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# Energy Condition Bounds in $f(R)$ Theories for Observational Confrontation: An Application to the Hu-Sawicki Model

Lucca Lopes Dias Santos

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Lucca Lopes Dias Santos


Dissertação de mestrado apresentada ao Instituto de Física da Universidade de Brasília como parte dos requisitos necessários à obtenção do título de Mestre em Física.

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# Abstract

The dark energy component in the Standard Model of Cosmology accounts for the recent accelerated expansion of the universe. An alternative approach involves geometric modifications to general relativity, leading to modified theories of gravity. When applied to a gravity theory, energy conditions impose constraints on the Ricci and energy-momentum tensors, which translate into inequalities. These constraints can be expressed in terms of cosmographic functions, such as the Hubble parameter, deceleration, jerk, and snap. In this work, we derive the equations of motion for a general f(R) gravity model and evaluate the energy conditions within the Hu-Sawicki f(R) theory, obtaining bounds for its parameters in terms of these cosmographic functions. These bounds provide a foundation for future comparisons with observational data, enabling the reconstruction of the cosmographic functions and the imposition of observational constraints on the Hu-Sawicki f(R) model parameters.

**Keywords:** Cosmology; energy conditions; f(R) theories; accelerated expansion.

# Resumo

No Modelo Padrão da Cosmologia, a energia escura é a responsável pela recente expansão acelerada do universo. Uma alternativa a essa abordagem envolve modificações geométricas na Relatividade Geral, levando às teorias modificadas da gravitação. As condições de energia, por sua vez, impõem restrições ao tensor de Ricci e ao tensor-energia momento, as quais se traduzem em inequações quando aplicadas a uma teoria gravitacional. Tais restrições podem ser expressas em termos de funções cosmográficas, como as funções de Hubble, desaceleração, jerk e snap. Neste trabalho, derivamos as equações de movimento para um modelo geral da gravitação f(R) e avaliamos as condições de energia para a teoria f(R) de Hu-Sawicki, obtendo vínculos para seus parâmetros em termos de tais funções cosmográficas. Esses vínculos fornecem uma base para futuras comparações com dados observacionais, possibilitando a reconstrução das funções cosmográficas e a imposição de restrições observacionais aos parâmetros do modelo f(R) de Hu-Sawicki

**Palavras-chave:** Cosmologia; condições de energia; teorias f(R); expansão acelerada.

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# 1 Introduction

The theory of General Relativity (GR) regards the gravitational interaction as a consequence of the spacetime geometry. In such a context, gravity is described by Einstein field equations, given by (Wald, 1984)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

which can alternatively be written as

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right). \tag{1.2}$$

The terms on the left-hand side are geometrical quantities: $G_{\mu\nu}$, $R_{\mu\nu}$, and $R$ are, respectively, the Einstein tensor, the Ricci tensor, and the Ricci scalar. The metric tensor $g_{\mu\nu}$, often called simply the 'metric', is used to define $R$ as the trace of $R_{\mu\nu}$ through the contraction $R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}$. The right-hand side, in turn, contains the energy-momentum tensor $T_{\mu\nu}$, which encapsulates information about the energy-related aspects of the universe's content that act as the source of gravity. The trace of $T_{\mu\nu}$ is given by $T = g^{\mu\nu}T_{\mu\nu}$, while $\kappa = \frac{8\pi G}{c^4}$ is the curvature-matter coupling constant, which depends on the gravitational constant $G$ and the speed of light $c$.

In general, $g_{\mu\nu}$ can be defined from the spacetime interval $ds^2$ in such a way that

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{1.3}$$

In this sense, the metric establishes the notion of distance in the geometric description of spacetime. For a homogeneous and isotropic universe, as considered throughout this work, it takes the form (Friedman, 1922; Lemaître, 1931; Robertson, 1935; Walker, 1937)

$$ds^2 = -dt^2 + a^2(t)[dr^2 + S_k^2(r)(d\theta^2 + \sin^2\theta d\theta^2)]. \tag{1.4}$$

This is the so-called Friedmann–Lemaître–Robertson–Walker (FLRW) metric. $a(t)$ is the scale factor, and it describes the evolution (expansion or contraction) of space through time, *i.e.*, it describes how the distance between two points changes in time. $S_k(r)$ indicates different functions depending on the value of the parameter $k$. The cases $k = 1, 0$ and $-1$ correspond to the functions $S_k(r) = (\sin(r), r, \sinh(r))$ and describe spherical, flat and hyperbolic universes, respectively. Eq. (1.4) also establishes the metric signature convention adopted in this work, namely $(-, +, +, +)$.

A particular class of energy-momentum tensor is that of a perfect fluid. A perfect fluid is fully characterized by its energy density $\rho$ and its pressure $p$, as seen in its rest frame, and its energy-momentum tensor does not display viscosity or heat conduction terms. It is also isotropic, so the pressure $p$ is the same in every direction. Its components are (Carroll, 2004)

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{1.5}$$

with $U^\mu$ being a timelike vector, normalized to unit, that represents the four-velocity of the fluid.

By following the reasoning presented in Ref.(Carroll, 2004), considering the FLRW metric, that the content in the universe behaves as a perfect fluid (at large scales) and associating Eqs. (1.1) and (1.5) leads to the first and second Friedmann equations, given respectively by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.6}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.7}$$

These relate the dynamics of the universe's evolution, represented by $a$ and its derivatives, to its content - the energy density $\rho$ and the pressure $p$. Cosmological models often consider that they are associated by the state equation

$$p = \omega\rho, \tag{1.8}$$

with $\omega$ being the equation of state parameter.

When describing the universe's content by the quantities $\rho$ and $p$ and assigning to them the relation given by Eq. (1.8), we take into account a set of different species (cosmological fluids), each with a different energy density $\rho_i$ and an associated parameter $\omega_i$, where the index $i$ labels the fluids. The different $\omega_i$ thus characterize each species, implying distinct evolution equations for the densities $\rho_i$. Matter and radiation stand as examples of species: matter consists of collisionless particles such that $p_M = 0$, implying $\omega_M = 0$; for radiation, in turn, it can be shown that $\omega_R = \frac{1}{3}$. Thus, if we consider any other species, *e.g.*, dark energy, there will be some specific $\omega_i$ related to it. In particular, for a dark energy described by a cosmological constant $\Lambda$, $\omega_\Lambda = -1$.

The model presented so far arises from making some considerations about certain features of the universe (*e.g.*, assuming homogeneity and isotropy) and from using GR as the theory of gravitation. Concerning its evolution, observational data indicates a late time accelerating expansion (Riess *et al.*, 1998; Perlmutter *et al.*, 1999; Schrabback *et al.*, 2010; Astier; Pain, 2012), and current candidates for explaining such phenomena consider either some unknown field or fluid (dark energy) or some extension of GR (extended theories).

One of the main goals of the Extended Theories of Gravity (ETGs) is to explain this recent accelerating stage through geometric modifications to GR. Among the ETGs, a widely explored class is that of $f(R)$ theories, which generalize GR by considering some nonlinear function of the Ricci scalar $R$ in the action used to obtain the field equations (Sotiriou, 2007; Sotiriou; Faraoni, 2010; Felice; Tsujikawa, 2010; Capozziello; Laurentis, 2011; Saridakis *et al.*, 2021). This approach raises the question of which theories lead to a proper cosmological description, as well as which parameters to use given a certain theory. In this context, the energy conditions provide a useful tool for constraining them.

The energy conditions stem from some basic considerations upon the physics of the model of interest. When applied to gravitation theories, they lead to bounds in the form of inequalities. These set relations between the parameters of the theory, and can thus be used to limit the $f(R)$ theories (Perez Bergliaffa, 2006; Lima; Vitenti; Rebouças, 2008; Penna-Lima; Vitenti; Rebouças, 2008; Bertolami; Sequeira, 2009; Capozziello; Lobo; Mimoso, 2015; Penna-Lima *et al.*, 2019). In Ref.(Penna-Lima *et al.*, 2019), the authors propose ways of writing such bounds in terms of observable cosmographic quantities, such as the Hubble function $H(z)$ and the deceleration function $q(z)$, for a general ETG. In Ref.(Perez Bergliaffa, 2006) Perez Bergliaffa do it, for the Weak Energy Condition bound equation, in the framework of $f(R)$ theories.

Among the broad range of $f(R)$ models, one often considered is the one proposed by Hu and Sawicki (Hu; Sawicki, 2007). This $f(R)$ theory aims to explain the accelerated expansion of the universe without requiring dark energy while satisfying solar system tests. It also includes the phenomenology of a $\Lambda$ - Cold Dark Matter ($\Lambda$CDM) cosmology as a limiting case. Due to its well-behaved properties, it has been widely explored in the literature (Hu; Sawicki, 2007; Oyaizu, 2008; Hu *et al.*, 2016; Vogt *et al.*, 2024; Kou; Murray; Bartlett, 2024)

In this work, we then explore the consequences of evaluating the energy conditions in the scope of a $f(R)$ theory. We follow the procedure set in Ref.(Penna-Lima *et al.*, 2019), first for a general $f(R)$ theory with minimal coupling and subsequently for the Hu-Sawicki model, finding the bound equations provided by assuming the Null, Strong, Weak, and Dominant Energy Conditions. We express these constraints in terms of $H(z)$, $q(z)$, $j(z)$ and $s(z)$ - respectively the Hubble, deceleration, jerk, and snap functions - and find expressions for $j(z)$ and $s(z)$ in terms of $q(z)$ and its derivatives.

Furthermore, the inequalities in the bound equations not just set limits to the parameters of the theory, but also provide a foundation for future comparison with observational data. Expressing them in terms of the observable quantities $H(z)$ and $q(z)$ enables the possibility of reconstructing the cosmographic functions from the data (Vitenti; Penna-Lima, 2015), which would also impose observational constraints on the Hu-Sawicki $f(R)$ model parameters.

In Ch.2 we present the concepts of geodesics and geodesic deviation, and also obtain Raychaudhuri's equation, as it plays an important role in the convergence conditions used for obtaining the Strong and Null Energy Conditions. In Ch. 3 we explore the energy conditions within the framework of General Relativity. In Ch.4 we compute the bounds in the context of extended theories, both for a general ETG (Penna-Lima *et al.*, 2019) and for $f(R)$ theories, including the Hu-Sawicki model. We present our conclusions and perspectives for future comparison with observational data in Ch.5.

# 2 Geodesics and the Raychaudhuri's Equation

The constraints imposed by some of the energy conditions in gravitational theories have implications that can be physically represented through their action on the trajectories followed by particles in spacetime. To this end, understanding the process of geodesic deviation and the evolution of the expansion for geodesics becomes necessary. In the following sections, we thus aim to address such topics, including the concepts of geodesics and geodesic deviation and their application in deriving Raychaudhuri's equation. In addition to the references explicitly cited in the text, throughout our discussion we will follow the reasoning presented in Refs.(Misner; Thorne; Wheeler, 1973), (Wald, 1984), (Carroll, 2004) and (d'Inverno; Vickers, 2022).

## 2.1 Geodesics

Physically, geodesics may be regarded as the trajectories followed by neutral particles moving subject only to the gravitational interaction. These particles are known as freely falling particles, and correspondingly trace the geodesics as their free-fall paths through spacetime.

In the geometric context, geodesics can be understood in different ways: more generally, they are the curves along which their tangent vectors are parallel-transported; specifically, for the Levi-Civita connection (which will be discussed in this section), they can also be regarded as the generalization of the notion of a straight line in curved spaces. In this sense, in General Relativity (GR), geodesics are also the paths of shortest distance between two points.

Parallel transport is the process of moving a tensor along a given path while keeping it constant. For instance, we can take a vector and move it along a curve. When comparing the vector to itself at infinitesimally close points of the curve, if it remains parallel and of equal length, then this vector is said to be parallel-transported along the curve (Schutz, 2009).

To examine the definitions of geodesics presented above, we can turn to the mathematical formalism and consider a curve $x^{\mu}(\lambda)$, parametrized by $\lambda$, with components such that $\mu \in (0,1,2,3)$. In such terms, its tangent vector is expressed by the derivative $\frac{dx^{\mu}}{d\lambda}$.

Let us first examine the definition of geodesic as the curve whose tangent vector is parallel-transported along itself. The constancy of the tangent vector requires that its components remain constant, *i.e.*,

$$\frac{d}{d\lambda}\left(\frac{dx^{\mu}}{d\lambda}\right) = 0. \tag{2.1}$$

Applying the chain rule, in flat space Eq. (2.1) would lead to

$$\frac{dx^{\mu}}{d\lambda}\partial_{\mu}\left(\frac{dx^{\mu}}{d\lambda}\right) = 0, \tag{2.2}$$

in which $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$.

The Principle of Minimal Coupling states that we should write the physical laws in a tensorial form in the simplest possible way when going from flat to curved spacetime, without adding unnecessary terms. In practice, this principle ensures the following identification concerning derivatives when transitioning to curved spacetime:

$$\partial_{\mu} \longrightarrow \nabla_{\mu}, \tag{2.3}$$

where $\nabla_{\mu}$ is the covariant derivative, whose expression varies depending on the type of index it acts upon. For instance, considering the vector $V^{\mu}$, with one contravariant index, its action is defined by

$$\nabla_{\mu}V^{\nu} = \partial_{\mu} + \Gamma^{\nu}_{\mu\rho}V^{\rho}, \tag{2.4}$$

while its action on covariant indices takes the form

$$\nabla_{\mu}V_{\nu} = \partial_{\mu} - \Gamma^{\rho}_{\mu\nu}V_{\rho}. \tag{2.5}$$

When acting on tensors with more than one index, for each index we add (or subtract) a new term, following Eqs. (2.4) and (2.5).

$\Gamma^{\nu}_{\mu\rho}$ is called the "connection coefficient" and acts as some correction factor. Ordinary derivatives do not behave in a covariant way when transforming between different coordinate systems, therefore to maintain the result invariant, we need to add a term to the ordinary derivative to compensate for the non-tensorial contribution that arises from its transformation. This term is precisely the connection, which leaves the covariant derivative invariant under coordinate transformations.

We can define different connections depending on the assumptions we take into account when building a gravitational theory. In GR, $\Gamma^{\nu}_{\mu\rho}$ takes a particular form due to some specific considerations, which will be discussed soon.

Under such considerations, by performing the transformation in Eq. (2.3) the condition in Eq. (2.1) reads

$$\frac{dx^{\mu}}{d\lambda}\nabla_{\mu}\left(\frac{dx^{\mu}}{d\lambda}\right) = 0, \tag{2.6}$$

from which it becomes possible to define the directional covariant derivative, given by

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu \ . \tag{2.7}$$

Thus, in terms of the directional derivative, the condition that the tangent vector is parallel-transported takes the form

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0 \ . \tag{2.8}$$

We can go further to find an explicit form of the geodesic equation by evaluating the covariant derivative in Eq. (2.6), resulting in

$$\begin{aligned} \frac{dx^\nu}{d\lambda}\nabla_\nu\left(\frac{dx^\mu}{d\lambda}\right) &= \frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} \\ &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}, \end{aligned} \tag{2.9}$$

which finally leads to the geodesic equation,

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \ . \tag{2.10}$$

From Eq. (2.10), we can indeed recover the equation of a straight line, in Euclidean space, by choosing Cartesian coordinates. With this particular set of coordinates, $\Gamma^\rho_{\mu\nu} = 0$ and the geodesic equation becomes just

$$\frac{d^2x^\rho}{d\lambda^2} = 0. \tag{2.11}$$

The definition of geodesics as the curves of the shortest distance between two points, in turn, arises through the variational method. That is, it is possible to look for stationary points of the length functional which will provide the curves with the minimum length[1].

Considering a differentiable curve on a manifold with metric $g_{\mu\nu}$, the length $s$ of the curve is defined from the spacetime interval in Eq. (1.3), from which it follows

$$s = \int\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda. \tag{2.12}$$

The definition of $s$ depends on the nature of each curve: whether it is spacelike, null, or timelike. The classification of a curve, in turn, depends on its tangent vector. For instance,

[1] Stationary points are the ones for which the variation of the action, which is a functional, is zero. The variation of the action being zero is precisely the necessary condition for it to have an extremum (Landau; Lifshitz, 1981). In this way, if we find the stationary points of the length functional, we will consequently obtain, in this case, the curves of minimal length.

a curve is timelike if its tangent vector is timelike everywhere, and the same applies for spacelike and null curves.

For spacelike curves, the length is defined as the spacetime interval in Eq. (2.12); for null curves, the length is 0, since the norm of its tangent vector is 0; for the timelike ones, the length is defined as the proper time $\tau$. By choosing a timelike curve to proceed, we see that the norm of the tangent vector is such that

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0. \tag{2.13}$$

So, in this case, it is necessary to change the sign in the argument of the square root of Eq. (2.12). We then obtain the length of a timelike curve, corresponding to the proper time functional,

$$\tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} d\lambda. \tag{2.14}$$

It is convenient to choose the parameter $\lambda$ to be the proper time $\tau$. This is a reasonable and sensible choice for timelike curves, as they represent the worldlines of massive bodies through spacetime. In this context, it is natural to use $\tau$ to parametrize their paths.

Under such considerations, performing the variation $\delta\tau$ and looking for stationary points yields the equation

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \ . \tag{2.15}$$

We see that this is the same as Eq. (2.10) if

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \ ; \tag{2.16}$$

*i.e.*, by looking for the path of shortest length we recover the geodesic equation as long as Eq. (2.15) is satisfied and $\lambda = \tau$. This amounts to choosing a particular connection - in this case, specifically the so-called Levi-Civita connection, represented by $\Gamma^\rho_{\mu\nu}$ (also called Christoffel symbol) and defined in Eq. (2.16).

The Levi-Civita connection can be defined on a manifold equipped with a metric if two conditions are satisfied: the connection must be symmetric in its last pair of indices,

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \ , \tag{2.17}$$

and must be metric-compatible, that is,

$$\nabla_\rho g_{\mu\nu} = 0 \ . \tag{2.18}$$

The Levi-Civita connection is the one adopted in the context of GR and, as discussed earlier, it is precisely the one that allows both definitions presented for geodesics. Since we will deal with extended theories still based on curvature functions, without assuming torsion[2], this will be the connection used throughout this work. We emphasize, however, that this is not the only possible situation. We can still build a theory in which the conditions for the definition of the Levi-Civita connection are not satisfied.

For instance, the Palatini formalism considers a non-metric connection, *i.e.*, a connection which is not related to $g_{\mu\nu}$ (Capozziello; Lobo; Mimoso, 2015). There are also some theories of gravitation whose connections do not satisfy Eq. (2.17), resulting in the presence of torsion, as in the case of the Teleparallel Equivalent of General Relativity or of the $f(T)$ theories, cases in which the construction of the theory itself leads to using a distinct connection, the so-called Weitzenböck connection (Andrade; Guillen; Pereira, 2002; Aldrovandi; Pereira, 2012). Hybrid theories as the $f(R,T)$ ones, which use generalized functions of both $R$ and $T$ (the torsion scalar)[3], are also examples of theories that requires distinct connections (Harko *et al.*, 2011).

It is also pertinent to point out some features regarding the nature of the parameter $\lambda$ since Eqs. (2.10) and (2.15) coincide only when $\lambda = \tau$. When building Eq. (2.15), we made a specific choice of using a timelike geodesic, case in which it is convenient to use the proper time $\tau$ as the parameter, as stated earlier. The point is that there is some freedom when choosing our parametrization.

Each value of $\lambda$ labels an event on the geodesics, and when doing so, we can choose different parameters. If we think of a test particle following its trajectory — for instance, a timelike geodesic — with a clock, this would be used to label each point on the geodesic with the time $\lambda = \tau$. However, this parametrization wouldn't be unique, since the choice of time origin and the units used to measure time are arbitrary. Then, if we choose the particular parameter $\tau$, we can generally also use some parameter $\lambda$ given by

$$\lambda = a\tau + b, \tag{2.19}$$

in which $b$ and $a$ are constants that respectively determine the zero and the unit of the parameter (Misner; Thorne; Wheeler, 1973).

$\lambda$ is called the "affine parameter" and we can see that Eq. (2.15) is invariant under transformations $\tau \to \lambda = a\tau + b$. Such a more general parameter would be necessary for

[2] The torsion tensor $T^{\rho}_{\ \mu\nu}$ can be defined as $\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}$, for any given connection. Since the Levi-Civita connection is symmetric, *i.e.*, $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$, then the torsion tensor vanishes and the connection is said to be 'torsion-free' (Carroll, 2004).

[3] In this specific example concerning the $f(R,T)$ theories, $T$ is regarded as the torsion scalar. Throughout the text, however, $T$ stands for the trace of the energy-momentum tensor $T_{\mu\nu}$.

dealing with null geodesics since the concept of proper time is not well defined in this situation.

It is also worth noticing that the geodesic equation can be written in an alternative way by using its tangent vector. In this case of a timeline geodesic, the tangent vector is the four-velocity $U^\mu$ of the body that traces it,

$$U^\mu = \frac{dx^\mu}{d\tau}. \tag{2.20}$$

In terms of $U^\mu$, the geodesic equation then reads

$$U^\mu \nabla_\mu U^\nu = 0. \tag{2.21}$$

This is an expression we will use in the following sections.

## 2.2 Geodesic Deviation

An important feature of curved spaces lies in how curvature determines the test bodies' paths. While in Euclidean geometry parallel lines remain parallel forever, in curved spaces curvature manifests itself by deviating the geodesic trajectories, causing them to accelerate toward or away from each other. We thus aim to find the geodesic deviation equation, which describes the relative acceleration between geodesics.

We start by considering a family of geodesic curves $\gamma_s(t)$, parameterized by $t$, that are initially parallel. Each $s \in \mathbb{R}$ labels a geodesic, with parameter $t$. This family of geodesics then defines a two-dimensional surface, since we can identify each point on the family of geodesics by the parameters $s$ and $t$, which play the role of coordinates. Consequently, the points of this surface may be represented by $x^\mu(s,t)$ — that is, each point will be located on a geodesic labeled by $s$, at a specific position on this geodesic, specified by $t$.

In this scenario, we can identify two vector fields: one related to each of the coordinates $s$ and $t$. The first of them is that of tangent vectors to the geodesics, indicated by $T = \frac{\partial}{\partial t}$. At each point $x^\mu(s,t)$, these tangent vectors have components

$$T^\mu = \left(\frac{\partial}{\partial t}\right)^\mu = \frac{\partial x^\mu}{\partial t}, \tag{2.22}$$

and since they are tangent to the geodesics, they automatically satisfy the geodesic equation, which when written in the fashion of Eq. (2.21) reads

$$T^\mu \nabla_\mu T^\nu = 0. \tag{2.23}$$

The second vector field, represented by $S^\mu$, is defined as

$$S^\mu = \left(\frac{\partial}{\partial s}\right)^\mu = \frac{\partial x^\mu}{\partial s}, \tag{2.24}$$

and indicates the displacement from one geodesic to another infinitesimally close one. That is, $S^\mu$ measures the deviation between neighboring geodesics and is thus called "deviation vector". Such quantities are depicted in figure 2.1.

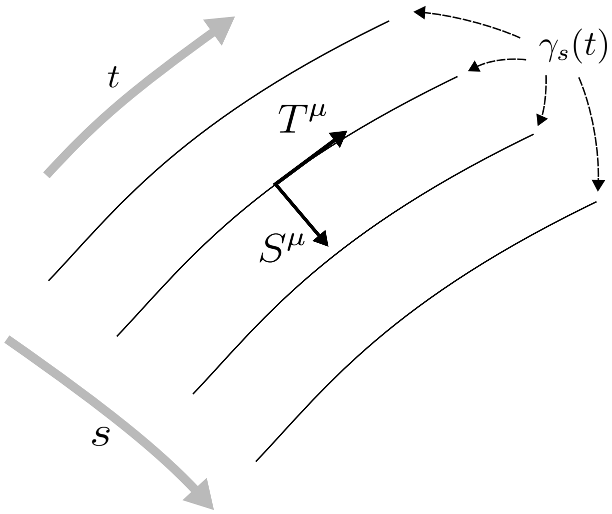


Figure 2.1 – Family of geodesic curves $\gamma_s(t)$, parametrized by $t$. $T^\mu$ and $S^\mu$ are, respectively, the tangent and the deviation vectors. Image based on an illustration found in (Carroll, 2004).

From $S^\mu$, we can then define the relative velocity of geodesics, $V^\mu$, as its directional covariant derivative with respect to $t$. Using Eq. (2.7), this is expressed as

$$V^\mu = \frac{DS^\mu}{dt} = T^\nu \nabla_\nu S^\mu. \tag{2.25}$$

As a directional derivative, $V^\mu$ represents the rate at which the displacement to a neighboring geodesic changes as we move along the geodesic.

Analogously, the directional derivative of $V^\mu$ with respect to $t$ provides the relative acceleration of geodesics, $A^\mu$. Again, by applying Eq. (2.7), we find

$$A^\mu = \frac{DV^\mu}{dt} = T^\nu \nabla_\nu V^\mu. \tag{2.26}$$

We now need to establish a connection between $A^\mu$ and the curvature to determine how the curvature of the manifold deviates the geodesics. To achieve this, it is useful to examine some properties of the commutator of $S^\mu$ and $T^\mu$.

Since these are derivative operators, they must satisfy the notion that tangent vectors can be regarded as directional derivatives when acting on scalar fields (Wald, 1984). That is, since $T = \partial/\partial t$, in Euclidean space we would expect the action of $T$ on some scalar $f$ to take the form

$$T(f)=\frac{\partial f}{dt}=\frac{\partial x^{\mu}}{dt}\frac{\partial f}{\partial x^{\mu}}=T^{\mu}\partial_{\mu}f. \tag{2.27}$$

In curved spaces, in turn, the Minimal Coupling Principle in Eq. (2.3) leads us to the expression

$$T(f)=T^{\mu}\nabla_{\mu}f, \tag{2.28}$$

which is just the directional covariant derivative $Df/dt$, according to Eq. (2.7).

Analogously, the action of $S$ on $f$ provides

$$S(f)=S^{\mu}\nabla_{\mu}f. \tag{2.29}$$

With Eqs. (2.28) and (2.29) it is now possible to find an expression for $[T,S]^{\mu}$. Since $T$ and $S$ are vectors, we expect the commutator $[T,S]$ to be also a vector. Consequently, following the same reasoning as in Eqs. (2.28) and (2.29), the action of $[T,S]$ on $f$ will be such that

$$[T,S]f=[T,S]^{\nu}\nabla_{\nu}f. \tag{2.30}$$

By expanding the left-hand side of Eq. (2.30), we find the expression

$$\begin{aligned}[T,S]f&=TS(f)-ST(f)\\&=T^{\mu}\nabla_{\mu}(S^{\nu}\nabla_{\nu}f)-S^{\mu}\nabla_{\mu}(T^{\nu}\nabla_{\nu}f)\\&=(T^{\mu}\nabla_{\mu}S^{\nu}-S^{\mu}\nabla_{\mu}T^{\nu})\nabla_{\nu}f,\end{aligned} \tag{2.31}$$

which, by comparison with Eq. (2.30), leads to

$$[T,S]^{\nu}=T^{\mu}\nabla_{\mu}S^{\nu}-S^{\mu}\nabla_{\mu}T^{\nu}. \tag{2.32}$$

Using Eqs. (2.22) and (2.24) for writing the components $T^{\mu}$ and $S^{\mu}$, and considering that the connections are symmetric according to Eq. (2.17), provides

$$\begin{aligned}T^{\mu}\nabla_{\mu}S^{\nu}-S^{\mu}\nabla_{\mu}T^{\nu}&=T^{\mu}\partial_{\mu}S^{\nu}+T^{\mu}\Gamma^{\nu}_{\mu\ \rho}S^{\rho}-S^{\mu}\partial_{\mu}T^{\nu}-S^{\mu}\Gamma^{\nu}_{\mu\rho}T^{\rho}\\&=\frac{\partial}{\partial t}\left(\frac{dx^{\mu}}{ds}\right)-\frac{\partial}{\partial s}\left(\frac{dx^{\mu}}{dt}\right)\\&=0.\end{aligned} \tag{2.33}$$

This leads to

$$T^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu T^\nu. \tag{2.34}$$

Such a property arises precisely from the facts that $T^\mu$ and $S^\mu$ are coordinate vector fields - that is, they can be regarded as basis vectors - and that the partial derivatives commute (Wald, 1984; Carroll, 2004; d'Inverno; Vickers, 2022).

By applying Eqs. (2.25) and (2.34) into Eq. (2.26), we obtain

$$\begin{aligned} A^\mu &= T^\rho \nabla_\rho (T^\nu \nabla_\nu S^\mu) = T^\rho \nabla_\rho (S^\nu \nabla_\nu T^\mu) \\ &= (S^\rho \nabla_\rho T^\nu) \nabla_\nu T^\mu + T^\rho S^\nu \nabla_\rho \nabla_\nu T^\mu. \end{aligned} \tag{2.35}$$

It is possible to express the derivative $\nabla_\rho \nabla_\nu T^\mu$ in terms of the Riemann tensor $R^\mu{}_{\xi\rho\nu}$ using the relation

$$[\nabla_\rho, \nabla_\nu] T^\mu = R^\mu{}_{\xi\rho\nu} T^\xi, \tag{2.36}$$

from which we get

$$\nabla_\rho \nabla_\nu T^\mu = \nabla_\nu \nabla_\rho T^\mu + R^\mu{}_{\xi\rho\nu} T^\xi. \tag{2.37}$$

The Riemann tensor is defined precisely by Eq. (2.36) to have components

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \tag{2.38}$$

Plugging Eq. (2.37) into Eq. (2.35) and using Eq. (2.23) finally yields the so-called geodesic deviation equation,

$$A^\mu = \frac{D^2 S^\mu}{dt^2} = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma. \tag{2.39}$$

This equation states that the relative acceleration between geodesics is indeed proportional to the curvature of the manifold, encoded into the Riemann tensor.

We again emphasize that the quantities discussed in this section, along with their behavior, are crucial for deriving Raychaudhuri's equation. This equation, in turn, will play a significant role in defining the Strong and Null Energy Conditions in the following chapters.

## 2.3 Raychaudhuri's Equation

In the last section, we derived Eq. (2.39), highlighting the dependence of the relative acceleration of geodesics on curvature. To achieve this, we considered a family of geodesics that were initially parallel. We can now move further and consider the more general case of a congruence of geodesics, *i.e.*, a family of geodesics in which they are not necessarily parallel. This is not only a physically reasonable scenario but also a configuration of particles that provides useful tools for evaluating the consequences of curvature and, therefore, of gravity itself.

A congruence is a family of curves in an open region of spacetime such that through each point in this region only one curve passes. Physically, a geodesic congruence refers to the paths of a set of particles (neutral test bodies) tracing its trajectories through spacetime, so that the paths do not cross (Wald, 1984; Carroll, 2004).

For the parallel configuration, the analysis of the geodesic deviation relied on the behavior of the deviation vector $S^{\mu}$. For a geodesic congruence, it will be necessary more than that to track the trajectories of the particles, since now there are more degrees of freedom along their motion. This accounts for the fact that they are now disposed along the three-dimensional space: we are considering a sphere of test particles and would like to describe the evolution of their geodesics.

To make this description, we can first consider a congruence of timelike geodesics. As stated before, in this case we can choose as parameter the proper time $\tau$, so the tangent vector introduced in Eq. (2.22) is thus identified as the four-velocity of the particle, $T^{\mu} = U^{\mu}$, and the normalization condition $U^{\mu}U_{\mu} = -1$ is satisfied.

The tool necessary for describing the evolution of the geodesics can already be found in Eq. (2.25). For timelike geodesics, the relative velocity $V^{\mu}$ takes the form

$$V^{\mu} = \frac{DS^{\mu}}{d\tau} = U^{\nu}\nabla_{\nu}S^{\mu}, \tag{2.40}$$

and Eq. (2.34) allows us to rewrite it as

$$\frac{DS^{\mu}}{d\tau} = (\nabla_{\nu}U^{\mu})S^{\nu} = B^{\mu}{}_{\nu}S^{\nu} \tag{2.41}$$

as long as we define the tensor (Wald, 1984; Carroll, 2004)

$$B^{\mu}{}_{\nu} = \nabla_{\nu}U^{\mu}. \tag{2.42}$$

If $S^{\mu}$ were parallel-transported, we would expect to have $\frac{DS^{\mu}}{d\tau} = 0$. Since this is not the case, and because $B^{\mu}{}_{\nu}$ appears as a coefficient for $S^{\mu}$ on the right-hand side of Eq. (2.41), it can be interpreted as a measure of how much $S^{\mu}$ fails to be parallel-transported. In other words, $B^{\mu}{}_{\nu}$ quantifies how much the geodesics deviate from remaining parallel. We can then turn to $B^{\mu}{}_{\nu}$ and its behavior to understand the evolution of geodesics, as such information is encoded in it.

An important feature of $B^{\mu}{}_{\nu}$ is related to the subspace it belongs to. We are describing spacetime as a manifold $M$. At any point $p \in M$ we can define the tangent space at $p$, represented by $T_pM$. This space is the set of all possible vectors at $p$ (not only the tangent ones, as its name would suggest). Moreover, the presence of the timelike geodesic congruence induces a vector field $U^\mu$. So, for each point $p \in M$, we can consider the subspace of $T_pM$ of all vectors normal to $U^\mu$ at that point, called normal subspace (Carroll, 2004).

The point is that $B_{\mu\nu} = g_{\mu\rho}B^\rho{}_\nu = \nabla_\nu U_\mu$ belongs to the normal subspace since it is "normal" to $U^\mu$ in its both indices. By evaluating the contraction regarding its first index, $U^\mu B_{\mu\nu} = U^\mu \nabla_\nu U_\mu$, we first note that, from the normalization on $U^\mu$, we have $\nabla_\nu(U^\mu U_\mu) = \nabla_\nu(-1) = 0$. Using this and considering also the metric compatibility, in Eq. (2.18), we thus obtain

$$\begin{gathered}\nabla_\nu(U^\mu U_\mu) = (\nabla_\nu U^\mu)U_\mu + (\nabla_\nu U_\mu)U^\mu = 2U^\mu \nabla_\nu U_\mu = 0 \\ \Rightarrow U^\mu \nabla_\nu U_\mu = 0.\end{gathered} \tag{2.43}$$

This leads to

$$U^\mu B_{\mu\nu} = U^\mu \nabla_\nu U_\mu = 0. \tag{2.44}$$

The contraction regarding the second index, $U^\nu B_{\mu\nu}$, is precisely the geodesic equation. Consequently, from Eq. (2.21), it is equal to zero, leading to

$$U^\nu B_{\mu\nu} = U^\nu \nabla_\nu U_\mu = 0. \tag{2.45}$$

Eqs. (2.44) and (2.45) shows that $B_{\mu\nu}$ has both indices projected orthogonal to $U^\mu$, implying that $B_{\mu\nu}$ itself does not have any component in the direction of $U^\mu$. Since the four-velocity is timelike, this property assigns a spatial character to the tensor $B_{\mu\nu}$ (Carroll, 2004; Albareti; Cembranos; Cruz-Dombriz, 2012).

As the vector of interest, $B_{\mu\nu}$, resides in the normal subspace, it is beneficial to define an object capable of projecting other vectors onto this subspace. Such an object enables the evaluation of scalars within the normal subspace and will be used for expressing quantities in it, as will be demonstrated shortly.

The object fulfilling this role is the projection operator, defined as

$$P_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \tag{2.46}$$

whose action is projecting any vector in $T_pM$ into the normal subspace. For instance, if we take some vector $W^\mu = W^\mu_{||} + W^\mu_\perp$, decomposed into a component $W^\mu_{||}$ parallel to $U^\mu$ and a component $W^\mu_\perp$ perpendicular to it, the action of $P_{\mu\nu}$ reads

$$P^\mu{}_\nu W^\nu = \delta^\mu_\nu W^\nu_{||} + U^\mu U_\nu W^\nu_{||} + \delta^\mu_\nu W^\nu_\perp + U^\mu U_\nu W^\nu_\perp. \tag{2.47}$$

Since $W_{\perp}^{\nu}$ is perpendicular to $U^{\nu}$, the contraction between them in the last term reads $U_{\nu}W_{\perp}^{\nu} = 0$. Furthermore, since $W_{||}^{\nu}$ is parallel to $U^{\nu}$, we can write $W_{||}^{\nu} = \alpha U^{\nu}$, with some proportionality constant $\alpha$, resulting in

$$
\begin{aligned}
P^{\mu}{}_{\nu}W^{\nu} &= \alpha\delta^{\mu}_{\nu}U^{\nu} + \alpha U^{\mu}U_{\nu}U^{\nu} + \delta^{\mu}_{\nu}W_{\perp}^{\nu} \\
&= \alpha U^{\mu} - \alpha U^{\nu} + W_{\perp}^{\mu} \\
&= W_{\perp}^{\mu},
\end{aligned}
\tag{2.48}
$$

in which we used the normalization condition for $U^{\mu}$. That is, only the perpendicular component is selected.

We now aim to analyze each component of $B_{\mu\nu}$ to better comprehend its influence on the congruence. To achieve this, we rely on certain tensor properties, particularly those associated with its symmetries.

For any tensor, we can symmetrize any number of its indices. This means to select the symmetric part of the tensor relative to these indices. For $B_{\mu\nu}$, its symmetric part is

$$
B_{(\mu\nu)} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}). \tag{2.49}
$$

That is, by changing the indices and adding them up, the antisymmetric components cancel out, while the symmetric ones add up. Analogously, with the antisymmetrization we take its antisymmetric part. In this case, the antisymmetric part of $B_{\mu\nu}$ is represented by

$$
B_{[\mu\nu]} = \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}). \tag{2.50}
$$

Then, we can decompose a tensor into symmetric and antisymmetric parts for any two indices by adding them. In fact, we see that by adding up Eqs. (2.49) and (2.50) we get

$$
B_{\mu\nu} = B_{(\mu\nu)} + B_{[\mu\nu]}. \tag{2.51}
$$

Since these quantities are tensors, they can be represented as matrices. Based on this, we can further decompose $B_{(\mu\nu)}$ into its trace and trace-free part. It is important to note that the trace of $B_{\mu\nu}$ lies within $B_{(\mu\nu)}$, as the trace components remain unchanged when the order of indices is swapped, reflecting their symmetry. Schematically, this decomposition takes the form

$$
B_{\mu\nu} = \{\text{trace of } B_{\mu\nu}\} + \{B_{(\mu\nu)} - \text{ trace of } B_{\mu\nu}\} + B_{[\mu\nu]}. \tag{2.52}
$$

The trace corresponding to $B_{\mu\nu}$ is given by $B^{\mu}{}_{\mu} = \nabla_{\mu}U^{\mu}$, but we can also express it in terms of $P_{\mu\nu}$ by noting that, from Eqs. (2.42) and (2.46),

$$
\begin{aligned}
P^{\mu\nu}B_{\mu\nu} &= P^{\mu}{}_{\alpha}g^{\alpha\nu}g_{\mu\beta}B^{\beta}{}_{\nu} \\
&= g^{\alpha\nu}g_{\mu\beta}(\delta^{\mu}_{\alpha} + U^{\mu}U_{\alpha})\nabla_{\nu}U^{\beta} \\
&= \nabla_{\mu}U^{\mu}.
\end{aligned} \tag{2.53}
$$

By denoting the trace as $\theta$, we can thus write it as $\theta = P^{\mu\nu}B_{\mu\nu} = \nabla_{\mu}U^{\mu}$. As $\theta$ is a scalar, it is necessary to associate it with some tensor to express the expansion of $B_{\mu\nu}$ in terms of it. Once $B_{\mu\nu}$ already lives in the normal subspace, we can thus use $P_{\mu\nu}$ and include the trace of $B_{\mu\nu}$ as $\theta P_{\mu\nu}$ in Eq. (2.52). However, as $P^{\mu\nu}P_{\mu\nu} = 3$, we take into account an normalization factor of $1/3$, and rewrite Eq. (2.52) as

$$
B_{\mu\nu} = \frac{1}{3}\theta P_{\mu\nu} + \left\{ B_{(\mu\nu)} - \frac{1}{3}\theta P_{\mu\nu} \right\} + B_{[\mu\nu]}. \tag{2.54}
$$

We then define the quantities

$$
\theta = P^{\mu\nu}B_{\mu\nu} = \nabla_{\mu}U^{\mu}, \tag{2.55}
$$

$$
\sigma_{\mu\nu} = B_{(\mu\nu)} - \frac{1}{3}\theta P_{\mu\nu}, \tag{2.56}
$$

$$
\omega_{\mu\nu} = B_{[\mu\nu]}, \tag{2.57}
$$

and finally find the expansion for $B_{\mu\nu}$ to be

$$
B_{\mu\nu} = \frac{1}{3}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{2.58}
$$

$\theta$ is the expansion and, as stated before, it corresponds to the trace of $B_{\mu\nu}$. As shown in Eq. (2.55), it is a divergence, so it measures the volume expansion (or contraction) of the sphere of particles. This expansion refers, as in the case of geodesic deviation, to infinitesimally nearby geodesics. $\sigma_{\mu\nu}$ stems for the shear. It is symmetric, since it comes from $B_{(\mu\nu)}$, and trace-free, as it is defined as the remaining part after subtracting the trace from $B_{(\mu\nu)}$. It describes how the shape of the initial sphere of particles distorts into an ellipsoid. $\omega_{\mu\nu}$ represents the rotation (or twist), and it is the antisymmetric part of $B_{\mu\nu}$. From Eq. (2.57), we see that $\omega_{\mu\nu} = \frac{1}{2}(\nabla_{\nu}U_{\mu} - \nabla_{\mu}U_{\nu})$. This is analogous to the components of a curl and, consequently, it measures the rotation of neighboring geodesics.

A useful property about $\sigma_{\mu\nu}$ that we will need when discussing the Strong and Null Energy Conditions is that as well as $B_{\mu\nu}$, in Eqs. (2.44) and (2.45), $\sigma_{\mu\nu}$ also has both of its indices projected orthogonally to $U^{\mu}$ (Venn; Agarwal; Vasak, 2024). In fact, using Eqs. (2.44) and (2.45) we see that

$$U^{\mu}\sigma_{\mu\nu} = \frac{1}{2}U^{\mu}B_{\mu\nu} + \frac{1}{2}U^{\mu}B_{\nu\mu} - \frac{1}{3}\theta U^{\mu}P_{\mu\nu} = 0,$$
$$U^{\nu}\sigma_{\mu\nu} = \frac{1}{2}U^{\nu}B_{\mu\nu} + \frac{1}{2}U^{\nu}B_{\nu\mu} - \frac{1}{3}\theta U^{\nu}P_{\mu\nu} = 0. \tag{2.59}$$

Looking back at Eq. (2.58), we see that each component in the decomposition describes some feature of the congruence behavior. We can determine the rate of change of these quantities by examining the evolution of the congruence itself, which is described by the covariant derivative of $B_{\mu\nu}$ along the geodesics,

$$\frac{DB_{\mu\nu}}{d\tau} = U^{\rho}\nabla_{\rho}B_{\mu\nu} = U^{\rho}\nabla_{\rho}\nabla_{\nu}U_{\mu}. \tag{2.60}$$

Using Eqs.(2.21) and (2.37), this leads to

$$\frac{DB_{\mu\nu}}{d\tau} = -B^{\sigma}{}_{\nu}B_{\mu\sigma} + R_{\lambda\mu\nu\sigma}U^{\sigma}U^{\lambda}. \tag{2.61}$$

This is the evolution equation of $B_{\mu\nu}$. From it, is possible to obtain an evolution equation for the expansion by taking its trace. By doing so, and using Eq. (2.58), results in

$$\begin{aligned}\frac{DB^{\nu}{}_{\nu}}{d\tau} &= -B^{\rho\mu}B_{\mu\rho} - R^{\nu}{}_{\lambda\nu\rho}U^{\rho}U^{\lambda}\\ &= -\frac{1}{9}\theta^2P^{\rho\mu}P_{\mu\rho} - \frac{1}{3}\theta P^{\rho\mu}\sigma_{\mu\rho} - \frac{1}{3}\theta P^{\rho\mu}\omega_{\mu\rho} - \frac{1}{3}\theta\sigma^{\rho\mu}P_{\mu\rho} - \sigma^{\rho\mu}\sigma_{\mu\rho} +\\ &\quad - \sigma^{\rho\mu}\omega_{\mu\rho} - \frac{1}{3}\theta\omega^{\rho\mu}P_{\mu\rho} - \omega^{\rho\mu}\sigma_{\mu\rho} - \omega^{\rho\mu}\omega_{\mu\rho} - R_{\lambda\rho}U^{\lambda}U^{\rho}.\end{aligned} \tag{2.62}$$

Taking into account that the contraction between a symmetric and an antisymmetric tensor is zero and expanding each term yields the following results:

- $\frac{1}{9}\theta^2P^{\rho\mu}P_{\mu\rho} = \frac{1}{3}\theta^2$
- $\frac{1}{3}\theta P^{\rho\mu}\sigma_{\mu\rho} = 0$
- $\frac{1}{3}\theta P^{\rho\mu}\omega_{\mu\rho} = 0$
- $\frac{1}{3}\theta\sigma^{\rho\mu}P_{\mu\rho} = 0$
- $\sigma^{\rho\mu}\sigma_{\mu\rho} = \sigma^{\mu\rho}\sigma_{\mu\rho}$
- $\sigma^{\rho\mu}\omega_{\mu\rho} = 0$
- $\theta\omega^{\rho\mu}P_{\mu\rho} = 0$
- $\omega^{\rho\mu}\sigma_{\mu\rho} = 0$
- $\omega^{\rho\mu}\omega_{\mu\rho} = -\omega^{\mu\rho}\omega_{\mu\rho}$

Furthermore, since $B^{\nu}{}_{\nu} = \theta$ is a scalar, its covariant derivative is the same as an ordinary one, and consequently $\frac{D\theta}{d\tau} = \frac{d\theta}{d\tau}$. These results lead to the evolution equation for $\theta$, the so-called Raychaudhuri's equation, given by

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^{\mu}U^{\nu}. \tag{2.63}$$

This will be very useful when stating the convergence condition for timelike geodesics, in the construction of the Strong Energy Condition.

For a null geodesic congruence, an analogous procedure leads to the evolution equation for the expansion. However, in this case, the tangent vector to the geodesics, $k^{\mu} = \frac{dx^{\mu}}{d\lambda}$, is null, and consequently $k^\mu k_\mu = 0$. That is, $k^\mu$ is normal to itself, preventing us from normalizing it and leading to problems in the definition of a three-dimensional space orthogonal to it, as we did in the timelike case with the normal subspace (Wald, 1984; Carroll, 2004).

To proceed, a possible approach is defining an auxiliary null vector $l^\mu$ and then setting the normalization conditions between $k^\mu$ and $l^\mu$. We choose $l^\mu$ to point in the opposite spatial direction in relation to $k^\mu$, and set the normalization condition

$$l^\mu k_\mu = -1. \tag{2.64}$$

Moreover, since $l^\mu$ is null, then $l^\mu l_\mu = 0$. We also require that $l^\mu$ be parallel-transported. *i.e.*,

$$k^\nu \nabla_\nu l^\mu = 0. \tag{2.65}$$

For instance, to an observer in some specific frame, these vectors would appear as $k^\mu = (1/\sqrt{2}, 0, 0, 1/\sqrt{2})$ and $l^\mu = (1/\sqrt{2}, 0, 0, -1/\sqrt{2})$, with components such that $k^3 = -l^3$. This is just a particular example since these components are frame-dependent. The point is that having $k^\mu$ and $l^\mu$ in hands enables us to define a two-dimensional space of vectors normal to both of them, denoted $T_\perp$. This is a subspace of "spatial" vectors in the sense that this normal space spans the two spatial directions orthogonal to $k^\mu$ and $l^\mu$ (which already span by themselves the remaining spatial direction). Again, the objects necessary to track the evolution of the null congruence live in $T_\perp$, in the same way that $B^\mu{}_\nu$ was found in the normal subspace to $U^\mu$ in the timelike case (Carroll, 2004).

The projection operator to $T_\perp$, for null geodesics, is defined as

$$Q_{\mu\nu} = g_{\mu\nu} + k_\mu l_\nu + k_\nu l_\mu. \tag{2.66}$$

As an example, we can turn to its action in a vector $W^\mu = W^\mu_{||l} + W^\mu_{||k} + W^\mu_\perp$, with a component parallel to $l^\mu$, $W^\mu_{||l} = \alpha l^\mu$, a component parallel to $k^\mu$, $W^\mu_{||k} = \beta k^\mu$, and a component perpendicular to both $l^\mu$ and $k^\mu$, $W^\mu_\perp$, with $\alpha$ and $\beta$ constants. We see that

$$\begin{aligned} Q^\nu{}_\mu W^\mu &= (\delta^\nu_\mu + k^\nu l_\mu + l^\nu k_\mu)(\alpha l^\mu + \beta k^\mu + W^\mu_\perp) \\ &= \alpha l^\nu + \beta^\nu + W^\nu_\perp + \beta k^\nu k^\mu l_\mu + \alpha l^\nu l^\mu k_\mu \\ &= W^\nu_\perp, \end{aligned} \tag{2.67}$$

that is, only the orthogonal component $W^\mu_\perp \in T_\perp$ survives.

Considering the deviation vector $S^\mu$ between infinitesimally nearby geodesics, following the same reasoning behind Eq. (2.34) leads us to write

$$k^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu k^\nu. \tag{2.68}$$

Thus, the relative velocity between geodesics reads

$$\frac{DS^{\mu}}{d\lambda} = k^{\nu}\nabla_{\nu}S^{\mu} = B^{\mu}{}_{\nu}S^{\nu}, \tag{2.69}$$

with

$$B^{\mu}{}_{\nu} = \nabla_{\nu}k^{\mu}, \tag{2.70}$$

analogously to Eq. (2.41). Again, $B^{\mu}{}_{\nu}$ measures how much $S^{\mu}$ failures to be parallel-propagated.

In the timelike case, $B_{\mu\nu}$ was decomposed, as displayed in Eq. (2.58), and its parts were used to explain the evolution of the congruence. But for the null congruence, only the portion of $B_{\mu\nu}$ projected into $T_{\perp}$ is enough for describing how the geodesics change.

Firstly, considering some vector $V^{\mu} \in T_{\perp}$, we have

$$Q^{\nu}{}_{\mu}V^{\mu} = (\delta^{\nu}_{\mu} + k^{\nu}l_{\mu} + l^{\nu}k_{\mu})V^{\mu} = V^{\nu}, \tag{2.71}$$

which together with Eq. (2.69) results in

$$\frac{DS^{\mu}}{d\lambda} = k^{\nu}\nabla_{\nu}(Q^{\mu}{}_{\rho}S^{\rho}). \tag{2.72}$$

Then, using the identity $k^{\rho}\nabla_{\rho}Q^{\mu}{}_{\nu} = 0$, we get

$$\frac{DS^{\mu}}{d\lambda} = Q^{\mu}{}_{\rho}k^{\nu}\nabla_{\nu}S^{\rho}, \tag{2.73}$$

and through Eqs. (2.68) and (2.71) we obtain

$$\frac{DS^{\mu}}{d\lambda} = Q^{\mu}{}_{\rho}B^{\rho}{}_{\nu}Q^{\nu}{}_{\sigma}S^{\sigma}. \tag{2.74}$$

If we define the projection of $B^{\rho}{}_{\nu}$ into $T_{\perp}$ to be

$$\hat{B}^{\mu}{}_{\sigma} = Q^{\mu}{}_{\rho}B^{\rho}{}_{\nu}Q^{\nu}{}_{\sigma}, \tag{2.75}$$

then we see that Eq. (2.69) is equivalent to

$$\frac{DS^{\mu}}{d\lambda} = \hat{B}^{\mu}{}_{\nu}S^{\nu}, \tag{2.76}$$

so only the portion of $B^{\mu}{}_{\nu}$ in $T_{\perp}$ is relevant.

Under such considerations, by following the same prescription for finding Eq. (2.58), we decompose $\hat{B}_{\mu\nu}$ into symmetric and antisymmetric parts, writing it in terms of the expansion $\theta$, the shear $\hat{\sigma}_{\mu\nu}$ and the expansion (or twist) $\hat{\omega}_{\mu\nu}$. These are defined in the same way as Eqs.(2.55), (2.56) and (2.57), but now in terms of $\hat{B}_{\mu\nu}$,

$$\theta = Q^{\mu\nu}\hat{B}_{\mu\nu}, \tag{2.77}$$

$$\hat{\sigma}_{\mu\nu} = \hat{B}_{(\mu\nu)} - \frac{1}{2}\theta Q_{\mu\nu}, \tag{2.78}$$

$$\hat{\omega}_{\mu\nu} = \hat{B}_{[\mu\nu]}, \tag{2.79}$$

leading to a decomposition of $\hat{B}_{\mu\nu}$ analogous to Eq. (2.58),

$$\hat{B}_{\mu\nu} = \frac{1}{2}\theta Q_{\mu\nu} + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu}. \tag{2.80}$$

Evaluating the directional derivative $D\hat{B}_{\mu\nu}/d\lambda$ and taking the trace of it yields the evolution equation for the expansion of null geodesics in the congruence,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu, \tag{2.81}$$

which is analogous to Eq. (2.63). The $1/2$ factor in the first term is because, in this case, $T_\perp$ is a two-dimensional space and consequently $Q^{\mu\nu}Q_{\mu\nu} = 2$, requiring this normalization factor.

Eq. (2.81) will be useful when stating the convergence condition for null geodesics, during the derivation of the Null Energy Condition.

# 3 Energy Conditions in General Relativity

Initially, both the metric $g_{\mu\nu}$ and the tensor $T_{\mu\nu}$ in Einstein's equation, Eq. (1.1), may be completely arbitrary. Nevertheless, we can make some physical assumptions that impose certain restrictions on $T_{\mu\nu}$ - these are the so-called energy conditions. Such conditions are limitations to the arbitrariness of the energy-momentum tensor and are given by bounds in the form of inequalities. These inequalities depend on the components of the energy-momentum tensor, so it is useful first to examine the class of energy-momentum tensor we will use in this work. In this section, we thus discuss how to classify the energy-momentum tensor. Subsequently, we introduce the energy conditions and show how to apply them to the extended theories of gravity. We will base our discussion on Refs.(Hawking; Ellis, 1973), (Misner; Thorne; Wheeler, 1973), (Wald, 1984) and (Carroll, 2004).

## 3.1 The Energy-momentum Tensor

The energy-momentum tensor $T^{\mu\nu}$ is the object that encodes all the information about the energy of the system, like stress, pressure, heat conduction, energy density, and so on. Physically, it can be defined as the flux of the four-momentum component $p^\mu$ across a surface $x^\nu$ = constant - an interpretation which will soon be discussed in this section.

Mathematically, when obtaining Einstein field equations through the variational method, $T^{\mu\nu}$ can also be defined in terms of the variation of the action of matter, $S_M$, with respect to $g^{\mu\nu}$ as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.1}$$

It is a symmetric tensor of rank two, so $T^{\mu\nu} = T^{\nu\mu}$ and, in matrix form, it can be represented as a two-dimensional matrix.

We may classify the energy-momentum tensor according to the nature of its eigenvectors according to Ref.(Hawking; Ellis, 1973). This classification is defined for components with contravariant indices, $T^{\mu\nu}$, and within the case in which one expresses them with respect to an orthonormal basis $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$, with $\boldsymbol{\xi}_1$ a timelike vector. That is, given the equation

$$T^\mu_{\ \nu} V^\nu = \lambda V^\nu, \tag{3.2}$$

in which $V^\mu$ and $\lambda$ are, respectively, an eigenvector of $T^\mu_{\ \nu}$ and its corresponding eigenvalue, each $T^{\mu\nu}$ is settled by the components $V^\mu$ (Stephani *et al.*, 2003; Santos; Alcaniz, 2005).

To use Eq. (3.2), we first need to find the components $T^{\mu}_{\ \nu}$. We can express them in terms of $T^{\mu\nu}$, the components of the tensor we aim to classify, by contracting it with $g_{\mu\nu}$. We then get the equation

$$T^{\mu}_{\ \nu} = T^{\mu\rho} g_{\rho\nu}. \tag{3.3}$$

By definition, the components $g_{\mu\nu}$ in Eq. (3.3) are given by the scalar product of the base vectors, which in this case are given by the vectors $\{\boldsymbol{\xi}_{\mu}\}$. Mathematically, this notion can be expressed by the relation

$$g_{\mu\nu} \equiv g(\boldsymbol{\xi}_{\mu}, \boldsymbol{\xi}_{\nu}) = g(\boldsymbol{\xi}_{\nu}, \boldsymbol{\xi}_{\mu}) = \boldsymbol{\xi}_{\mu} \cdot \boldsymbol{\xi}_{\nu}. \tag{3.4}$$

That is to say, $g_{\mu\nu}$ depends both on the components of the base vectors and on the way the scalar product between these vectors is defined. Despite that, it is always possible to choose an local orthonormal basis (the one made by unitary vectors orthogonal to each other) such that $g(\boldsymbol{\xi}_{\mu}, \boldsymbol{\xi}_{\nu}) = 0$, if $\mu \neq \nu$, and that $g(\boldsymbol{\xi}_{\mu}, \boldsymbol{\xi}_{\mu}) = \pm 1$ (Wald, 1984).

Consequently, the components of the metric tensor take the form

$$g_{\mu\nu} = \mathrm{diag}(-1, ..., -1, +1, ..., +1), \tag{3.5}$$

so that they have $\frac{1}{2}(n - s)$ negative terms and $\frac{1}{2}(n + s)$ positive terms in its diagonal. $s$ is the signature of the metric (in this case, the number of positive elements on the diagonal of the metric, minus the number of negative ones) and $n$ is the dimension of the manifold used to represent spacetime (Hawking; Ellis, 1973).

Since the classification of $T^{\mu\nu}$ is defined for the case in which it is expressed with respect to the orthonormal basis $\{\boldsymbol{\xi}_{\mu}\}$, the metric $g_{\mu\nu}$ in Eq. (3.3) can be reduced to the metric of Minkowski spacetime, $\eta_{\mu\nu}$, by considering Eq. (3.5). As in our basis there is one timelike vector, $\boldsymbol{\xi}_0$, and three spacelike vectors, $\boldsymbol{\xi}_i$, and that we adopt a positive signature for the metric, we have $s = +2$; besides that, since there are four vectors in the basis, we have $n = 4$. In this way, we can choose $\{\boldsymbol{\xi}_{\mu}\}$ such that

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

As a consequence, according to the Eq. (3.3), we can obtain the components $T^{\mu}_{\ \nu}$:

$$
\begin{aligned}
T^0_{\ 0} &= T^{0\rho}\eta_{\rho 0} = T^{00}\eta_{00} = -T^{00};\\
T^0_{\ i} &= T^{0\rho}\eta_{\rho i} = T^{0i}\eta_{ii} = T^{0i};\\
T^i_{\ 0} &= T^{i\rho}\eta_{\rho 0} = T^{i0}\eta_{00} = -T^{i0};\\
T^i_{\ j} &= T^{i\rho}\eta_{\rho j} = T^{ij}\eta_{jj} = T^{ij}.
\end{aligned} \tag{3.7}
$$

These, in turn, give rise to the matrix representation

$$
T^\mu_{\ \nu} = \begin{pmatrix} -T^{00} & T^{01} & T^{02} & T^{03} \\ -T^{10} & T^{11} & T^{12} & T^{13} \\ -T^{20} & T^{21} & T^{22} & T^{23} \\ -T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}, \tag{3.8}
$$

and, as a result, Eq. (3.2) can be written as

$$
\begin{pmatrix} -T^{00} & T^{01} & T^{02} & T^{03} \\ -T^{10} & T^{11} & T^{12} & T^{13} \\ -T^{20} & T^{21} & T^{22} & T^{23} \\ -T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} = \lambda \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}. \tag{3.9}
$$

Therefore, depending on the chosen eigenvectors, we can find its eigenvalues, as well as determine the components $T^{\mu\nu}$.

There are four canonical forms in which the components $T^{\mu\nu}$ may be expressed with respect to the basis $\{\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. The one of greatest interest for the present work is the so-called 'type I': when the energy-momentum tensor has one timelike eigenvector, $\boldsymbol{U} = \boldsymbol{\xi}_0$, which coincides with the timelike basis vector and has components $U^\mu$, and three spacelike vectors $\boldsymbol{V}_i$, with components $V^\mu_i$.

Under such conditions, if we choose the set of vectors (Lobo, 2017)

$$
\begin{aligned}
\boldsymbol{U} &= (-1, 0, 0, 0),\\
\boldsymbol{V}_1 &= (0, 1, 0, 0),\\
\boldsymbol{V}_2 &= (0, 0, 1, 0),\\
\boldsymbol{V}_3 &= (0, 0, 0, 1),
\end{aligned} \tag{3.10}
$$

and apply them to Eq. (3.2), for the eigenvector $\boldsymbol{U}$ we then obtain the expression $T^\mu_{\ \nu} U^\nu = \lambda_0 U^\nu$, whose matrix representation is given by Eq. (3.9):

$$\begin{pmatrix} -T^{00} & T^{01} & T^{02} & T^{03} \\ -T^{10} & T^{11} & T^{12} & T^{13} \\ -T^{20} & T^{21} & T^{22} & T^{23} \\ -T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_0 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} T^{00} \\ T^{10} \\ T^{20} \\ T^{30} \end{pmatrix} = \begin{pmatrix} -\lambda_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.11}$$

Hence we find the components

$$\begin{aligned} T^{00} &= -\lambda_0, \\ T^{01} = T^{02} &= T^{03} = 0. \end{aligned} \tag{3.12}$$

Following the same procedure to the others eigenvectors, $\boldsymbol{V}_i$, we obtain the system

$$\begin{aligned} T^{ii} &= \lambda_i, \\ T^{i0} &= 0, \\ T^{ij} &= 0, \quad i \neq j. \end{aligned} \tag{3.13}$$

From Eqs. (3.12) and (3.13), we see that the eigenvalues of $T^{\mu}_{\ \nu}$ with respect to the eigenvectors $\boldsymbol{U}$ and $\boldsymbol{V}_i$ are, respectively, $\lambda_0 = -T^{00}$ and $\lambda_i = T^{ii}$. We also note that the matrix representation in Eq. (3.8) displays these eigenvalues in its diagonal, while all the other components are zero. Therefore, type I tensors are the ones whose matrix representation can be diagonalized (Martín-Moruno; Visser, 2021).

To give the eigenvalues $\lambda_\mu$ some physical meaning, we turn to the interpretation of the components $T^{\mu\nu}$ in terms of which they are written. As stated before, each component $T^{\mu\nu}$ is defined as the flux of the $\mu$ component of the four-momentum across a surface of constant $x^\nu$. Under such consideration, $T^{00}$ is the flux of the zero component of the four-momentum across a surface $x^0 = t =$ constant. This is the energy density, defined as $\rho$, measured by an observer whose worldline at a given point has unit tangent vector (four-velocity) with components $U^\mu$. The components $T^{ij}$, in turn, are the flux of momentum, along each spatial direction $\boldsymbol{\xi}_i$, across surfaces of constant $x^i$. These are the principal pressures, given by $p_i$, in these spatial directions (Schutz, 2009).

Consequently, from Eqs. (3.12) and (3.13), we identify the eigenvalues as $\lambda_0 = -T^{00} = -\rho$ and $\lambda_i = T^{ii} = p_i$. Thus, the energy-momentum tensors of type I, with contravariant indices, are given by (Hawking; Ellis, 1973; Lobo, 2017)

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}. \tag{3.14}$$

It is worth noticing that the energy-momentum tensor of a perfect fluid, displayed in Eq. (1.5), is a special case of the Type I tensor in Eq. (3.14) with $p_1 = p_2 = p_3 = p$.

There are also three other canonical forms of the energy-momentum tensor, provided by different sets of eigenvalues. Type II tensors, for instance, have two lightlike eigenvectors, given by $\xi_0 + \xi_1 = (1,1,0,0)$, and two spacelike eigenvectors, while type III tensors have three lightlike eigenvectors, given by $\xi_0 + \xi_1 = (1,1,0,0)$, and one spacelike eigenvector. Type IV tensors, in turn, do not present causal eigenvectors, *i.e.*, timelike or null vectors, presenting only spacelike eigenvectors or eigenvectors with complex components (Lobo, 2017; Martín-Moruno; Visser, 2021).

Each one of these energy-momentum tensor types has a different form, with different components in comparison to the ones displayed in Eq. (3.14), as well as a distinct set of eigenvalues. As a consequence, they also provide a different set of constraint equations when the energy conditions are considered. Nevertheless, most classical and semi-classical fields are of the type I form, including the ones with a perfect fluid energy-momentum tensor. In this context, since we will later assume homogeneity and isotropy and describe the spacetime content as a perfect fluid, we turn our attention to the energy-momentum tensors of type I.

We emphasize again that type I tensors are those whose matrix representation can be diagonalized, which means finding an observer in the rest frame of the field associated with it (Martín-Moruno; Visser, 2021). This is precisely the case for the energy-momentum tensor of a perfect fluid. In this work, we will deal with the FLRW metric, displayed in Eq.(1.4). As it presents elements distinct from unity in its diagonal, this metric leads to an energy-momentum tensor that, in general, takes the form of Eq.(1.5), in contrast to Eq.(3.14). For instance, according to Eq.(1.5) the energy-momentum tensor of a perfect fluid for the FLRW metric displays the component $T^{11} = g^{11}p$, while from Eq.(3.14) we find $T^{11} = p_1$. Despite this, as pointed out before, in a locally comoving inertial frame, *i.e.*, for some observer "moving with the fluid", the energy-momentum tensor in Eq.(1.5) reduces to $T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$ (Schutz, 1970). This corresponds to Eq. (3.14) with $p_1 = p_2 = p_3 = p$ and, in this way, this classification applies to more general metrics than the one in Eq. (3.6), including also the FLRW metric.

## 3.2 The Energy Conditions

### 3.2.1 The Weak Energy Condition

Broadly speaking, the Weak Energy Condition (WEC) states that the local energy density must be non-negative. It conveys the physically reasonable idea that an observer would expect to measure either a positive amount of energy density or no energy density at all, while it excludes the possibility of measuring a negative quantity of it. Then, considering an observer following a timelike world-line whose unit tangent vector is $t^\mu$ (which corresponds to the observer's four-velocity), the energy density of matter is given by $T_{\mu\nu}t^\mu t^\nu$ and the WEC reads

$$T_{\mu\rho}t^{\mu}t^{\nu} \geq 0. \tag{3.15}$$

The vector $t^{\mu}$ is normalized in such a way that $t^{\mu}t_{\mu} = -1$, leading to

$$\begin{aligned} t^{\mu}t_{\mu} = g_{\mu\nu}t^{\mu}t^{\nu} = g_{00}(t^0)^2 + g_{ii}(t^i)^2 = -1 \\ \Rightarrow (t^0)^2 = 1 + g_{ii}(t^i)^2, \end{aligned} \tag{3.16}$$

since we are considering a metric in the form of the FLRW one, in Eq. (1.4), with $g_{00} = -1$ (when $c = 1$).

For a perfect fluid, in which case $T_{\mu\nu}$ takes the form of Eq. (1.5), from Eq. (3.15) the WEC reads

$$T_{00}(t^0)^2 + T_{ii}(t^i)^2 \geq 0. \tag{3.17}$$

Furthermore, Eq. (1.5) also provides the components

$$\begin{aligned} T_{00} &= \rho, \\ T_{ii} &= pg_{ii}, \end{aligned} \tag{3.18}$$

as well as the trace

$$T \equiv T^{\mu}_{\ \mu} = -\rho + 3p. \tag{3.19}$$

By plugging the components $T_{00}$ and $T_{ii}$, from Eq. (3.18), and the component $(t^0)^2$, from Eq. (3.16), into Eq. (3.17), we get

$$\rho + g_{ii}(\rho + p)(t^i)^2 \geq 0. \tag{3.20}$$

Since $(t^i)^2$ are positive arbitrary numbers, the only way the inequality can be satisfied is if

$$\rho \geq 0, \tag{3.21}$$

$$\rho + p \geq 0. \tag{3.22}$$

These are the WEC bounds, in GR, for a perfect fluid.

### 3.2.2 The Dominant Energy Condition

The Dominant Energy Condition (DEC) states that, for any future directed, timelike vector $t^{\mu}$, the vector $-T^{\mu}_{\ \nu}t^{\nu}$ should be non-spacelike. Mathematically, since we are assuming a positive signature for $g_{\mu\nu}$, $-T^{\mu}_{\ \nu}t^{\nu}$ can't have a positive norm (it can't be spacelike), *i.e.*,

$$T_{\mu\rho}T^{\rho}_{\ \nu}t^{\mu}t^{\nu} \leq 0. \tag{3.23}$$

$-T^{\mu}_{\ \nu}t^{\nu}$ is the energy-momentum four-current density of matter seen for an observer with four-velocity $t^{\mu}$, and its physical meaning can be understood through an analogy with current densities in electromagnetism.

For a flow of charge in a three-dimensional region, the volume current density is given by (Griffiths, 2023)

$$\mathbf{J} = \rho\mathbf{v}, \tag{3.24}$$

if we take $\rho$ as the volume charge density[1] and $\mathbf{v}$ as its velocity. The vector $-T^{\mu}_{\ \nu}t^{\nu}$, a four-current density, is then a generalization of the current density $\mathbf{J}$, with $\rho$ being replaced by the energy-momentum tensor itself, and $\mathbf{v}$ being replaced by the four-velocity $t^{\mu}$.

We can also interpret the DEC based on the nature of the four-current density of matter, which is a vector. If we think in the lightcone of some event in spacetime, $-T^{\mu}_{\ \nu}t^{\nu}$ will lie within this lightcone since it cannot be spacelike. In this sense, DEC implies that matter flows slower than light, *i.e.*, that nothing can move faster than light.

For a perfect fluid, Eq. (3.23) reads

$$T_{\mu\rho}T^{\rho}_{\ \nu}t^{\mu}t^{\nu} = T_{00}T^{0}_{\ 0}(t^{0})^{2} + T_{11}T^{1}_{\ 1}(t^{1})^{2} + T_{22}T^{2}_{\ 2}(t^{2})^{2} + T_{33}T^{3}_{\ 3}(t^{3})^{2} \leq 0. \tag{3.25}$$

From Eq. (1.5), we find that $T^{0}_{\ 0} = -\rho$ and that $T^{1}_{\ 1} = T^{2}_{\ 2} = T^{3}_{\ 3} = p$. By plugging these components into Eq. (3.25) and using Eqs. (3.16) and (3.18), we get

$$T_{\mu\rho}T^{\rho}_{\ \nu}t^{\mu}t^{\nu} = -\rho^{2} + p^{2}g_{ii}(t^{i})^{2} = -\rho^{2} + (-\rho^{2} + p^{2})g_{ii}(t^{i})^{2} \leq 0. \tag{3.26}$$

Again, as $(t^{i})^{2}$ are arbitrary, the inequality is fulfilled only if both $-\rho \leq 0$ and $-\rho^{2} + p^{2} \leq 0$, which respectively lead to the conditions

$$\rho \geq 0, \tag{3.27}$$

$$\rho \geq |p| \ \ (\text{or } -\rho \leq p \leq \rho). \tag{3.28}$$

Equations (3.27) and (3.28) are the bounds imposed by the DEC, in GR, for a perfect fluid. Furthermore, we can see that DEC is equivalent to WEC, with the additional constraint that the pressure does not exceed the energy density in magnitude.

[1] In this specific example, the $\rho$ in Eq. (3.24) is not necessarily the same as the energy density $T^{00}$, since it refers specifically to the volume charge density, despite being represented by the same letter.

### 3.2.3 The Strong Energy Condition

The Strong Energy Condition (SEC) can be physically regarded as a statement of the attractiveness of gravity. Such interpretation follows from Raychaudhuri's equation for timelike geodesics, displayed in Eq. (2.63).

Considering a timelike geodesic congruence, such geodesics are going to converge if its expansion $\theta$ decreases, which implies $d\theta/d\tau \leq 0$ (Hawking; Ellis, 1973). From Eq. (2.63), we see that such derivative depends on four terms: the first one, $-\frac{\theta^2}{3}$, is negative because $\theta^2 > 0$.

The second one, $-\sigma_{\mu\nu}\sigma^{\mu\nu}$, is also negative because $\sigma_{\mu\nu}\sigma^{\mu\nu} > 0$. This last inequality arises from the fact that $U^\mu$ is projected orthogonally with respect to both indices of $\sigma_{\mu\nu}$, according to Eq. (2.59). In this sense, both indices of $\sigma_{\mu\nu}$ are spacelike since they do not have components in the direction of the timelike $U^\mu$, and consequently $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$ (Carroll, 2004; Albareti; Cembranos; Cruz-Dombriz, 2012; Venn; Agarwal; Vasak, 2024).

The third term, $\omega_{\mu\nu}\omega^{\mu\nu}$, is null due to the nature of the congruence, which is hypersurface orthogonal as a requirement for being globally defined. According to Frobenius' theorem, the necessary and sufficient condition for $U^\mu$ being hypersurface orthogonal is (Wald, 1984)

$$U_{[\mu}\nabla_\nu U_{\rho]} = 0. \tag{3.29}$$

By evaluating the left-hand side of it, we obtain

$$U_{[\mu}\nabla_\nu U_{\rho]} = \frac{1}{6}(U_\mu B_{[\nu\rho]} + U_\nu B_{[\rho\mu]} + U_\rho B_{[\mu\nu]}) \tag{3.30}$$

$$= \frac{1}{6}(U_\mu \omega_{\nu\rho} + U_\nu \omega_{\rho\mu} + U_\rho \omega_{\mu\nu}), \tag{3.31}$$

which is 0 if the vorticity vanishes, *i.e*, $\omega_{\mu\nu} = 0$. As the condition for hypersurface orthogonality requires Eq. (3.29) to be satisfied, then $\omega_{\mu\nu} = 0$ and therefore $\omega_{\mu\nu}\omega^{\mu\nu} = 0$. This is consistent with a universe described by the FLRW metric in Eq.(1.4), as the universe's content in this scenario is characterized by an irrotational fluid. Indeed, there are no terms in Eq.(1.4) related to rotation.

Under such considerations, to have $\frac{d\theta}{d\tau} \leq 0$, the fourth term in Eq. (2.63), $-R_{\mu\nu}t^\mu t^\nu$, must be negative. This leads to the timelike convergence condition, given by

$$R_{\mu\nu}t^\mu t^\nu \geq 0. \tag{3.32}$$

By using Eq. (1.2), we see that this condition is fulfilled if the energy-momentum satisfies the inequality

$$T_{\mu\nu}t^{\mu}t^{\nu} + \frac{T}{2} \geq 0. \tag{3.33}$$

which is the Strong Energy Condition (SEC).

We note that the first term in Eq. (3.33), $T_{\mu\nu}t^{\mu}t^{\nu}$, is just the WEC, and therefore is nonnegative. In this sense, demanding SEC is equivalent to requiring that the pressure exerted by matter, encoded in $T$, will not take so large negative values as would be necessary to make the inequality negative. From Eq. (3.33), we also see that WEC implies SEC and, for this reason, the latter is a more strict condition than the former. As stated before, SEC also implies the nondiverging effect of matter on timelike geodesics.

For a perfect fluid, using Eqs. (3.16), (3.18) and (3.19), SEC can be written as

$$T_{\mu\nu}t^{\mu}t^{\nu} + \frac{T}{2} = \rho(t^0)^2 + pg_{ii}(t^i)^2 - \frac{1}{2}(\rho + 3p) \tag{3.34}$$

$$= \frac{1}{2}(\rho + 3p) + (\rho + p)g_{ii}(t^i)^2 \geq 0. \tag{3.35}$$

Since $(t^i)^2$ are arbitrary positive numbers, each term must be nonnegative to the inequality to be satisfied. This leads to the following equations for SEC in the scope of GR:

$$\rho + 3p \geq 0, \tag{3.36}$$

$$\rho + p \geq 0. \tag{3.37}$$

Comparing the bounds for WEC, in Eqs. (3.21) and (3.22), and the bounds for SEC, in Eqs. (3.36) and (3.37), we see that they are indeed satisfied provided that $\rho \geq 0$ and that there are no large negative pressures (equal or larger than $-\frac{\rho}{3}$).

Cosmological observations indicate that SEC is not fulfilled, due to the recent accelerated expansion of the universe (Riess *et al.*, 1998; Perlmutter *et al.*, 1999; Schrabback *et al.*, 2010; Astier; Pain, 2012). The main current attempts to explain such behavior are to assume the presence of some fluid with negative pressure (dark energy) and to consider geometrical extensions of GR. The SEC validity and the role of its bound equations will be revisited in the context of alternative theories in the next section.

### 3.2.4 The Null Energy Condition

Analogously to the SEC case, the Null Energy Condition (NEC) follows from a convergence condition for null geodesics. From Eq. (2.81), a nondiverging evolution for the expansion of null geodesics requires $\frac{d\theta}{d\lambda} \leq 0$. To ensure that this inequality is satisfied, the right-hand side of Eq. (2.81) must be nonnegative.

By following the same arguments as in the timelike case, the terms depending on $\theta$, $\hat{\sigma}_{\mu\nu}$ and $\hat{\omega}_{\mu\nu}$ are nonpositive or null. Similarly, the last term should be nonnegative, then leading to the null convergent condition, which reads

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0. \tag{3.38}$$

From Eq. (1.2), we thus find the Null Energy Condition:

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0. \tag{3.39}$$

Equation (3.39) is valid as NEC can be regarded as a limiting case of WEC. Specifically, if Eq. (3.15) holds for all timelike $t^{\mu}$, then by continuity, Eq. (3.39) will also hold for all null $k^{\mu}$. In this sense, WEC implies that matter has a nondiverging effect on null geodesics, since NEC arises from it by a continuity argument. Moreover, using a similar continuity argument, NEC can also follow from SEC. If Eq. (3.33) is valid, then by continuity, Eq. (3.39) is also valid.

To find an explicit bound equation for NEC in terms of $\rho$ and $p$, it is necessary to take into account the nature of $k^{\mu}$: since it is a null vector, its magnitude is 0, so

$$\begin{aligned} k^{\mu}k_{\mu} = g_{\mu\nu}k^{\mu}k^{\nu} = g_{00}(k^0)^2 + g_{ii}(k^i)^2 = 0 \\ \Rightarrow (k^0)^2 = -\frac{g_{ii}}{g_{00}}(k^i)^2 = g_{ii}(k^i)^2. \end{aligned} \tag{3.40}$$

We are again considering a metric along the lines of the FLRW one, in Eq. (1.4), and taking $g_{00} = -1$.

Applying Eqs. (3.18) and (3.40) into Eq. (3.39), we find

$$T_{\mu\nu}k^{\mu}k^{\nu} = T_{00}(k^0)^2 + t_{ii}(k^i)^2 \tag{3.41}$$

$$= (\rho + p)g_{ii}(k^i)^2 \geq 0, \tag{3.42}$$

which leads to the bound equation for NEC within the framework of GR,

$$\rho + p \geq 0. \tag{3.43}$$

This is a weaker requirement than the other conditions since the inequality in Eq. (3.43) is already covered by WEC, DEC and SEC in Eqs. (3.22), (3.28) and (3.37), respectively, without any other extra requirement.

It is also worth noticing that both SEC and NEC, as results of the convergence conditions, are conditions on $R_{\mu\nu}$, in contrast to WEC and DEC, which constrain the energy-momentum tensor directly. This property of the energy conditions, as well as its implications for the bound equations, will play an important role in the nature of the bounds for extended theories of gravity.

### 3.2.5 Applying the Energy Conditions to Constrain the Dark Energy Equation of State Parameter $\omega$

In this work, we aim to use the energy condition bounds to constrain extended theory models. Before doing that, there is a direct application of these constraints that we can use to illustrate the physical consequences of the energy conditions.

The considerations made up to that point about $T_{\mu\nu}$ led us to energy condition bounds that generally depend on $\rho$ and $p$. As often assumed in Cosmology, these quantities are related by the equation of state in Eq. (1.8) (Dodelson; Schmidt, 2020). For perfect fluids obeying this relation, the bounds can be written solely in terms of the energy density $\rho$ and the equation of state parameter $\omega$, leading to explicit constraints on the parameter $\omega$ itself. These constraints are theoretical limits, according to each energy condition, to the possible values of $\omega$ for the components of the universe. In particular, there are some considerations concerning these bounds we can make about dark energy.

Observational data points for a current phase of accelerated expansion of the universe (Riess *et al.*, 1998; Perlmutter *et al.*, 1999; Astier; Pain, 2012). Within the framework of General Relativity, dark energy is often taken as the component of the universe necessary for justifying such behavior: it is some fluid (or substance) with negative pressure, that is, with a repulsive nature, leading to the accelerated expansion (and therefore implying $\ddot{a} > 0$) (Frieman; Turner; Huterer, 2008; Dodelson; Schmidt, 2020).

The requirement of a positive acceleration can be related to a negative pressure if we look at the second Friedmann equation, in Eq. (1.7), from which we get

$$\ddot{a} = -\frac{4\pi G}{3} a(\rho + 3p). \tag{3.44}$$

Imposing the expansion condition, $\ddot{a} > 0$, this leads to

$$p < -\frac{\rho}{3}, \tag{3.45}$$

resulting in

$$\omega_{DE} < -\frac{1}{3} \tag{3.46}$$

if we take into account Eq. (1.8), with the subscript 'DE' standing for 'dark energy'. In other words, having a repulsive nature and, consequently, being characterized by the parameter $\omega_{DE} < -1/3$ can be regarded as the property that initially defines dark energy (Frieman; Turner; Huterer, 2008). In the $\Lambda$CDM scenario, a cosmological constant corresponds to $\omega_{DE} = -1$. However, recent observational data suggests that $\omega_{DE}$ varies in time, as we will see further.

Under such considerations, let us turn to the implications of the energy conditions on $\omega_{DE}$, by applying Eq. (1.8). For WEC, Eq. (3.22) implies

$$\rho + p \geq 0 \Rightarrow \omega_{DE} \geq -1, \tag{3.47}$$

in addition to the requirement of a non-negative energy density.

This is under the limit imposed by the Friedmann equation for dark energy since it allows $\omega_{DE} < -1/3$. However, it constrains $\omega_{DE}$ to the lower limit of $\omega_{DE} = -1$, while it does not define an upper limit. The same happens for NEC, based on Eq. (3.39), except for the restriction on negative energy densities.

DEC, in turn, is more restrictive. From Eq. (3.28), it follows

$$-\rho \leq p \leq \rho \Rightarrow -1 \leq \omega_{DE} \leq 1, \tag{3.48}$$

resulting not only in the lower limit of $-1$, as WEC did, but also imposing the upper limit of $\omega_{DE} = 1$. We then see that DEC also agrees on the condition for dark energy since it allows $\omega_{DE} < -1/3$. However, it is more restrictive than WEC and NEC.

On the other hand, for SEC the bound in Eq. (3.36) provides

$$\rho + 3p \geq 0 \Rightarrow \omega_{DE} \geq -\frac{1}{3}, \tag{3.49}$$

contradicting Eq. (3.46). That is, SEC is not satisfied due to the accelerated expansion of the universe. This makes sense since we deduced SEC by considering the convergence condition for timelike geodesics, based on the attractive nature of gravity. However, the accelerated expansion arises from the presence of some fluid with negative pressure and, consequently, with a repulsive nature, if we think of dark energy as its cause.

Previously, observational data suggested a constant $\omega_{DE}$ close to $-1$ (Frieman; Turner; Huterer, 2008), thus favoring the ΛCDM model. However, recent BAO measurements from the Dark Energy Spectroscopic Instrument (DESI), associated with SNe Ia and CMB data suggest a time-evolving $\omega_{DE}$. That is, a parameter changing with time presented a best fit to the data, no longer favoring the ΛCDM description and suggesting dynamical dark energy (Karim *et al.*, 2025).

# 4 Energy Conditions in Extended Theories of Gravity

We now aim to apply the energy conditions presented in the previous sections to extended theories of gravity and then find the generalized bound equations analogous to those provided by Einstein's General Relativity. We first consider a general extended theory and, subsequently, $f(R)$-type theories. We also address the specific case of the Hu-Sawicki model, writing the bounds in terms of the Hubble, deceleration, jerk, and snap functions.

## 4.1 Bounds for General Extended Theories

The field equations within the framework of GR are given by Eq. (1.1), whose geometric part is encapsulated in the Einstein tensor, $G_{\mu\nu}$. Starting from this point, we can consider a class of extended theories that generalize Einstein equations by adding geometrical information in the form of a tensor $H_{\mu\nu}$. Under such considerations, following the procedure presented by Penna-Lima et al. (Penna-Lima *et al.*, 2019), we first examine the general case for which the field equations take the form

$$g_1(\psi^i)(G_{\mu\nu} + H_{\mu\nu}) = 8\pi G g_2(\psi^i)T_{\mu\nu}. \tag{4.1}$$

Eq. (4.1) is written in natural units, so the speed of light is $c = 1$. $\psi^i$ are the fields that contribute to the dynamics of the theory, such as curvature invariants or scalar fields, while $g_1(\psi^i)$ and $g_2(\psi^i)$ are the generalized coupling factors to the matter fields (Capozziello; Lobo; Mimoso, 2015). $H_{\mu\nu}$, in turn, represents the additional geometric terms that modify the theory and, as a geometric quantity, it depends on the fields $\psi^i$. We can recover GR by taking $H_{\mu\nu} = 0$ (no geometrical addition) and $g_1(\psi^i) = g_2(\psi^i) = 1$.

Before moving on, we emphasize that in this work we are interested in theories whose field equations can be written specifically in the form of Eq. (4.1), since there are theories that arise from geometric modifications in the action but whose field equations do not necessarily follow Eq. (4.1). The Palatini formalism, for instance, depends on both the metric $g_{\mu\nu}$ and a non-metric connection, and therefore yields distinct field equations for each of these quantities. The field for the variation with respect to the non-metric connection does not depend on $T_{\mu\nu}$, so it does not displays the right hand-side of Eq. (4.1) (Oliveira, 2010; Jr *et al.*, 2025).

Given the nature of the extended theory we aim to investigate in this work (the Hu-Sawicki $f(R)$ theory), we focus on theories with minimal coupling, meaning those where the explicit curvature-matter coupling, expressed by $g_2(\psi^i)$, satisfies $g_2(\psi^i) = 1$. In practice, minimal coupling implies the absence of direct explicit associations between geometric quantities and matter terms in the field equation. For instance, if we take $\psi = R$ and $g_2(R) = f(R) \neq 1$, the term $8\pi G f(R) T_{\mu\nu}$ would appear on the right-hand side of the equation, where curvature ($f(R)$) and matter ($T_{\mu\nu}$) would then be directly coupled in the field equation. On the other hand, choosing $g_2(\psi^i) = 1$ allows for the decoupling of curvature and matter terms, placing each type of term on separate sides of the equation, as seen in Eq. (4.2).

In this scenario, Eq. (4.1) leads to the field equation

$$g_1(\psi^i)(G_{\mu\nu} + H_{\mu\nu}) = 8\pi G T_{\mu\nu}. \tag{4.2}$$

We can then use it to find the bounds provided by the energy conditions using Eqs. (3.15), (3.23), (3.32), and (3.38).

To compute the inequality relative to SEC, we first need to find an expression for $R_{\mu\nu}$, since according to Eq. (3.32) SEC is defined in terms of it. By using the definition of $G_{\mu\nu}$ from Eq. (1.1), we obtain the expression

$$g_1(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + H_{\mu\nu}) = 8\pi G T_{\mu\nu}, \tag{4.3}$$

from which we can express $R_{\mu\nu}$ as (Penna-Lima *et al.*, 2019)

$$R_{\mu\nu} = \frac{8\pi G}{g_1} T_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} - H_{\mu\nu}. \tag{4.4}$$

Fulfilling SEC requires performing a contraction between $R_{\mu\nu}$ and two timelike vectors, in the form $R_{\mu\nu} t^\mu t^\nu$. Considering that we're using a positive signature for $g_{\mu\nu}$ and that the vectors $t^\mu$ are normalized such that $t^\mu t_\nu = -1$, from Eq. (4.4) we get

$$\begin{aligned} R_{\mu\nu} t^\mu t^\nu &= \frac{8\pi G}{g_1} T_{\mu\nu} t^\mu t^\nu + \frac{1}{2} R g_{\mu\nu} t^\mu t^\nu - H_{\mu\nu} t^\mu t^\nu \\ &= \frac{8\pi G}{g_1} T_{\mu\nu} t^\mu t^\nu - \frac{1}{2} R - H_{\mu\nu} t^\mu t^\nu. \end{aligned} \tag{4.5}$$

Besides that, it is also possible to write the Ricci scalar, defined as $R \equiv g^{\mu\nu} R_{\mu\nu}$, in terms of the traces $H \equiv g^{\mu\nu} H_{\mu\nu} = H^\mu{}_\mu$ and $T \equiv g^{\mu\nu} H_{\mu\nu} = T^\mu{}_\mu$ by contracting both sides of Eq. (4.2) with $g^{\mu\nu}$. It follows that

$$R = H - \frac{8\pi G}{g_1} T. \tag{4.6}$$

Applying Eq. (4.6) to Eq. (4.5) thus provides

$$R_{\mu\nu} = \frac{8\pi G}{g_1} T_{\mu\nu} + \frac{1}{2} H g_{\mu\nu} - \frac{1}{2} \frac{8\pi G}{g_1} T g_{\mu\nu} - H_{\mu\nu}. \tag{4.7}$$

We can now perform the contraction imposed by the SEC in Eq. (3.32), ultimately resulting in the generalized SEC condition:

$$\frac{8\pi G}{g_1}\left(T_{\mu\nu}t^{\mu}t^{\nu}+\frac{T}{2}\right)-\left(H_{\mu\nu}t^{\mu}t^{\nu}+\frac{H}{2}\right)\geq 0. \tag{4.8}$$

We emphasize that it is possible to recover the GR case, displayed in Eq. (3.33), when $H_{\mu\nu}=H=0$ and $g_1=1$.

Initially, Eq. (4.8) is a statement of the attractiveness of gravity, since according to Raychaudhuri's equation the contraction $R_{\mu\nu}t^{\mu}t^{\nu}$ must be nonnegative for gravity to remain attractive. In this general case, we see that even if the matter part of the equation is negative, *i.e.*, if the terms with $T_{\mu\nu}$ and $T$ contribute negatively to it, the condition in Eq. (4.8) can be still fulfilled due to the geometric terms containing $H_{\mu\nu}$ and $H$. A negative contribution from $T_{\mu\nu}$ and $T$ could be interpreted as matter fields with negative pressure, such as the presence of dark energy (Capozziello; Lobo; Mimoso, 2015).

Despite that, the accelerated expansion indicated by cosmological observations implies the violation of SEC, as will be discussed in the next section. However, as pointed out by Penna-Lima et al. (Penna-Lima *et al.*, 2019), Eq. (4.8) may be used to understand how this violation occurs. If the matter terms are such that

$$\frac{8\pi G}{g_1}\left(T_{\mu\nu}t^{\mu}t^{\nu}+\frac{T}{2}\right)\geq 0, \tag{4.9}$$

then the violation, and consequently the acceleration in the expansion of the universe, would be due solely to the geometrical terms of Eq. (4.8) - the ones with $H_{\mu\nu}$ and $H$.

In this sense, to explain the acceleration of the Universe without requiring any additional fluid the condition in Eq. (4.9) must be fulfilled. As mentioned before, another scenario would be the one with a dark energy in the form of matter fields with negative pressure. In this case, Eq. (4.9) would be violated, and the accelerated expansion would not only be due to the geometric terms but also to the matter content itself.

Similarly, we can find the generalized NEC bound equation by applying Eq. (4.4) to Eq. (3.38). The contraction of $R_{\mu\nu}$ with two null vectors $k^{\mu}$ provides the expression

$$\left(\frac{8\pi G}{g_1}T_{\mu\nu}+\frac{1}{2}Rg_{\mu\nu}-H_{\mu\nu}\right)k^{\mu}k^{\nu}\geq 0. \tag{4.10}$$

Since for null vectors $k^{\mu}k_{\mu}=0$, we find the following generalized NEC equation:

$$\frac{8\pi G}{g_1}T_{\mu\nu}k^{\mu}k^{\nu}-H_{\mu\nu}k^{\mu}k^{\nu}\geq 0. \tag{4.11}$$

Analogously to the SEC case, we can recover the RG results in Eq. (3.39) when $H_{\mu\nu}=0$ and $g_1=1$.

From Eqs. (4.8) and (4.11) we see that the SEC and NEC impose restrictions not only to $T_{\mu\nu}$ but also to the additional geometrical terms $H_{\mu\nu}$ and $H$. This dependence on the extended gravity terms comes from the definition of these conditions, which are given in terms of $R_{\mu\nu}$. Since $R_{\mu\nu}$ depends on $H_{\mu\nu}$, as shown by equation Eq. (4.4), we can naturally expect the presence of the modified gravity terms ($h_t$ and $h_s$) in the new bounds. This is not the case of GR, for which the SEC and NEC depend only on $T_{\mu\nu}$ and $T$, as we can see in Eqs. (3.33) and (3.39) (Penna-Lima *et al.*, 2019).

On the other hand, WEC and DEC are, by definition, direct restrictions on $T_{\mu\nu}$, as displayed respectively in Eqs. (3.15) and (3.23), so initially they do not depend on $H_{\mu\nu}$ (and consequently on $h_t$ and $h_s$). Then, when we write the bounds in terms of $\rho$ and $p$, they remain the same as in GR, since they do not depend on the functions $h_t$ and $h_s$. Consequently, in terms of $\rho$ and $p$, the bound equations for WEC and DEC are identical to those in GR and are respectively given by Eqs. (3.15) and (3.23). However, we emphasize that this equivalence holds only when the bounds are expressed in terms of $\rho$ and $p$. Soon we will find explicit expressions for $\rho$ and $p$ which may depend on $h_t$ and $h_s$, so when we write then in terms of these functions such dependence implies bounds that actually differ from those in GR, as we will see further.

An alternative approach for dealing with the geometric extension would be taking Eq. (4.1) and rearranging it as

$$G_{\mu\nu} = 8\pi G g_2 \left( \frac{T_{\mu\nu}}{g_1} - \frac{H_{\mu\nu}}{8\pi G g_2} \right) = 8\pi G g_2 T^{\text{eff}}_{\mu\nu}, \tag{4.12}$$

with

$$T^{\text{eff}}_{\mu\nu} = \frac{T_{\mu\nu}}{g_1} - \frac{H_{\mu\nu}}{8\pi G g_2}. \tag{4.13}$$

In this scenario, $T^{\text{eff}}_{\mu\nu}$ is taken as an effective energy-momentum tensor that encapsulates all the geometric modifications in the theory. From that, we could then calculate an effective energy density $\rho^{\text{eff}}$ and an effective pressure $p^{\text{eff}}$. After that, we would apply the energy conditions to the effective tensor $T^{\text{eff}}_{\mu\nu}$, writing the bounds in terms of the effective quantities $\rho^{\text{eff}}$ and $p^{\text{eff}}$, instead of the usual $\rho$ and $p$ (as we will see in the following section).

However, as pointed out by the authors in Refs.(Capozziello; Lobo; Mimoso, 2015) and (Penna-Lima *et al.*, 2019), this would imply that we are considering the $H_{\mu\nu}$ contribution as a kind of fictitious fluid, arising from geometry rather than from the energy-matter content of the universe (as in the case of the usual $T_{\mu\nu}$). It would then be misleading to apply the standard energy conditions from GR to these effective quantities, not only because of the way they are initially defined (based on considerations of the behavior of $T_{\mu\nu}$ and the convergence conditions), but also because this would suggest we are considering the existence of some extra fluid as the cause of the expansion of the universe.

Conversely, we are considering conditions (bounds) that the extended theories need to fulfill to explain the accelerated expansion of the universe without requiring the existence of extra fluids. That is, we assume that if such a theory can explain the accelerated expansion based on geometric modifications, it wouldn't be necessary to require the presence of extra fluids. For this reason, in this work, we follow the approach of maintaining $H_{\mu\nu}$ in the geometric part of the field equation, as in Eq. (4.1), and when necessary apply the energy conditions on $T_{\mu\nu}$, and not on effective quantities.

## 4.2 Bounds for a Homogeneous and Isotropic Universe

In this work, we study the energy conditions in the context of a homogeneous and isotropic universe, which is described by the Friedman-Lamaître-Robertson-Walker (FLRW) metric in Eq. (1.4). From the FLRW metric, using Eqs. (2.16) and (2.38), and considering that $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, we find the the non-zero components of $R_{\mu\nu}$ to be

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}, \\ R_{ij} &= \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}\right] g_{ij}, \end{aligned} \tag{4.14}$$

leading to

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right]. \tag{4.15}$$

That is, the Ricci tensor is a diagonal tensor whose components are functions of the time $t$ times the corresponding component of $g_{\mu\nu}$. Under such considerations, since $R_{\mu\nu}$ is given in terms of $H_{\mu\nu}$ by Eq. (4.4), we then assume that $H_{\mu\nu}$ also takes a diagonal form with components (Penna-Lima *et al.*, 2019)

$$H_{\mu\nu} = (h_t(t)g_{00}, h_s(t)g_{ij}). \tag{4.16}$$

$h_t(t)$ and $h_s(t)$ are functions of time, and it is convenient to write the components $H_{\mu\nu}$ in terms of such functions because, in this fashion, all the additional geometrical information is encoded in them.

At first, $H_{\mu\nu}$ not necessarily must take this form; from Eq. (4.4) we see that $H_{\mu\nu}$ depends not only on $R_{\mu\nu}$ and $g_{\mu\nu}$ but also on $T_{\mu\nu}$, in a way that an anisotropic energy momentum-tensor would lead to non-zero off-diagonal components. However, once we assume homogeneity and isotropy, it is reasonable to also assume the validity of Eq. (4.16). Moreover, based on these assumptions, we also assign a perfect fluid energy-momentum tensor to the matter fields, *i.e.*, we take $T_{\mu\nu}$ to be the one in Eq. (1.5).

We can now find the Friedmann equations for the ETGs by following an analogous procedure to the GR's one, but this time using Eq. (4.7). For instance, the $R_{00}$ component takes the form

$$R_{00} = \frac{8\pi G}{g_1} T_{00} + \frac{1}{2} H g_{00} - \frac{1}{2}\frac{8\pi G}{g_1} T g_{00} - H_{00}. \tag{4.17}$$

From Eq. (4.14), we already have an expression for $R_{00}$, while from Eq. (4.16) we also have the component $H_{00} = h_t(t) g_{00}$. Eq. (1.5), in turn, provides $T_{00} = \rho$, and Eq. (1.4) sets $g_{00} = -1$. We can then use these quantities and rewrite Eq. (4.17) as

$$-3\frac{\ddot{a}}{a} = \frac{8\pi G}{g_1}\rho - \frac{1}{2} H + \frac{1}{2}\frac{8\pi G}{g_1} T - H_{00}. \tag{4.18}$$

We now just need to find expressions for the traces $T$ and $H$. The former can be deduced from Eq. (1.5) by contracting it with $g^{\mu\nu}$,

$$\begin{aligned} T \equiv T^{\mu}_{\ \mu} &= (\rho + p) U^{\mu} U_{\nu} + p\delta^{\mu}_{\mu} \\ &= -\rho - p + 4p \\ &= 3p - \rho. \end{aligned} \tag{4.19}$$

while the later comes analogously from a contraction between $g^{\mu\nu}$ and $H_{\mu\nu}$, whose components we already know through Eq. (4.16). We then have:

$$\begin{aligned} H \equiv H^{\mu}_{\ \mu} &= H_{\mu\nu} g^{\mu\nu} \\ &= h_t g_{00} g^{00} + h_s g_{ij} g^{ij} \\ &= h_t + 3h_s. \end{aligned} \tag{4.20}$$

Plugging Eqs. (4.19) e (4.20) into Eq. (4.17) yields

$$-3\frac{\ddot{a}}{a} = \frac{8\pi G}{g_1}\rho - \frac{1}{2}(h_t + 3h_s) + \frac{1}{2}\frac{8\pi G}{g_1}(-\rho + 3p) + h_t, \tag{4.21}$$

leading us to one of the Friedmann equations for the ETGs (Penna-Lima *et al.*, 2019),

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}(h_t - 3h_s) = \frac{4\pi G}{g_1}(\rho + 3p). \tag{4.22}$$

The usual Friedmann equation (Eq. (1.7)) can be recovered from Eq. (4.22) by making $h_t = h_s = 0$ and $g_1 = 1$, which is equivalent to taking no geometric modifications at all.

From Eq. (4.7), the component $R_{ij}$ reads

$$R_{ij} = \frac{8\pi G}{g_1} T_{ij} + \frac{1}{2} H g_{ij} - \frac{1}{2}\frac{8\pi G}{g_1} T g_{ij} - H_{ij}, \tag{4.23}$$

while from Eq. (1.5) we have $T_{ij} = pg_{ij}$. Eqs. (4.14), (4.16), (4.19) and (4.20) already give us expressions for $R_{ij}$ (in terms of the scale factor $a$ and its derivatives), $H_{ij}$, $T$ and $H$, respectively, so by plugging these quantities into Eq. (4.23) we get

$$\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] g_{ij} = \frac{8\pi G}{g_1} p g_{ij} + \frac{1}{2}(h_t + 3h_s) g_{ij} - \frac{1}{2}\frac{8\pi G}{g_1}(-\rho + 3p) g_{ij} - h_s g_{ij}. \tag{4.24}$$

Equation (4.22) provides an expression for $\ddot{a}/a$. By using it, we find the other Friedman equation for the ETGs to be (Penna-Lima *et al.*, 2019)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{h_t}{3} = \frac{8\pi G}{3g_1}\rho, \tag{4.25}$$

from which we can recover the RG case when $h_t = h_s = 0$ and $g_1 = 1$.

Once provided Eqs. (4.22) and (4.25), it is possible to find expressions for $\rho$ and $p$ in terms of $h_t$ and $h_s$. From Eq. (4.25) we can straightforwardly write

$$\rho = \frac{g_1}{8\pi G}\left[3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} - h_t\right], \tag{4.26}$$

while an expression for $p$ arises from Eq. (4.22) by isolating $p$ and applying Eq. (4.26):

$$p = \frac{g_1}{8\pi G}\left[-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + h_s\right]. \tag{4.27}$$

To write the bound equations for each energy condition in terms of $h_t(t)$ and $h_s(t)$, it is necessary to take into account a timelike vector $t^\mu$ and a null vector $k^\mu$, in terms of which they are defined. These behave according to Eqs. (3.16) and (3.40), respectively. Using these components, we can write each term of the sum in Eq. (4.11):

$$\begin{aligned}\frac{8\pi G}{g_1} T_{\mu\nu} k^\mu k^\nu - H_{\mu\nu} k^\mu k^\nu &= \left(\frac{8\pi G}{g_1} T_{00} - H_{00}\right)(k^0)^2 + \left(\frac{8\pi G}{g_1} T_{ii} - H_{ii}\right)(k^i)^2 \\ &= \left[\frac{8\pi G}{g_1}(T_{00} + p) - (H_{00} + h_s)\right] g_{ii}(k^i)^2 \geq 0,\end{aligned} \tag{4.28}$$

in which we used Eqs. (1.5) and (4.16) when writing $T_{ii}$ and $H_{ii}$, respectively. Consequently, we obtain the following equation for NEC (Penna-Lima *et al.*, 2019):

$$\frac{8\pi G}{g_1}(\rho + p) + h_t - h_s \geq 0. \tag{4.29}$$

This is a generalization of Eq. (3.43), *i.e.*, of the GR case, which can be recovered by making $h_t = h_s = 0$.

The SEC bound equation arises from evaluating the summation terms in Eq. (4.8):

$$
\begin{aligned}
\frac{8\pi G}{g_1}\left(T_{\mu\nu}t^\mu t^\nu + \frac{T}{2}\right) - \left(H_{\mu\nu}t^\mu t^\nu + \frac{H}{2}\right) &= \left[\frac{8\pi G}{g_1}T_{00} - H_{00}\right](t^0)^2 + \left[\frac{8\pi G}{g_1}T_{ii} - H_{ii}\right](t^i)^2 + \\
&+ \frac{8\pi G}{g_1}\frac{T}{2} - \frac{H}{2} \\
&= \left[\frac{8\pi G}{g_1}(T_{00} + p) - (H_{00} + h_s)\right] g_{ii}(t^i)^2 + \\
&+ \frac{8\pi G}{g_1}\left(T_{00} + \frac{T}{2}\right) - \left(H_{00} + \frac{H}{2}\right) \geq 0 \qquad (4.30)
\end{aligned}
$$

The terms $(t^i)^2$ are arbitrary non-negative numbers, provided that the normalization condition is fulfilled. SEC thus holds when both the term within square brackets, acting as the coefficient of $(t^i)^2$, and the terms adding to it are independently non-negative. The former is the same as the one in Eq. (4.28) and again implies Eq. (4.29), while the latter results in the following generalized SEC equation:

$$
\frac{8\pi G}{g_1}(\rho + 3p) + h_t - 3h_s \geq 0. \qquad (4.31)
$$

From Eq. (4.31), it is possible to get back Eq. (3.36) by choosing $h_t = h_s = 0$, thereby recovering the GR results.

WEC and DEC, in turn, are direct restrictions on $T_{\mu\nu}$, as can be noted from Eqs. (3.15) and (3.23). For this reason, for any ETG the bound equations for these two conditions are expressed in terms of $\rho$ and $p$ in the same way as in GR (although the expressions for such quantities change in the context of some ETG, as we will see ahead), namely Eqs. (3.21) and (3.22), for WEC, and Eqs. (3.27) and (3.28), for DEC.

We can thus summarize the bound equations found up to this point as

**NEC:**
$$
\frac{8\pi G}{g_1}(\rho + p) + h_t - h_s \geq 0, \qquad (4.32)
$$

**SEC:**
$$
\frac{8\pi G}{g_1}(\rho + 3p) + h_t - 3h_s \geq 0 \quad \text{and} \qquad (4.33)
$$
$$
\frac{8\pi G}{g_1}(\rho + p) + h_t - h_s \geq 0, \qquad (4.34)
$$

**WEC:**
$$
\rho \geq 0 \quad \text{and} \qquad (4.35)
$$
$$
\rho + p \geq 0, \qquad (4.36)
$$

**DEC:**
$$
\rho \geq 0 \quad \text{and} \qquad (4.37)
$$
$$
-\rho \leq p \leq \rho. \qquad (4.38)
$$

Thus, given any ETG with field equations in the form of Eq.(4.2), we can find the bounds once we find expressions for $h_t(t)$ and $h_s(t)$, since they depend only on these functions and on $\rho$ and $p$, which are given by Eqs.(4.26) and (4.27), respectively.

Since we aim to confront these bounds with observational data, it is convenient to write them in terms of quantities we can measure or reconstruct from the data. In practice, we can not measure the scale factor $a(t)$ directly (Vitenti; Penna-Lima, 2015), but useful quantities can be found from its series expansion. Evaluating the series around the present-time, $t_0$, such that $a_0 \equiv a(t_0)$, we have (Visser, 2004)

$$\begin{aligned} a(t) = \ & a_0 + \dot{a}|_{t=t_0}(t-t_0) + \frac{1}{2!}\ddot{a}|_{t=t_0}(t-t_0)^2 + \frac{1}{3!}\dddot{a}|_{t=t_0}(t-t_0)^3 + \frac{1}{4!}a^{(4)}\Big|_{t=t_0}(t-t_0)^4 + ... \\ = \ & a_0 + a\frac{\dot{a}}{a}\Big|_{t=t_0}(t-t_0) + \frac{a}{2!}\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2}H^2\Big|_{t=t_0}(t-t_0)^2 + \frac{a}{3!}\frac{\dddot{a}}{a}\frac{a^3}{\dot{a}^3}H^3\Big|_{t=t_0}(t-t_0)^3 + \\ & + \frac{a}{4!}\frac{a^{(4)}}{a}\frac{a^4}{\dot{a}^4}H^4\Big|_{t=t_0}(t-t_0)^4 + ... \\ = \ & a_0\left[1 + H_0(t-t_0) - \frac{1}{2}q_0H_0^2(t-t_0)^2 + \frac{1}{3!}j_0H_0^3(t-t_0)^3 + \frac{1}{4!}s_0H_0^4(t-t_0)^4 + ...\right], \end{aligned} \tag{4.39}$$

in which we have used the definitions of the Hubble ($H$), deceleration ($q$), jerk ($j$), and snap ($s$) functions[1], given respectively by (Carroll, 2004; Bertolami; Sequeira, 2009)

$$H(t) = \frac{\dot{a}}{a}, \tag{4.40}$$

$$q(t) = -\frac{1}{H^2}\frac{\ddot{a}}{a}, \tag{4.41}$$

$$j(t) = \frac{1}{H^3}\frac{a^{(3)}}{a}, \tag{4.42}$$

$$s(t) = \frac{1}{H^4}\frac{a^{(4)}}{a}. \tag{4.43}$$

Superscript indices in parenthesis indicate the order of the derivative. That is, $a^{(3)}$ stands for $\dddot{a}$, while $a^{(4)} = \ddddot{a}$.

Since the redshift is defined by

$$z = \frac{a_0}{a} - 1, \tag{4.44}$$

which depends on $a(t)$ and is monotonically increasing (since we are describing an universe always in expansion), we can use $z$ as a time variable and then describe the evolution of the universe with it (Dodelson; Schmidt, 2020). For this reason, we are able to write $H(t) = H(z)$, $q(t) = q(z)$, $j(t) = j(z)$ and $s(t) = s(z)$.

[1] This expansion can be performed around any time $t$. The most common is to do it at the present-day $t_0$. In this case, $H_0$ is the so-called Hubble constant, $q_0 = q(t_0)$, $j_0 = j(t_0)$ and $s_0 = s(t_0)$.

The quantities displayed in Eqs. (4.40) to (4.43) are the ones we can reconstruct from observational data, as we will discuss in Sec. 4.5. We can thus rewrite the energy conditions using them: from Eqs. (4.40) and (4.44), we get equations for $a$,

$$a = \frac{a_0}{1+z}, \tag{4.45}$$

and $\dot{a}$,

$$\dot{a} = \frac{a_0 H}{1+z}, \tag{4.46}$$

while from Eqs. (4.41) and (4.46) we get an equation for $\ddot{a}$,

$$\ddot{a} = -\frac{a_0 q H^2}{1+z}. \tag{4.47}$$

Following the same procedure, from Eqs. (4.42) and (4.43) we obtain equations for $a^{(3)}$ and $a^{(4)}$, respectively

$$a^{(3)} = \frac{a j H^3}{1+z} \quad \text{and} \tag{4.48}$$

$$a^{(4)} = \frac{a s H^4}{1+z}. \tag{4.49}$$

Starting with the NEC case, we can apply the expressions for $\rho$ (Eq. (4.26)) and $p$ (Eq. (4.27)) into Eq. (4.29). Hence, we get

$$\frac{\dot{a}^2}{a} + \frac{k}{a} - \ddot{a} \geq 0. \tag{4.50}$$

By applying Eqs. (4.45), (4.46) and (4.47), and considering the present-day curvature density parameter, defined as (Carroll, 2004)

$$\Omega_k^0 = -\frac{k}{(a_0 H_0)^2}, \tag{4.51}$$

we finally get the NEC inequality:

$$[1+q(z)]E(z)^2 - \Omega_k^0 (1+z)^2 \geq 0, \tag{4.52}$$

in which $E(z) = H(z)/H_0$ stands for the normalized Hubble function.

Analogously, for the SEC case we apply the equations for $\rho$ and $p$, *i.e.*, Eqs. (4.26) and (4.27), into Eq. (4.31), which provides

$$-\frac{\ddot{a}}{a} \geq 0. \tag{4.53}$$

Using Eq. (4.41), we can thus express the SEC bound equation in terms of $q(z)$:

$$q(z) \geq 0. \tag{4.54}$$

The WEC, for its part, is expressed by Eqs. (4.35) and (4.36). The application of Eq. (4.26) into the first of them results in

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} - h_t \geq 0. \tag{4.55}$$

Then, by applying Eqs. (4.45), (4.46) and (4.51) to it we obtain the first WEC bound equation,

$$\frac{h_t}{3H_0^2} \leq E(z)^2 - \Omega_k^0(1+z)^2, \tag{4.56}$$

which we call WEC 1.

Following the same procedure, the application of Eqs. (4.26) and (4.27) into Eq. (4.36) provides the expression

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} - h_t - 2\frac{\ddot{a}}{a} + h_s - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} \geq 0, \tag{4.57}$$

which leads to the second bound equation for WEC, here called WEC 2:

$$\frac{h_t - hs}{2H_0^2} \leq -\Omega_k^2(1+z)^2 + E(z)^2(1+q). \tag{4.58}$$

The DEC, in turn, is embodied into Eqs. (4.37) and (4.38). The former is equal to the first bound found for WEC (Eq. (4.35)), while the latter provides two possible inequalities. The first one, $-\rho \leq p$, results in the second bound found for WEC (Eq. (4.36)), in such a way that the new information due to the DEC is contained in the second inequality of Eq. (4.38), *i.e.*, $\rho - p \geq 0$. By applying Eqs. (4.26) and (4.27) to it, we find

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} - h_t + 2\frac{\ddot{a}}{a} - h_s + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \geq 0. \tag{4.59}$$

Then, using Eqs. (4.45), (4.46) and (4.47), we consequently get the following expression for DEC:

$$\frac{h_t + h_s}{6H_0^2} \leq \frac{E(z)^2(2-q) - 2\Omega_k^0(1+z)^2}{3}. \tag{4.60}$$

Summarizing the results found so far, we have the following set of unambiguous conditions for ETGs, obtained respectively from Eq. (4.29), (4.31), (4.35), (4.36) and (4.38) (Penna-Lima *et al.*, 2019):

$$
\begin{aligned}
&\textbf{NEC:} && [1+q(z)]E(z)^2 - \Omega_k^0(1+z)^2 \geq 0, \\
&\textbf{SEC:} && q(z) \geq 0, \\
&\textbf{WEC 1:} && \frac{h_t}{3H_0^2} \leq E(z)^2 - \Omega_k^0(1+z)^2, \\
&\textbf{WEC 2:} && \frac{h_t - hs}{2H_0^2} \leq -\Omega_k^2(1+z)^2 + E(z)^2(1+q), \\
&\textbf{DEC:} && \frac{h_t + h_s}{6H_0^2} \leq \frac{E(z)^2(2-q) - 2\Omega_k^0(1+z)^2}{3}.
\end{aligned}
\tag{4.61}
$$

When there are no extra geometric contributions, $H_{\mu\nu} = 0$ and, consequently, $h_t = h_s = 0$, the modified field equations (Eq. (4.1)) reduce to the Einstein field equation (Eq. (1.1)), so we would expect the bounds in Eq. (4.61) to also reduce to their GR equivalents. By taking $h_t = h_s = 0$, Eq. (4.61) take the form

$$
\begin{aligned}
&\textbf{NEC (GR):} && q(z) - \Omega_k^0 \frac{(1+z)^2}{E(z)^2} \geq -1, \\
&\textbf{SEC (GR):} && q(z) \geq 0, \\
&\textbf{WEC 1 (GR):} && E(z)^2 \geq \Omega_k^0(1+z)^2, \\
&\textbf{DEC (GR):} && q(z) + 2\Omega_k^0 \frac{(1+z)^2}{E(z)^2} \leq 2.
\end{aligned}
\tag{4.62}
$$

These are indeed the bounds for each energy condition in the GR case, as pointed out by Penna-Lima *et al.* (Penna-Lima; Vitenti; Rebouças, 2008). The WEC 2 equation in Eq. (4.61) reduces to the same as the NEC one, so it is not displayed in Eq. (4.62) to avoid redundancy.

We note that the bounds for NEC and SEC, respectively Eqs. (4.52) and (4.54), do not depend on $h_t$ and $h_s$, and remain the same in Eq. (4.62). These are the same equations as the ones provided for NEC and SEC in the scope of GR (Penna-Lima; Vitenti; Rebouças, 2008), so we see that NEC and SEC do not depend on geometric modifications of the theory. The WEC and DEC equations, in turn, depend on the modified functions.

This behavior of the bound equations is due to the nature of the procedure we are following: we aim to reconstruct the geometry of the theory by the confrontation of such equations with observational data; to do this, we found an expression for $R_{\mu\nu}$ in terms of $H_{\mu\nu}$. But the contributions of $h_t$ and $h_s$ from $R_{\mu\nu}$ cancel out with the ones coming from $\rho$ and $p$ in the conditions in the Ricci tensor (NEC and SEC). The conditions directly on $T_{\mu\nu}$ (WEC and DEC), in turn, preserve the contributions from $\rho$ and $p$.

Furthermore, it is worth noticing that, from Eq. (4.53), the bound equation for the SEC in terms of the scale factor $a$ reads $\ddot{a} \leq 0$. The accelerated expansion of the universe, however, implies $\ddot{a} \geq 0$ (Frieman; Turner; Huterer, 2008). In this sense, saying that cosmological observations indicate that SEC is violated stands for saying that they point to a positive value for the expansion's acceleration.

## 4.3 Bounds for $f(R)$ Theories

We can follow the procedure performed in the previous section for a specific type of extended theory - the so-called $f(R)$ theories - since we are interested in analyzing the Hu-Sawicki $f(R)$ model. To do that, it is first necessary to obtain its field equations through the variational principle.

Within the scope of GR, it is possible to obtain the Einstein field equations (displayed in Eq. (1.1)) using the Lagrangian formulation. We can find such equations through the principle of least action by varying the action

$$S = S_{EH} + S_M. \tag{4.63}$$

$S_{EH}$ is the Einstein-Hilbert action, expressed by (Hawking; Ellis, 1973; Wald, 1984; Carroll, 2004)

$$S_{EH} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R, \tag{4.64}$$

in which $g = \det(g_{\mu\nu})$. $S_M$, in turn, is the action corresponding to matter, whose argument is the Lagrangian density for matter, $\mathcal{L}_m$. $S_M$ gives rise to the matter part of Einstein equations, *i.e.*, the right-hand side of Eq. (1.1), while $S_{EH}$ gives rise to the geometrical part of the field equations, *i.e.*, the left-hand side of Eq. (1.1).

Under such considerations, the so-called $f(R)$ theories are the ones that generalize the argument of $S_{HE}$, the scalar curvature $R$, to some non-linear function of $R$, resulting in the generalized action (Sotiriou, 2007; Sotiriou; Faraoni, 2010; Capozziello; Laurentis, 2011; Saridakis *et al.*, 2021)

$$S_{f(R)} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}f(R). \tag{4.65}$$

It is worth noticing that there are different approaches when considering the generalized function $f(R)$. In part of the literature, the $f(R)$ function also encapsulates the scalar $R$, which gives rise to the Einstein part of the extended field equation, while in some works $f(R)$ stands only for the generalized terms. Throughout this work, it will sometimes become useful to treat the sum of $R$ and the nonlinear extension as a single function, in which cases we will make such a definition explicit.

Despite the different ways to define the function $f(R)$, these theories constitute a specific class of the most general extended theories presented in the previous section, whose generalized term in the field equations, given by the tensor $H_{\mu\nu}$, comes from the function $f(R)$. In this context, we can apply the variational method to find the field equations corresponding to $S_{f(R)}$ and afterward identify $H_{\mu\nu}$ for this particular case by comparing these field equations to the general ones, given by Eq. (4.2). Once we do this, it is possible to find the bound equations for any $f(R)$ theory through Eqs. (4.29), (4.31), (4.35), (4.36) and (4.38) - or, equivalently, through the set of inequalities in Eq. (4.61).

We begin by examining models with action of the form (Bertolami; Sequeira, 2009)

$$S = \int \sqrt{-g}\left[\frac{1}{2\kappa}f_1(R) + f_2(R)\mathcal{L}_m\right] d^4x. \tag{4.66}$$

Here, for convenience we wrote the function $f_1(R)$ in such a way it encapsulates both $R$ and the geometric extension $f(R)$. For instance, $f_1(R)$ can take the form $f_1(R) = R + f(R)$, with $f(R)$ the nonlinear extension. $f_2(R)$, for its part, represents a direct coupling to $\mathcal{L}_m$, while $\kappa$ is the coupling constant that, in S.I. units, by comparing Eq. (4.66) with Eqs. (4.64) and (4.65), takes the value

$$\kappa = \frac{8\pi G}{c^4}. \tag{4.67}$$

In this scenario, we thus have a non-minimal curvature-matter coupling, since in general $f_2(R) \neq 1$. To recover the GR case, *i.e.*, Eq. (4.64), it would be necessary to take $f_2(R) = 1$ and $f_1(R) = R$.

By varying $S$ in Eq. (4.66) and using the variational principle, we then obtain

$$\delta S = 0 \Rightarrow \delta \int f_1(R)\sqrt{-g}d^4x + 2\kappa\delta \int f_2(R)\mathcal{L}_m\sqrt{-g}d^4x = 0. \tag{4.68}$$

Performing the variation with respect to $g^{\mu\nu}$, it can be shown that the first term in Eq. (4.68) can be written as[2] (Capozziello; Laurentis, 2011)

$$\delta \int \sqrt{-g}f_1 d^4x = \int \sqrt{-g}\left[f_1'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f_1 - \nabla_\mu\nabla_\nu f_1' + g_{\mu\nu}\Box f_1'\right]\delta g^{\mu\nu}, \tag{4.69}$$

while for the second term, we have

$$\begin{aligned}\delta \int f_2\mathcal{L}_m\sqrt{-g}d^4x &= \int \{\sqrt{-g}\mathcal{L}_m\delta f_2 + f_2\delta(\sqrt{-g}\mathcal{L}_m)\}d^4x \\ &= \int \left\{\sqrt{-g}[f_2'\mathcal{L}_m R_{\mu\nu} - \nabla_\mu\nabla_\nu(f_2'\mathcal{L}_m) + g_{\mu\nu}\Box(f_2'\mathcal{L}_m)]\delta g^{\mu\nu} + f_2\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right\} d^4x,\end{aligned} \tag{4.70}$$

[2] For a step-by-step demonstration of this variation, see Appendix A.

in which we performed the variation $\sqrt{-g}\mathcal{L}_m\delta f_2$ from the first to the second line. In Eq. (4.69), $\Box$ is the d'Alambertian, defined as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, and $'$ indicates the derivative with respect to $R$.

By plugging Eqs. (4.69) and Eq. (4.70) into Eq. (4.68), we obtain the equation

$$\int \sqrt{-g}[f_1'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f_1 - \nabla_\mu\nabla_\nu f_1' + g_{\mu\nu}\Box f_1' + 2\kappa f_2'\mathcal{L}_m R_{\mu\nu} + \\ - 2\kappa\nabla_\mu\nabla_\nu(f_2'\mathcal{L}_m) + 2\kappa g_{\mu\nu}\Box(f_2'\mathcal{L}_m) + 2\kappa\frac{1}{\sqrt{-g}}f_2\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}]\delta g^{\mu\nu}d^4x = 0, \tag{4.71}$$

leading to the expression

$$\left(f_1' + 2\kappa\mathcal{L}_m f_2'\right)R_{\mu\nu} - \frac{1}{2}f_1 g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\left(f_1' + 2\kappa\mathcal{L}_m f_2'\right) = -\kappa f_2\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}. \tag{4.72}$$

By defining the operator

$$\triangle_{\mu\nu} \equiv \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box, \tag{4.73}$$

and the energy-momentum tensor according to Eq. (3.1), which in terms of the Lagrangian of matter $\mathcal{L}_m$ reads

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{4.74}$$

we finally obtain the field equation relative to the action in Eq. (4.66) (Bertolami; Sequeira, 2009):

$$\left(f_1' + 2\kappa\mathcal{L}_m f_2'\right)R_{\mu\nu} - \frac{1}{2}f_1 g_{\mu\nu} - \triangle_{\mu\nu}\left(f_1' + 2\kappa\mathcal{L}_m f_2'\right) = \kappa f_2 T_{\mu\nu}. \tag{4.75}$$

Before moving on, we notice that when the operator $\triangle_{\mu\nu}$ is applied to some generic function $h(R, \mathcal{L}_m)$ of $R$ and $\mathcal{L}_m$, considering the FLRW metric in Eq. (1.4), it takes the form (for a positive metric signature)

$$\triangle_{\mu\nu}h(R,\mathcal{L}_m) = (\partial_\mu\partial_\nu + g_{\mu\nu}\partial_0\partial_0)h - (\Gamma^0_{\mu\nu} - 3Hg_{\mu\nu})\partial_0 h, \tag{4.76}$$

in which $H$ is the Hubble function, defined in Eq. (4.40). This expression will soon be useful.

It is convenient to write Eq. (4.75) in a way that makes the $G_{\mu\nu}$ tensor explicit, as in Eq. (4.2), so we can identify $H_{\mu\nu}$ for this $f(R)$ model. Subtracting and adding the term $\left(f_1' + \frac{2}{k}\mathcal{L}_m f_2'\right)\frac{1}{2}Rg_{\mu\nu}$ to the left-hand side of Eq. (4.75) gives rise to $G_{\mu\nu}$, defined in Eq. (1.1), leading to

$$\left(f_1' + 2\kappa\mathcal{L}_m f_2'\right) G_{\mu\nu} + \left(f_1' + 2\kappa\mathcal{L}_m f_2'\right) \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} f_1 g_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) \left(f_1' + 2\kappa\mathcal{L}_m f_2'\right) = -\kappa f_2 T. \tag{4.77}$$

Rearranging the terms in Eq. (4.77) thus yields

$$\begin{aligned}(f_1' + 2\kappa\mathcal{L}_m f_2')\Big\{G_{\mu\nu} - \frac{1}{f_1' + 2\kappa\mathcal{L}_m f_2'}\Big[\frac{1}{2}[f_1 - (f_1' + 2\kappa\mathcal{L}_m f_2')R]g_{\mu\nu} + \\ + \Delta_{\mu\nu}(f_1' + 2\kappa\mathcal{L}_m f_2')\Big]\Big\} = \kappa f_2 T_{\mu\nu}.\end{aligned} \tag{4.78}$$

Equation (4.78) is the same as Eq. (4.75); it is just written in the form of Eq. (4.2). For theories with no explicit curvature-matter couplings - *i.e.*, couplings between the Ricci scalar $R$ and the matter Lagrangian $\mathcal{L}_m$ (or the energy-momentum tensor $T_{\mu\nu}$) -, the $f_2(R)$ function in Eq. (4.75) is some constant (we can see that from Eq. (4.66), in which $f_2(R)$ is the function that directly couples these quantities). Then, by taking $f_2(R) = 1$, Eq. (4.75) reads[3]

$$f_1' \left\{G_{\mu\nu} - \frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1' R) g_{\mu\nu} + \Delta_{\mu\nu} f_1'\right]\right\} = \kappa T_{\mu\nu}. \tag{4.79}$$

By comparing Eqs.(4.2) and (4.79), we can respectively identify $g_1$ and $H_{\mu\nu}$ as[4]

$$g_1 = f_1' \tag{4.80}$$

and

$$H_{\mu\nu} = -\frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1' R) g_{\mu\nu} + \Delta_{\mu\nu} f_1'\right]. \tag{4.81}$$

Again, we are considering minimal coupling since the class of theories whose bounds we aim to analyze present minimal curvature-matter coupling. We already made this restriction when dealing with general ETGs by choosing $g_2 = 1$ in Eq. (4.2) - in that case, the coupling explicitly appeared between the field elements (as the energy-momentum tensor $T_{\mu\nu}$), while in Eq. (4.66) we have the coupling in the action elements (the matter Lagrangian from which $T_{\mu\nu}$ comes from), in a way that $g_2 = f_2(R)$[5].

From Eqs. (4.76) and (4.81), the component $H_{00}$ reads

[3] We are free to choose $f_2(R)$ to be any constant $f_2 \neq 1$, but in such case $f_2$ would be incorporated into $\kappa$, and we would define a new $\kappa' = \kappa f_2$. For this reason, we choose $f_2(R) = 1$, for convenience.

[4] For a derivation of the $g_1$ and $H_{\mu\nu}$ expressions for a non-minimal coupling $f(R)$ theory, see Appendix B.

[5] We chose to maintain the notation $f_2(R)$ in Eq. (4.66) (and not directly calling it $g_2(R)$) as, typically in the literature, modifications of this type in the action are referred to as $f(R)$ functions - hence the name $f(R)$ theories. Analogously, in Eq. (4.1) we did not set $g_2(\psi_i) = f_2(R)$ as, in that equation, $g_2(\psi_i)$ is not necessarily a function of $R$, since in that case we were dealing with a more general situation.

$$H_{00} = -\frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1'R) + 3\frac{\dot{a}}{a}\partial_0 f_1'\right] g_{00}. \tag{4.82}$$

We used Eq. (4.76) for evaluating $\Delta_{00} f_1'$, as well as the fact that $\Gamma^0_{00} = 0$ according to Eq. (2.16) together with the FLRW metric in equation Eq. (1.4). Then, comparing Eqs.(4.16) and (4.82) leads us to identify $h_t(t)$ as

$$h_t(t) = -\frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1'R) + 3\frac{\dot{a}}{a}\partial_0 f_1'\right] = \left[-\frac{1}{2}\frac{f_1}{f_1'} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} - 3\frac{\dot{a}}{a}\frac{\partial_0 f_1'}{f_1'}\right], \tag{4.83}$$

expression for which we also considered Eq. (4.15) when writing $R$ in terms of the scalar factor $a$ and its derivatives (assuming a flat universe, $k = 0$).

Analogously, for the $H_{ii}$ component Eq. (4.81) provides

$$H_{ij} = -\frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1'R) + \partial_0\partial_0 f_1' - \frac{1}{2}\frac{1}{g_{ij}}(\partial_0 g_{ij})(\partial_0 f_1') + 3\frac{\dot{a}}{a}\partial_0 f_1'\right] g_{ij}. \tag{4.84}$$

Again, by comparing it with Eq. (4.16) allows us to identify $h_s(t)$ as

$$\begin{aligned} h_s(t) &= -\frac{1}{f_1'}\left[\frac{1}{2}(f_1 - f_1'R) + \partial_0\partial_0 f_1' - \frac{1}{2}\frac{1}{g_{ij}}(\partial_0 g_{ij})(\partial_0 f_1') + 3\frac{\dot{a}}{a}\partial_0 f_1'\right] \\ &= -\frac{1}{2}\frac{f_1}{f_1'} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} - \frac{1}{f_1'}\partial_0\partial_0 f_1' - \frac{2}{f_1'}\frac{\dot{a}}{a}\partial_0 f_1'. \end{aligned} \tag{4.85}$$

In Eq. (4.85) we one more time used Eqs.(2.16) and (4.76), as well as the FLRW metric in Eq. (1.4), for which $\Gamma^0_{ij} = \frac{1}{2}\partial_0 g_{ij}$ and $\frac{1}{2}\frac{1}{g_{ij}}\partial_0 g_{ij} = \frac{\dot{a}}{a}$.

Applying Eqs. (4.83) and (4.85) into Eqs. (4.26) and (4.27), we find equations for the energy density $\rho$ and the pressure $p$ in terms of the scale factor $a$, the extended function $f_1(R)$, as well of its derivatives:

$$\rho = \frac{1}{8\pi G}\left[-3\frac{\ddot{a}}{a}f_1' + 3\frac{\dot{a}}{a}\partial_0 f_1' + \frac{f_1}{2} + 3\frac{k}{a^2}f_1'\right], \tag{4.86}$$

$$p = \frac{1}{8\pi G}\left[\frac{\ddot{a}}{a}f_1' + 2\frac{\dot{a}^2}{a^2}f_1' - 2\frac{\dot{a}}{a}\partial_0 f_1' - \partial_0\partial_0 f_1' - \frac{f_1}{2} + \frac{k}{a^2}f_1'\right]. \tag{4.87}$$

In the present work we aim to deal with the case of a flat universe, for which $k = 0$. Furthermore, Eqs. (4.86) and (4.87) are written in natural units, *i.e.*, $c = 1$. By taking the case $k = 0$ and recovering the S.I. units, so the $c$ factors explicitly show up[6], we get

[6] A step-by-step procedure of how to recover the S.I. units from the natural ones is provided in Appendix C.

$$\rho = \frac{c^4}{8\pi G}\left[-\frac{3}{c^2}\frac{\ddot{a}}{a}f_1' + \frac{3}{c^2}\frac{\dot{a}}{a}\partial_0 f_1' + \frac{f_1}{2}\right], \tag{4.88}$$

$$p = \frac{c^4}{8\pi G}\left[\frac{1}{c^2}\frac{\ddot{a}}{a}f_1' + \frac{2}{c^2}\frac{\dot{a}^2}{a^2}f_1' - \frac{2}{c^2}\frac{\dot{a}}{a}\partial_0 f_1' - \frac{1}{c^2}\partial_0\partial_0 f_1' - \frac{f_1}{2}\right]. \tag{4.89}$$

Analogously, in S.I. units the $h_t(t)$ and $h_s(t)$ functions in Eqs. (4.83) and (4.85) read

$$h_t(t) = -\frac{c^4}{2}\frac{f_1}{f_1'} + 3c^2\frac{\ddot{a}}{a} + 3c^2\frac{\dot{a}^2}{a^2} - 3c^2\frac{\dot{a}}{a}\frac{\partial_0 f_1'}{f_1'}, \tag{4.90}$$

$$h_s(t) = -\frac{c^4}{2}\frac{f_1}{f_1'} + 3c^2\frac{\ddot{a}}{a} + 3c^2\frac{\dot{a}^2}{a^2} - \frac{c^2}{f_1'}\partial_0\partial_0 f_1' - \frac{2c^2}{f_1'}\frac{\dot{a}}{a}\partial_0 f_1'. \tag{4.91}$$

We can now use Eqs. (4.88), (4.89), (4.90) and (4.91) to find the bound equations for each energy condition: we use Eq. (4.35) for WEC 1, Eq. (4.36) for WEC 2 and the second inequality in Eq. (4.38) for DEC[7]. By doing so, and considering that $k = 0$, such that $\Omega_k^0 = 0$, we obtain

**NEC:** $$\frac{\ddot{a}a}{\dot{a}^2} \leq 1, \tag{4.92}$$

**SEC:** $$\frac{\ddot{a}}{a} \leq 0, \tag{4.93}$$

**WEC 1:** $$-3\frac{\ddot{a}}{a}f_1' + 3\frac{\dot{a}}{a}\partial_0 f_1' + \frac{c^2}{2}f_1 \geq 0, \tag{4.94}$$

**WEC 2:** $$-2\frac{\ddot{a}}{a}f_1' + \frac{\dot{a}}{a}\partial_0 f_1' + 2\frac{\dot{a}^2}{a^2}f_1' - \partial_0\partial_0 f_1' \geq 0, \tag{4.95}$$

**DEC:** $$-4\frac{\ddot{a}}{a}f_1' - 2\frac{\dot{a}^2}{a^2}f_1' + 5\frac{\dot{a}}{a}\partial_0 f_1' + \partial_0\partial_0 f_1' + c^2 f_1 \geq 0. \tag{4.96}$$

These are the bound equations of the energy conditions for $f(R)$ theories with minimal coupling in a homogeneous and isotropic universe. It is worth noticing again that the NEC and SEC equations in Eqs. (4.92) and (4.93) are the same as the corresponding inequalities in Eq. (4.62), in the case $\Omega_k^0 = 0$. That is, NEC and SEC bounds remain the same as in GR.

With these bound equations, given any $f_1(R)$ function we can find the inequalities corresponding to each energy condition in terms of the scale factor $a$ and its derivatives. For instance, choosing $f(R) = R$ would lead to the GR bounds. In the next sections, we will use such expressions with a $f_1(R)$ function of interest, and use the inequalities to constrain the theory's parameters.

[7] An (equivalent) alternative to using these equations would be applying Eqs. (4.90) and (4.91) to the set of bounds in Eq.(4.61).

## 4.4 A Specific Case: the Hu-Sawicki $f(R)$ Theory

We now turn our attention to a specific class of $f(R)$ theory: the so-called Hu-Sawicki model (Hu; Sawicki, 2007), which aims to explain the recent accelerated expansion of the universe without requiring extra fluids like dark energy. This model has been widely explored in the literature - as in Refs.(Oyaizu, 2008; Hu *et al.*, 2016; Kou; Murray; Bartlett, 2024; Vogt *et al.*, 2024) - and fulfills the conditions for a well-behaved $f(R)$ theory, hence our interest in constraining its parameters. Throughout this section, we compute the bounds in Eqs. (4.92) to (4.96) for the Hu-Sawicki function, both in terms of the scale factor $a$ and its derivatives, and in terms of observable quantities as the Hubble function $H(z)$ and the deceleration parameter $q(z)$. After that, we consider an approximation of the Hu-Sawicki function and accordingly compute the bounds, again in terms of $a$ and the observable quantities.

### 4.4.1 Computing the Bound Equations

The Hu-Sawicki is a particular $f(R)$ theory whose extension on $R$ takes the f4orm (Hu; Sawicki, 2007)

$$f_{HS}(R) = -m^2 \frac{\alpha (R/m^2)^n}{\beta (R/m^2)^n + 1}. \tag{4.97}$$

$\alpha$ and $\beta$ are dimensionless parameters, and $n > 0$. The coefficient $m^2$ sets the scale of the theory and can be conveniently chosen as (Hu; Sawicki, 2007; Hu *et al.*, 2016)

$$m^2 = \frac{\kappa \bar{\rho}_0}{3}, \tag{4.98}$$

with $\kappa$ defined as in Eq. (4.67), which in natural units reads $\kappa = 8\pi G$.

$\bar{\rho}_0$ stands for the average density of matter at the present time and reads (Carroll, 2004)

$$\bar{\rho}_0 = \Omega_{m0} \frac{3H_0^2}{\kappa}, \tag{4.99}$$

in which $\Omega_{m0}$ and $H_0$ are respectively the mass density parameter and the Hubble function at the present-day. By plugging it into Eq. (4.98) we then find

$$m^2 \equiv H_0^2 \Omega_{m0}. \tag{4.100}$$

Such equation is expressed in natural units. By converting it to S.I. units[8] results is

$$m^2 = (8315\text{Mpc})^{-2} \left( \frac{\Omega_{m0} h^2}{0.13} \right). \tag{4.101}$$

[8] A step-by-step procedure on how to perform such conversion in found in Appendix C.

$h$ is defined in terms of $H_0$ by the relation $H_0 = 100\,h$ km/s/Mpc, and expressing the factor 0.13 explicitly arises from considering $\Omega_{m0} \approx 0.25$ and $H_0 \approx 72 km/s/Mpc$ - values constrained by the Wilkinson Microwave Anisotropy Probe (WMAP) observations (Hinshaw *et al.*, 2013). These lead to $\Omega_{m0}h^2 \approx 0.13$.

As stated above, Hu-Sawicki $f(R)$ gravity aims to explain the accelerated expansion of the universe at late times without requiring dark energy, simultaneously satisfying solar-system tests and including $\Lambda$CDM phenomenology as a limiting case. In general, there are some conditions an $f(R)$ theory should satisfy to be compatible with $\Lambda$CDM features and to have observationally acceptable properties. In particular, they should respect (Hu; Sawicki, 2007)

$$\begin{aligned} &\lim_{R\to\infty} f(R) = \text{const.}, \\ &\lim_{R\to 0} f(R) = 0. \end{aligned} \tag{4.102}$$

The function $f_{HS}(R)$ satisfies such conditions, as from Eq. (4.97) we indeed see that $\lim_{R\to 0} f_{HS} = 0$ and that

$$\lim_{R\to\infty} f_{HS}(R) = -\frac{m^2\alpha}{\beta}. \tag{4.103}$$

The choice of $m^2$ plays an important role in the theory. In the first place, it sets the extension $f_{HS}$ to the same unit as $R$: since $m^2$ has units of $1/(\text{distance})^2$, it makes $f_{HS}$ compatible to be added to $R$ in the action[9], as in Eq. (4.66). Moreover, as a multiplicative factor, $m^2$ also sets the scale of the theory, and choosing it as in Eq. (4.101) allows us to make a useful approximation, as we will soon see.

Then, by applying Eq. (4.97) into Eqs. (4.92) to (4.96) and using Eq. (4.15) (with $k = 0$), we can find the bounds for each energy condition in terms of $a$ and its derivatives. Again, the NEC and SEC bounds - Eqs. (4.92) and (4.93), respectively - remain the same as they do not depend on the geometric modifications coming from $f_{HS}$. Considering thus a function $f_1 = R + f_{HS}$, we find its first derivative to be

$$\frac{df_1}{dR} = f_1' = 1 + n\alpha\beta \frac{\left(\frac{R}{m^2}\right)^{2n-1}}{\left[1 + \beta\left(\frac{R}{m^2}\right)^n\right]^2} - n\alpha \frac{\left(\frac{R}{m^2}\right)^{n-1}}{1 + \beta\left(\frac{R}{m^2}\right)^n}. \tag{4.104}$$

By applying it and Eq. (4.97) to Eqs. (4.94), (4.95) and (4.96), we respectively obtain the bound equations for WEC 1 (Eq. (4.35)), WEC 2 (Eq. (4.36)) and DEC (the second inequality in Eq. (4.38)). As stated before, these equations are given in terms of the scale factor $a$ and

9 More details about the units of $R$ and other quantities in the field equations can be found in Appendix C.

its derivatives, and are provided in appendix D. We can then use Eqs. (4.46), (4.47), (4.48) and (4.49) to express them in terms of the Hubble $H(z)$, deceleration $q(z)$, jerk $j(z)$, and snap $s(z)$ functions. Thus, defining

$$\eta(z) = \frac{H^2(z)(1-q(z))}{c^2m^2} = \frac{c^2E^2(z)(1-q(z))}{\Omega_m}, \tag{4.105}$$

and using Eqs. (4.46) to (4.49) leads for the bound equation for WEC 1:

$$\frac{1}{(q-1)^2\left(\beta(6\eta)^n+1\right)^3}\alpha c^2m^2(6\eta)^n\Big\{\beta(6\eta)^n\Big(n(n+1)j-2\left(n^2+n+1\right)+q\left((n-2)q-n(n+2)+4\right)\Big)$$
$$-\beta^2 36^n(q-1)^2\eta^{2n}-n^2j+nj+n^2q+2n^2+nq^2-2nq-2n-q^2+2q-1\Big\}+6H^2\geq 0. \tag{4.106}$$

Doing the same for WEC 2 results in the bound equation

$$\frac{\alpha 6^n nH^6\eta^{n-1}(-j+q+2)^2\Big(\beta\left(-2^{n+2}\right)3^n\left(n^2-1\right)\eta^n+n^2+\beta^2 36^n(n+1)(n+2)\eta^{2n}-3n+2\Big)}{c^2m^2\left(\beta 6^n\eta^n+1\right)^2}$$
$$+2c^2\eta m^2H^2(q+1)\Big(6\eta+6^n\eta^n(12\beta\eta-\alpha n)+\beta^2 6^{2n+1}\eta^{2n+1}\Big)$$
$$+\frac{\alpha 6^n nH^4\eta^n\left(\beta 6^n(n+1)\eta^n-n+1\right)\left(j-q(q+9)-s-8\right)}{\beta 6^n\eta^n+1}\geq 0 \tag{4.107}$$

while for DEC we find the bound equation

$$\frac{\alpha 6^n nH^4\eta^n\left(\beta 6^n(n+1)\eta^n-n+1\right)\left(5j+q(q+3)+s-4\right)}{\beta 6^n\eta^n+1}\geq \alpha c^4m^4 6^{n+1}\eta^{n+2}\left(\beta 6^n\eta^n+1\right)$$
$$+\frac{\alpha 6^n nH^6\eta^{n-1}(-j+q+2)^2\Big(\beta\left(-2^{n+2}\right)3^n\left(n^2-1\right)\eta^n+n^2+\beta^2 36^n(n+1)(n+2)\eta^{2n}-3n+2\Big)}{c^2m^2\left(\beta 6^n\eta^n+1\right)^2}$$
$$+2c^2\eta m^2H^2\Big(6^n\eta^n\left(-24\beta\eta+\alpha(-n)+2q(6\beta\eta+\alpha n)\right)+\beta^2 6^{2n+1}\eta^{2n+1}(q-2)+6\eta(q-2)\Big) \tag{4.108}$$

Eqs. (4.106), (4.107) and (4.108) are given in terms of cosmographic quantities, and we can estimate such values from observational data. However, there are still some considerations we can make to simplify $f_{HS}(R)$ in Eq. (4.97) and consequently obtain simpler equations.

### 4.4.2 Taking an Approximation: the Case of High Curvatures

A feature concerning Eq. (4.97) we can still take into account is the order of magnitude of the factor $(m^2/R)$. As stated above, the choice of $m^2$ sets the scale of the theory, and defining it properly allows us to take a high curvature regime of $R$ concerning $m^2$, *i.e.*, $R \gg m^2$, which is useful since it allows the theory to remain stable at high curvatures (Hu; Sawicki, 2007).

Furthermore, in the case of $R \gg m^2$, we can take the limit $m^2/R \to 0$ and expand Eq. (4.97) as a power series up to the first order. Taking this limit is possible for present-day values (corresponding to a redshift $z = 0$) if $m^2$ is defined as in Eq. (4.101) (Hu *et al.*, 2016). The values of these quantities at the present-day are indeed such that $R_0 \gg m^2$ (Hu; Sawicki, 2007; Oyaizu, 2008).

Under such considerations, it seems like a reasonable procedure to perform the power series expansion of $f_{HS}$. First, we can make the factor $m^2/R$ explicit in $f_{HS}$ by rewriting Eq. (4.97) as

$$f_{HS}(R) = -m^2 \frac{\alpha}{\beta + \left(\frac{m^2}{R}\right)^n}. \tag{4.109}$$

By defining the new variable

$$x = \frac{m^2}{R}, \tag{4.110}$$

we can regard $f_{HS}$ as a function of $x$,

$$f_{HS}(x) = -m^2 \frac{\alpha}{\beta + x^n} = -\frac{m^2\alpha}{\beta}\left(\frac{1}{1 + x^n/\beta}\right) \tag{4.111}$$

and then perform the expansion in terms of it.

For some $|y| < 1$, the function $\frac{1}{1-y}$ can be expressed as a power series given by (Stewart, 2012)

$$\frac{1}{1-y} = \sum_{m=0}^{\infty} y^m. \tag{4.112}$$

By setting $y = -\frac{x^n}{\beta}$, we can then write the power series for $\frac{1}{1+x/\beta}$ in Eq. (4.111), whose first terms read

$$\frac{1}{1 + \frac{x^n}{\beta}} = 1 - \frac{x^n}{\beta} + \ldots \tag{4.113}$$

In the limit $x = \frac{m^2}{R} \to 0$, as we are dealing with a small $x$, we can perform the expansion up to first order and consider only the terms displayed explicitly in Eq. (4.113). By doing so, Eq. (4.111) reads

$$\lim_{x\to 0} f_{HS}(x) \approx -\frac{m^2\alpha}{\beta}\left(1-\frac{x^n}{\beta}\right) = -\frac{\alpha m^2}{\beta} + \frac{\alpha m^2}{\beta^2}x^n. \tag{4.114}$$

Writing it back in terms of $R$ by means of Eq. (4.110) thus yields

$$\lim_{\frac{m^2}{R}\to 0} f_{HS}(R) \approx -\frac{\alpha m^2}{\beta} + \frac{\alpha m^2}{\beta^2}\frac{m^{2n}}{R^n}. \tag{4.115}$$

We can go further and look for a convenient way of expressing the first order term in the expansion. To do this, we take the derivative $f'_{HS} = \frac{df_{HS}}{dR}$. If we consider the definition of $x$ in terms of $R$, applying the chain rule results in

$$\frac{df_{HS}}{dR} = \frac{df_{HS}}{dx}\frac{dx}{dR}. \tag{4.116}$$

From Eq. (4.110), the derivative of $x$ reads $\frac{dx}{dR} = -\frac{m^2}{R^2}$, and from Eq. (4.111) we find the derivative of $f_{HS}$ to be

$$\frac{df_{HS}}{dx} = \alpha n m^2 \frac{x^{n-1}}{(\beta+x^n)^2}. \tag{4.117}$$

Using this result, as well as Eq. (4.110), provides

$$\frac{df_{HS}}{dR} = -\frac{\alpha n x^{n+1}}{\beta^2 + 2\alpha x^n + x^{2n}} = -\frac{\alpha n}{\beta^2 x^{-(n+1)} + 2\beta x^{-1} + x^{n-1}}. \tag{4.118}$$

Once we are dealing with small values of $x$ and considering that $n > 0$, we can take only the dominant term $\beta^2 x^{-(n+1)}$ and write $\frac{df_{HS}}{dR}$ as

$$\frac{df_{HS}}{dR} = -\frac{\alpha n}{\beta^2}x^{n+1}. \tag{4.119}$$

Then, by expressing it back in terms of $R$, from Eq. (4.110) we obtain (Hu; Sawicki, 2007)

$$\frac{df_{HS}}{dR} = -\frac{m^2\alpha}{\beta^2}\frac{nm^{2n}}{R^{n+1}}. \tag{4.120}$$

With this equation, we can write the factor $\frac{\alpha m^2}{\beta^2}$ in Eq. (4.115) in terms of $df_{HS}/dR$. When doing this, we must remember that the expansion is taken in the limit $\frac{m^2}{R} \to 0$, so we evaluate the derivative at the present background curvature $R_0$. If we define $f_{R0} \equiv \frac{df_{HS}}{dR}(R_0)$, we thus have

$$f_{R0} = -\frac{m^2\alpha}{\beta^2}\frac{nm^{2n}}{R_0^{n+1}} \Rightarrow \frac{\alpha m^2}{\beta^2} = -\frac{f_{R0}}{n}\frac{R_0^{n+1}}{m^{2n}}. \tag{4.121}$$

Again, we emphasize that taking the limit $\frac{m^2}{R} \to 0$ regards for evaluating the variable $x$, and hence $f_{HS}(R)$ itself, at $R_0$, since as stated before present epoch values for these quantities yields $R_0 \gg m^2$.

By plugging Eq. (4.121) into Eq. (4.115), we find the expansion for $f_{HS}(R)$ up to first order in terms of $R_0$ and $f_{R0}$ (Vogt *et al.*, 2024):

$$\lim_{\frac{m^2}{R}\to 0} f_{HS}(R) \approx -m^2\frac{\alpha}{\beta} - \frac{f_{R0}R_0^{n+1}}{nR^n} \tag{4.122}$$

In this way, we can describe the theory taking $f_{R0}$ as a parameter instead of the amplitude $\alpha/\beta^2$ (Vogt *et al.*, 2024). This is a useful expression once taking this approximation leads to an evolution similar to the one of a $\Lambda$CDM model (Hu; Sawicki, 2007).

We can now find the bound equations in terms of $f_{R0}$ by using the expanded function in Eq. (4.122). Considering thus a function $f_1 = R + f_{HS}$, with $f_{HS}$ as displayed in Eq. (4.122), we find its first derivative to be

$$\frac{df_1}{dR} = f_1' = 1 + f_{R0}\left(\frac{R_0}{R}\right)^{n+1}. \tag{4.123}$$

Applying it to Eqs. (4.94), (4.95) and (4.96), we respectively obtain the bound equations for WEC 1 (Eq. (4.35)), WEC 2 (Eq. (4.36)) and DEC (the second inequality in Eq. (4.38)). Such equations are given in terms of $a$ and its derivatives and are displayed in Appendix D.

We can once more use Eqs. (4.46), (4.47), (4.48) and (4.49) and express it in terms of the dynamical quantities $H(z)$, $q(z)$, $j(z)$ and $s(z)$. For WEC 1 we obtain

$$\begin{aligned}&-\frac{f_{R0}6^{-n}R_0^{n+1}\left(n(n+1)j + q((n+1)q - n(n+2) - 2) - 2n(n+1) + 1\right)\left(-\frac{H^2(q-1)}{c^2}\right)^{-n}}{n(q-1)^2}\\&-\frac{\alpha m^2}{\beta} + \frac{6H^2}{c^2} \geq 0\end{aligned} \tag{4.124}$$

For WEC 2, the bound equation takes the form

$$\begin{aligned}&\frac{c^2 f_{R0}6^{-n-1}R_0^{n+1}}{(q-1)^3}\left(-\frac{H^2(q-1)}{c^2}\right)^{-n}\Big(-(n+1)j\big((2n+5)q + 4n + 7\big) + (n+1)(n+2)j^2\\&+ q\big((n+1)s + n(4n+11) + 9\big) + (n-1)q^3 + \big(n(n+11) + 12\big)q^2 - (n+1)s + 4n(n+1) - 2\Big)\\&+ 2H^2(q+1) \geq 0\end{aligned} \tag{4.125}$$

while for DEC we find

$$\frac{6^{-n}}{\beta n(q-1)}\left(-\frac{H^2(q-1)}{c^2}\right)^{-n}\Bigg\{-\beta c^2 f_{R0}R_0^{n+1}\Big(-n(n+1)j\big((2n-1)q+4n+13\big)+n(n+1)(n+2)j^2$$
$$+n(n+1)(q-1)s+q\Big(q\big((n+2)(n+3)q+(n-1)n(n+6)-18\big)+n\big(n(4n+5)+9\big)+18\Big)\Big)$$
$$-2\beta c^2 f_{R0}(n+3)\big(2n(n+1)-1\big)R_0^{n+1}-6^{n+1}n(q-1)^3\left(-\frac{H^2(q-1)}{c^2}\right)^n\left(\alpha c^2m^2+2\beta H^2(q-2)\right)\Bigg\}\geq 0 \tag{4.126}$$

It is worth noticing that in Eqs. (4.124), (4.125) and (4.126) the quantities $H(z)$, $q(z)$, $j(z)$ and $s(z)$ are functions of $z$, and consequently can be evaluated for different values of $z$. Although the approximation in Eq. (4.122) was evaluated considering the regime $R_0 \gg m^2$, *i.e.*, for present day values, this limit also implies $R \gg m^2$ for the expansion history, since $R$ is a monotonically decreasing function (Oyaizu, 2008). We can express $R$ in terms of $H$ and its first derivative (Hu; Sawicki, 2007), and since $H$ decreases when we consider positive energy densities (Carroll, 2004), $R$ also acquires a decreasing behavior. Consequently, the approximation holds for past values of $R$.

Furthermore, there is one more constraint we can impose on the theory. As pointed out by Hu and Sawicki, the accelerated expansion provided by this theory approximates the ΛCDM one if (Hu; Sawicki, 2007)

$$\frac{\alpha}{\beta}\approx 6\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}=6\omega, \tag{4.127}$$

with

$$\omega \equiv \frac{\Omega_{\Lambda 0}}{\Omega_{m0}}. \tag{4.128}$$

$\Omega_{\Lambda 0}$ and $\Omega_{m0}$ are, respectively, the present-day dark energy density parameter and the matter density parameter (Kou; Murray; Bartlett, 2024).

Eq. (4.122) then takes the form

$$f_{HS}\approx -6\omega m^2-\frac{f_{R0}R_0^{n+1}}{nR^n}, \tag{4.129}$$

and its derivative remains the same. Now the freedom of the theory relies on the only two free parameters $f_{R0}$ and $n$, justifying the use of $f_{R0}$.

Following the same process and using the bounds in Eqs. (4.94), (4.95) and (4.96) for the expanded function in Eq. (4.129), and then writing it in terms of $H(z)$, $q(z)$, $j(z)$ and $s(z)$ with Eqs. (4.40), (4.41), (4.42) and (4.43) yields the bound equations for WEC 1,

$$- \frac{f_{R0} 6^{-n} R_0^{n+1} (n(n+1)j + q((n+1)q - n(n+2) - 2) - 2n(n+1) + 1) \left(-\frac{H^2(q-1)}{c^2}\right)^{-n}}{n(q-1)^2}$$
$$- 6\omega m^2 + \frac{6H^2}{c^2} \geq 0 \tag{4.130}$$

and for DEC,

$$\frac{6^{-n}}{n(q-1)} \left(-\frac{H^2(q-1)}{c^2}\right)^{-n} \Bigg\{ c^2 f_{R0} n(n+1) j R_0^{n+1} ((2n-1)q + 4n + 13) - c^2 f_{R0} n(n+1)(n+2) j^2 R_0^{n+1}$$
$$- c^2 f_{R0} R_0^{n+1} (n(n+1)(q-1)s + q(q((n+2)(n+3)q + (n-1)n(n+6) - 18) + n(n(4n+5) + 9) + 18)$$
$$+ 2(n+3)(2n(n+1) - 1)) - 2^{n+2} 3^{n+1} n(q-1)^3 \left(-\frac{H^2(q-1)}{c^2}\right)^n \left(3c^2 m^2 \omega + H^2(q-2)\right) \Bigg\} \geq 0 \tag{4.131}$$

The equation for WEC 2 remains the same as Eq. (4.125), since it does not depend on $f_{HS}$, just on its derivative. Besides that, we emphasize that we can recover the GR case by making $\alpha = 0$, which implies $f_{R0} = 0$. In this case, Eqs.(4.130), (4.125) and (4.131) reduce to the ones in Eq. (4.62), for the case $k = 0$.

For the particular case $n = 1$ often considered in the literature, the WEC 1 bound in Eq. (4.130) leads to

$$f_{R0} \geq \frac{36}{c^4 R_0^2} \left[ \frac{H^2(q-1)^3(c^2 m^2 \omega^2 - H^2)}{2j + (q-3)(2q+1)} \right]. \tag{4.132}$$

The WEC 2 bound in Eq. (4.125), in turn, reads

$$f_{R0} \geq -\frac{36H^4}{c^4 R_0^2} \left[ \frac{(q+1)(q-1)^4}{-3 - 3j^2 + j(7q+11) + s - q(12 + 12q + s)} \right], \tag{4.133}$$

while the DEC bound in Eq. (4.131) takes the form

$$f_{R0} \geq \frac{36}{c^4 R_0^2} \left[ \frac{(q-1)^4 [3c^2 m^2 \omega + H^2(q-2)]}{3j^2 - j(q+1) + 3(2q+1)(4 + q(q-2)) + s(q-1)} \right]. \tag{4.134}$$

## 4.5 Dealing with Observable Quantities

Our concern in writing the bound equations in terms of $H(t)$, $q(t)$, $j(t)$, and $s(t)$ relies on the fact that these are functions that can be obtained from observable quantities. In practice, what cosmological measurements provide are quantities such as distances, and not directly the scale factor $a(t)$ (Vitenti; Penna-Lima, 2015). Hence, it is important to write the bounds regarding quantities we can measure (or reconstruct from data) to constrain the theory's parameters.

Two of these quantities are the normalized Hubble function (Vitenti; Penna-Lima, 2015; Penna-Lima *et al.*, 2019)

$$E(z) = \frac{H(z)}{H_0} = \exp \int_0^z \frac{1+q(z')}{1+z'} dz', \tag{4.135}$$

and the comoving distance

$$D(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}. \tag{4.136}$$

Both depend on the deceleration function, given by (Vitenti; Penna-Lima, 2015)

$$q(z) = \frac{1+z}{H(z)} \frac{dH(z)}{dz} - 1. \tag{4.137}$$

Vitenti and Penna-Lima introduced a method to estimate $q(z)$ which is independent both on the theory of gravitation and on how you specify the matter content in the universe, once it just assumes homogeneity and isotropy. This method aims to directly reconstruct $q(z)$ by using a segmented function that allows controlling its complexity from a simple linear case to a more complex interpolation through the use of splines with many knots (Vitenti; Penna-Lima, 2015). So, writing the bounds in terms of $q(z)$ and $H(z)$ (which is related to $E(z)$ by Eq. (4.135)) enables the possibility of comparing the results to observational data.

The inequalities we found from the application of the energy conditions - Eqs. (4.106) to (4.108), Eqs.(4.124) to (4.126), and Eqs. (4.130) and (4.131) - are written in terms of $H(z)$, $q(z)$, $j(z)$ and $s(z)$. This is a useful way of expressing the bounds as we can write $j(z)$ and $s(z)$ in terms of $q(z)$ and its derivatives.

From Eqs. (4.40) and (4.45) we find that the relation between a derivative in the redshift $z$ and the time coordinate $t$ is

$$dz = -(1+z)H dt. \tag{4.138}$$

Then, computing $\frac{dq}{dz}$ and using Eqs. (4.41), (4.42) and (4.138) we find the following expression for $j(z)$:

$$j(z) = (1+z)\frac{dq}{dz} + 2q^2 + q. \tag{4.139}$$

Analogously, performing the derivative $\frac{dj}{dz}$ leads to

$$s(z) = -(1+z)\frac{dj}{dz} - 3jq - 2j. \tag{4.140}$$

Using Eq. (4.139) for applying the derivative, in turn, provides

$$s(z) = -(1+z)^2\frac{d^2q}{dz^2} - (4+7q)(1+z)\frac{dq}{dz} - 6q^3 - 7q^2 - 2q. \tag{4.141}$$

That is, we see that we can indeed express $j(z)$ and $s(z)$ in terms of $q(z)$, a quantity we can estimate. Consequently, the bounds are subject to the possibility of observational confrontation.

# 5 Conclusion and Future Perspectives

In this work, we constrained the parameters of ETGs by imposing the energy conditions. We obtained the bound equations, in the form of inequalities, for a general $f(R)$ theory and also for a particular case - the Hu-Sawicki model - by following the general procedure in Ref.(Penna-Lima *et al.*, 2019). We then expressed these inequalities in terms of the cosmographic quantities $H(z)$ (Hubble function), $q(z)$ (deceleration function), $j(z)$ (jerk), and $s(z)$ (snap). We also computed expressions for $j(z)$ and $s(z)$ in terms of $q(z)$ and its derivatives.

For computing the bounds for a generic $f(R)$ theory we first computed its field equations. We achieved this by applying the procedure performed in Ref.(Capozziello; Lobo; Mimoso, 2015) to the general action proposed in Ref.(Bertolami; Sequeira, 2009), thus obtaining Eq. (4.75) (Bertolami; Sequeira, 2009). We then rearranged the field equations in the fashion of the ones proposed by Penna-Lima *et al.* in Ref.(Penna-Lima *et al.*, 2019), as displayed in Eq.(4.78), and assumed a minimal curvature-matter coupling, leading to Eq.(4.79).

Imposing the energy conditions to it resulted in a set of bounds, in the form of inequalities, in Eqs. (4.92) to (4.96). The bounds concerning NEC and SEC, respectively Eqs. (4.92) and (4.93), were the same as in GR, as can be seen in Eq. (4.62). This was already expected, as these bounds do not depend on the geometric modifications, as pointed out by Eqs. (4.52) and (4.54) (Penna-Lima *et al.*, 2019).

This is a consequence of the fact that we are attributing the extension of the theory to its geometric part, not including geometric modifications to an effective energy-momentum tensor (see the discussion regarding Eq. (4.12) onwards at the end of Sec. 4.1). Consequently, at first sight the bounds applied directly on $R_{\mu\nu}$, standing for the geometry of the theory, present terms related to the extension.

As NEC and SEC are at first restrictions on $R_{\mu\nu}$, applying the energy conditions to it thus led to equations that depend on the modified functions $h_t$ and $h_s$, as well on $\rho$ and $p$, as showed in Eqs. (4.29) and (4.31). Despite that, the contribution of $\rho$ and $p$ cancels out the extended functions in the bounds, as they also depend on $h_t$ and $h_s$ (see Eqs. (4.26) and (4.27)), leading to the same result as in GR. On the other hand, WEC and DEC are restrictions on $T_{\mu\nu}$, therefore depending only on $\rho$ and $p$, and yielding explicit forms that depend on the extended functions $h_t$ and $h_s$ (Eqs. (4.56), (4.58) and (4.60)). Such discussion is also addressed in Ref.(Penna-Lima *et al.*, 2019).

When we applied the bounds to the Hu-Sawicki model, we found that the ones concerning WEC 1, WEC 2 and DEC, respectively Eqs. (4.94), (4.95) and (4.96), presented higher order derivatives of the scale factor $a$ (as presented in details in appendix D). This is a consequence of the application of second-order derivatives to $f'(R)$ in the field equation for a general $f(R)$ theory, displayed in Eq. (4.75) and computed in detail in Appendix A. $f'(R)$ is a function of $R$, and since $R$ depends on first and second derivatives of $a$ (see Eq. (4.15)), taking second derivatives of it led to third and fourth derivatives of $a$.

As a consequence, by expressing these bounds in terms of the observable dynamical quantities, at first in Eqs. (4.94), (4.95) and (4.96), we found them to depend on the jerk $j(z)$ and snap $s(z)$ functions. These are precisely defined in terms of the third and second derivatives of $a$ in Eqs. (4.42) and (4.43), respectively. This is a different scenario in comparison to GR, whose bounds can be expressed only in terms of $E(z)$ (or, equivalently, $H(z)$) and $q(z)$.

After assuming the high-curvature regime of $R$ in relation to $m^2$, the bounds for the Hu-Sawicki model took the form of Eqs. (4.124), (4.125) and (4.126). These take $f_{R0}$ as a parameter of the theory. This showed up as a useful approach after considering the condition in Eq. (4.127), required to satisfy the behavior of a ΛCDM expansion, as pointed out by the authors in Ref.(Hu; Sawicki, 2007). Such conditions provided the bounds in Eqs. (4.130) and (4.131). By taking it, the only two remaining free parameters are $f_{R0}$ and $n$.

Our concern in writing these inequalities in terms of $H(z)$, $q(z)$, $j(z)$, and $s(z)$ relies on the interest of future comparisons to observational data. As pointed out by Vitenti and Penna-Lima (Vitenti; Penna-Lima, 2015), it is possible to reconstruct quantities such as $q(z)$ and $E(z)$ (related to $H(z)$ by Eq. (4.135)). Our equations are given in terms of such functions, as well as of the jerk and snap functions. We also found expressions for $j(z)$ and $s(z)$ - Eqs. (4.139) and (4.141) - in terms of $q(z)$ and its derivatives, showing that is indeed possible to write the bounds in terms of quantities we can estimate, even for higher order derivatives of the scale factor.

This sets the base for the possibility of future comparison with observational data. As the next steps, we aim to use Type Ia supernovae samples, cosmic-chronometer [$H(z)$] measurements, and baryon acoustic oscillation data points to reconstruct these functions. By doing so, we hope to constrain the theory's parameters, as $f_{R0}$ and $n$, using the observational data.

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# Appendices

# Appendix A – Field Equations for a General $f(R)$ Theory

Let us compute the field equations for a general $f(R)$ theory whose action is

$$S_{f(R)} = \int dx^4 \sqrt{-g} f(R), \tag{A.1}$$

in which $f(R)$ is some non-linear function of the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. To do so, we will follow the procedure by Capozziello and De Laurentis (Capozziello; Laurentis, 2011).

By performing the variation of the action, we get

$$\begin{aligned} \delta S &= \delta \int d^4x \sqrt{-g} f(g^{\mu\nu}R_{\mu\nu}) \\ &= \int d^4x [(\delta\sqrt{-g}) f(R) + \sqrt{-g} f'(R)(\delta R_{\mu\nu})g^{\mu\nu} + \sqrt{-g} f'(R) R_{\mu\nu}\delta g^{\mu\nu}]. \end{aligned} \tag{A.2}$$

The prime denotes differentiation with respect to $R$, in such a way that $f'(R) = \frac{df(R)}{dR}$.

The variation $\delta\sqrt{-g}$ is given by (Carroll, 2004)

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}, \tag{A.3}$$

so the first term in Eq. (A.2) takes the form

$$(\delta\sqrt{-g}) f(R) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} f(R). \tag{A.4}$$

Consequently, the variation reads

$$\delta \int d^4x \sqrt{-g} f(R) = \int d^4x \sqrt{-g} \left[ f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) \right] \delta g^{\mu\nu} + \int d^4x \sqrt{-g} f'(R) g^{\mu\nu} \delta R_{\mu\nu}. \tag{A.5}$$

Let us now focus on the second term and work out the variation $\delta R_{\mu\nu}$: the variation of the Riemann tensor is given by (Carroll, 2004)

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}), \tag{A.6}$$

from which we obtain the variation of $R_{\mu\nu}$ insofar as $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$:

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu}). \tag{A.7}$$

Equation (A.5) displays the quantity $g^{\mu\nu}\delta R_{\mu\nu}$, so we can use the previous result to compute it. Taking this contraction and using metric compatibility leads to

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\sigma[g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\mu\lambda})] \\ &= \nabla_\sigma W^\sigma, \end{aligned} \tag{A.8}$$

if we define

$$W^\sigma = g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\mu\lambda}). \tag{A.9}$$

Defining $W^\sigma$ will be useful for better visualization of the next steps we are about to take. Writing the variation of the second term in Eq. (A.5) in terms of $W^\sigma$ and integrating by parts yields

$$\begin{aligned} \int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} &= \int d^4x\sqrt{-g}f'(R)\nabla_\sigma W^\sigma \\ &= \int d^4x\nabla_\sigma[\sqrt{-g}f'(R)W^\sigma] - \int d^4x\nabla_\sigma[\sqrt{-g}f'(R)]W^\sigma \\ &= -\int d^4x\nabla_\sigma[\sqrt{-g}f'(R)]W^\sigma. \end{aligned} \tag{A.10}$$

From the second to the third equality we discarded the total divergence term, since we are considering that at infinity the fields vanish.

We now look for an explicit expression for $W^\sigma$. Writing the Levi-Civita connection explicitly from Eq. (2.16), the first term in Eq. (A.9) yields

$$\begin{aligned} g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) &= g^{\mu\nu}\delta\left[\frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\right] \\ &= g^{\mu\nu}\frac{1}{2}\delta(g^{\sigma\rho})(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + g^{\mu\nu}\frac{1}{2}g^{\sigma\rho}[\partial_\mu\delta(g_{\nu\rho}) + \partial_\nu\delta(g_{\rho\mu}) - \partial_\rho\delta(g_{\mu\nu})], \end{aligned} \tag{A.11}$$

since $\delta(\partial_\rho g_{\mu\nu}) = \partial_\rho(\delta g_{\mu\nu})$ (d'Inverno; Vickers, 2022).

For convenience, we can compute the variations of the terms in Eq. (A.11) in the local inertial frame, since we are dealing with tensorial equations, which must be valid in different frames. In the local inertial frame, the connections vanish, and ordinary derivatives are equivalent to the covariant derivatives, *i.e.*,

$$\partial_\rho g_{\mu\nu} = \nabla_\rho g_{\mu\nu} = 0, \tag{A.12}$$

as we are also assuming metric compatibility and using Eq. (2.18). As a consequence, Eq. (A.11) reduces to

$$g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) = g^{\mu\nu}\frac{1}{2}\delta(g^{\sigma\rho})(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{A.13}$$

Applying to it the identities (Carroll, 2004)

$$\begin{aligned} \delta g^{\alpha\beta} &= -g^{\mu\alpha}g^{\nu\beta}\delta g_{\mu\nu} \quad \text{and} \\ \delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}, \end{aligned} \tag{A.14}$$

and considering that the metric commutes with the derivatives due to Eq. (A.12), we thus obtain

$$g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) = \frac{1}{2}\partial^\sigma(g_{\alpha\beta}\delta g^{\alpha\beta}) - \partial^\nu(g_{\nu\alpha}\delta g^{\alpha\sigma}). \tag{A.15}$$

Following the same procedure for the second term in Eq. (A.9) leads to

$$g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\mu\lambda}) = -\frac{1}{2}\partial^\sigma(g_{\alpha\beta}\delta g^{\alpha\beta}), \tag{A.16}$$

in such a way that plugging this together with Eq. (A.15) into Eq. (A.9) results in

$$W^\sigma = \partial^\sigma(g_{\mu\nu}\delta g^{\mu\nu)} - \partial^\mu(g_{\mu\nu}\delta g^{\sigma\nu}). \tag{A.17}$$

Once we have an explicit expression for $W^\sigma$, we can write Eq. (A.10) in terms of it:

$$\begin{aligned} \int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = &\int d^4x\nabla_\sigma[\sqrt{-g}f'(R)]\partial^\mu(g_{\mu\nu}\delta g^{\sigma\nu}) + \\ &- \int d^4x\nabla_\sigma[\sqrt{-g}f'(R)]\partial^\sigma(g_{\mu\nu}\delta g^{\mu\nu}). \end{aligned} \tag{A.18}$$

Integrating by parts again and considering Eq. (A.12) - both for using metric compatibility and exchanging covariant derivatives for ordinary ones -, we obtain

$$\int d^4x\sqrt{-g}f'(R)g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\{g_{\mu\nu}\partial^\sigma\partial_\sigma[\sqrt{-g}f'(R)] - g_{\sigma\nu}\partial^\sigma\partial_\mu[\sqrt{-g}f'(R)]\}\delta g^{\mu\nu}. \tag{A.19}$$

Applying the derivatives gives rise to terms depending on derivatives of $\sqrt{-g}$, *i.e.*,

$$\partial\sigma\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\partial_\sigma g. \tag{A.20}$$

For its time, $\partial_\sigma g$ can be expressed as[1]

1 A didactic exposition of the demonstration of this property is available at https://web.phys.ntnu.no/~mika/week6.pdf.

$$\partial_\sigma g = g g^{\mu\nu} \partial_\sigma g_{\mu\nu}, \tag{A.21}$$

which, by using again Eq. (A.12), leads to

$$\partial_\sigma \sqrt{-g} = 0. \tag{A.22}$$

As a consequence, all terms depending on such derivatives in Eq. (A.19) vanish. The remaining terms, after changing back to covariant derivatives (again according to Eq. (A.12)), read

$$\int d^4x \sqrt{-g} f'(R) g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} [g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R)] \delta g^{\mu\nu}, \tag{A.23}$$

in which $\Box = \nabla^\sigma \nabla_\sigma$ is the D'Alambertian.

We can now go back to the total variation at Eq. (A.5) and rewrite it as

$$\delta \int d^4x \sqrt{-g} f(R) = \int \sqrt{-g} \left[ f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) \right] \delta g^{\mu\nu}, \tag{A.24}$$

which is just the same as Eq. (4.69).

By using the variational principle, this corresponds to a field equation (Capozziello; Laurentis, 2011)

$$f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = 0. \tag{A.25}$$

If we also consider the matter part of the action, represented by $S_m$, we can write

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} f(R) + \mathcal{L}_m \right], \tag{A.26}$$

with $\mathcal{L}_m$ standing for the Lagrangian of matter and $\kappa$ for the curvature-matter coupling. By performing the variation and following the same steps taken to find Eq. (A.24) results in the field equation

$$f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f'(R) = \kappa T_{\mu\nu}. \tag{A.27}$$

$T_{\mu\nu}$ is the energy-momentum tensor, defined as (Carroll, 2004)

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{A.28}$$

An alternative way of writing Eq. (A.27) would be explicitly applying the covariant derivatives to $f'(R)$. This would lead to the expression (Buchdahl, 1970; Kerner, 1982)

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + f''(R)(\nabla_\mu\nabla_\nu R - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta R) + \\ + f'''(R)(\nabla_\mu R\nabla_\nu R - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha R\nabla_\beta R) = \kappa T_{\mu\nu}, \tag{A.29}$$

which is completely equivalent to Eq. (A.27).

In Eq. (4.66) we deal with an action that depends on the functions $f_1(R)$ and $f_2(R)$. Its variation is given by

$$\delta S = \int d^4x \left[\frac{1}{2\kappa}\delta(\sqrt{-g}f_1) + \delta(f_2\mathcal{L}_m)\right]. \tag{A.30}$$

The variation of the first term results exactly in Eq. (A.24), and we can just replace $f_1(R)$ by any function. For instance, in this work we considered the function $f_1(R) = R + f_{HS}(R)$, with $f_{HS}(R)$ defined by Eq. (4.97). For the second term, we can follow an analogous procedure: the variation $f_2\delta\mathcal{L}_m$ gives rise to the $T_{\mu\nu}$ term, while the variation $\mathcal{L}_m\delta f_2$ leads to the same terms from when we were varying $\delta f(g^{\mu\nu}R_{\mu\nu})$ throughout this appendix (up to the $\mathcal{L}_m$ factor), as displayed in Eq. (4.70).

# Appendix B – $h_t(t)$ and $h_s(t)$ Functions for a $f(R)$ Theory with non-minimal Coupling

A $f(R)$ theory with non-minimal coupling admits an explicit matter-curvature coupling. Comparing Eqs. (4.1) and (4.78), we can identify the couplings $g_1$ and $g_2$ respectively as

$$g_1 = f_1' + 2\mathcal{L}_m k^{-1} f_2' \tag{B.1}$$

and

$$g_2 = f_2, \tag{B.2}$$

while the $H_{\mu\nu}$ tensor reads

$$H_{\mu\nu} = -\frac{f_2}{f_1' + \frac{2}{k}\mathcal{L}_m f_2'}\left[\frac{1}{2}\left(\frac{f_1}{f_2} - \frac{f_1' + 2\mathcal{L}_m k^{-1} f_2'}{f_2}R\right) g_{\mu\nu} + \frac{1}{f_2}\Delta_{\mu\nu}\left(f_1' + \frac{2}{k}\mathcal{L}_m f_2'\right)\right]. \tag{B.3}$$

We can recover Eqs. (4.80) and (4.81) by requiring $f_2 = 1$. Furthermore, Eq. (B.3) shows that, in general, $H_{\mu\nu}$ includes not only information concerning the geometric fields but also information about the matter fields as a consequence of the explicit curvature-matter coupling. On the other hand, if there is minimal coupling, consequently there are no $\mathcal{L}_m$ terms on $H_{\mu\nu}$, as in the case of Eq. (4.81).

# Appendix C – Converting Natural Units to SI Units

Natural units are the ones in which the speed of light $c$ is set to unit, *i.e.*, $c = 1$. Eqs. (4.86) and (4.87) are written in natural units, and we would like to write them in S.I. units to recover the $c$ factors explicitly and then evaluate such quantities, as well as the energy condition bounds, using observational data.

To transform the units of $\rho$ and $p$, we first need to know the units of the quantities in terms of which they are defined. Looking at Eqs. (4.86) and (4.87), we see that they depend on $f_1(R)$ and its derivatives $f_1(R)' = \frac{\partial f_1}{\partial R}$, as well on time derivatives $\partial_0 f_1'$. They also depend on the scale factor $a$ and its time derivatives.

From Eq. (4.75), we see that $f_1(R)$ is the function of $R$ that appears in the modified action. It depends both on $R$ itself and on the geometric extension of the theory - for instance, it can take the form $f_1(R) = R + f(R)$, in which $f(R)$ is some nonlinear function of $R$. Since $f_1(R)$ is a function that also depends linearly on $R$, and since we expect it to lead to field equations that extend the Einstein ones by adding some geometric modification to it (as in Eq. (4.1)), then $f_1(R)$ must necessarily have units of $R$. So, if we use brackets to indicate the units of each quantity[1], we must have $[f_1(R)] = [R]$.

We can find the units of $R$ from the Einstein field equations. In S.I units, the right-hand side of Eq. (1.1) reads $\frac{8\pi G}{c^4}T_{\mu\nu}$. From Newton's Universal Law of Gravitation, $G$ has units of $[G] = \frac{[F][\text{length}]^2}{[\text{mass}]^2}$ ($F$ stands for force). Since in S.I. units $[\text{length}] = \text{m}$ (m stands for meters), $[\text{mass}] = \text{kg}$, and $[F] = [\text{mass}][\text{acceleration}] = [\text{mass}]\frac{[\text{length}]}{[\text{time}]^2} = \frac{\text{kg m}}{\text{s}^2}$, we have $[G] = \frac{\text{m}^3}{\text{kgs}^2}$. For its part, $T_{\mu\nu}$ has units of energy density (remember that $T^{00} = \rho$), in such a way that $[T_{\mu\nu}] = \frac{[\text{Joule}]}{[\text{length}]^3} = \frac{\text{kg}}{\text{ms}^2}$. Once $[c^{-4}] = \frac{\text{s}^4}{\text{m}^4}$, we see that the right-hand side of Einstein equations has units of $\left[\frac{8\pi G}{c^4}T_{\mu\nu}\right] = \frac{1}{\text{m}^2}$. To ensure consistency, both sides of Eq. (1.1) must have the same units and, consequently, the term $\frac{1}{2}Rg_{\mu\nu}$ has units of $\text{m}^{-2}$.

The metric $g_{\mu\nu}$ can be defined by (Carroll, 2004)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{C.1}$$

in which $ds^2$ is the spacetime interval. Since $[ds] = [\text{length}]$ and $[dx^\mu] = [\text{length}]$, we have

$$[\text{length}]^2 = [g_{\mu\nu}][\text{length}]^2, \tag{C.2}$$

[1] We adopt such convention specifically throughout this appendix. In the other chapters, we assign their usual function - delimiting factors in equations - to brackets.

*i.e.*, $g_{\mu\nu}$ is dimensionless. Consequently, to fulfill the condition $[\frac{1}{2}Rg_{\mu\nu}] = [\mathrm{m}^{-2}]$, we find that $[R] = [\mathrm{m}^{-2}]$.

As a consequence, the derivative $\frac{\partial f_1}{\partial R}$ is dimensionless, since $\left[\frac{\partial f_1}{\partial R}\right] = \frac{[f_1]}{[R]} = \frac{\mathrm{m}^{-2}}{\mathrm{m}^{-2}} = 1$. Time derivatives have units of $t^{-1}$, as $\left[\frac{\partial}{\partial t}\right] = \frac{1}{[\text{time}]} = \frac{1}{t}$, so applying it to $f_1'$ results in $\left[\frac{\partial f_1'}{\partial t}\right] = t^{-1}$. Analogously, since $a$ is dimensionless, taking time derivatives of it adds units of $t^{-1}$, in such a way that $[\dot{a}] = \left[\frac{\partial a}{\partial t}\right] = \frac{1}{[\text{time}]} = \frac{1}{\mathrm{s}}$ and $[\ddot{a}] = \left[\frac{\partial^2 a}{\partial t^2}\right] = \frac{1}{[\text{time}]^2} = \frac{1}{\mathrm{s}^2}$.

Under such considerations, looking at Eq. (4.26) we see that the units of each term are

$$\begin{aligned}
[\rho] = [\text{energy density}] = \left[\frac{J}{\mathrm{m}^3}\right] &= \frac{\mathrm{kg}}{\mathrm{ms}^2}, \\
[p] = [\text{pressure}] = \left[\frac{F}{A}\right] &= \frac{\mathrm{kg}}{\mathrm{ms}^2}, \\
\left[\frac{1}{8\pi G}\right] = [G^{-1}] &= \frac{\mathrm{kgs}^2}{\mathrm{m}^3}, \\
\left[-3\frac{\ddot{a}}{a}f_1'\right] = \left[\frac{\partial^2 a}{\partial t^2}\right] &= \frac{1}{\mathrm{s}^2}, \\
\left[3\frac{\dot{a}}{a}\partial_0 f_1'\right] = \left[\frac{\partial a}{\partial t}\right]\left[\frac{\partial f_1'}{\partial t}\right] &= \frac{1}{\mathrm{s}^2}, \\
\left[\frac{f_1}{2}\right] &= \frac{1}{\mathrm{m}^2}.
\end{aligned} \tag{C.3}$$

We can transform from natural units to S.I. units by using appropriate multiplicative factors (Myers, 2016). They depend on the quantities we are dealing with but, in general, we have

$$[\text{natural units}] \to [\text{multiplicative factor}][\text{natural units}] = [\text{S.I. units}]. \tag{C.4}$$

This multiplicative factor is given by[2] $\hbar^{\beta+\gamma}c^{\beta-2\alpha}$ (Myers, 2016), in which $\hbar$ is the reduced Planck constant and $\alpha$, $\beta$ and $\gamma$ are obtained from the units we are transforming by the relation $(\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma)$, for S.I. units. For instance, the factors by which we need to multiply quantities with units of mass (kg), length (m), and time (s) in natural units to obtain the S.I. ones are, respectively, $c^{-2}$, $\hbar c$ and $\hbar$.

For instance, since in S.I. units $[G^{-1}] = \frac{\mathrm{kgs}^2}{\mathrm{m}^3} = \mathrm{kgm}^{-3}\mathrm{s}^2$, then $\alpha = 1$, $\beta = -3$ and $\gamma = 2$. Consequently, when transforming the units of $G^{-1}$ from natural units to the S.I. ones we would need to use the factor $\hbar^{\beta+\gamma}c^{\beta-2\alpha} = (\hbar c^5)^{-1}$, *i.e.*,

$$[G^{-1}]_{\text{natural units}} \to (\hbar c^5)^{-1}[G^{-1}]_{\text{natural units}} = [G^{-1}]_{\text{S.I. units}}. \tag{C.5}$$

[2] Specifically in this appendix $\alpha$ and $\beta$ stand for the powers in the multiplicative factor for transforming units. In Ch. 4, however, we use $\alpha$ and $\beta$ as parameters of the Hu-Sawicki $f(R)$ function, in Eq. (4.97).

Analogously, given the units of $\rho$ in Eq. (C.4), the multiplicative factor for energy density is $(\hbar c)^{-3}$.

These multiplicative factors are useful since they allow us to set a correspondence between the natural and S.I. units for each quantity. We can take Einstein's field equations as an example: in natural units, they read

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{C.6}$$

whose units are such that

$$[G_{\mu\nu}] = 8\pi [G T_{\mu\nu}]. \tag{C.7}$$

Based on the discussion above, $G_{\mu\nu}$ units are $[G_{\mu\nu}] = \mathrm{m}^{-2}$, yielding a multiplicative factor $(\hbar c)^{-2}$. For its part, $[G] = \frac{\mathrm{m}^3}{\mathrm{kgs}^2}$, leading to a factor $\hbar c^5$, while $T_{\mu\nu}$ units are $[T_{\mu\nu}] = \frac{\mathrm{kg}}{\mathrm{ms}^2}$, with a corresponding factor $(\hbar c)^{-3}$. We can thus transform Eq. (C.7) into S.I. units by multiplying each term by its respective factor, resulting in

$$\frac{1}{\hbar^2 c^2}[G_{\mu\nu}] = 8\pi \frac{\hbar c^5}{\hbar^3 c^3}[G T_{\mu\nu}], \tag{C.8}$$

and, after rearranging the factors, in

$$\frac{1}{c^4}[G_{\mu\nu}] = 8\pi [G T_{\mu\nu}]. \tag{C.9}$$

It is worth noticing again that we are dealing with the units of these quantities when using brackets. Eq. (C.9) tells us that "the units of the right-hand side of Einstein's equations are equivalent to the units of the left-hand side divided by $c^4$". Therefore, to make the units on both sides compatible, we must divide the right-hand side by the factor $c^4$, ensuring that its units match those of the left-hand side. In S.I. units, Einstein's equations thus take the form:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{C.10}$$

Following this same reasoning for the energy density $\rho$, we see that the units of the terms and factors in Eq. (4.26) are such that

$$[\rho] = \left[\frac{1}{8\pi G}\right]\left(\left[-3\frac{\ddot{a}}{a}f_1'\right] + \left[3\frac{\dot{a}}{a}\partial_0 f_1'\right] + \left[\frac{f_1}{2}\right]\right), \tag{C.11}$$

Computing the appropriate factors for each one of them using the quantities in Eq. (C.4), we can rewrite it in S.I. units as

$$\hbar^{-3}c^{-3}[\rho] = \hbar^{-1}c^{-5}\left[\frac{1}{8\pi G}\right]\left(\hbar^{-2}\left[-3\frac{\ddot{a}}{a}f_1'\right] + \hbar^{-2}\left[3\frac{\dot{a}}{a}\partial_0 f_1'\right] + \hbar^{-2}c^{-2}\left[\frac{f_1}{2}\right]\right), \tag{C.12}$$

which leads to the expression

$$c^{-3}[\rho] = \left[\frac{1}{8\pi G}\right]\left(c^{-5}\left[-3\frac{\ddot{a}}{a}f_1'\right] + c^{-5}\left[3\frac{\dot{a}}{a}\partial_0 f_1'\right] + c^{-7}\left[\frac{f_1}{2}\right]\right). \tag{C.13}$$

Each term on the right-hand side of Eq. (C.13) must match the left-hand side one. For instance, to make the $c^{-5}\left[-3\frac{\ddot{a}}{a}f_1'\right]$ term compatible to $c^{-3}[\rho]$, we should multiply the term $-3\frac{\ddot{a}}{a}f_1'$ in Eq. (4.86) by $c^2$ . Analogously, the terms corresponding to the second and third terms on the right-hand side of Eq. (C.13) must be multiplied by the factors $c^2$ and $c^4$, respectively. This leads to the expression for $\rho$ in S.I. units, given by

$$\rho = \frac{c^4}{8\pi G}\left(-\frac{3}{c^2}\frac{\ddot{a}}{a}f_1' + \frac{3}{c^2}\frac{\dot{a}}{a}\partial_0 f_1' + \frac{f_1}{2}\right). \tag{C.14}$$

Following the same procedure, we find that the expression for $p$ in S.I. units reads

$$p = \frac{c^4}{8\pi G}\left(\frac{1}{c^2}\frac{\ddot{a}}{a}f_1' + \frac{2}{c^2}\frac{\dot{a}^2}{a^2}f_1' - \frac{2}{c^2}\frac{\dot{a}}{a}\partial_0 f_1' - \frac{1}{c^2}\partial_0\partial_0 f_1' - \frac{f_1}{2}\right). \tag{C.15}$$

For recovering the $c$ factors in the expressions for $h_t$ and $h_s$ - Eqs. (4.83) and (4.85), respectively -, it is necessary first to find which are the units of these functions so we can apply the appropriate multiplicative factors. From Eq. (4.26), we see that

$$8\pi G\rho = f_1'\left(3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} - h_t\right), \tag{C.16}$$

that is, since $f_1'$ is dimensionless, $h_t$ has units of

$$[h_t] = [8\pi G\rho] = \frac{\mathrm{m}^3}{\mathrm{kgs}^2}\frac{\mathrm{kg}}{\mathrm{ms}^2} = \frac{\mathrm{m}^2}{\mathrm{s}^4}, \tag{C.17}$$

yielding a multiplicative factor $c^2/\hbar^2$.

So, in which concern to units, Eq. (4.83) reads

$$[h_t(t)] = \left[-\frac{1}{2}\frac{f_1}{f_1'}\right] + \left[3\frac{\ddot{a}}{a}\right] + \left[3\frac{\dot{a}^2}{a^2}\right] - \left[3\frac{\dot{a}}{a}\frac{\partial_0 f_1'}{f_1'}\right]. \tag{C.18}$$

Looking at the units of each term according to Eq. (C.4), we can apply the appropriate multiplicative factors to rewrite it in S.I. units as

$$\frac{c^2}{\hbar^2}[h_t(t)] = \frac{1}{\hbar^2 c^2}\left[-\frac{1}{2}\frac{f_1}{f_1'}\right] + \frac{1}{\hbar^2}\left[3\frac{\ddot{a}}{a}\right] + \frac{1}{\hbar^2}\left[3\frac{\dot{a}^2}{a^2}\right] - \frac{1}{\hbar^2}\left[3\frac{\dot{a}}{a}\frac{\partial_0 f_1'}{f_1'}\right]. \tag{C.19}$$

Then, comparing the terms on both sides of the equation we find the factors that make the units compatible. These lead to the equation for $h_t$ in S.I. units:

$$h_t(t) = -\frac{c^4}{2}\frac{f_1}{f_1'} + 3c^2\frac{\ddot{a}}{a} + 3c^2\frac{\dot{a}^2}{a^2} - 3c^2\frac{\dot{a}}{a}\frac{\partial_0 f_1'}{f_1'}. \tag{C.20}$$

Following the same prescription for $h_s$ yields, in S.I. units, the expression

$$h_s(t) = -\frac{c^4}{2}\frac{f_1}{f_1'} + 3c^2\frac{\ddot{a}}{a} + 3c^2\frac{\dot{a}^2}{a^2} - \frac{c^2}{f_1'}\partial_0\partial_0 f_1' - \frac{2c^2}{f_1'}\frac{\dot{a}}{a}\partial_0 f_1'. \tag{C.21}$$

Another situation in which we had to change from natural to S.I. units consisted in expressing the $m^2$ parameter for the Hu-Sawicki function when going from Eq. (4.100) to Eq. (4.101). We can apply the same procedure to make this conversion but, before doing that, it is worth noticing that we are now faced with the unpleasant coincidence regarding the parameter $m^2$ and the unit for distance 'meter' (m), as they are expressed by the same letter. To avoid confusion, we highlight that when referring to the parameter, we will always use brackets, *i.e.*, $[m^2]$, while the unit 'meter' will be written without brackets. Furthermore, the parameter will be denoted with a crafted '$m$', while for meters we will use the usual 'm'.

That said, we can proceed by looking what are the units of the parameter, that is, $[m^2]$. Since the Hu-Sawicki function in Eq. (4.97) will be added to $R$ in the action of the gravitation theory, then it must present the same units, in such a way that $[f_{HS}(R)] = [R] = \mathrm{m}^{-2}$. Consequently, the parameter will have units $[m^2] = \mathrm{m}^{-2}$, as $\alpha$ and $\beta$ (the ones from Eq. (4.97)) are dimensionless and, with this units, so is $R/m^2$. We then associate to it a multiplicative conversion factor $(\hbar c)^{-2}$.

Looking at the right-hand side of Eq. (4.100), we have the dimensionless $\Omega_{m0}$ and the Hubble parameter with units $[H_0^2] = \frac{1}{\mathrm{s}^2}$. This yields a multiplicative factor $\hbar^{-2}$. Putting all this together, we find

$$\frac{1}{(\hbar c)^{-2}}[m^2] = \frac{1}{\hbar^{-2}}[\Omega_{m0}H_0^2] \Rightarrow [m^2] = c^2[\Omega_{m0}H_0^2], \tag{C.22}$$

so we need to multiply the left-hand side by $c^2$, which leads to

$$m^2 = \frac{1}{c^2}\Omega_{m0}H_0^2, \tag{C.23}$$

in S.I. units.

Writing $H_0 = 100h$ km/s/Mpc and changing from 'km' to 'm' gives

$$m^2 = \frac{\Omega_{m0}h^2}{c^2} \times 10^{10}\ \frac{\mathrm{m}^2}{\mathrm{s}^2\mathrm{Mpc}^2}. \tag{C.24}$$

WMAP observations suggests $\Omega_{m0} \approx 0.25$ and $H_0 \approx 72$ km/s/Mpc (Hinshaw *et al.*, 2013), leading to $\Omega_{m0}h^2 \approx 0.13$. We can then rewrite Eq. (C.24) by making this factor explicit:

$$m^2 = \frac{1}{c^2}\left(\frac{\Omega_{m0}h^2}{0.13}\right) 0.13 \times 10^{10}\ \frac{\mathrm{m}^2}{\mathrm{s}^2\mathrm{Mpc}^2}. \tag{C.25}$$

If we apply the explicit value for the speed of light, *i.e.*, $c$ = 299,792,458 m/s, we then get

$$m^2 = (8315\ \mathrm{Mpc})^{-2} \left( \frac{\Omega_{m0} h^2}{0.13} \right), \tag{C.26}$$

which is precisely the result in Eq. (4.101).

# Appendix D – Bound Equations in Terms of the Scale Factor $a(t)$ and its Time Derivatives

## D.1 Bounds for the Hu-Sawicki Function without Approximation

Using Eqs. (4.97) and (4.104), and defining

$$\xi = \frac{\dot{a}^2 + a\ddot{a}}{c^2 m^2 a^2}, \tag{D.1}$$

from Eq. (4.94) we then obtain the expression for WEC 1 (Eq. (4.35)):

$$\begin{aligned} &\frac{c^2\alpha m^2}{2\beta}\left[\frac{1}{\beta(6\xi)^n+1}-1\right]+\frac{3\left(a\ddot{a}+\dot{a}^2\right)}{a^2}-\frac{\ddot{a}\left(6-\frac{\alpha 6^n n\xi^{n-1}}{[\beta(6\xi)^n+1]^2}\right)}{2a}+\\ &-\frac{\alpha m^2 2^{n-1}3^n n\dot{a}\left(-a^2a^{(3)}+2\dot{a}^3-a\dot{a}\ddot{a}\right)\xi^n\left[\beta(n+1)(6\xi)^n-n+1\right]}{\left(a\ddot{a}+\dot{a}^2\right)^2\left[\beta(6\xi)^n+1\right]^3}\geq 0. \end{aligned} \tag{D.2}$$

Using Eqs. (4.46), (4.47), (4.48) and (4.49) to express $a(t)$ and its derivatives in terms of the cosmographic functions $H(t)$, $q(t)$, $j(t)$ and $s(t)$ leads to Eq.(4.106).

Analogously, from Eq. (4.95) we get the bound equation for WEC 2 (Eq. (4.36)):

$$
\begin{aligned}
&\frac{(6\xi)^n}{a^2}\Bigg(24\beta\dot{a}^8+2c^2m^2n\big(2n(n+1)-1\big)\alpha a^2\dot{a}^6+48\beta a\ddot{a}\dot{a}^6\\
&+a^3\ddot{a}\Big(-c^2m^2n\big(n(4n+11)+9\big)\alpha\dot{a}^2-48\beta\ddot{a}^2\Big)\dot{a}^2\\
&+a^4\Big(-24\beta\ddot{a}^4+c^2m^2n\big(n(n+11)+12\big)\alpha\dot{a}^2\ddot{a}^2-c^2m^2n(n+1)(4n+7)\alpha\dot{a}^3a^{(3)}\Big)\\
&+c^2m^2n\alpha a^5\Big(-\big((n-1)\ddot{a}^3\big)+(n+1)(2n+5)\dot{a}a^{(3)}\ddot{a}-(n+1)\dot{a}^2a^{(4)}\Big)\\
&+c^2m^2n(n+1)\alpha a^6\Big((n+2){a^{(3)}}^2-\ddot{a}a^{(4)}\Big)\Bigg)\\
&+\frac{3^{2n+1}4^{n+1}\beta^2\left(\dot{a}^2-a\ddot{a}\right)\left(\dot{a}^2+a\ddot{a}\right)^3\xi^{2n}}{a^2}-\frac{6c^2m^2n^3\alpha\left(-2\dot{a}^3+a\ddot{a}\dot{a}+a^2a^{(3)}\right)^2}{\beta\left((6\xi)^n\beta+1\right)^2}\\
&+\frac{2}{\beta a^2}\Bigg(6\beta\dot{a}^8-4c^2m^2n^2(3n+1)\alpha a^2\dot{a}^6+12\beta a\ddot{a}\dot{a}^6\\
&+a^3\ddot{a}\Big(c^2m^2n^2(12n+11)\alpha\dot{a}^2-12\beta\ddot{a}^2\Big)\dot{a}^2\\
&+a^4\Big(-6\beta\ddot{a}^4-c^2m^2n^2(3n+11)\alpha\dot{a}^2\ddot{a}^2+c^2m^2n^2(12n+11)\alpha\dot{a}^3a^{(3)}\Big)\\
&+c^2m^2n^2\alpha a^5\Big(\ddot{a}^3-(6n+7)\dot{a}a^{(3)}\ddot{a}+\dot{a}^2a^{(4)}\Big)+c^2m^2n^2\alpha a^6\Big(\ddot{a}a^{(4)}-3(n+1){a^{(3)}}^2\Big)\Bigg)\\
&+\frac{2c^2m^2n^2\alpha}{(6\xi)^n\beta^2+\beta}\Bigg(4(6n+1)\dot{a}^6-(24n+11)a\ddot{a}\dot{a}^4-(24n+11)a^2a^{(3)}\dot{a}^3\\
&+a^2\Big((6n+11)\ddot{a}^2-aa^{(4)}\Big)\dot{a}^2+(12n+7)a^3\ddot{a}a^{(3)}\dot{a}-a^3\Big(\ddot{a}^3+aa^{(4)}\ddot{a}-3(2n+1)a{a^{(3)}}^2\Big)\Bigg)\geq 0
\end{aligned}
\tag{D.3}
$$

in which superscript indices in parenthesis indicate the order of the derivative. That is, $a^{(3)}$ stands for $\dddot{a}$, while $a^{(4)}=\ddddot{a}$. Again, using Eqs. (4.46), (4.47), (4.48) and (4.49) leads to Eq.(4.107).

We note that Eq. (D.3) depends on third and fourth-order derivatives of the scale factor $a(t)$. These terms arise from the terms $\partial_0\partial_0 f_1'$ in Eqs. (4.95) and (D.4). The derivative $f_1'$ in Eq. (4.104) depends on $R$, and from Eq. (4.15) this is given not only in terms of $a$, but also in terms of $\dot{a}$ and $\ddot{a}$. Consequently, taking second-order time derivatives of it results in factors of $a^{(3)}$ and $a^{(4)}$.

Following the same procedure and applying $f_{HS}$ into Eq. (4.96) leads to the bound equation for DEC (the second inequality in Eq. (4.38)):

$$
\begin{aligned}
&\frac{1}{a^2\left(\dot{a}^2+a\ddot{a}\right)\left(\beta(6\xi)^n+1\right)^2}\Bigg\{(6\xi)^n\Bigg(96\beta\dot{a}^8+336\beta a\ddot{a}\dot{a}^6\\
&+2a^2\Big(216\beta\ddot{a}^2-c^2m^2\big(n(2(n-4)n+5)+3\big)\alpha\dot{a}^2\Big)\dot{a}^4\\
&+a^3\ddot{a}\Big(c^2m^2\big(n(n(4n-5)+9)-18\big)\alpha\dot{a}^2+240\beta\ddot{a}^2\Big)\dot{a}^2\\
&+a^4\Big(48\beta\ddot{a}^4-c^2m^2\big((n-6)n(n+1)+18\big)\alpha\dot{a}^2\ddot{a}^2+c^2m^2(n-1)n(4n-13)\alpha\dot{a}^3a^{(3)}\Big)\\
&+c^2m^2\alpha a^5\Big(-\big((n-3)(n-2)\ddot{a}^3\big)+n\big(-2n^2+n+1\big)\dot{a}a^{(3)}\ddot{a}-(n-1)n\dot{a}^2a^{(4)}\Big)\\
&-c^2m^2(n-1)n\alpha a^6\Big((n-2)a^{(3)2}+\ddot{a}a^{(4)}\Big)\Bigg)\\
&+2^{2n+1}9^n\beta\Bigg(72\beta\dot{a}^8+252\beta a\ddot{a}\dot{a}^6+a^2\Big(c^2m^2\big(8n^3-10n-9\big)\alpha\dot{a}^2+324\beta\ddot{a}^2\Big)\dot{a}^4\\
&+a^3\ddot{a}\Big(c^2m^2\big(-8n^3+9n-27\big)\alpha\dot{a}^2+180\beta\ddot{a}^2\Big)\dot{a}^2\\
&+a^4\Big(36\beta\ddot{a}^4+c^2m^2\big(2n^3+6n-27\big)\alpha\dot{a}^2\ddot{a}^2+c^2m^2n\big(13-8n^2\big)\alpha\dot{a}^3a^{(3)}\Big)\\
&+c^2m^2\alpha a^5\Big((5n-9)\ddot{a}^3+\big(4n^3+n\big)\dot{a}a^{(3)}\ddot{a}+n\dot{a}^2a^{(4)}\Big)\\
&+c^2m^2n\alpha a^6\Big(2\big(n^2-1\big)a^{(3)2}+\ddot{a}a^{(4)}\Big)\Bigg)\xi^{2n}\\
&+216^n\beta^2\Bigg(96\beta\dot{a}^8+336\beta a\ddot{a}\dot{a}^6+2a^2\Big(216\beta\ddot{a}^2-c^2m^2\big(n(2n(n+4)+5)+9\big)\alpha\dot{a}^2\Big)\dot{a}^4\\
&+a^3\ddot{a}\Big(c^2m^2\big(n(n(4n+5)+9)-54\big)\alpha\dot{a}^2+240\beta\ddot{a}^2\Big)\dot{a}^2\\
&+a^4\Big(48\beta\ddot{a}^4-c^2m^2\big((n-1)n(n+6)+54\big)\alpha\dot{a}^2\ddot{a}^2+c^2m^2n(n+1)(4n+13)\alpha\dot{a}^3a^{(3)}\Big)\\
&+c^2m^2\alpha a^5\Big(\big(n(n+5)-18\big)\ddot{a}^3-n\big(2n^2+n-1\big)\dot{a}a^{(3)}\ddot{a}+n(n+1)\dot{a}^2a^{(4)}\Big)\\
&-c^2m^2n(n+1)\alpha a^6\Big((n+2)a^{(3)2}-\ddot{a}a^{(4)}\Big)\Bigg)\xi^{3n}\\
&+6^{4n+1}\beta^3\left(\dot{a}^2+a\ddot{a}\right)^3\Big(-c^2m^2\alpha a^2+2\beta\ddot{a}a+4\beta\dot{a}^2\Big)\xi^{4n}\\
&+12\left(\dot{a}^2+a\ddot{a}\right)^3\left(2\dot{a}^2+a\ddot{a}\right)\Bigg\}\geq 0
\end{aligned}
\tag{D.4}
$$

Once more, Eqs. (4.46), (4.47), (4.48) and (4.49) lead to Eq.(4.108).

## D.2 Bounds for the Expanded Hu-Sawicki Function

Applying Eqs. (4.122) and (4.123) to Eq. (4.94) leads to the equation for WEC 1 (Eq. (4.35)) in terms of the scale factor $a(t)$ and its derivatives:

$$
\begin{aligned}
&-\frac{\alpha c^2 m^2}{\beta}+\frac{1}{\left(a\ddot{a}+\dot{a}^2\right)^2}\left\{\frac{6\left(\dot{a}^3+a\dot{a}\ddot{a}\right)^2}{a^2}-\frac{c^2 f_{R0}6^{-n}R_0^{n+1}}{n}\left(\frac{a\ddot{a}+\dot{a}^2}{c^2a^2}\right)^{-n}\times\right.\\
&\left.\times\left[(n+1)a^2\ddot{a}^2+(1-2n(n+1))\dot{a}^4+n(n+1)a^2a^{(3)}\dot{a}+(n(n+2)+2)a\dot{a}^2\ddot{a}\right]\right\}\geq 0
\end{aligned}
\tag{D.5}
$$

Using Eqs. (4.46), (4.47), (4.48) and (4.49) to express $a(t)$ and its derivatives in terms of $H(t)$, $q(t)$, $j(t)$ and $s(t)$ leads to Eq.(4.124).

Doing the same for WEC 2 (Eq. (4.36)), by using Eq. (4.95), results in

$$
\begin{aligned}
&\frac{6^{-n}}{c^2a^4}\left(\frac{a\ddot{a}+\dot{a}^2}{c^2a^2}\right)^{-n-1}\left\{-2c^2 f_{R0}(2n(n+1)-1)a^2R_0^{n+1}\dot{a}^6+c^2 f_{R0}(n(4n+11)+9)a^3R_0^{n+1}\dot{a}^4\ddot{a}\right.\\
&+2^{n+2}3^{n+1}\left(\dot{a}^2-a\ddot{a}\right)\left(a\ddot{a}+\dot{a}^2\right)^3\left(\frac{a\ddot{a}+\dot{a}^2}{c^2a^2}\right)^n-c^2 f_{R0}(n+1)a^6R_0^{n+1}\left((n+2)a^{(3)2}-a^{(4)}\ddot{a}\right)\\
&+c^2 f_{R0}a^4R_0^{n+1}\dot{a}^2\left((n+1)(4n+7)a^{(3)}\dot{a}-(n(n+11)+12)\ddot{a}^2\right)\\
&\left.+c^2 f_{R0}a^5R_0^{n+1}\left((n-1)\ddot{a}^3+(n+1)a^{(4)}\dot{a}^2-(n+1)(2n+5)a^{(3)}\dot{a}\ddot{a}\right)\right\}\geq 0
\end{aligned}
\tag{D.6}
$$

Again, using Eqs. (4.46), (4.47), (4.48) and (4.49) leads to Eq.(4.125).

From Eq. (4.96), the bound equation for DEC (the second inequality in Eq. (4.38)) reads

$$
\begin{aligned}
&c^2\left(-\frac{f_{R0}6^{-n}R_0^{n+1}\left(\frac{a\ddot{a}+\dot{a}^2}{c^2a^2}\right)^{-n}}{n}-\frac{\alpha m^2}{\beta}\right)+\frac{2a\ddot{a}+4\dot{a}^2}{a^2}+\frac{c^2 f_{R0}6^{-n-1}R_0^{n+1}}{(a\ddot{a}+\dot{a}^2)^3}\left(\frac{a\ddot{a}+\dot{a}^2}{c^2a^2}\right)^{-n}\times\\
&\times\left\{2\left(2n(n+4)+5\right)\dot{a}^6-\left(n(4n+5)+9\right)a\dot{a}^4\ddot{a}+(n+1)a^4\left((n+2)a^{(3)2}-a^{(4)}\ddot{a}\right)\right.\\
&+a^2\dot{a}^2\left((n-1)(n+6)\ddot{a}^2-(n+1)(4n+13)a^{(3)}\dot{a}\right)\\
&\left.+a^3\left(-\left((n+5)\ddot{a}^3\right)-(n+1)a^{(4)}\dot{a}^2+\left(2n^2+n-1\right)a^{(3)}\dot{a}\ddot{a}\right)\right\}\geq 0
\end{aligned}
\tag{D.7}
$$

Using Eqs. (4.46), (4.47), (4.48) and (4.49), we obtain Eq.(4.126).