuzzy space-time consisting of cells of volume of the order of $(2\pi\beta)^2$ depending on the units of $J^{\mu\nu}$ and the dimension of the space. As a concrete example consider a two-dimensional phase space described by the coordinates (x, p). Upon the promotion of the classical coordinates to quantum coordinates (X, P), which are non-commutative by construction, one cannot measure simultaneously X and P to some arbitrary precision, which means that we can think of this space divided in *Bohr cells* of volume $2\pi\beta$. If we consider a classical phase-space with finite volume, the quantum analogue of it would have a finite number of cells, and any function in such space could be represented as a finite collection of numbers and could be denoted by a matrix. So the natural non-commutative generalization of the algebra of functions over a two-dimensional phase-space with the restriction in its total volume is the non-commutative algebra of matrices[13].

Following the natural approach, one could use this non-commutative algebra as a tool for quantization, and in this context we must analyze the alternatives for this type of quantization. The standard formulation that uses operators in Hilbert space follows from a collective effort of some of the most brilliant minds of the last century culminating in an axiomatization of the quantum theory (see [13]) setting in stone the early rules for the quantization schemes hereafter. After these early times other formalisms took place such as the path integral formulation and the phase-space formulation, which we used extensively in our previous works and which is based on Wigner's quasi-probability distribution function in phase space, WF for short, and Weyl's correspondence between quantum operators and ordinary c-number phase-space functions. It relies on the star-product, that was fully understood in [12] by Groenewold together with Moyal, which maps products of operators that act in some Hilbert space to a non-commutative product of functions on the phase space, giving an alternative procedure to achieve the quantization.

[13] See [8] for further concrete examples and applications of this formalism.

2.2.1 The phase-space quantization

From the references [10] and [11] we can define the WF and construct the star product in a convenient way, facilitating the quantization of some phase-space functions. This is related to the formalism from our recent paper, which allowed to achieve the full construction of the quantum AdS_2. Now we start by defining the WF as

$$f(x,p) := \frac{1}{2\pi}\int dy\ \psi^*\left(x-\frac{\hbar}{2}y\right)e^{-iyp}\psi\left(x+\frac{\hbar}{2}y\right)\ , \tag{2.11}$$

for $\psi(x)$ some measurable function. If $\psi(x) \in L^2(\mathbb{R})$, i.e. if ψ is a Lebesgue square-integrable complex-valued function on $\mathbb{R}$ satisfying $|\psi|^2= 1$, obviously the WF is normalized

$$\begin{aligned}\int dp\ dx f(x,p) &= \frac{1}{2\pi}\int dy \int dp\ dx\ \psi^*\left(x-\frac{\hbar}{2}y\right)e^{-iyp}\psi\left(x+\frac{\hbar}{2}y\right)\ ,\\ &= \int dy\ dx\ \psi^*\left(x-\frac{\hbar}{2}y\right)\delta(y)\psi\left(x+\frac{\hbar}{2}y\right)\ ,\\ &= \int dx\ |\psi(x)|^2 = 1\ .\end{aligned} \tag{2.12}$$

In the classical limit as $\hbar \to 0$, it reduces to the probability density in coordinate space. The usual x- or p-projection leads to probability densities in momentum or coordinate space. WF cannot be interpreted as a probability distribution, it is therefore a quasi-probability distribution because it can assume negative values for an arbitrary open set in the phase-space, but it leads to correct position and momentum probability distributions given by quantum mechanics, replacing the wave-function in this formulation. It also provides the integration measure for functions on phase space that represent classical quantities in general. These functions are associated to ordered operators upon quantization through the Weyl's correspondence.

The *Weyl correspondence* is the association of a quantum-mechanical operator $W(g)$ in a given ordering prescription to a classical c-number Fourier transformed function $g(x,y)$ on phase-space. This correspondence reads

$$W(g) = \mathfrak{G}(\mathfrak{x},\mathfrak{y}) = \frac{1}{(2\pi)^2}\int dy\ dx\ d\alpha\ d\beta\ g(x,y)\exp\left(i\alpha(\mathfrak{y}-y)+i\beta(\mathfrak{x}-x)\right)\ , \tag{2.13}$$

where $g(x,y)$ is the corresponding phase-space function, and $\mathfrak{x}$ and $\mathfrak{y}$ are the respective quantum operators associated to x and y. The ordering prescription requires that an arbitrary operator written as a power series of $\mathfrak{x}$ and $\mathfrak{y}$ be ordered in a completely symmetrized expression by use of Heisenberg's commutation relations, $[\mathfrak{x},\mathfrak{y}] = i\hbar$. Finally, Groenewold worked out how two classical c-number functions $f(x,y)$ and $g(x,y)$ must compose in order to yield the product of operators $\mathfrak{G}$ and $\mathfrak{F}$:

$$\mathfrak{G}\mathfrak{F} = \frac{1}{(2\pi)^2}\int d\alpha\ d\beta\ dx\ dy\ \exp(i\alpha(\mathfrak{y}-y)+i\beta(\mathfrak{x}-x))(f\star g)(x,y)\ , \tag{2.14}$$

here $\star$ stands for the star product. This is the original definition of the star product and it enables the formulation of quantum mechanics in the phase-space. The star product is an associative pseudo-differential deformation of the ordinary product of phase-space c-number functions. It is defined as

$$\star := \exp\left[\frac{ih}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_y - \overleftarrow{\partial}_y\overrightarrow{\partial}_x)\right] . \tag{2.15}$$

It can be written in an expanded form as

$$F(x,y)\star G(x,y) = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \epsilon^{i_1 j_1}\ldots\epsilon^{i_n j_n}(\partial_{i_1}\ldots\partial_{i_n}F)(\partial_{j_1}\ldots\partial_{j_n}G) , \tag{2.16}$$

where $i,j = 1,2$ for $x^1 = x, x^2 = y$ and the matrices ϵ^{ij} are the Levi-Civita symbols of rank two. Since it involves exponential of derivatives, it can be easily evaluated through translation of function arguments

$$F(x,y)\star G(x,y) := F\left(x + \frac{i\hbar}{2}\overrightarrow{\partial}_y,\ y - \frac{i\hbar}{2}\overrightarrow{\partial}_x\right)G(x,y) . \tag{2.17}$$

If one uses the Fourier representation of the star product as an integral kernel

$$\begin{aligned} F\star G(x,y) =& \frac{1}{(\hbar\pi)^2}\int dy'\ dy''\ dx'\ dx''\ f(x',y')g(x'',y'') \\ & \times \exp\left(-\frac{2i}{\hbar}\Big(y(x'-x'') + y'(x''-x) + y''(x-x')\Big)\right) . \end{aligned} \tag{2.18}$$

If one needs to calculate multiple star products it is worthwhile to remember that the expression in the exponent is twice the area of the phase-space triangle determined by the points (x,p), (x',p'), and (x'',p'') which simplifies the calculation of the composition of star products.

2.3 The commutative AdS_2/CFT_1 correspondence - The massless, massive and interacting cases

Here we will follow our previous work and present this topic in a similar way (see [18]). The AdS/CFT correspondence is a conjecture introduced by Juan Maldacena in 1997 which states that a certain type II-B superstring theory on $AdS_5\times S^5$ is dual to a highly symmetric $\mathcal{N}=4$ super Yang-Mills theory in the large N limit. Maldacena demanded that in the 't Hooft limit coupling be large compared with r dependent term in the metric in units of string length, turning the metric of a type II-B super-gravity into

$$ds^2 = \frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + \ell^2 d\Omega_5^2 . \tag{2.19}$$

The form of the metric shows that near the horizon the supergravity solution is $AdS_5\times S^5$ with the lenght scale ℓ playing the role of the "radius" of the five-sphere and the 'radius' of AdS^5. When analyzing the main features of this conjecture, we must consider firstly, that it is a strong-weak coupling duality, secondly, that it is non perturbative in the string coupling and also in the Yang-Mills coupling g_{YM} and, lastly, it is a classical-quantum duality because classical supergravity is conjectured to be dual to a quantum gauge theory (the corrections are suppressed by powers of $1/N$, see [15]). The general correspondence formula is

$$\int e^{-iS_{AdS}[\phi]} D\phi_0 = \left\langle \exp \int d^d x \mathcal{O}(x)\phi_0(x) \right\rangle \ , \tag{2.20}$$

where $\mathcal{O}$ denotes the conformal primary operators[14] on the boundary and the left integral is over all fields whose the assymptotic boundary values are ϕ_0. In the classical limit one can make the saddle-point approximation and find that

$$S_{AdS}[\phi_0] = W_{CFT}[\phi_0] \ , \tag{2.21}$$

where S_{AdS} is the classical on-shell action of an AdS theory and W_{CFT} is the effective action given by minus the logarithm of the right hand side of (2.20). Since the AdS metric is divergent on the boundary, one expect that the classical action is also divergent and in order to extract any meaningful physical information from it one must renormalize the on-shell action by adding counter terms which cancel the infinities, giving

$$S_R = W_{CFT} \ . \tag{2.22}$$

Here S_R stands for the renormalized on-shell action for AdS. Any field theory on the AdS space has a corresponding counterpart on the CFT side. This includes gravity, as the boundary value of gravitons couples to the energy-momentum tensor which is an standard feature of any CFT. Thus, the AdS/CFT correspondence is an important tool for formulating non-trivial CFT's, as well as extracting information about the physics on the AdS space.

We will follow the usual prescription for the AdS/CFT correspondence which is that the connected correlation functions for operators $\mathcal{O}$ spanning the CFT are generated by the field theory action on the asymptotical AdS space, in which the the fields $\phi_0(t)$ are sources for the operators $\mathcal{O}(t)$. Specifically we will follow some well-determined steps, i) Defining a suitable action and extremizing it, ii) Using the AdS propagators to find regular solutions expressed in terms of the boundary fields, iii) Substituting the aforementioned solutions in the action in order to identify this with the generating functional of the n-point connected correlation functions for the operators associated with the boundary fields, iv) Lastly, we will calculate the correlation functions of interest. The first application of this prescription will be done for the massless scalar field.

[14]For a good overview of CFT's and all the background necessary to deeply understand the technical part of the conjecture we refer to [15], [16] and [17].

2.3.1 Massless Case

In this section and in the subsequent ones we will use the superscript (0) to refer to the commutative scalar fields. We then have the massless scalar field denoted as $\Phi^{(0)}(z,t)$ on the $EAdS_2$ (Euclidean AdS) and the following action

$$S[\Phi^{(0)}] = \frac{1}{2}\int_0^\infty dz \int_{-\infty}^\infty dt \Big[(\partial_z\Phi^{(0)})^2 + (\partial_t\Phi^{(0)})^2\Big] \ . \tag{2.23}$$

Taking the variation of (2.23) and extremizing it with respect to the Dirichlet boundary conditions $\delta\Phi^{(0)}|_{z=0^+}= 0$, one finds the field equation

$$\Box\Phi^{(0)} = \Big(\partial_z^2 + \partial_t^2\Big)\Phi^{(0)} \ = 0. \tag{2.24}$$

We use the boundary-to-bulk propagator, the details in [14], to express the solution in terms of the boundary[15] field $\phi_0(t)$

$$\Phi^{(0)}(z,t) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{z\phi_0}{z^2+(t-t')^2}dt' \ . \tag{2.25}$$

After the substitution of the solution (2.25) in the action (2.23) the only non-vanishing term is the boundary one

$$S[\Phi^{(0)}[\phi_0]] = -\frac{1}{2\pi}\int_{-\infty}^\infty dt \int_{-\infty}^\infty dt' \frac{\phi_0(t)\phi_0(t')}{(t-t')^2} \ . \tag{2.26}$$

In order to apply the AdS/CFT correspondence, one should identify (2.26) with the generating functional of the n-point correlation functions for the operator $\mathcal{O}$ for which ϕ_0 is the source field. We can summarize this prescription in the equation below

$$\left\langle \prod_{i=1}^n \mathcal{O}(t_i)\right\rangle \ = \prod_{i=1}^n \delta_{t_i}\Big(S[\Phi^{(0)}[\phi_0]]\Big)\Big|_{\phi_0=0} , \tag{2.27}$$

where δ_{t_i} represents the action of the functional derivative with respect to the function $\phi_0(t_i)$. As a trivial example, we have the two-point function

$$\langle \mathcal{O}(t)\mathcal{O}(t')\rangle = \delta_t\delta_{t'}\left[-\frac{1}{2\pi}\int_{-\infty}^\infty dt \int_{-\infty}^\infty dt' \frac{\phi_0(t)\phi_0(t')}{(t-t')^2}\right] = -\frac{1}{2\pi}\frac{1}{(t-t')^2} \ . \tag{2.28}$$

Now we proceed to analyse the massive and interacting cases using a similar strategy.

2.3.2 Massive Case

For the free real massive scalar field we add the mass term to the action (2.23) to obtain a new action

$$S[\Phi^{(0)}] = \frac{1}{2}\int_0^\infty dz \int_{-\infty}^\infty dt \left[(\partial_z\Phi^{(0)})^2 + (\partial_t\Phi^{(0)})^2 + \left(\frac{m\ell}{z}\Phi^{(0)}\right)^2\right] \ . \tag{2.29}$$

[15] Remember that the boundary value of the field is obtained by taking $z \to 0$ satisfying $\phi_0(t) = \Phi^{(0)}(0,t)$.

Following the procedure applied in [7], [14] and [19] we find that the non-vanishing dominant solution of (2.29) near the boundary $z \to 0^+$ is

$$\Phi^{(0)}(z \to 0^+, t) = z^{\Delta_-}\phi_0(t) \ , \tag{2.30}$$

for $\Delta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (m\ell)^2}$, satisfying the Breitenlohner-Freedman bound (see [20]) which comes from the condition that the theory must be free of normalizable negative energy states granting the theory some consistency. Using the boundary-to-bulk propagator we find the regular solutions of (2.29)

$$\Phi^{(0)}(z,t) = \frac{\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} dt' \left(\frac{z}{z^2 + (t-t')^2}\right)^{\Delta_+} \phi_0(t') \ , \tag{2.31}$$

for $\nu = \sqrt{\frac{1}{4} + (m\ell)^2}$. Substituting the solutions (2.31) into (2.29) we find the on-shell action

$$S[\Phi^{(0)}[\phi_0]] = -\frac{\Delta_+\Gamma(\Delta_+)}{2\sqrt{\pi}\Gamma(\nu)} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \frac{\phi_0(t')\phi_0(t'')}{|t'-t''|^{2\Delta_+}} \ , \tag{2.32}$$

which is valid for any boundary point satisfying $t' \neq t''$. Applying (2.27) for the two-point function, we find that

$$\langle \mathcal{O}(t')\mathcal{O}(t'') \rangle = \delta_{t'}\delta_{t''}\Big[S[\Phi^{(0)}[\phi_0]]\Big] = -\frac{\Delta_+\Gamma(\Delta_+)}{2\sqrt{\pi}\Gamma(\nu)} \frac{1}{|t-t'|^{2\Delta_+}} \ . \tag{2.33}$$

which is the 2-point function of the conformal field theory, for Δ_+ the conformal dimension (see [17]), note that if we set $\Delta_+ = 1$ which corresponds to the massless case we obtain (2.28). Now we proceed to the interacting case.

2.3.3 Interacting case

In order to have a non-trivial n-point function for $n > 2$, we need to consider an interacting scalar field. So, we add a cubic term to (2.29) to get

$$S[\Phi^{(0)}] = \frac{1}{2}\int_0^\infty dz \int_{-\infty}^{\infty} dt \left[(\partial_z\Phi^{(0)})^2 + (\partial_t\Phi^{(0)})^2 + \left(\frac{m\ell}{z}\Phi^{(0)}\right)^2 + \frac{2\lambda}{3z^2}(\Phi^{(0)})^3 \right] \ , \tag{2.34}$$

for some real parameter λ. Extremizing the action yields the field equation

$$\Big(\Box - (m\ell)^2\Big)\Phi^{(0)} = \lambda(\Phi^{(0)})^2. \tag{2.35}$$

Assuming the same asymptotic behavior of the massive case we use the boundary-to-bulk and bulk-to-bulk propagators denoted as $K(z,\ t;\ t')$ and $G(z,\ t;\ z'\ t')$ respectively to construct the regular solutions. They are defined from the expressions below

$$\Phi^{(0)}(z,t) = \int_{\mathbb{R}} dt' K(t;t')\phi_0(t') \ ,$$

$$\Big\{ -\Big(\partial_z^2 + \partial_t^2\Big) + \Big(\frac{m\ell}{z}\Big)^2 \Big\} G(z,t;z',t') = \delta(z-z')\delta(t-t') \ .$$

From the equations above one can calculate the propagators (see [14])

$$K(z,\ t;\ t') := \frac{\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\Delta_+ - \frac{1}{2})}\left(\frac{z}{z^2 + (t-t')^2}\right)^{\Delta} ,$$

$$G(\eta) = \frac{\Gamma(\Delta_+)}{\sqrt{\pi}\Gamma(\Delta_+ - \frac{1}{2})(2\Delta_+ - 1)}\left(\frac{\eta}{2}\right)^{\Delta_+} {}_2F_1\left(\frac{\Delta_+}{2},\ \frac{\Delta_+ + 1}{2};\ \Delta_+ + \frac{1}{2}; \eta^2\right) ,$$

for $\eta = \frac{2zz'}{z^2 z'^2 + (t-t')^2}$. In the interaction case, the on-shell action has a bulk as well as a boundary contribution and they will be denoted as

$$S[\Phi^0] = S_{bdy} + S_{blk} = -\frac{1}{2}\int_{-\infty}^{\infty} dt\ \Phi^{(0)}\partial_z\Phi^{(0)}\Big|_{z=0} + \frac{\lambda}{3}\int_0^{\infty}\int_{-\infty}^{\infty}\left(\frac{\Phi^{(0)}}{z^{2/3}}\right)^3 dt\ dz\ . \tag{2.36}$$

To proceed, we have to construct the regular solutions. aThis can be done only pertubatively in the parameter λ. To the second order in λ, the result is (see [14] for details)

$$\begin{aligned}
&\Phi^{(0)}(z,t) = \int_{-\infty}^{\infty} dt'\ K(z,\ t;\ t')\phi_0(t') \\
&- \lambda\int_0^{\infty}\int_{-\infty}^{\infty}\frac{G(z,\ t;\ z',\ t')}{z'^2}dt\ dz\int_{\mathbb{R}^2} dt_1 dt_2 K(z',\ t';\ t_1)K(z',\ t';\ t_2)\phi_0(t_1)\phi_0(t_2) \\
&+ O(\lambda^2)\ .
\end{aligned} \tag{2.37}$$

Plugging this in (2.36), we obtain a term that leads exactly to the two-point function (2.33) plus a non-trivial contribution to the three-point function. Using (2.27), we finally obtain the three-point function[16]

$$\langle\mathcal{O}(t_1)\mathcal{O}(t_2)\mathcal{O}(t_3)\rangle = \frac{\lambda\Gamma(\Delta_+/2)^3\Gamma\Big((3\Delta_+ - 1)/2\Big)}{2\pi\Gamma(\nu)^3|t_1 - t_2|^{\Delta_+}|t_1 - t_3|^{\Delta_+}|t_2 - t_3|^{\Delta_+}}\left(\frac{3\Delta_+}{2\nu} + 2\right)\ . \tag{2.38}$$

2.4 The non-commutative AdS_2/CFT_1 correspondence

In this section we will review the results obtained in our previous paper and give all the background needed to justify our new approach. We will follow the lines of [6] and [7], which are the main references for this part of the work. In the non-commutative AdS/CFT correspondence, we will replace the geometry on the gravity side of the correspondence by a non-commutative version of the Euclidean AdS_2 and by the general belief that the quasiclassical regime of quantum gravity should appear as a QFT on some non-commutative background[17], we will try to find some natural quantum gravitational corrections due to non-commutativity of the background. Another point of interest is to verify the claim that the AdS/CFT correspondence is exact and holds also at the quantum level and this will be verified analyzing the non-commutative effects on the explicit application of the prescription for the correspondence.

[16] All the details of this calculation can be found in [21].

[17] This belief is supported by multiple arguments, see [27] for instance.

2.4.1 Euclidean AdS_2 and its non-commutative analogue

We start with the canonical coordinates (x, y) defined in the section (2.1). Together with the embedding (2.7) we introduce the Killing vectors that are written as

$$K^0 = \partial_x \ , \quad K^1 = \frac{1}{\ell} e^{-x} y \partial_x - X^2 \partial_y \ , \quad K^2 = \frac{1}{\ell} e^{-x} y \partial_x - X^1 \partial_y \ . \tag{2.39}$$

These vector fields satisfiy the Killing equation $\mathcal{L}_K g = 0$ where $\mathcal{L}_K$ is the Lie derivative with respect to K and g is the metric for the $EAdS_2$ given by

$$ds^2 = \frac{1}{z^2}\Big(dz^2 + dt^2\Big) \ . \tag{2.40}$$

These vector fields generate the $SO(2, 1)$ isometry group. This can be seen either by the direct calculation using (2.39) or by observing that in terms of the natural Poisson structure, where X are the embedding coordinates. The action of any Killing vector on a function of X, $\phi(X)$, is given by $K^\mu\Big(\phi(X)\Big) = \{X^\mu, \phi(X)\}$ leading to the relations below

$$\begin{aligned} \{X^\mu, X^\nu\} &= \epsilon^{\mu\nu\gamma} X_\gamma \ , \\ [K^\mu, K^\nu] &= \epsilon^{\mu\nu\gamma} K_\gamma \ , \end{aligned} \tag{2.41}$$

where the curly brackets stands for the Poisson Bracket and the straight ones for the commutator and the ambient metric tensor used to raise and lower the Greek indices is $\eta_{\mu\nu} = \text{diag}(1, 1, -1)$, giving $X^\mu X_\mu = (X^0)^2 + (X^1)^2 - (X^2)^2 = -\ell^2$, for ℓ_0 being some real scale parameter. So, as usual the action of the Killing vector fields on the embedding coordinates is equivalent to the Poisson bracket between them $(K^\mu X^\nu) = \{X^\mu, X^\nu\} = \epsilon^{\mu\nu\rho} X_\rho$. Another important fact to consider is that when we take the boundary limit $z \to 0$ of the Killing vector fields $K^\pm = K^2 \pm K^1$ denoted in (2.39) we obtain the generators for the global conformal symmetries on the boundary[18]

$$K^-|_{z=0} = -\partial_t \ , \qquad K^0|_{z=0} = -t\partial_t \ , \qquad K^+|_{z=0} = -t^2\partial_t \ . \tag{2.42}$$

Following the usual procedure for quantization[19] of Poisson manifolds, we replace the three embedding coordinates X^μ by Hermitian operators[20] on some Hilbert space, satisfying the analogues of the equations in (2.41) and promoting Poisson brackets to commutation relations

$$\hat{X}^\mu \hat{X}_\mu = -\ell^2 \mathbb{1} \ , \qquad [\hat{X}^\mu, \hat{X}^\nu] = i\alpha\epsilon^{\mu\nu\rho} \hat{X}_\rho \ , \tag{2.43}$$

where α stands for the parameter that 'measures' the non-commutativity (it has units of length). To recover the commutative AdS_2 we just take the commutative limit $\alpha \to 0$ and $\ell \to \ell_0$. We introduce now a new operator that will be of great importance later in our work

$$\hat{r} = \hat{z}^{-1} = \frac{1}{\ell}(\hat{X}^1 - \hat{X}^2) \ , \tag{2.44}$$

[18]They generate respectively the translation, dilatations and special conformal transformations on the boundary (see [17]).

[19]For more details about this straightforward method see [28] and [29].

[20]These new introduced operators defining the non-commutative AdS_2 generate the $\mathfrak{so}(2, 1)$ algebra.

this operator is the quantum analogue of the radial coordinate defined in section 2 and we will obtain its spectrum and eigenfunctions in the following sections. In the classical case the boundary of AdS_2 is achieved by taking the coordinates to the limit $r \to \infty$, i.e., $z \to 0$, so in the non-commutative theory the expectation value of $\hat{r}$ is expected to become arbitrarily large as we approach to the boundary and this limit will be very clear after we specify the Hilbert space of our theory.

2.4.2 States and discrete series representation

The states of our theory belong to the unitary irreducible representations of the universal cover[21] $\mathcal{U}\Big(SU(1,1)\Big)$, which are given by the principal, supplemental and discrete series representations[22] and are usually labeled by two parameters ϵ_0 and k. Taking as a basis the eigenvectors[23] of $\hat{X}^2$, defining $\hat{X}_\pm = \hat{X}^1 \pm i\hat{X}^0$ and imposing that these eigenvectors are orthonormal we find that

$$\begin{aligned} \hat{X}_+|\epsilon_0, k, m\rangle &= -\alpha\ c_m\ |\epsilon_0, k, m+1\rangle\ , \\ \hat{X}_-|\epsilon_0, k, m\rangle &= -\alpha\ c_{m-1}\ |\epsilon_0, k, m-1\rangle\ , \\ \hat{X}^2|\epsilon_0, k, m\rangle &= -\alpha\ (\epsilon_0 + m)\ |\epsilon_0, k, m\rangle\ , \\ \hat{X}_\mu\hat{X}^\mu|\epsilon_0, k, m\rangle &= -\alpha^2\ k(k+1)\ |\epsilon_0, k, m\rangle\ , \end{aligned} \tag{2.45}$$

where the coefficient c_m is ensuring the orthonormality of this eigenbasis

$$c_m = \sqrt{(k + \epsilon_0 + m + 1)(\epsilon_0 - k + m)}\ . \tag{2.46}$$

It is clear from the action of the Casimir operator in the eigenbasis $|\epsilon_0,\ k,\ m\rangle$ that $k(k+1) = \frac{\ell^2}{\alpha^2}$ and one can trivially conclude that the commutative limit is obtained when $k \to \pm\infty$. Now we turn to the radial operator, one can easily calculate its expectation value

$$\langle\epsilon_0,\ k,\ m|\ \hat{r}\ |\epsilon_0,\ k,\ m\rangle = -\frac{\langle\hat{X}^2\rangle}{\ell} = \frac{\alpha(\epsilon_0 + m)}{\ell}\ , \tag{2.47}$$

and it is clear by the last equation that the boundary of the non-commutative AdS_2 space is reached when we take $m \to \infty$. Since we want to construct the $EAdS_2$ we need $\ell \in \mathbb{R}$ and from [28] we conclude that the principal and supplemental series are not suitable[24] for our analysis. For the discrete series $D^\pm(k)$, k can be any negative number which implies that ℓ will be real and the

[21] The universal cover of a connected topological space A is a simply connected space B with a surjective projection $p : B \to A$ that is locally a homeomorphism.

[22] See [30] for a detailed explanation and derivation of all representations.

[23] Here we will denote these eigenvectors as $|\epsilon_0,\ k,\ m\rangle$ for $m \in \mathbb{Z}$.

[24] The principal series has $k = -\frac{1}{2} - i\rho$ for $\rho \in \mathbb{R}$ implying that ℓ is imaginary. The supplemental series has $k \in \mathbb{R}$ but it is constrained by $k \in \big(-\frac{1}{2}, 0\big)$ making again ℓ imaginary and since we cannot take $k \to \infty$ this case doesn't have a commutative limit. Both cases correspond in some limit to the Lorentzian version of AdS_2, and purely quantum case.

Casimir operator will be negative for any $k < -1$. Since both limits for $k \to \pm\infty$ exists yielding the $EAdS_2$ in both cases we can define the series $D^{\pm}$ by the allowed values of m (positive or negative integers) which are related to the two distinct hyperboloids of the $EAdS_2$.

Using the main results and formulas of [31] we will apply the generalized Laguerre polynomials to construct a differential representation for the embedding coordinates for the discrete representations $D^{\pm}(k)$. Beginning with $D^{+}(k)$ and setting the lowest state as $|-k,\ k,\ 0\rangle$, or $|k,\ 0\rangle$ for brevity, which is anihhilated by $\hat{X}_-$ since we are assuming $\epsilon_0 = -k > 0$, we expand the eigenvectors of the radial operator $|r, k\rangle_+$ in terms of the $\hat{X}^2$ eigenbasis

$$|r, k\rangle_+ = \sum_{m=0}^{\infty} \psi^+_{k,m}(r)|k, m\rangle \ . \tag{2.48}$$

Writing the radial operator in terms of the raising and lowering operators

$$\hat{r} = \frac{1}{2\ell}(\hat{X}_+ - \hat{X}_- - 2\hat{X}^2) \ , \tag{2.49}$$

one can write the eigenvalue equation

$$\hat{r}|r, k\rangle = \frac{1}{2\ell}(\hat{X}_+ - \hat{X}_- - 2\hat{X}^2)|r, k\rangle = r|r, k\rangle \ . \tag{2.50}$$

Using the equations (2.45) we get to the following

$$\begin{aligned} \frac{\ell r}{\alpha}\psi^+_{k,m}(r) = &-\sqrt{(m+1)(m-2k)}\psi^+_{m+1}(r) \\ &- \sqrt{m(m-1-2k)}\psi^+_{m-1}(r) + 2(k-m)\psi^+_{k,m}(r) \ , \end{aligned} \tag{2.51}$$

which is the recursion relation for the generalized Laguerre polynomials for $m > 0$ if we make

$$\psi^+_{k,m}(r) = \sqrt{\frac{m!}{(m-2k-1)!}} L_m^{-2k-1}\left(\frac{2\ell r}{\alpha}\right) \ . \tag{2.52}$$

The domain of the generalized Laguerre polynomials agrees with the restrictions of our theory ($r \geq 0$) and the boundary occurs at $r \to \infty$, this means that these representations picks one of the boundaries of the commutative $EAdS_2$.

We will find a differential representation for the quantum coordinates $\hat{X}^\mu$. The orthogonality conditions of $L^\alpha_m(x)$ read

$$\int_{\mathbb{R}_+} dx\ x^\beta e^{-x} L^\beta_m(x) L^\beta_n(x) = \frac{\delta_{n,m}}{m!}\Gamma(m+\beta+1) \ . \tag{2.53}$$

Defining

$$C_m := \sqrt{\frac{m!}{(m-2k-1)!}} \ , \tag{2.54}$$

and using (2.52) in (2.53) we get the following

$$\int_{\mathbb{R}_+} dr\ \left(\frac{2\ell}{\alpha}\right)^{-2k} e^{-2\ell r/\alpha} r^{-2k-1} \frac{\psi^+_{k,m}(r)}{C_m}\frac{\psi^+_{k,n}(r)}{C_n} = \frac{\delta_{n,m}}{m!}(m-2k-1)! \ . \tag{2.55}$$

We can rewrite the last expression as

$$\int_{\mathbb{R}} dr\ u^{+}_{k,m}(r) u^{+}_{k,n}(r) = \delta_{m,n}\ , \tag{2.56}$$

where

$$u^{+}_{k,m}(r) = \left(\frac{2\ell}{\alpha}\right)^{-k} e^{-\ell r/\alpha} r^{-k-1/2}\ \psi^{+}_{k,m}(r)\ , \tag{2.57}$$

is the orthonormal basis for $L^2(\mathcal{R}_+, dr)$. Finally to get a representation of the differential operator $\hat{D} = (\hat{r} - r)$ satisfying $\hat{D}\psi^{+}_{k,m}(r) = 0$ we must use the differential equation that defines the generalized Laguerre polynomials

$$x\frac{d^2}{dx^2} L^{\beta}_{m}(x) + (\beta + 1 - x)\frac{d}{dx} L^{\beta}_{m}(x) + m L^{\beta}_{m}(x) = 0\ . \tag{2.58}$$

Upon a series of substitutions and and using $x = 2\ell r/\alpha$ and $\beta = -2k - 1$ it is not hard to find that

$$\left(\frac{\alpha(k+\frac{1}{2})^2}{2\ell r} + \frac{\ell r}{2\alpha} + \frac{\alpha(m-k)}{2\ell} - \frac{d}{dr}\Big(\frac{\alpha r}{2\ell}\frac{d}{dr}\Big)\right) u^{+}_{k,m}(r) = 0\ . \tag{2.59}$$

Now if we compare the eigenvalue equation (2.45) for $\hat{X}^2$ with (2.59) we conclude that this is the differential representation π^k of $\hat{X}^2$ on $L^2(\mathbb{R}_+, dr)$ simply by multiplying the equation (2.59) by $-\alpha$. To find the representations of the other operators lets calculate $\pi^k([\hat{r}, \hat{X}^2])$

$$\Big(r\pi^k(\hat{X}^2) - \pi^k(\hat{X}^2) r\Big)[\psi(r)] = \pi^k([\hat{r}, \hat{X}^2])[\psi(r)]\ , \tag{2.60}$$

using $[\hat{r}, \hat{X}^2] = \frac{1}{\ell}[\hat{X}^1, \hat{X}^2] = \frac{i\alpha}{\ell}$ we find that

$$\frac{\alpha^2}{2\ell}\left(r\frac{d}{dr}\left[r\frac{d\psi}{dr}\right] - \frac{d}{dr}\left[r\psi + r^2\frac{d\psi}{dr}\right]\right) = \frac{i\alpha}{\ell}\pi^k(\hat{X}^0)[\psi(r)]\ . \tag{2.61}$$

which simplifies to

$$\pi^k(\hat{X}^0) = i\alpha\left(r\frac{d}{dr} + \frac{1}{2}\right)\ . \tag{2.62}$$

doing the same procedure for $\hat{X}^1$ using the commutator between X^2 and X^0 we finnaly obtain

$$\pi^k(\hat{X}^2) = -\frac{\alpha^2}{2\ell}\left(\frac{(k+\frac{1}{2})^2}{r} + \frac{\ell^2 r}{\alpha^2} - \frac{d}{dr}\Big(r\frac{d}{dr}\Big)\right)\ , \tag{2.63}$$

$$\pi^k(\hat{X}^1) = -\frac{\alpha^2}{2\ell}\left(\frac{(k+\frac{1}{2})^2}{r} - \frac{\ell^2 r}{\alpha^2} - \frac{d}{dr}\Big(r\frac{d}{dr}\Big)\right)\ . \tag{2.64}$$

These operators act on $L^2(\mathbb{R}_+, dr)$, the space of square-integrable functions on the half real line. Replacing $r = e^x$ we can recover the linear operators $\tilde{\pi}(\hat{X}^\mu)$ that act on $L^2(\mathbb{R}, dx)$ spanned by functions of the set $\{f(x) = e^{x/2}\psi(e^x)\}$. In terms of the self-adjoint operators $\hat{x}$ and $\hat{y}$, that act as $\hat{x}f(x) = xf(x)$ and $\hat{y} = -i\alpha f'(x)$ clearly satisfying

$$[\hat{x}, \hat{y}] = i\alpha\mathbb{1}\ . \tag{2.65}$$

Representing the operators $\hat{X}^\mu$ with respect to the new operators $\hat{x}$ and $\hat{y}$ acting on $L(\mathcal{R}, dx)$, we have

$$\tilde{\pi}^k(\hat{X}^0) = -\hat{y}\ , \tag{2.66}$$

$$\tilde{\pi}^k(\hat{X}^1) = -\frac{1}{2\ell}\hat{y}e^{\hat{x}}\hat{y} - \frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}} + \frac{\ell}{2}e^{\hat{x}} \ , \tag{2.67}$$

$$\tilde{\pi}^k(\hat{X}^2) = -\frac{1}{2\ell}\hat{y}e^{\hat{x}}\hat{y} - \frac{\alpha^2}{2\ell}k(k+1)e^{-\hat{x}} - \frac{\ell}{2}e^{\hat{x}} \ . \tag{2.68}$$

Since the operators $\hat{x}$ and $\hat{y}$ satisfy the canonical commutation relations they can be mapped to their respective symbols on the Moyal-Weyl plane spanned by coordinates (x, y). This mapping is an isomorphism and by the Weyl correspondence, the product of functions of the operators $\mathfrak{F}\mathfrak{G}(\hat{x}, \hat{y})$ is mapped to the star product on the Moyal-Weyl plane $\mathcal{F}(x, y)\star\mathcal{G}(x, y)$ defined by (2.17) with $\alpha \to \hbar$. The symbols of $\tilde{\pi}^k(\hat{X}^\mu)$ are denoted by $\mathcal{X}^\mu$ and take the following form

$$\mathcal{X}^0 = -y \ , \tag{2.69}$$

$$\mathcal{X}^1 = -\frac{1}{2\ell}y \star e^{-x} \star y - \frac{\alpha^2}{2\ell}k(k+1)e^{-x} + \frac{\ell}{2}e^{x} \ , \tag{2.70}$$

$$\mathcal{X}^2 = -\frac{1}{2\ell}y \star e^{-x} \star y - \frac{\alpha^2}{2\ell}k(k+1)e^{-x} - \frac{\ell}{2}e^{x} \ . \tag{2.71}$$

Of course, these functions satisfy the same defining relations of the embedding coordinates of $EAdS_2$ when mapping the usual point-wise product to the star product on the Moyal-Weyl plane.

$$\mathcal{X}^\mu \star \mathcal{X}_\mu = -\ell^2 \ , \tag{2.72}$$

$$[\mathcal{X}^\mu, \mathcal{X}^\nu]_\star = \mathcal{X}^\mu \star \mathcal{X}^\nu - \mathcal{X}^\nu \star \mathcal{X}^\mu = i\alpha\epsilon^{\mu\nu\rho}\mathcal{X}_\rho \ . \tag{2.73}$$

Clearly, taking $\alpha \to 0$ we recover the point-wise product and, as explained before, the leading term of the α expansion in the star commutator is the Poisson bracket for (x, y) coordinates. For some calculations we will need to to introduce the non-commutative analogue of the FG coordinates, see section 2.1. Starting with $[\hat{x}, \hat{y}] = i\alpha\hat{I}$, we want to define new operators as functions of $\hat{x}$ and $\hat{y}$ in parallel with (2.8) satisfying the ordering prescription as follows:

$$\hat{t} = \frac{1}{2\ell}\left(\hat{y}e^{-\hat{x}} + e^{-\hat{x}}\hat{y}\right) \ , \qquad \hat{z} = e^{-\hat{x}} \ . \tag{2.74}$$

We can calculate the commutator of the new operators using the fact that x and y are canonically conjugate. One can easily find by induction

$$[\hat{x}^n, \hat{y}] = n\hat{x}^{n-1}[\hat{x}, \hat{y}] \ , \tag{2.75}$$

by expanding in Taylor's series

$$[f(\hat{x}), \hat{y}] = \left[\sum_{n=1}^{\infty}\frac{\partial^n f}{\partial x^n}\frac{\hat{x}^{n-1}i\alpha}{(n-1)!}, \hat{y}\right] = i\alpha\frac{\partial f}{\partial x}(x)\Big|_{x=\hat{x}} \ . \tag{2.76}$$

With this useful result, we can calculate the commutator of $\hat{z}$ and $\hat{t}$. Using $[f(\hat{x}), g(\hat{x})] = 0$ we get

$$[\hat{z}, \hat{t}] = \frac{1}{2l_0}([e^{-\hat{x}}, \hat{y}e^{-\hat{x}}] + [e^{-\hat{x}}, e^{-\hat{x}}\hat{y}]) = -\frac{i\alpha\hat{z}^2}{\ell} \ . \tag{2.77}$$

In order to verify if the mapping to the Moyal-Weyl plane preserves this commutator, we must calculate the Moyal-commutator of the symbols z and t. Denoting as $\star_{x,y}$ the star product for the canonical coordinates and $\star_{z,t}$ the transformed star product, one can calculate

$$[z,t]_{\star_{x,y}}(x,y) = z(x,y)\star_{x,y} t(x,y) - t(x,y)\star_{x,y} z(x,y) \ , \tag{2.78}$$

applying the definition of the star product

$$[z,t]_{\star_{x,y}}(x,y) = \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i\alpha}{2}\right)^n \epsilon^{i_1 j_1}\dots\epsilon^{i_n j_n}(\partial_{i_1}\dots\partial_{i_n})e^{-x}(\partial_{j_1}\dots\partial_{j_n})\Big(\frac{y}{\ell}e^{-x}\Big) \ , \tag{2.79}$$

and calculating up to $\mathcal{O}(1)$

$$[z,t]_{\star_{x,y}}(x,y) \simeq e^{-2x}y - e^{-2x}y = 0 \ , \tag{2.80}$$

the term proportional to α is

$$[z,t]_{\star_{x,y}}(x,y) \simeq \frac{i\alpha}{2}\left[\partial_x(e^{-x})\partial_y\Big(\frac{e^{-x}y}{\ell}\Big) + \partial_y\Big(\frac{e^{-x}y}{\ell}\Big)\partial_x(e^{-x})\right] = -\frac{e^{-2x}i\alpha}{\ell} \ , \tag{2.81}$$

the term proportional to α^2 consist in products of two derivatives acting on z and t, clearly for any y-derivative acting on z the respective term will be zero. Since all terms have at least one y derivative on z, all of them are zero except the term that has two x-derivatives on z, but it's clear that $\partial_y^2 t(x,y) = 0$. With this analysis, it's clear that the only non-vanishing term of the commutator is

$$[z,t]_{\star_{x,y}}(x,y) = -\frac{e^{-2x}i\alpha}{\ell} = -\frac{i\alpha}{\ell}z^2 \ , \tag{2.82}$$

which is equivalent to (2.77). One can easily write $\star_{z,t}$ in terms of (z,t) (see [18]). Up to $\mathcal{O}(\alpha)$ the result is

$$\star_{z,t} = 1 - \frac{i\alpha}{2}\left(\overleftarrow{\partial_z}z^2\overrightarrow{\partial_t} - \overleftarrow{\partial_t}z^2\overrightarrow{\partial_z}\right) + \mathcal{O}\left(\alpha^2\right) \ . \tag{2.83}$$

Now we proceed to construct the Killing vectors of the theory. From (3.21) the action of the isometries of AdS_2 on a scalar field can be obtained by taking the Poisson bracket of this field with respect to the embedding coordinates. In the non-commutative case for a function $\hat{\Phi}$ the action of the $SO(2,1)$ isometry group will induce an infinitesimal variation of the form

$$\delta_{nc}\hat{\Phi} = \epsilon_\mu(\hat{K}^\mu\hat{\Phi}) = i\epsilon_\mu[\hat{X}^\mu,\hat{\Phi}] \ , \tag{2.84}$$

for some infinitesimal parameter ϵ_μ. A natural step is map these Killing vectors to the Moyal-Weyl plane. From now on, the functions without $\wedge$ will denote the symbols of the Killing vectors. Then, the equation above becomes

$$\delta_{nc}\Phi = \epsilon_\mu(K^\mu_\star\Phi) = i\epsilon_\mu[\mathcal{X}^\mu,\Phi]_\star \ , \tag{2.85}$$

where $(K^\mu_\star\Phi)$ is the symbol of $(\hat{K}^\mu\hat{\Phi})$. Following some straightforward calculations (see [18]) we get to the deformed Killing vectors

$$\begin{aligned} K^-_\star &= -\ell e^x\Delta_y \qquad , \qquad K^0_\star = \partial_x \\ K^+_\star &= \frac{e^{-x}}{\ell}\left(2y\partial_x S_y + \Big(y^2+\ell^2+\frac{\alpha^2}{4}(1-\partial_x^2)\Big)\right) \ , \end{aligned} \tag{2.86}$$

for

$$\Delta_y \Phi(x,y) = \frac{2}{\alpha} \sin\left(\frac{\alpha}{2}\partial_y\right) \Phi(x,y) \ ,$$
$$S_y \Phi(x,y) = \cos\left(\frac{\alpha}{2}\partial_y\right) \Phi(x,y) \ . \tag{2.87}$$

In the commutative limit, these differential operators agree with (2.39) and they indeed satisfy the $\mathfrak{so}(2,1)$ algebra. $K^0_\star$ is the same as K^0, while the others are deformations containing infinite polynomials in ∂_y. Writing the equations above in terms of FG coordinates we get

$$K^-_\star = -\frac{\ell}{z}\Delta_t \quad , \quad K^0_\star = -t\partial_t - z\partial_z$$
$$K^+_\star = -2t(t\partial_t + z\partial_z)S_t + \frac{\ell}{z}\left(t^2 + \left(1 + \frac{\alpha^2}{4\ell^2}\right)z^2\right)\Delta_t - \frac{\alpha^2 z}{4\ell}(t\partial_t + z\partial_z)^2\Delta_t \ . \tag{2.88}$$

In the near boundary limit, $z \to 0_+$, (2.88) gives exactly the commutative expressions for the conformal generators (2.41). This shows that the $ncAdS_2$ is asymptotically AdS_2. Now we may try, in principle, to apply the AdS/CFT correspondence. In the next sections we will explore briefly the massless, massive and interacting non-commutative scalar fields, since these cases are deeply discussed in [4], [6], [7] and [18] we will not derive all equations and just proceed to analyse the main results of these papers.

2.5 The non-commutative correspondence - Massless case

We obtain the field equation from the action principle imposing Dirichlet boundary conditions[25]. By writing the commutative action in terms of Poisson brackets we get

$$S[\Phi^{(0)}] = \frac{1}{2\ell}\int_{AdS_2} d\mu \{X^\mu \Phi^{(0)}\}\{X_\mu \Phi^{(0)}\} \ , \tag{2.89}$$

where $d\mu$ is the invariant integration measure on AdS_2. In our current work we will give a direct proof that the natural generalization of the action written above is given by

$$S_{nc}[\hat{\Phi}] = \frac{1}{2\ell}\mathrm{Tr}[\hat{X}^\mu, \hat{\Phi}][\hat{X}_\mu, \hat{\Phi}] \ , \tag{2.90}$$

where Tr denotes the trace operation. Mapping this action to the Moyal-Weyl plane gives

$$S_{nc}[\Phi] = \frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} [\mathcal{X}^\mu, \Phi]_\star \star [\mathcal{X}_\mu, \Phi]_\star dx \ dy \ . \tag{2.91}$$

Now we choose the set of coordinates in which the action above will be written. As we discussed above in the FG coordinates, as one approaches the boundary, the Lagrangian loses all the non-commutative corrections[26] but retains a scaling factor in the coordinate t

$$\mathcal{L}_{nc}|_{z\to 0} = k(\alpha)^2(\partial_t\Phi)^2 + (\partial_z\Phi)^2 \ , \tag{2.92}$$

[25] This can be done because it is proven that there isn't any non-commutative correction to the boundary term from the variations of the action (see [6] and [7]).

[26] This can be easily seen by noting that $\Delta_t\Phi \to \frac{z}{\ell}\partial_t\Phi|_{z=0}$ and $S_t\Phi \to \Phi|_{z=0}$.

for $k(\alpha) = \sqrt{1 + \frac{\alpha^2}{4\ell^2}}$. This $k(\alpha)$ term will reappear from various different considerations through our study and its meaning will be clear after some insightful considerations. From now it suffices to see that Φ satisfies the equation for a massless scalar field on an asymptotically AdS_2 space. Upon taking the variation of the action all the boundary terms vanish (see [6]) and the field equation in the bulk reduces to

$$[\mathcal{X}^\mu,\ [\mathcal{X}_\mu, \Phi]_\star]_\star = 0 \ . \tag{2.93}$$

The first result we analyze can be found in [6], after applying some standard techniques to solve the field equations up to the leading order in $O(\alpha^2)$ and after the application of the prescription for the AdS_2/CFT_1 correspondence the conclusion is that the n-point correlation functions of quantum mechanical operators $\mathcal{O}(t)$ on the one-dimensional boundary remain conformal and just get rescaled, at least in the leading order in α^2. For the two-point function we get

$$\langle \mathcal{O}(t)\mathcal{O}(t')\rangle = -\frac{1}{\pi}\left(1 + \frac{\alpha^2}{8\ell^2}\right)\frac{1}{(t-t')^2} + O(\alpha^4) \ . \tag{2.94}$$

Moreover, in a subsequent paper (see [4]) the authors of [6] find an exact solution to the field equations by quantizing the expressions of the coordinates[27] z and t in terms of the embedding coordinates X^μ as shown below

$$\hat{z} = (\hat{X}^2 - \hat{X}^0)^{-1} \ , \qquad \hat{t} = -\frac{1}{2}[\hat{z}, \hat{X}^1]_+ \ . \tag{2.95}$$

where $[-,-]_+$ stands for the anti-commutator. They also conclude that the appearance of the non-trivial deformation factor $k(\alpha)$ is of utmost importance for the consistency of the algebraic relations defining the isometries of the embedding coordinates. This factor can be derived also by the use of the approach presented further, which relies on a completely different formalism to achieve the construction of the quantum AdS_2 surface. Following [4] and applying the prescription of the correspondence one can show

$$\langle \mathcal{O}(t)\mathcal{O}(t')\rangle = -\frac{k(\alpha)}{\pi}\frac{1}{(t-t')^2} = -\frac{1}{\pi}\left(1 + \frac{\alpha^2}{8\ell^2}\right)\frac{1}{(t-t')^2} + O(\alpha^4) \ . \tag{2.96}$$

which completely agrees[28] (up to order α^2) with the results obtained in [6], [7] and [18].

[27] In [4] the authors use a different signature for the ambient metric tensor η_{ab} which is diag$(-,\ +,\ +)$. This detail doesn't change any important fact since the groups $SO(2,1)$ and $SO(1,2)$ are isomorphic, but it changes signs in some equations and could be rather cumbersome to find such small differences in both works.

[28] One should note that there is a small mistake in the derivation of the two-point function in [6] which is addressed in [7].

2.6 The non-commutative correspondence - Massive case

One of the key features of the approach is that we are interested in preserving all isometries[29] of AdS_2 undeformed, i. e. when making the transition to the non-commutative setup we want to keep all $\mathfrak{so}(2,1)$ isometries untouched and this is important since it allows us to aim at a purely algebraic approach. In [7] we start by the natural assumption that the action can be written as

$$S_{nc}[\hat{\Phi}] = -\frac{1}{2\ell}\mathrm{Tr}\Big\{[\hat{X}^\mu,\hat{\Phi}][\hat{X}_\mu,\hat{\Phi}] - (\alpha\ell m_0\hat{\Phi})^2\Big\}\ , \tag{2.97}$$

where m_0 stands for the mass of the scalar field and we assume that the mass doesn't undergo any deformation in the quantization process. Upon quantization the field equation obtained from the variation of the action is

$$K^\mu_\star K_{\star\mu}\Phi := \frac{1}{\alpha^2}[\mathcal{X}^\mu,[\mathcal{X}_\mu,\ \Phi]_\star]_\star = (m_0\ell)^2\Phi\ . \tag{2.98}$$

These non-commutative Killing vectors are exactly the ones in (2.88) and, as we know, they preserve the $\mathfrak{so}(2,1)$ symmetry and two of them get deformed on passing to the quantum case. In order to simplify finding perturbative (first order in $\frac{\alpha^2}{\ell^2}$) solutions to the field equation we constructed an operator $U(\alpha)$ which maps the set of commutative Killing vectors on its non-commutative counterpart modulo some corrections (see [7]). After writing the field equation in terms of FG coordinates and doing the similarity transformation $U(\alpha)\mathcal{L}U^{-1}(\alpha)$ where $\mathcal{L}$ is the non-commutative Laplacian, we get the following

$$\left(\mathcal{L}^{(0)} - \frac{\alpha^2}{8\ell^2}z^4\partial_t^4 + O\Big(\frac{\alpha^4}{\ell^4}\Big)\right)\Phi_U(z,t) = (m\ell)^2\Phi_U(z,t), \tag{2.99}$$

where $\mathcal{L}^{(0)}$ is the commutative Laplacian and $\Phi_U(z,t) = U(\alpha)\Phi(z,t)$ is the transformed scalar field. Using some of the results from [32] and using some facts already used in the massless case the on-shell action takes the following form

$$\begin{aligned} S_{nc}[\Phi[\phi_0]] = &-\frac{1}{2}\left(1+\frac{3\alpha^2}{32\ell^2}\right)\int_{\mathbb{R}^3} dt\ dt'\ dt'' K^U_{nc}(z,t;t')\partial_z K^U_{nc}(z,t;t'')|_{z=0}\phi_0(t')\phi_0(t'') \\ &+\ O\Big(\frac{\alpha^4}{\ell^4}\Big)\ , \end{aligned} \tag{2.100}$$

where K^U_{nc} is the non-commutative boundary-to-bulk propagator defined as

$$K^U_{nc}(z,t;t') = UK_{nc}(z,t;t')\ .$$

In order to find the two-point function one must expand it in powers of $\left(\frac{\alpha}{\ell}\right)^2$

$$\langle\mathcal{O}(t)\mathcal{O}(t')\rangle = \langle\mathcal{O}(t)\mathcal{O}(t')\rangle^{(0)} + \frac{\alpha^2}{\ell^2}\langle\mathcal{O}(t)\mathcal{O}(t')\rangle^{(1)} + O\Big(\frac{\alpha^4}{\ell^4}\Big)\ , \tag{2.101}$$

[29] Since all the derivations done in [6] and [7] are carried over perturbative methods one could argue that all the findings are just artifacts coming from the first order approximations. By assuring that the full symmetries go undeformed after the quantization we can suppose that might exist a way of constructing exact solutions for this problem, and this is done in [4] as discussed before.

and in order to determine the first order correction we can evaluate all the integrals coming from the on-shell action (see [33] for some additional techiniques). Finally we find that

$$\langle\mathcal{O}(t)\mathcal{O}(t')\rangle^{(1)} = \frac{\Gamma(\Delta_+)}{32\sqrt{\pi}\Gamma(\Delta_+ - \frac{1}{2})}\left\{\frac{8}{3}\left(\Delta_+^2 - \frac{1}{4}\right)\left(\Delta_+ - \frac{3}{2}\right) - 3\Delta_+\right\}\frac{1}{|t-t'|^{2\Delta_+}} \ , \tag{2.102}$$

and for the special case of a massless scalar field we have $\Delta_+ = 1$ which reduces the equation above to

$$\langle\mathcal{O}(t)\mathcal{O}(t')\rangle^{(1)} = -\frac{1}{8\pi}\frac{1}{|t-t'|^2} \ , \tag{2.103}$$

in conformity with (2.96).

2.7 The non-commutative correspondence - Interacting case

Now we analyze the correspondence for the interacting case and the first step to achieve this goal is to generalize the equation (2.34). Using our standard strategy and after mapping the trace to the integral on the Moyal-Weyl plane, we find that the non-commutative interacting action is

$$S_nc[\Phi] = -\frac{1}{2\ell\alpha^2}\int_{\mathbb{R}^2} dx\ dy\Big\{[\mathcal{X}^\mu,\Phi]_\star \star [\mathcal{X}_\mu,\Phi]_\star - (\alpha\ell m_0)\Phi\star\Phi - \frac{2}{3}\alpha^2\lambda\Phi\star\Phi\star\Phi\Big\} \tag{2.104}$$

which gives the following field equation

$$\mathcal{L}\Phi - (\ell m)^2\Phi = \lambda\Phi\star\Phi \ . \tag{2.105}$$

We already solved the free commutative (and non-commutative) theory given by $\lambda = 0$, now we apply the same perturbative procedure assuming that λ is small and substituting the commutative Green's functions by their non-commutative analogues. For the non-commutative boundary-to-bulk and bulk-to-bulk propagators we have for $K(z,t;t')$ the commutative propagator

$$\begin{gathered} K_{nc}(z,t;t') = K_{nc}^U(z,t;t') + \frac{\alpha^2}{\ell^2}\mathcal{D}_{z,t}K(z,t;t') + O(\alpha^4) \ , \\ \Big[(U\mathcal{L}U^{-1} - (\ell m)^2)\Big]G_{nc}^U = -z^2\delta(z-z')\delta(t-t') \end{gathered} \tag{2.106}$$

for $\mathcal{D}_{z,t} = \frac{z^2}{96}(9 + 4t\partial_t + 6z\partial_z)\partial_t^2 + \frac{3}{32}z\partial_z$ and the bulk-to-bulk propagator is determined perturvatively as a solution of the differential equation denoted above. The perturbative solution for Φ of first order in λ is

$$\begin{aligned} \Phi(z,t) = &\int_{\mathbb{R}} K_{nc}(z,t;t')\phi_0(t')dt' - \lambda\int_{\mathbb{R}^3\times\mathbb{R}_+}\frac{dz'dt'dt_1dt_2}{z'^2}U_{z,t}^{-1}G_{nc}^U(z,t;z',t')U_{z',t'} \\ &\times [K_{nc}(z',t';t_1)\star K_{nc}(z',t';t_2)]\phi_0(t_1)\phi_0(t_2) + O(\lambda^2) \ . \end{aligned} \tag{2.107}$$

After substituting this solution in (2.104) we get the on-shell action and in this case, as in the commutative one in (2.38), we find two terms, a boundary one and a bulk one. After applying the prescription for the correspondence we find the perturbative expression for the three-point function

$$\begin{aligned}\langle O(t_1)O(t_2)O(t_3)\rangle \; &= \frac{\lambda}{2}\int_{\mathbb{R}^2}\frac{dz\; dt}{z^2}\Bigg\{\frac{1}{2\Delta_+-1}\left(1+\frac{3\alpha^2}{32\ell^2}\right)U_{z,t}K_{nc}^{(1)}(z,t) \\ \times U_{z,t}[K_{nc}^{(2)}\star K_{nc}^{(3)}](z,t) &+ \frac{2}{3}[K^{(1)}\star K_{nc}^{(2)}\star K_{nc}^{(3)}](z,t)\Bigg\} + \text{permutations} + O(\alpha^4)\;,\end{aligned} \tag{2.108}$$

for $K_{nc}(z,t) = K_{nc}(z,t;t_i)$. Since the last integral is very hard to calculate explicitly, we verified if the three-point function for $ncAdS_2$ preserves the same set of undeformed conformal invariances[30] found in the commutative case. It was easy to show that the non-commutative three-point function transforms as the commutative one for silmultaneous scalings on t_i's i.e.

$$\langle O(\mu t_1)O(\mu t_2)O(\mu t_3)\rangle = \mu^{-3\Delta_+}\langle O(t_1)O(t_2)O(t_3)\rangle\;. \tag{2.109}$$

As a final step we demonstrated the invariance of the three-point function under simultaneous translations. We reffer to [7] and [18] for a deeper analysis of the calculations done in order to find the final result. Under translations of the type t_i+a most of the terms in (2.108) present non-trivial transformations. After the use of some standard techniques (see [18]) we showed that all terms in (2.108) are translationally invariant. So we can conclude that the non-commutative three-point function shares the symmetries of its commutative counterpart and, as the consequence, it should have the following form

$$\langle O(t_1)O(t_2)O(t_3)\rangle \; = \frac{\Big(1+c\alpha^2\Big)\lambda\Gamma(\Delta_+/2)^3\Gamma\Big((3\Delta_+-1)/2\Big)}{2\pi\Gamma(\nu)^3|t_1-t_2|^{\Delta_+}|t_1-t_3|^{\Delta_+}|t_2-t_3|^{\Delta_+}}\left(\frac{3\Delta_+}{2\nu}+2\right)\; + O(\alpha^3). \tag{2.110}$$

where the coefficient c can be calculated by solving (2.108), which is a non-trivial task and imposed technical constraints to our final conclusion. By the implicit study of the transformation properties of (2.108) we verified that its conformal behavior is retained after the quantization and it points out to a possible continuation of this project by the explicit calculation of the coefficient c. This result also motivated us to try to find another way of quantizing the AdS_2 that could lead to an exact solution of the proposed problem. This was done in [4] and in the next chapter we will develop a method that will yield the same result and expand the possible applications of it.

[30] We will not demonstrate that the invariance under special conformal transformations holds in this case. One could argue that since we are analysing the correlator of the same fields, which has the same conformal dimension, it suffices to show that the three-point function is well behaved for scalings and translations.

3 The geometry of the Quantum AdS_2

The previous chapter provides a comprehensive summary of the results obtained so far. We presented a concrete method for the study of the main properties of the AdS_2/CFT_1 correspondence for scalar fields and studied the massless, massive and interacting theory on a non-commutative background. In this context, our quantization was designed to preserve all symmetries of the embedding keep the algebra of Killing vector fields undeformed. This requirement enables us to extend all crucial findings from the commutative theory to our non-commutative setup.

In this part of the thesis we want to generalize the example of the non-commutative Catenoid studied in section 1.3.1. example of this formalism for a simple classical surface. We apply the same formalism to construct Quantum AdS_2. This nomenclature is justified because we require that the generators of the algebra, employed to define our embedding coordinates, are elements of a Weyl algebra for some set of commutation relations. Consequently, the surface is inherently quantum by the construction. We define the Lie algebra of derivations on this space and introduce the concept of basis vectors, inverse elements, and positive elements. Additionally, we establish a homeomorphism that allows us to represent certain functions in this algebra as formal power series.[31] Subsequently, we construct all geometric entities necessary for defining a concrete quantum manifold, establishing a module over this manifold and proving the linear independence of its basis elements. Various sets of coordinates are introduced to verify the consistency of our setup. Following comprehensive consistency checks, we proceed to define the non-commutative analogue of a metric on both the ambient space and the manifold itself. We demonstrate that most metric coefficients undergo quantum corrections, reducing to the commutative metric as $\hbar \to 0$. One interesting detail is that when we apply this procedure, the resulting metric is Hermitian, which brings another layer of complexity to future calculations as we will discuss in following sections. After these steps we apply our setup to alternative coordinate systems, providing some examples of how we can construct metric coefficients in these situations.

As a concluding step, we construct the tangent space of the quantum AdS_2 as a module structure of our non-commutative algebra. Using the Fefferman-Graham coordinates as our local set of coordinates, and the ambient coordinates to construct the surface itself, we define the non-commutative covariant derivative and Christoffel symbols, which are then used to calculate the non-commutative Riemann tensor in these coordinates. Following this, we rigorously study the Ricci scalar within our framework, justifying the need for some caution to avoid ambiguities that are expected to arise when analyzing quantum surfaces. We prove a theorem relating the Laplace operator found in this new formalism to another operator derived using a different approach used

[31] We define these series this way because we are not worried about any notion of convergence.

in the previous chapter. Additionally, we construct the non-commutative analogue of the Killing equation to verify whether the Killing vector fields preserve the symmetries of the $EAdS$ after quantization. We also discuss the existence of non-commutative integration using eigenfunctions, showing that by solving the Laplace equation in this setting, one could find non-commutative functions that satisfy this requirement. We conclude this chapter by discussing some unfinished steps and potential further developments that could be pursued using the framework introduced in this thesis.

3.1 The Quantum AdS_2

As we saw in the catenoid example, much of the formalism introduced by the non-commutative calculus can be applied to a class of surfaces and in this section we will try apply it without initially worrying about the details regarding the module structure that we will define soon, but in the following subsections we will discuss a more rigorous way of obtaining these results. In order to differentiate this approach from the Poisson quantization employed in the previous chapter we will denote the non-commutative parameter as $\hbar$, which should not be confused with the Planck's constant.

3.1.1 Parametrization

Recall the section 2.1, where we defined the commutative Euclidean AdS_2 in terms of embedding coordinates $X^\mu,\ \mu = 1,\ 2,\ 3$, spanning the three-dimensional Minkowski space with the ambient metric tensor $\eta = \mathrm{diag}(1, 1, -1)$. The constraint equation was

$$X^\mu X_\mu = (X^1)^2 + (X^2)^2 - (X^3)^2 = -\ell_0^2 \ , \tag{3.1}$$

with $\ell_0^2 > 0$ being a scale parameter. One of the parametrizations for the hyperboloid, which will be useful, is given by

$$\overrightarrow{X} = \Big(- v,\ -\frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \sinh u,\ -\frac{1}{2\ell_0} e^{-u} v^2 - \ell_0 \cosh u \Big) \ , \tag{3.2}$$

where $(u, v) \in \mathbb{R}^2$. As before, we attach a Poisson bracket to the AdS manifold respecting the expected relations of the isometry group $SO(2, 1)$, which can be related to the global conformal symmetry on the boundary of this space. For the canonically conjugate coordinates[32] u, v the

[32]This set of coordinates has the following property $\{u, v\} = 1$. This pair of coordinates have the same properties of the set (x, y) introduced in the section 2.1.

Poisson bracket can be expressed by the following relation

$$\{X^\mu, X^\nu\} = \epsilon^{\mu\nu\rho} X_\rho \ ,$$

which can be readly demonstrated

$$\begin{aligned}
\{X^0, X^1\} &= \partial_u X^0 \partial_v X^1 - \partial_v X^0 \partial_u X^1 = \partial_u \Big(-\frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \sinh u \Big) = \frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \cosh u \\
&= -X^2 = \epsilon^{012} \eta_{22} X^2 \ , \\
\{X^1, X^2\} &= \partial_u X^1 \partial_v X^2 - \partial_v X^1 \partial_u X^2 = \Big(-\frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \cosh u \Big) \Big(-\frac{1}{\ell_0} e^{-u} v \Big) - \\
&\Big(-\frac{1}{\ell_0} e^{-u} v \Big) \Big(-\frac{1}{2\ell_0} e^{-u} v^2 - \ell_0 \sinh u \Big) = -e^{-u} v (\sinh u + \cosh u) = -v \\
&= X^0 = \epsilon^{120} \eta_{00} X^0 \ , \\
\{X^2, X^0\} &= \partial_u X^2 \partial_v X^0 - \partial_v X^2 \partial_u X^0 = -\partial_u \Big(-\frac{1}{2\ell_0} e^{-u} v^2 - \ell_0 \cosh u \Big) = -\frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \sinh u \ , \\
&= X^1 = \epsilon^{201} \eta_{11} X^1 \ ,
\end{aligned}$$

Here η and ϵ are defined as $\eta = \mathrm{diag}(1, 1, -1)$ and $\epsilon^{012} = 1$. Now we start from the Weyl algebra $W_\hbar$ (see the discussion in section 1.1) consisting of Hermitian generators U and V satisfying

$$[U, V] = i\hbar \mathbb{1} \ .$$

We construct an algebra generated by V, e^U and e^{-U}, denoting them by V, Y and Y^{-1} respectively. The generators of this algebra represent the set of non-commutative local coordinates that will be used to construct a non-commutative differential calculus over the $ncAdS_2$. In order to ensure the cannonical commutation relations we form a two-sided ideal $\mathcal{I}$ generated by the following relations:

$$\begin{aligned}
&YY^{-1} = \mathbb{1} \ , \\
&YV = VY + i\hbar Y \ ,
\end{aligned} \tag{3.3}$$

where the second relation in (3.3) is clearly corresponding to the canonical commutation relation. Now we define

Definition 3.1. *Let us denote by* $\mathbb{C}[V, Y, Y^{-1}]$ *the free associative unital algebra on the letters* V, Y, Y^{-1} *and let* $\mathcal{I}$ *be the two-sided ideal generated by the relations* (3.3). *We define the algebra* $\mathcal{C}_\hbar$ *as the quotient algebra*

$$\mathcal{C}_\hbar = \mathbb{C}[V, Y, Y^{-1}]/\mathcal{I} \ . \tag{3.4}$$

If one uses the Diamond lemma [39] and the reduction system employed in [2] in order to remove any ambiguity in the system, one can define a basis for $\mathcal{C}_\hbar$ as

$$E^{ij} = V^i Y^j \ , \tag{3.5}$$

with $i \in \mathbb{N}_0$, $j \in \mathbb{Z}$, and $(Y^{-1})^j = Y^{-j}$. Clearly we can turn $\mathcal{C}_\hbar$ into a *-algebra by defining an involution operation by requiring the Hermiticity of V, Y and Y^{-1}. Clearly, the set of relations generating $\mathcal{I}$ is invariant under involution. To check the existence of derivations in $\mathcal{C}_\hbar$, we must find linear derivative[33] operators that satisfy: i) the Leibnitz rule, ii) preserves the relations that generate the two-sided ideal $\mathcal{I}$. For our purposes it suffices that the derivations satisfy

$$\partial_V V = \mathbb{1} \ , \qquad \partial_V Y = 0 \ ,$$
$$\partial_U V = 0 \ , \qquad \partial_U Y = Y \ .$$

The extension to a general element of the algebra is given by the Leibnitz rule. One can check the consistency of this definition applying the derivation defined above to the expressions (3.3). For instance

$$\partial_U(YV - VY - i\hbar Y) = \partial_U(Y)V - V\partial_U(Y) - i\hbar\partial_U(Y) = YV - VY - i\hbar Y = 0 \ ,$$

$$\partial_V(YV - VY - i\hbar Y) = Y\partial_V(V) - \partial_V(V)Y - i\hbar\partial_V(Y) = Y - Y = 0 \ ,$$

while requiring that $\partial_u(YY^{-1}) = \partial_u(Y^{-1}Y) = 0$ gives: $\partial_U Y^{-1} = -Y^{-1}\partial_V Y^{-1} = 0$. We now state a series of propositions and lemmas following [2], the proofs can be found there.

Proposition 3.1. *The algebra $\mathcal{C}_\hbar$ has no zero divisors.*

Proposition 3.2. *For every $a, b \in \mathcal{C}_\hbar$ there exists $p, q \in \mathcal{C}_\hbar$ such that*

$$ap = bq \ ,$$

and at least one of p and q is non-zero.

Proposition 3.3. *Let $Z_\hbar(V)$ be the commutative sub-algebra of $\mathcal{C}_\hbar$ generated by $\mathbb{1}$ and V and define a homeomorphism of commutative algebras $\psi : Z_\hbar(V) \to C^\infty(\mathbb{C})$ via*

$$\psi(\mathbb{1}) = 1 \ , \qquad \psi(V) = v \ .$$

The subset $Z^+_\hbar = \{x[V] \in Z_\hbar(V) \ : \ |\psi(x)[v]| > 0 \ \forall \ v \in \mathbb{C}\}$ is a multiplicative set.

In order to construct an sub-algebra where all elements are invertible, we must guarantee the existence of a non-trivial localization by means of the universal property. This is done in [2] and we will refer to this paper for any additional information regarding the details of some definitions and propositions used here. The notation for the aforementioned localization is $(\mathcal{C}_\hbar)_{Z^+_\hbar(V)}$, and we will denote it as $\mathcal{C}_\hbar$ in future sections. We will make the distinction if needed in order to avoid any confusion. Now it is easy to show that for every $x \in Z^+_\hbar(V)$ and $x_i \in \mathbb{C}$ satisfying

$$x[V] = \sum x_i V^i \ ,$$

[33]We are more interested in the inner derivations $\mathrm{Inn}(W_\hbar)$, which come from $ad_g(f) = [f, g]$ for $[\ ,\ ]$ being the abstract Lie bracket from the algebra. Since in the commutative case the adjoint action always yields zero, we have in this case only the outer derivations, which are defined as $\mathrm{Out}(W_\hbar) := \mathcal{L}(W_\hbar)/\mathrm{Inn}(W_\hbar)$ where $\mathcal{L}(W_\hbar)$ stands for the space of all linear endomorphism of $W_\hbar$ satisfying the Leibnitz rule.

we have

$$Yx[V] = \sum x_i Y V^i = \sum x_i (V + i\hbar \mathbb{1})^i Y = x[V + i\hbar\mathbb{1}]Y \ ,$$

$x[V + i\hbar\mathbb{1}]$ clearly belongs to $Z_\hbar^+(V)$. One can show the same for $Yx^{-1}[V]$, thus with these results we conclude that any element $a \in \mathcal{C}_\hbar$ can always be written as

$$a[U] = \sum_{k\in\mathbb{Z}} a_k[V]Y^k \ ,$$

with $a_k \in \mathcal{F}_\hbar^+(V) \subset (\mathcal{C}_\hbar)_{Z_\hbar^+}$ being generated by $Z_\hbar(V)$ and the inverses of the elements from $Z_\hbar^+(V)$.

In the next subsection we will construct the Riemannian geometry over the non-commutative AdS_2 using a module structure for it.

3.1.2 The module structure

We start this subsection by defining the non-commutative manifolds that are determined by our choice of embedding coordinates X^μ. Then we apply this to the construction of the $ncAdS_2$.

Definition 3.2. *Let $\{X^\mu\}$ be a set of n elements $X^\mu \in \mathcal{A}$. A triple $\Sigma = (\mathcal{A},\ \mathfrak{g},\ \{X^1,\ \dots\ ,X^n\})$ where $(\mathcal{A}, \mathfrak{g})$ is a Lie pair with all elements of the set $\{X^\mu\}$ being hermitian, is called an* ***embedded non-commutative manifold****.*

Now we introduce a free right $\mathcal{C}_\hbar$-module M by defining it to be $M = (\mathcal{C}_\hbar)^3$ and we will set up the Lie pair structure in later steps. Now to define the quantum AdS_2 as an embedded non-commutative manifold we can use the classical AdS embedding coordinates promoting the commuting canonical coordinates to the non-commutative ones and imposing the symmetric ordering for V

$$\overrightarrow{X} = \left(-V \ , \ -\frac{1}{2\ell_0}VY^{-1}V + \frac{\ell_0}{2}(Y - Y^{-1}) \ , \ -\frac{1}{2\ell_0}VY^{-1}V - \frac{\ell_0}{2}(Y + Y^{-1}) \right) \ . \tag{3.6}$$

To see if this embedding satisfies (3.1), we calculate, for $\mu = \overline{0,2}$

$$\begin{aligned}
X^\mu X^\nu \eta_{\mu\nu} &= (X^0)^2 + (X^1)^2 - (X^2)^2 \ , \\
&= V^2 + \left(\frac{1}{4}(VY^{-1}V)^2 - \frac{1}{4}\Big(VY^{-1}VY - (VY^{-1})^2 + YVY^{-1}V - (Y^{-1}V)^2\Big)\right) \\
&- \frac{1}{4}\left((VY^{-1}V)^2 + \frac{1}{4}\Big(VY^{-1}VY + (VY^{-1})^2 + YVY^{-1}V + (Y^{-1}V)^2\Big)\right) + \\
&+ \frac{\ell_0^2}{4}\left(\Big(Y + Y^{-1}\Big)^2 - \Big(Y + Y^{-1}\Big)^2\right) \ , \\
&= V^2 - \frac{1}{2}\Big(V(VY^{-1} - i\hbar Y^{-1})Y + (VY + i\hbar Y)Y^{-1}V\Big) - \ell_0^2\mathbb{1} \ , \\
&= -\mathbb{1}\ell_0^2 \ ,
\end{aligned}$$

which implies that

$$X^\mu X_\mu = -\ell_0^2 \mathbb{1} \ , \tag{3.7}$$

for $\eta = \mathrm{diag}(+,+,-)$. We also must verify if this set of coordinates has the $\mathfrak{su}(1,1)$ Lie bracket structure. For instance, one can directly calculate

$$[X^0, X^1] = \Big[-V,\ \frac{1}{2\ell_0} V Y^{-1} V + \frac{\ell_0}{2}(Y + Y^{-1}) \Big] = i\hbar \epsilon^{012} \eta_{22} X^2 = -i\hbar X^2 \ .$$

After some straightforward calculations is trivial to verify that in general

$$[X^\mu, X^\nu] = i\hbar \epsilon^{\mu\nu\rho} X_\rho \ ,$$

as expected. Returning to the classical case one can verify that the commutative vectors that span the tangent space of AdS_2 are

$$\begin{aligned} \phi_u &= \partial_u \overrightarrow{X} = \Big(0\ ,\ \frac{1}{2\ell_0} e^{-u} v^2 + \ell_0 \cosh(u)\ ,\ \frac{1}{2\ell_0} e^{-u} v^2 - \ell_0 \sinh(u) \Big) \ , \\ \phi_v &= \partial_v \overrightarrow{X} = \Big(-1\ ,\ -\frac{1}{\ell_0} e^{-u} v\ ,\ -\frac{1}{\ell_0} e^{-u} v \Big) \ . \end{aligned} \tag{3.8}$$

In order to obtain the non-commutative analogues we should apply the derivative operators defined previously to the embedding operators X. For our choice of ordering, the only term that slightly changes its structure is

$$\frac{1}{2\ell_0} \partial_V (V Y^{-1} V) = \frac{1}{2\ell_0} \Big(V Y^{-1} + Y^{-1} V \Big) \ .$$

From now on, we will use the greek indices for $\mu = 0, 1,\ 2$ to denote the module indices and the roman $i = U,\ V$ denote the local coordinates. We will also denote the basis of the right free module $\mathcal{X}(\mathcal{C}_\hbar)$[34] of rank 2 by $\{\Phi_i\}$ and a tangent vector can be obtained as the linear combination

$$\Phi_i = \hat{e}_\mu \Phi_i^\mu \ , \tag{3.9}$$

where $\Phi_i^\mu \in \mathcal{C}_\hbar$ and $\hat{e}_\mu$ is the natural basis of $(\mathcal{C}_\hbar)^3$ defined as

$$\begin{aligned} \hat{e}_0 &= (\mathbb{1}, 0, 0) \ , \\ \hat{e}_1 &= (0, \mathbb{1}, 0) \ , \\ \hat{e}_2 &= (0, 0, \mathbb{1}) \ . \end{aligned}$$

Φ_i^μ can be obtained by the direct calculation from the embedding coordinates from (3.6)

$$\Phi_i^\mu = \partial_i X^\mu \ . \tag{3.10}$$

[34]This module can be thought as the tangent space of the non-commutative AdS_2.

Doing the direct computation we finally obtain

$$\begin{aligned}
\Phi_U^0 &= 0 \ , \\
\Phi_U^1 &= \frac{1}{2\ell_0}(VY^{-1}V) + \frac{\ell_0}{2}(Y + Y^{-1}) \ , \\
\Phi_U^2 &= \frac{1}{2\ell_0}(VY^{-1}V) - \frac{\ell_0}{2}(Y - Y^{-1}) \ , \\
\Phi_V^0 &= -\mathbb{1} \ , \\
\Phi_V^1 &= \Phi_V^2 = -\frac{1}{2\ell_0}(Y^{-1}V + VY^{-1}) \ .
\end{aligned} \tag{3.11}$$

In order to prove that $\{\Phi_i\}$ is a basis we must show that for $a, b \in \mathcal{C}_\hbar$ the expression $a\Phi_U + b\Phi_V = 0$ implies that a and b are equal to zero, but it is easily shown by noting that $a\Phi_U^0 + b\Phi_V^0 = 0$ implies that $b = 0$, since b vanishes using any of the remaining relations one can show that $a = 0$. Now we introduce the Lie algebra $\mathfrak{g}$ generated by the set of the inner hermitian derivations

$$\partial_0(F) = \frac{1}{i\hbar}[X_0,\ F] \ , \qquad \partial_1(F) = \frac{1}{i\hbar}[X_1,\ F] \ , \qquad \partial_2(F) = \frac{1}{i\hbar}[X_2,\ F] \ , \tag{3.12}$$

for $F \in \mathcal{C}_\hbar$ and we will use $\eta_{\mu\nu}$ and its inverse to raise and lower the greek indices. We can also use complex coordinates Λ and Λ^* (see [40]) to define a new set of complex derivations, as done in (1.13), and proceed with the previous construction. In order to define the abstract Lie bracket in $\mathfrak{g}$ one should observe that

$$\partial_\mu\Big(\partial_\nu(F)\Big) = \frac{1}{(i\hbar)^2}[X_\mu \ , [X_\nu, F]] \ , \tag{3.13}$$

for an arbitrary differentiable function F in $\mathcal{C}_\hbar$. Using the fact that the operator commutator should satisfy the Jacobi identity, we get

$$\partial_\mu\Big(\partial_\nu(F)\Big) = -\frac{1}{(i\hbar)^2}\Big([F \ , [X_\mu, X_\nu]] + [X_\mu \ [F, X_\nu]]\Big) = \frac{1}{i\hbar}\epsilon_{\mu\nu}{}^{\rho}[X_\rho, F] + \partial_\nu(\partial_\mu(F)) \ ,$$

which trivially gives the Lie bracket structure for $\mathfrak{g}$

$$[\partial_\mu \ , \partial_\nu]_{\mathfrak{g}} = \epsilon_{\mu\nu}{}^{\rho}\partial_\rho \ . \tag{3.14}$$

We can also introduce a Lie algebra $\mathfrak{h}$ generated by the ∂_U and ∂_V.With it we define the associated Lie pair $(\mathcal{C}_\hbar, \mathfrak{h})$ to our non-commutative manifold and we will define a map $\varphi \ : \ \mathfrak{h} \to (\mathcal{C}_\hbar)^3$ as

$$\varphi(\partial_i) = \partial_i \overrightarrow{X} = \hat{e}_\mu \partial_i X^\mu = \hat{e}_\mu \Phi_i^\mu \tag{3.15}$$

for $\partial_i \in \mathfrak{h}$ and note that we could apply this map to any element of $\mathfrak{h}$ and obtain a representation of it in the module $(\mathcal{C}_\hbar)^3$. Clearly the right module generated by the image of φ has the aforementioned $\mathcal{X}(\mathcal{C}_\hbar)$ as its submodule and will be called the module of vector fields on our embedded non-commutative manifold, denoted by $T\Sigma$. We state some useful definitions from [45]

Definition 3.3. *An embedded non-commutative manifold Σ is called* ***regular*** *if $(T\Sigma,\ h)$ is a regular hermitian module, where h is a hermitian form from definition 1.15.*

In view of some propositions and theorems from [45] we aim to demonstrate that $T\Sigma$ is regular by asserting the existence of an hermitian form h such that $e_\mu h^{\mu\nu} h_{\nu\beta} = e_\beta$ which implies that should exist a hermitian connection on $(T\Sigma,\ h)$. In the following section we will define the dual-module structure and a hermitian form that will play the role of the metric tensor in our setting.

3.2 The non-commutative metric

Using the algebraic gadgets introduced in the first chapter we will define a hermitian form that plays the role of the quantum AdS_2 metric. Firstly let M^* be the dual module of $M = (\mathcal{C}_\hbar)^3$. For a canonically right $\mathcal{C}_\hbar$-module M its dual is a canonically left $\mathcal{C}_\hbar$-module M by means of the following property

$$(a \cdot W)[mb] = aW[m]b \ ,$$

where $a, b \in \mathcal{C}_\hbar$, $W \in M^*$, the element inside the square bracket refer to the functional property of W over an element $m \in M$ and the multiplication by juxtaposition in the right side is the usual $\mathcal{C}_\hbar$ multiplication. In our case, M is a free module of rank 3 and its dual space will also be a free module of rank 3 with a set of basis dual-vectors $\{\hat{\omega}^\mu\}$ with $\mu = \overline{0,2}$. Now we introduce a hermitian form $g : M \times M \to \mathcal{C}_\hbar$ satisfying

$$\begin{aligned} &(1)\ g(m,n) = g(n,m)^* \\ &(2)\ g(m_1 a + m_2, nb) = a^* g(m_1, n)b + g(m_2, n)b \ , \end{aligned} \tag{3.16}$$

for $a, b \in \mathcal{C}_\hbar$ and $m, m_1, m_2, n \in M$. In order to have a concrete realisation of g we must determine its coefficients $g_{\mu\nu}$ and we will do it as follows

$$g_{\mu\nu} = g(\hat{e}_\mu, \hat{e}_\nu) = \eta_{\mu\nu} \mathbb{1} \ .$$

With this we have our hermitian form g as

$$g(U, V) = g(\hat{e}_\mu U^\mu\ , \hat{e}_\nu V^\nu) = (U^\mu)^* \eta_{\mu\nu} V^\nu = (U^\mu)^* V_\mu \ , \tag{3.17}$$

for $U, V \in M$ (not to be confused with the coordinates U, V from the last section). Let $\hat{g} : M \to M^*$ be an associate map given by

$$\hat{g}(m)[n] = g(m, n) \ ,$$

for $m, n \in M$. We construct $\hat{g}$ explicitly by defining

$$\hat{g}(U) = \hat{g}(\hat{e}_\mu U^\mu) = (U^\mu)^* \eta_{\mu\nu} \hat{\omega}^\nu \ . \tag{3.18}$$

Following the construction done in [45] we can use the inverse of $\hat{g}$ to define g^{-1}. First, define the inverse of $\hat{g}$ as

$$\hat{g}^{-1}(W) = \hat{g}^{-1}(W_\mu \hat{\omega}^\mu) = \hat{e}_\mu \eta^{\mu\nu} (W_\nu)^* \ ,$$

for $W \in M^*$. Using the properties of the involution one could easily verify that the definition above yields the correct result, by direct inspection we get to the following

$$\hat{g}^{-1}\Big(\hat{g}(\hat{e}_\mu U^\mu)\Big) = \hat{g}^{-1}\Big((U^\mu)^*\eta_{\mu\nu}\hat{\omega}^\nu\Big) = \hat{e}_\rho\eta^{\rho\nu}\eta_{\mu\nu}(U^\mu)^{**} = \hat{e}_\rho\delta^\rho_\mu U^\mu = U \ ,$$

for $U = \hat{e}_\mu U^\mu$. Now we define $g^{-1} : M^* \times M^* \to \mathcal{C}_\hbar$ as

$$g^{-1}(W,T) = W\Big(\hat{g}^{-1}(T)\Big) = W\Big(\hat{e}_\mu\eta^{\mu\nu}(T_\nu)^*\Big) = W_\rho\hat{\omega}^\rho(\hat{e}_\mu)\eta^{\mu\nu}(T_\nu)^* = W_\mu\eta^{\mu\nu}(T_\nu)^* \ , \tag{3.19}$$

where we used the dual basis property $\hat{\omega}^\mu(\hat{e}_\nu) = \delta^\mu_\nu$. One can easily check that g^{-1} is a hermitian form on the left module M and on the right module M^*. We can also calculate g explicitly as a function of the local coordinates U and V by direct substitution of the basis elements $\Phi_i \in \mathcal{X}(\mathcal{C}_\hbar)$ of the last subsection, we get to the following after some straightforward calculations

$$g(\Phi_U, \Phi_V) = -V - \frac{i\hbar}{2}\mathbb{1} \ , \tag{3.20}$$

$$g(\Phi_V, \Phi_U) = -V + \frac{i\hbar}{2}\mathbb{1} \ , \tag{3.21}$$

$$g(\Phi_V, \Phi_V) = \mathbb{1} \ , \tag{3.22}$$

$$g(\Phi_U, \Phi_U) = V^2 + \ell_0^2\mathbb{1}, \tag{3.23}$$

in terms of coordinates u, v comparing this with the commutative metric tensor induced on the surface, which is given by

$$ds^2 = dx^\mu dx_\mu = (\ell_0^2 + v^2)du^2 + dv^2 - 2vdudv \ ,$$

shows us that only cross terms of the non-commutative analogue of the metric receive a non-commutative correction. In the last calculation we used the vector fields defined in (3.2). One can also verify that g and g^{-1} are in fact Hermitian if we treat g as a classical metric tensor since it only depends on functions of V and the operations can be carried as in the commutative case. In the following subsection we will introduce a new set of local coordinates in which the metric takes a simple form, enabling us to write the non-commutative covariant derivatives in a compact and simple way.

3.3 The Fefferman-Graham coordinates

Now we introduce the last set of coordinates we will analyze, they are the non-commutative analogues to the FG coordinates for AdS_2, and they are obtained for the commutative case by making

$$z = e^{-u} \qquad \text{and} \qquad t = \frac{1}{\ell_0}ve^{-u} \quad \text{with} \quad \{z,t\} = \frac{z^2}{\ell_0} \ , \tag{3.24}$$

which gives a new set of derivations

$$\partial_u = -z\partial_z - t\partial_t \ , \qquad \text{and} \qquad \partial_v = \frac{z}{\ell_0}\partial_t \ . \tag{3.25}$$

From now on we will denote

$$R = Y \quad , \; Z = Y^{-1} = R^{-1} \; , \qquad \text{and} \qquad T = \frac{1}{2\ell_0}\Big(VY^{-1} + Y^{-1}V\Big) \tag{3.26}$$

and assume that the ideal generated by the relations (3.3) also has a new set of relations

$$TZ = ZT + \frac{i\hbar}{\ell_0}Z^2 \; , \; ZR \quad = \mathbb{1} \; , \; RT = TR + \frac{i\hbar}{\ell_0}\mathbb{1} \; . \tag{3.27}$$

Using this new set of generators, we define the derivatives under respect to R and T

$$\begin{aligned} \partial_T(A) &= -\frac{\ell_0}{i\hbar}[A, R], \\ \partial_R(A) &= \frac{\ell_0}{i\hbar}[A, T] \; . \end{aligned} \tag{3.28}$$

In order to verify the consistency of the definition above, one should show that the set of derivatives respects the ideal generated by the relations (3.27), from (3.27) and (3.28), one immediately has

$$\begin{aligned} \partial_T(T) &= \mathbb{1} \; , \qquad \partial_T(R) = 0 \; , \\ \partial_R(T) &= 0 \; , \qquad \partial_R(R) = \mathbb{1} \; . \end{aligned} \tag{3.29}$$

Checking the relations defining the ideal we can get, for instance

$$\begin{aligned} &\partial_T\Big(TZ - ZT - \frac{i\hbar}{\ell_0}Z^2\Big) = Z - Z = 0 \; , \\ &\partial_R\Big(TZ - ZT - \frac{i\hbar}{\ell_0}Z^2\Big) = Z[Z,T] + [Z,T]Z + \frac{2i\hbar}{\ell_0}Z^3 = -\frac{2i\hbar}{\ell_0}Z^3 + \frac{2i\hbar}{\ell_0}Z^3 = 0 \; , \end{aligned}$$

and the other relations can be easily verified. It is also easy to see that these derivatives are indeed Hermitian. In order to write the coordinates X^μ of the $ncAdS_2$ in the coordinates R and T we first note that

$$V = \frac{\ell_0}{2}\Big(TR + RT\Big) \; ,$$

and we use it to construct

$$X^0 = -V = -\frac{\ell_0}{2}\Big(TR + RT\Big) \; . \tag{3.30}$$

Now taking the previously constructed X^1 and X^2 in coordinates U, V one can obtain the embedding in terms of the new coordinates. Firstly note that

$$\begin{aligned} VY^{-1}V &= \frac{\ell_0^2}{4}\Big(T^2R + TRT + RTZTR + RT^2\Big) \; , \\ RTZTR &= \frac{\hbar^2}{\ell_0^2}Z + \frac{i\hbar}{\ell_0}T + T^2R = \frac{\hbar^2}{\ell_0^2}Z + TRT \; . \end{aligned}$$

With these results, one can show that

$$-\frac{1}{2\ell_0}VY^{-1}V = -\frac{1}{2\ell_0}\Big(\ell_0^2 TRT + \frac{\hbar^2}{4}Z\Big) \; ,$$

where we used

$$T^2R = TRT - \frac{i\hbar}{\ell_0}T \qquad \text{and} \qquad RT^2 = TRT + \frac{i\hbar}{\ell_0}T \; .$$

Now we apply these results on the coordinates X^1 and X^2, which gives

$$\begin{aligned} X^0 &= -\frac{\ell_0}{2}\Big(TR + RT\Big) \ , \\ X^1 &= -\frac{\ell_0}{2}\Big(TRT + \Big(1 + \frac{\hbar^2}{4\ell_0^2}\Big)Z - R\Big) \ , \\ X^2 &= -\frac{\ell_0}{2}\Big(TRT + \Big(1 + \frac{\hbar^2}{4\ell_0^2}\Big)Z + R\Big) \ . \end{aligned} \tag{3.31}$$

We can now calculate the vector fields on the free right module $(\hat{\mathcal{C}}_\hbar)^3$ generated by the derivatives of $\overrightarrow{X}$ with respect to R and T. Applying the derivative operators defined in (3.28) in the coordinates (3.31) we get to the following

$$\begin{aligned} \Phi^0_T &= -\ell_0 R \ , \\ \Phi^1_T &= \Phi^2_T = -\frac{\ell_0}{2}\Big(TR + RT\Big) \ , \\ \Phi^0_R &= -\ell_0 T \ , \\ \Phi^1_R &= -\frac{\ell_0}{2}\left(-\left(1 + \frac{\hbar^2}{4\ell_0^2}\right)Z^2 + T^2 - \mathbb{1}\right) \ , \\ \Phi^2_R &= -\frac{\ell_0}{2}\left(-\left(1 + \frac{\hbar^2}{4\ell_0^2}\right)Z^2 + T^2 + \mathbb{1}\right) \ . \end{aligned} \tag{3.32}$$

To further construct the geometrical objects over the non-commutative AdS_2 one could analyse the full set of coordinates by adding the elements $W \text{and} J$ to the set of generators, with the set of commutation relations satisfying

$$\begin{aligned} [T, Z] &= \frac{i\hbar}{\ell_0}Z^2 \ , \quad [R, T] = \frac{i\hbar}{\ell_0}\mathbb{1} \ , \\ [W, R] &= \frac{i\hbar}{\ell_0}W^2 \ , \quad [Z, W] = \frac{i\hbar}{\ell_0}WZ^2W \ , \end{aligned} \tag{3.33}$$

with the inverse[35] elements $R = Z^{-1}$ and $W = T^{-1}$ as well as one could construct J, that is canonically conjugate to Z satisfying

$$[J, Z] = \frac{i\hbar}{\ell_0}\mathbb{1} \ , \tag{3.34}$$

for $J = RTR$. We will not follow this path because, in order to define a proper affine connection, we would have to use the Lie algebra of derivations $\mathfrak{g}$ for our full set of coordinates, which would lead to some ambiguities, in order to avoid this we will only use the complex algebra $\mathbb{C}[Z, R, T]$ modulo the ideals defined by the relations in (3.27) to construct $\mathcal{C}_\hbar$. Clearly with the full set of local coordinates we can define the set of inner derivations $\mathcal{I} \subset der(\mathcal{C}_\hbar)$ for some function $A \in \mathcal{C}_\hbar$ as follows:

$$\begin{aligned} \partial_T(A) &= -\frac{\ell_0}{i\hbar}[A, R] \ , \qquad \partial_R(A) = \frac{\ell_0}{i\hbar}[A, T] \ , \\ \partial_Z(A) &= -\frac{\ell_0}{i\hbar}[A, J] \ , \qquad \partial_J(A) = \frac{\ell_0}{i\hbar}\ [A, Z], \end{aligned} \tag{3.35}$$

note that the relation for the derivative with respect to Z follows from the derivative of a function with respect to the inverse variable, for instance $\partial_A F(A^{-1}) = -A^{-1}F'(A^{-1})A^{-1}$. Recovering the

[35]In order to have well defined inverse elements in the algebra, we should make sure to add to the ideal (3.27) the relations $AB - \mathbb{1} = 0$, for any pair of inverse elements A and B.

parametric equation for the $ncAdS_2$ written with respect to R and T we can write it compactly as

$$\overrightarrow{X} = -\frac{\ell_0}{2}\Big(TR + RT,\ TRT + k^2 Z - R,\ TRT + k^2 Z + R\Big) \ , \tag{3.36}$$

for $k = \sqrt{1 + \frac{\hbar^2}{4\ell_0^2}}$, and denote the tangent vectors to the $ncAdS_2$ surface in a similar way

$$\begin{aligned}
\hat{\Phi}_T &= \hat{e}_\alpha \Phi_T^\alpha = -\frac{\ell_0}{2}\Big(2R,\ TR + RT,\ TR + RT\Big) \ , \\
\hat{\Phi}_R &= \hat{e}_\alpha \Phi_R^\alpha = -\frac{\ell_0}{2}\Big(2T,\ -k^2 Z^2 + T^2 - \mathbb{1},\ -k^2 Z^2 + T^2 + \mathbb{1}\Big) \ .
\end{aligned} \tag{3.37}$$

Using the formalism above we are able to directly calculate from the definitions established earlier the matrix elements of g with respect to the coordinates Z, T, R:

$$\begin{aligned}
&g(\hat{\Phi}_R,\ \hat{\Phi}_R) = k^2 \ell_0^2 Z^2, \\
&g(\hat{\Phi}_R,\ \hat{\Phi}_T) = g(\hat{\Phi}_T,\ \hat{\Phi}_R)^* = -\frac{i\hbar\ell_0}{2}\mathbb{1} \ , \\
&g(\hat{\Phi}_T,\ \hat{\Phi}_T) = \ell_0^2 R^2.
\end{aligned} \tag{3.38}$$

This shows explicitly that our metric isn't only symmetric but Hermitian, as expected. When we constructed the metric for the coordinates U and V, as well as to the other set R, Z and T, it was easy to note that g was defined for a canonically right $\mathcal{C}_\hbar$-module M for the first and the second entries. We now introduce the elements of the opposite algebra $\mathcal{C}_\hbar^{op}$ in which the module construction implied earlier would have a swap from right to left and vice versa. The sided multiplication properties of some arbitrary elements $A, B \in \mathcal{C}_\hbar$ are defined by

$$\begin{aligned}
A^l(B) &= AB \ , \qquad A^l \in \mathcal{C}_\hbar \ , \\
A^r(B) &= BA \ , \qquad A^r \in \mathcal{C}_\hbar^{op} \ .
\end{aligned} \tag{3.39}$$

One can also note that $(BC)^r F = FBC = C^r(FB) = C^r B^r F$ which implies that the opposite algebra acts as an anti-involution and for real elements it is equivalent to the usual involution. In order to define the relations between the algebra and its opposite in a more rigorous way we will use the notion of enveloping algebra that is introduced in [47]. For any algebra $\mathcal{A}$ we define the enveloping algebra $\mathcal{A}^e$ as

$$\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{op} \ , \tag{3.40}$$

it is easy to see that a $\mathcal{A}$-bimodule M_b can be considered a left $\mathcal{A}^e$-module. There are natural homomorphisms from $\mathcal{A}$ and $\mathcal{A}^{op}$ to $\mathcal{A}^e$ if we take

$$\begin{aligned}
A^l \in \mathcal{A} &\mapsto A \otimes_{\mathbb{C}} \mathbb{1}_{\mathcal{A}^{op}} \in \mathcal{A}^e \ , \\
B^r \in \mathcal{A}^{op} &\mapsto \mathbb{1}_{\mathcal{A}} \otimes_{\mathbb{C}} B \in \mathcal{A}^e \ ,
\end{aligned}$$

where the subscript of the tensor product symbol denotes the common subfield or algebra that the product is taken over. As an simple example consider the set of derivations (3.35) written as elements of $\mathcal{A}^e$

$$\begin{aligned}
\partial_T F &= -\frac{\ell_0}{i\hbar}\Theta\Big((\mathbb{1}\otimes R - R\otimes\mathbb{1})F\Big) \ , \quad \partial_R F = \frac{\ell_0}{i\hbar}\Theta\Big((\mathbb{1}\otimes T - T\otimes\mathbb{1})F\Big) \ , \\
\partial_Z F &= -\frac{\ell_0}{i\hbar}\Theta\Big((\mathbb{1}\otimes J - J\otimes\mathbb{1})F\Big) \ , \quad \partial_J F = \frac{\ell_0}{i\hbar}\Theta\Big((\mathbb{1}\otimes Z - Z\otimes\mathbb{1})F\Big) \ .
\end{aligned} \tag{3.41}$$

where we treated $\mathcal{A}^e$ as a right $\mathcal{A}$-module and we also introduced a non-linear[36] map $\Theta : \mathcal{A}^e \to \mathcal{A}$ that takes an element of $\mathcal{A}^e$ and associates it with an element in $\mathcal{A}$ that carries the information of the specific multiplication order, for instance, using $n = A \otimes_{\mathbb{C}} B$ and $C \in \mathcal{A}$ the map do the following

$$nC = (A \otimes_{\mathbb{C}} B)C \mapsto ACB \in \mathcal{A} .$$

The differential calculus $\Omega^*(\mathcal{A})$ can be extended in a natural way to the enveloping algebra by taking

$$\Omega^*(\mathcal{A}^e) = \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^*(\mathcal{A}^{op}) = \Big(\Omega^*(\mathcal{A})\Big)^e , \tag{3.42}$$

where the exterior derivative satisfy $d(A \otimes B) = dA \otimes B + A \otimes dB$ for $A \in \mathcal{A}$ and $B \in \mathcal{A}^{op}$. Now we introduce a identification that will be useful: $(\mathbb{1} \otimes_{\mathbb{C}} \xi) \otimes_{\mathbb{C}} \eta \mapsto \eta \otimes_{\mathcal{A}} \xi$ which implies that $(\mathcal{A} \otimes_{\mathbb{C}} \Omega^1(\mathcal{A}^{op})) \otimes_{\mathcal{A}^e} M \simeq M \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ for a $\mathcal{A}$-bimodule M and with these results we can construct a suitable space to define our connections and covariant derivatives

$$\Omega^1(\mathcal{A}^e) \otimes_{\mathcal{A}^e} M = (\Omega(\mathcal{A}) \otimes_{\mathcal{A}} M) \oplus (M \otimes_{\mathcal{A}} \Omega(\mathcal{A})) . \tag{3.43}$$

We will use this space when we define the non-commutative connection in AdS_2 and the non-commutative Killing vector fields. If we define as g_r metric for the right $\mathcal{A}$-module structure and as g_l the metric for the left $\mathcal{A}$-module structure, it is easy to show that $g_l(\Phi_R, \Phi_T) = g_r(\Phi_T, \Phi_R)$ as elements of $\mathcal{A}$ and in order to obtain a symmetric g without the complex off-diagonal terms we can define $g_s := \frac{1}{2}(g_r + g_l)$ where g_s stands for the symmetric one with zero off-diagonal terms. Clearly $\det(g) = \ell_0^4 \mathbb{1}$ and since all elements of the metric are R dependent we can treat it commutatively, note also that the determinant of g_s receive a quantum correction since the off-diagonal terms cancel and we carry the $\mathcal{O}(\hbar^2)$ correction inside $k(\hbar)$. As g can act by the left over some element A of the algebra, we must guarantee that the metric action transforms correctly. If we remember that when changing the side of the action we get some extra terms with derivatives now we must impose a specific ordering to fulfill our purposes in constructing such geometrical formalism. The chosen ordering will be the symmetric one for any quadratic term in the polynomial ring which has only one dependence in the generators Z, R and T. For the odd terms or cross terms in the polynomial ring we use the symmetrization of such element. The formal definition of the symmetric mapping $\mathcal{S} \ : \ \mathcal{A} \to \mathcal{A}^e$ is the following

$$\mathcal{S}(A_1 A_2 \dots A_n) = \frac{1}{n} \sum_{\sigma(A_I)}^{n} \tau^l(A_i, \dots, A_j) \otimes \tau^r(A_k, \dots, A_l) , \tag{3.44}$$

where the τ function refer to the specific symmetrization of the A_i's in a such way that we don't get elements of the form $\prod_I A_I \otimes \mathbb{1}$ or $\mathbb{1} \otimes \prod_I A_I$ for the quadratic terms also obeying the symmetric property for the odd ones, and this specific symmetrization guarantees the hermiticity of a general element obtained from the action of $\mathcal{S}$ over $\mathcal{A}$. The function σ is just the set of possible permutations

[36] This map distributes over addition but in general $\Theta\Big((mB)C\Big) \neq \Theta(mB)C$ for $B, C \in \mathcal{A}$ and $m \in \mathcal{A}^e$.

in some index set I. For instance we apply this mapping in some terms that will appear in the following sections

$$\mathcal{S}(TR) = \frac{1}{2}\Big((TR)^l + (TR)^r\Big) = \frac{1}{2}(TR \otimes \mathbb{1} + \mathbb{1} \otimes RT) \ , \qquad \mathcal{S}(T^2) = T \otimes T \ .$$

Applying this map to the components of g_s we get

$$\begin{aligned} \mathcal{S}\Big(g_s(\Phi_R,\ \Phi_R)\Big) &= k^2\ell_0^2 Z^l Z^r = k^2\ell_0^2(Z \otimes Z) \ , \\ \mathcal{S}\Big(g_s(\Phi_R,\ \Phi_T)\Big) &= \mathcal{S}\Big(g_s(\Phi_T,\ \Phi_R)^*\Big) = 0 \ , \\ \mathcal{S}\Big(g_s(\Phi_T,\ \Phi_T)\Big) &= \ell_0^2 R^r R^l = \ell_0^2(R \otimes R) \ . \end{aligned} \tag{3.45}$$

For g we can note also that most of the coefficients got a non-commutative correction and we recover the usual AdS^2 metric when we take $\hbar \to 0$. Since we defined the field of functions with inverses, we can also denote the metric and its inverse in the following way[37]

$$g(\Phi_a, \Phi_b) = \begin{pmatrix} \ell_0^2 k^2 Z^2 & -\frac{i\hbar\ell_0}{2} \\ \\ \frac{i\hbar\ell_0}{2} & \ell_0^2 R^2 \end{pmatrix} , \tag{3.46}$$

and

$$g^{-1}(\Phi^a, \Phi^b) = \frac{\mathbb{1}}{\ell_0^4} \begin{pmatrix} \ell_0^2 R^2 & \frac{i\hbar\ell_0}{2} \\ \\ -\frac{i\hbar\ell_0}{2} & \ell_0^2 k^2 Z^2 \end{pmatrix} . \tag{3.47}$$

It is easy to see that g_s is obtained by doing the following

$$g_s(\Phi_a, \Phi_b) = \frac{1}{2}\Big(g_r(\Phi_a, \Phi_b) + g_l(\Phi_a, \Phi_b)\Big) \ = \begin{pmatrix} \ell_0^2 k^2 Z^2 & 0 \\ \\ 0 & \ell_0^2 R^2 \end{pmatrix} . \tag{3.48}$$

Now we are ready to construct the covariant derivative and the connection coefficients for the Quantum AdS_2.

3.4 The Levi-Civita Connection

In this section we will derive the non-commutative connection using two strategies that should yield the same result. Firstly, we will introduce some definitions and from now on we denote as $\mathfrak{g}$ as the complex Lie algebra generated inner derivations $\partial \in \mathrm{Der}(\hat{\mathcal{C}}_\hbar)$.

[37] The precise definition of the determinant of the metric can be derived from the fact that the matrix algebra of commutative functions in $\mathcal{C}_\hbar(R)$ have the usual determinant 2-form well defined, with this property one can easily show that $\det[g] = \ell_0^4\mathbb{1}$, by applying the usual definition of determinant, since all metric coefficients are functions of R, Z and $\mathbb{1}$ and hence commutative, making the metric non-singular in the whole $ncAdS_2$.

Definition 3.4. *Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let M be a right $\mathcal{A}$-module. A* ***connection*** *on M is a map $\mathfrak{g} \times M \to M$ such that*

$$(1)\ \nabla_{\partial}(m+n) = \nabla_{\partial}m + \nabla_{\partial}n\ ,$$

$$(2)\ \nabla_{\lambda\partial_1+\beta\partial_2}m = \lambda\nabla_{\partial_1}m + \beta\nabla_{\partial_2}m\ ,$$

$$(3)\ \nabla_{\partial}(ma) = (\nabla_{\partial}m)a + m\partial(a)\ ,$$

for $m, n \in M$; $\partial, \partial_1, \partial_2 \in \mathfrak{g}$; $\lambda, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$.

Definition 3.5. *A connection ∇ on a right hermitian $\mathcal{A}$-module is called* ***hermitian*** *if*

$$\partial h(m,n) = h(\nabla_{\overline{\partial}}m, n) + h(m, \nabla_{\partial}n)\ ,$$

for all $m, n \in M$ and $\partial, \overline{\partial} \in \mathfrak{g}$. We say that an hermitian connection is ***compatible*** *with the hermitian form h.*

The notion of torsion freeness will be introduced utilizing the map from (3.15), as the following definition states

Definition 3.6. *A connection ∇ in a non-commutative embedded manifold is called* ***torsion free*** *if for the map from (3.15) the connection satisfy the following*

$$\nabla_{\partial_\alpha}\varphi(\partial_\beta) - \nabla_{\partial_\beta}\varphi(\partial_\alpha) = \varphi([\partial_\alpha, \partial_\beta])\ , \tag{3.49}$$

for all $\partial_\alpha, \partial_\beta \in \mathfrak{g}$ and if the connection is compatible with an hermitian form h we call it a ***Levi-Civita connection****.*

It is proven in [45] that given a free hermitian module with an orthogonal projection defined in it, any hermitian connection of the free module induces a hermitian connection on the corresponding projective module, which implies that every regular hermitian module has a hermitian connection and therefore, from the theorem (4.6), it is guaranteed that exists a Levi-Civita connection on every regular embedded manifold. Following this procedure we will define a projection for the $ncAdS_2$ and after this we will verify the properties of this projection and use it to construct our connection for the ambient coordinates and the local coordinates. It is necessary to stress out that for each case we will use a different Lie algebra of derivations $\mathfrak{g}$ which will result in a different construction for the connection coefficients. As a final step we will compare both results with its commutative counterparts.

3.4.1 Connection in ambient coordinates

One of our objectives in this chapter is to find an analogue to the non-commutative Ricci scalar. If we remember the commutative case, following [44] one can find that the Poisson structure introduced in the 2-Riemannian manifold (Σ, g) embedded in the m-dimensional Riemannian manifold $(M, \overline{g})$, allow us to construct[38] a pair of orthogonal projections $\Pi^i_j = \delta^i_j - P^i_j$ for $P^i_j = \{x^i, x^k\}\overline{g}_{kl}\{x^m, x^l\}\overline{g}_{mj}$ satisfying $P, \Pi \in \mathrm{End}(T_pM)$. In this setting the Ricci scalar $\mathcal{R}$ is obtained from the equation $\mathcal{R} = P^{jl}P^{ik}R_{ijkl}$, we will find two non-commutative analogues of this equation. We start by constructing the endomorphisms P and Π directly from the commutative definition

$$\begin{aligned} P^\mu{}_\nu &= -\frac{1}{\ell_0^2\hbar^2}[X_\nu, X_\gamma][X^\mu, X^\gamma] = -\frac{1}{\ell_0^2\hbar^2}\Big(X_\nu X_\gamma - X_\gamma X_\nu\Big)\Big(X^\mu X^\gamma - X^\gamma X^\mu\Big) \\ &= -\frac{1}{\ell_0^2}\Big(\delta^\mu_\nu\delta^\sigma_\rho - \delta^\mu_\rho\delta^\sigma_\nu\Big)X^\rho X_\sigma = \delta^\mu_\nu \mathbb{1} + \frac{1}{\ell_0^2}X^\mu X_\nu \ , \end{aligned}$$

now we must define them properly from the strategy employed in the commutative case and show that they are indeed projectors, we do this in the following

Proposition 3.4. The endomorphism Π of $(\mathcal{C}_\hbar)^3$ is a projector and it is defined as

$$\Pi(U) = \hat{e}_\mu \Pi^\mu{}_\nu U^\nu = -\frac{1}{\ell_0^2}\hat{e}_\mu X^\mu X_\nu U^\nu \ , \tag{3.50}$$

for $U \in (\mathcal{C}_\hbar)^3$ and X^μ the embedding coordinates defined in (3.7).

Proof: We can easily prove that Π is indeed a projector by applying it twice over $U \in (\mathcal{C}_\hbar)^3$

$$\Pi^2(U) = \hat{e}_\mu \Pi^\mu{}_\nu \Pi^\nu{}_\rho U^\rho = \frac{1}{\ell_0^4}\hat{e}_\mu X^\mu (X_\nu X^\nu) X_\rho U^\rho = -\frac{1}{\ell_0^2}\hat{e}_\mu X^\mu X_\rho U^\rho = \hat{e}_\mu \Pi^\mu{}_\rho U^\rho = \Pi(U) \ ,$$

where we used (3.7). □

We can construct Π explicitly as a formal matrix

$$(\Pi^\mu{}_\nu) = -\frac{1}{\ell_0^2}\begin{pmatrix} (X^0)^2 & X^0X^1 & -X^0X^2 \\ X^1X^0 & (X^1)^2 & -X^1X^2 \\ X^2X^0 & X^2X^1 & -(X^2)^2 \end{pmatrix} \ . \tag{3.51}$$

It is easy to see that $\Pi^\mu{}_\nu = (\Pi_\nu{}^\mu)^*$ and we can apply the projector over the basis elements $\{\hat{e}_\mu\}$ which gives $\Pi(\hat{e}_\mu) = \overrightarrow{X}X_\mu$, hinting to the fact that the projection yields as image a rank 1 module. In the classical geometry the complementary projection of Π, that is defined as $P = \mathbb{1} - \Pi$, characterizes the module of sections of the tangent bundle, we will refer to it as $T\mathcal{C}_\hbar = P\Big((\mathcal{C}_\hbar)^3\Big)$ and this allow us to define a finitely generated projective module given by

$$(\mathcal{C}_\hbar)^3 = T\mathcal{C}_\hbar \oplus \mathcal{N}_\hbar \ ,$$

where $\mathcal{N}_\hbar = \Pi\Big((\mathcal{C}_\hbar)^3\Big)$. One could try to identify $\mathcal{X}(\mathcal{C}_\hbar)$ with $T\mathcal{C}_\hbar$ but we should verify if $T\mathcal{C}_\hbar$ is

[38] In the following construction we are considering that $\sqrt{g}/\rho^2 = 1$. The full construction can be found in [44] and one can find a little mistake in the indices at the definition of $D^i{}_j$. Here in my construction we consider the corrected case.

spanned by Φ_i given in (3.37). We can write it explicitly as

$$(P^\mu{}_\nu) = \frac{1}{\ell_0^2}\begin{pmatrix} \ell_0^2\mathbb{1} + (X^0)^2 & X^0X^1 & -X^0X^2 \\ X^1X^0 & \ell_0^2\mathbb{1} + (X^1)^2 & -X^1X^2 \\ X^2X^0 & X^2X^1 & \ell_0^2\mathbb{1} - (X^2)^2 \end{pmatrix} . \tag{3.52}$$

Proposition 3.5. The endomorphisms Π and P are orthogonal projectors in the sense of (1.19) with respect to the hermitian form g from (3.18).

Proof: One easily checks that

$$g(\Pi(U), V) = -\frac{1}{\ell_0^2} g(\hat{e}_\mu X^\mu X_\nu U^\nu, \hat{e}_\rho V^\rho) = -\frac{1}{\ell_0^2}(U^\nu)^* X_\nu X^\mu V^\rho \eta_{\mu\rho} = -\frac{1}{\ell_0^2}(U^\nu)^* \eta_{\nu\lambda}(X^\lambda X_\rho V^\rho) ,$$
$$g(\Pi(U), V) = g(\hat{e}_\nu U^\nu, -\frac{1}{\ell_0^2}\hat{e}_\lambda X^\lambda X_\rho V^\rho) = g(U, \Pi(V)) ,$$

for $U,\ V \in (\mathcal{C}_\hbar)^3$ and by the right-linearity of g it is easy to show that P is also orthogonal. □

Using the proposition (2.9) from [45] it is guaranteed that $(T\mathcal{C}_\hbar, g|_{T\mathcal{C}_\hbar})$ is a regular hermitian module. It is clear that we can generate $T\mathcal{C}_\hbar$ with the vectors $\tilde{e}_\nu = \hat{e}_\mu P^\mu{}_\nu$ and the space $\mathcal{N}_\hbar$ is the equivalent of the classical normal subspace and analogously it is simple[39] to prove that $\mathcal{N}_\hbar$ is generated by $\overrightarrow{X} = (X^0, X^1, X^2)$ with $\Pi(\overrightarrow{X}) = \overrightarrow{X}$. Now we write explicitly the basis $\{\tilde{e}_\mu\}$

$$\begin{aligned} \tilde{e}_0 &= \frac{1}{\ell_0^2}\Big(\ell_0^2\mathbb{1} + (X^0)^2,\ X^1X^0,\ X^2X^0\Big) , \\ \tilde{e}_1 &= \frac{1}{\ell_0^2}\Big(X^0X^1,\ \ell_0^2\mathbb{1} + (X^1)^2,\ X^2X^1\Big) , \\ \tilde{e}_2 &= \frac{1}{\ell_0^2}\Big(-X^0X^2, -X^1X^2,\ \ell_0^2\mathbb{1} - (X^2)^2\Big) , \end{aligned} \tag{3.53}$$

Following the proposition (3.8) from [45] we will show that the non-commutative analogue of the tangent space of the hyperboloid, that can be generated by $e_0 = (0, -z, -y)$, $e_1 = (z, 0, x)$ and $e_2 = (-y, x, 0)$, receive a non-comutative correction. First we define the non-commutative elements that generate the module $T\mathcal{C}_\hbar$ by $e_\lambda = \frac{1}{\ell_0}\epsilon_\lambda{}^{\nu\rho}\tilde{e}_\nu X_\rho$, these elements carry the $\mathfrak{su}(1,1)$ symmetry which can be seen easily using the map defined in (3.15) for the embedding coordinates that obey the same Lie algebra structure, as one could also use the $\mathfrak{so}(3)$ symmetry to construct the tangent space of the classical sphere. It is obvious that $\{e_\mu\} \in T\mathcal{C}_\hbar$ since these elements are linear combinations of the projected basis $\{\tilde{e}_\mu\}$ which belongs to the tangent space by definition. Calculating $\{e_\mu\}$ explicitly we get

$$\begin{aligned} e_0 = -\frac{1}{\ell_0}\Big(\tilde{e}_2X^1 + \tilde{e}_1X^2\Big) &= -\frac{1}{\ell_0}\Big(\hat{e}_2X^1 + \hat{e}_1X^2\Big) - \frac{i\hbar}{\ell_0^3}\overrightarrow{X}X_0 , \\ e_1 = \frac{1}{\ell_0}\Big(\tilde{e}_0X^2 + \tilde{e}_2X^0\Big) &= \frac{1}{\ell_0}\Big(\hat{e}_0X^2 + \hat{e}_2X^0\Big) - \frac{i\hbar}{\ell_0^3}\overrightarrow{X}X_1 , \\ e_2 = -\frac{1}{\ell_0}\Big(\tilde{e}_0X^1 - \tilde{e}_1X^0\Big) &= -\frac{1}{\ell_0}\Big(\hat{e}_0X^1 - \hat{e}_1X^0\Big) - \frac{i\hbar}{\ell_0^3}\overrightarrow{X}X_2 , \end{aligned} \tag{3.54}$$

[39] The proof follows a similar path as is done in [45] p.12.

Proposition 3.6. The transformation rules between the sets of basis vectors $\{e_\mu\}$ and $\{\tilde{e}_\mu\}$ are

$$\tilde{e}_\mu = -\frac{1}{\ell_0}\Big(\epsilon_\mu{}^{\nu\rho} e_\nu X_\rho + i\hbar e_\mu\Big)\ , \qquad e_\mu = \frac{1}{\ell_0}\epsilon_\mu{}^{\nu\rho}\tilde{e}_\nu X_\rho\ . \tag{3.55}$$

Proof: The second relation follow from the definition of $\{e\}$. The first one can be obtained when we calculate

$$\begin{aligned}\frac{1}{\ell_0}\epsilon_\mu{}^{\nu\rho} e_\nu X_\rho = \frac{1}{\ell_0^2}\epsilon_\mu{}^{\nu\rho}(\epsilon_\nu{}^{\sigma\beta}\tilde{e}_\sigma X_\beta)X_\rho = \frac{1}{\ell_0^2}\Big(\delta^\sigma_\mu\eta^{\rho\beta} - \delta^\beta_\mu\eta^{\rho\sigma}\Big)\tilde{e}_\sigma X_\beta X_\rho \\ = \frac{1}{\ell_0^2}\Big(\tilde{e}_\mu\eta^{\rho\beta}X_\beta X_\rho - \tilde{e}_\sigma X_\mu X^\sigma\Big) = -\tilde{e}_\mu + \hat{e}_\alpha P^\alpha{}_\rho \Pi^\rho{}_\mu - \frac{i\hbar}{\ell_0^2}\epsilon_\mu{}^{\sigma\gamma}\tilde{e}_\sigma X_\gamma\ ,\end{aligned}$$

where we used the bracket of X_μ and X^ρ and by applying the orthogonality of P and Π the last expression implies that the first relation is true. □

The last proposition means that, as in the fuzzy sphere case, the tangent space for $ncAdS_2$ has 3 generators which are mapped to 2 if we take $\hbar \to 0$ in the commutative limit, as expected. In order to calculate the metric coefficients for the basis $\{e_\mu\}$ we must know its action over $\{\tilde{e}_\mu\}$ since

$$g(e_\mu, e_\nu) = \frac{1}{\ell_0^2}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\alpha\beta}X_\sigma g(\tilde{e}_\rho, \tilde{e}_\alpha)X_\beta\ ,$$

and we calculate it directly using $P_\rho{}^\gamma = \Big(\delta^\gamma_\rho + \frac{1}{\ell_0^2}X_\rho X^\gamma\Big) = \Big(P^\gamma{}_\rho\Big)^*$

$$\tilde{g}_{\rho\alpha} = g(\tilde{e}_\rho, \tilde{e}_\alpha) = P_\rho{}^\gamma\eta_{\gamma\lambda}P^\lambda{}_\alpha = P_{\rho\alpha}\ .$$

Proposition 3.7. The hermitian form g applied in the basis elements $\{e_\mu\}$ yields the respective restriction of g in $T\mathcal{C}_\hbar$ and gives as result

$$g(e_\mu,\ e_\nu) = P_{\nu\mu} - \frac{\hbar^2}{\ell_0^2}\Pi_{\mu\nu}\ . \tag{3.56}$$

satisfying $(g_{\mu\nu})^* = g_{\nu\mu}$.

Proof: By direct inspection and using

$$\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\alpha\beta} = \eta_{\mu\nu}\eta^{\rho\beta}\eta^{\alpha\sigma} - \eta_{\mu\nu}\eta^{\alpha\rho}\eta^{\sigma\beta} + \delta^\rho_\nu\delta^\alpha_\mu\eta^{\beta\sigma} - \delta^\rho_\nu\delta^\beta_\mu\eta^{\alpha\sigma} - \eta^{\rho\beta}\delta^\alpha_\mu\delta^\sigma_\nu + \eta^{\rho\alpha}\delta^\beta_\mu\delta^\sigma_\nu\ ,$$

we get the following expression for the components of g in the new basis

$$g(e_\mu, e_\nu) = \frac{1}{\ell_0^2}\Big[\eta_{\mu\nu}\Big(X^\sigma P_{\beta\sigma}X^\beta - X^\sigma P_\rho{}^\rho X_\sigma\Big) + X^\beta P_{\nu\mu}X_\beta - X^\alpha P_{\nu\alpha}X_\mu - X_\nu P_{\beta\mu}X^\beta + X_\nu P_\rho{}^\rho X_\mu\Big]\ ,$$

note that if you commute the projector with the embedding coordinates with lower indices, which gives $[X_\alpha, P_{\beta\gamma}] = \frac{i\hbar}{\ell_0^2}\Big(\epsilon_{\alpha\beta}{}^\lambda X_\lambda X_\gamma + \epsilon_{\alpha\gamma}{}^\lambda X_\beta X_\lambda\Big) = -i\hbar(\epsilon_{\alpha\beta}{}^\lambda\Pi_{\lambda\gamma} + \epsilon_{\alpha\gamma}{}^\lambda\Pi_{\beta\lambda})$, you can get some elements that simplify by the orthogonality of Π and P. We show below the explicit calculations for every term separately

$$\begin{aligned} i)\quad \frac{1}{\ell_0^2}\eta_{\mu\nu}X^\alpha P_{\beta\alpha}X^\beta &= -\eta_{\mu\nu}\Pi^{\alpha\beta}P_{\beta\alpha} + \frac{1}{\ell_0^2}X^\alpha\eta_{\mu\nu}[P_{\beta\alpha}\ , X^\beta] = \frac{i\hbar}{\ell_0^4}\eta_{\mu\nu}X^\alpha\epsilon_\alpha{}^{\beta\sigma}X_\beta X_\sigma\ , \\ &= \frac{\hbar^2}{\ell_0^4}\eta_{\mu\nu}X^\alpha X_\alpha = -\frac{\hbar^2}{\ell_0^2}\eta_{\mu\nu}\ .\end{aligned}$$

Following the last step, we can calculate each term individually as follows

$$ii)\quad -\frac{1}{\ell_0^2}X^\sigma P_\rho{}^\rho X_\sigma\eta_{\mu\nu} = -\frac{2}{\ell_0^2}\eta_{\mu\nu}X^\sigma X_\sigma = 2\eta_{\mu\nu}\ ,$$

$$\begin{aligned}
iii)\quad \frac{1}{\ell_0^2}X^\beta P_{\nu\mu}X_\beta &= \frac{1}{\ell_0^2}P_{\nu\mu}X^\beta X_\beta + \frac{1}{\ell_0^2}[X^\beta, P_{\nu\mu}]X_\beta = -P_{\nu\mu} + \frac{ih}{\ell_0^4}\Big(\epsilon^\beta{}_\nu{}^\sigma X_\sigma X_\mu + \epsilon^\beta{}_\mu{}^\sigma X_\nu X_\sigma\Big)X_\beta\ ,\\
&= -P_{\nu\mu} + \frac{i\hbar}{\ell_0^4}\bigg(\epsilon_\nu{}^{\sigma\beta}X_\sigma\Big(X_\beta X_\mu + i\hbar\epsilon_{\mu\beta}{}^\rho X_\rho\Big) + X_\nu\Big(\epsilon_\mu{}^{\sigma\beta}X_\sigma X_\beta\Big)\bigg)\ ,\\
&= -P_{\nu\mu} + \frac{\hbar^2}{\ell_0^4}\Big(2X_\nu X_\mu + \eta_{\nu\gamma}\eta^{\rho\alpha}\delta^{\gamma\sigma\beta}_{\mu\beta\alpha}X_\sigma X_\rho\Big)\ ,\\
&= -P_{\nu\mu} - \frac{2\hbar^2}{\ell_0^2}\Pi_{\nu\mu} - \frac{\hbar^2}{\ell_0^4}\Big(\eta_{\mu\nu}\eta^{\rho\sigma} - \delta^\rho_\nu\delta^\sigma_\mu\Big)X_\sigma X_\rho\ ,\\
&= -P_{\nu\mu} - \frac{\hbar^2}{\ell_0^2}\Big(2\Pi_{\nu\mu} + \Pi_{\mu\nu} - \eta_{\mu\nu}\Big)\ ,
\end{aligned}$$

$$iv)\quad -\frac{1}{\ell_0^2}X^\alpha P_{\nu\alpha}X_\mu = -\frac{1}{\ell_0^2}[X^\alpha, P_{\nu\alpha}]X_\mu = -\frac{i\hbar}{\ell_0^4}\epsilon^\alpha{}_\nu{}^\sigma X_\sigma X_\alpha X_\mu = -\frac{\hbar^2}{\ell_0^4}X_\nu X_\mu = \frac{\hbar^2}{\ell_0^2}\Pi_{\nu\mu}\ ,$$

$$v)\quad -\frac{1}{\ell_0^2}X_\nu P_{\beta\mu}X^\beta = -\frac{1}{\ell_0^2}X_\nu[P_{\beta\mu}, X^\beta] = -\frac{i\hbar}{\ell_0^4}X_\nu\epsilon_\mu{}^{\beta\sigma}X_\beta X_\sigma = -\frac{\hbar^2}{\ell_0^4}X_\nu X_\mu = \frac{\hbar^2}{\ell_0^2}\Pi_{\nu\mu}\ ,$$

$$vi)\quad \frac{1}{\ell_0^2}X_\nu P_\rho{}^\rho X_\mu = -\frac{2}{\ell_0^2}X_\nu X_\mu = 2\Pi_{\nu\mu}\ ,$$

canceling all terms and simplifying the resulting expression we prove (3.56) and $g_{\mu\nu}$ is

$$g_{\mu\nu} = 2\eta_{\mu\nu} - P_{\nu\mu} - \frac{\hbar^2}{\ell_0^2}\Pi_{\mu\nu} - 2\Pi_{\nu\mu} = P_{\nu\mu} - \frac{\hbar^2}{\ell_0^2}\Pi_{\mu\nu}\ .$$

If we use the fact[40] that $\Pi_{\mu\nu} = (\Pi_{\nu\mu})^*$ with the same holding for P it is easy to verify that

$$\Big(g(e_\mu,\ e_\nu)\Big)^* = (g_{\mu\nu})^* = g_{\nu\mu}\ \square$$

The last proposition shows that if we want an orthogonal projector in order to define the connection properly, we should make it using some linear combination of Π's and P's, but before doing it, we will show in the next proposition that exists an suitable inverse metric which is also orthogonal and can be written with respect to the projectors.

Proposition 3.8. The hermitian form g^{-1} restricted to $T\mathcal{C}_\hbar$ can be indirectly constructed and has the following form $g^{\mu\nu} = \Big(1 + \frac{\hbar^2}{\ell_0^2}\Big)\eta^{\mu\nu} + \Pi^{\nu\mu}$ satisfying the equation $e_\rho = e_\mu g^{\mu\nu}g_{\nu\rho}$ where it can be considered the inverse of the metric $g_{\mu\nu}$ in the projection space.

Proof: In order to prove the proposition, one can verify these relations using (3.55)

$$\begin{aligned}
e_\mu\Pi^{\nu\mu} &= -\frac{i\hbar}{\ell_0^2}e_\mu\epsilon^{\nu\mu\sigma}X_\sigma = \frac{i\hbar}{\ell_0}\eta^{\mu\nu}\Big(\tilde{e}_\mu + \frac{i\hbar}{\ell_0}e_\mu\Big)\ ,\\
e_\mu P^{\nu\mu} &= \Big(1 + \frac{\hbar^2}{\ell_0^2}\Big)\eta^{\mu\nu}e_\mu - \frac{i\hbar}{\ell_0}\eta^{\mu\nu}\tilde{e}_\mu\ ,\\
\tilde{e}_\mu\Pi^{\nu\mu} &= -\frac{i\hbar}{\ell_0}\eta^{\mu\nu}e_\mu\ ,\\
\tilde{e}_\mu P^{\nu\mu} &= \eta^{\mu\nu}\Big(\tilde{e}_\mu + \frac{i\hbar}{\ell_0}e_\mu\Big)\ ,
\end{aligned}\tag{3.57}$$

[40] This is a fact only for projectors with indices in same level, both up or down.

with the relations above we will show that the operator $g^{\mu\nu} = \Big(1+\frac{\hbar^2}{\ell_0^2}\Big)\eta^{\mu\nu} + \Pi^{\nu\mu}$ satisfies the following equation $e_\mu g^{\mu\nu} g_{\nu\rho} = e_\rho$. Since e_μ and $\tilde{e}_\mu$ belong to $T\mathcal{C}_\hbar$ the action of Π over them is zero and we used this fact to calculate the relations in (3.57), using this we first show that

$$e_\mu g^{\mu\nu} = e_\mu\Big(1+\frac{\hbar^2}{\ell_0^2}\Big)\eta^{\mu\nu} + \frac{i\hbar\eta^{\mu\nu}}{\ell_0}\Big(\tilde{e}_\mu + \frac{i\hbar}{\ell_0}e_\mu\Big) = e^\nu + \frac{i\hbar}{\ell_0}\tilde{e}^\nu \ ,$$

applying it to $g_{\mu\nu}$ one get

$$\Big(e^\nu + \frac{i\hbar}{\ell_0}\tilde{e}^\nu\Big)g_{\nu\rho} = \Big(1+\frac{\hbar^2}{\ell_0^2}\Big)e_\rho - \frac{i\hbar}{\ell_0}\tilde{e}_\rho + \Big(\frac{i\hbar}{\ell_0}\tilde{e}_\rho - \frac{\hbar^2}{\ell_0^2}e_\rho\Big) = e_\rho \ ,$$

as we intended to show. To show the reasoning behind this result we write explicitly the product of the restriction of the metric and its inverse

$$g^{\mu\nu}g_{\nu\rho} = \Big[\Big(1+\frac{\hbar^2}{\ell_0^2}\Big)\eta^{\mu\nu}\mathbb{1} + \Pi^{\nu\mu}\Big]\Big(P_{\rho\nu} - \frac{\hbar^2}{\ell_0^2}\Pi_{\nu\rho}\Big) \ ,$$

now we just simplify further the expression obtained and in order to help the reader to verify some intermediate steps we write the expression below which is obtained after some straightforward calculations

$$g^{\mu\nu}g_{\nu\rho} = \Big(1+\frac{\hbar^2}{\ell_0^2}\Big)P_\rho{}^\mu - \frac{\hbar^2}{\ell_0^2}\Big(1+\frac{\hbar^2}{\ell_0^2}\Big)\Pi^\mu{}_\rho + \frac{1}{\ell_0^2}[X^\mu, X_\rho] - \frac{\hbar^2}{\ell_0^2}P_\rho{}^\mu + \frac{\hbar^2}{\ell_0^2}\Big(1+\frac{\hbar^2}{\ell_0^2}\Big)\Pi^\mu{}_\rho$$

and after some simple algebra we get to the following

$$g^{\mu\nu}g_{\nu\rho} = P_\rho{}^\mu + \frac{1}{\ell_0^2}[X^\mu, X_\rho] = P^\mu{}_\rho \ ,$$

and this is expected because the projector act as an identity operator over the generators of the projected space giving trivially $e_\mu P^\mu{}_\rho = e_\rho$ as was shown previously. □

Now we turn our attention to the connection in $T\mathcal{C}_\hbar$ and verify how it compose with the projectors Π and P, the following proposition (see [45]) elucidates this.

Proposition 3.9. *Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let ∇ be a hermitian connection on the free hermitian module $(\mathcal{A}^n,\ h)$. If $P\ :\ \mathcal{A}^n \to \mathcal{A}^n$ is an orthogonal projection, then $P\circ\nabla$ is a hermitian connection on $\Big(p(\mathcal{A}^n),\ g|_{P(\mathcal{A}^n)}\Big)$.*

Now we will explicitly construct the hermitian right connection and with this goal we start by defining its coefficients

$$\nabla_{\partial_\mu}\hat{e}_\nu = \hat{e}_\rho\Gamma^\rho_{\mu\nu} \ , \tag{3.58}$$

for our case $\Gamma^\rho_{\mu\nu} \in \mathcal{C}_\hbar$ and $\hat{e}$ some basis for $(\mathcal{C}_\hbar)^3$. The action of the connection defined previously in an arbitrary element of $(\mathcal{C}_\hbar)^3$ is

$$\nabla_\mu\Big(\hat{e}_\nu U^\nu\Big) = \hat{e}_\rho\Gamma^\rho_{\mu\nu}U^\nu + \hat{e}_\nu\partial_\mu U^\nu \ , \tag{3.59}$$

for $\nabla_\mu = \nabla_{\partial_\mu}$ and $U^\nu \in \mathcal{C}_\hbar$. Using the fact that for our case all the elements of $\mathfrak{g}$ are hermitian we can apply the hermiticity condition to our connection and obtain

$$\partial_\alpha g(\hat{e}_\mu, \hat{e}_\nu) = g(\nabla_\alpha \hat{e}_\mu, \hat{e}_\nu) + g(\hat{e}_\mu, \nabla_\alpha \hat{e}_\nu) = \left(g_{\nu\rho}\Gamma^\rho_{\alpha\mu}\right)^* + g_{\mu\rho}\Gamma^\rho_{\alpha\nu} \ , \tag{3.60}$$

for the general case $g_{\mu\rho}\Gamma^\rho_{\alpha\nu}$ could have real and imaginary parts, if we denote[41] $\Im(g_{\mu\rho}\Gamma^\rho_{\alpha\nu})$ as $\gamma_{(\alpha)\mu\nu}$ it is easy to see that if use the hermiticity of g to show that $\Re(g_{\mu\rho}\Gamma^\rho_{\alpha\nu}) = \Re(g_{\nu\rho}\Gamma^\rho_{\alpha\mu}) = \frac{1}{2}\partial_\alpha g_{\mu\nu}$ one get to the following

$$\partial_\alpha g_{\mu\nu} = 2\Re(g_{\nu\rho}\Gamma^\rho_{\alpha\mu}) + i\left(\gamma_{(\alpha)\mu\nu} - \gamma^*_{(\alpha)\nu\mu}\right) \ ,$$

and in order to cancel the imaginary part we can impose that $\gamma^*_{(\alpha)\nu\mu} = \gamma_{(\alpha)\mu\nu}$ as well as define the connection coefficients using the inverse of g giving as result

$$\Gamma^\rho_{\alpha\mu} = \frac{1}{2}g^{\rho\lambda}\partial_\alpha g_{\lambda\mu} + i g^{\rho\lambda}\gamma_{(\alpha)\lambda\mu} \ . \tag{3.61}$$

Upon setting a suitable $\gamma_{(\alpha)\rho\mu} \in \mathcal{C}_\hbar$ we should have a well defined metric connection ∇^γ, where for a different choice of γ one get a different connection. Using the corollary (3.7) from [45] we can construct explicitly an hermitian connection for the tangent space $T\mathcal{C}_\hbar$ as the composition of the projector P and the connection ∇^γ as can be seen below

$$\nabla_\alpha \tilde{e}_\mu = P \circ \nabla^\gamma_\alpha(\tilde{e}_\mu) = \nabla^\gamma_\alpha\left(\hat{e}_\rho P^\rho{}_\mu\right) = \tilde{e}_\rho \partial_\alpha P^\rho{}_\mu + \tilde{e}_\rho \tilde{g}^{\rho\lambda}\left(\frac{1}{2}\partial_\alpha \tilde{g}_{\lambda\sigma} + i\gamma_{(\alpha)\lambda\sigma}\right)P^\sigma{}_\mu \ , \tag{3.62}$$

where P is the projector defined in (3.52) and metric $\tilde{g}_{\mu\nu} = g(\tilde{e}_\mu, \tilde{e}_\nu)$ which is the respective hermitian form that turns the projected space defined by $P_\mu{}^\nu$ into an hermitian module.

Proposition 3.10. *The projected connection ∇^γ acting upon the basis sets $\{e_\mu\}$ and $\{\tilde{e}_\nu\}$ of $T\mathcal{C}_\hbar$ gives*

$$\nabla_\alpha \tilde{e}_\nu = \frac{1}{\ell_0} e_\alpha X_\nu + i\tilde{e}_\rho \eta^{\rho\lambda}\left(\gamma_{(\alpha)\lambda\nu} + \frac{1}{\ell_0^2}\gamma_{(\alpha)\lambda\sigma}X^\sigma X_\nu\right) \ ,$$
$$\nabla_\alpha e_\mu = \frac{1}{\ell_0}\tilde{e}_\alpha X_\mu + \frac{i}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\tilde{e}_\beta \gamma_{(\alpha)}{}^\beta{}_\lambda X_\rho + \frac{i\hbar}{\ell_0^3}\tilde{e}_\beta\left(i\gamma_{(\alpha)}{}^{\beta\sigma} + \epsilon_\alpha{}^{\beta\sigma}\right)X_\sigma X_\mu \ ,$$

for e and $\tilde{e}$ from (3.55).

Proof: We start calculating the first term from the upper expression

$$\tilde{e}_\rho \partial_\alpha P^\rho{}_\mu = \frac{\tilde{e}_\rho}{i\ell_0^2\hbar}\left(X^\rho[X_\alpha, X_\mu] + [X_\alpha, X^\rho]X_\mu\right) = \frac{1}{\ell_0^2}\tilde{e}_\rho \epsilon_\alpha{}^{\rho\sigma}X_\sigma X_\mu = \frac{1}{\ell_0}e_\alpha X_\mu \ ,$$

where we used (3.55) and the fact that

$$\tilde{e}_\rho X^\rho = \hat{e}_\beta\left(\delta^\beta_\rho + \frac{1}{\ell_0^2}X^\beta X_\rho\right)X^\rho = \hat{e}(X^\beta - X^\beta) = 0 \ ,$$

[41]The index (α) inside parenthesis is to emphasize that it carry different symmetry properties in comparison to the other pair of indices.

simillarly one can show that the same is valid for $\{e_\mu\}$. Applying $\tilde{g}^{\mu\nu} = P^{\mu\nu}$ and $\tilde{g}_{\mu\nu} = P_{\mu\nu}$ one get to the following expression

$$\begin{aligned}
\nabla^\gamma_\alpha \tilde{e}_\mu &= \frac{1}{\ell_0} e_\alpha X_\mu + \tilde{e}_\rho P^{\rho\lambda}\Big(\frac{1}{2\ell_0^2}(\epsilon_{\alpha\lambda}{}^\theta X_\theta X_\sigma + \epsilon_{\alpha\sigma}{}^\theta X_\lambda X_\theta) + i\gamma_{(\alpha)\lambda\sigma}\Big)P^\sigma{}_\mu \ , \\
&= \frac{1}{\ell_0} e_\alpha X_\mu + \Big(-\frac{1}{2}\eta^{\lambda\rho}\tilde{e}_\rho \epsilon_{\alpha\lambda}{}^\theta \Pi_{\theta\sigma} + i\tilde{e}_\rho \gamma_{(\alpha)}{}^\rho{}_\sigma\Big)\Big(\delta^\sigma_\mu - \Pi^\sigma{}_\mu\Big) \ , \\
&= \frac{1}{\ell_0} e_\alpha X_\mu + i\tilde{e}_\rho\Big(\gamma_{(\alpha)}{}^\rho{}_\mu - \gamma_{(\alpha)}{}^\rho{}_\sigma \Pi^\sigma{}_\mu\Big) \ ,
\end{aligned}$$

using that $\tilde{e}_\rho P^{\rho\lambda} = \tilde{e}_\rho \eta^{\rho\lambda}$ and the orthogonality of Π and P. To calculate the remaining expression we must use the result of the first one, we start by writing the full expression as

$$\begin{aligned}
\nabla^\gamma_\alpha e_\mu = \nabla^\gamma_\alpha\Big(\frac{1}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\tilde{e}_\lambda X_\rho\Big) &= \frac{1}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\Big[(\nabla^\gamma_\alpha \tilde{e}_\lambda)X_\rho + \tilde{e}_\lambda \partial_\alpha X_\rho\Big] \ , \\
&= \frac{1}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\Big[\Big[\frac{1}{\ell_0} e_\alpha X_\lambda + i\tilde{e}_\theta \eta^{\theta\beta}\Big(\gamma_{(\alpha)\beta\lambda} - \gamma_{(\alpha)\beta\sigma}\Pi^\sigma{}_\lambda\Big)\Big]X_\rho + \tilde{e}_\lambda \epsilon_{\alpha\rho}{}^\sigma X_\sigma\Big] \ , \\
&= -\frac{1}{\ell_0}\tilde{e}_\alpha X_\mu - \frac{i\hbar\tilde{e}_\beta}{\ell_0}\epsilon_\alpha{}^{\beta\theta}\Pi_{\theta\mu} + \frac{i\tilde{e}_\beta}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\gamma_{(\alpha)}{}^\beta{}_\sigma X_\rho - i\tilde{e}_\beta \frac{i\hbar}{\ell_0}\gamma_{(\alpha)}{}^\beta{}_\sigma \Pi^\sigma{}_\mu \ , \\
&= -\frac{1}{\ell_0}\tilde{e}_\alpha X_\mu + \frac{i\tilde{e}_\beta}{\ell_0}\epsilon_\mu{}^{\lambda\rho}\gamma_{(\alpha)}{}^\beta{}_\lambda X_\rho - \frac{i\hbar}{\ell_0}\tilde{e}_\beta\Big(i\gamma_{(\alpha)}{}^{\beta\sigma} + \epsilon_\alpha{}^{\beta\sigma}\Big)\Pi_{\sigma\mu} \ ,
\end{aligned}$$

which finishes the proof. □

Using the results from the last proposition, we have two natural choices for $\gamma_{(\alpha)\beta\sigma}$ in order to simplify the expressions obtained. The first choice is the trivial $\gamma_{(\alpha)\beta\sigma} = 0$ and the second is $\gamma_{(\alpha)\beta\sigma} = i\epsilon_{\alpha\beta\sigma}$ which clearly satisfies $(\gamma_{(\alpha)\beta\sigma})^* = -i\epsilon_{\alpha\beta\sigma} = i\epsilon_{\alpha\sigma\beta} = \gamma_{(\alpha)\sigma\beta}$ in order to fulfill the construction employed earlier. Setting $\gamma = 0$ and acting upon an arbitrary $U = \hat{e}_\nu U^\nu \in (\mathcal{C}_\hbar)^3$ we get to the following

$$\nabla^0_\alpha\Big(\hat{e}_\nu U^\nu\Big) = \hat{e}_\rho \frac{1}{2}\Big(\frac{i\hbar}{\ell_0^4}X^\rho X_\alpha X_\nu - \epsilon_\alpha{}^{\rho\theta}\Pi_{\theta\nu}\Big)U^\nu + \hat{e}_\nu \partial_\alpha U^\nu \ , \tag{3.63}$$

where we substituted (3.61) in (3.59) and obtained for $\gamma = 0$

$$\Gamma^\rho_{\alpha\nu} = \frac{1}{2}P^{\rho\lambda}(\partial_\alpha P_{\lambda\nu}) = -\frac{1}{2}\Big(\epsilon_\alpha{}^{\rho\theta}\Pi_{\theta\nu} - \frac{i\hbar}{\ell_0^4}X^\rho X_\alpha X_\nu\Big) \ .$$

Doing the same calculation for $\gamma_{(\alpha)\beta\sigma} = i\epsilon_{\alpha\beta\sigma}$ we obtain

$$\nabla^\epsilon_\alpha\Big(\hat{e}_\nu U^\nu\Big) = \hat{e}_\rho \frac{1}{2}\Big[\Big(\frac{i\hbar}{\ell_0^4}X^\rho X_\alpha X_\nu - \epsilon_\alpha{}^{\rho\theta}\Pi_{\theta\nu}\Big) - P^{\rho\lambda}\epsilon_{\alpha\lambda\nu}\Big] + \hat{e}_\nu \partial_\alpha U^\nu \ ,$$

with

$$\Gamma^\rho_{\alpha\nu} = \frac{1}{2}P^{\rho\lambda}(\partial_\alpha P_{\lambda\nu}) + iP^{\rho\lambda}\gamma_{(\alpha)\lambda\nu} = -\frac{1}{2}\Big(\epsilon_\alpha{}^{\rho\theta}\Pi_{\theta\nu} - \frac{i\hbar}{\ell_0^4}X^\rho X_\alpha X_\nu\Big) \ - P^{\rho\lambda}\epsilon_{\alpha\lambda\nu} \ .$$

In the upcoming section of this work, we will apply a similar procedure to identify a well-suited connection for local coordinates. Additionally, our aim is to establish correlations between the results obtained in this section and the new ones, comparing the equivalence of the connections. Following this, we will utilize the definition of an embedded non-commutative manifold and leverage findings from [45] to formulate our Levi-Civita connection. The primary objective is to define the curvature tensor for both sets of coordinates and calculate the Ricci scalar, subsequently comparing the results from these two approaches.

3.4.2 Connection in local coordinates

Building on the groundwork explored in the previous section, our focus now shifts towards constructing a connection for the space $\mathcal{X}(\mathcal{C}_\hbar)$. This connection will be expressed with respect to the basis Φ_i^α utilizing the hermitian right metric constructed in (3.46). To achieve this, we commence defining a set of arbitrary Christoffel symbols that will properly characterize the connection. Firstly we define the action of the connection over a right $\mathcal{C}_\hbar$-module explicitly for a element $A \in (\mathcal{C}_\hbar)^3$ such that $A = \hat{e}_\alpha A^\alpha$ for $A^\alpha = \Phi_i^\alpha A^i$, with $A^i \in \mathcal{C}_\hbar$ and $\hat{e}_\alpha \Phi_i^\alpha = \hat{\Phi}_i$ a basis of $\mathcal{X}(\hat{\mathcal{C}}_\hbar)$

$$\nabla_{\partial_a} A = \nabla_a(\hat{e}_\alpha \Phi_i^\alpha A^i) = \hat{e}_\alpha \Phi_i^\alpha \left(\partial_a A^i + \Gamma_{aj}^i A^j\right) \ , \tag{3.64}$$

where we used the image of the homeomorphism φ from (3.15) in order to define properly the action of the connection over the vector $\hat{e}_\alpha \Phi_i^\alpha$ as our connection coefficients. Returning to the coordinates R, T and Z, we will try to construct the relations for Γ_{bc}^a following the usual commutative construction. We will take for granted the following properties for ∇ assuming it to be $i)$ metric compatible and $ii)$ torsion-free which is a reasonable ansatz for the structure of the connection coefficients. These properties can be translated to the following equations

$$\begin{aligned} i)\ \partial_i g(\Phi_a,\ \Phi_b) &= g(\nabla_i \Phi_a,\ \Phi_b) + g(\Phi_a,\ \nabla_i \Phi_b) \ , \\ ii)\ \nabla_{\partial_\alpha}\varphi(\partial_\beta) - \nabla_{\partial_\beta}\varphi(\partial_\alpha) &= \varphi([\partial_\alpha, \partial_\beta]) \ , \end{aligned} \tag{3.65}$$

as we will see below, some of these properties will imply a different structure for the connection coefficients. Using the map φ we can calculate the hermitian form g_{ab}

$$g_{ab} = g(\varphi(\partial_a), \varphi(\partial_b)) = g(\hat{e}_\alpha \Phi_a^\alpha, \hat{e}_\beta \Phi_b^\beta) = g\Big(\hat{e}_\alpha(\partial_a X^\alpha), \hat{e}_\beta(\partial_b X^\beta)\Big) = (\partial_a X^\alpha)^* g(\hat{e}_\alpha, \hat{e}_\beta)(\partial_b X^\beta) \ ,$$

for $\overrightarrow{X} = \hat{e}_\alpha X^\alpha$ being the ambient coordinates of the $ncAdS_2$. Following the last construction one can use the hermiticity of g_{ab} to construct the connection coefficients directly from the action of the elements of $\mathfrak{g}$ in the hermitian form g as follows

$$\partial_a g(\Phi_b, \Phi_c) = g(\Phi_b,\ \Phi_d)\Gamma_{ac}^d + \Big(g(\Phi_c,\ \Phi_d)\Gamma_{ab}^d\Big)^* \ ,$$

in the following calculations we will denote $g(\varphi(\partial_a), \varphi(\partial_b))$ as g_{ab}. Using the hermiticity of g and defining $\tilde{\Gamma}_{(a)ij} = g_{ik}\Gamma_{aj}^k = \frac{1}{2}\partial_a g_{ij} + i\sigma_{(a)ij}$, one can find that if

$$\partial_a g_{ij} = \tilde{\Gamma}_{(a)ij} + \tilde{\Gamma}_{(a)ji}^* \qquad \Longrightarrow \qquad \sigma_{(a)ij} = \sigma_{(a)ji}^* \ ,$$

additionally in light of the theorem (4.6) from [45] implying that $\sigma_{(a)bc}$ must have some inner symmetry in its indices in order to define a Levi-Civita connection, for instance, taking all $\sigma_{(a)bc} = 0$ one gets

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \partial_a g_{db} \ . \tag{3.66}$$

If we try to calculate explicitly each Γ^c_{ab} for this case we find the following set of equations, for instance

$$\partial_T \Big(g_{RR} \Big) \implies 0 = \Re \Big(g_{RR} \Gamma^R_{TR} + g_{RT} \Gamma^T_{TR} \Big) \ ,$$

and it is straightforward to show that

$$\begin{aligned}
\partial_R \Big(g_{RR} \Big) &\implies -\ell_0^2 k^2 Z^3 = \Re \Big(g_{RR} \Gamma^R_{RR} + g_{RT} \Gamma^T_{RR} \Big) \ , \\
\partial_T \Big(g_{TT} \Big) &\implies 0 = \Re \Big(g_{TR} \Gamma^R_{TT} + g_{TT} \Gamma^T_{TT} \Big) \ , \\
\partial_R \Big(g_{TT} \Big) &\implies \ell_0^2 R = \Re \Big(g_{TR} \Gamma^R_{RT} + g_{TT} \Gamma^T_{RT} \Big) \ , \\
\partial_R \Big(g_{TR} \Big) &\implies 0 = g_{TR} \Gamma^R_{RR} + g_{TT} \Gamma^T_{RR} + \Big(g_{RR} \Gamma^R_{RT} + g_{RT} \Gamma^T_{RT} \Big)^* \ , \\
\partial_T \Big(g_{TR} \Big) &\implies 0 = g_{TR} \Gamma^R_{TR} + g_{TT} \Gamma^T_{TR} + \Big(g_{RR} \Gamma^R_{TT} + g_{RT} \Gamma^T_{TT} \Big)^* \ .
\end{aligned} \tag{3.67}$$

The solution to the system of non-commutative equations above is equivalent to apply (3.66) to each combination of indices as we do below

$$\begin{aligned}
\Gamma^R_{RR} &= \frac{1}{2} \Big(g^{RR} \partial_R g_{RR} + g^{RT} \partial_R g_{TR} \Big) = -k^2 Z \ , \\
\Gamma^R_{RT} &= \Big(g^{RR} \partial_R g_{RT} + g^{RT} \partial_R g_{TT} \Big) = \frac{i\hbar}{2\ell_0} R \ , \\
\Gamma^R_{TR} &= \Big(g^{RR} \partial_T g_{RR} + g^{RT} \partial_T g_{TR} \Big) = 0 \ , \\
\Gamma^R_{TT} &= \Big(g^{RR} \partial_T g_{RT} + g^{RT} \partial_T g_{TT} \Big) = 0 \ , \\
\Gamma^T_{TT} &= \Big(g^{TR} \partial_T g_{RT} + g^{TT} \partial_T g_{TT} \Big) = 0 \ , \\
\Gamma^T_{RT} &= \Big(g^{TR} \partial_R g_{RT} + g^{TT} \partial_R g_{TT} \Big) = k^2 Z \ , \\
\Gamma^T_{RR} &= \Big(g^{TR} \partial_R g_{RR} + g^{TT} \partial_R g_{TR} \Big) = \frac{i\hbar k^2}{2\ell_0} Z^3 \ , \\
\Gamma^T_{TR} &= \Big(g^{TR} \partial_T g_{RR} + g^{TT} \partial_T g_{TR} \Big) = 0 \ .
\end{aligned} \tag{3.68}$$

Plugging the set of Christoffel Symbols above in (3.67) satisfy the full set of equations, as expected. If we compare the non-commutative set of connection coefficients with the non-zero commutative analogues $\Gamma^r_{rr} = -\Gamma^t_{rt} = -\Gamma^t_{tr} = -z$ and $\Gamma^r_{tt} = -r^3$, we observe that the set of coefficients obtained above does not converge to the commutative limit as we take $\hbar \to 0$, this can be explained by remembering that we have a family of Levi-Civita connections associated with an non-commutative manifold, we can also consider the fact that we haven't used yet the torsion free condition in our setup. Since we know the symmetry properties of $\sigma_{(a),bc}$ we must modify slightly these coefficients in order to obtain the correct commutative limit. Writing the coefficients for $\sigma \neq 0$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \partial_a g_{db} + i g^{cd} \sigma_{(a)db} \ , \tag{3.69}$$

implying that we must choose wisely eight hermitian elements of $\mathcal{C}_\hbar$. Analyzing the non-converging coefficient Γ^R_{TT} we can impose to it the following form $\Big(\alpha(\hbar)-1\Big)R^3$ for some real valued function $\alpha(\hbar)$, this imposition will guarantee the correct commutative limit implying that

$$\Gamma^R_{TT} = i\frac{R^2}{\ell_0^2}\sigma_{(T)RT} - \frac{\hbar}{2\ell_0^3}\sigma_{(T)TT} \ = \Big(\alpha(\hbar)-1\Big)R^3 \ ,$$

And from this equation, we can use the torsion free condition and solve the system of sixteen equations and unknowns to find the correct converging set of Christoffel Symbols. This would be clearly a cumbersome task and we will not follow this path, we instead rewrite our objects using a projector defined as an endomorphism $p: \ (\mathcal{C}_\hbar)^3 \to (\mathcal{C}_\hbar)^3$ which satisfies $p\Big((\mathcal{C}_\hbar)^3\Big) = \mathcal{X}(\mathcal{C}_\hbar)$, this will prove to be very useful to circumvent the aforementioned nuisance. We will also denote the projector as

$$p(A) = p(\hat{e}_\nu A^\nu) = \hat{e}_\mu p^\mu{}_\nu A^\nu = \hat{e}_\mu \Phi^\mu_a g^{ab}(\Phi^\beta_b)^*\eta_{\beta\nu}A^\nu \ , \tag{3.70}$$

for g^{ab} the inverse of the hermitian metric constructed for $\mathcal{X}(\mathcal{C}_\hbar)$. The coefficients $p^\mu{}_\nu$ can be calculated directly from

$$p^\mu{}_\nu = \Phi^\mu_a g^{ab}(\Phi^\beta_b)^*\eta_{\beta\nu} \ , \tag{3.71}$$

and can be used to prove that p is indeed a projector, namely

$$\begin{aligned} p^2(U) &= \hat{e}_\mu p^\mu{}_\nu p^\nu{}_\alpha U^\alpha = \hat{e}_\mu \Phi^\mu_a g^{ab}(\Phi^\beta_b)^*\eta_{\beta\nu}\Phi^\nu_c g^{cd}(\Phi^\sigma_d)^*\eta_{\sigma\alpha}U^\alpha = \hat{e}_\mu \Phi^\mu_a g^{ab} g_{bc} g^{cd}(\Phi^\sigma_d)^*\eta_{\sigma\alpha}U^\alpha \ , \\ &= \Phi^\mu_a \delta^a_c g^{cd}(\Phi^\sigma_d)^*\eta_{\sigma\alpha} = \hat{e}_\mu \Phi^\mu_a g^{ad}(\Phi^\sigma_d)^*\eta_{\sigma\alpha}U^\alpha = \hat{e}_\mu p^\mu{}_\alpha U^\alpha = p(U) \ . \end{aligned}$$

As an example solving for $\mu = \nu = 0$ we obtain

$$\begin{aligned} p^0{}_0 &= \Phi^0_b g^{ab}(\Phi_{b0})^* = \Phi^0_R g^{RR}(\Phi_{R0})^* + \Phi^0_R g^{RT}(\Phi_{T0})^* + \Phi^0_T g^{TR}(\Phi_{R0})^* + \Phi^0_T g^{TT}(\Phi_{T0})^* \ , \\ &= TR^2T + \left(1 + \frac{3h^2}{4\ell_0^2}\right) \ , \end{aligned}$$

where we used the program from the appendix A to help in the long calculations. Doing the same for the other combination of indices we get

$$
\begin{aligned}
p^0{}_1 &= \frac{1}{2}\Big[TRTRT - RTR + \frac{i\hbar}{2\ell_0}TRT + \Big(1+\frac{5\hbar^2}{4\ell_0^2}\Big)T + \frac{3i\hbar k^2}{2\ell_0}Z + \frac{i\hbar}{2\ell_0}R\Big] \ , \\
p^0{}_2 &= \frac{1}{2}\Big[-TRTRT - RTR - \frac{i\hbar}{2\ell_0}TRT - \Big(1+\frac{5\hbar^2}{4\ell_0^2}\Big)T - \frac{3i\hbar k^2}{2\ell_0}Z + \frac{i\hbar}{2\ell_0}R\Big] \ , \\
p^1{}_0 &= \frac{1}{2}\Big[TRTRT - RTR - \frac{i\hbar}{2\ell_0}TRT + \Big(1+\frac{7\hbar^2}{4\ell_0^2}\Big)T + \frac{i\hbar k^2}{2\ell_0}Z - \frac{i\hbar}{2\ell_0}R\Big] \ , \\
p^2{}_0 &= \frac{1}{2}\Big[-TRTRT - RTR + \frac{i\hbar}{2\ell_0}TRT - \Big(1+\frac{7\hbar^2}{4\ell_0^2}\Big)T - \frac{i\hbar k^2}{2\ell_0}Z - \frac{i\hbar}{2\ell_0}R\Big] \ , \\
p^1{}_1 &= \frac{1}{2}\Big[\frac{T^2R^2T^2}{2} - TR^2T + \Big(1+\frac{5\hbar^2}{4\ell_0^2}\Big)T^2 + \Big(\frac{1}{2} - \frac{\hbar^2}{4\ell_0^2} - \frac{\hbar^4}{32\ell_0^4}\Big)Z^2 + \frac{R^2}{2} + \\
&+ \Big(\frac{i\hbar}{2\ell_0} + \frac{i\hbar^3}{4\ell_0^3}\Big)(ZT+TZ) - \frac{i\hbar}{\ell_0}(RT+TR) + \Big(1-\frac{\hbar^2}{4\ell_0^2}\Big)\mathbb{1}\Big] \ , \\
p^1{}_2 &= \frac{1}{2}\Big[-\frac{T^2R^2T^2}{2} - \Big(\frac{1}{2}+\frac{5\hbar^2}{4\ell_0^2}\Big)T^2 + \Big(-\frac{1}{2} + \frac{\hbar^2}{4\ell_0^2} + \frac{3\hbar^4}{32\ell_0^4}\Big)Z^2 + \frac{R^2}{2} + \frac{i\hbar}{\ell_0}ZT + \\
&+ \frac{3i\hbar}{\ell_0}TR - \frac{i\hbar^3}{4}\ell_0^3 ZT - \frac{3\hbar^2}{2\ell_0^2}\mathbb{1}\Big] \ , \\
p^2{}_1 &= \frac{1}{2}\Big[\frac{T^2R^2T^2}{2} + \Big(1+\frac{5\hbar}{4\ell_0^2}\Big)T^2 + \Big(\frac{1}{2} - \frac{\hbar^2}{4\ell_0^2} - \frac{3\hbar^4}{32\ell_0^4}\Big)Z^2 - \frac{R^2}{2} - \frac{i\hbar}{\ell_0}ZT + \\
&+ \frac{i\hbar^3}{4\ell_0^3} + \frac{3i\hbar}{\ell_0}TR - \frac{3\hbar^2}{2\ell_0^2}\Big] \ , \\
p^2{}_2 &= \frac{1}{2}\Big[-\frac{T^2R^2T^2}{2} - TR^2T - \Big(1+\frac{5\hbar^2}{4\ell_0^2}\Big)T^2 - \Big(\frac{1}{2} - \frac{\hbar^2}{4\ell_0^2} - \frac{3\hbar^4}{32\ell_0^4}\Big)Z^2 - \frac{R^2}{2} - \\
&- \Big(\frac{i\hbar}{\ell_0} + \frac{i\hbar^3}{4\ell_0^3}\Big)ZT - \frac{i\hbar}{\ell_0}(RT+TR) + k^2\mathbb{1}\Big] \ ,
\end{aligned}
\tag{3.72}
$$

We can also show that the projector p is orthogonal with respect to the hermitian form g from (3.18) by direct inspection

$$
\begin{aligned}
g(p(U),V) &= g\Big(\hat{e}_\sigma \Phi_a g^{ab}(\Phi_b^\beta)^* \eta_{\alpha\beta} U^\alpha, \hat{e}_\gamma V^\gamma\Big) = (U^\alpha)^* \Phi_b^\beta g^{ba} (\Phi_a^\sigma)^* \eta_{\alpha\beta}\eta_{\gamma\sigma} V^\gamma \ , \\
&= (U^\alpha)^* \eta_{\alpha\beta}\Big(\Phi_b^\beta g^{ba}(\Phi_a^\sigma)^*\eta_{\sigma\gamma}\Big)V^\gamma = g(\hat{e}_\alpha U^\alpha, \hat{e}_\beta \Phi_b^\beta g^{ba}(\Phi_a^\sigma)^*\eta_{\sigma\gamma}V^\gamma) = g(U, p(V)) \ .
\end{aligned}
$$

If we compose the projection p with the connection from (3.69), denoting it by $\nabla = p \circ \tilde{\nabla}$ where $\tilde{\nabla}$ refer to the connection defined in (3.69), and impose the torsion free condition, we can finally define a Levi-Civita connection on $\mathcal{X}(\mathcal{C}_\hbar)$ starting from

$$
\nabla_a U = (p \circ \tilde{\nabla}_a)U = p\Big(\hat{e}_\alpha \partial_a U^\alpha + \hat{e}_\mu\Big(\frac{1}{2}\eta^{\mu\nu}\partial_a \eta_{\nu\rho} + i\eta^{\mu\nu}\sigma_{(a)\nu\rho}\Big)U^\rho\Big) \ ,
$$

using $\partial_a \eta_{\mu\nu} = 0$ and applying the projected connection to the basis of $\mathcal{X}(\mathcal{C}_\hbar)$, we get to the following

$$
\nabla_a \hat{\Phi}_d = \hat{\Phi}_b g^{bc}\Big(\eta_{\alpha\beta}(\Phi_c^\alpha)^* \partial_a \Phi_d^\beta + i(\Phi_c^\gamma)^* \sigma_{(a)\gamma\lambda}\Phi_d^\lambda\Big) \ ,
$$

where from now on we will denote $(\Phi_c^\gamma)^* \sigma_{(a)\gamma\lambda}\Phi_d^\lambda$ as $\tilde{\sigma}_{(a)cd}$. Now we apply the torsion free condition using $[\partial_R, \partial_T] = 0$ and $\partial_a X^\alpha = \Phi_a^\alpha$

$$
\begin{aligned}
\Big(\nabla_a \varphi(\partial_b) - \nabla_b \varphi(\partial_a)\Big)\hat{e}_\alpha X^\alpha &= \hat{\Phi}_c g^{cd}\Big(\eta_{\alpha\beta}(\Phi_d^\alpha)^* \partial_a \Phi_b^\beta + i\tilde{\sigma}_{(a)db} - \eta_{\alpha\beta}(\Phi_d^\alpha)^*\partial_b \Phi_a^\beta - i\tilde{\sigma}_{(b)da}\Big) \ , \\
&= \hat{\Phi}_c g^{cd}\Big(\eta_{\alpha\beta}(\Phi_d^\alpha)^*[\partial_a, \partial_b]X^\beta + i(\tilde{\sigma}_{(a)db} - \tilde{\sigma}_{(b)da})\Big) = 0
\end{aligned}
$$

since the commutator of derivatives is zero, this implies an additional constraint in the indices of $\tilde{\sigma}$ meaning that $\tilde{\sigma}_{(a)bc} = \tilde{\sigma}_{(c)ab}$ and considering the following $\sigma_{(a)ij} = \sigma^*_{(a)ji}$, we conclude that $\tilde{\sigma}_{(a)bc}$ must be symmetric in all three indices, giving 4 choices for our case. In particular, one could choose $\tilde{\sigma} = 0$ for all combinations of indices, it would still be a Levi-Civita connection. Calculating for this case we get

$$\nabla_a(\hat{\Phi}_b U^b) = \hat{\Phi}_b \partial_a U^b + \hat{\Phi}_c g^{cd} (\Phi^\alpha_d)^* \eta_{\alpha\beta} \partial_a \Phi^\beta_b \; U^b \ , \tag{3.73}$$

for $\Gamma^c_{ab} = g^{cd}(\Phi^\alpha_d)^* \eta_{\alpha\beta} \partial_a \Phi^\beta_b$. Calculating each coefficient directly one gets to the following

$$\begin{aligned}
\Gamma^R_{RR} &= \Big(g^{RR}(\Phi^\alpha_R)^* + g^{RT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_R \Phi^\beta_R = -k^2 Z \ , \\
\Gamma^R_{RT} &= \Big(g^{RR}(\Phi^\alpha_R)^* + g^{RT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_R \Phi^\beta_T = \frac{i\hbar}{2\ell_0} R \ , \\
\Gamma^R_{TT} &= \Big(g^{RR}(\Phi^\alpha_R)^* + g^{RT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_T \Phi^\beta_T = -R^3 \ , \\
\Gamma^R_{TR} &= \Big(g^{RR}(\Phi^\alpha_R)^* + g^{RT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_T \Phi^\beta_R = \frac{i\hbar}{2\ell_0} R \ , \\
\Gamma^T_{TR} &= \Big(g^{TR}(\Phi^\alpha_R)^* + g^{TT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_T \Phi^\beta_R = k^2 Z \ , \\
\Gamma^T_{RT} &= \Big(g^{TR}(\Phi^\alpha_R)^* + g^{TT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_R \Phi^\beta_T = k^2 Z \ , \\
\Gamma^T_{RR} &= \Big(g^{TR}(\Phi^\alpha_R)^* + g^{TT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_R \Phi^\beta_R = \frac{i\hbar}{2\ell_0} k^2 Z^3 \ , \\
\Gamma^T_{TT} &= \Big(g^{TR}(\Phi^\alpha_R)^* + g^{TT}(\Phi^\alpha_T)^*\Big)\eta_{\alpha\beta}\partial_T \Phi^\beta_T = \frac{i\hbar}{2\ell_0} R \ ,
\end{aligned}$$

and can be easily seen by the expressions above that this set of Christoffel symbols have the correct commutative limit. Choosing another hermitian elements for $\tilde{\sigma}$ satisfying the symmetry of indices, one could find another Levi-Civita connection. We will use $\tilde{\sigma} = 0$ in order to calculate the Ricci Scalar. As a last consideration, if one wants to use the opposite algebra to define the connection for the left module of $\mathcal{X}(\mathcal{C}_\hbar)$ the natural way of doing it is just defining the left connection as

$$\Big(\nabla_{\partial_a} X\Big)^l = \Big[\Big(\partial_a X^d + \Gamma^d_{ab} X^b\Big)\Phi_d\Big]^l \ , \tag{3.74}$$

with the right analogue being all connections we defined before belonging to the opposite algebra. We can also define the symmetric connection over $\mathcal{A}^e$, seeing it as an left module, following the same procedure we employed to construct the symmetric hermitian form g^s

$$\nabla^s_{\partial_a} X = \frac{1}{2}\Big(\nabla^r_{\partial_a} + \nabla^l_{\partial_a}\Big) X \ .$$

Additionally we can generalize the usual definition of the Laplace-Beltrami operator for a Levi-Civita connection $\Delta(F) = |g|^{-1/2}\partial_\mu\Big(|g|^{1/2} g^{\mu\nu}\partial_\nu F\Big)$ by using $|g| = \ell_0^4 \mathbb{1}$ for the non-commutative case and applying the symmetric mapping to the metric. This must be done because when we define the Laplacian one should raise the index in one of the covariant derivatives, and in order to fulfill our construction of the dual moduli for differential forms, it must act from the left instead, making necessary the use of right and left algebra simultaneously to guarantee the correct transformation property of the resulting Laplacian. This can be translated as

$$\Delta(F) := \partial_a\Big(g^{ab}\partial_b(F)\Big) \ , \tag{3.75}$$

for $F \in \hat{\mathcal{C}}_\hbar$, $g^{ab} = (g_{ab})^{-1}$ the inverse metric constructed in (3.47) and a, b being the coordinates R and T. Now we apply the symmetric mapping to the inverse metric following the procedure we did for g^s

$$\begin{aligned} \mathcal{S}\Big(g^{-1}(\Phi^R, \Phi^R)\Big) &= \frac{1}{\ell_0^2} R \otimes R = \frac{1}{\ell_0^2} R^l R^r \ , \\ \mathcal{S}\Big(g^{-1}(\Phi^T, \Phi^T)\Big) &= \frac{k(\hbar)^2}{\ell_0^2} Z \otimes Z = \frac{k(\hbar)^2}{\ell_0^2} Z^l Z^r \ . \end{aligned} \tag{3.76}$$

It is easy to see that the cross terms vanish since they are proportional to the identity and from the commutator of the generators of $\mathfrak{g}$ we get $\partial_R(\partial_T F) = \partial_T(\partial_R F)$. In order to differentiate elements of $\mathcal{C}_\hbar^e$ we use the function Θ to define how to do it

$$\partial_A[(B \otimes C)F] := \partial_A \Theta\Big((B \otimes C)F\Big) = \Theta\Big(((\partial_A B) \otimes C + B \otimes (\partial_A C) + B \otimes C\partial_A)F\Big) \ ,$$

which implies that $\partial_A(B \otimes C) := (\partial_A B) \otimes C + B \otimes (\partial_A C) + (B \otimes C)\partial_A$ when considering the enveloping algebra as a left module with the map Θ to $\mathcal{C}_\hbar$. Now we can explicitly calculate the non-zero terms of (3.75)

$$\begin{aligned} \partial_R\Big(g^{-1}(\Phi^R, \Phi^R)\partial_R F\Big) &= \frac{1}{\ell_0^2}\Big(R^l R^r \partial_R^2 F + (R^l + R^r)\partial_R F\Big) \ , \\ \partial_T\Big(g^{-1}(\Phi^T, \Phi^T)\partial_R F\Big) &= \frac{1}{\ell_0^2} k(\hbar)^2 Z^l Z^r \partial_T^2 F \ . \end{aligned} \tag{3.77}$$

With these tools we will prove the following theorem that relates the Laplacian in both ambient and local coordinates, giving us the possibility of defining the integration over $(\mathcal{C}_\hbar)^3$ by using the analogues of eigenfunctions of the surface.

Theorem 3.1. *The* ***non-commutative Laplacian*** *defined in (3.75) acting on an arbitrary function $F \in \mathcal{C}_\hbar$ satisfies the following equation*

$$\Delta(F) := \partial_a\Big(\mathcal{S}\Big(g^{ab}\partial_b(F)\Big)\Big) = \frac{2}{\hbar^2}\Big(\frac{1}{\ell_0^2}(X^\nu)^l (X^\mu)^r \eta_{\mu\nu} + \mathbb{1}\Big)F \ , \tag{3.78}$$

for X^μ the ambient coordinates of the ncAdS$_2$ and $\eta_{\mu\nu} = diag(1, 1, -1)$.

Proof: We will try to show that the Laplacian (3.75) constructed using the coordinates (3.2) can be written as

$$(X^\mu)^l (X^\nu)^r \eta_{\mu\nu} F = \frac{\hbar^2 \ell_0^2}{2}\Big(\Delta(F) - \frac{2}{\hbar^2}F\Big) \ . \tag{3.79}$$

The proof for this claim will be provided in the following steps. First we directly calculate the Laplacian using the metric elements, for instance, as said before the cross terms vanish since

$$\partial_R\Big(g^{-1}(\Phi^R, \Phi^T)\partial_T(F)\Big) + \partial_T\Big(g^{-1}(\Phi^T, \Phi^R)\partial_R(F)\Big) = F[T, R] + [R, T]F = 0 \ ,$$

which is expected since the derivatives commute. For the other terms we get

$$\partial_R\Big(g^{-1}(\Phi^R,\Phi^R)\partial_R(F)\Big)=\frac{1}{\ell_0^2}\partial_R\Big(R^lR^r\partial_R(F)\Big)\ ,$$
$$\partial_T\Big(g^{-1}(\Phi^T,\Phi^T)\partial_T(F)\Big)=\frac{k^2}{\ell_0^2}Z^lZ^r\partial_T^2(F)\ .$$

Expanding this we get a messy expression that can be further simplified using (3.33)

$$\begin{aligned}\Delta(F)&=-\frac{1}{\hbar^2}\Big[R[F,T]R,\ T\Big]-\frac{k^2}{\hbar^2}Z\Big[[F,\ R],\ R\Big]Z=-\frac{1}{\hbar^2}\Big(k^2\Big(Z[F,R]\\&-[F,R]Z\Big)+R[F,T]RT-TR[F,T]R\Big)=-\frac{1}{\hbar^2}\Big(k^2\Big(ZFR-2F-RFZ\Big)\\&+\Big(RFTRT-RTFRT-TRFTR+TRTFR\Big)\Big)\ ,\end{aligned}$$

reordering the symmetric terms of type $RTFRT$ we obtain a simplification

$$\begin{aligned}&\frac{1}{2}\Big(RT+TR\Big)F\Big(RT+TR\Big)=\frac{1}{2}\Big(RTFRT+TRFTR+RTFTR+TRFRT\Big)\\&=\frac{1}{2}\Big(2RTFRT+2TRFTR+\frac{i\hbar}{\ell_0}\Big(FTR-FRT\Big)\Big)=RTFRT+TRFTR+\frac{\hbar^2}{2\ell_0^2}F\ .\end{aligned}$$

Substituting the above expression in the main one and multiplying the Laplacian by $2\hbar^2$, we get to the following

$$\begin{aligned}2\hbar^2\Delta(F)&=-2k^2\Big(ZFR-RFZ\Big)+4k^2F-2(RFTRT+TRTFR)\\&+\Big(RT+TR\Big)F\Big(RT+TR\Big)-\frac{\hbar^2}{\ell_0^2}F\ .\end{aligned}\tag{3.80}$$

One can also verify

$$\begin{aligned}&\Big((X^1)^l(X^1)^r-(X^2)^l(X^2)^r\Big)F=-\frac{\ell_0^2k^2}{2}\Big(ZFR-RFZ-2F\Big)\\&-\frac{\ell_0^2}{2}\Big(TRTFR+RFTRT\Big)\ ,\end{aligned}$$

and also

$$(X^0)^l(X^0)^rF=\frac{\ell_0^2}{4}(RT+TR)F(RT+TR)\ .$$

If we use the fact that $(X^\nu)^l(X^\mu)^r\eta_{\mu\nu}F=\Big((X^0)^l(X^0)^r+(X^1)^l(X^1)^r-(X^2)^l(X^2)^r\Big)F$ and dividing the whole equation (3.80) by $\frac{4}{\ell_0^2}$ we get

$$\frac{\hbar^2\ell_0^2}{2}\Delta(F)=\Big((X^\nu)^l(X^\mu)^r\eta_{\mu\nu}+\ell_0^2\mathbb{1}\Big)F\ ,$$

which can be rearranged and yelds the final result

$$\Delta(F)=\frac{2}{\hbar^2}\Big(\frac{1}{\ell_0^2}(X^\nu)^l(X^\mu)^r\eta_{\mu\nu}+\mathbb{1}\Big)F\ .\tag{3.81}$$

This expression for the quantum Laplacian agrees with [4] where it is obtained by a different approach. □

The previous step is of big importance in our work since it connects the two different formalisms employed, one from my work defining the whole differential calculus over the $ncAdS$ in order to construct an well suited Laplacian and the other formalism coming from [4] where they use the construction that comes from the deformation of the Poisson structure of the Euclidean AdS. Now that we have our first important theorem we will use the Levi-Civita connections for both local and ambient coordinates to pursuit a natural following step that is to define a curvature following the usual relations from the commutative differential geometry. We will start by giving a simple definition to it and we will proceed calculating it until we find the Ricci Scalar for the $ncAdS_2$.

3.5 Non-commutative curvature

We will define the non-commutative curvature $\mathcal{R} : \mathfrak{g}^2 \times \mathcal{X}(\hat{\mathcal{C}}_\hbar) \longrightarrow \mathcal{X}(\hat{\mathcal{C}}_\hbar)$ as a right acting operator using the Levi-Civita connection constructed in the last section. The expressions for the Riemann curvature in ambient and local coordinates, respectively, and their components are

$$\begin{aligned} \mathcal{R}(\partial_\mu, \partial_\nu)e_\alpha &= \Big(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu - \nabla_{[\partial_\mu,\partial_\nu]}\Big)e_\alpha = e_\lambda \mathcal{R}^\lambda{}_{\alpha\mu\nu} \ , \\ \mathcal{R}(\partial_a, \partial_b)\Phi_c &= (\nabla_{\partial_a}\nabla_{\partial_b} - \nabla_{\partial_b}\nabla_{\partial_a})\Phi_c \ = \Phi_d \mathcal{R}^d_{cab} \ , \end{aligned} \tag{3.82}$$

the derivatives used belong to $\mathfrak{g}$ and depend of the choice of coordinates, we also considered that $\tilde{e}_\rho$ is the basis of $T\mathcal{C}_\hbar$ coming from its Lie algebra properties and lastly $\Phi_c \in \mathcal{X}(\hat{\mathcal{C}}_\hbar)$ and a, b, c and d being the coordinates R and T. Note that from the definition above it is easy to see that $\mathcal{R}(\partial_\mu, \partial_\nu)e_\alpha = -\mathcal{R}(\partial_\nu, \partial_\mu)e_\alpha$ and from the torsion free condition, we can show that

$$\begin{aligned} &\mathcal{R}(\partial_\mu, \partial_\nu)e_\alpha + \mathcal{R}(\partial_\alpha, \partial_\mu)e_\nu + \mathcal{R}(\partial_\nu, \partial_\alpha)e_\mu = \nabla_\mu\varphi([\partial_\nu, \partial_\alpha]) + \nabla_\nu\varphi([\partial_\alpha, \partial_\mu]) + \nabla_\alpha\varphi([\partial_\mu, \partial_\nu]) - \\ &\quad \nabla_{[\partial_\mu,\partial_\nu]}e_\alpha - \nabla_{[\partial_\nu,\partial_\alpha]}e_\mu - \nabla_{[\partial_\alpha,\partial_\mu]}e_\nu = \varphi([\partial_\mu, [\partial_\nu, \partial_\alpha]]) + \varphi([\partial_\nu, [\partial_\alpha, \partial_\mu]]) + \varphi([\partial_\alpha, [\partial_\mu, \partial_\nu]]) = 0 \ , \end{aligned}$$

where we used the Jacobi identity and the fact that $e_\mu = \varphi(\partial_\mu)$ from (3.15), leading us to the conclusion that the first Bianchi identity holds for the non-commutative case. The last result shows that In the first part of this section we will state some useful propositions in order to prove the main result of this chapter. After we introduce all small results we will show that these two constructions are equivalent. To start our endeavor we consider $\gamma_{(\alpha)\beta\lambda} = 0$ and we analyze the curvature for the ambient coordinates showing that

$$\nabla^0_\alpha e_\mu = \nabla_\alpha e_\mu = \frac{1}{\ell_0}\Big(\epsilon_\mu{}^{\rho\sigma}(\nabla_\alpha\tilde{e}_\rho)X_\sigma + \epsilon_\mu{}^{\rho\sigma}\epsilon_{\alpha\sigma}{}^\gamma\tilde{e}_\rho X_\gamma\Big) = -\frac{1}{\ell_0}\Big(\tilde{e}_\alpha + \frac{i\hbar}{\ell_0}e_\alpha\Big)X_\mu = \frac{1}{\ell_0^2}\epsilon_\alpha{}^{\rho\sigma}e_\rho X_\sigma X_\mu \ ,$$

with this we can calculate separately each term of the Riemann curvature

$$\begin{aligned} \nabla_\mu(\nabla_\nu e_\alpha) &= \frac{1}{\ell_0^4}\Big[\epsilon_\nu{}^{\gamma\rho}\epsilon_\mu{}^{\beta\theta}e_\beta X_\theta X_\gamma X_\rho X_\alpha + \ell_0^2\epsilon_\nu{}^{\gamma\rho}e_\gamma\Big(\epsilon_{\mu\rho}{}^\sigma X_\sigma X_\alpha + \epsilon_{\mu\alpha}{}^\sigma X_\rho X_\sigma\Big)\Big] \ , \\ \nabla_\nu(\nabla_\mu e_\alpha) &= \frac{1}{\ell_0^4}\Big[\epsilon_\mu{}^{\gamma\rho}\epsilon_\nu{}^{\beta\theta}e_\beta X_\theta X_\gamma X_\rho X_\alpha + \ell_0^2\epsilon_\mu{}^{\gamma\rho}e_\gamma\Big(\epsilon_{\nu\rho}{}^\sigma X_\sigma X_\alpha + \epsilon_{\nu\alpha}{}^\sigma X_\rho X_\sigma\Big)\Big] \ , \\ \nabla_{[\partial_\mu,\partial_\nu]}e_\alpha &= \frac{1}{\ell_0^2}\epsilon_{\mu\nu}{}^\rho\epsilon_\rho{}^{\gamma\sigma}e_\gamma X_\sigma X_\alpha \ , \end{aligned} \tag{3.83}$$

and after some straightforward calculations we get to the following expression

$$\begin{aligned}\mathcal{R}(\partial_\mu,\partial_\nu)e_\alpha = \frac{1}{\ell_0^4}\Big[& i\hbar e_\beta\Big(\epsilon_\nu{}^{\beta\sigma}X_\sigma X_\mu + \epsilon_{\mu\nu}{}^{\sigma}X_\sigma X^\beta - \epsilon_\mu{}^{\beta\sigma}X_\sigma X_\nu\Big)X_\alpha + e_\nu\Big(- \ell_0^2 X_\mu X_\alpha \\ &+ \ell_0^4\eta_{\mu\alpha}\mathbb{1} + i\hbar\epsilon_\mu{}^{\sigma\rho}X_\sigma X_\rho X_\alpha + i\hbar\ell_0^2\epsilon_{\alpha\mu}{}^{\sigma}X_\sigma\Big) - e_\mu\Big(\ell_0^2 X_\nu X_\alpha - \ell_0^4\eta_{\nu\alpha}\mathbb{1} \\ &+ i\hbar\epsilon_\nu{}^{\sigma\rho}X_\sigma X_\rho X_\alpha + i\hbar\ell_0^2\epsilon_{\alpha\nu}{}^{\sigma}X_\sigma\Big) + i\hbar\ell_0^2\Big(\epsilon_\mu{}^{\beta\sigma}e_\beta X_\sigma\eta_{\nu\alpha} - \epsilon_\nu{}^{\beta\sigma}e_\beta X_\sigma\eta_{\mu\alpha} \\ &+ \epsilon_{\mu\nu}{}^{\sigma}e_\alpha X_\sigma\eta_{\nu\alpha}\Big)\Big] \ ,\end{aligned} \tag{3.84}$$

which can be further simplified to

$$\mathcal{R}(\partial_\mu,\partial_\nu)e_\alpha = \frac{1}{\ell_0^2}\Big(e_\nu\Big(\eta_{\mu\alpha} + \frac{1}{\ell_0^2}X_\alpha X_\mu + \frac{\hbar^2}{\ell_0^4}X_\mu X_\alpha\Big) - e_\mu\Big(\eta_{\nu\alpha} + \frac{1}{\ell_0^2}X_\alpha X_\nu + \frac{\hbar^2}{\ell_0^4}X_\nu X_\alpha\Big)\Big) \ ,$$

and using the definition of $g(e_\mu, e_\nu)$ from the proposition (3.7) we find that the curvature has the following form

$$\mathcal{R}(\partial_\mu,\partial_\nu)e_\alpha = -\frac{1}{\ell_0^2}\Big(e_\mu g(e_\nu,e_\alpha) - e_\nu g(e_\mu,e_\alpha)\Big) = \frac{1}{\ell_0^2}(e_\nu g_{\mu\alpha} - e_\mu g_{\nu\alpha}) \ , \tag{3.85}$$

in order to write it as (3.82) we use the fact that $e_\mu = e_\lambda P^\lambda{}_\mu$ which allows us to rewrite the expression

$$e_\lambda R^\lambda{}_{\alpha\mu\nu} = \frac{1}{\ell_0^2}e_\lambda\Big(P^\lambda{}_\nu g_{\mu\alpha} - P^\lambda{}_\mu g_{\nu\alpha}\Big) \ ,$$

and upon setting $\mathcal{R}_{\rho\alpha\mu\nu} = g(e_\rho, \mathcal{R}(\partial_\mu,\partial_\nu)e_\alpha)$ we get to the expression

$$\mathcal{R}_{\rho\alpha\mu\nu} = \frac{1}{\ell_0^2}(g_{\rho\nu}g_{\mu\alpha} - g_{\rho\mu}g_{\nu\alpha}) \ , \tag{3.86}$$

which is the non-commutative analogue of the Riemann curvature tensor, keeping the same symmetries and structure of the commutative counterpart[42]. Now we turn our attention to the curvature tensor calculated in the local coordinates. Since the generators of the Lie algebra for local coordinates commutes with each other the term $\nabla_{[\partial_a,\partial_b]}$ will be zero and the curvature will take the simpler form seen in (3.82). Now we proceed to calculate all non-zero terms of the non-commutative right acting Riemann tensor $R^d{}_{cab}$ using

$$\begin{aligned}\nabla_a(\nabla_b\hat{\Phi}_c) &= \nabla_a\Big(\hat{\Phi}_e\Gamma^e_{bc}\Big) = \hat{\Phi}_d\Big(\partial_a\Gamma^d_{bc} + \Gamma^d_{ae}\Gamma^e_{bc}\Big) \ , \\ \nabla_b(\nabla_a\hat{\Phi}_c) &= \nabla_b\Big(\hat{\Phi}_e\Gamma^e_{ac}\Big) = \hat{\Phi}_d\Big(\partial_b\Gamma^d_{ac} + \Gamma^d_{be}\Gamma^e_{ac}\Big) \ ,\end{aligned}$$

where we impose that $\tilde{\sigma}_{(a)bc} = 0$ for all combinations of a, b and c clearly satisfying the symmetry constraint for this set of indices. Now by direct inspection we find the non-zero elements

$$\begin{aligned}\mathcal{R}^R_{TTR} &= -\mathcal{R}^R_{TRT} = R^2\Big(1 - \frac{\hbar^2}{2\ell_0^2}\Big) \ , \\ \mathcal{R}^R_{RRT} &= -\mathcal{R}^R_{RTR} = \frac{i\hbar\mathbb{1}}{\ell_0}\Big(\frac{1}{2} + k(\hbar)^2\Big) \ , \\ \mathcal{R}^T_{TRT} &= -\mathcal{R}^T_{TTR} = \frac{i\hbar\mathbb{1}}{\ell_0}\Big(-\frac{1}{2} + k(\hbar)^2\Big) \ , \\ \mathcal{R}^T_{RRT} &= -\mathcal{R}^T_{RTR} = Z^2k(\hbar)^2\Big(1 + \frac{\hbar^2}{2\ell_0^2}\Big) \ .\end{aligned} \tag{3.87}$$

[42]We use the program constructed in the appendix (A.1) to explicitly calculate each of the 81 terms of this tensor in order to verify these properties. The full construction can be found in the github file provided there.

Now that we have the whole set of ingredients that will be needed to prove the main theorem of this section, we state it below.

Theorem 3.2. *The* ***non-commutative Ricci scalar*** *will be denoted as* $\mathcal{R}$ *and can be calculated for the ambient and local coordinates, respectively, using the following expressions*

$$\mathcal{R} := g^{\mu\rho}\Big(\mathcal{R}_{\rho\alpha\mu\nu}\Big)g^{\alpha\nu} = \mathcal{R}^{\rho}{}_{\alpha\rho\nu}g^{\alpha\nu} \ ,$$
$$\mathcal{R} := g^{ab}\Big(\mathcal{R}^{c}{}_{acb}\Big) \ ,$$

yielding the same result for the non-commutative Ricci scalar $\mathcal{R} = -\Big(\frac{2}{\ell_0^2} + \frac{\hbar^2}{\ell_0^4}\Big)\mathbb{1}$ *showing that the curvature scalar receive a non-commutative correction for these specific ordering choices.*

Proof: To prove this result we already have the most important tools calculated, we just need to explicit calculate the Ricci scalar from the definitions above. We start calculating it for the ambient coordinates

$$\mathcal{R} = \frac{1}{\ell_0^2}g^{\mu\rho}\Big(g_{\rho\nu}g_{\mu\alpha} - g_{\rho\mu}g_{\nu\alpha}\Big)g^{\alpha\nu} = \frac{1}{\ell_0^2}\Big(P^{\mu}{}_{\nu}P_{\mu}{}^{\nu} - P^{\mu}{}_{\mu}P_{\nu}{}^{\nu}\Big) = \mathcal{R}^{\mu}{}_{\alpha\mu\nu}g^{\alpha\nu} \ ,$$

to progress in the demonstration we must use that the trace $P^{\alpha}{}_{\alpha}$ is 2 from the definition of $P_{\mu\nu}$ and using $P^{\mu}{}_{\nu}P_{\mu}{}^{\nu} = P^{\mu\nu}P_{\mu\nu}$ we simplify the expression above finding that

$$\begin{aligned}\mathcal{R} &= \frac{\mathbb{1}}{\ell_0^2}\Big(-4 + \Big(\eta^{\mu\nu} + \frac{1}{\ell_0^2}X^{\mu}X^{\nu}\Big)\Big(\eta_{\mu\nu} + \frac{1}{\ell_0^2}X_{\mu}X_{\nu}\Big)\Big) = \frac{\mathbb{1}}{\ell_0^2}\Big(-6 + \delta^{\mu}_{\mu} + \frac{1}{\ell_0^4}X^{\mu}X^{\nu}X_{\mu}X_{\nu}\Big) \ , \\ &= -\frac{3}{\ell_0^2}\mathbb{1} + \frac{1}{\ell_0^6}\Big(X^{\nu}X^{\mu} + i\hbar\epsilon^{\mu\nu\rho}X_{\rho}\Big)X_{\mu}X_{\nu} = -\frac{2}{\ell_0^2}\mathbb{1} + \frac{i\hbar}{\ell_0^6}\epsilon^{\rho\mu\nu}X_{\rho}X_{\mu}X_{\nu} \ ,\end{aligned}$$

using $\epsilon^{\rho\mu\nu}X_{\rho}X_{\mu}X_{\nu} = -i\hbar X^{\nu}X_{\nu} = i\hbar\ell_0^2\mathbb{1}$ we can finnaly calculate the non-commutative Ricci scalar for the ambient coordinates

$$\mathcal{R} = -\frac{2}{\ell_0^2}\mathbb{1} - \frac{\hbar^2}{\ell_0^4}\mathbb{1}$$

Now we calculate the Ricci scalar for the local coordinates. Using (3.47) and (3.87) we get to the following

$$\mathcal{R} = g^{RR}\mathcal{R}^{c}{}_{RcR} + g^{RT}\mathcal{R}^{c}{}_{RcT} + g^{TR}\mathcal{R}^{c}{}_{TcR} + g^{TT}\mathcal{R}^{c}{}_{TcT} \ ,$$

by direct inspection it is easy to show that for the ambient coordinates we also have that

$$\mathcal{R} = -\frac{2}{\ell_0^2}\mathbb{1} - \frac{\hbar^2}{\ell_0^4}\mathbb{1} \ ,$$

this result is expected since in [44] the author obtain an non-commutative correction for the Ricci scalar of the fuzzy sphere, when analysing the $ncAdS_2$ we see that there are a lot of similarities between these two non-commutative spaces and this is seen in the fact that we get also a non-commutative correction here. □

Using the opposite algebra we could define the analogue left acting Differential geometry, doing the same calculations it is not hard to show that the Ricci scalar for the left-module structure is

$$\mathcal{R}^l = -\frac{\mathbb{1}}{\ell_0^2}\Big(2 - \frac{\hbar^2}{\ell_0^2}\Big) \ .$$

Introducing the enveloping algebra structure and the symmetric mapping from (3.44) we find that the symmetric Ricci Scalar doesn't receive any non-commutative correction

$$\mathcal{R}^s = \frac{1}{2}\mathcal{S}\Big(\mathcal{R}^l + \mathcal{R}^r\Big) = \mathcal{S}\Big(-\frac{2}{\ell_0^2}\mathbb{1}\otimes\mathbb{1}\Big)$$

which is the result we would obtain if we used the metric g^s to calculate $\mathcal{R}$. We can also note that, for the local coordinates, if we change ordering we use to define the Ricci scalar, for instance $\mathcal{R} = g^{ba}\Big(\mathcal{R}^c{}_{acb}\Big)$ the Ricci scalar for this specific ordering doesn't receive a non-commutative correction, implying that in the construction of our mathematical structure over the ambient coordinates we could define these objects in such a way that they relate to the symmetric map construction without the use of the left module structure. At first glance this result could sound strange, but it is a well known fact that when calculating the non-commutative scalar curvature one expect to find non-unique results that depends on the choice of ordering in the definition of the geometrical objects used (see [51]), and since these ambiguites are expected to arise, our work is successful in explaining what is the correct ordering prescription in order to remove the non-commutative correction found in the result for the local coordinates. For the ambient coordinates we analyse the result obtained in [49] for a pseudo-Riemannian calculi, as defined in the paper itself, we conclude that the non-uniqueness of the Ricci scalar is a direct consequence of the fact that we haven't restricted our definitions to the real case, as is done in [49]. In addition to this, we also must find an pseudo-inverse metric $\hat{g}^{ab}$ of g_{ab} such that, for a Hermitian element $H \in \mathcal{C}_\hbar$ this inverse metric satisfies $\hat{g}^{ab}g_{bc} = g_{cb}\hat{g}^{ba} = \delta^a_c H$, and following this definition, it is not hard to prove that the unique scalar curvature $\mathcal{R}$ is obtained from the following equation

$$\mathcal{R} = H^{-1}\hat{g}^{\rho\mu}\mathcal{R}_{\rho\alpha\mu\nu}\hat{g}^{\alpha\nu}H^{-1} \ .$$

We will not find this pseudo-inverse in my thesis, but I intend to further investigate the existence of it in future developments from our current work. In the next section we proceed in our construction of the Riemmanian geometry of the $ncAdS_2$ by defining the Killing vector fields for the metric we found and used so far. In conclusion, we demonstrate that the Killing vector fields in our formalism are related to those in references [4], [6], [7], and [48]. The underlying Lie algebra symmetry is crucial in connecting these two seemingly different approaches.

3.6 Killing Vector Fields

A natural first step in the construction of a non-commutative Killing vector field is the definition of a Lie derivative written with respect to the local coordinates. From now on, we will perform all calculations for the left-acting Killing vector field, utilizing the suitable left-module structure employed earlier. Similarly, one can perform the same calculations analogously for the right-acting Killing vector field.

Definition 3.7. *Let $\mathcal{T}$ be a non-commutative left-tensor field over the algebra $\mathcal{C}_\hbar$, we define the* ***non-commutative Lie derivative operator*** *for the local coordinates (R, T, Z) as $\left(\mathcal{L}^{(nc)}_X \mathcal{T}\right)^{a_1 \cdots a_n}_{b_1 \cdots b_m}$ as the symmetrization of the left and right acting Lie derivatives*

$$\mathcal{L}^{(nc)}(\mathcal{T}) := \frac{1}{2}\Big(\mathcal{L}^r(\mathcal{T}) + \mathcal{L}^l(\mathcal{T})\Big) ,$$

where we follow the usual left/right distinction and use the The left(right) Lie derivative defined below

$$\begin{aligned} \mathcal{L}_X\left(\mathcal{T}^{a_1 \cdots a_n}_{b_1 \cdots b_m}\right) &:= X^c\left(\partial_c \mathcal{T}^{a_1 \cdots a_n}_{b_1 \cdots b_m}\right) - (\partial_c X^{a_1})\mathcal{T}^{c \cdots a_n}_{b_1 \cdots b_m} - \ldots \\ -(\partial_c X^{a_n})\mathcal{T}^{a_1 \cdots c}_{b_1 \cdots b_1} &+ (\partial_{b_1} X^c)\mathcal{T}^{a_1 \cdots a_n}_{c \cdots b_m} + \ldots + (\partial_{b_m} X^c)\mathcal{T}^{a_1 \cdots a_n}_{b_1 \cdots c} . \end{aligned} \tag{3.88}$$

for $X = X^c\partial_c$ an general element of the left $\mathfrak{g}$-module.

Applying the Lie derivative defined above to the metric written with respect to the local coordinates yields the non-commutative Killing equation $\mathcal{L}_K\left(g_{\mu\nu}\right) = 0$ that can be explicitly written as

$$X^a\Big(\partial_a g(\Phi_b, \Phi_c)\Big) + \Big(\partial_b X^a\Big)g(\Phi_a, \Phi_c) + \Big(\partial_c X^a\Big)g(\Phi_b, \Phi_a) = 0 , \tag{3.89}$$

for the left Killing vector field satisfying $K = X^R\partial_R + X^T\partial_T$. Now we state the following theorem about Killing vectors

Theorem 3.3. *The symmetric map applied to the solution of the Killing vector field equation (3.89) satisfies the following relation*

$$K_\mu(F) = -\frac{1}{i\hbar}[X_\mu, F] ,$$

for $F \in \mathcal{C}_\hbar$, K_μ the Killing vector field and $X_\mu = X^\nu\eta_{\mu\nu}$ the ambient coordinates for the metric $\eta_{\mu\nu} = diag(1, 1, -1)$.

Proof: Our objective is to find the elements X^R and X^T of $K = X^R\partial_R + X^T\partial_T$, we start solving (3.89) making $b = c = R$

$$X^a\Big(\partial_a g(\Phi_R, \Phi_R)\Big) + \Big(\partial_R X^a\Big)g(\Phi_a, \Phi_R) + \Big(\partial_R X^a\Big)g(\Phi_R, \Phi_a) = 0 ,$$

which gives as result the equation $X^R Z = \partial_R X^R$, now we add the equations for the cross terms $a \neq b$

$$X^a\Big(\partial_a g(\Phi_R, \Phi_T)\Big) + \Big(\partial_R X^a\Big)g(\Phi_a, \Phi_T) + \Big(\partial_T X^a\Big)g(\Phi_R, \Phi_a) = 0 ,$$
$$X^a\Big(\partial_a g(\Phi_T, \Phi_R)\Big) + \Big(\partial_T X^a\Big)g(\Phi_a, \Phi_R) + \Big(\partial_R X^a\Big)g(\Phi_T, \Phi_a) = 0 ,$$

to get to the second equation $\partial_R X^T R^2 + k^2 \partial_T X^R Z^2 = 0$ and lastly for $a = b = T$ we find the third equation for our system

$$X^a \Big(\partial_a g(\Phi_T, \Phi_T)\Big) + \Big(\partial_T X^a\Big) g(\Phi_a, \Phi_T) + \Big(\partial_T X^a\Big) g(\Phi_T, \Phi_a) = 0 \ ,$$

giving as result $X^R + \partial_T X^T R = 0$.Now in order to solve the following system of equations

$$X^R Z - \partial_R X^R = 0 \ ,$$
$$\partial_R X^T R^2 + k^2 \partial_T X^R Z^2 = 0 \ ,$$
$$X^R + \partial_T X^T R = 0 \ ,$$

we can use the ansatz coming from the commutative case, we will show that the following operators satisfy the set of equations

$$\begin{aligned} X^R &= R + 2TR \ , \\ X^T &= -T + k^2 Z^2 - T^2 \ . \end{aligned} \tag{3.90}$$

First we apply the ansatz for X^R in the first equation, which gives

$$\partial_R X^R = \mathbb{1} + 2T = (R + 2TR)Z = X^R Z \ ,$$

clearly satisfying it. For the second equation we calculate separately the following terms

$$\partial_R X^T = -2k^2 Z^3 \ ,$$
$$\partial_T X^R = 2R \ ,$$

now we apply these results to the second equation of the system to get

$$\partial_R X^T R^2 = -2k^2 Z = -(2k^2 R) Z^2 = -k^2 \partial_T X^R Z^2$$

which satisfies the second equation of the system. The last equation is easily satisfied as can be seen by taking the derivative

$$(\partial_T X^T) R = -(\mathbb{1} + 2T) R = -(R + 2TR) = -X^R \ .$$

We can do solve the right acting Killing vector equation by using the adjoint of the ansatz above. Now we join both the left and right parts getting as result a candidate for non-commutative Killing vector field

$$K^{(nc)} = X^R \partial_R + X^T \partial_T \ , \tag{3.91}$$

with X^R and X^T expressed as

$$\begin{aligned} X^R =& \frac{1}{2}\Big[\Big(R^l + R^r\Big) + 2\Big(T^l R^l + T^r R^r\Big)\Big] \ , \\ X^T =& -\frac{1}{2}\Big[\Big(T^r + T^l\Big) - k^2\Big((Z^2)^l + (Z^2)^r\Big) + \Big((T^2)^l + (T^2)^r\Big)\Big] \ . \end{aligned} \tag{3.92}$$

Now, as the last step, we must verify if K transforms correctly with the choice of ordering for the quadratic terms. It is easy to see if we act (multiply as an element of $\mathcal{C}^e_\hbar$) K over some element

$A \in \mathcal{C}_\hbar$ without considering the symmetric map the manipulation of the terms will give raise to an additional term proportional to $-\frac{\ell_0^2}{\hbar^2}(\partial_J^2 - \partial_R^2)A$ and it will cause K to not transform correctly. To avoid this we apply the symmetric map $\mathcal{S}$ over X^T and X^R getting the final form for the non-commutative Killing vector field

$$\begin{aligned} X^R =& \frac{1}{2}\left[\left(R^l + R^r\right) + 2\left(T^l R^l + T^r R^r\right)\right] , \\ X^T =& -\frac{1}{2}\left[\left(T^r + T^l\right) - 2k^2 Z^l Z^r + 2T^l T^r\right] . \end{aligned} \tag{3.93}$$

Now we introduce a new set of operators $K = K_0 + K_1 + K_2$ defined as

$$\begin{aligned} K_0 =& \frac{1}{2}\left[\left(R^l + R^r\right)\partial_R - \left(T^r + T^l\right)\partial_T\right] , \\ K_1 =& \frac{1}{2}\left[\left(T^l R^l + T^r R^r\right)\partial_R + \left(k^2 Z^l Z^r - T^r T^l + \mathbb{1}\right)\partial_T\right] , \\ K_2 =& \frac{1}{2}\left[\left(T^l R^l + T^r R^r\right)\partial_R + \left(k^2 Z^l Z^r - T^r T^l - \mathbb{1}\right)\partial_T\right] , \end{aligned} \tag{3.94}$$

and upon acting them over a function $F(R,T) = F$ we find that, for K_0 as an simple example

$$\begin{aligned} K_1(F) =& \frac{\ell_0}{2i\hbar}\Big(R[F,T] + [F,T]R + T[F,R] + [F,R]T\Big) , \\ =& \frac{\ell_0}{2i\hbar}\Big([F, TR + RT]\Big) = -\frac{1}{i\hbar}[F, X_0] . \end{aligned}$$

The same calculation can be done for K_2 and K_3 and they will therefore lead us to the conclusion that, in fact, the Killing vectors found by our formalism are directly associated with the ones found in [4] using the quantization of the Poisson algebraic structure, namely

$$K_\mu(F) = -\frac{1}{i\hbar}[F, X_\mu] , \tag{3.95}$$

which is the result we intended to prove. $\square$

If we remember that the map defined in (3.15) uses the Lie algebra underlying the symmetry of the $ncAdS_2$, the expression obtained above can be interpreted as the direct consequence of fact that Killing vector fields are the manifestation of symmetries in the context of the non-commutative surfaces. With this result could also try to verify the non-commutative AdS/CFT correspondence for the massive and interacting case writing the desired action with respect to the local coordinates since we have an exact form for the Killing vector fields. In the next section we will finish our analysis of the geometry of the $ncAdS_2$ as a quantum surface finding a way to integrate functions defined over it using the non-commutative eigenfunctions that we will define, we will also show that it is possible to decompose the functions over $ncAdS_2$ as linear combinations of this kind of eigenfunctions.

3.7 Euclidean AdS_2 surface eigenfunctions and non-commutative integration

In this section I try to construct a well defined way of integrating functions over the quantum AdS_2 following a simmilar path as found in [48], we will define functional that should map the zeroth term of the formal power expansion of the functions F in $\mathcal{C}_\hbar$ as a linear combination of non-commutative eigenfunctions. As a starting point, one could try to solve the action for the commutative Laplacian in a arbitrary function defined on $EAdS_2$

$$\begin{aligned}\Delta f(r,t) &= \frac{1}{\sqrt{g}}\partial_i\Big(g^{ij}\sqrt{g}\;\partial_j f(r,t)\Big) = \frac{1}{\ell_0^2}\Big[\partial_r\Big(r^2\partial_r f(r,t)\Big) + r^{-2}\partial_t^2 f(r,t)\Big] \ ,\\ \Delta f(r,t) &= \frac{r^2}{\ell_0^2}\partial_r^2 f(r,t) + \frac{2r}{\ell_0^2}\partial_r f(r,t) + (r\ell_0)^{-2}\partial_t^2 f(r,t) \ ,\end{aligned} \tag{3.96}$$

where we used that

$$g^{ij} = \frac{1}{\ell_0^2}\begin{pmatrix} r^2 & 0 \\ 0 & r^{-2}\end{pmatrix} \ , \qquad \sqrt{g} = \ell_0^2 \ .$$

we can assume that the solution for the equation $\Delta f(r,t) = 0$ have the form

$$f(r,t) = r^\alpha R(r^\beta)T(t) \ , \tag{3.97}$$

for arbitrary α and β we could impose some restrictions to these coefficients in order to find a simple solution. Setting them to $\alpha = 0$ and $\beta = -1$ we find a simple set of solutions for the separeted ODE's. For the coordinate t we get as a solution the superposition of all plane waves for the parameter $\lambda^2 > 0$, showing explicitly the translation invariance in the t direction

$$f(r,t) = \frac{1}{2\pi}\int_{\mathbb{R}} R_\lambda\left(\frac{1}{r}\right)e^{i\lambda t}d\lambda \ , \tag{3.98}$$

with $R_\lambda(\frac{1}{r})$ being the solution in "momentum" space. Solving for $R_\lambda\left(\frac{1}{r}\right)$ one can get to the following

$$r^4 R''_\lambda\left(\frac{1}{r}\right) + 2r^3 R'_\lambda\left(\frac{1}{r}\right) - \lambda^2 R_\lambda\left(\frac{1}{r}\right) = 0 \implies R_\lambda(r) = a_\lambda\cosh\left(\frac{\lambda}{r}\right) + ib_\lambda\sinh\left(\frac{\lambda}{r}\right), \tag{3.99}$$

to have well behaved solutions in the limits $r \to 0$ and $r \to \infty$ we must impose some conditions in the constants a_λ and b_λ. If we define

$$\alpha_\lambda = \frac{a_\lambda + ib_\lambda}{2} \ ,$$

it is easy to see that the general solution for $\Delta f(r,t)$ is of the type

$$f(r,t) = A\left(\frac{1}{r} + it\right) + B\left(\frac{1}{r} - it\right) \tag{3.100}$$

with $A(\xi)$ and $B(\xi^*)$ for $\xi = \frac{1}{r} + it$ and

$$\begin{aligned}A(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} \alpha_\lambda \exp\left[\lambda\Big(\frac{1}{r} + it\Big)\right]d\lambda \ ,\\ B(\xi^*) &= \frac{1}{2\pi}\int_{\mathbb{R}} \alpha^*_\lambda \exp\left[-\lambda\Big(\frac{1}{r} - it\Big)\right]d\lambda \ .\end{aligned} \tag{3.101}$$

One can also verify that the solution to the Laplacian (3.96) is (3.97) if one rewrites it as

$$\partial_\xi \partial_{\xi^*} f(\xi, \xi^*) = 0 \implies f(\xi, \xi^*) = A(\xi) + B(\xi^*) \ .$$

Now we introduce the total integral of a function $F(r,t)$ on $EAdS_2$ with respect to the induced metric g_{ij} as

$$I(F) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\ell_0^2}{2\pi} F_\lambda \left(\frac{1}{r}\right) e^{i\lambda t} d\lambda dr dt \ . \tag{3.102}$$

We say that a function $F(r,t)$ is *integrable* if $I(F)$ exists, where we used (3.97) to write $F(r,t)$ as an integral. Now we turn our attention to the non-commutative case and prove the last theorem of this thesis.

Theorem 3.4. *A function $F(R,T) \in \mathcal{C}_\hbar$ will be called* ***integrable*** *over the $ncAdS_2$ if $F(R,T)$ can be written as $F(R,T) = F_+(R,T) + F_-(R,T)$ for*

$$F_\pm(R,T) = \frac{\xi_\pm}{2\pi} \int_{\mathbb{R}} \exp\Big(\pm \frac{i\lambda T}{2}\Big) \left(\frac{\pm\frac{\hbar\lambda}{2\ell_0} + R}{\pm\frac{\hbar\lambda}{2\ell_0} - R}\right)^{\frac{k\ell_0}{\hbar}} \exp\Big(\pm \frac{i\lambda T}{2}\Big) d\lambda \ ,$$

for some real parameter λ, $k^2 = k(\hbar)^2 = 1 + \frac{\hbar^2}{4\ell_0^2}$ and $\xi_\pm$ being some complex coefficients.

Proof: First we will assume that the solution can be expressed as a superposition of the analogues of plane waves in the quantum surface by defining the non-commutative Fourier transform as follows

$$F(R,T) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{T}_\lambda(T) \mathcal{R}_\lambda(R) \mathcal{T}_\lambda(T) d\lambda \tag{3.103}$$

for some real parameter λ and for $\mathcal{T}(T) = \exp(i\lambda T/2)$. Now we start solving the non-commutative Laplace equation

$$\Delta(F(R,T)) = \frac{1}{\ell_0^2} \Big(\partial_R \Big(R \partial_R(F) R\Big) + k^2 Z \partial_T^2(F) Z\Big) \ , \tag{3.104}$$

to simplify the expression above we must use the following properties

$$R\mathcal{T}_\lambda = \mathcal{T}_\lambda R + \frac{i\hbar}{\ell_0} \partial_T(\mathcal{T}_\lambda) = \mathcal{T}_\lambda \Big(R - \frac{\hbar\lambda}{2\ell_0} \mathbb{1}\Big) \ ,$$
$$Z\mathcal{T}_\lambda = \mathcal{T}_\lambda Z + [Z, \mathcal{T}_\lambda] = \mathcal{T}_\lambda Z \Big(\mathbb{1} - \frac{\hbar\lambda}{2\ell_0} Z\Big)^{-1} \ ,$$

where the last step can be calculated using $[Z, \mathcal{T}_\lambda] = -\frac{i\hbar}{\ell_0} Z \partial_T(\mathcal{T}_\lambda) Z = \frac{\hbar\lambda}{2\ell_0} Z \mathcal{T}_\lambda Z$. Organizing these terms we rewrite the non-commutative Laplacian as

$$\partial_R \Big(R \partial_R(F) R\Big) = \mathcal{T}_\lambda \partial_R \Big(\Big(R - \frac{\hbar\lambda}{2\ell_0} \mathbb{1}\Big) \partial_R(\mathcal{R}_\lambda) \Big(R + \frac{\hbar\lambda}{2\ell_0} \mathbb{1}\Big)\Big) \mathcal{T}_\lambda \ ,$$

where the exponents in $\mathcal{T}_\lambda$ could be moved to the outer parts of the expression since they commute with the derivative with respect to R. One can note that the inner terms are all R dependent we then could merge the inner terms of this expression to get

$$\partial_R \Big(R \partial_R(F) R\Big) = \mathcal{T}_\lambda \partial_R \Big(\Big(R^2 - \frac{\hbar^2\lambda^2}{4\ell_0^2} \mathbb{1}\Big) \partial_R(\mathcal{R}_\lambda)\Big) \mathcal{T}_\lambda \ . \tag{3.105}$$

For the other term we use the remaining properties

$$Z\partial_T^2(F)Z = -k^2\lambda^2\mathcal{T}_\lambda\Big(\mathbb{1} - \frac{\hbar^2\lambda^2}{4\ell_0^2}Z^2\Big)^{-1}\mathcal{R}_\lambda\mathcal{T}_\lambda \ .$$

By taking $\Delta\Big(F(R,T)\Big) = 0$ the radial equation becomes

$$\frac{d}{dR}\Big(\Big(R^2 - \frac{\hbar^2\lambda^2}{4\ell_0^2}\mathbb{1}\Big)\mathcal{R}_\lambda^{'}(R)\Big) - k^2\lambda^2 Z\Big(\mathbb{1} - \frac{\hbar^2\lambda^2}{4\ell_0^2}Z^2\Big)^{-1}Z\mathcal{R}_\lambda = 0 \ , \tag{3.106}$$

and after some manipulations[43] we finally get to

$$\frac{d}{dr}\Big(\Big(R^2 - \frac{\hbar^2\lambda^2}{4\ell_0^2}\Big)\mathcal{R}_\lambda^{'}\Big) - \frac{k^2\lambda^2}{R^2 - \frac{\hbar^2\lambda^2}{4\ell_0^2}}\mathcal{R}_\lambda = 0 \ . \tag{3.107}$$

If one look carefully, the equation above resembles the general Legendre equation for the commutative case. We write it below

$$(1-x^2)y'' - 2xy' + \Big(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\Big)y = 0 \tag{3.108}$$

and its solution can be expressed in terms of the hypergeometric function for $|1-z|< 2$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\Big(\frac{1+z}{1-z}\Big)^{\mu/2} {}_2F_1\Big(-\nu, \nu+1 \ ; \ 1-\mu; \ \frac{1-z}{2}\Big) \ . \tag{3.109}$$

For the case when $\nu = 0$ the hypergeometric function becomes 1 and the legendre equation yelds eight different solutions. If we extend this analysis to the non-commutative case, we must choose the solution that have the correct commutative limit and as done in [48] which gives as a final result

$$P_0^{-\mu}(z) = \frac{1}{\Gamma(1+\mu)}\Big(\frac{1-z}{1+z}\Big) \ . \tag{3.110}$$

In order to show this using another approach, we will solve the equation (3.107) directly as we did in the commutative case. First consider the EDO

$$\frac{d}{dx}\Big((x^2-a^2)y'(x)\Big) - \frac{b^2}{x^2-a^2}y(x) = 0$$

now we do a change of variables and apply the same strategy used in (3.96) to achieve the solution

$$y(x) = C_1\cosh\left(\frac{b}{a}\tanh^{-1}\left(\frac{x}{a}\right)\right) + iC_2\sinh\left(\frac{b}{a}\tanh^{-1}\left(\frac{x}{a}\right)\right) \ . \tag{3.111}$$

Using the logarithm form of the inverse hyperbolic tangent function as $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ and expressing $\xi = \frac{C_1+iC_2}{2}$ we get to the following

$$y(x) = \xi\left(\frac{a+x}{a-x}\right)^{\frac{b}{2a}} + \xi^*\left(\frac{a-x}{a+x}\right)^{\frac{b}{2a}} \ , \tag{3.112}$$

for some coefficients ξ and ξ^*, we can also apply the solution written above to (3.107) because all the functions and steps used in solving the EDO are well defined in our setting and by simply

[43] The inversion of the terms is well defined because we constructed the whole field of fractions and functions with inverses in the previous section and it guarantee the existence of an inverse term.

taking $x = R$, $b = \pm k\lambda$ and $a = \pm\frac{\hbar\lambda}{2\ell_0}$ where the choice of sign is related to the sign of λ one get two distinct solutions that have the form

$$\mathcal{R}_{\pm\lambda}(R) = \xi_\pm \left(\frac{\pm\frac{\hbar\lambda}{2\ell_0} + R}{\pm\frac{\hbar\lambda}{2\ell_0} - R}\right)^{\frac{k\ell_0}{\hbar}} \quad . \tag{3.113}$$

where $\xi_\pm$ are the respective coefficients obtained from the manipulations of ξ and ξ^* in (3.112) . Now we can write the full solution for $F(R,T)$

$$F(R,T) = F_+(R,T) + F_-(R,T) \ , \tag{3.114}$$

for

$$F_\pm(R,T) = \frac{\xi_\pm}{2\pi}\int_{\mathbb{R}} e^{\pm i\lambda T/2}\left(\frac{\pm\frac{\hbar\lambda}{2\ell_0} + R}{\pm\frac{\hbar\lambda}{2\ell_0} - R}\right)^{\frac{k\ell_0}{\hbar}} e^{\pm i\lambda T/2} d\lambda \ . \tag{3.115}$$

This final solution resembles the commutative one and is equal to the solution found in [48]. With this last construction we finish our construction of the Riemannian geometry of the quantum AdS_2 adding to its analysis a wide range of tools that will help us to formalize some results obtained in [4], [6], [7], [18] and [48] justifying why some steps of these papers following a more rigorous approach. In the next chapter, we will discuss potential future developments that could arise from the results obtained in this thesis, presented in a simple and non rigorous manner.

4 Future developments from this work

This chapter provides a final disccussion on the possible topics one could try to analyse using the techniques and tools provided by our work on the geometry of quantum surfaces. We expose a concrete and solid method for the study of the main properties of the AdS_2 as a quantum surface and we think that this can be extrapolated to other surfaces as the Fuzzy sphere (see [43], [44], [45] for some examples of applications) and to the study of field equations and field theories over these surfaces. We start by discussing the possibility of expanding our analysis to an analogue of the Einstein-Hilbert equation, exposing some questions about the well definiteness of the integral, the choice for some of the objects in the integral and the meaning of the result obtained in our setting. If we achieve a consistent definition for the non-commutative analogue of the Einstein-Hilbert action we could also consider the possibility of applying it in the construction of the Jackiw-Teitelboim gravity upon assuming that the scalar field considered is a non-commutative one, and if we follow the steps we took in order to calculate the fields for the non-commutative AdS_2/CFT_1 correspondence, as shown in the chapter 2, we could verify if the non-commutative field equations have some additional properties or quantum corrections. After we also discuss the application of the framework constructed in this thesis to the case of a spinor field, we do this in unrigorously but from this discussion we will observe a lot of details that we could explore in other papers in the future.

4.1 Einstein-Hilbert Action

We could try to define a non-commutative analogue for the Einstein-Hilbert action S_{EH} using the commutative analogue as an educated guess and for this we will define all its constituents individually. Firstly consider the determinant of an non-commutative metric. There is no unique-way of defining what is an suitable non-commutative determinant and simultaneously guarantee that it obeys all laws regarding determinants of commuting variables. As our candidate consider the definition below for the non-commutative left acting determinant, for $A \in GL(\mathcal{C}_\hbar, n)$

$$det^l(A) = \frac{1}{n!}\epsilon_{i_1\ \dots\ i_n}\epsilon_{j_1\ \dots\ j_n}\mathrm{Sym}\Big(\prod_{k=1}^{n} a_{i_k j_k}\Big)\ , \tag{4.1}$$

where a_{ij} are the matrix elements of A and the Sym operation is defined by

$$\mathrm{Sym}(A_1 \dots A_n) = \frac{1}{n!}\sum_{k=1}^{n}\mathrm{Perm}\Big(A_1 \dots A_n\Big)\ ,$$

and can be understood as the sum of all permutations over the generators producing all A_i elements. The determinant defined above doesn't obey the usual rule $det(AB) = det(A)det(B)$, but it obey an slightly different version of this property

$$\mathrm{Sym}\Big(det(AB)\Big)[F] = \mathrm{Sym}\Big(det(A)det(B)\Big)[F] \ , \tag{4.2}$$

for F some element of $\mathcal{C}_\hbar$ and the square bracket means that the action over F is done after the permutation. We should take into account two aspects of this definition, the first one is that as we will be using the determinant inside a continuous function and this function will be inside a trace that will make the role of a non-commutative integration we expect that the symmetrization and other ordering problems regarding to action elements will not be so annoying. The second aspect is that the non-commutative entrys of our metric are all R-dependent, so we could treat it commutatively without much to concern, but as we want to give a rigorous prescription for other types of surfaces we will consider the general case instead.

Our square-root function would obey the properties below for $A, B \ \in \mathcal{C}_\hbar$ and $\alpha \in \mathbb{C}$

$$\begin{aligned}
&(i)\ \sqrt{\alpha AB} = \sqrt{\alpha}\sqrt{A}\sqrt{B} \ , \\
&(ii)\ \sqrt{A}\sqrt{A} = A \ , \\
&(iii)\ \frac{d}{dA}\Big(\sqrt{A}\Big) = \frac{1}{2}\Big(\sqrt{A}\Big)^{-1} \ ,
\end{aligned} \tag{4.3}$$

using the definition for the derivative of the inverse function constructed in [1] one can calculate the derivatives of the square-root up to an arbitrary order and define a suitable formal Taylor series expansion. With all of this we will define the Einstein-Hilbert Action for the quantum AdS_2 as follows

$$S_{EH} := \frac{c^4}{16\pi G}\mathrm{Tr}\Big[\Big(R - 2\Lambda\Big)\sqrt{det^l(g_s)}\Big] \ , \tag{4.4}$$

We can also consider another strategy to verify what is the non-commutative analogue of the field equations, if we construct a non-commutative contracted Bianchi identity, using the first identity we obtained when we defined the non-commutative curvature tensor, we could explore a rigorous way to make the following contraction

$$\nabla_\mu \mathcal{R}^\mu{}_{\nu\alpha\beta} + \nabla_\nu \mathcal{R}_{\alpha\beta} - \nabla_\alpha \mathcal{R}_{\nu\beta} = 0 \ ,$$

following the fact that, for a pseudo-Riemannian real calculus over $ncAdS_2$, one would expect the scalar curvature $\mathcal{R}$ to be Hermitian (see [49]), we can extrapolate from the equation above and attempt to find the non-commutative Einstein tensor arising from the modified contracted identity. Several questions emerge: Should the trace here have the same structure as the one defined in the Arlind paper? Should we construct it directly from the eigenfunctions? Since, within the trace, we can swap the order of elements in our non-commutative algebra while varying the action, the ordering will be ignored. Another crucial question is: What is the interpretation of the surface elements of this integral in the non-commutative setting? If we aim to derive the non-commutative Einstein field equations from this action, we must first address these questions. Furthermore, we intend to apply this formalism to Jackiw-Teitelboim gravity, a solvable model for quantum black holes that is well understood in the commutative case, as seen in [50]. This model should be the natural next step to verify the consistency of the framework developed in my thesis.

4.2 The Spin connection and the Dirac operator

As a naive tentative I will try to define not too rigorously the volume form[44], the spin connection and the Dirac operator in the quantum AdS_2. We start by denoting as $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k\mathfrak{g}$ for $T^k\mathfrak{g} = \bigotimes_{i=1}^{k} \mathfrak{g}$ being the k-th tensor power of the algebra $\mathfrak{g}$ seen as a vector field over $\hat{\mathcal{C}}_\hbar$ spanned by the derivations ∂_R and ∂_T. We introduce the exterior algebra by considering the two sided ideal $\mathcal{I}_T$ defined by the following relation

$$a \otimes b + b \otimes a + a \otimes a + b \otimes b - (a+b) \otimes (a+b) \ ,$$

for a and $b \in \mathfrak{g}$. Now we define the exterior algebra[45] $\Lambda(\mathfrak{g})$

$$\Lambda(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}_T \ .$$

With these definitions we introduce the exterior derivative d of elements $F \in \hat{\mathcal{C}}_\hbar$

$$dF = \frac{\ell_0}{i\hbar}\Big([F,T]dR - [F,R]dT\Big) \ , \tag{4.5}$$

for $dR, dT \in \Lambda^1(\mathfrak{g})$ the space of 1-forms over $\mathfrak{g}$ satisfying the duality condition $dI(\partial_J) = \delta_{IJ}$ with the indices I, J being the coordinates R and T. Now we rewrite the metric $g(\Phi_a, \Phi_b)$ as

$$g = \ell_0^2 k(\hbar)^2 R^{-2} dR \otimes dR + \frac{i\hbar\ell_0}{2}\Big(dT \otimes dR - dR \otimes dT\Big) + \ell_0^2 R^2 dT \otimes dT \ .$$

Now consider the bi-linear form $h : \Lambda^1(\mathfrak{g}) \times \Lambda^1(\mathfrak{g}) \longrightarrow \hat{\mathcal{C}}_\hbar$ defined as

$$h_{abcd}(\Omega^c_i dX^i, \Xi^d_j dX^j) = (\Omega^c_a)^* \Xi^d_b \delta_{cd}, \tag{4.6}$$

for $\Omega,\ \Xi \in \Lambda^1(\mathfrak{g})$, a and b being the indices in $\Lambda^1(\mathfrak{g})$. Now we introduce the non-commutative dual basis $\{\Theta^a\}$ satisfying

$$\Theta^a = E^a_c dX^c \ , \tag{4.7}$$

[44] I know that there is a formalization of this construction following the universal calculus discussed in the chapter 6 of [8], but in these sections I'm trying to follow closely the formalism introduced in [1-3] and see if we can generalize some steps that are well known from commutative Riemannian geometry.

[45] We can also define the product $\wedge$ induced by the tensor product $\otimes$ satisfying the following relation for the canonical surjection $\pi : T(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$

$$a \wedge b = \pi(x \otimes y) \ ,$$

for $a, b \in \Lambda(\mathfrak{g})$, $x, y \in T(\mathfrak{g})$ and $\pi(x) = a$ and $\pi(y) = b$. This will be the usual exterior product.

for c being the coordinates R and T and $E_a^c \in \hat{\mathcal{C}}_\hbar$ the non-commutative vielbein. We proceed to construct the vielbeins by using the following relation

$$g(\Phi_a, \Phi_b) := g_{ab} = h_{abcd}(\Theta^c, \Theta^d) = (E_a^c)^* E_b^d \delta_{cd} \ , \tag{4.8}$$

which gives the following system of equations

$$\begin{aligned} \ell_0^2 k(\hbar)^2 R^{-2} &= |E_R^1|^2 + |E_R^2|^2 \ , \\ \ell_0^2 R^2 &= |E_T^1|^2 + |E_T^2|^2 \ , \\ \frac{i\hbar\ell_0}{2} &= (E_T^1)^* E_R^1 + (E_T^2)^* E_R^2 \ , \\ -\frac{i\hbar\ell_0}{2} &= (E_R^1)^* E_T^1 + (E_R^2)^* E_T^2 \ . \end{aligned} \tag{4.9}$$

The solution for the system above is then given by

$$E_R^1 = \ell_0 R^{-1} \ , \quad E_R^2 = \frac{i\hbar}{2} R^{-1} \ , \quad E_T^1 = 0 \ , \quad E_T^2 = \ell_0 R \ . \tag{4.10}$$

We can also verify the converse, for e_b^a the inverse of E_a^b

$$g_{ab} (e_c^a)^* e_d^b = \delta_{cd} \tag{4.11}$$

which gives

$$e_1^R = \frac{R}{\ell_0} \ , \quad e_2^R = 0 \ , \quad e_1^T = -\frac{i\hbar}{2\ell_0^2} R^{-1} \ , \quad e_2^T = \frac{R^{-1}}{\ell_0} \ . \tag{4.12}$$

Now we can construct explicitly the dual basis

$$\begin{aligned} \Theta^1 &= E_R^1 dR + E_T^1 dT = \ell_0 R^{-1} dR \ , \\ \Theta^2 &= E_R^2 dR + E_T^2 dT = \frac{i\hbar}{2} R^{-1} dR + \ell_0 R dT \ . \end{aligned} \tag{4.13}$$

which satisfies

$$g_{ij} dX^i \otimes dX^j = \delta_{ab} (\Theta^a)^* \otimes \Theta^b \ . \tag{4.14}$$

Now we introduce the set of gamma matrices $\{\gamma^a\}$ that satisfies the twisted anti-commutator defined below

$$[\gamma^a, \gamma^b]_+ = (\gamma^a)^* \gamma^b + \gamma^b (\gamma^a)^* = 2 g^{ab} \mathbb{1} \tag{4.15}$$

where they should be explicitly[46]

$$\gamma^R = \frac{R}{\ell_0} \sigma_1 \ , \qquad \gamma^T = -\frac{R^{-1}}{\ell_0} \left(\frac{i\hbar}{2\ell_0} \sigma_1 - \sigma_2 \right) \ , \tag{4.16}$$

these matrices satisfies the following commutation relations

$$[\gamma^R, \gamma^R] = [\gamma^T, \gamma^T] = 0 \ , \qquad [\gamma^R, \gamma^T] = -[\gamma^T, \gamma^R] = \frac{2i}{\ell_0^2} \sigma^3 \tag{4.17}$$

[46] One can directly verify (2.72) for these gamma matrices. The conjugation in the definition of the anti-commutator is needed to guarantee the correct anti-commutation relations.

for $[\gamma^a, \gamma^b] := \gamma^a\gamma^b - \gamma^b\gamma^a$. Since we are considering the Euclidean case we can obtain the local gamma matrices $\hat{\gamma}$ which are transformed by the vielbein fields E^a_b. These gamma matrices satisfy $[\hat{\gamma}^i, \hat{\gamma}^j]_+ = 2\delta^{ij}\mathbb{1}$ and can be explicitly constructed as

$$\begin{aligned}\hat{\gamma}^1 &= \hat{\gamma}_1 = E^1_R\gamma^R + E^1_T\gamma^T = \sigma_1 \ , \\ \hat{\gamma}^2 &= \hat{\gamma}_2 = E^2_R\gamma^R + E^2_T\gamma^T = \sigma_2 \ .\end{aligned} \tag{4.18}$$

We proceed to calculate the vielbein one-form

$$\begin{aligned}e &= e_a dX^a = \sigma_i E^i_a dX^a = \Big(\sigma_1 E^1_R + \sigma_2 E^2_R\Big) dR + \Big(\sigma_1 E^1_T + \sigma_2 E^2_T\Big) dT \ , \\ e &= \ell_0 R^{-1}\left(\sigma_1 + \frac{i\hbar}{2\ell_0}\sigma_2\right) dR + \ell_0 R \sigma_2 dT \ .\end{aligned} \tag{4.19}$$

Calculating the exterior derivative of the vielbein one-form we obtain

$$de = \ell_0 \sigma_2 dR \wedge dT \ . \tag{4.20}$$

Now we apply the no-torsion condition to find the spin connection one-form $\omega = \omega_a dX^a$

$$\begin{aligned}& de + \omega \wedge e + e \wedge \omega = 0 \ , \\ & \ell_0\sigma_2 dR \wedge dT = \Big([\omega_T, e_R] + [e_T, \omega_R]\Big) dR \wedge dT\end{aligned} \tag{4.21}$$

Since de depends on σ_2 and the right hand side of the equation above depends on the commutators with e_R and e_T, this implies that

$$\omega = \frac{\hbar R^{-1}}{4\ell_0}\sigma_3 dR + \frac{R}{2i}\sigma_3 dT \tag{4.22}$$

Now we find a candidate for the Dirac Operator

$$\begin{aligned}D &= \gamma^i(\partial_i + \omega_i) = \gamma^R(\partial_R + \omega_R) + \gamma^T(\partial_T + \omega_T) \ , \\ D &= \left(\frac{R}{\ell_0}\partial_R + \frac{1}{2\ell_0} - \frac{i\hbar}{2\ell_0^2}R^{-1}\partial_T\right)\sigma_1 + \left(\frac{1}{\ell_0}R^{-1}\partial_T + \frac{i\hbar}{4\ell_0^2}\right)\sigma_2 \ .\end{aligned} \tag{4.23}$$

As a next step, we must verify the consistency of the Dirac operator obtained above and find a proper chirality operator that commutes with it. There are numerous details and definitions that need to be provided to make the construction rigorous. However, before delving into these details, we should address certain questions. For example, why do we obtain a twisted anticommutator relation? We suspect that this arises because our metric is not only symmetric but also Hermitian. Would this issue persist if we consider using the metric g_s? Utilizing g_s would clearly add an overall scale factor to some terms of the dual basis, but it would significantly simplify our calculations. The final point to consider is whether the properly defined Dirac operator would agree with the operator found in [4], for example. By comparing with their approach, we can verify the consistency of the prescription applied above. These are some ideas and results that I might pursue in future papers, expanding on the findings of this thesis.

5 Conclusion

As discussed in the introduction of this thesis, the primary goal of this research is to explore an alternative approach to constructing the geometric properties of the $ncAdS_2$ as a quantum surface. This involves analyzing its commutative limit, symmetries, and other attributes that could shed light on the results obtained through deformation quantization and various perturbative methods. During the development of the mathematical framework necessary for this endeavor, we applied the methods from [1], [2], [43], [49] to the $ncAdS_2$ and extended our investigation to include structures such as Killing vector fields and potential non-commutative eigenfunctions. In recent years, numerous topological and geometric aspects of quantum surfaces have been studied, with some properties of classical surfaces being either generalized or not to the non-commutative setting using advanced analytical tools from pseudo-Riemannian calculus (e.g., [49]). Several papers ([52] - [55]) have calculated the scalar curvature for certain non-commutative surfaces, defining it as a specific term in the asymptotic expansion of the heat kernel, similar to classical Riemannian geometry. However, these works typically start with a spectral triple, where the metric is implicitly defined by the Dirac operator. This approach does not clarify whether a bilinear form representing the metric corresponding to the Dirac operator exists, nor does it address the existence of structures like the Levi-Civita connection.

Given this context, a simpler approach becomes appealing, one that defines a module along with a bilinear form and develops the necessary conditions to make this framework both well-defined and comprehensible. While the existence of a Levi-Civita connection is not always assured in this setup, the original authors have established the conditions for its uniqueness, as well as for the existence of curvature and the Ricci scalar. Utilizing this framework, I constructed a suitably structured module over a non-commutative unital algebra that could represent the $ncAdS_2$ and examined the geometric aspects of this surface.

Using the metric found and applying the pseudo-Riemannian formalism to our case I found the Levi-Civita connection, which is non-unique in our setting, and without loss of generality, we choose the simpler one where we set the respective coefficients $\gamma_{(\alpha)\mu\nu}$ and $\sigma_{(a)ij}$ equal to zero, giving the following non-commutative Christoffel symbols as result

$$\Gamma^{\rho}_{\alpha\nu} = \frac{1}{2}P^{\rho\lambda}(\partial_\alpha P_{\lambda\nu}) = -\frac{1}{2}\Big(\epsilon_\alpha{}^{\rho\theta}\Pi_{\theta\nu} - \frac{i\hbar}{\ell_0^4}X^\rho X_\alpha X_\nu\Big) \ ,$$

$$\Gamma^{c}_{ab} = g^{cd}(\Phi^{\alpha}_{d})^{*}\eta_{\alpha\beta}\partial_a\Phi^{\beta}_{b} \ .$$

Using the metric we defined we determined the correct braiding function that would have to be imposed over the ordering of elements of the metric in order to have objects that transform correctly for our case. This is a well known fact, that some ambiguites arise in the ordering implied and one should set a suitable braiding function in order to remove some of these ambiguities. In our case, the chosen function was the symmetric mapping defined in (3.44) and when applied to our calculations it gives as consequence results that agree with the ones found in [4], [7] and [48]. One of this results is proven in the theorem (3.1) which shows that the non-commutative Laplacian satisfies the following equation

$$\Delta(F) := \partial_a\Big(\mathcal{S}\Big(g^{ab}\partial_b(F)\Big)\Big) = \frac{2}{\hbar^2}\Big(\frac{1}{\ell_0^2}(X^\nu)^l(X^\mu)^r\eta_{\mu\nu} + \mathbb{1}\Big)F \ .$$

Subsequently, I focused on the curvature and the Ricci scalar. Utilizing the developed framework, we proved the first Bianchi identity and established several symmetry properties of the Riemann tensor. We derived a closed form for the tensor, demonstrating that when expressed in a specific basis, it retains the form of the commutative Riemann tensor in the ambient coordinates. This result was also extended to local coordinates. Based on these findings, we defined the Ricci scalar with a particular ordering and discovered that by altering this ordering, we could avoid introducing any non-commutative corrections

$$\mathcal{R} = -\frac{\mathbb{1}}{\ell_0^2}\Big(2 - \frac{\hbar^2}{\ell_0^2}\Big) \ .$$

As can be found in [44], the author defined the Ricci scalar in such a way that it showed non-commutative corrections up to the fourth power of the non-commutative parameter. This suggests that similar corrections might be expected in our case. However, we also found that applying the same construction to the right module structure and symmetrizing it yields a Ricci scalar that matches its commutative counterpart, as one can see below

$$\mathcal{R} = \frac{1}{2}\Big(\mathcal{R}^l + \mathcal{R}^r\Big) = g^{ba}\Big(\mathcal{R}^c{}_{acb}\Big) = -\frac{2}{\ell_0^2}\mathcal{S}(\mathbb{1}\otimes\mathbb{1}) \ ,$$

This indicates that an appropriate ordering prescription can be applied to the Ricci scalar, as defined with respect to the ambient coordinates, to eliminate these corrections. Moreover, by altering the ordering of indices in the definition of the Ricci scalar in local coordinates, we obtained a scalar devoid of non-commutative corrections. These ambiguities arise from the fact that we haven't constructed a real pseudo-Riemannian calculus on the *ncAdS* surface, leading to some results being unclear. This issue will be addressed in future work, as outlined in the concluding section of this thesis.

Building on the possibilities introduced by applying the framework developed in this thesis to classical geometric objects, we sought to define a Killing vector field using the previously defined metric. By employing the symmetric map, we demonstrated that the Killing vector field, which solves the Killing equation (3.89), is equivalent to the traditional construction that relates the action of classical Killing vectors to the Poisson bracket with respect to the ambient coordinates.

In the non-commutative setting, this yields the expression:

$$K_\mu(F) = -\frac{1}{i\hbar}[F, X_\mu] \ .$$

This result represents an advancement over previous findings, such as those in [18], as it explicitly describes the Killing vectors with respect to local non-commutative coordinates. This explicit formulation enables the determination of additional related objects and opens up new avenues for exploration. As a further demonstration of the efficacy of this formalism, we utilized the Laplacian constructed earlier to identify the non-commutative functions that solve the homogeneous case. These functions were associated with non-commutative eigenfunctions, which can be employed to integrate functions over the $ncAdS_2$

$$F(R,T) = \frac{\xi_+}{2\pi}\int_{\mathbb{R}} e^{i\lambda T/2}\left(\frac{\frac{\hbar\lambda}{2\ell_0}+R}{\frac{\hbar\lambda}{2\ell_0}-R}\right)^{\frac{k\ell_0}{\hbar}} e^{i\lambda T/2}d\lambda + \frac{\xi_-}{2\pi}\int_{\mathbb{R}} e^{-i\lambda T/2}\left(\frac{-\frac{\hbar\lambda}{2\ell_0}+R}{-\frac{\hbar\lambda}{2\ell_0}-R}\right)^{\frac{k\ell_0}{\hbar}} e^{-i\lambda T/2}d\lambda \ .$$

In conclusion, the developments and discoveries presented in this thesis provide a foundation for investigating how classical gauge and field theories behave over non-commutative surfaces. Additionally, they offer insights into potential non-commutative corrections that could arise from the application of this specific mathematical framework.

5.1 Final considerations and Acknowledgements

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Appendix

A Non-commutative Calculator

In this appendix, we explain how the non-commutative calculator works. The calculator was sent together with the main text of this thesis to the evaluation committee, it will be made avaliable on github as soon as all intended updates are applied to the main program. We also discuss how to configure it for use on your PC. When performing extensive calculations in non-commutative algebraic settings, small mistakes in manipulations can turn a simple commutative calculation into a complex and error-prone task. To address this, I have initiated the development of a program with a specific data structure that will enable the use of this tool for calculations involving algebraic elements, functions, and vectors being also possible to extend it to tensors. This will provide an invaluable tool for verifying the accuracy of the results obtained.

We begin this appendix by explaining the main functions of this tool and how to set up the algebra in the program, including the necessary information to ensure it works correctly. In the second part, we describe how we used the calculator in this work, providing examples and presenting additional results not explicitly included in this thesis. These results can be demonstrated directly using the provided tool.

A.1 Data Structure and Useful Functions

The objective of the program is to explicitly calculate the steps of a calculation, perform necessary simplifications, and save these steps in a LaTeX file. This file can then be copied into a user's main LaTeX document for further use. With this goal in mind, we begin by defining the main *Algebra* (as a class) and introducing the fundamental elements of our data structure, the *Monomial*.

The elements of the *Algebra* class are the foundation of our data structure. We initialize an algebra by setting its main attributes at the beginning of the program. Below, we outline the most important attributes required for proper initialization:

- To initialize an *Algebra*, we need to specify the list of algebra elements. These elements, represented as strings in a list, are the symbols that generate the algebra. We also need to declare the dimension of the algebra.

- If we want to define a module structure using copies of the algebra, we must declare the signature. This is done by setting `ALGEBRA.signature = []`, where the list contains the chosen signature for our algebra. In our case, we used the list `[1, 1, -1]`.

- We must also define the pairs of inverse elements, if they exist, using the method *definverses*. We have to set the commutation relations between generators of the algebra we declared using the method *set_commutation_relation*. In this method, we have to input the order in which we are calculating the commutator as the first entry. For example, if we want to declare $[T, Z]$, we must input `"TZ"` as the first entry. The second entry should be a list with the elements of the monomial that is the result of the commutator. As a last step, we declare the derivatives using the method *set_derivative*, where the first entry is the variable in which the derivative is being calculated and the second entry is the resulting monomial.

As an example, we provide a fragment of the main Python file used in this thesis. Here we declare the algebra of elements R, T, Z, W, x, y, z, where x, y, and z stand for X^0, X^1, X^2, and T is the inverse of W.

```python

# Inicialization - Declaring elements, algebra, inverses and commutation relations.
elements = ["R", "T", "Z", "W","x","y","z"]
dimension = 3
algebra = Algebra(elements, dimension)
algebra.signature = [1,1,-1]
algebra.definverses("R", "Z")
algebra.definverses("T", "W")
algebra.set_commutation_relation("TZ", monomial_to_list(Monomial("ZZ", 1, 1, -1, 1, 1)))
algebra.set_commutation_relation("RY", monomial_to_list(Monomial("ZZ", 1, 1, -1, 1, 1)))
algebra.set_commutation_relation("WR", monomial_to_list(Monomial("WW", 1, 1, -1, -1, 1)))
algebra.set_commutation_relation("WZ", monomial_to_list(Monomial("WZZW", 1, 1, -1, -1, 1)))
algebra.set_commutation_relation("RT", monomial_to_list(Monomial("", 1, 1, -1, 1, 1)))
algebra.set_commutation_relation("xy", monomial_to_list(Monomial("z", 1, 1, 0, -1, 1)))
algebra.set_commutation_relation("yz", monomial_to_list(Monomial("x", 1, 1, 0, 1, 1)))
algebra.set_commutation_relation("zx", monomial_to_list(Monomial("y", 1, 1, 0, 1, 1)))
algebra.set_derivative("R", Monomial("T", -1, -1, 1, 1, 1))
algebra.set_derivative("T", Monomial("R", -1, -1, 1, -1, 1))
algebra.set_derivative("x", Monomial("x", -1, -1, 0, -1, 1))
```

```
algebra.set_derivative("y", Monomial("y", -1, -1, 0, -1, 1))
algebra.set_derivative("x", Monomial("z", -1, -1, 0, 1, 1))
```

Now we outline how the calculations are made and the data structure we used to create this calculator. A monomial is an element of the *Monomial* class and has the following attributes:

- The "words" associated with elements from the algebra are registered as a string and declared as the first initialization parameter of a *Monomial* element. This is because we are working with elements from a non-commutative algebra, and the string data type is non-commutative by definition. Each letter of a word is taken from the set of generators of the algebra associated with the monomial. For an arbitrary monomial `MONO`, the command `MONO.monomial` refers to the string associated with the generators of an algebra element.
- The second parameter required to initialize a *Monomial* element is the power of the imaginary unit. In all calculations, this will be taken modulo 4, and its power needs to be set initially. For an arbitrary monomial `MONO`, the power of the imaginary unit is referred to as `MONO.im` and can yield any integer between 0 and 3.
- The third parameter required for the initialization of a monomial is the power of the $\hbar$ constant. In the first version of the calculator, I defined the *Monomial* class using only $\hbar$ and ℓ_0 as the constants of our theory. This imposed a constraint if we wanted to track what happens with other constants in the calculation, and it prevented us from using these new constants in future calculations. To solve this problem, in the second version, I added a list to the third entry. This list of constants should be defined where we declare the `ALGEBRA` we are using. This list also includes constants, symbols, scale factors, and other relevant elements for our calculations. These symbols will appear explicitly in all calculations. For an arbitrary monomial `MONO`, the list of symbols can be obtained using the command `MONO.sym_list`, with each entry referring to the power of each constant declared in the definition of the main algebra we are working with. In the old version, the third entry only referred to the power of $\hbar$ and the fourth entry to the power of the scaling factor ℓ_0.
- The last two parameters of our object `MONOMIAL` are the numerator and the denominator of a possible fraction we could obtain in some calculations.

Consider, as an example of the construction above, the monomial defined in the older version of the calculator[47] as `MONO1 = Monomial('T', 1, 2, -2, 1, 2)`, this monomial is equivalent

[47] From this point forward, whenever we refer to a monomial in this appendix, we are referring to the older version, as it was predominantly used in my work.

to MONO1 $= \frac{i\hbar^2}{2\ell_0^2}T$, and as a last example, MONO2 = Monomial(' ', -1, -1, 2, 4, 3) $= \frac{4\ell_0^2}{3i\hbar}\mathbb{1}$, where we used the empty string as the unit of our algebra.

In the main file, we defined the multiplication from the right and from the left between elements of the class *Monomial*. The sum of monomials gives rise to an element of the class *Expression*. An element of the *Expression* class has only two needed parameters: the first one is a list of monomials in which the sum is applied between them, and the second is the line index, which starts at 0 by default and will be used to track how many operations are executed in the expression between lines of the calculations. This will only be functional in the newer version of the code. As an example, if we execute the sum of MONO1 with MONO2 from the last example, we should obtain the following set of outputs: MONO1 + MONO2 = Expression([Monomial('T', 1, 2, -2, 1, 2), Monomial(' ', -1, -1, 2, 4, 3)], 0) = $\left(\frac{i\hbar^2}{2\ell_0^2}T + \frac{4\ell_0^2}{3i\hbar}\mathbb{1}\right)$. The parentheses are implied since we also defined the multiplication of an *Expression* by elements of the class *Monomial*, by constants, and by other elements of the class *Expression*.

When working with the module structure introduced in the main text of the thesis, we should use the elements of the class *Vector*. To define an element of this class, we should declare as the first parameter the algebra we are using for our calculations and as the second parameter a list of elements. In our case, this list has a length of 3 since the rank of our module is also 3. To demonstrate this structure, we declare here some objects of the thesis in the data structure we chose:

```
# Defining some elements of the thesis in the data structure we choose

X0 = Expression([Monomial("TR",0,0,1,-1,2),Monomial("RT",0,0,1,-1,2)],0)
X1 = Expression([Monomial("TRT",0,0,1,-1,2),Monomial("Z",0,0,1,-1,2),Monomial("Z"
    ,0,2,-1,-1,8),Monomial("R",0,0,1,1,2)],0)
X2 = Expression([Monomial("TRT",0,0,1,-1,2),Monomial("Z",0,0,1,-1,2),Monomial("Z"
    ,0,2,-1,-1,8),Monomial("R",0,0,1,-1,2)],0)
R0 = differentiate(algebra, "R", X0)
R1 = differentiate(algebra, "R", X1)
R2 = differentiate(algebra, "R", X2)
T0 = differentiate(algebra, "T", X0)
T1 = differentiate(algebra, "T", X1)
T2 = differentiate(algebra, "T", X2)
X = Vector(algebra,[X0,X1,X2])
X_R = Vector(algebra,[R0,R1,R2])
X_T = Vector(algebra, [T0,T1,T2])
```

In this last fragment of code, we first introduced the components of the vector $\overrightarrow{X}$. After this, we differentiated them with respect to R and T, respectively, and defined the vectors from these components. One can find these objects in the equations (3.2) and (3.32) of the main text.

We finish this subsection by explaining the most useful function of this program, the function *solve(ALGEBRA, EXPRESSION)*. This function is mainly used to simplify an *Expression* by finding some anagrams between the words of the monomials contained in it, applying the rules defined in the main `ALGEBRA`, and executing the simplification of terms that have the same powers of the constants and the same combinations of letters. For instance, if an element has the word `"RTR"` in its defining monomial and another element has the word `"RRT"`, the function *solve()* will swap the second and third letters of the monomial using the built-in function *swap()*, resulting in an expression with the simplification of the terms with the same word `"RTR"` and the remaining terms coming from the commutation of the elements T and R. As a concrete example, suppose we define the following expressions

```
EX1=Expression([Monomial("TTR", 0, 0, 0, 1, 1), Monomial("T", 1, 1, -1, -1, 2)], 0)
EX2=Expression([Monomial("TRT", 0, 0, 0, 1, 1), Monomial("T", 1, 1, -1, 1, 3)], 0)
```

If we execute the command `solve(ALGEBRA, EX1+EX2)`, the program will simplify the expression through the following steps:

$$
\begin{aligned}
&\Big(TTR - \frac{i\hbar}{2\ell_0}T\Big) + \Big(TRT + \frac{i\hbar}{3\ell_0}T\Big) \ , \\
&\Big(TTR + T(TR + [R,T]) - \frac{i\hbar}{6\ell_0}T\Big) \ , \\
&\Big(2TTR + \frac{i\hbar}{\ell_0}T - \frac{i\hbar}{6\ell_0}T\Big) \ , \\
&\Big(2TTR + \frac{5i\hbar}{6\ell_0}T\Big)
\end{aligned}
$$

If we execute the command `print(solve(ALGEBRA, EX1 + EX2))`, we get as our output the simplified expression for the set of operations above, explicitly:

```
Expression = [Monomial("TTR", 0, 0, 0, 2.0, 1.0), Monomial("T", 1, 1, -1, 5.0, 6.0)]
```

where the numerator and denominator in the monomials are converted to the type *float* to avoid some problems.

In the next subsection, we will demonstrate how to perform direct calculations using the provided tool. We will verify some results found in the main text of this thesis.

A.2 Examples of Application

We start this subsection by demonstrating equation (3.1). In the following calculations, we will omit the quotation marks around the strings defining the word in elements of the *Monomial* class. Using the vectors declared in the last subsection, we also introduce the function `inprod(ALGEBRA, VECTOR1, VECTOR2)`, which calculates the inner product between two vectors defined using the signature declared when we initialize the main `ALGEBRA`. From the code

```
#Calculating the inner product X^\mu X_\mu
XX = inprod(algebra, X,X)
print("XX = ",XX)
```

the output is:

```
XX = Expression = [Monomial( ,0,0,2,-1.0,1.0), Monomial(ZT,1,1,1,-1.0,4.0),
Monomial(ZT,1,3,-1,-1.0,16.0), Monomial(RTZZ,1,1,1,1.0,4.0),
Monomial(RTZZ,1,3,-1,1.0,16.0), Monomial(ZZ,0,2,0,1.0,2.0),
Monomial(ZZ,0,4,-2,1.0,8.0)]
```

The output provided isn't fully simplified. To manipulate further the elements from the expression, we can use the method `EXPRESSION.swap(ALGEBRA, MONOMIAL, INDEX1, INDEX2)`, which swaps two neighboring indices in some `MONOMIAL` that belongs to the list of monomials of an element of the class `EXPRESSION`. The first step of the calculation is:

$$
\begin{aligned}
X^\mu X_\mu &= X^0 X_0 + X^1 X_1 - X^2 X_2 \ , \\
&= \Big(-\ell_0^2 \mathbb{1} - \frac{i\hbar\ell_0}{4} ZT - \frac{i\hbar^3}{16\ell_0} ZT + \frac{i\hbar\ell_0}{4} RTZZ + \frac{i\hbar^3}{16\ell_0} RTZZ + \frac{\hbar^2}{2} ZZ + \frac{\hbar^4}{8\ell_0^2} ZZ \Big) \ ,
\end{aligned}
$$

Now we use the commutation relation of $[R, T]$ in the fourth and fifth elements of the expression above with the following code:

```
#Swaping the first two letters of the fourthand fifth elements from the last
    expression

XX.swap(algebra, 3, 0, 1)
XX.swap(algebra, 4, 0, 1)
print("XX = ",XX)
```

which gives as output:

```
XX = Expression = [Monomial( ,0,0,2,-1.0,1.0), Monomial(ZT,1,1,1,-1.0,4.0),
```

```
Monomial(ZT,1,3,-1,-1.0,16.0), Monomial(TRZZ,1,1,1,1.0,4.0),
Monomial(TRZZ,1,3,-1,1.0,16.0), Monomial(ZZ,0,2,0,1.0,2.0),
Monomial(ZZ,0,4,-2,1.0,8.0), Monomial(ZZ,2,2,0,1.0,4.0),
Monomial(ZZ,2,4,-2,1.0,16.0)]
```

The operations employed were:

$$
\begin{aligned}
&= \Big(-\ell_0^2 \mathbb{1} - \frac{i\hbar\ell_0}{4} ZT - \frac{i\hbar^3}{16\ell_0} ZT + \frac{i\hbar\ell_0}{4} RTZZ + \frac{i\hbar^3}{16\ell_0} RTZZ + \frac{\hbar^2}{2} ZZ + \frac{\hbar^4}{8\ell_0^2} ZZ \Big) \ , \\
&= \Big(-\ell_0^2 \mathbb{1} - \frac{i\hbar\ell_0}{4} ZT - \frac{i\hbar^3}{16\ell_0} ZT + \frac{i\hbar\ell_0}{4} \Big(TR + \frac{i\hbar}{\ell_0} \mathbb{1} \Big) ZZ + \frac{i\hbar^3}{16\ell_0} \Big(TR + \frac{i\hbar}{\ell_0} \mathbb{1} \Big) ZZ + \frac{\hbar^2}{2} ZZ + \frac{\hbar^4}{8\ell_0^2} ZZ \Big) \ ,
\end{aligned}
$$

It is easy to see that the calculator applied the distributive rule and gathered all elements together. To finish the calculation, we just apply the function *solve()* to the last expression.

```
#Using the function 'solve' to simplify the last expression

print("XX = ", solve(algebra, XX))
```

The output of this query is: `XX = Expression = [Monomial ( ,0,0,2,-1.0,1.0)]`, which is the expected result. The set of operations done in the last step were:

$$
\begin{aligned}
&= \Big(-\ell_0^2 \mathbb{1} + \frac{i\hbar}{4\ell_0} [T, Z] + \frac{i\hbar^3}{16\ell_0} [T, Z] + \frac{\hbar^2}{4} ZZ + \frac{\hbar^4}{16\ell_0^2} ZZ \Big) \ , \\
&= \Big(-\ell_0^2 \mathbb{1} - \frac{\hbar^2}{4} ZZ - \frac{\hbar^4}{16\ell_0^2} ZZ + \frac{\hbar^2}{4} ZZ + \frac{\hbar^4}{16\ell_0^2} ZZ \Big) \ , \\
&= \Big(-\ell_0^2 \mathbb{1} \Big) \ .
\end{aligned}
$$

We write below the full code needed for this specific calculation:

```
#The set of operations needed to obtain the final result

XX = inprod(algebra, X,X)
XX.swap(algebra, 3, 0, 1)
XX.swap(algebra, 4, 0, 1)
solve(algebra, XX)
print("XX = ",XX)
```

As a final example, we demonstrate how one can use the provided program to easily calculate the non-commutative Christoffel Symbol Γ^R_{TR} using the formula (3.73). I will not write explicitly all the steps needed to calculate it manually; only the important steps will be considered. First, note that:

$$
\Gamma^R_{TR} = \Big(g^{RR} (\Phi^\alpha_R)^* + g^{RT} (\Phi^\alpha_T)^* \Big) \eta_{\alpha\beta} \partial_T \Phi^\beta_R = (\Phi^\alpha_a g^{aR})^* \eta_{\alpha\beta} \partial_T \Phi^\beta_R = g(\hat{e}_\alpha \Phi^\alpha_a g^{aR}, \hat{e}_\beta \partial_T \Phi^\beta_R) \ ,
$$

now we declare all variables using the definitions from the thesis and run the following script:

```python
#The full script to calculate the desired Christoffel Symbol

GRR = Monomial("RR",0,0,-2,1,1)
GTR = Monomial("",1,1,-3,-1,2)
V1 = Vector(algebra,[R1*GRR+T1*GTR, R2*GRR+T2*GTR, R3*GRR+T3*GTR])
V2 = Vector(algebra, [differentiate(algebra,"T",R1), differentiate(algebra,"T",R2),
    differentiate(algebra,"T",R3)])
Cris = inprod(algebra, V1, V2)
print(solve(algebra, Cris))
```

Where the counting for the ambient indices is not $\overline{0,2}$, it is $\overline{1,3}$ instead. GRR and GTR are the coefficients from the metric for local coordinates, $R\mu$ and $T\mu$ are Φ^{μ}_{R} and Φ^{μ}_{T} respectively, and the vectors $V1$ and $V2$ are the vectors from the main expression that we are taking the inner products of. The output of the last script is: `Expression = [Monomial (R,1,1,-1,1.0,2.0)]`, which gives the correct result $\Gamma^{R}_{TR} = \frac{i\hbar}{2\ell_0}R$. Here it wasn't necessary to swap any index or execute other calculations before running the *solve* operation, but we used the function `differentiate(ALGEBRA, VARIABLE, TERM)` which differentiates the TERM with respect to VARIABLE using the rules from the ALGEBRA.

Lastly, I want to point out that these examples are just a small part of all the functions implemented in the calculator I created to verify some lengthy equations encountered during the writing of this thesis. In the provided link, I have posted the main file, which can be run on any PC, and I have explained there all the functionalities not presented here.

A.3 Future Upgrades

As a natural addition to the functionalities of the non-commutative calculator, one should consider adding tensorial operations. Before doing this, I will list below all the upgrades intended to be added to the program to make it more suitable for the analysis of non-commutative surfaces.

- Add the functionality of the module structure as a specific class type.
- Add more sets of possible manipulations for each term.
- Add the differentiation of the algebra and its opposite counterpart. This will likely be added as a boolean variable, which will be verified before any calculation.
- Add the tensor product structure as a class type.

- Add the possibility to operate the indices following the set of rules from the algebra.
- Create a function that converts all steps used by the `solve` function to a LaTeX file.
- Generalize the structure developed here to apply to any non-commutative surface we wish to analyze further.

In this appendix, we have developed and utilized a non-commutative calculator to perform and verify complex algebraic computations within non-commutative algebraic settings. This tool has been instrumental in reducing errors in manual calculations and providing a structured approach to handle algebraic manipulations involving non-commutative elements. We started by discussing the core functionalities of the non-commutative calculator, including the data structures and functions that form its foundation. The *Algebra* class and its associated elements, such as the *Monomial*, *Expression*, and *Vector* classes, were defined and their roles in the calculator explained in detail. The initialization process for these structures, including the specification of algebra elements, dimensions, commutation relations, and derivatives, was elaborated upon to give a comprehensive understanding of how to set up and use the calculator.

Through several examples, we demonstrated the practical applications of this tool. We showed how to define algebraic structures, perform operations, and simplify expressions. The step-by-step breakdown of calculations highlighted the efficacy of the calculator in managing non-commutative terms and ensuring accurate results. By utilizing functions such as *solve()* and *swap()*, we showcased the ability to automate and verify intricate algebraic processes. The examples provided, such as the combination of expressions and the inner product calculations, also with the operations showcased in the main file provided illustrates the robustness of the calculator. We verified results found in the main text of the thesis, reinforcing the validity and reliability of the computational tool developed.

In conclusion, the non-commutative calculator represents now an small advancement in computational algebra, but with further development, it has the potential to become a highly robust tool for solving complex problems. It provides a powerful means to manage the intricacies of non-commutative algebraic calculations, ensuring both accuracy and efficiency. This tool not only supports current research but also paves the way for future explorations and applications in the study of non-commutative surfaces.

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