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Um convite à Teoria Modular: Entropia e Progressos Recentes

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"After almost half a century of existence,
the main question about quantum field theory
seems still to be: What does it really describe?
And not yet: Does it provide a good description
of nature?"

Swieca, *Jorge André*

Resumo

Este trabalho apresenta algumas propriedades analíticas do operador Hamiltoniano modular, com base em avanços recentes na área. Nos últimos anos, contribuições significativas foram feitas, e este trabalho procura fornecer uma visão geral dos resultados e aplicações mais notáveis da teoria modular. Dada a falta de uma interpretação geométrica para o fluxo modular no caso massivo, é natural concentrar-se na derivação de suas propriedades analíticas. Seguindo uma abordagem numérica recente, encontramos que a primeira derivada do Hamiltoniano modular em relação à massa é divergente. Para lidar com essa não-diferenciabilidade, investigamos a estrutura analítica do núcleo de um operador relevante. Este núcleo também está relacionado à parte não-diagonal da transformação de Bogoliubov que conecta duas representações com massas diferentes. As propriedades analíticas dessas transformações de Bogoliubov são exploradas utilizando transformações de Mellin e funções G de Meijer. No entanto, encontramos que a primeira derivada dos coeficientes de Bogoliubov em relação à massa se comporta de forma bem comportada. Como o núcleo do operador estudado deslocaliza os campos, ele aparece naturalmente em conjunto com projetores. Concluímos que a divergência da primeira derivada do Hamiltoniano modular está enraizada em considerações mais profundas da análise funcional. Esta afirmação sugere uma investigação futura sobre as propriedades desses projetores.

Palavras-chave: Teoria modular; Tomita-Takesaki; Entropia; Função G de Meijer; Teoria Algébrica de Campos Quânticos.

Áreas do conhecimento: Física; Física Matemática; Teoria Algébrica de Campos Quânticos.

Abstract

This work presents some analytical properties of the modular Hamiltonian operator, building on recent advancements in the field. In recent years, significant contributions have been made, and this paper aims to provide an overview of the most notable results and applications of modular theory. Given the lack of a geometric interpretation for the modular flow in the massive case, it is natural to focus on deriving its analytical properties. By following a recent numerical approach, we find that the first derivative of the modular Hamiltonian with respect to mass is divergent. To address this non-differentiability, we investigate the analytical structure of the kernel of a relevant operator. This kernel is also linked to the non-diagonal part of the Bogoliubov transformation that connects two representations with different masses. The analytical properties of these Bogoliubov transformations are explored using Mellin transformations and Meijer G-functions. However, we find that the first derivative of the Bogoliubov coefficients with respect to mass is well-behaved. Since the studied kernel of the operator delocalizes the fields, it naturally appears in conjunction with projectors. We conclude that the divergence of the modular Hamiltonian's first derivative is rooted in deeper functional analysis considerations. This assertion prompts a future investigation into the properties of these projectors.

Contents

Chapter 1

Introduction

Consider the Klein-Gordon equation,

$$(\Box + m^2)\Phi(x,t) = (\partial_t^2 - \partial_x^2 + m^2)\Phi(x,t) = 0. \tag{1.1}$$

Taking only real solutions, one has the massive Hermitian scalar free field. At first glance, we could imagine the field behaving as a usual function defined at each spacetime *point*. For each spacetime point, some information would be associated. In fact, the solutions are operator-valued distributions [1]. This means that the field is always smeared with some test function that localizes it in a spacetime *region*. There does not exist such a concept as a field at a point. If one desires to quantize it, it is expected that the Hilbert space associated with it will be influenced by those test functions mentioned.

By the same arguments for the nonexistence of a field at a point, we cannot guarantee that the Energy-Momentum Tensor is positive. A positive definite local energy is incompatible with the notion of a field being an operator-valued distribution [2]. If one assumes that this is not a problem in Quantum Field Theory, it is possible to invoke conjectures and conditions placing bounds on the Energy-Momentum Tensor of the field [3]. For instance, one can introduce the Bekenstein bound [4]. The Bekenstein bound states that the entropy of a matter system localized in a spacetime region is bounded by the mass-energy of the field and the radius of the smallest sphere that fits around the matter system.

Also, there is the Quantum Null Energy Condition [5], which states a lower bound on the Energy-Momentum Tensor in terms of the second variation in a null direction of the entropy of a region. As we can see, there is always a necessity for defining the field smeared over a spacetime region. A first conclusion is that there is a necessity for defining the entropy of a vector, or even better, the entropy of a wave. In this way, we can analyze the Energy-Momentum Tensor density in the framework of distributions. Besides this, the problem of UV divergence in Quantum Field Theory arises when evaluating entropy.

Starting with Cauchy data of a Hermitian scalar free field on a time slice, the field, the momentum field, and the Hilbert space of one-particle states are defined. The quantization procedure is divided into two maps [6]. The first quantization map relates an open spacetime region to a closed real subspace of the one-particle space. The second quantization map relates this closed real subspace of the one-particle space to a von Neumann algebra on the Fock space.

Introduce a standard subspace of a Hilbert space, i.e., a closed, real linear subspace of the Hilbert space which is cyclic and separating. Hence, the quantization process associates the von Neumann algebra of a spacetime region with the von Neumann algebra of the standard subspace associated with that spacetime region.

In 1967, during an operator algebras conference, M. Tomita distributed a preprint on modular theory, and R. Haag presented a talk on the characterization of equilibrium states in quantum statistical mechanics at infinite volume, extending the Gibbs condition to the KMS (Kubo-Martinez-Schwinger) condition [7].

With respect to the standard subspace, one can define the anti-linear Tomita operator. The Tomita operator S can be decomposed into a polar form

$$S = J\Delta^{1/2}. \tag{1.2}$$

The operator J is an anti-unitary involution, i.e., it maps elements of a standard subspace to its symplectic complement. The operator Δ is the modular operator, the intrinsic unitary evolution operator. In this way, the Tomita operator maps vectors from a standard subspace to a complementary region and evolves them while respecting the KMS condition. The KMS condition states that the evolution of a state is dictated by the thermal time flow.

Hence, the so important Tomita-Takesaki theorem associates a flow given by the modular operator Δ to an algebra of operators. One can write

$$\Delta^{it} = e^{-itK}, \tag{1.3}$$

where K, the *modular Hamiltonian*, is the generator of an intrinsic evolution. If one knows the modular Hamiltonian of the theory, one knows the dynamics of the theory. Only a few cases are known and, in particular, for the double cone and a massive field, it is completely unknown due to the lack of a geometrical interpretation of the modular flow.

Also, if the modular Hamiltonian is known, one can introduce the Araki-Uhlmann formula for the entropy of a vector $k \in \mathcal{H}$ with respect to a closed real

linear subspace H [8, 9, 10],

$$S_k = \mathrm{Im}(k, P_H i \log \Delta_H k). \tag{1.4}$$

These operators P_H are projection operators that cut the Cauchy data. Also, in Quantum Field Theory, we have an infinite number of degrees of freedom, which makes relative entropy important. Due to the lack of a geometrical interpretation in the massive case, it is natural to try to extract analytical properties of the modular Hamiltonian.

The dissertation is divided into three parts. In the first one, Part I, the concept of Cauchy data, the representation of an algebra, and the previously mentioned quantization maps are defined. The notion of relative entropy and the bounds on the Energy-Momentum Tensor are also discussed.

Part II focuses on the Tomita-Takesaki theorem and how the known results on modular flow are established.

Part III aims to present recent approaches to the unresolved massive case. The notion of a standard subspace is introduced, along with the definition of wave entropy for known cases, and the discussion of recent numerical approaches.

As a result, it is observed that for the massive modular Hamiltonian, despite being unknown, its first derivative with respect to the mass diverges at mass zero. This suggests that although it is continuous, it cannot be defined perturbatively. The final chapter of this work attempts to trace the origin of this divergence.

Chapter 2

Algebra of Local Observables

2.1 Spacetime

The purpose of this section is to introduce the concept of time-evolving phenomena, which arise from conditions on a slice of spacetime. Just as in classical physics, particularly in Newton's second law, one seeks to describe time evolution from initial conditions. This is achieved through the concept of a *Cauchy hypersurface*, which, due to Einstein's causality, leads to the refined notion of the *domain of dependence*.

The following construction is based on Rejzner [11, chap. 2] and Witten [12] and can be extended, if desired, to any general relativity textbook. Let us begin with the notion of spacetime.

Definition 2.1.1 *A spacetime is a pair (M, g), where M is a smooth manifold and g is a smooth Lorentzian metric in a sense of smooth tensor field $g \in \Gamma(T^*M \otimes T^*M)$ such that for every $p \in M$, g_p is a symmetric non-degenerate bilinear form. Here M is a four dimensional manifold, and it is required the metric g to be of Lorentzian signature $(+,-,-,-)$.*

Assume the manifold M to be time-oriented, which means that at each point in M, there exist a preferred notion of what represents a *future* and *past* timelike directions.

Definition 2.1.2 *Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$, let $\dot{\gamma}$ be the vector tangent to the curve. We say γ is*

- *Timelike, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,*
- *Spacelike, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,*
- *Lightlike (null), if $g(\dot{\gamma}, \dot{\gamma}) = 0$,*
- *Causal, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.*

Given a global timelike vector field u on M, a causal curve γ is called *future-directed* if the vector tangent $\dot{\gamma}$ projected on u direction is positive all along γ, i.e. $g(u, \dot{\gamma}) > 0$. The curve γ is *past-directed* if $g(u, \dot{\gamma}) < 0$.

Definition 2.1.3 *Let $p \in M$ be a point in a time-oriented spacetime*

- *$J^{\pm}(p)$ is defined to be the set of all points in M which can be connected to p by a future(+)/past(−)-directed causal curve $\gamma : I \to M$ so that $x = \gamma(inf\, I)$,*
- *The set $J^{+}(p)$ is called the causal future and $J^{-}(p)$ the causal past of p,*
- *The future (past) of a subset $B \subset M$ is defined by*

$$J^{\pm}(B) = \bigcup_{p \in B} J^{\pm}(p). \tag{2.1}$$

Finally, one of the most important statement on physics can be made. *Principle of causality*: an event happening at a point p can be influenced only by the events in $J^{-}(p)$ and the consequences of the event p can influence only the events in $J^{+}(p)$.

Definition 2.1.4 *A subset $A \subset M$ is called past(future)-compact if $A \bigcap J^{\pm}(p)$ is compact for all $p \in M$.*

Consider now, causal paths from a point q to a point p in its causal future. Such a path will lie within a subset of spacetime that we will call the *causal diamond*, or the *double cone*, D_q^p. This diamond is the intersection of the causal future of q with the causal past of q.

Essential points to consider:

1. The space of causal points from q to p is in a suitable sense compact,
2. Causality is essential here.

Causality guarantees that a sequence of paths converges, since it constraints velocity to be less or equal to the speed of light. Hence, if p is in the causal future of q, the space of causal paths from q to p is compact.

Definition 2.1.5 *Two subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ in M are called causally (or spacetime) separated if they cannot be connected by a causal curve, i.e., for all $x \in \overline{\mathcal{O}}_1$ one has $J^{\pm}(x) \bigcap \overline{\mathcal{O}}_2 = \emptyset$. The overline here means the closure of a set.*

Definition 2.1.6 *The causal complement $\mathcal{O}^{\perp}$ (or $\mathcal{O}'$) of $\mathcal{O}$ is defined as the largest open set in M that is causally separated from $\mathcal{O}$.*

From the principle of causality, events happening at spacelike separated points cannot influence each other.

Consider a hypersurface (submanifold of co-dimension 1) S which has the property of nearby points in S to be spacelike separated. A typical example is the so-called time slice at $t = 0$. In a Minkowski space with metric $ds^2 = dt^2 - d\vec{x}^2$ so that the metric induced at $t = 0$ is $ds^2 = -d\vec{x}^2$, a Euclidean one. In addition to the above, S needs to be *achronal*. This means there is no timelike path in M connecting distinct points $p, q \in S$. If S is of codimension 1 in M, it follows that there is no causal path from p to q. A curve γ in M passing through $r \in M$ is called extendible, since can be indefinitely made into past and future. After remove $r \in M$, γ splits in two causal curves, γ_1 and γ_2, in which γ_1 is inextendible in the future and γ_2 is inextendible in the past.

Definition 2.1.7 *A Cauchy hypersurface or initial value hypersurface in M is an achronal spacelike hypersurface S with the property that if p is a point in M which is not in S then every inextendible causal path γ through p intersects S.*

Definition 2.1.8 *A spacetime M with a Cauchy hypersurface S is said to be globally hyperbolic.*

If p is to the future of S, so every sufficiently extended past going causal path through p meets S. What one will observe at p can be predicted from a knowledge of what there was on S. An inextendible causal path $\gamma \subset M$ will always intersect S in precisely one point. Since S is achronal there is no such p, $p' \in S$ that γ intersects, because it would imply the existence of a path connecting p and p' in S, which is not possible.

If p is a point just slightly to the future of $q \in S$, then any inextendible causal path through p will meets S. This is because a very small neighborhood of q can be approximated by a small open set in Minkowski space, with S approximately the spacelike hyperplane $t = 0$.

Definition 2.1.9 *Domain of dependence $\mathcal{D}_S$ of S consist of all points $p \in M$ with the property that every inextendible causal curve through p meets S.*

The domain of dependence is the largest region in M in which the physics can be predicted from a knowledge of initial conditions on S. The Cauchy hypersurface S

divides $\mathcal{D}_S$ into a *future* and *past*, which are known as the future and past domains of dependence of S, denoted $\mathcal{D}_s^+$ and $\mathcal{D}_s^-$. The boundary of the closure of $\mathcal{D}_S$ is called the Cauchy horizons, H_S.

2.2 Quantum Theory

Now, we turn our attention to the mathematical foundations of quantum theory. The physics is left aside, and we will concern just with the notions of bounded and unbounded operators, conditions on the metric, Hermitian operators and other concepts which are widely used to define the physical quantities. The construction of bounded and unbounded operators is based on Bogolubov et al. [13, chap. 1].

Consider first a *complex linear space* $\mathcal{X}$ and a function ω that associates a pair of elements in $\mathcal{X}$ to a complex number. The function ω is a hermitian form if the following holds

$$\omega(u,v) = \overline{\omega(u,v)}. \tag{2.2}$$

The function $\omega(u,v)$ defines a scalar product of $u, v \in \mathcal{X}$ if it is hermitian and non-degenerate, i.e., if $\omega(u,v) = 0 \,\forall\, v \in \mathcal{X}$ then $u = 0$. The scalar square of $\omega(u,u) \,\forall\, u \in \mathcal{X}$ is a real number. The ω is called non negative-definite if the scalar square of any vector is non-negative

$$\omega(u,u) \geq 0, \quad \forall u \in \mathcal{X}, \tag{2.3}$$

and positive definite if also holds

$$\omega(u,u) = 0, \quad \text{only if} \quad u = 0. \tag{2.4}$$

A space $\mathcal{X}$ with a positive-definite hermitian form $\omega(u,v)$ is called a complex *pre-Hilbert space*. Every pre-Hilbert space is a normed space with the norm

$$||u|| = \sqrt{\omega(u,u)}. \tag{2.5}$$

A complete (every fundamental sequence on a normed space converges) pre-Hilbert space $\mathcal{H}$ is called a Hilbert space.

For Φ and $\Psi \in \mathcal{X}$, these vectors are said to be orthogonal if their scalar product (Φ, Ψ) is zero. Let X be a subset on a Hilbert space $\mathcal{H}$, X is total if the linear span of X is everywhere dense in $\mathcal{H}$, i.e. the closure of this linear span is the whole of

$\mathcal{H}$. Every closed linear subspace $\mathcal{H}_1$ of a Hilbert space $\mathcal{H}$ is itself a Hilbert space.

The set

$$\mathcal{H}_1^{\perp} = \{\Phi \in \mathcal{H} : (\Phi, \Psi) = 0 \,\forall\, \Psi \in \mathcal{H}_1\} \tag{2.6}$$

is called the orthogonal complement of $\mathcal{H}_1$. The following holds

1. $\mathcal{H}_1^{\perp}$ is a closed complement of $\mathcal{H}_1$,
2. $\mathcal{H}_1 \bigcap \mathcal{H}_1^{\perp} = 0$,
3. Every vector on $\mathcal{H}$ can be uniquely represented in the form $\Phi = \Phi_1 + \Phi_2$ where $\Phi_1 \in \mathcal{H}_1, \Phi_2 \in \mathcal{H}_1^{\perp}$.

It follows that the Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ in the form

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}. \tag{2.7}$$

In the decomposition $\Phi = \Phi_1 + \Phi_2$, the vectors on the right-hand side are called the (orthogonal) projections of the vector Φ onto the subspace $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ respectively.

Consider a linear operator A defined on a certain dense linear *domain* D_A of a Hilbert space $\mathcal{H}$ and taking values in $\mathcal{H}$

$$\mathcal{H} \supset D_A \ni \Phi \to A\Phi \in \mathcal{H}. \tag{2.8}$$

The operator A is said to be bounded (in D_A) if the square of the norm

$$||A\Phi||^2 = (A\Phi, A\Phi) \tag{2.9}$$

is bounded when $\Phi \in D_A$ and $||\Phi|| \leq 1$.

The least upper bound of $||A\Phi||$ as Φ runs through the intersection of D_A and the unit sphere is called the norm of the operator A and is denoted by $||A||$

$$||A|| = \sup_{||\Phi||=1,\, \Phi \in D_A} ||A\Phi||. \tag{2.10}$$

If in the above right-hand side is going to infinity, then the operator is called unbounded.

Bounded operators can be defined everywhere on $\mathcal{H}$ and we do not refer to their domain without loss of generality. For unbounded operators we cannot do

the same, it is essential to indicate their domain.

Definition 2.2.1 *The graph $\Gamma(A)$ of an operator A is the set of all pairs $(\Phi, A\Phi)$, where $\Phi \in D_A$ and $A\Phi \in \mathcal{H}$.*

In addition, the graph is a closed subset of $\mathcal{H} \oplus \mathcal{H}$. The domain D_A of a closed and unbounded operator A is everywhere dense in $\mathcal{H}$. The operator B is an extension of A (A $\subset B$) if $\Gamma(A) \subset \Gamma(B)$, i.e., if $D_A \subset D_B$ and $A\Phi = B\Phi \,\forall\, \Phi \in D_A$.

A naive way to define the adjoint operator is

$$(\Psi, A\Phi) = (A^*\Psi, \Phi), \tag{2.11}$$

without reference to the domain of A and A^*, which is fine for A bounded, and we can suppose in this case $D_A = \mathcal{H}$ and furthermore that the adjoint exist and is defined on the whole of $\mathcal{H}$.

In the case of unbounded operators, the adjoint operator does not always exist. If D_{A^*} is dense in $\mathcal{H}$, then we say that the adjoint operator A^* exists. A necessary and sufficient condition for an operator A to possess and adjoint is that A should have a closure in $\mathcal{H}$.

If $\{\Phi_n\}$ is a convergent sequence of vectors in D_A then $\{A\Phi_n\}$ either converges or has no accumulation point in $\mathcal{H}$. Strictly speaking, it means that the possibility of two different subsequences converging to different limits is excluded, and in this case the closure of A is equal to A^{**}. An operator A is said to be symmetric or Hermitian if $A \subset A^*$, that is, if

$$(A\Psi, \Phi) = (\Psi, A\Phi), \quad \Phi, \Psi \in D_A. \tag{2.12}$$

If $A = A^*$ (further, D_A and D_{A^*} are equal) then we say the operator is self-adjoint.

A linear operator U defined on the whole of a Hilbert space $\mathcal{H}$ and with range coinciding with $\mathcal{H}$ is said to be unitary if it preserves the scalar product

$$(U\Phi, U\Psi) = (\Phi, \Psi), \quad \Phi, \Psi \in \mathcal{H}. \tag{2.13}$$

Definition 2.2.2 *An operator E such that $E = E^* = E^2$ is called projector.*

Clearly E behaves like the identity operator in $E\mathcal{H}$ and like the zero operator in $(1 - E)\mathcal{H}$. Also $E\mathcal{H}$ and $(1 - E)\mathcal{H}$ are mutually orthogonal closed subspaces in $\mathcal{H}$.

A nice realization of the above operators construction on a Hilbert space is that it can be viewed as an algebra of operators. The realization of an algebra

of operators as $C^* - algebra$ is given by Rejzner [11], Bertozzini [14], Haag [7] and also the applications to non-commutative geometry, quantum gravity theory, particle physics and deformation quantization are therein contained.

Definition 2.2.3 *Define a complex unital algebra $\mathcal{U}$ if there exists an element $\mathbb{1} \in \mathcal{U}$ such that $\mathbb{1}A = A\mathbb{1} = A$ for all $A \in \mathcal{U}$.*

Also the algebra $\mathcal{U}$ is abelian (commutative) if $ab = ba$ for all $a, b \in \mathcal{U}$.

Definition 2.2.4 *A complex algebra $\mathcal{U}$ is called an involutive algebra (or a $* - algebra$) if it is equipped with an involution, i.e. a conjugate linear map $* : \mathcal{U} \to \mathcal{U}$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{U}$.*

One can also introduce the norm concept to the algebraic setting as a powerful tool to construct the topology.

Definition 2.2.5 *A normed algebra $\mathcal{U}$ is a normed vector space whose norm $||\cdot||$ and any $A, B \in \mathcal{U}$ vectors satisfies*

$$||AB|| \leq ||A||||B||. \tag{2.14}$$

If $\mathcal{U}$ is unital, then it is a normed algebra if in addition $||\mathbb{1}|| = 1$.

Definition 2.2.6 *A Banach space is a normed vector space equipped with the norm-induced topology that is complete with respect to this topology. A Banach (unital) algebra is a Banach space and a normed (unital) algebra with respect to the same norm.*

Definition 2.2.7 *An involutive complex unital algebra $\mathcal{U}$ is called a $C^* - algebra$ if $\mathcal{U}$ is a Banach space with a norm $a \mapsto ||a||$ such that $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$, for all $a, b \in \mathcal{U}$.*

As defined, a Hilbert space $\mathcal{H}$ is a complex vector space with a Hermitian inner product such that the norm induced by this product makes $\mathcal{H}$ into a Banach space. With the definitions above, it is clear the realization that a subset of a Hilbert space can be raised to an algebraic subspace. But one must care with the topology induced, specially when operators are under concern.

The set of all bounded linear operators acting in a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Also, follows that the space of bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space forms a $C^* - algebra$. The most evident topology in $\mathcal{H}(\mathcal{B})$ is provided by the norm of operators.

Given $\mathcal{B}(\mathcal{H})$ a normed linear space, the *norm* or *uniform topology* is determined by specifying a $\epsilon - neighborhood$ of $A, B \in \mathcal{B}(\mathcal{H})$ such that $||A - B|| < \epsilon$. The closure of a set $S \subset \mathcal{B}(\mathcal{H})$ in this topology means that we add to S all elements which are limits of uniformly converging Cauchy sequences in S. A nice fact is that if a $* - alegbra$ is uniformly closed then is called $C^* - algebra$.

The *strong topology* on $\mathcal{B}(\mathcal{H})$ is defined in terms of *seminorms*. Pick an arbitrary vector $\Psi \in \mathcal{H}$ and a sequence of operators A_n, then one have the vector norm $||A_n\Psi||$. The sequence is strongly convergent if for every $\Psi \in \mathcal{H}$ the sequence of vectors $\Psi_n = A_n\Psi$ is strongly convergent, i.e., if $||\Psi_n - \Psi_m|| \to 0$ as $n, m \to \infty$.

The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is obtained via a system of seminorms $||(\Phi, A\Psi)||$. For a sequence of operators A_n, the weak operator topology is established if all matrix elements $(\Phi, A_n\Psi)$ between arbitrary vectors state converges.

Definition 2.2.8 *A weakly closed $* - subalgebra$ of $\mathcal{B}(\mathcal{H})$ which contains the unit operator will be called a von Neumann algebra, denoted by $\mathcal{R}$.*

Definition 2.2.9 *For any subset $S \subset \mathcal{B}(\mathcal{H})$, is the commutant*

$$S' = \{x \in \mathcal{B}(\mathcal{H}) : sx = xs \, \forall \, s \in S\}, \tag{2.15}$$

the set of all bounded operators on a Hilbert space $\mathcal{H}$ commuting with elements of S.

If S is a self-adjoint subset, i.e., S is a $* - subalgebra$, then S' is a $C^* - algebra$ of $\mathcal{B}(\mathcal{H})$ that is closed.

Definition 2.2.10 *A von Neumann algebra $\mathcal{R}$ on a Hilbert space $\mathcal{H}$ is a $* - subalgebra$ of $\mathcal{B}(\mathcal{H})$, such that*

$$\mathcal{R} = (\mathcal{R}')' = \mathcal{R}''. \tag{2.16}$$

The below theorem is known as the *bicommutant theorem*.

Theorem 2.2.1 *Let $\mathcal{R}$ be unital $* - subalgebra$ of $\mathcal{B}(\mathcal{H})$, then following conditions are equivalent:*

- $\mathcal{R} = \mathcal{R}''$,
- *$\mathcal{R}$ is closed on the weak operator topology,*
- *$\mathcal{R}$ is closed on the strong operator topology.*

Denoting the *smallest* von Neumann algebra containing $\mathcal{R}_1$ and $\mathcal{R}_2$ by $\mathcal{R}_1 \vee \mathcal{R}_2$ and $\mathcal{R}_1 \wedge \mathcal{R}_2$ the *largest* von Neumann algebra containing $\mathcal{R}_1$ and $\mathcal{R}_2$. Thus one has

$$\mathcal{R}_1 \vee \mathcal{R}_2 = (\mathcal{R}_1 \cup \mathcal{R}_2)'', \tag{2.17}$$

and

$$(\mathcal{R}_1 \vee \mathcal{R}_2)' = \mathcal{R}_1' \wedge \mathcal{R}_1' \wedge \mathcal{R}_2' = \mathcal{R}_1' \cap \mathcal{R}_2'. \tag{2.18}$$

At this point, we consider states and representations. Define a *state* ω which associates to an observable A a real number $\omega(A)$ obtained by averaging the results of measurements of A for state prepared to be in the state ω. In the following, one can assume as the algebraic approach to *quantum theory*:

- A physical system is defined by its unital $C^* - algbera\ \mathcal{U}$,
- States are identified with positive normalized linear functional on $\mathcal{U}$, i.e., $\omega(A^*A) \geq 0$ for all $A \in \mathcal{U}$ and $\omega(\mathbb{1}) = 1$.

Definition 2.2.11 *A state on an involutive algebra $\mathcal{U}$ is a linear functional ω, such that*

$$\omega(A^*A) \geq 0, \quad \omega(\mathbb{1}) = 1. \tag{2.19}$$

Observables are self adjoint elements of $\mathcal{U}$ and possible measurement results for an observable A are characterized by its spectrum $\sigma(A)$.

Definition 2.2.12 *The spectrum $\sigma(A)$ of $A \in \mathcal{U}$ is the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda\mathbb{1})$ has no inverse in $\mathcal{U}$.*

Abstract elements of an involutive algebra $\mathcal{U}$ are realized as operators on some Hilbert space by a choice of a representation, [see 15, chap. 2].

Definition 2.2.13 *A representation of an involutive unital algebra $\mathcal{U}$ is a unital $* -$ homomorphism π into the algebra of linear operators on a dense subspace K of a Hilbert space $\mathcal{H}$.*

In particular, a representation of a $C^* - algebra\ \mathcal{U}$ is a unital $* - homomorphism$ $\pi : \mathcal{U} \to \mathcal{B}(\mathcal{H})$. A representation is called faithful if $Ker(\pi) = \{0\}$. It is called irreducible if there are no non-trivial subspaces of $\mathcal{H}$ invariant under $\pi(\mathcal{U})$.

Definition 2.2.14 *Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a $C^* - algebra\ \mathcal{U}$ are called unitarily equivalent, if $U\pi_1(A) = \pi_2(A)U$ holds for all $A \in \mathcal{U}$ with some unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$.*

Let $\mathcal{H}_1$ be a subspace of $\mathcal{H}$. The subspace $\mathcal{H}_1$ is invariant, or stable, under π if $\pi(A)\mathcal{H}_1 \subseteq \mathcal{H}_1$ for all $A \in \mathcal{U}$. If $\mathcal{H}_1$ is a closed subspace of $\mathcal{H}$ and $P_{\mathcal{H}_1}$ is the orthogonal projector with range $\mathcal{H}_1$ then the invariance of $\mathcal{H}_1$ under π implies that

$$P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1} = \pi(A)P_{\mathcal{H}_1}, \quad A \in \mathcal{U}. \tag{2.20}$$

Hence,

$$\begin{aligned} \pi(A)P_{\mathcal{H}_1} &= (P_{\mathcal{H}_1}\pi(A^*)P_{\mathcal{H}_1})^* \\ &= (\pi(A^*)P_{\mathcal{H}_1})^* \\ &= P_{\mathcal{H}_1}\pi(A), \end{aligned} \tag{2.21}$$

for all $A \in \mathcal{U}$. That means $P_{\mathcal{H}_1}$ commutes with each of the representations $\pi(A)$. Conversely, this commutation property implies that $\mathcal{H}_1$ is invariant under π. Hence, $\mathcal{H}_1$ is invariant under π if and only if

$$\pi(A)P_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(A), \quad A \in \mathcal{U}. \tag{2.22}$$

Furthermore, we may deduce that if $\mathcal{H}_1$ is invariant under π and if π_1 is defined by,

$$\pi_1(A) = P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1}, \tag{2.23}$$

then $(\mathcal{H}_1, \pi_1)$ is a representation of $\mathcal{U}$.

If $\mathcal{H}_1$ is invariant under π then its orthogonal complement $\mathcal{H}_1^{\perp}$ is also invariant. Setting $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$ one can define a second subrepresentation $(\mathcal{H}_2, \pi_2)$ by $\pi_2(A) = P_{\mathcal{H}_2}\pi(A)P_{\mathcal{H}_2}$. But $\mathcal{H}$ has a direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and each operator $\pi(A)$ decomposes as a direct sum $\pi(A) = \pi_1(A) \oplus \pi_2(A)$. Thus, we can write $\pi = \pi_1 \oplus \pi_2$ and $(\mathcal{H}, \pi) = (\mathcal{H}_1, \pi_1) \oplus (\mathcal{H}_2, \pi_2)$.

Consider the set of bounded linear operators $\mathcal{B}(\mathcal{H})$, we say $\mathcal{B}(\mathcal{H})$ acts non-degenerately if for set of vectors $\{\Psi\}$ such that $A\Psi = 0$ for any $A \in \mathcal{B}(\mathcal{H})$ is given by the zero element $\{0\}$. An important class of such a case is the class of cyclic representations. Define a vector Ω in a Hilbert space $\mathcal{H}$ to be cyclic for a set of bounded operators $\mathcal{B}(\mathcal{H})$ if the set linear span $\{A\Omega,\ A \in \mathcal{B}(\mathcal{H})\}$ is dense in $\mathcal{H}$.

Definition 2.2.15 *A cyclic representation of a $C^* - algebra\ \mathcal{U}$ is defined to be a triple $(\mathcal{H}, \pi, \Omega)$ where $(\mathcal{H}, \Omega)$ is a representation of $\mathcal{U}$ and Ω is a vector in $\mathcal{H}$ which is cyclic for π in $\mathcal{H}$.*

Suppose the nondegenerate representation $(\mathcal{H}, \pi)$ for the $C^* - algebra\ \mathcal{U}$. Then

follows that π is the direct sum of a family of cyclic representation.

We say a set $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ is defined to be irreducible if the only invariants subspaces, closed or not, are $\{0\}$ and $\mathcal{H}$.

Proposition 2.2.1 *For the selfadjoint set of bounded operators $\mathcal{B}(\mathcal{H})$ on the Hilbert space, the following are equivalent*

1. *$\mathcal{B}(\mathcal{H})$ is irreducible,*
2. *The commutant $\mathcal{B}(\mathcal{H})'$ of $\mathcal{B}(\mathcal{H})$, consists of multiples of the identity operator*
3. *Every nonzero vector $\Psi \in \mathcal{H}$ is cyclic for $\mathcal{B}$ in $\mathcal{H}$, or $\mathcal{B} = 0$ and $\mathcal{H} = \mathbb{C}$.*

If U is a unitary operator on $\mathcal{H}$ and we introduce π_U by $\pi_U(A) = U\pi(A)U^*$ then $(\mathcal{H}, \pi_U)$ is a second representation. Two representations $(\mathcal{H}_1, \pi)$ and $(\mathcal{H}_2, \pi_2)$ are equivalent or unitarily equivalent, if there exists a unitary operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that

$$\pi_1(A) = U\pi_2(A)U^*, \tag{2.24}$$

for all $A \in \mathcal{U}$. Equivalence of π_1 and π_2 is denoted by $\pi_1 \simeq \pi_2$.

Let ω_1 and ω_2 be positive linear functionals over the $C^* - algebra\, \mathcal{U}$. It follows that $\omega_1 + \omega_2$ is a positive linear functional and $||\omega_1 + \omega_2|| = ||\omega_1|| + ||\omega_2||$. In particular, the states over $\mathcal{U}$ form a convex subset of the dual of $\mathcal{U}$. The concept of a state to be convex is clear from the following: there is a state $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ positive for $0 \leq \lambda \leq 1$ and $||\omega|| = \lambda||\omega_1|| + (1-\lambda)||\omega_2|| = 1$, also one has the property that $\omega \geq \lambda\omega_1$ and $\omega \geq (1-\lambda)\omega$. Call a state of *pure* whenever it cannot be written as a convex combination of other states.

The algebra $\mathcal{U}$ is a Banach space and with the aid of the state ω it may be converted into a pre-Hilbert space by introduction of the positive semi-definite scalar product

$$(A, B) = \omega(A^*B). \tag{2.25}$$

Define $\mathcal{I}_\omega$ by

$$\mathcal{I}_\omega = \{A;\, A \in \mathcal{U},\, \omega(A^*A) = 0\}, \tag{2.26}$$

as the left ideal of $\mathcal{U}$. Since $I \in \mathcal{I}_\omega$ and $A \in \mathcal{U}$ one have that

$$0 \leq \omega((AI)^*\, AI) \leq ||A||^2\omega(I^*I) = 0, \tag{2.27}$$

then one can conclude $AI \in \mathcal{I}_\omega$.

Now we can define the *strict pre-Hilbert space* by the equivalence class

$$\psi_A = \{\hat{A},\ \hat{A} = A + I,\ I \in \mathcal{I}_\omega\}, \tag{2.28}$$

provided with operations,

$$\begin{aligned} &\psi_A + \psi_B = \psi_{A+B}, \\ &\alpha\psi_A = \psi_{\alpha A}, \\ &(\psi_A, \psi_B) = \omega(A^* B), \end{aligned} \tag{2.29}$$

and the scalar product is independent of the particular representation class used in its definition. A nice property of the strict pre-Hilbert space is that it can be completed in a sense to be linearly embedded as a dense subspace of a Hilbert space preserving the scalar product. The completion is defined as the representation space $\mathcal{H}_\omega$. The representation $\pi_\omega(A)$ acts on the dense subspace of $\mathcal{H}_\omega$ formed by vectors ψ_B for $B \in \mathcal{U}$ in the following manner

$$\pi_\omega(A)\psi_B = \psi_{AB}, \tag{2.30}$$

which is independent of the representation used for the class ψ_B, as one can see,

$$\pi_\omega(A)\psi_{B+I} = \psi_{AB+AI} = \psi_{AB} = \pi_\omega(A)\psi_B, \tag{2.31}$$

for $I \in \mathcal{I}_\omega$. Each $\pi_\omega(A)$ is a linear operator

$$\begin{aligned} \pi_\omega(A)(\lambda\psi_B + \psi_C) &= \pi(A)\psi_{\lambda B+C} = \psi_{\lambda AB+AC}, \\ &= \lambda\psi_{AB} + \psi_{AC}, \\ &= \lambda\pi_\omega(A)\psi_B + \pi_\omega(A)\psi_C. \end{aligned} \tag{2.32}$$

Also $\pi_\omega(A)$ has a bounded closure

$$\begin{aligned} ||\pi_\omega(A)\psi_B||^2 &= (\psi_{AB}, \psi_{AB}), \\ &= \omega(B^* A^* AB), \\ &\leq ||A||^2 \omega(B^* B), \\ &= ||A||^2 ||\psi_B||^2. \end{aligned} \tag{2.33}$$

If $\mathcal{U}$ contains the identity we define Ω_ω by

$$\Omega_\omega = \psi_{\mathbb{1}}, \tag{2.34}$$

and this gives the correct identification of ω:

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (\psi_{\mathbb{1}}, \psi(A)) = \omega(A). \tag{2.35}$$

The set $\{\pi_\omega(A)\Omega_\omega,\ A \in \mathcal{U}\}$ is exactly the dense set of equivalence classes $\{\psi_A,\ A \in \mathcal{U}\}$ and hence Ω_ω is cyclic for $(\mathcal{H}_\omega, \pi_\omega)$. By construction the set $\pi_\omega(\widetilde{\mathcal{U}})\Omega_\omega = \pi_\omega(\mathbb{C}\mathbb{1} + \mathcal{U})\Omega_\omega$ is dense and thus the cyclicity of Ω_ω, for $\pi_\omega(\mathcal{U})$, follows if Ω_ω is in the closure of $\pi_\omega(\mathcal{U})\Omega_\omega$.

Finally, the *GNS construction theorem* can be announced:

Theorem 2.2.2 *Let ω be a state over the $C^* - algebra\ \mathcal{U}$. It follows that exists a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{U}$ such that*

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega), \tag{2.36}$$

for all $A \in \mathcal{U}$ and consequently, $||\Omega_\omega||^2 = ||\omega|| = 1$. Moreover, the representation is unique up to unitary equivalence.

For the proof of uniqueness consider a second representation $(\Omega'_\omega, \pi'_\omega\Omega'_\omega)$ such that

$$\omega(A) = (\Omega'_\omega, \pi'_\omega(A)\Omega'_\omega), \tag{2.37}$$

for all $A \in \mathcal{U}$, then must exist a unitary operator from $\mathcal{H}_\omega$ onto $\mathcal{H}'_\omega$ such that

$$U^{-1}\pi'_\omega(A)U = \pi_\omega(A), \tag{2.38}$$

for all $A \in \mathcal{U}$, and

$$U\Omega_\omega = \Omega'_\omega. \tag{2.39}$$

Which is established by defining U through

$$U\pi_\omega(A)\Omega_\omega = \pi'_\omega(A)\Omega'_\omega, \tag{2.40}$$

and by noticing that

$$\begin{aligned}(U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega) &= (\pi'_\omega(A)\Omega'_\omega, \pi'_\omega(B)\Omega'_\omega),\\ &= \omega(A^*B),\\ &= (\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega).\end{aligned} \tag{2.41}$$

Thus U preserves the scalar product. Also, the closure of U is unitary and has all desired algebraic properties.

2.3 Local Quantum Field Theory

How is it possible to develop an entirely new theory that unifies the previous two sections? The first section focuses on causality and the way interactions can be described, while the second deals with measurements, observables, and a space known as Hilbert space.

Motivated by the idea that a measurement in one spacetime region cannot influence a measurement in another spacetime region that is causally separated from the former, one seeks to construct an algebra of operators that aligns with the spacetime framework introduced above.

The first *naïve* assumption one might make is that the algebra of operators associated with the Cauchy initial data region is the same as the algebra associated with its domain of dependence [see 16]. Additionally, it is important to consider only bounded operators, ensuring that they can be multiplied without any complications and are well-defined on the entire Hilbert space. Otherwise, as previously discussed, unbounded operators may be defined on different dense subspaces of the Hilbert space, leading to inconsistencies.

Worth to recall the assumption that, in a general spacetime, one can always consider the neighborhoods of a point to looks like the Minkowski spacetime (locality principle or just manifold property, partition of unity, if one prefer). Consider a double cone $D_x^y \equiv \mathcal{O} \subset \mathbb{R}^d$. Introduce a quasi-local $C^* - algebra\ \mathcal{U}$, and assign to each (nonempty) double cone $\mathcal{O} \subset \mathbb{R}^d$ a $C^* - algebra\ \mathcal{U}(\mathcal{O}) \subset \mathcal{U}$ which are called the *local algebra*. The collection (net) of local algebras must satisfy the following *axioms*:

1. Generating property: $\mathcal{U} = \overline{\cup_{\mathcal{O}}\mathcal{U}(\mathcal{O})}^{||.||}$ where the union is over the set of all double cones, i.e., *inductive limit*. The overline is the completion in the norm topology,

2. Isotony: for any pair of double cones $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{U}(\mathcal{O}_1) \subset \mathcal{U}(\mathcal{O}_2)$,

3. Causality: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated, then $[\mathcal{U}(\mathcal{O}_1), \mathcal{U}(\mathcal{O}_2)] = \{0\}$,

4. Poincare covariance: there is a (norm) continuous linear representation α_g of the Poincare group $\mathcal{P}_+^\uparrow$ in $\mathcal{U}$, such that $\alpha_g(\mathcal{U}(\mathcal{O})) = \mathcal{U}(g\mathcal{O})$ for any open bounded region $\mathcal{O}$ and all $g \in \mathcal{P}_+^\uparrow$, where the action of $g \in \mathcal{P}_+^\uparrow$ over a region $\mathcal{O}$ is given by $g\mathcal{O} = \{\Lambda x + a; x \in \mathcal{O}\}$,

5. Vacuum: there is a pure state ω in $\mathcal{U}$ invariant under all α_g. Then in its GNS representation $(\pi, \mathcal{H}, \Omega)$ the linear representation α_g is implemented by a positive energy unitary representation of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$ in the sense of $U(g)\pi(A)U^*(g) = \pi(\alpha_g(A))$ for all $A \in \mathcal{U}$ and all $g \in \mathcal{P}_+^\uparrow$ (the notion of positive energy is to be given below),

6. Time slice: the algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region. This physically correspond to the well-posedness of an initial value problem, i.e., we only need to determine observables in some small time interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ to reconstruct the full algebra.

Positive energy means that the representation is strongly continuous and the infinitesimal generators P^μ of the transformation subgroup (i.e., $U(0, a) = e^{ip^\mu a_\mu}$) have their spectral projections on the closed forward light cone $\widetilde{V}_+ = \{p \in \mathbb{R}^d; p \cdot p > 0 \,\text{and}\, p^0 > 0\}$.

As usual, since we have a unique and invariant state, we want to consider the collection of $C^* - algebras$ $\pi(\mathcal{U}(\mathcal{O})) \subset \mathcal{B}(\mathcal{H})$ acting on the vacuum Hilbert space.

It is important, as discussed before, to define the topology of operators. We want some statistical interpretation such that considering a physical quantity, for their *n measurements*, we associate a sequence $a_1, a_2, \ldots a_n$ of operators in which the matrix $(\psi, a_n\xi)$ converge to the corresponding matrix element $(\psi, a\xi)$ of the operator a. Then it is of interest to incorporate the von Neumann algebra, which takes the weak operator topology. Consider the von Neumann algebra on $\mathcal{O}$ such that $\mathcal{R}(\mathcal{O}) = \pi(\mathcal{U}(\mathcal{O}))''$. The double prime corresponds to the double commutant which coincides with the weak closure.

Theorem 2.3.1 *(Reeh-Schlieder) The vacuum vector Ω is cyclic for any algebra $\pi(\mathcal{U}(\mathcal{O}))''$ corresponding to any $(nonempty)$ open region. Moreover, if $\mathcal{O}'$ is also open and nonempty, then Ω is also separating for $\pi(\mathcal{U}(\mathcal{O}))''$.*

We can state the above *Reeh-Schlieder* theorem in terms of local Cauchy data [16]. Consider a restriction to an arbitrary small open set $\mathcal{V} \subset \Sigma$, for a complete Cauchy surface Σ. Also consider a small neighborhood $\mathcal{U}_\mathcal{V}$ of $\mathcal{V}$ in spacetime. Then restrict the states to be created from a vacuum in $\mathcal{U}_V$ still suffices to generate the same $\mathcal{H}_0$ that the vacuum sector of Hilbert space. Assume $\overline{\mathcal{V}} \subset \Sigma$ to be the closure of the set $\mathcal{V}$ and $\mathcal{V}'$ disjoint of $\mathcal{V}$. The Reeh-Schlieder theorem applies equally well to $\mathcal{V}$ or to $\mathcal{V}'$, as they are both nonempty sets in the initial value surface Σ.

If an operator a supported in $\mathcal{U}_\mathcal{V}$ annihilates the vacuum, then also annihilates any vector state created from the vacuum which is supported in $\mathcal{U}_{\mathcal{V}'}$, but theses are dense in $\mathcal{H}_0$. The consequence is that a is separating for both algebras in $\mathcal{V}$ and $\mathcal{V}'$. The theorem then says if a state is cyclic for one of these algebras then is separating for the other, so one have that the vacuum is cyclic and separating for both algebras defined in $\mathcal{V}$ and $\mathcal{V}'$.

Now it is of interest to see how one can define hermitian scalar field, or even better, distribution operator-valued defined on spacetime regions. First let us define the space of Schwartz functions which will delimit where the operator is defined.

Definition 2.3.1 *Denote $S(\mathbb{R}^d, \mathbb{R})$ the space of test functions, i.e., the Schwartz space of real, smooth and exponentially decreasing functions at infinity.*

Next is defined the *one-particle Hilbert space*. Consider the d-tuples $x = (x^0, \mathbf{x})$ and $p = (p^0, \mathbf{p})$.

Definition 2.3.2 *The Hilbert space* $\mathbf{h}$ *of one-particle states of mass $m > 0$ and zero spin is made up of the square integrable functions on the mass shell hyperboloid $H_m = \{p \in \mathbb{R};\quad p^2 = m^2,\quad p^0 > 0\}$ with the invariant measure $d\mu(p) = \Theta(p^0)\delta(p^2 - m^2)d^d p$ where Θ is the usual Heaviside step function and δ is the usual Dirac delta distribution.*

The one-particle Hilbert space and the scalar product can be defined respectively as follows,

$$\begin{aligned} \mathbf{h} &= L^2\left(\mathbb{R}^{d-1}, \frac{d^{d-1}p}{\omega}\right), \\ < f, g >_{\mathbf{h}} &= \int_{\mathbb{R}^{d-1}} \frac{d^{d-1}p}{\omega} f^*(\vec{p}) g(\vec{p}), \end{aligned} \tag{2.42}$$

the dispersion relation $\omega = \sqrt{\vec{p}^2 + m^2}$ is defined and always hold.

The Schwartz space carries a representation of the Poincare group $\mathcal{P}_+^\uparrow$ given by $f \mapsto f_{(\Lambda,a)} \in \mathcal{P}_+^\uparrow$, with $f_{(\Lambda,a)}(x) = f(\Lambda(x-a))$ for any $(\Lambda, a) \in \mathcal{P}_+^\uparrow$. The one particle Hilbert space carries a unitary representation of $\mathcal{P}_+^\uparrow$ given by $(u(\Lambda,a)f)(p) = e^{ipa} f(\Lambda^{-1}p)$ for any $f \in \mathbf{h}$ and $(\Lambda, a) \in \mathcal{P}_+^\uparrow$.

Definition 2.3.3 *Denote now $\mathcal{H}$ the Fock Hilbert space as the direct sum of the symmetric tensor powers of one-particle Hilbert space* $\mathbf{h}$*:*

$$\mathcal{H} = \oplus_{n=0}^{\infty} \mathbf{h}^{\otimes n, sym}. \tag{2.43}$$

For each $f \in \mathbf{h}$ define operator $a(f)$ and $a^*(f)$, on $\mathcal{H}$ by initially setting $a(f)\psi^{(0)} = 0$, $a^*(f)\psi^{(0)} = f$, where $\psi^{(0)} \in \mathbf{h}^0 = \mathbb{C}$ and

$$\begin{aligned} a(f)(f_1 \otimes f_2 \otimes \ldots \otimes f_n) &= n^{1/2} < f, f_1 > f_2 \otimes f_3 \otimes \ldots \otimes f_n, \\ a^*(f)(f_1 \otimes f_2 \otimes \ldots \otimes f_n) &= (n+1)^{1/2} f \otimes f_1 \otimes f_2 \otimes \ldots \otimes f_n. \end{aligned} \tag{2.44}$$

Let $\Omega = (1, 0, 0, \ldots)$ be the zero-particle state called the vacuum state. The vectors

$$\psi(f) = a^*(f)\Omega, \tag{2.45}$$

identify with elements of the one-particle space $\mathbf{h}$ and hence $a^*(f)$ *creates* a particle in the state f. Similarly, $a(f)$ reduces the number of particles.

The rest of the section will be devoted to describe the *first* and *second quantization map* [6]. The former map means the *canonical commutation relations* (*CCR*) defined with respect to the creation and annihilation operators. The latter map is in the sense of canonical commutation relations of the scalar hermitian fields and its conjugated momentum. A nice example on what will be defined is describe in the following.

Example. Consider $\mathbf{h} = L^2(\mathbb{R}^2)$, then $\mathcal{H}(\mathbf{h})$consists of sequences $\{\psi^{(n)}\}_{n\geq 0}$ of functions of n variables $x_i \in \mathbb{R}^d$ totally symmetric. The action of the annihilation and creation operators is given by

$$\begin{aligned} (a(f)\psi)^n(x_1, \ldots, x_n) &= (n+1)^{1/2} \int dx \overline{f(x)} \psi^{(n+1)}(x, x_1, \ldots, x_n), \\ (a^*(f)\psi)^n(x_1, \ldots, x_n) &= (n)^{-1/2} \sum_{i=1}^{n} f(x_i) \psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n), \end{aligned} \tag{2.46}$$

where $\hat{x}_i$ denotes the variable to be omitted. Note the maps

$$f \mapsto a(f), \quad f \mapsto a^*(f), \tag{2.47}$$

are antilinear and linear, respectively. One may also introduce operator-valued distributions, i.e., fields $a(x)$ and $a^*(x)$, such that

$$a(f) = \int dx\, \overline{f(x)} a(x), \quad a^*(f) = \int dx\, f(x) a^*(x), \tag{2.48}$$

and the action is given by

$$\begin{aligned} &(a(x)\psi)^{(n)}(x_1, \ldots, x_n) = (n+1)^{1/2} \psi^{(n+1)}(x, x_1, \ldots, x_n), \\ &(a^*(x)\psi)^{(n)}(x_1, \ldots, x_n) = (n)^{-1/2} \sum_{i=1}^{n} \delta(x - x_i)\, \psi^{(n-1)}(x_1, \ldots, \hat{x}_i, \ldots, x_n). \end{aligned} \tag{2.49}$$

The first quantization map will be related to the first CCR identities and the second quantization map will be related to the second CCR identities above.

The assignment $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ is determined by the composition of two different maps

$$\begin{aligned} &\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) \subset \mathbf{h}, \\ &K \subset \mathbf{h} \to \mathcal{R}(K) \subset \mathcal{B}(\mathcal{H}), \end{aligned} \tag{2.50}$$

called the first and second quantization maps, respectively. A region $\mathcal{O}$ is called causally complete if $\mathcal{O} \equiv \mathcal{O}''$.

For any closed linear subspace $K \subset \mathbf{h}$ we define its symplectic complement as

$$K' = \{h \in \mathbf{h}, \quad Im < h, k >_{\mathbf{h}} = 0, \quad \forall\, k \in K\}. \tag{2.51}$$

Consider the real dense embedding $E : S(\mathbb{R}^d, \mathbb{R}) \to \mathbf{h}$ as follows

$$(Ef)(\vec{p}) = (2\pi)^{\frac{1}{2}} \hat{f}\Big|_{H_m}(\vec{p}) = (2\pi)^{\frac{1}{2}} \hat{f}(p^0, \vec{p}), \tag{2.52}$$

where $\hat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{ip\cdot x} d^d x$ is the usual Fourier transform. Such embedding is Poincaré invariant, i.e. $E(f_{(\Lambda,a)}) = u(\Lambda, a) E(f)$. Now, one can use the above embedding to associate functions on the Schwartz functions $S(\mathbb{R}^d, \mathbb{R})$ with vectors on the one-particle Hilbert space.

The first quantization map is an assignment $\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) \subset \mathbf{h}$, where

$K(\mathcal{O})$ is a real closed linear subspace. Define the map action by

$$\mathcal{O} \subset \mathbb{R}^d \to K(\mathcal{O}) = \overline{\{E(f) : f \in S(\mathbb{R}^d, \mathbb{R}),\, supp(f) \subset \mathcal{O}\}} \subset \mathbf{h}. \tag{2.53}$$

Following, we define the embedding $W : \mathbf{h} \to \mathcal{B}(\mathcal{H})$ by the Weyl unitaries $W(h)$

$$W(h) = e^{i(a(h)+a^*(h))}, \tag{2.54}$$

which satisfies the CCR

$$W(h_1)W(h_2) = e^{-i\, Im<h_1,h_2>_{\mathbf{h}}} W(h_1 + h_2), \tag{2.55}$$

$$W(h)^* = W(-h). \tag{2.56}$$

A Poincare unitary $U(\Lambda, a)$ acts covariant on a Weyl operator according to

$$U(\Lambda, a)W(h)U(\Lambda, a)^* = W(u(\Lambda, a)h), \tag{2.57}$$

$$W(h)\Omega = e^h. \tag{2.58}$$

The second quantization map is an assignment $K \subset \mathbf{h} \to \mathcal{R}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$, for the set of real closed linear subspace of $\mathbf{h}$ to the set of von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. It is defined by

$$K \in \mathbf{h} \to \mathcal{R}(K) = \{W(k),\, k \in K\}'' \subset \mathcal{B}(\mathcal{H}). \tag{2.59}$$

The net of local algebras $\mathcal{O} \subset \mathbb{R}^d \to \mathcal{R}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ of the free Hermitian scalar field is defined as the composition of the first and second quantization map, i.e.,

$$\mathcal{R}(\mathcal{O}) = \mathcal{R}(K(\mathcal{O})), \tag{2.60}$$

which satisfies the previous section axioms and also the *Haag duality*, [see 7]:

$$K(\mathcal{O}') = K(\mathcal{O})', \tag{2.61}$$

$$\mathcal{R}(K') = \mathcal{R}(K)'. \tag{2.62}$$

For $f \in S(\mathbb{R}^d, \mathbb{R})$, the field operator $\phi(f)$ is defined through the relation

$$W(E(f)) = e^{i\phi(f)} = W(f). \tag{2.63}$$

Now we want to describe the initial conditions on the Cauchy data surface,

and it is done by the time slice axiom described before. Suppose a time slice at a diamond, in special at $t = 0$ time. Also, it is of interest to provide the statement that the local algebra at a time slice is the same in its domain of dependence.

First decompose the one-particle Hilbert space into two $\mathbb{R}-$linear closed subspace $\mathbf{h} = \mathbf{h}_\phi \oplus_{\mathbb{R}} \mathbf{h}_\pi$

$$\mathbf{h}_\phi = \{h \in \mathbf{h},\, h(\vec{p}) = h(-\vec{p})\}, \mathbf{h}_\pi = \{h \in \mathbf{h},\, h(\vec{p}) = -h(-\vec{p})\}. \tag{2.64}$$

By construction, each $h \in \mathbf{h}$ can be uniquely written as $h = h_\phi + h_\pi$

$$h_\phi(\vec{p}) = \frac{h(\vec{p}) + h^*(-\vec{p})}{2}, \quad h_\pi(\vec{p}) = \frac{h(\vec{p}) - h^*(\vec{p})}{2}. \tag{2.65}$$

Also holds the following useful relations

$$Im < h_\phi, h'_\phi >= Im < h_\pi, h'_\pi >= Re < h_\phi, h_\pi >= 0, \tag{2.66}$$

for all $h_\phi, h'_\phi \in \mathbf{h}_\phi$ and $h_\pi, h'_\pi \in \mathbf{h}_\pi$.

Consider the real dense embedding $E_{\phi,\pi} : S(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathbf{h}_{\phi,\pi}$, such that

$$(E_\phi f) = \hat{f}(\vec{p}), \quad \text{and} \quad (E_\pi f)(\vec{p}) = ip^0\, \hat{f}(\vec{p}), \tag{2.67}$$

where $\hat{f}(\vec{p}) = (2\pi)^{-\frac{d-1}{2}} \int_{\mathbb{R}^{d-1}} f(\vec{x}) e^{-i\vec{p}\cdot\vec{x}}\, d^{d-1}x$.

From now on, identify functions on $S(\mathbb{R}^{d-1}, \mathbb{R})$ with vectors on $\mathbf{h}_\phi, \mathbf{h}_\pi$ through these embeddings. The map E_ϕ, (or E_π) is actually defined on a bigger class of test functions, namely $H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$ [or $H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})$], i.e.,

$$E_\phi : H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathbf{h}_\phi, \tag{2.68}$$

$$E_\pi : H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}) \to \mathbf{h}_\pi, \tag{2.69}$$

where $H^\alpha(\mathbb{R}^{d-1}, \mathbb{R})$ is the real Sobolev space of order α. We have that actually when defined on the whole Sobolev space it generates the whole one-particle Hilbert space, $E_\phi(H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathbf{h}_\phi$ and $E_\pi(H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R})) = \mathbf{h}_\pi$. For each $h_\phi \in \mathbf{h}_\phi$ and $h_\pi \in \mathbf{h}_\pi$ and using $W(h) = e^{i(a(h)+a^*(h))}$, we define the Weyl unitaries again

$$W_\phi(h_\phi) = W(h_\phi), \quad W(h_\pi) = W(h_\pi), \tag{2.70}$$

which satisfies the CCR in the Weyl form

$$W_\phi(h_\phi)W_\pi(h_\pi)W_\phi(h'_\phi)W_\pi(h'_\pi) = W_\phi(h_\phi + h'_\phi)W_\pi(h_\pi + h'_\pi)e^{2iIm<h'_\phi,h_\pi>_{\mathbf{h}}}, \quad (2.71)$$

$$W^*_\phi(h_\phi) = W_\phi(-h_\phi), \quad (2.72)$$

$$W^*_\pi(h_\pi) = W_\pi(-h_\pi). \quad (2.73)$$

The field operator at a fixed time $\phi(f_\phi)$ and its canonical conjugate momentum field $\pi(f_\pi)$ are defined through the formulas

$$W_\phi(E_\phi(f_\phi)) = e^{i\phi(f_\phi)} = W_\phi(f_\phi), \quad (2.74)$$

$$W_\pi(E_\pi(f_\pi)) = e^{i\pi(f_\pi)} = W_\pi(f_\pi). \quad (2.75)$$

Now to finish we define the local algebras properly and the quantization maps at fixed time.

Consider a spatially complete region $\mathcal{C} \subset \mathbb{R}^{d-1}$ and its (open) space complement $\mathcal{C}' = \mathbb{R}^{d-1} - \overline{\mathcal{C}}$. The first quantization map is given by

$$\mathcal{C} \subset \mathbb{R}^{d-1} \to K_\phi(\mathcal{C}) = \overline{\{E_\phi(f) : f \in S(\mathbb{R}^{d-1}, \mathbb{R}),\, supp(f) \subset \mathcal{C}\}} \subset \mathbf{h}_\phi, \quad (2.76)$$

$$\mathcal{C} \subset \mathbb{R}^{d-1} \to K_\pi(\mathcal{C}) = \overline{\{E_\pi(f) : f \in S(\mathbb{R}^{d-1}, \mathbb{R}),\, supp(f) \subset \mathcal{C}\}} \subset \mathbf{h}_\pi. \quad (2.77)$$

The following can be realized as well

$$K_\phi(\mathcal{C}) = \{E_\phi(f) : f \in H^{-\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}), supp(f) \subset \overline{\mathcal{C}}\}, \quad (2.78)$$

$$K_\pi(\mathcal{C}) = \{E_\pi(f) : f \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}, \mathbb{R}), supp(f) \subset \overline{\mathcal{C}}\}. \quad (2.79)$$

For the second quantization map, associates for each pair $K_\phi \subset \mathbf{h}_\phi$ and $K_\pi \subset \mathbf{h}_\pi$ of $\mathbb{R}$-linear closed subspace a von Neumann algebra

$$(K_\phi, K_\pi) \to \mathcal{R}(K_\phi, K_\pi) = \{W_\phi(k_\phi)W_\pi(k_\pi) : k_\phi \in K_\phi, k_\pi \in K_\pi\}'' \subset \mathcal{B}(\mathcal{H}). \quad (2.80)$$

The net of local algebras $\mathcal{C} \subset \mathbb{R}^{d-1} \to \mathcal{R}_0(\mathcal{C}) \subset \mathcal{B}(\mathcal{H})$ of the free Hermitian scalar field at a fixed time is then defined as the composition of the first and second quantization map, i.e.,

$$\mathcal{R}_0(\mathcal{C}) = \mathcal{R}_0(K_\phi(\mathcal{C}), K_\pi(\mathcal{C})). \quad (2.81)$$

Also, holds the following relations,

$$\begin{aligned}
&\mathcal{R}_0(\mathcal{C}_1) \subset \mathcal{C}_2, \quad \textit{if} \quad \mathcal{C}_1 \subset \mathcal{C}_2, \\
&\mathcal{R}_0(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathcal{R}_0(\mathcal{C}_1) \vee \mathcal{R}_0(\mathcal{C}_2), \\
&\mathcal{R}_0(\mathcal{C}') = \mathcal{R}_0(\mathcal{C})', \\
&\mathcal{R}_0(\mathbb{R}^{d-1}) = \mathcal{B}(\mathcal{H}).
\end{aligned} \tag{2.82}$$

Given any spatially complete region $\mathcal{C} \subset \mathbb{R}^{d-1}$, we define $\mathcal{O}_\mathcal{C} \subset \mathbb{R}^d$ its domain of dependence. Then the following relation holds,

$$K(\mathcal{C}_\mathcal{O}) = K_\phi(\mathcal{C}) \otimes_\mathbb{R} K_\pi(\mathcal{C}), \tag{2.83}$$

and hence the desired claim in the beginning of the section is now established:

$$\mathcal{R}_0(\mathcal{C}) = \mathcal{R}(\mathcal{O}_\mathcal{C}). \tag{2.84}$$

A final example is of importance.

Example. Given $f \in S(\mathbb{R}^d, \mathbb{R})$, define

$$F(x) = \int_\mathbb{R}^d \Delta(x-y) f(y) d^d y, \tag{2.85}$$

where $\Delta(x) = -i(2\pi)^{-(d-1)} \int_\mathbb{R}^d e^{ip\cdot x} \delta(p^2 - m^2) sign(p^0) d^d p$. Indeed one can see $(\Box + m^2)F = 0$ and we can take its initial Cauchy data at $x^0 = 0$ through

$$f_\phi(\vec{x}) = -\frac{\partial F}{\partial x^0}(0, \vec{x}), \quad f_\pi(\vec{x}) = F(0, \vec{x}). \tag{2.86}$$

Finally, also can be shown $f_\phi, f_\pi \in S(\mathbb{R}^d, \mathbb{R})$ and

$$E(f) = E_\phi(f_\phi) + E_\pi(f_\pi). \tag{2.87}$$

Moreover, since $F(x) = 0$ if $x \in supp(f)'$, then we have $supp(f) \subset \mathcal{O}_\mathcal{C}$ and $supp(f_\phi), supp(f_\pi) \subset \mathcal{C}$.

Chapter 3

Entropy and Energy Conditions

Since the algebra of a finite region is defined, it is important to describe the so-called *microstates* contained within it. This is linked, in a sense, to the information associated with a vector or a wave. However, as will be demonstrated, there are numerous complications arising from the degrees of freedom, energy localization, boundedness, and unboundedness.

3.1 Relative Entropy

First, we shall define entropy with respect to a Hilbert space and try to relate the concept of distinguishability of two different states as in [17].

Let A be a subsystem of interest with a Hilbert space $\mathcal{H}_A$ and B the whole rest with a Hilbert space $\mathcal{H}_B$. We can build a whole new Hilbert space $\mathcal{H}_{AB}$ from the product of $\mathcal{H}_A$ with $\mathcal{H}_B$:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{3.1}$$

Consider the case of vectors ψ_{AB} from $\mathcal{H}_{AB}$ that can be written as a product of unit vectors $\psi_A \in \mathcal{H}_A$ and $\psi_B \in \mathcal{H}_B$. Any measure made on ψ_A can be evaluated completely without make any mention to ψ_B. In the cases it cannot be done, in a sense of states ψ_{AB} that cannot be written as product of states ψ_A with ψ_B then it is said to be a generic pure state *entangled*. For any pure state, one can write

$$\psi_{AB} = \sum_i \sqrt{p_i}\psi^i_A \otimes \psi^i_B, \tag{3.2}$$

where ψ^i_A and ψ^i_B are orthonormal by assumption,

$$(\psi^i_A, \psi^j_A) = (\psi^i_B, \psi^j_B) = \delta^{ij}, \tag{3.3}$$

and that $p_i > 0$.

Suppose an operator O_A on $\mathcal{H}_A$. The expectation value is

$$(\psi_{AB}, O_A \otimes \mathbb{1}\psi_{AB}) = \sum_i p_i (\psi^i_A, O_A \psi^i_A). \tag{3.4}$$

One can use the density matrix definition,

$$\rho_A = \sum_i p_i \left|\psi^i_A\right\rangle \left\langle\psi^i_A\right|, \tag{3.5}$$

to write the $(\psi_{AB}, O_A \otimes \mathbb{1}\psi_{AB})$ as

$$Tr_{\mathcal{H}_A} \rho_A O_A. \tag{3.6}$$

From the definition of the density matrix ρ one can see that it is hermitian and positive semi definite and also has trace 1.

Now let us introduce the *von Neumann entropy* of a density matrix ρ_A defined as

$$S(\rho_A) = -Tr\rho_A \log \rho_A. \tag{3.7}$$

Worth to notice that since the trace represents the class of a representation, the density matrix is invariant under a unitary transformation. Then one can diagonalize ρ_A as follows,

$$\rho_A = \sum_i p_i \left|\psi^i_A\right\rangle \left\langle\psi^i_A\right|, \tag{3.8}$$

therefore,

$$S(\rho_A) = -\sum_i p_i \log p_i. \tag{3.9}$$

Since the $p'_i s$ represents probability and the states ψ^i_A are orthonormal, the von Neumann entropy is bounded

$$S(\rho_A) \geq 0, \tag{3.10}$$

with the equality for a pure state, only one of p^i are 1 and the rest are zero. The upper bound can be seen considering a system with K states,

$$S(\rho_A) \leq \log K, \tag{3.11}$$

with the equality only if

$$\rho_A = \frac{1}{K} diag(1, 1, \ldots, 1). \tag{3.12}$$

In this case, A is in a maximally mixed state. For a system AB in a pure state $\psi_{AB} = \sum_i \sqrt{p_i}\psi_A^i \otimes \psi_B^i \in \mathcal{H}_A \otimes \mathcal{H}_B$ then $S(\rho_A) = S(\rho_B)$ in a sense of they have the same probability and consequently the entropy S_{AB} vanishes.

Let us move forward and start to define ways to distinguish or quantify the entanglement. First, suppose a general case in which A and B has an amount of information. Also suppose A can make measurements on B and vice versa.

Definition 3.1.1 *The quantum conditional entropy is given by*

$$S(A|B) = S_{AB} - S_B, \tag{3.13}$$

which represents the entropy that remains in B once A is known.

The conditional entropy can be negative, for instance if the system AB is an entangled pure state. The entropy of the system vanishes, $S_{AB} = 0$, but as the system B is in a mixed state, $S_B > 0$, and in this case one have $S(A|B) < 0$. The quantity $S_{AB} - S_B$ is the information unknown for B after make a measurement. Also, it is of interest to know the inverse, i.e., quantify how much is known.

Definition 3.1.2 *The quantum mutual information given by*

$$I(A;B) = S_A - S_{AB} + S_B. \tag{3.14}$$

That measures how much is known about A after a measurement in B.

The quantum mutual information is non-negative

$$I(A;B) \geq 0. \tag{3.15}$$

Moreover $I(A;B) = 0$ if and only if the density matrix factorizes

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{3.16}$$

And now the most important concept of this section arise, the relative entropy.

Definition 3.1.3 *For ρ and σ two density matrices of the same Hilbert space. The quantum relative entropy can be defined by*

$$S(\rho||\sigma) = Tr\,\rho(\log\rho - \log\sigma), \tag{3.17}$$

and it quantifies the difference between two probability distributions ρ and σ.

If σ is put on a diagonal form, nothing says ρ is diagonal on the same basis. Denote ρ_D the diagonal elements of ρ. Using that $Tr\,\rho\,\log\sigma = Tr\,\rho_D\log\sigma$, the difference $S(\rho||\sigma) - S(\rho_D||\sigma)$ can be written as

$$S(\rho||\sigma) = S(\rho_D||\sigma) + S(\rho_D) - S(\rho). \tag{3.18}$$

But follows from the entropy concavity property that if one diagonalize a density matrix its entropy increase

$$S(\rho_D) - S(\rho) > 0, \tag{3.19}$$

and the right-hand side of (3.18) is positive, guaranteeing the *relative entropy positivity*. Also, if $\sigma = diag(q_1,\ldots,q_n)$ and $\rho_D = (p_1,\ldots,p_n)$, then,

$$S(\rho_D||\sigma) = \sum_i p_i(\log p_i - \log q_i), \tag{3.20}$$

being zero only if $\rho = \sigma$ and one have a lower bound for the relative entropy,

$$S(\rho||\sigma) \geq 0. \tag{3.21}$$

Considering a Canonical ensemble such that σ is a *Gibbs thermal density matrix* at some fixed temperature $T = 1/\beta$ and a Hamiltonian H for the system,

$$\sigma = \frac{\exp\{-\beta H\}}{Z}. \tag{3.22}$$

So, $\log\sigma = -\beta H - \log Z$ and therefore the relative entropy between any density matrix ρ and σ is

$$\begin{aligned} S(\rho||\sigma) &= Tr\,\rho(\log\rho - \log\sigma) \\ &= -S(\rho) + Tr\,\rho(\beta H + \log Z), \\ &= \beta(E(\rho) - TS(\rho)) + \log Z, \end{aligned} \tag{3.23}$$

where the average energy computed in the density matrix ρ is

$$E(\rho) = Tr\,\rho H. \tag{3.24}$$

One can then define the *free energy* as

$$F(\rho) = E(\rho) - TS(\rho), \tag{3.25}$$

and the $\log Z$ term is independent of ρ and gives a constant that ensure that $S(\sigma||\sigma) = 0$. So

$$S(\rho||\sigma) = \beta(F(\rho) - F(\sigma)), \tag{3.26}$$

relates the relative entropy with the difference of the available energy for the system described by the respective density matrix. It holds for any evolution of the system which preserves thermal equilibrium at a fixed temperature β, for instance time. The density matrix σ is mapped into itself, but it maps ρ to a generally different density matrix ρ'. The consequence is that the relative entropy can only go down under an evolution, i.e.,

$$S(\rho||\sigma) \geq S(\rho'||\sigma), \tag{3.27}$$

and therefore

$$F(\rho) \geq F(\rho'). \tag{3.28}$$

As well the evolution that preserves thermal equilibrium can only reduce the free energy.

Let us apply the above construction to a drastically different situation: a black hole. The density matrices can be seen as describing the inside and outside of the black hole, and it is of interest to understand how the information can be detailed in both regions. The example of a black hole is of interest in what follows because of the notion of the split of spacetime, i.e., the split into the inside and outside of a region.

Consider a black hole of some mass M and area (in the sense of a boundary) A. If something with a huge amount of entropy, like a box, falls inside a black hole, then its information is completely erased from the universe. This statement completely goes against the second law (SL) of thermodynamics. Based on the fact that the area of a black hole is entirely proportional to its mass, there is an *Area Theorem* which states that the black hole event horizon never decreases with time. Since the mass of the box falling inside the black hole increases its mass, its area will increase respectively. But since, until now, things only enter and never leave the black hole, the black hole area, not emitting particles, can never have its area decreasing. In terms of information (entropy), when an object falls inside a black hole, to an outside observer, the information is lost forever. A naive example is that of a hot cup of tea put inside a fridge; after some time, the fridge and the cup of tea will be cold enough, and the "hot" information is lost. What happened was

that the observer forgot that the fridge warms the kitchen, and there is a balance guaranteeing the SL of thermodynamics. Based on this, there is the Bekenstein black hole entropy, suggesting that the black hole by itself has entropy which is proportional to its area,

$$S_{BH} = \frac{A}{4}. \tag{3.29}$$

Moreover, Bekenstein proposed that the SL of thermodynamics is valid only when the entropy of matter exterior to the black hole and the black hole entropy are taken into consideration:

$$\begin{aligned} S_{gen} &= S_{BH} + S_{out}, \\ S_{gen} &= \frac{A}{4} + S_{out}. \end{aligned} \tag{3.30}$$

The last statement is called the generalized second law (GSL) of thermodynamics, which can be stated as follows:

$$S_{gen} = S_{in} + S_{out}. \tag{3.31}$$

From the first law of thermodynamics, the black hole must have a temperature proportional to its entropy and mass. By the notion of a black body, if a black body has a nonzero temperature, then it radiates. But, until now, a black hole does not radiate, i.e., no matter escapes from its event horizon. Then, there is Hawking radiation, asserting that a distant observer will detect a thermal spectrum of particles coming from the black hole at a certain temperature. This occurs only very far away from the black hole. Now, a radiating black hole will lose mass, shrink, and disappear, just as the fridge information would do if there were no motors. Also, a certain box of mass and size can have arbitrary entropy as one desires, and the GSL could still be violated. From the idea of an object of any amount of entropy falling inside a black hole, it would increase the black hole area, and one may conclude that the black hole can have any undefined amount of information, and it should have no dependence on the energy stored in the black hole.

Bekenstein then came up with a nice inequality concerning a universal bound for the entropy. The GSL implies the Bekenstein bound:

$$S_{matter} \leq \lambda 2\pi\, E\, R, \tag{3.32}$$

where E is the mass-energy, R is the radius of the smallest sphere that fits around the matter system, and λ is some constant.

A nice proof of the Bekenstein bound is given by Casini, and it is in terms of the relative entropy of the density matrix of some region V, and the inequality follows for some local form of energy. This local form of energy is by no means obvious to evaluate, and some general concept of *time evolution* needs to be defined.

3.2 Bekenstein Bound

Here will be reproduced the Casini's proof on the Bakenstein bound [see 4].

Consider a spatial region V lying on a Cauchy surface. Associate to V an algebra of operators $\mathcal{A}(V)$. Suppose the Hilbert space can be decomposed as a tensor product $\mathcal{H} = \mathcal{H} \otimes \mathcal{H}_{-V}$, where $-V$ is the complement on the Cauchy surface of V. Now define the reduced density matrix,

$$\rho_V = Tr_{-V}\, \rho, \tag{3.33}$$

where the trace is over the Hilbert space $\mathcal{H}_{-V}$. The entropy then is given by,

$$S(\rho_V) = -Tr\, \rho_V \, \log \rho_V, \tag{3.34}$$

which is divergent by a couple of reasons. The first reason we can mention is due to the degree of freedom, which in usual quantum field theory is infinity and consequently the von Neumann algebra is of type III, see Appendix A, the traceless one. The second divergence reason that we can mention is due to the cloud of vacuum fluctuations, which is present for any state. For the first divergence kind, one can consider a situation of a cutoff theory. For the second divergence kind, just subtract from the system entropy the entropy in the vacuum state $\rho_V^0 = Tr_{-V} \left|0\right\rangle \left\langle 0\right|$ corresponding to the same region V. The well-defined entropy is then

$$S_V = S(\rho_V) - S(\rho_V^0) \tag{3.35}$$

Now let us analyze how the energy on V can be defined. The energy of a localized state ρ_V cannot be computed, since the Hamiltonian operator is defined in the whole Hilbert space, while the local density matrices act on the operator algebra generated by the observable localized in V. A localized definition of

energy is needed. Take the local vacuum density matrix to be like

$$\rho_V^0 = \frac{e^{-K}}{Tr\, e^{-K}}. \tag{3.36}$$

The Hermitian and positive definite matrix ρ_V^0 can always be written in this way since it has no zero eigenvalues (except when V is the whole space, where it becomes the vacuum state projector). The Hermitian and dimensionless operator K is the local Hamiltonian for V. We suppose here the state on V to be thermal with respect to a given notion of time. We hypothesize that exist an automorphism which takes care of the evolution of the system at a fixed temperature, and a Hamiltonian can be defined with respect to this automorphism. The reduced density matrix is of the form

$$\rho_V = \frac{e^{\beta E}}{Tr\, e^{-\beta E}}, \tag{3.37}$$

with E the time translation operator and β the inverse temperature. Comparing with the vacuum density matrix (3.36), the Hamiltonian is given by,

$$K = \beta E. \tag{3.38}$$

For example, one have this kind of Hamiltonian for a black hole in a Hartle-Hawking state. For a Schwarzschild one, it is given by $K = 8\pi GME$, where E is the energy operator eigenvalue as measured by asymptotic observers, and M the black hole mass. The energy operator must be integrated over the exterior of the black hole region. Now, it is a matter of fact the equation (3.36) does not care with some replacement $K \to K + c$ for c a constant. Then is made a new kind of vacuum subtracting

$$K_V = Tr(K\rho_V) - Tr(K\rho_V^0). \tag{3.39}$$

The Bekenstein bound now reads

$$S_V \leq K_V, \tag{3.40}$$

which is,

$$S(\rho_V) - S(\rho_V^0) \leq Tr\, K\rho_V - Tr\, K\rho_V^0. \tag{3.41}$$

Taking the trace of (3.36), we have,

$$K = -\log \rho_V^0 - \log Tr\, e^{-K}. \tag{3.42}$$

Using now the fact the density matrices has trace one, $Tr\,\rho_V = Tr\,\rho_V^0 = 1$, the bound is written as,

$$Tr(\rho_V \log \rho_V) - Tr(\rho_V \log \rho_V^0) \geq 0. \tag{3.43}$$

The final result is the statement of the positivity of the relative entropy (3.21) for $S(\rho_V|\rho_V^0)$ between local density matrices corresponding to the state of the system and the vacuum, both reduced to V.

The proof is based on the difficult process of localization. It relates the coherent states as perturbation of the vacuum and the vacuum itself and how much distinguishable they are depends on how much energy (effort) one wants to use to localize it, as well, by providing more spatial room to distinguish them [see 3].

3.3 Quantum Null Energy Condition

Considering two neighboring points and a congruence of curves, in a sense of a family of curves, one through each point of a neighborhood subspace. Define a vector representing the separating of corresponding points in neighboring curves. Then one can evaluate, via Lie derivative, the distance of the corresponding points on neighboring curves with respect to the tangent vector of null geodesics. It is obtained the Raychaudhuri equation for timelike geodesics [18, 5],

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b, \tag{3.44}$$

where θ is the trace of the rate of separation expansion, D is the spacetime dimension, R_{ab} the Ricci tensor and σ the shear.

Suppose now a null vector W and from the Einstein equation

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 8\pi T_{ab}, \tag{3.45}$$

one can assert, [18, ch. 4], that the *energy density is always measured as non-negative by any observer*. But only for the following case:

$$T_{ab}W^aW^b \geq 0, \quad iff \quad R_{ab}W^aW^b \geq 0 \quad \forall\Lambda. \tag{3.46}$$

The right-hand side of the condition above is called *null curvature condition*. Finally, the *classical focusing theorem* is announced [5].

Theorem 3.3.1 *In a spacetime satisfying the null curvature condition, the expansion is non-increasing at all regular (except where two neighbors curves intersects, the so-called caustic points) points of a surface orthogonal null congruence:*

$$\frac{d\theta}{d\lambda} \leq 0. \tag{3.47}$$

Basically light rays always get focused, never anti-focused by matter.

There are now two problems. The first resides in quantum field theory. Take, for instance, a free field $\phi(x)$ and its energy-momentum density tensor $T^{\mu\nu}(x)$, for which the vacuum expectation value is zero. Although the energy operator P^0 is self-adjoint and positive, $T^{00}(x)$ is not positive, even when smeared out in space and time with a positive test function. Consider, for instance, the expectation value of a product of fields at the same spacetime point. It is also shown to be an unavoidable phenomenon for any local quantum field theory [see 2]. The second problem resides in black hole theory. The event horizon is a null surface, and the expansion θ can be expressed in terms of the area. It follows that the black hole entropy (which is related to the area) is always increasing. If one considers Hawking radiation, the black hole will lose mass and evaporate, contradicting the second law of thermodynamics. To fix this, one considers the generalized second law of thermodynamics (3.31). It remains to obtain a quantum focusing conjecture, which will lead to a quantum version of energy positivity.

First, consider a σ spacelike codimension-2 surface that splits a Cauchy surface Σ into two portions Σ_{inside} and Σ_{out} so that σ need not be connected nor compact. Choose out of four null hypersurfaces orthogonal to σ the one which terminated by caustic. For each point y of σ exist one generator of N. Take λ to be an affine parameter along the generator such that $\lambda = 0$ on σ and λ is increasing away from σ. Basically now one have a coordinate system (λ, y) for N. Define a function $V(y) \geq 0$ which is a slice in N and consists of generators y for each $\lambda = V$, in such a way, a surface is defined, and the entropy is defined by $S[V(y)]$. Consider a second slice which differs from the first only in a neighborhood of generators near a given y_1, forming an infinitesimal area A

$$V_\varepsilon(y) = V(y) + \varepsilon\eta_{y_1}(y), \tag{3.48}$$

which $\eta_{y_1} = 1$ in an infinitesimal neighborhood of area A around y_1 and is zero otherwise. The quantum expansion is now defined by the derivative of the

generalized entropy after the deformation, $V_\varepsilon(y)$,

$$\Theta[V(y), y_1] = \lim_{A\to 0} \frac{4G\hbar}{A} \frac{dS_{gen}}{d\varepsilon}\bigg|_{y_1} = \frac{4G\hbar}{\sqrt{g(y_1)}} \frac{\delta S_{gen}}{\delta V(y_1)}. \tag{3.49}$$

It is with $\sqrt{g}$ being the metric induced by the area element. Using now the generalized entropy formula (3.31) one can write,

$$\Theta = \theta + \frac{4G\hbar}{A} S'_{out}, \tag{3.50}$$

quantifying some change of the quantum area and relates to the classical geometric change of area that would be happening plus some change in the boundary between what is *out* and *in*. The *Quantum Focusing Conjecture (QFC)* becomes

$$0 \geq \Theta' = \theta' + \frac{4G\hbar}{A}(S''_{out} - S'_{out}\theta). \tag{3.51}$$

Choosing some congruence with tangent vector k^a, such that the shear and congruence both vanish at some point, one obtain

$$< T_{kk} > \geq \frac{\hbar}{2\pi A} S''_{out}, \tag{3.52}$$

by using (3.44) and (3.45). For the limit $\hbar \to 0$, the positivity of energy $< T_{kk} > \geq 0$ is given at any point. For the limit of $\hbar$ considerable, it is obtained the QFC implies the *Quantum Null Energy Condition (QNEC)*

$$< T_{kk} > \geq \frac{\hbar}{2\pi A} S'', \tag{3.53}$$

which does not depend on G and is entirely a statement about quantum field theory. The second variation of the entropy of a region places a lower bound on the energy-momentum tenor null component of a field localized there. At the moment is the closest to an important connection between locally, energy density and information [5, 3]. The knowledge of all information in a given region places a lower bound on the energy in the complementary region.

3.4 Motivation to the Modular Hamiltonian

In Casini's proof of the Bekenstein bound, an operator K was considered, being the Hamiltonian operator for a specific region V. For the relative entropy calculation, a problem was then faced: the degree of freedom of quantum field theory turns the von Neumann entropy into a traceless quantity. To better understand this, consider a physical system of volume V with N particles inside. Consequently, we have a finite number of degrees of freedom. Consider again K to be the Hamiltonian describing the entropy S. The mean value of an observable a when S is in an equilibrium state is, as usual,

$$\omega(a) = \frac{1}{Z} Tr(e^{-\beta K} a), \tag{3.54}$$

where β is the inverse temperature and $Z = Tr(e^{-\beta K})$ is the partition function. The entire statistical interpretation and Casini's proof work well for the above consideration of a finite number of particles.

Now, extend the number of particles N and the volume V to infinity in such a way that the density N/V is fixed. The partition function (just as in quantum field theory) becomes an undefined quantity. A new way to characterize the equilibrium state is needed to investigate the QNEC, for instance, in the quantum field theory scenario.

Another fact very interesting is that the time flow in quantum theory for an observable A is given by,

$$A_t = \alpha_t(A) = e^{itH_0} A e^{-itH_0}, \tag{3.55}$$

and the density (which we can, and we shall associate it for a state of the algebra) is

$$\rho[A] = Tr[A\rho] = \omega(A). \tag{3.56}$$

The relation between a Gibbs state ρ_0 and H_0 is

$$\rho_0 = N e^{-\beta H_0}. \tag{3.57}$$

The correlation amplitude can be written as

$$\begin{aligned}\omega_\beta((\alpha_t A)B) &= Z^{-1}Tr(e^{\beta H}e^{iHt}Ae^{-iHt}B) = Z^{-1}Tr(Be^{iH(t+i\beta)}Ae^{-iHt}) \\ &= Z^{-1}Tr(Be^{iH(t+i\beta)}Ae^{-iHt}e^{-i\beta H}e^{i\beta H}) \\ &= \omega_{beta}(Be^{iH(t+i\beta)}Ae^{-iH(t+i\beta)}) \\ &= \omega_\beta(B\alpha_{t+i\beta}A).\end{aligned} \tag{3.58}$$

Then we can state the *KMS equilibrium condition*: a state ω from a certain algebra is a KMS (Kubo-Martin-Schwinger) state at inverse temperature $\beta = 1/k_bT$ (k_b the usual Boltzmann constant and T the temperature) with respect to some one-parameter group of automorphism in the algebra if the function $f(t) = \omega(B(\gamma_t A))$ is analytic in the strip $0 < Im\, t < \beta$. The condition of analyticity implies $\omega((\gamma_t A)B) = \omega(B(\gamma_{t+i\beta}A))$.

The consequences of the above construction ares various. As Carlos Rovelli in the *thermal time hypothesis* study says: *in a quantum system with an infinite number of degrees of freedom, what we generally measure is the effect of small perturbations around a thermal state*. Suppose a free scalar field. The propagator is given by

$$F(t) = \langle 0|\, \phi(\vec{x},t)\phi(\vec{x},0)\, |0\rangle = \omega_0(\gamma_t(\phi(\vec{x},0))\phi(\vec{x},0)). \tag{3.59}$$

The ω_0 is the vacuum sector over the field algebra and γ_t is the time flow being the one from KMS condition as described before. The evolution of a prepared state A to a state B, B being a perturbation of the state A, is dictated by the thermal time flow. In order to better understand, consider a gas in a room. There is a preferred Lorentz frame in which the average momentum of the gas is at rest. For the others frame, there is a thermal bath happening. That is the relation between Lorentz time (the one with no thermal bath) and other time flow with thermal bath.

There is a very important theorem, the *Tomita-Takesaki* theorem, which associates a specific flow to an algebra of operators acting in a Hilbert space with a preferred cyclic and separating vector (usually the vacuum). The flow is given by a modular operator Δ. We can write $\Delta = e^{-K}$ (note that this is the exact expression used in Casini's proof of the Bekenstein bound), and $U(t) = e^{-itK}$, where K is the modular Hamiltonian, a self-adjoint operator whose spectrum will generally extend from $-\infty$ to $+\infty$ and has a zero eigenvalue for the cyclic and separating eigenvector. Then, one can say that the KMS condition generalizes Gibbs states, which describe the equilibrium condition for infinite systems. Also, it is worth

writing $K = -\log(\Delta)$ and calling it the generator of an intrinsic evolution, where the positivity of energy is now the *KMS* condition.

Everything seems nice, but what if one wants to consider the above construction of entanglement, entropy, and energy of states in a local region? Associate a von Neumann algebra $A(\mathcal{O})$ to the spacetime region $\mathcal{O}$ with a cyclic and separating vacuum sector. Additionally, one can assign to $\mathcal{O}$ the modular group associated with the time flow and, consequently, a modular Hamiltonian $\log(\Delta_{\mathcal{O}})$. As discussed before, the measurement of a state can be seen as the measurement of perturbation from the vacuum. Hence, it is of interest to quantify how distinguishable a state (perturbed from the vacuum) is from the vacuum itself. A formula for this has already been given, i.e., the relative entropy (3.17). But, as discussed, in general quantum field theory, the von Neumann algebra is of type III (see Appendix A), and the trace does not exist. Now arises a very important formula, the *Araki-Uhlmann* relative entropy between two faithful normal states on the von Neumann algebra:

$$S(\phi||\psi) = -\langle\phi|\log\Delta|\psi\rangle = S(\rho_\phi||\rho_\psi). \tag{3.60}$$

The problem of infinity in the calculation of relative entropy is avoided, but a new problem of how to obtain the exact expression for the modular Hamiltonian is introduced. The interest in how we can define the entropy of a wave packet (real solutions of the Klein-Gordon equation) is natural. But how can one define the entropy carried by a wave? What about the energy-entropy conditions stated in the previous sections? Can they be defined? There are many questions surrounding this construction, and the remainder of the chapters will be dedicated to addressing them.

Chapter 4

Modular Hamiltonian Theory

It is relevant to build such an intrinsic (time) evolution, which encodes the thermal states. We proceed to describe the thermal flow associated to these states. Also, the Araki-Uhlmann relative entropy formula is derived.

4.1 The Tomita-Takesaki Theorem

Consider the following density matrix ρ,

$$\rho_\beta = \frac{e^{-\beta H}}{Tr(e^{-\beta H})} = \frac{e^{-\beta H}}{Z}, \tag{4.1}$$

to be thermal with respect to the Hamiltonian H at inverse temperature β. Now consider the time-evolved two-point function of the thermal state. For instance, it quantifies the projection of a system prepared in the state B to be *measured* in the quantum excitation of A. In some sense, after a time evolution (in the Heisenberg picture sense) of initial state A, how likely it is to be evolved to the state B. A natural question to be answered in the future is *which time parameter is this?*

The time evolved two-point function of the thermal state is then given by

$$< a(t)\, b >_\beta = < e^{iHt} a e^{-iHt} b >_\beta = \frac{Tr(e^{-(\beta - it)H} a e^{-iHt} b)}{Z}. \tag{4.2}$$

The *Kubo, Schwinger and Martinez* observation is that we can relate the well behavior of the Hamiltonian H and the finite trace question with the analytic continuation from it to more general complex z. If H is bounded, then the following

$$F(z) = \frac{Tr(e^{-(\beta - z)H} a e^{-zH} b)}{Z} \tag{4.3}$$

is an analytic function of the entire complex plane. But if H is only bounded from below, one have the following cases

1. $Re(z) < 0 : e^{-zH}$ diverges,
2. $Re(z) > 0 : e^{-(\beta - z)H}$ diverges.

The *KMS* observation is then translated to a vertical strip of the complex plane given by $0 \leq Re(z) \leq \beta$, which delimits the two-point function analytic range.

A problem arises at this moment, because thermal functions only can be computed at finite volume [see 19]. In some sense, at large volume the vacuum fluctuations will be considerable and this is what is on concern in the Casini's proof of the Bekenstein bound when the vacuum subtraction is made. But, there, a finite volume is considered. Before, it was only used that the thermal density matrix ρ can be computed in finite volume by a path integral on $W \times I$ where W correspond to a finite spatial lattice and I is the analytic range $0 \leq \tau \leq \beta$. Then one just subtract the free density energy from T_{00} volume integral to subtract out the vacuum fluctuations. This procedure suffices to obtain a normalized density matrix at finite volume.

Introduce the *Thermal Double Field* construction such that the thermal double field resulting can be seen as a *purification* of the thermal density matrix. First, double the degrees of freedom such that from the usual Hilbert space $\mathcal{H}$ one have now the doubling Hilbert space $\mathcal{H}_l \otimes \mathcal{H}_r$ which is preferable to see the *left Hilbert space* $\mathcal{H}_l$ to be just the complex conjugation of the *right Hilbert space* $\mathcal{H}_r$ and consequently the left Hilbert space is just the time reversal conjugate to the right system. Suppose the Hamiltonian with eigenstate ψ_i and eigenvalue E_i. Hence, the double thermofield is Ψ_{TFD}, a vector in $\mathcal{H}_r \otimes \mathcal{H}_l$ defined as,

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |i\rangle_l \otimes |i\rangle_r , \tag{4.4}$$

i.e., a pure state in the double thermal system with right (and left) density matrix being indistinguishable from a thermal state as follows

$$\rho_r = Tr_{\mathcal{H}_l} |\Psi_{TFD}\rangle \langle\Psi_{TFD}| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_r \langle i|_r = \frac{1}{Z} e^{-\beta H_r}. \tag{4.5}$$

To avoid the problem of vacuum fluctuations in H_r and H_l, on volume V when $V \to \infty$ which causes the divergence, we can say that for a and b acting on $\mathcal{H}$ exist a Hamiltonian $H = H_r - H_l$ acting on $\mathcal{H}_r \otimes \mathcal{H}_l$ and a state Ψ_{TFD} in $\mathcal{H}_r \otimes \mathcal{H}_l$. Also,

one have the following relations

$$a_r \left|\Psi_{TFD}\right\rangle = e^{-\beta H/2} a_l^\dagger \left|\Psi_{TFD}\right\rangle, \tag{4.6}$$

$$a_l \left|\Psi_{TFD}\right\rangle = e^{-\beta H/2} a_r^\dagger \left|\Psi_{TFD}\right\rangle, \tag{4.7}$$

$$\tag{4.8}$$

such that

$$H \left|\Psi_{TFD}\right\rangle = 0. \tag{4.9}$$

Now it is possible to write the partition function of a thermal field in the form of a vacuum-vacuum transition amplitude and the zero temperature case will be a special case. For more details about double thermal field and its application, see Das [20] and Guo [21]. After all the motivations above, now one can define the *KMS condition*, the desire conditions for the total Hilbert space $\mathcal{H}$, the von Neumann algebra $\mathcal{A}$ and the states needs to satisfy to be well-behaved. The rest of the section is devoted to construct the Tomita-Takesaki theory and is strongly based on the very recent approach due to Sorce [22].

Definition 4.1.1 *Let $\mathcal{H}$ be a Hilbert space, $|\Omega\rangle$ a state, and $\mathcal{A}$ a von Neumann algebra. Let H be a self adjoint, possibly unbounded operator. The state $|\Omega\rangle$ is said to satisfy the KMS condition with respect to $\mathcal{A}$ and H at inverse temperature β if the following hold.*

1. *H generates a symmetry of the state for all $t \in \mathbb{R}$ such that*

$$e^{-iHt} \left|\Omega\right\rangle = \left|\Omega\right\rangle, \tag{4.10}$$

2. *H generates an automorphism of the algebra such that for all $t \in \mathbb{R}$ and for all $a \in \mathcal{A}$ one have*

$$e^{iHt} a e^{-iHt} \in \mathcal{A}, \tag{4.11}$$

3. *The two-point functions of $\mathcal{A}$ in the state $|\Omega\rangle$ look thermal with respect to the flow generated by H: for all $a, b \in \mathcal{A}$ the function*

$$F(it) = \left\langle\Omega\right| e^{iHt} a e^{-iHt} b \left|\Omega\right\rangle = \left\langle\Omega\right| a e^{-iHt} b \left|\Omega\right\rangle, \tag{4.12}$$

admits a bounded analytic continuation to the vertical strip $0 \leq Re(z) \leq \beta$ and on

the right boundary of this strip the analytic continuation is given by

$$\begin{aligned} F(\beta + it) &= \langle \Omega | \, b e^{iHt} a e^{-iHt} \, | \Omega \rangle \\ &= \langle \Omega | \, b e^{iHt} a \, | \Omega \rangle \, . \end{aligned} \tag{4.13}$$

Finally, the thermal nature is expressed as a property of the analytic structure of the two-point function. We can notice that there is a requirement that for any $a \in \mathcal{A}$ there is $b^\dagger \in \mathcal{A}'$, yielding to establish: $S\mathcal{A} \, |\Omega\rangle = \mathcal{A}' \, |\Omega\rangle$. That is the motivation to define an anti-linear operator S_0, on the domain $\mathcal{D}_{S_0} = \mathcal{A} \, |\Omega\rangle$, by,

$$S_0 x \, |\Omega\rangle = x^\dagger \, |\Omega\rangle \, , \quad x \in \mathcal{A}. \tag{4.14}$$

Assume $|\Omega\rangle$ to be cyclic and separating for $\mathcal{A}$ and consider the densely defined operators

$$S_0 \left(a \, |\Omega\rangle \right) = a^\dagger \, |\Omega\rangle \, , \tag{4.15}$$

$$F_0 \left(a' \, |\Omega\rangle \right) = (a')^\dagger \, |\Omega\rangle \, . \tag{4.16}$$

For any $a' \in \mathcal{A}$, and any $a \, |\Omega\rangle$ in the domain of S_0, one can have

$$\left\langle S_0 a \Omega \middle| a' \Omega \right\rangle = \left\langle a^\dagger \Omega \middle| a' \Omega \right\rangle = \left\langle (a')^\dagger \Omega \middle| a \Omega \right\rangle . \tag{4.17}$$

So $S_0^\dagger$ can act on $a' \, |\Omega\rangle$ and its action is determined by

$$\left\langle a \Omega \middle| S_0^\dagger a' \Omega \right\rangle = \left\langle a \Omega \middle| (a')^\dagger \Omega \right\rangle . \tag{4.18}$$

Uniquely fixing $S_0^\dagger$ to act (antilinearly) on $a' \, |\Omega\rangle$ as $S_0^\dagger (a' \, |\Omega\rangle) = (a')^\dagger \, |\Omega\rangle$ implies $S_0^\dagger \supseteq F_0$. From the fact above, there is a dense subspace of $\mathcal{H}$ on which the adjoint $S_0^\dagger$ of S_0 is defined, we conclude that S_0 is preclosed. Moreover, $S_0^\dagger$ is defined on the domain of F_0, which is $\mathcal{A}' \, |\Omega\rangle$ and is dense. By symmetry the converse is also true and $F_0^\dagger \supseteq S_0$ and then F_0 is preclosed as well. Denote their closures by S and F, such that $F \subseteq S^\dagger$ and $S \subseteq F^\dagger$.

Recall that for a closed operator T, with domain $\mathcal{D}(T)$, to *affiliate* to $\mathcal{A}$ one just need that T commutes with any element of $\mathcal{A}'$, the commutant of $\mathcal{A}$ [see 23, cap. 2]. Suppose now $|\Omega\rangle$ in the domain of T and T', also suppose $|\psi\rangle$ in the domain of S

such that

$$\begin{aligned} |\psi\rangle &= T\,|\Omega\rangle\,, \\ S\,|\psi\rangle &= T^{\dagger}\,|\Omega\rangle\,. \end{aligned} \tag{4.19}$$

Assume at first sight T and T' to be affiliated to $\mathcal{A}$ such that it act on $a'\,|\Omega\rangle$ as

$$Ta'\,|\Omega\rangle = (a')T\,|\Omega\rangle = (a')\,|\psi\rangle \tag{4.20}$$

and

$$T^{\dagger}a'\,|\Omega\rangle = a'T^{\dagger}\,|\Omega\rangle = a'S\,|\psi\rangle\,. \tag{4.21}$$

Define the operators α_0 and β_0 on the domain $\mathcal{A}'\,|\Omega\rangle$ by

$$\begin{aligned} \alpha_0 a'\,|\Omega\rangle &= a'\,|\psi\rangle\,, \\ \beta_0 a'\,|\Omega\rangle &= a'S\,|\psi\rangle\,. \end{aligned} \tag{4.22}$$

For $b' \in \mathcal{A}'$, one have

$$\begin{aligned} \langle b'\Omega|\,\alpha_0\,|a'\Omega\rangle &= \langle b'\Omega|a'\psi\rangle \\ &= \Big\langle (a')^{\dagger}\Omega\Big|\psi\Big\rangle \\ &= \Big\langle F(b')^{\dagger}a'\Omega\Big|\psi\Big\rangle \\ &= \Big\langle F^{\dagger}\psi\Big|(b')^{\dagger}\Omega\Big\rangle\,. \end{aligned} \tag{4.23}$$

But since $F^{\dagger} \supseteq S$, and $|\psi\rangle$ is in the domain of S one have

$$\begin{aligned} \langle b'\Omega|\,\alpha_0\,|a'\Omega\rangle &= \Big\langle S\psi\Big|(b')^{\dagger}a'\Omega\Big\rangle \\ &= \langle b'S\psi|a'\Omega\rangle \\ &= \langle \beta_0 b'\Omega|a'\Omega\rangle\,. \end{aligned} \tag{4.24}$$

The above gives the inclusion $\alpha_0^{\dagger} \supseteq \beta_0$ and by symmetry $\beta^{\dagger} \supseteq \alpha_0$. Then α_0 is preclosed and denote by T its closure. To prove effectively that T is affiliated to $\mathcal{A}$,

$$\begin{aligned} Ta'b'\,|\Omega\rangle &= \alpha_0 a'b'\,|\Omega\rangle = a'b'\,|\psi\rangle = a'\alpha_0 b'\,|\Omega\rangle \\ &= a'Tb'\,|\Omega\rangle\,. \end{aligned} \tag{4.25}$$

So T commutes with any element of $\mathcal{A}'$ and by definition is affiliated to $\mathcal{A}$. Also,

notice that $|\Omega\rangle$ is in the domain of both T and $T^\dagger$. The final conclusion is the following

$$ST\,|\Omega\rangle = S\alpha_0\,|\Omega\rangle = S\,|\psi\rangle = \beta_0\,|\Omega\rangle = T^\dagger\,|\Omega\rangle\,. \tag{4.26}$$

Hence, we can start from the fact T is closed and affiliated to $\mathcal{A}$. Since T is closed, we can make the polar decomposition as $T = u|T|$. Considering $T_n = u|T|_n$ to be defined by projections of $|T|$ on the spectral subspace $[0, n]$, each T_n is in $\mathcal{A}$, so each $T_n\,|\Omega\rangle$ is in the domain of S and maps to $T_n^\dagger = u^\dagger|T^\dagger|_n\,|\Omega\rangle$. The spectral theorem guarantees these sequences converges

$$\begin{aligned} &T_n\,|\Omega\rangle \to T\,|\Omega\rangle\,, \\ &ST_n\,|\Omega\rangle \to T^\dagger\,|\Omega\rangle\,. \end{aligned} \tag{4.27}$$

The final conclusion is that since S is a closed operator, this implies that $T\,|\Omega\rangle$ is in the domain of S, with $ST\,|\Omega\rangle = T^\dagger\,|\Omega\rangle$.

Suppose $|\psi\rangle$ to be in the domain of $F^\dagger$ and using the result that $T\,|\psi\rangle = |\psi\rangle \in \mathcal{D}_{F^\dagger}$ one can conclude $|\psi\rangle$ is in the domain of S. But it is a fact that $|\psi\rangle \in \mathcal{D}_S$ so then $S = F^\dagger$. Moreover, $\mathcal{D}_{S^2} = \mathcal{D}_S$ and follows that $S^2 = \mathbb{1}_{\mathcal{D}_S}$. Now worth to note that since S is a closed, densely defined, conjugate-linear operator, and $F = S^\dagger$, it follows that FS and SF are positive self-adjoint linear operators [24]. From the previous domain characterization, S and F has the range as the same as its domain, is invertible , and coincides with its inverse.

Define the polar decomposition of S to be

$$S = J\Delta^{1/2} = \Delta^{-1/2}J, \tag{4.28}$$

where $\Delta^{1/2}$ is a positive, self-adjoint given by $\Delta^{1/2} = \sqrt{SS^\dagger}$ and J is an antilinear partial isometry whose support is $supp(J) = Ker(S)^\perp$. Since $S^2 = \mathbb{1}_{\mathcal{D}_S}$, the domain of S is equal to the image of S, the kernel of S is trivial and J is supported on all Hilbert space.

Also one have

$$\mathbb{1}_{\mathcal{D}_S} = J\Delta^{1/2}J\Delta^{1/2}, \tag{4.29}$$

since the kernel of Δ is the same as the kernel of S, so Δ is invertible on its domain and follows that

$$J\Delta^{1/2}J \supseteq \Delta^{-1/2}. \tag{4.30}$$

If one take an arbitrary $|\psi\rangle$ in the domain of $J\Delta^{1/2}J$ then $J\,|\psi\rangle$ is in the domain of

$\Delta^{1/2}$, which is equal to the domain of S. Must exist an arbitrary $|\eta\rangle$ in the domain of S such that

$$J|\psi\rangle = S|\eta\rangle = J\Delta^{1/2}|\eta\rangle, \tag{4.31}$$

and it is possible to conclude $|\psi\rangle = \Delta^{1/2}|\eta\rangle$. Hence, every vector which is in the domain of $J\Delta^{1/2}J$ is in the image of $\Delta^{1/2}$ and in the domain of $\Delta^{-1/2}$ so that

$$J\Delta^{1/2}J = \Delta^{-1/2}. \tag{4.32}$$

Now one can observe the following,

$$\Delta^{-1/2} = J\Delta^{1/2}J = J(JJ^\dagger)\Delta^{1/2}J = J^2(J^\dagger\Delta^{1/2}J). \tag{4.33}$$

By uniqueness of the polar decomposition of the closed operator $\Delta^{-1/2}$, one must have $J^2 = \mathbb{1}$. Also follows that

$$F = S^\dagger = \Delta^{-1/2}J^\dagger = \Delta^{1/2}J = J\Delta^{-1/2}. \tag{4.34}$$

To summarize, the notations are as follows

- S is the *Tomita operator*,
- J is the *modular conjugation*,
- Δ is the *modular operator*.

Suppose the modular operator to be written as $\Delta^{1/2} = e^{-K/2}$ such that $a, b \in \mathcal{A}$ obey

$$\begin{aligned}\left\langle b^\dagger\Omega\middle|a\Omega\right\rangle &= \left\langle Sb\Omega\middle|Sa^\dagger\Omega\right\rangle \\ &= \left\langle J\Delta^{1/2}b\Omega\middle|J\Delta^{1/2}a^\dagger\Omega\right\rangle \\ &= \left\langle e^{-K/2}a^\dagger\Omega\middle|e^{-K/2}b\Omega\right\rangle.\end{aligned} \tag{4.35}$$

Take the vector $T|\Omega\rangle$ in the domain of e^{-H} and the action of e^{-H} is given by

$$\left\langle a^\dagger\Omega\middle|\, e^{-H}|T\Omega\right\rangle = \left\langle T^\dagger\Omega\middle|a\Omega\right\rangle. \tag{4.36}$$

But also one have

$$\begin{aligned}\left\langle a^\dagger\Omega\right| \Delta \left|T\Omega\right\rangle &= \left\langle \Delta^{1/2}a^\dagger\Omega\middle|\Delta^{1/2}T\Omega\right\rangle \\ &= \left\langle T^\dagger\Omega\middle|a\Omega\right\rangle .\end{aligned} \tag{4.37}$$

Since $a^\dagger \left|\Omega\right\rangle$ are dense in the Hilbert space, one have the vector equation

$$e^{-H}\left|T\Omega\right\rangle = \Delta \left|T\Omega\right\rangle . \tag{4.38}$$

So if a vector is in the domain of Δ, it is also in the domain of e^{-H} and their action agree. The final conclusion is

$$\Delta = e^{-H}. \tag{4.39}$$

The most important theorem of the section can be now announced.

Theorem 4.1.1 *(Tomita-Takesaki) Let $\mathcal{A}$ be a von Neumann algebra, $\left|\Omega\right\rangle$ a cyclic and separating vector, Δ and J as above. For all $a \in \mathcal{A}$ one have,*

$$e^{iKt}ae^{-iKt} = \Delta^{-it}a\Delta^{it} \in \mathcal{A}, \tag{4.40}$$

i.e., the following holds for all real t

$$J\mathcal{A}J = \mathcal{A}', \tag{4.41}$$

$$\Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}, \tag{4.42}$$

$$\Delta^{it}\mathcal{A}'\Delta^{-it} = \mathcal{A}'. \tag{4.43}$$

One wants to find the evolution operator which obeys the analytic structure as defined before. The states $a^\dagger \left|\Omega\right\rangle$ and $b\left|\Omega\right\rangle$ are in the domain of Δ^w for all $0 \leq Re(w) \leq 1/2$ so that the following

$$w \mapsto \left\langle \Delta^{\overline{w}}a^\dagger\Omega\middle|\Delta^w b\Omega\right\rangle , \tag{4.44}$$

is bounded and analytic in the strip $0 \leq Re(w) \leq 1/2$. Consequently the following

$$F(z) = \left\langle \Delta^{\overline{z}/2}a^\dagger\Omega\middle|\Delta^{z/2}b\Omega\right\rangle , \tag{4.45}$$

is bounded and analytic in the strip $0 \leq z \leq 1$. On the left strip one have

$$\begin{aligned} F(it) &= \left\langle \Delta^{-it/2} a^\dagger \Omega \middle| \Delta^{it/2} \Omega \right\rangle, \\ &= \langle \Omega | \, a \Delta^{it} b \, | \Omega \rangle \,, \\ &= \langle \Omega | \, a e^{-iKt} b \, | \Omega \rangle \,. \end{aligned} \tag{4.46}$$

On the right strip one have as well

$$\begin{aligned} F(1+it) &= \left\langle \Delta^{1/2} \Delta^{-it/2} a^\dagger \Omega \middle| \Delta^{1/2} \Delta^{it/2} b \Omega \right\rangle, \\ &= \left\langle \Delta^{1/2} (\Delta^{-it/2} a^\dagger \Delta^{it/2}) \Omega \middle| \Delta^{1/2} (\Delta^{it/2} b \Delta^{-it/2}) \Omega \right\rangle, \\ &= \left\langle (\Delta^{it/2} b^\dagger \Delta^{-it/2}) \Omega \middle| (\Delta^{-it/2} a \Delta^{it/2}) \Omega \right\rangle, \\ &= \left\langle b^\dagger \middle| \, \Delta^{-it} \, | a \rangle \right., \\ &= \langle \Omega | \, b e^{iKt} a \, | \Omega \rangle \,. \end{aligned} \tag{4.47}$$

So the function $F(z)$ provides an analytic continuation of the two-point function satisfying the KMS condition. Recall that for a von Neumann algebra $\mathcal{A}$, we have an automorphism $\gamma_t(\mathcal{A}) = e^{itH} \mathcal{A} e^{-itH}$ for t a real parameter and H the generator. The final conclusion is that the thermal flow $\alpha_t(\mathcal{A}) = e^{i\beta t H} \mathcal{A} e^{-i\beta t H}$ given by the modular group which satisfy the necessary bound to the Hamiltonian to exist is exactly the time evolution group $\gamma_{\beta t}(\mathcal{A})$ with the time parameter rescaled by β which is associated to the thermal temperature, i.e. the following holds

$$\alpha_t = \gamma_{\beta t}, \tag{4.48}$$

since the Tomita-Takesaki theorem holds. The *modular Hamiltonian* operator is defined to be

$$K = -\log \Delta. \tag{4.49}$$

4.2 Araki-Uhlmann Relative Entropy Formula

The usual definition for the entropy of a given density matrix associated to some region can be written in the von Neumann fashion way,

$$S(\rho) = -tr(\rho \log \rho). \tag{4.50}$$

In fact it faces many problems due to the vacuum fluctuations. Motivated by this, one can introduce the relative entropy. Given two density matrices ρ_ψ and ρ_ϕ associated to two different states ψ and ϕ the relative entropy is

$$S(\rho_\phi|\rho_\psi) = tr(\rho_\psi \log \rho_\psi) - tr(\rho_\psi \log \rho_\phi), \tag{4.51}$$

and as discussed it measures the distinguishability of these states. But in continuum, e.g., quantum field theory, the trace is not defined. Strictly speaking, the von Neumann algebra associated to the region is of type III. The construction that follows is due Araki [8, 9] that shown a nice way to define the relative entropy. In Araki way, the relative entropy is not defined under trace class but purely in terms of the *relative modular* operator and the *modular automorphism*.

Start defining a $*-$algebra M of matrices to which A, B and ρ belong. Define a linear functional ω on M which is positive $\omega(x^*x) \geq 0 \, \forall \, x \in M$ and that can be written in terms of a density matrix $\rho_\omega \in M$ as follows

$$\omega(x) = tr\,(\rho_\omega x), \quad x \in M. \tag{4.52}$$

Take the linear functional to be normal faithful positive: $\omega(x^*x) = 0$ if $x = 0$, i.e., $\rho_\omega \geq 0$ and it is in fact satisfied if one consider $\rho_\omega = e^A$. Also follows the usual definition for a density matrix: $tr\, e^A B = \omega(B)$.

Introduce now Ψ and Φ to be cyclic and separating vectors of M on a Hilbert space H. Let $S_{\Phi,\Psi}$ be the relative Tomita operator, as before it is an antilinear operator defined on $M\Psi$ as follows

$$S_{\Phi,\Psi} x\Psi = x^*\Phi, \quad x \in M. \tag{4.53}$$

Also one have that $S_{\Phi,\Psi} S_{\Psi,\Phi} = 1$. The absolute square defines the relative modular operator

$$\Delta_{\Phi,\Psi} = (S_{\Phi,\Psi})^* S_{\Phi,\Psi}. \tag{4.54}$$

Given a vector state Ψ, the relative modular operator $\Delta_{\Phi,\Psi}$ depends only on the normal faithful positive linear functional

$$\omega(x) = (\Phi, x\Phi), \quad x \in M, \tag{4.55}$$

and does not depend on the representative vector Φ.

Once the Tomita-Takesaki theorem is established, for a given $x \in M$ one can

introduce the modular automorphism σ_t^ω

$$\sigma_t^\omega(x) \equiv (\Delta_{\Phi,\Psi})^{it} x (\Delta_{\Phi,\Psi})^{-it} \in M, \tag{4.56}$$

for all real t and σ_t^ω is a continuous one-parameter group of automorphism of M. By its structure, the automorphism depends only on ω.

Now introduce the polar decomposition,

$$S_{\Psi,\Psi} = J_\Psi (\Delta_\Psi)^{1/2}. \tag{4.57}$$

The operator J_Ψ is antiunitary involution such that $(J_\Psi f, J_\Psi g) = (f, g)$ and $(J_\Psi)^2 = 1$. For all $x \in M$,

$$j_\Psi(x) = J_\Psi x J_\Psi \in M'. \tag{4.58}$$

Where the M' is defined as the commutant of M.

Define the positive cone V_Ψ as follows,

$$V_\Psi = \overline{\{(\Delta_\Psi)^{1/4} x \Psi, \quad \forall\, x \in M : \omega(x^* x) \geq 0\}}. \tag{4.59}$$

Some properties of the cone are given below.

- $(f, g) \geq 0 \,\forall\, g \in V_\Psi$ iff $f \in V_\Psi$,
- $\forall\, \Phi \in V_\Psi$ and $x \in M$
$$x j_\Psi(x) \Phi \in V_\Psi, \tag{4.60}$$
- The set of vectors $x j_\Psi(x) \Psi \,\forall\, x \in M$ is dense in V_Ψ,
- For all vectors $\Phi \in V_\Psi$ is cyclic iff it is separating,
- For all vectors $\Phi \in V_\Psi$, $J_\Phi = J_\Psi$ and $V_\Phi = V_\Psi$,
- Given a cyclic and separating Φ exist a unitary u' in M' such that $V_\Phi = u' V_\Psi$ and $J_\Phi = u' J_\Psi (u')^*$ and
$$S_{\Phi,\Psi} = u' J_\Psi (\Delta_{\Phi,\Psi})^{1/2}. \tag{4.61}$$

From the independence of the functional ω on the representative vector when it is defined for some cone V one obtain that in fact it has a unique representative

vector $\xi(\omega) \in V$ such that

$$\omega(x) = (\xi(\omega), x\xi(\omega)). \tag{4.62}$$

In that way for a faithful ω, $\xi(\omega)$ is given by

$$\xi(\omega) = (\Delta_{\Phi,\Psi})^{1/2}\Psi. \tag{4.63}$$

Now introduce the inner product for M as

$$< \eta(x), \eta(y) >= tr(x^* y), \tag{4.64}$$

such that

$$\begin{aligned} \pi(x)\eta(y) &\equiv \eta(xy), \\ \pi'(x)\eta(y) &\equiv \eta(yx). \end{aligned} \tag{4.65}$$

Here $\pi(M)$ is isomorphic to M and π' is the generator of $\pi(M)'$.

Let ρ_Ψ and ρ_ω density matrices as

$$\omega(x) = tr(\rho_\omega x). \tag{4.66}$$

Let Ψ be $\eta(\rho_\psi^{1/2})$. Then for $x \in M$ one have

$$\begin{aligned} &\Delta_{\Phi,\Psi}\eta(x) = \eta(\rho_\omega x \rho_\Psi^{-1}), \\ &J\eta(x) = \eta(x^*), \\ &V = \eta(M^+), \\ &\xi(\omega) = \eta(\rho_\omega^{1/2}), \\ &\sigma_t^\omega(\pi(x)) = \pi(\rho_\omega x \rho_\omega^{-1}). \end{aligned} \tag{4.67}$$

Finally the relative entropy in Araki's way can be defined

$$S(\omega|\Psi) = -(\Psi, (\log \Delta_{\Phi,\Psi})\Psi). \tag{4.68}$$

Strictly speaking, the cone introduction is needed to guarantee the constraint that the relative entropy is always positive or zero.

Chapter 5

Known Results

5.1 The Rindler Wedge

The first proper expression for the modular Hamiltonian was given by Bisognano and Wichmann [1]. At first glance, they were concerned with the duality condition, [see 7],

$$\mathcal{R}'(\mathcal{O}) = \mathcal{R}(\mathcal{O}'), \tag{5.1}$$

where the prime denotes the commutant of the von Neumann algebra associated with the region, as well as the spacelike separated region associated with it. The above duality condition is deeply connected to the question of whether the duality condition alone is a necessary condition to prove the TCP-theorem [25, 26]. In particular, the region considered in Bisognano and Wichmann's paper was the so-called wedge-shaped region W_R of Minkowski space. Given a boost generator in the Poincaré group, one can identify the boost with a unitary operator that acts on the Hilbert space. Then, if one ensures that the boost is strongly continuous on a specific strip and is also an analytic function on the open interior of the strip, one obtains a statement similar to the KMS condition. Furthermore, one arrives at the final conclusion of Bisognano and Wichmann: the geometrical interpretation of the modular operator is given by the Poincaré boost, and the modular conjugation operator J is the CTP operator. In fact, the modular conjugation is identified from the fact that one has a rotation by an angle π over the x^3 space component, i.e., one is identifying the right wedge with the left wedge and vice versa, such that $J\mathcal{R}_R J = \mathcal{R}_L$ holds for $\mathcal{R}_R$ and $\mathcal{R}_L$ being the right and left wedge von Neumann algebras, respectively. In what follows, some specific comments on the construction will be given, particularly regarding the analytic continuation and the wedge of the edge problem.

The Minkowski space coordinates are given by $x = (x^1, x^2, x^3, x^4)$ such that $x \cdot y = x^4 y^4 - x^1 y^1 - x^2 y^2 - x^3 y^2$. Define the right wedge and the left wedge to be $W_R = \{x | x^3 > |x^4|\}$ and $W_L = \{x | x^3 < -|x^4|\}$ respectively, and also for

the causal complement one have $W_R^c = \overline{W_L}$ and $W_L^c = \overline{W_R}$. The $\Lambda = \Lambda(M, y)$ are elements of the Poincaré group $\overline{L_0}$ such that for any x it acts as rotation and translation $\Lambda x = \Lambda(M, y)x = Mx + y$. The separable Hilbert space, $\mathcal{H}$ which carries a strongly continuous unitary representation $\Lambda \to U(\Lambda)$ of the Poincaré group $\overline{L_0}$. Given the translation subgroup, $T(x) = U(I, x)$ the spectral condition says $T(x) = U(I, x) = \int e^{i\,x\cdot p}\,\mu(d^4p)$ such that the support of the spectral measure μ is contained in the closed forward light cone $\overline{V}_+$. Introduce the vacuum vector Ω such that is uniquely defined by $U(\Lambda)\Omega = \Omega$.

Since one is interested in a local Hermitian scalar field, the fields are to be an operator-valued tempered distribution. Assume (X, D, Ω) for X operators defined on $D \subset \mathcal{H}$ and $\mathcal{R}(\mathbb{R}^4)$ being the algebra of operators of (X, D, Ω). For all element of the space of test functions $\mathcal{S}(\mathbb{R}^4)$ there is a mapping into $\mathcal{R}(\mathbb{R}^4)$ defined by $\phi[f]$,

$$\phi[f] = \int d^4x\, f(x)\phi(x). \tag{5.2}$$

Let R be any subset of Minkowski space. Denote $\mathcal{R}(R)$ to be the linear span of the identity operator and all the operators $(\phi[f], D_1)$ such that D_1 is a linear manifold and dense in the Hilbert space $\mathcal{H}$, also the support of the test function $supp(f) \subset R$ is contained in the subset R.

An important step is to characterize the spectral condition in such a way the *timelike tube theorem* [27, 28] will make sense, i.e., one wants to characterize some kind of equivalence between the local region algebra to either its domain of dependence or its timelike envelope made up of timelike curves with same endpoints. After such characterization, it will be clear that the velocity transformation plays the modular operator role.

The unitary representation $x \to T(x)$ of the translation group can be extended to a representation of the semigroup of all complex translation $z = x + iy$ with x and y real, in some sense an analytic completion, also one have $y \in \overline{V}_+$, and the extension is

$$T(z) = \int e^{ipz}\,\mu(d^4p) = e^{iz\cdot P}, \tag{5.3}$$

where the operator-valued function $T(z)$ satisfies $||T(z)|| = 1$ and is a strongly continuous function of z on *closed forward imaginary tube* $\overline{V}_{+i} = \{z | Im(z) \in \overline{V}_+\}$. It is important to note also that the function $T(z)$ is analytic in the sense of the uniform topology on the open forward imaginary tube V_{+i}, which implies that the vector-valued function $T(z)\psi$ of z is strongly analytic on V_{+i} for any $\psi \in \mathcal{H}$. For

$f \in \mathcal{S}(\mathbb{R}^{4n})$, one have

$$T(z)\phi\{f\}\Omega = \phi\{f_z\}\Omega, \tag{5.4}$$

such that the equality follows from the existence of $f_z \in \mathcal{S}(\mathbb{R}^{4n})$ so that

$$\tilde{f}_z(p_1,\ldots,p_n) = \tilde{f}(p_1,\ldots,p_n)\exp\left\{(iz\cdot\sum_{r=1}^{n}p_r)\right\}, \tag{5.5}$$

for $(p_1,\ldots,p_n) \in V_n$, where V_n is the subset of $\mathbb{R}^{4n}$ defined by

$$V_n = \{(p_1,\ldots,p_n)|\sum_{r=1}^{n}p_r \in \overline{V}_+, \quad k=1,\ldots,n\}, \tag{5.6}$$

and $\tilde{f}$ is the Fourier transform of f

$$\tilde{f}(p_1,\ldots,p_n) = \int d^4x_1\ldots d^4x_n\, f(x_1,\ldots,x_n)\exp\left\{(i\sum_{r=1}^{n}x_rp_r)\right\}. \tag{5.7}$$

Define a function $E(p;z)$ of the real four-vector p and the complex four-vector z by

$$E(p;z) = u_0(p\cdot p)u_0(p^{(4)})\exp\{i\cdot p\}, \tag{5.8}$$

in some sense u_0 have the theta function role, it is equal to one for positive argument and equal to zero for negative arguments. Hence, in some sense, is guaranteed that $p^2 \geq 0$ and for the four component it obeys $p^{(4)} \geq 0$ such that $E(p;z) = \exp\{iz\cdot p\}$ for $p \in \overline{V}_+$. Follows that for f_z its Fourier transform obeys,

$$\tilde{f}_z(p_1,\ldots,p_n) = E(p;z)\tilde{f}(p_1,\ldots,p_n), \quad p = \sum_{r=1}^{n}p_r, \tag{5.9}$$

for any $z \in \overline{V}_{+i}$.

Now that the spectral condition is well established, rest to implement the analicity condition. Let T_n be the open tube region in $4n-$*dimensional* complex space $\mathbb{C}^{4n}$, regarded as the direct sum of n replicas of the complex Minkowski space,

$$T_n = \{(z_1,\ldots,z_n)\,|\,z_k \in V_{+i},\, k=1,\ldots,n\}. \tag{5.10}$$

Consider a $n-$*tuplet* of test functions $\{f_k|f_k \in \mathcal{S}(\mathbb{R}^4),\, k=1,\ldots,n\}$. The vector

$$\beta(z_1,\ldots,z_n) = T(z_1)\phi[f_1]T(z_2)\phi[f_2]\ldots T(z_n)\phi[f_n]\Omega, \tag{5.11}$$

is well-defined for all $(z_1,\ldots,z_n) \in \overline{T}_n$, also for $\beta(z_1,\ldots,z_n) = \phi\{f\}\Omega$, denote $f = f(x_1,\ldots,x_n;z_1,\ldots,z_n)$ such that its Fourier transform is given by

$$\tilde{f}(p_1,\ldots,p_n;z_1,\ldots,z_n) = \Pi_{k=1}^{n}\tilde{f}_k(p_k)E\bigg(\sum_{r=k}^{n} p_r;z_k\bigg). \tag{5.12}$$

Also the vector valued function $\beta(z_1,\ldots,z_n)$ of $(z_1,\ldots,z_n)$ is strongly continuous on the closed tube $\overline{T}_n$ and analytic on the open tube T_n. Given the open tube T_n definition, for each $n \geq 1$, let E_n be the set of all functions $f(x_1,\ldots,x_n;z_1,\ldots,z_n)$ defined for $(x_1,\ldots,x_n) \in \mathbb{R}^{4n}$ and $(z_1,\ldots,z_n) \in T_n$ and such that $f \in \mathcal{S}(\mathbb{R}^{4n})$ and such that

$$\tilde{f}(p_1,\ldots,p_n;z_1,\ldots,z_n) = \exp\left\{ i\sum_{r=1}^{n}\sum_{r=k}^{n} z_k \cdot p_k \right\}, \tag{5.13}$$

for all $(p_1,\ldots,p_n) \in V_n$. The set E_n is nonempty, and contains the function

$$\tilde{f}_0(p_1,\ldots,p_n;z_1,\ldots,z_n) = \Pi_{k=1}^{n}E\bigg(\sum_{r=k}^{n} p_r;z_k\bigg). \tag{5.14}$$

With respect to the set E_n, there is a *unique* vector-valued function $\phi(z_1,\ldots,z_n)$ on T_n, defined by

$$\phi(z_1,\ldots,z_n) = \varphi\{f\}\Omega. \tag{5.15}$$

Now is clear that the vector-valued function $\phi(z_1,\ldots,z_n)$ is strongly continuous on T_n. To see that the vector valued function $\phi(z_1,\ldots,z_n)$ is analytic on, T_n consider some function $g(x) \in \mathcal{D}(\mathbb{R}^4)$ such that $\tilde{g}(0) = 1$. Let $\lambda > 1$ such that $\beta(z_1,\ldots,z_n;\lambda)$ with all the β properties previous presented. Consider $f_k(x) = \lambda^4 g(\lambda x)$ for $k = 1,\ldots,n$. On T_n, the vector-valued function follows to be an analytic function. Now from the property of convolution for tempered distributions and the fact ϕ is strong continuous on T_n, $\beta(z_1,\ldots,z_n;\lambda)$ tends to $\phi(z_1,\ldots,z_n)$ as λ tends to infinity uniformly on any closed polydisc contained in T_n and hence ϕ is analytic on T_n. Note, the vector ϕ might be written as

$$\phi(z_1,\ldots,z_n) = \varphi(z_1)\varphi(z_1+z_2)\ldots\varphi(z_1+z_2+\ldots+z_n)\Omega. \tag{5.16}$$

Now is clear that the conditions for the vector-valued functions to be analytic and strongly continuous inside the forward light cone is established. In order to define

the Poincare group action, just take any element $\Lambda = \Lambda(M, x) \in \overline{L}_0$. One have

$$U(\Lambda)\phi(z_1, \ldots, z_n) = \phi(Mz_1 + x, Mz_2, \ldots, Mz_n). \tag{5.17}$$

Also for any $(z_1, \ldots, z_n) \in T_n$ the vector $\phi(z_1, \ldots, z_n)$ is an analytic vector for the Lie algebra of the group $U(\overline{L}_0)$.

Consider now all the velocity transformation $V(e_3, t)$ in the $3 - direction$ given by the one-parameter Abelian subgroup $\{V(e_3, t) | t \in \mathbb{R}^1\}$. The $V(e_3, t)$ have a representation by

$$V(e_3, t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t \\ 0 & 0 & \sinh t & \cosh t \end{bmatrix}. \tag{5.18}$$

By Stone's theorem there exists a unique self-adjoint operator (K_3, D_k) such that

$$V(t) = U(V(e_3, t), 0) = \exp\{-itK_3\}, \quad \forall t \in \mathbb{R}^1. \tag{5.19}$$

Now consider the analytic continuation of the function $V(t)$ to the complex plane such that (K_3, D_K) corresponds to a representation of the additive group of all complex numbers: $\tau \to \exp\{-i\tau K_3\} = V(\tau)$. The spectral resolution is

$$V(\tau) = \exp\{-i\tau K_3\} = \int \exp\{-i\tau s\}\mu_k(ds), \tag{5.20}$$

the domain of the closed operator $V(\tau)$ depends only on $Im(\tau)$. For any complex $\tau = \rho + i\lambda$, with ρ and λ real, let $D_V(\lambda)$ be the linear manifold such that $(V(\tau), D_V(\lambda))$ is closed and normal. Let now $\lambda \neq 0$. Then $D_V(\lambda)$ is a core for all operators $(V(\tau), D_V(Im(\tau)))$ such that $0 \leq Im(\tau)/\lambda \leq 1$. If $\psi \in D_V(\lambda)$, then the vector valued function $V(\tau)\psi$ of τ is well-defined, strongly continuous and bounded on the closed strip $0 \leq Im(\tau)/\lambda \leq 1$ and an analytic function τ on the interior of the strip. Consider now the action of $V(\tau)$ on the vectors $\phi(z_1, \ldots, z_n)$. Let $z = x + iy$, with x and y real, and z any complex number. Then one have

$$z(\tau) = V(e_3, \tau)z, \tag{5.21}$$

such that for $\tau = i\lambda$, which is the one that matters, one have

$$\begin{aligned} z^1(i\lambda) &= x^1 + iy^1, \qquad z^2(i\lambda) = x^2 + iy^2, \\ z^3(i\lambda) &= (x^3\cos\lambda - y^4\sin\lambda) + i(y^3\cos\lambda + x^4\sin\lambda), \\ z^4(i\lambda) &= (x^4\cos\lambda - y^3\sin\lambda) + i(y^4\cos\lambda + x^3\sin\lambda). \end{aligned} \tag{5.22}$$

Now let $(z_1,\ldots,z_n)$ be an n-tuple of complex four-vectors $z_k = x_k + iy_k$, where x_k and y_k are real, for the right wedge one put the condition $y_k^1 = y_k^2 = 0$, $y_k^4 > |y_k^3|$, for $k = 1,\ldots,n$. Now restricting x_k to the right wedge W_R: $x^3 > |x^4|$, then $Im(z(i\lambda)) \in V_+$ for all $\lambda \in [0,\pi/2]$. Thence, since the tube T_n is made up of $(z_1,\ldots,z_n)$ such that $z_k \in V_{+i} = \{z|Im(z) \in V_+\}$ for all $k = 1,\ldots,n$. Is clear that $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$ for all λ on the closed interval $[0,\pi/2]$. Recall now that the vector valued functions $\phi(z_1,\ldots,z_n)$ is an analytic function of $(z_1,\ldots,z_n)$ on the tube T_n. Is clear that if one consider some open neighborhood N (in the complex λ plane) of $[0,\pi/2]$ such that $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$for all $\lambda \in N$, the vector $\phi(z_1(i\lambda),\ldots,z_n(i\lambda))$ is well-defined and regarded as a function of λ is an analytic function. Also follows that $\phi(z_1,\ldots,z_n) \in D_V(Im(i\lambda))$ for $\lambda \in N$, hence, $\phi(z_1,\ldots,z_n)$ is in the domain $D_V(\pi/2)$ and

$$V(i\lambda)\phi(z_1,\ldots,z_n) = \phi(z_1(i\lambda),\ldots,z_n(i\lambda)), \tag{5.23}$$

for all $\lambda \in [0,\pi/2]$.

A nice Bisognano & Wichmann realization is that the above consideration on velocity transformation defined on the strip can be related to a mapping between regions. Consider the *involutory mapping* $x \to \mathcal{J}x$ of Minkowski space onto itself, defined by,

$$\begin{aligned} \mathcal{J}x &= -R(e_3,\pi)x, \\ \mathcal{J}(x^1,x^2,x^3,x^4) &= (x^1,x^2,-x^3,-x^4), \end{aligned} \tag{5.24}$$

where here $R(e_3,\pi)$ denotes the rotation by angle π about the $3-axis$. Hence, we have now a mapping between the right wedge W_R to the left Wedge W_L. Also, one can realize that the definition of the involutory mapping can be defined by

$$\mathcal{J} = V(e_3, i\pi). \tag{5.25}$$

Now one just need to see how the involutory mapping act on fields.

Assume the existence of an antiunitary involution Θ_0, interpreted as an inversion transformation with respect to the origin in Minkowski space. The operator Θ_0 satisfies

$$\begin{aligned} &\Theta^2 = \mathcal{I}, \quad \Theta_0\Omega = \Omega, \quad \Theta_0 U(M,x)\Theta_0 = U(M,-x), \\ &\Theta_0\phi(x)\Theta_0 = \phi(-x). \end{aligned} \tag{5.26}$$

Introduce now another antiunitary involution J, such that

$$J = U(R(e_3,\pi),0)\Theta_0 = \Theta_0 U(R(e_3,\pi),0), \tag{5.27}$$

is implementing both mapping between two spacelike regions and also the mapping the charge sector into its conjugate sector. Follows now the properties of the involution J

$$J^2 = \mathcal{I}, \quad J\Omega = \Omega, \quad JU(M,x)J = U(\mathcal{J}M\mathcal{J}, \mathcal{J}x), \tag{5.28}$$

which also preserves the domain, $JD_1 = D_1$, such that

$$J\phi[f]J = \phi[f^j]^* \quad \text{on } D_1, \tag{5.29}$$

for any test function $f \in S(\mathbb{R}^4)$ and where $f^j(x) = f(\mathcal{J}x)$.

Let us now consider the involution act on well-defined operators. For the case of a (real) velocity transformation in the 3-direction one have

$$JV(t)J = V(t), \text{ all real } t, \tag{5.30}$$

also held the following properties

$$\begin{aligned} &JD_K = D_K, \quad J(K_3, D_K)J = -(K_3, D_K), \\ &JD_V(\lambda) = D_V(-\lambda), \quad J(V(\tau), D_V(\lambda))J = (V(\tau^*), D_V(-\lambda)), \end{aligned} \tag{5.31}$$

for any complex $\tau = \rho + i\lambda$, ρ and λ real. The mapping J is an object on Hilbert space that act as $\mathcal{J}$ on Minkowski space. Note that if $supp(f) \subset W_R$, then $supp(f^j) \subset W_L$, and vice versa. The final conclusion is that conjugation with J thus maps operators locally associated with the right wedge W_R into operators

locally associated with the left wedge W_L. Also holds the following

$$JU(\Lambda)J = U(\Lambda), \text{ all } \Lambda \in \overline{L}_0(W_R), \tag{5.32}$$

where here $\overline{L}_0(W_R)$ is the group of all Poincaré transformation which maps W_R onto W_R.

Let $\mathcal{R}$ be a von Neumann algebra of operators on a separable Hilbert space which has a cyclic and separating vector Ω, and let $\mathcal{R}'$ be its commutant. Then there exist a unique antiunitary involution J, and a unique self-adjoint operator (K, D_K), satisfying the following relations

$$J\Omega = \Omega, \quad \Omega \in D_K, \quad K\Omega = 0; \tag{5.33a}$$
$$J\mathcal{R}J = \mathcal{R}'; \tag{5.33b}$$
$$JD_K = D_K, \quad J(K, D_K)J = (-K, D_K); \tag{5.33c}$$
$$\exp\{-itK\}\mathcal{R}\exp\{itK\} = \mathcal{R}, \tag{5.33d}$$
$$\exp\{-itK\}\mathcal{R}'\exp\{itK\} = \mathcal{R}', \tag{5.33e}$$

for all real t, and the one-parameter group of unitary operators $\exp\{-itK\}$ is thus, acting by conjugation, a group of automorphism of $\mathcal{R}$ and of $\mathcal{R}'$.

Now is clear that $\exp\{2\pi K\}$ is the Tomita operator Δ and consequently one can identify the operator K by $2\pi K = \ln \Delta$. Also, finally, the Haag duality is established $\mathcal{R}_c(W_R) = \mathcal{R}_c(W_L)'$.

5.2 The Lightcone and the massless Fields

To fathom the Hislop and Longo [29] result concerning the modular operator expression for the double cone case, it is reasonable to comprehend first the Buchholz [30] result on what concern the modular operator for the lightcone. Both constructions have a massless scalar field under consideration.

Buchholz was interested mainly on the problem of minimal and local conditions to characterize the interacting and non-interacting fields. Under the same mathematical arguments of the section 5.1, one can invoke the Poincaré group representation and the spectrum condition to establish the asymptotic fields ϕ^{in} and ϕ^{out} and then state that there is interaction if these two fields do not coincide.

Also, one can invoke the *Huyghen's Principle* in the first algebraic form

$$\mathcal{F}^{out}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O}_+)', \tag{5.34}$$

where $\mathcal{O}$ is any bounded or unbounded region with future $\mathcal{O}_+$ and $\mathcal{F}^{out}(\mathcal{O})$ is the smallest von Neumann algebra containing all algebras $\mathcal{F}^{out}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}$. Also, it is useful to state the Huyghen's principle in the second form

$$[\phi^{in}(x), \phi(y)] = [\phi^{out}(y), \phi(x)] = 0, \quad (x-y) \in V_+, \tag{5.35}$$

in few words (*actio in distans*) the commutator between fields and asymptotic fields vanishes at spacelike distances.

The first part of the criterion found by Buchholz, which doesn't deal with the algebraic tool, says that the local fields ϕ on the Fock space does not interact *if*

$$[\phi(x), \phi(y)] = 0, \quad (x-y) \in V_+ \cup V_-. \tag{5.36}$$

Once the Huyghen's principle is under consideration one just need to recall the fact the vacuum vector is separating for $(\phi^{in} - \phi^{out})$ and also ϕ^{in} and ϕ^{out} create both the same one-particle state from the vacuum: $\phi^{in}(x)\Omega = \phi^{out}(x)\Omega$. Therefore $\phi^{in} = \phi^{out}$. The *if* condition here is essential, one have the criterion that the local fields at timelike distance do not interact if their commutator vanishes.

To see the *only if* part start from the fact concerning net of von Neumann algebras: $\mathcal{F}^{in}(V_-) \subseteq \mathcal{F}(V_-)$, $\mathcal{F}^{out}(V_+) \subseteq \mathcal{F}(V_+)$ and because of Huygens principle $\mathcal{F}(V_-) \subseteq \mathcal{F}^{in}(V_+)'$, $\mathcal{F}(V_+) \subseteq \mathcal{F}^{out}(V_-)'$. Establish the *timelike duality*

$$\mathcal{F}^{in}(V_+)' = \mathcal{F}^{in}(V_-), \tag{5.37}$$

to see that indeed

$$\mathcal{F}(V_-) = \mathcal{F}^{in}(V_-), \quad \mathcal{F}(V_+) = \mathcal{F}^{out}(V_+). \tag{5.38}$$

The condition to have no interaction now reads as

$$\mathcal{F}(V_-) = \mathcal{F}^{in}(V_-) = \mathcal{F}^{out}(V_-) = \mathcal{F}^{out}(V_+)' = \mathcal{F}(V_+)'. \tag{5.39}$$

One can now consider an arbitrary translation by a factor $a \in \mathbb{R}$ such that $C_\pm \subset V_\pm + a$. Thence follows that $\mathcal{F}(C_-) \subset \mathcal{F}(C_+)'$ and using weak additivity, see [7],

one can ensure (5.38). Rest now to establish the condition of no interaction, i.e., the timelike duality. Here essentially appears the Tomita-Takesaki theory because of the Bisognano and Wichmann result that was obtained from the work concerning the same kind of duality that is under concern here.

Start recalling the definition of the Tomita's operator: $SF^{in}\Omega = F^{in*}\Omega$ for $F^{in} \in \mathcal{F}^{in}(V_+)$. Take the polar decomposition as usual to be $S = J\Delta^{1/2}$. The main result of the Tomita-Takesaki theory is the desired equality

$$J\mathcal{F}^{in}(V_+)J = \mathcal{F}^{in}(V_+)'. \tag{5.40}$$

As in the Bisognano and Wichmann construction one have $J = W^{in}\theta^{in}$ where W^{in} is an operator with eigenvalue ± 1 corresponding if there is an odd or even number of particles and θ^{in} is the TCP operator. From the fact $J\phi^{in}(x)J = -\phi^{in}(-x)$ the desired duality is established

$$\mathcal{F}^{in}(V_+)' = J\mathcal{F}^{in}(V_+)J = \mathcal{F}^{in}(V_-). \tag{5.41}$$

Consider the dilation $D(\lambda)$ which leaves the vacuum vector invariant and act on ϕ^{in} as

$$D^{in}(\lambda)\phi^{in}(x)D^{in}(\lambda)^{-1} = e^{\lambda}\phi^{in}(e^{\lambda}x), \tag{5.42}$$

so that the dilation is continuous, unitary and $\lambda \in \mathbb{R}$. By Stone's theorem one can introduce a selfadjoint generator for the group of these operators and so λ can be complex according. Now one have

$$D^{in}(\lambda)\phi^{in}(x_1)\dots\phi^{in}(x_n)\Omega = e^{\lambda n}\phi^{in}(e^{\lambda}x_1)\dots\phi^{in}(e^{\lambda}x_n)\Omega, \tag{5.43}$$

requiring the configuration to be on the forward lightcone V_+ one can extend this expression analytically to $0 \geq Im\lambda \geq \pi$. Now is clear that for $\lambda \to i\pi$ one obtain

$$\begin{aligned} &D^{in}(i\pi)\phi^{in}(x_1)\dots\phi^{in}(x_n)\Omega \\ &\quad = (-1)^n\phi^{in}(-x_1)\dots\phi^{in}(-x_n)\Omega \\ &\quad = (-1)^n\phi^{in}(-x_n)\dots\phi^{in}(-x_1)\Omega, \end{aligned} \tag{5.44}$$

now just apply the $W^{in}\theta^{in}$ operator to get

$$W^{in}\theta^{in}D^{in}(i\pi)\phi^{in}(x_1)\dots\phi^{in}(x_n)\Omega = (\phi^{in}(x_1)\dots\phi^{in}(x_n))^*\Omega. \tag{5.45}$$

Which is exactly what is expected from S operator.

The final conclusion is that $S = W^{in}\theta^{in}D^{in}(i\pi)$ such that $J = W^{in}\theta^{in}$ and $\Delta^{1/2} = D^{in}(i\pi)$ because of the uniqueness of the polar decomposition. The modular flow is related to the dilation from the conformal transformations. By the same arguments, the modular Hamiltonian is proportional to the generator of dilations.

5.3 The Massless Double Cone

The main goal of this section is to make a complete exposition (under the tools obtained up to now) on what concern the double cone case. One should keep in mind that just like the others two previous sections, the modular operator was obtained as a gift from the geometrical insight concerning the *massless* scalar field.

Start from an observation. For free fields, the local algebra for $\mathcal{O}$ being the double cone region is determined by the value of the field and its time derivative on the time slice $t = 0$ hypersurface. The Cauchy data for such local algebra $\mathcal{R}(\mathcal{O})$ is mapped by the modular conjugation operator such that $\mathcal{R}(\mathcal{O}^c) = \mathcal{R}(\mathcal{O})'$.

Consider $1+1$ dimensions and take the time zero hypersurface to be an interval $(-1, 1)$. The inversion map $x \to x^{-1}$ sends the interval $(-1, 1)$ onto $(-\infty, -1) \cup (1, \infty)$. A nice fact is that the interval $(-1, 1)$ is preserved by the one-parameter group of coordinate transformation $x \to x'(\lambda)$ given by $(x' - 1)(x' + 1)^{-1} = e^{\lambda}(x - 1)(x + 1)^{-1}$ for $\lambda \in \mathbb{R}$. Just like in the previous sections, one can take the analytic continuation for $\lambda = i\pi$ to obtain $x \to x'(i\pi) = x^{-1}$.

Consider the field operator $\phi[f]$, essentially self-adjoint on the dense domain $D_0 \subset \Gamma(\mathcal{H})$. Here $\Gamma(\mathcal{H})$ denotes the boson Fock space constructed from $\mathcal{H} = L^2[H_0, \theta(p_0)\delta(p^2)d^4p]$ where $H_0 = \{p \in \mathbb{R}^4 | p_0 > 0, p^2 = 0\}$ is the mass-shell hyperboloid. The Cauchy data is given by $\phi|_{x_0=0} = f$ and $\partial_0\phi|_{x_0=0} = g$.

Let $\mathcal{O}$ denote the double cone on the Minkowski spacetime $\mathbb{R}^{d+1}$ with base on the open unit ball B centered at the origin in the time zero hyperplane $\mathbb{R}^d$. Consider the one-parameter group of conformal transformation that preserves $\mathcal{O}$, namely

$$(u, v) \mapsto ((Z(u, \lambda), Z(v, \lambda))), \tag{5.46}$$

where

$$Z(z_i, \lambda) = \frac{(1 + z_i) - e^{-\lambda}(1 - z_i)}{(1 + z) + e^{-\lambda}(1 - z_i)} = \frac{g(z_i, \lambda)}{f(z_i, \lambda)}, \tag{5.47}$$

with

$$f(z_i,\lambda)=\frac{(1+z_i)+e^{-\lambda}(1-z_i)}{2},\quad g(z_i,\lambda)=\frac{(1+z_i)-e^{-\lambda}(1-z_i)}{2}. \tag{5.48}$$

The z_i, with $i=1,2$, defined in such a way it simplifies

$$\begin{aligned} z_1 &= x_0+r=u,\\ z_2 &= x_0-r=v,\\ r &= |\vec{x}|=\sqrt{x_1^2+\dots x_d^2}. \end{aligned} \tag{5.49}$$

Now let $V(\lambda)$ with $\lambda\in\mathbb{R}$, be the linear operator on $\mathcal{H}$ determined by

$$\begin{aligned} V(\lambda)\phi[f]\Omega &= \phi[f_\lambda]\Omega\\ &= \gamma(u,v,\lambda)\phi(Z(u,\lambda),Z(v,\lambda))\Omega. \end{aligned} \tag{5.50}$$

The double cone is invariant under the mapping $f\to f_\lambda$ and the above defines $V(\lambda)$ on a dense domain of $\mathcal{H}$. The cocycle γ is given by

$$\gamma(u,v,\lambda)=F(u,\lambda)F(-v,-\lambda),\quad f(z,\lambda)=f^{-D}(z,\lambda), \tag{5.51}$$

where $D=(d-1)/2$. Follows that

$$\gamma(u,v,\lambda)=2^6[(1+u)+e^{-\lambda}(1-u)]^{-3}[(1-v)+e^{\lambda}(1+v)]^{-3}, \tag{5.52}$$

then one can see that indeed for $\lambda=i\pi$

$$\begin{aligned} \gamma(u,v,i\pi) &= 2^6[(1+u)-1+u]^{-3}[(1-v)-1-v]^{-3}\\ &= -\frac{1}{(x^0-r^2)^3}. \end{aligned} \tag{5.53}$$

The above realization (due [29, 31]) is great because if one take the relativistic ray inversion map $\rho(x)=-x(x^2)^{-1}$ for $x^2=x_0^2-|\vec{x}|^2$ and consider the $T=\{x|x^2>0\}$ and $S=\{x|x^2<0\}$ as sets of timelike and spacelike vectors respectively. For $f\in D(T\cup S)$, the test function transforms like $f_\rho(x)=I(x)x^2f[\rho(x)]=-(x^2)^{-3}f[\rho(x)]$ where $I(x)=-(x^2)^{-4}$ is the Jacobian of ρ. So is a nice fact that the one-parameter group of conformal transformation led to the geometrical interpretation of the modular operator for the double cone: $\Delta_B^{-i\lambda}=V(2\pi\lambda)$.

Indeed, we can go further and compute $K_0=\frac{d}{d\lambda}V(\lambda)|_{\lambda=0}$, the generator of V

[31]. First note

$$\begin{aligned} &Z(z,0) = z, \quad Z'(z,0) = \frac{g'}{f} - \frac{g}{f^2}f' = \frac{1-z^2}{2}, \\ &f(z,0) = 1, \quad g(z,0) = z, \\ &f'(z,0) = -\frac{1-z}{2}, \quad g'(z,0) = \frac{1-z}{2}. \end{aligned} \tag{5.54}$$

Also, for the cocycle follows,

$$\begin{aligned} &\gamma(u,v,0) = 1, \quad F'(z,s)|_{\lambda=0} = \frac{D}{2}(1-z), \\ &\gamma'(u,v,0) = F'(u,s)|_{\lambda=0} - F'(-v,-s)|_{\lambda=0} = -\frac{D}{2}(u+v) = -Dx_0. \end{aligned} \tag{5.55}$$

Following, define the derivative,

$$\begin{aligned} (V(\lambda)\phi)(u,v)' =& \gamma'(u,v,\lambda)\phi(Z(u,\lambda),Z(v,\lambda)) \\ &+ \gamma(u,v,\lambda)(\partial_u\phi(Z(u,\lambda),Z(v,\lambda)))Z'(u,\lambda) \\ &+ \partial_v\phi(Z(u,\lambda),Z(v,\lambda))Z'(v,\lambda), \end{aligned} \tag{5.56}$$

to finally define the generator $(K_0\phi)(u,v)$, such that,

$$\begin{aligned} (K_0\phi)(u,v) = (V(\lambda)\phi)(u,v)'|_\lambda &= -\frac{D}{2}(u+v)\phi(u,v) + \frac{1}{2}\partial_u\phi(u,v)(1-u^2) \\ &\quad + \frac{1}{2}\partial_v\phi(u,v)(1-v^2) \\ &= -\frac{D}{2}(u+v)\phi - \frac{1}{2}u^2\partial_u\phi - \frac{1}{2}\partial_v\phi + \frac{1}{2}\partial_u\phi + \frac{1}{2}\partial_v\phi. \end{aligned} \tag{5.57}$$

Now, recall $u = x_0 + r$ and $v = x_0 - r$, such that,

$$\partial_u = \frac{1}{2}(\partial_0 + \partial_r), \quad \partial_v = \frac{1}{2}(\partial_0 - \partial_r), \tag{5.58}$$

so that

$$\begin{aligned}(K_0\phi)(x_0,r) &= -\frac{1}{4}((x_0+r)^2(\partial_0+\partial_r)\phi+(x_0-r)^2(\partial_0-\partial_r))\phi - Dx_0\phi+\frac{1}{2}\partial_0\phi\\ &= -\frac{1}{2}(x_0+r^2)\partial_0\phi - x_0r\partial_r\phi - Dx_0\phi+\frac{1}{2}\partial_0\phi\\ &= \frac{1}{2}(1-(x_0^2-r^2))\partial_0\phi - x_0r\partial_r\phi - Dx_0\phi.\end{aligned} \tag{5.59}$$

Now, just impose the time slice $x_0 = 0$ condition to obtain,

$$(K_0\phi)|_{x_0=0} = \frac{1}{2}(1-r^2)\partial_0\phi|_{x_0=0}, \tag{5.60}$$

$$(\partial_0 K\phi)|_{x_0=0} = \frac{1}{2}(1-r^2)\partial_r^2\phi - r\partial_r\phi - D\phi\,|_{x_0=0}. \tag{5.61}$$

Finally, we can write

$$\log\Delta = 2\pi K_0. \tag{5.62}$$

5.4 The Role of the Second Quantization

From the three previous sections one can realize that the known cases for the modular Hamiltonian theory, the modular operator is obtained majority from geometrical arguments and also the authors are usually concerning some other problem, like duality and interactions conditions, but ended up finding the so important expression for the modular operator. Hence, it is of interest in describing the modular operator under arguments of the CCR algebra, Weyl representation of the algebra and the second quantization, i.e., under the language of the section 2.3. It was mainly done by [32, 33, 34]. The main points and realizations are about to be reproduced in what follows of this section.

Recall from section 2.3 the complete Hilbert space $\mathcal{H} = L^2 = L^2(\mathbb{R}^3, d^3p/(\vec{p}^2 + m^2)^{1/2})$ with $m \geq 0$. From that complete Hilbert space, construct the symmetric Fock space

$$\begin{aligned}\mathcal{H}_s &= \bigoplus_{n=0}^{\infty}\mathcal{H}_s^{(n)} = \left(\bigoplus_n \mathcal{H}\right)_s,\\ \mathcal{H}_s^{(0)} &= \mathbb{R}.\end{aligned} \tag{5.63}$$

Define the Segal field operator,

$$\phi_s(\tilde{f}) = \frac{1}{\sqrt{2}}[a(\tilde{f}) + a^*(\tilde{f})], \tag{5.64}$$

such that $a^*(\tilde{f}) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}$ and $a(f)$ is its adjoint.

Consider the closed subset of L^2,

$$\begin{aligned} K &= \{f \in L^2 | \overline{\tilde{f}(-p)} = \tilde{f}(p)\}, \\ K' &= \{f \in L^2 | \overline{\tilde{f}(-p)} = -\tilde{f}(p)\}. \end{aligned} \tag{5.65}$$

In fact, K and K' are not (complex) subspaces of L^2. Define the real Hilbert space L of $L^2(\mathbb{R}^d)$ considered as the real linear space with the real scalar product,

$$(\tilde{f}_1, \tilde{f}_2)_L = Re(\tilde{f}_1, \tilde{f}_2)_{L^2}. \tag{5.66}$$

Also, K and K' are orthogonal closed subspaces of L and satisfy

$$K' = \beta K = K^{\perp}, \tag{5.67}$$

where β is the operator of multiplication by the imaginary unit. Then, if $\tilde{f} \in L^2$, there is a unique decomposition

$$\tilde{f} = \tilde{g} + i\tilde{h}, \quad \tilde{g} \in K, \tilde{h} \in K, \tag{5.68}$$

which implies

$$\begin{aligned} \phi_s(\tilde{f}) &= \phi_s(\tilde{g}) + \phi_s(i\tilde{h}) \\ &= \phi(\tilde{g}) + \pi(\tilde{h}), \end{aligned} \tag{5.69}$$

where

$$\phi(\tilde{g}) = \frac{1}{\sqrt{2}}[a^*(\tilde{g}) + a(\tilde{g})], \tag{5.70}$$

$$\pi(\tilde{h}) = \frac{1}{\sqrt{2}} i[a^*(\tilde{h}) - a(\tilde{h})]. \tag{5.71}$$

Introduce the wave equation,

$$(\Box + m^2)\Phi(x) = 0. \tag{5.72}$$

The function $\Phi(x)$ is uniquely determined by the knowledge of its initial condition $a(x) = \dot{\Phi}(0,x)$ and $b(x) = \Phi(0,x)$. Now the following holds

$$\tilde{f}(p) = \tilde{g}(p) + i\tilde{h}(p), \tag{5.73}$$

and

$$\tilde{a}(p) = -\tilde{g}(p), \quad \tilde{b}(p) = (p^2 + m^2)^{-1/2}\tilde{h}(p). \tag{5.74}$$

Then easily follows that L can be identified with the real Hilbert space L. Define $\mathcal{H} = \mathcal{H}_\phi^{(m)} \oplus \mathcal{H}_\pi^{(m)}$, where $\mathcal{H}_\phi^{(m)}$ and $\mathcal{H}_\pi^{(m)}$ are the real Hilbert spaces obtained by completing the space $S_r(\mathbb{R}^3)$ of the real C^∞ functions of rapid decrease with respect to the scalar product

$$\begin{aligned} (f,g)_\phi &= \int \tilde{f}(p)\tilde{g}(p)\,\omega^{-1}\,d^3p, \\ (f,g)_\pi &= \int \tilde{f}(p)\tilde{g}(p)\,\omega\,d^3p. \end{aligned} \tag{5.75}$$

The correspondence is now the following

$$\tilde{f} = \tilde{g} + i\tilde{h} \in L = K \oplus \beta K \to < -\tilde{g}, \omega^{-1}\tilde{h} > \in \mathcal{H} = \mathcal{H}_\phi^{(m)} \oplus \mathcal{H}_\pi^{(m)}. \tag{5.76}$$

The multiplication by ω^{-1} is a unitary operator from $\mathcal{H}_\phi^{(m)}$ onto $\mathcal{H}_\pi^{(m)}$, then one can write

$$\mathcal{H} = \mathcal{H}_\phi^{(m)} \oplus \mathcal{H}_\pi^{(m)}. \tag{5.77}$$

It means that if $f \in S_r(\mathbb{R}^4)$ and $suppf \subset \mathcal{O} \subset \mathbb{R}^4$, the function $\Phi(x)$ has initial condition $a(x) = \dot{\Phi}(0,x)$ and $b(x) = \Phi(0,x)$ which satisfies

$$supp\, a,\ supp\, b \subset (\mathcal{O}')' \cap S_{x_0=0}, \tag{5.78}$$

where $S_{x_0=0}$ is the three-dimensional hypersurface at x_0 fixed and $\mathcal{O}'$ is the causal complement of $\mathcal{O}$.

Now is time to deal to the von Neumann algebra for the above construction. Take a linear subspace L of H, associate with L the von Neumann algebra $\mathcal{R}(\mathcal{L})$ defined by

$$\mathcal{R}(L) = \{\exp\{i\phi_s(h)\} | h \in L\}''. \tag{5.79}$$

Analogous, if K_1 and K_2 are two linear subspaces of K, define

$$\mathcal{R}(K_1, K_2) = \{\exp\{i\phi(g)\}\exp\{i\pi(h)\}|g \in K_1,\, h \in K_2\}''. \tag{5.80}$$

If L is a linear subspace of H and K_1 and K_2 are two linear subspaces of K such that $L\tilde{K}_1 \oplus \beta K_2$ imply that

$$\mathcal{R} = \mathcal{R}(K_1, K_2). \tag{5.81}$$

Also define

$$H_\alpha^c = \beta H_\alpha^\perp, \quad \mathcal{R}(H_\alpha)^c = \mathcal{R}(H_\alpha)', \tag{5.82}$$

such that the duality now reads

$$\mathcal{R}(K_1, K_2)' = \mathcal{R}(K_2^\perp, K_1^\perp). \tag{5.83}$$

Now if one take a closed region $\mathcal{O} \subset \mathbb{R}^4$ an as open set and the subspace $L = D_r(\mathcal{O})/S_r^0(\mathbb{R}^4)$, where now one can associate to the region a class of test functions, $D_r(\mathcal{O}) = \{\phi \in C_r^\infty(\mathbb{R}^4)\,|supp\phi \text{ is a compact set contained in } \mathcal{O}\}$ and $S_r^0(\mathbb{R}^4) = \{h \in S_r(\mathbb{R}^4)\,|(h,h)_H = 0\}$. The algebra $\mathcal{R}(L)$, which is also the same as $\mathcal{R}(\mathcal{O})$ has the physical interpretation of the algebra of the observable, which can be measured in the spacetime region $\mathcal{O}$.

Consider $\mathcal{O} = C(B)$ for $B \subset \mathbb{R}^3$ is an open set and

$$C(B) = \{x \in \mathbb{R}^4 | (x-y)^2 < 0,\, y \in S_0 \cap B^c\}. \tag{5.84}$$

If one consider also $L = D_r(C(B))|S_r^0(\mathbb{R}^4)$ follows that $\overline{L} = \mathcal{H}^{(m)}(B) \oplus \mathcal{H}_\pi^{(m)}(B)$,

$$F_{\phi,\pi}^{(m)}(B) = \overline{\phi \in D_r(B)}^{\mathcal{H}_{\phi,\pi}^{(m)}}, \tag{5.85}$$

then

$$\mathcal{R}(C(B)) = \mathcal{R}(\mathcal{H}_\phi^{(m)}(B), \omega\mathcal{H}_\pi^{(m)}(B)). \tag{5.86}$$

Now is time to consider the coherent vectors and operators using Weyl unitaries. Consider the coherent vectors

$$e^x = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} x^{\otimes_n}. \quad x \in H, \tag{5.87}$$

such that it form a total set. Define the vacuum vector state to be $\Omega = e^0 \in e^H$. Then the Weyl unitarires $W(x)$ are defined for $x \in H$ on e^H, by their action on the

vacuum Ω and by their canonical commutation relations (CCR) as follows

$$W(x)e^0 = e^{-\frac{1}{4}||x||^2}e^{ix/\sqrt{2}}, \quad x \in H, \tag{5.88}$$

$$W(x)W(y) = e^{-\frac{i}{2}Im<x,y>}W(x+y), \quad x,y \in H. \tag{5.89}$$

Now as before, for any closed subspace K of H, define a von Neumann algebra

$$\mathcal{R}(K) = \{W(h)|h \in K\}'', \tag{5.90}$$

on e^H, the second quantized algebra of K.

Definition 5.4.1 *The real closed subspace $K \leq_{\mathbb{R}} H$ is standard if the following conditions are fulfilled*

- *$K + iK$ is dense in H,*
- *$K \cap iK = \{0\}$.*

Also one have the following theorem concerning the vacuum vector state and the standard property.

Theorem 5.4.1 *The vacuum vector state e^0 is cyclic and separating for $\mathcal{R}(K)$, in some sense it means the $\mathcal{R}(K)$ is in standard form with respect to the vacuum, if and only if K is standard.*

The symplectic complement of the standard subspace K is the standard subspace $K' = \{h' \in H : Im < k, h' >= 0 \, \forall k \in K\}$, also K is called a factor (or factorial) if $K \cap K' = \{0\}$, which is equivalent to $\mathcal{R}(K)$ being a factor von Neumann algebra, i.e. $\mathcal{R} \cap \mathcal{R}(K)' = \mathbb{C}1$. In this case, K is equipped with a non-degenerate form $\sigma(h,k) = Im < h,k >$ for $h,k \in K$, and for K it becomes a (real) symplectic space.

Let A be a closed densely defined linear operator on H with domain $D(A)$. Then

$$e^A : e^H \to e^H \tag{5.91}$$

is the closure of the linear operator acting on linear combinations of coherent vectors with exponent in $D(A)$ such that

$$\Gamma(A)e^h = e^A e^h = e^{Ah}, \tag{5.92}$$

the exponentiation preserves selfadjointness, passivity, unitarity but not boundness. The $\Gamma(A)$ is the closure of the operator on e^H defined on the linear span of coherent vectors e^h, $h \in D(A)$.

Now just recall that for a von Neumann algebra $\mathcal{R}$ in standard form with respect to the vacuum, there are associated the Tomita operators S, J, Δ, where S is the closure of the operator S_0 such that

$$\begin{aligned} &S_0 : \mathcal{R}\Omega \to \mathcal{R}\Omega, \\ &S_0 A\Omega = A^*\Omega, \end{aligned} \tag{5.93}$$

and $S = J\Delta^{1/2}$ is its polar decomposition. So then for the standard subspace K associate a closed, densely defined conjugate linear operator,

$$\begin{aligned} s :& K + iK \to K + iK, \\ & h + ik \to h - ik, \quad h, k \in K. \end{aligned} \tag{5.94}$$

Moreover, if $s = j\delta^{1/2}$ is the polar decomposition of s, they satisfy,

$$j^* = j = j^{-1}, \quad j\delta = \delta^{-1}j, \quad jK = K', \quad \delta^{it}K = K, t \in \mathbb{R}. \tag{5.95}$$

The operators $J = \Gamma(j)$, $\Delta = \Gamma(\delta)$ are respectively the modular conjugation and the modular operator of $A(K)$ with respect to Ω.

Consider the complex Hilbert space $H = L^2(\mathbb{R}^D)$. The Sobolev spaces $H_m^\alpha = H_m^\alpha(\mathbb{R}^3)$ are the completion of $D(\omega_m^\alpha) \subset L^2(\mathbb{R}^3)$ such that ω_m^α is the multiplication operator by $\omega_m(p)^\alpha = (m^2 + p^2)^{\alpha/2}$ in Fourier form and H_m^α is defined under the convergence of the norm $||x||_{\alpha,m} = ||\omega_m^\alpha x||$. The following duality holds

$$(H_m^\alpha)^* = H_m^{-\alpha}, \quad \forall \alpha \in \mathbb{R}, \tag{5.96}$$

so that there is an inner product defined for both H_m^α and $H_m^{-\alpha}$

$$< f, g > \in H_m^\alpha \times H_m^{-\alpha} \mapsto \int_{\mathbb{R}} \overline{\hat{f}} g. \tag{5.97}$$

In fact, the usual Sobolev space definition is for $m = 1$ so that $H_m^\pm(\mathbb{R})$ are totally equivalent with respect to the mass m because the Sobolev space is defined with respect to the convergence of $(p^2 + 1)^{\alpha/2}$ in such a way a rescale on p is always possible (to better details see [31, 33]).

Now it is of interest to study the local Sobolev space. Consider an open, non-empty subset of $\mathbb{R}^3$ denoted by $\mathcal{O}$ and $\mathcal{O}'$ its complement with respect to the interior. Since $\mathcal{O}$ and $\mathcal{O}'$ is non-empty, $H_m(\mathcal{O})$ is the closed, real linear subspace of the complex Hilbert space. For each region $\mathcal{O}$ contained in $\mathbb{R}^3$, define

$$H_m^\alpha(\mathcal{O}) = \overline{L^2(\mathcal{O}) \cap D(\omega_m^\alpha)}^{||\cdot||_{\alpha,m}}. \tag{5.98}$$

Which makes clear that one have

$$H_m(\mathcal{O}) = H_m^{1/2}(\mathcal{O}) \oplus H_m^{-1/2}(\mathcal{O}). \tag{5.99}$$

But happens that for $m = 0$, the Sobolev space $H_0^\alpha(\mathbb{R}^3)$ is different from $H_m^\alpha(\mathbb{R}^3)$ when $m > 0$ and $\alpha \neq 0$.

The great result [32, 33] is that if the region $\mathcal{O}$ in $\mathbb{R}^3$ is bounded, and $\alpha > -3/2$, then

$$H_m^\alpha(\mathcal{O}) \cong H_0^\alpha(\mathcal{O}), \tag{5.100}$$

that is they are the same vector space with equivalent norms. For large p, which correspond to small distance, it is clear the convergence in $\mathcal{O}$ stills. For small p, which correspond to large distance, one could expect some noise from outside of $\mathcal{O}$, but in fact $\mathcal{O}$ is bounded and the result holds.

Proposition 5.4.1 *[Figliolini and Guido [32], Longo and Morsella [31]] If $\mathcal{O} \subset \mathbb{R}^3$ is bounded, and $\alpha = \pm 1/2$, then*

$$H_m^\alpha(\mathcal{O}) \cong H_0^\alpha(\mathcal{O}), \tag{5.101}$$

that is, they are the same vector space with equivalent norms, more precisely,

(i) $||f||_{1/2,0} \leq ||f||_{1/2,m} \leq c(m,\mathcal{O})||f||_{1/2,0}$

(ii) $||f||_{-1/2,m} \leq ||f||_{-1/2,0} \leq c(m,\mathcal{O})||f||_{-1/2,m}$

Under the same arguments of p and distances, it is clear that the first inequality of the two items holds. For the second inequality, it is sufficient to consider small $p's$.

The following real subspaces of $L^2(\mathbb{R}^3)$ is given by

$$H_\phi = \omega^{-1/2} H_m^{-1/2}(\mathcal{O}), \quad H_\pi = \omega^{1/2} H_m^{1/2}(\mathcal{O}), \tag{5.102}$$

so that one can define

$$K_{\sigma/2}(\mathcal{O}) = \omega^{\sigma/2} H^{\sigma/2}(\mathcal{O}), \tag{5.103}$$

for $\sigma = \pm 1$ and follows the definition of the local subspace $K_m(\mathcal{O})$ written in terms of these spaces,

$$K_m(\mathcal{O}) = \omega^{-1/2} H^{-1/2}_{m,\mathbb{R}}(\mathcal{O}) + i\omega^{1/2} H^{1/2}_{m,\mathbb{R}}(\mathcal{O}). \tag{5.104}$$

The unitary equivalence between $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ and $K_m + iK_m$ is defined by

$$H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}) \ni < f, g > \mapsto (\omega^{-1/2} f - i\omega^{1/2} g) \in K_m + iK_m. \tag{5.105}$$

In some sense a unitary operator is defined

$$i_m = \begin{bmatrix} 0 & \omega_m^{-1} \\ -\omega_m & 0 \end{bmatrix}, \tag{5.106}$$

since one have $\omega_m^{1/2} : H_m^{1/2} \to H_m^{-1/2}$, given by

$$\hat{(\omega_m} f)(p) = \sqrt{p^2 + m^2}\hat{f}(p). \tag{5.107}$$

Proposition 5.4.2 *If $\mathcal{O}$ has C^1 boundary, the interior, and the complement of the interior of $\mathcal{O}$ is non-empty, then follows that $K(\mathcal{O})$ is standard in $L^2(\mathbb{R}^3)$.*

Recall the necessary conditions to be standard (5.4.1). Before to verify, worth to note the following remark [32].

$$(H^{\alpha}(\mathcal{O}))^0 = H^{-\alpha}(\mathcal{O}^c), \tag{5.108}$$

$$D_\sigma = H^{\sigma/2}(\mathcal{O}) + H^{\sigma/2}(\mathcal{O}^c) \text{ is dense in } H^{\sigma/2}(\mathbb{R}^3) \text{ if } \sigma = \pm 1, \tag{5.109}$$

$$\omega \text{ is antilocal, i.e. } suppf \subset \mathcal{O}, supp(\omega f) \subset \mathcal{O} \Rightarrow f = 0, \tag{5.110}$$

$$K(\mathcal{O}) = ReK_{-1/2}(\mathcal{O}) \oplus_{\mathbb{R}} iReK_{+1/2}(\mathcal{O}), \tag{5.111}$$

$$(H^{1/2}(\mathcal{O}))^0 = H^{-1/2}(\mathcal{O}^c) \Rightarrow (K_{\sigma/2}(\mathcal{O}))^{\perp} = K_{-\sigma/2}(\mathcal{O}^c), \tag{5.112}$$

$$K_{\sigma/2}(\mathcal{O}) \cap K_{\sigma/2}(\mathcal{O}^c) = \omega^{\sigma/2}(H^{\sigma/2}(\mathcal{O}) \cap H^{\sigma/2}(\mathcal{O}^c)) = \{0\}. \tag{5.113}$$

Now is clear that $K_{1/2}(\mathcal{O}) \cap K_{-1/2}(\mathcal{O}) = \{0\}$ which follows that $K_{1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O})$ is dense in $L^2(\mathbb{R}^3)$. Also, one would have $K(\mathcal{O}) \cap iK(\mathcal{O})$ if and only if there are $h_\sigma, k_\sigma \in ReK_{\sigma/2}(\mathcal{O})$ such that $h_+ + ih_- = ik_+ - k_-$. But also one can write $h_+ - ik_+ = -(k_- + ih_-)$ resulting that these objects are elements

of the sets $K_{1/2}(\mathcal{O}), K_{-1/2}(\mathcal{O})$ respectively. A necessary condition then is that $K(\mathcal{O}) \cap iK(\mathcal{O}) = \{0\}$ if and only if $K_{1/2}(\mathcal{O}) \cap K_{-1/2}(\mathcal{O}) = \{0\}$. Moreover, one can write [32]

$$\begin{aligned} K(\mathcal{O}) + iK(\mathcal{O}) =& ReK_{-1/2}(\mathcal{O}) + iReK_{1/2}(\mathcal{O}) \\ &+ iReK_{-1/2}(\mathcal{O}) + ReK_{+1/2}(\mathcal{O}) = K_{1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O}). \end{aligned} \tag{5.114}$$

Since now the complete identification of the standard subspace of the Hilbert space is made, one is ready to start to make the identification of the Tomita operator δ_m (or even better: s) associated with the standard space $K_m(\mathcal{O})$.

Starting from $L^2(\mathbb{R}^3)$ one can introduce a multiplication operator $\chi_{\mathcal{O}}$ which localizes in $\mathcal{O}$ and follows the interest to define a similar operator for the $H^{\sigma/2}(\mathbb{R}^3)$. Define the operator P_σ which acts on $D_\sigma = H^{\sigma/2}(\mathcal{O}) + H^{\sigma/2}(\mathcal{O}^c)$ by

$$\begin{aligned} &P_\sigma : D_\sigma \subset H^{\sigma/2}(\mathbb{R}^3) \to H^{\sigma/2}(\mathbb{R}^3), \\ &P_\sigma|_{H^{\sigma/2}(\mathcal{O})} = \mathbb{I}, \qquad P_\sigma|_{H^{\sigma/2}(\mathcal{O}^c)} = 0. \end{aligned} \tag{5.115}$$

Recall now the antilocality of ω: $\omega H^{1/2}(\mathcal{O}) \cap H^{-1/2}(\mathcal{O}) = \{0\}$. Playing with the previous antilocality and with the $\sigma = \pm 1$ one can define the operator

$$\mathcal{F}_\sigma = \omega^{-\sigma} D_{-\sigma} \cap H^{\sigma/2}(\mathcal{O}). \tag{5.116}$$

Now make the identification

$$\begin{aligned} &\omega^{-\sigma} D_{-\sigma}|_{\mathcal{O}} = \omega^{-\sigma} H^{-\sigma/2}(\mathcal{O}), \\ &\omega^{-\sigma} D_{-\sigma}|_{\mathcal{O}^c} = \omega^{-\sigma} H^{-\sigma/2}(\mathcal{O}^c). \end{aligned} \tag{5.117}$$

From the antilocality, we have that after the acting of the operator $\mathcal{F}$ the only non-zero elements are in $\omega^{-\sigma} D_{-\sigma}|_{\mathcal{O}^c} \cap H^{\sigma/2}(\mathcal{O}) = \omega^{-\sigma} H^{-\sigma/2}(\mathcal{O}^c) \cap H^{\sigma/2}(\mathcal{O})$ and it is of interest to take the only functions of the set $H^{\sigma/2}(\mathcal{O})$ that satisfy the previous intersection, i.e., that are also elements of the set $\omega^{-\sigma} H^{-\sigma/2}(\mathcal{O}^c)$. In some sense, it is of interest in defining the so important operator A_σ as follows,

$$A_\sigma : \mathcal{F}_\sigma \subset H^{\sigma/2}(\mathcal{O}) \to H^{-\sigma/2}(\mathcal{O}), \quad A_\sigma = P_{-\sigma} \omega^\sigma|_{H^{\sigma/2}(\mathcal{O})}. \tag{5.118}$$

Also, follows a theorem concerning A_σ.

Theorem 5.4.2 *A_σ is a densely defined closed operator and*

$$(A_\sigma)^* = A_{-\sigma}. \tag{5.119}$$

Inherently one have a complete characterization of $H^{\sigma/2}(\mathcal{O})$ from the $H^{\sigma/2}(\mathbb{R}^3)$ using ω^σ, for both $\sigma = \pm 1$, and recalling these are orthogonal subspaces which one can define the Cauchy data, the consequences are enormously (as will be discussed) and will have concrete implications under the norms.

Now recalling from (5.114) introduce the Tomita operator s associated with the standard space $K(\mathcal{O})$. As it was discussed, there is a unitary equivalence between $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ and $K_m + iK_m = K_{1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O})$. Given $(f,g) \in H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ one have

$$T = \frac{1}{\sqrt{2}}(\omega^{-1/2} \oplus -i\omega^{1/2}), \tag{5.120}$$

which acts on (f,g) as

$$(f,g) \mapsto \frac{1}{\sqrt{2}}(\omega^{-1/2}f - i\omega^{1/2}g) \in K_{1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O}). \tag{5.121}$$

In fact, the domain of the Tomita operator is $D(s) = K_{1/2}(\mathcal{O}) + K_{-1/2}(\mathcal{O})$. Thence follows that $D(s)$ equipped with the graph norm $||\cdot||_{g(s)}$ of s is isometrically isomorphic to $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ via the unitary operator T which can be checked from the isometry

$$2\omega^{\sigma/2} : H^{\sigma/2}(\mathcal{O}) \to (D(s), ||\cdot||_{g(s)}). \tag{5.122}$$

Let $h + ik \in H^{\sigma/2}(\mathcal{O})$, with h, k real, then

$$\frac{1}{2}(||\omega^{\sigma/2}(h+ik)||_{g(s)})^2 = ||h||_\sigma^2 + ||k||_\sigma^2 = ||h+ik||_\sigma^2, \tag{5.123}$$

so that is true that given elements on $H^{\sigma/2}(\mathcal{O})$ space the unitary operator T gives an element on the standard space $K_\sigma(\mathcal{O})$ which is the domain of the Tomita operator s.

Now, if one introduce in $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ the quadratic form defined by the operator,

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}i\begin{pmatrix} 0 & -\omega \\ \omega^{-1} & 0 \end{pmatrix} \tag{5.124}$$

the map

$$T : (H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}), (\,\cdot\,, N\,\cdot\,)_0) \to H \tag{5.125}$$

results an isometry with dense range and now one can successfully to associate H with $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ with respect to the scalar product below to be shown. Consider $< f, g >, < h, k > \in H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ then under the identification of T is obtained

$$(< f, g >, N < h, k >)_m = \frac{1}{2}((h, f) + (k, g) + i[(k, \omega^{-1} f) - (h, \omega g)]), \tag{5.126}$$

where $(\,\cdot\,, \cdot\,)_m$ is the standard L^2 norm.

But in fact, only the operator N is not enough (or even better, minimal) to define the desired quadratic form. Recall the graph norm. Given a standard subspace $K \subset H$, introduce $K + iK$, the graph norm of s is

$$< x, y >_s = < x, y > + < x, s^* s y >, \quad x, y \in K + iK. \tag{5.127}$$

Follows that, given $f \in H^{-1/2}(\mathcal{O})$ and $g \in H^{1/2}(\mathcal{O})$, we can define $x = \omega^{-1/2} f - i\omega^{1/2} g \in K(\mathcal{O})$. Now the graph norm of s is

$$\begin{aligned} < x, x >_s = (||x||_{g(s)})^2 &= ||f||^2_{-1} + ||g||^2_{+1} \\ &= < x, x > + < x, s^* s x > \\ &= < x, x > + < x, \delta x > \\ &= < x, (\mathbb{I} + \delta) x > \end{aligned} \tag{5.128}$$

It is of interest in introducing a *zoo of projectors* in such a way one have the minimal conditions to take any test function and define the above construction, i.e., the established norm. Also, the most important motivation for the following definitions is the fact that

$$\delta = s^* s, \tag{5.129}$$

so that since s acts on $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ one have that s^* acts on $H^{-1/2}(\mathcal{O}^c) \oplus H^{1/2}(\mathcal{O}^c)$, where $\mathcal{O}^c$ means the causal complement of the region $\mathcal{O}$.

It is straightforward the introduction of the operator $\frac{1}{2}R$ such that given $(f \oplus g) \in H^{-1/2}(\mathbb{R}^3) \oplus H^{1/2}(\mathbb{R}^3)$ one obtain the product in the whole Hilbert space

(actually elements of $K(\mathcal{O})$ but in fact is dense),

$$(f \oplus g, \frac{1}{2} R\,(f \oplus g)) = (x, x). \tag{5.130}$$

Recalling the definition of N on $H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$ (5.124), (5.126) one can write

$$R = \begin{pmatrix} Q_- & 0 \\ 0 & Q_+ \end{pmatrix} \begin{pmatrix} 1 & -i\omega \\ i\omega^{-1} & 1 \end{pmatrix} \begin{pmatrix} Q_- & 0 \\ 0 & Q_+ \end{pmatrix} = \begin{pmatrix} 1 & -iQ_-\omega Q_+ \\ iQ_+\omega^{-1}Q_- & 1 \end{pmatrix}. \tag{5.131}$$

Where $dom(Q_\pm) = H^{\pm\sigma/2}(\mathcal{O}) + H^{\pm\sigma/2}(\mathcal{O}^c)$ is just the projection on $H^{\pm 1/2}(\mathcal{O}) < H^{\pm 1/2}(\mathbb{R}^3)$. Now the action of $s^*s = \delta$ is clear and well-defined. For the graph norm one have now

$$(x, (\mathbb{I} + \delta)x)_0 = (f \oplus g, \frac{1}{2} R(\mathbb{I} + \delta)(f \oplus g)) = (||x||_{g(s)})^2. \tag{5.132}$$

So then is expected that $\frac{1}{2}R(\mathbb{I} + \delta) = 1$.

Definition 5.4.2 *Let B be thself-adjointnt operator defined by*

$$B : \mathcal{F}_{-1/2} \oplus \mathcal{F}_{+1/2} \subset H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O}) \to H^{-1/2}(\mathcal{O}) \oplus H^{1/2}(\mathcal{O})$$

$$B = \begin{pmatrix} 0 & iA_{+1} \\ -iA_{-1} & 0 \end{pmatrix}. \tag{5.133}$$

Now if one define

$$\delta = \frac{B+1}{B-1}, \tag{5.134}$$

is clear that

$$\mathbb{I} + \delta = \frac{2B}{B-1}, \tag{5.135}$$

and follows that $\frac{1}{2}R\frac{2B}{B-1} = 1$ is the same as

$$\begin{aligned} RB &= B - 1, \\ 1 &= B(1 - R), \\ -1 = B(R-1) &= (R-1)B. \end{aligned} \tag{5.136}$$

Now is clear that

$$-B(R - \mathbb{I}) = \begin{pmatrix} A_+Q_+\omega^{-1}Q_- & 0 \\ 0 & A_-Q_-\omega Q_+ \end{pmatrix}, \tag{5.137}$$

must equal the unity $\mathbb{I}$. Just take $f, g \in H^{-1/2}(\mathcal{O})$ and recall that $A_- = \chi_{\mathcal{O}}\omega^{-1}\chi_{\mathcal{O}}$. Since one is already inside $\mathcal{O}$ all the projectors $\chi_{\mathcal{O}}$ and $Q_\pm$ will act as corrections to $\omega^{\pm 1}$ mess. Follows that

$$(f, A_+Q_+\omega^{-1}Q_-g)_{-1} = (A_-f, g)_0 = (\omega^{-1}f, g)_0 = (f, g)_{-1}, \tag{5.138}$$

$$A_+Q_+\omega^{-1}Q_- = \mathbb{I}\Big|_{H^{-1/2}(\mathcal{O})}. \tag{5.139}$$

Theorem 5.4.3 *[32] The operator δ is strongly continuous with respect to mass.*

First, follows the continuity of the operator $\omega_\lambda\omega_m^{-1}$ as $\lambda \to m$. Consider $f \in H^{\frac{1}{2}}_m(\mathcal{O})$,

$$\begin{aligned} \left|\left|\left(1 - \frac{\omega_\lambda^{1/2}}{\omega_m^{1/2}}\right)f\right|\right|^2_{1/2,\,m} &= ||(\omega_\lambda^{1/2} - \omega_m^{1/2})f||_{L^2} \\ &\leq (\sqrt{m} - \sqrt{\lambda})^2||f||^2_{\frac{1}{2}}. \end{aligned} \tag{5.140}$$

But since B^{-1} is well-defined, it is continuous on $L^2(\mathbb{R}^3)$. Hence, follows $B^{-1} = \frac{\delta_m - 1}{\delta_m + 1}$ is strongly continuous with respect to mass, and the same must hold for the operator δ.

From that section we can suppose $\delta_m = \delta_0 + m^2\,\delta'$ since $\lim_{m\to 0} m^2\delta' \to 0$. But this is a strong assertion and before such a try in writing this, let us analyze the modern approaches to the Tomita operator and the modular operator.

Chapter 6

The Role of the Second Quantization II

6.1 The Standard subspace

The main starting point of this section is the realization based on the Figliolini and Guido theorem. It is known that the vacuum vector state in the second quantization is cyclic and separating if the subspace associated with the von Neumann algebra is standard. Then, it is clear that having a nice description of the standard subspace is of interest. For more details and discussions about the following results, see [35, 36].

First, let H be a *real vector space* and β its *symplectic form* on H, such that β is real, bilinear, and antisymmetric. Now, one has a tuple (H, β) defined as the *real linear space* H equipped with a symplectic form β.

Consider the Klein-Gordon equation $(\Box + m^2)\Phi = 0$. The solutions satisfy the appropriate equal-time canonical commutation relations. One has a tuple of Cauchy data at the zero time slice and a mapping from the tuple to the space of Klein-Gordon solutions. For that case, a symplectic form is identified with respect to the conserved four-vector current.

Now if we introduce the α to be real scalar product on H, it is said to be compatible if

$$\beta(h,k)^2 \leq \alpha(h,h)\alpha(k,k), \quad h,k \in H, \tag{6.1}$$

holds.

Finally, from the above, a one-particle structure $(\mathcal{H}, \kappa)$ can be defined. The $(\mathcal{H}, \kappa)$ is the one-particle structure on H associated with the compatible scalar product α. In fact, $\mathcal{H}$ is a *complex Hilbert space* and $\kappa : H \to \mathcal{H}$ is a *real linear map* such that

- $Re(\kappa(h_1), \kappa(h_2)) = \alpha(h_1, h_2)$
  $Im(\kappa(h_1), \kappa(h_2)) = \beta(h_1, h_2)$
- $\kappa(H) + i\kappa(H)$ is dense in $\mathcal{H}$

for $h_1, h_2 \in H$. Also, κ is injective, i.e., if $\kappa(h) = 0$ for some $h \in H$ then $h = 0$. A completion $\overline{H}$ for H is always possible such that $\overline{k} : \overline{H} \to \mathcal{H}$ and define a one-particle structure for $\overline{H}$.

Now a very nice theorem can be stated.

Theorem 6.1.1 *Given H with a compatible scalar product α.*

- *There exist a $(\mathcal{H}, \kappa)$ on H associated with α*
- *It is unique such that if exist $(\mathcal{H}', \kappa')$ another one-particle structure on H, exist a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that the following commutes*

It is a nice theorem because it makes it possible to identify different one-particle structures such that the existence of a unitary operator connecting the different structures is guaranteed. In fact, one also has the guarantee of the existence of intertwines between the different flows generated by the dynamical operator for the local Cauchy data. The uniqueness is also a result of the choice of the time evolution operator for the classical phase space.

Let $H \subset \mathcal{H}$ a closed, real linear subspace of $\mathcal{H}$. By the Riesz lemma [see 35], there exist a unique bounded real linear operator D_H on H such that

$$\beta(h, k) = \alpha(h, D_H k), \quad h, k \in H, \tag{6.2}$$

Also, holds

$$||D_H|| \leq 1, \quad D_H^* = -D_H. \tag{6.3}$$

Recall that $\alpha(\,\cdot\,,\,\cdot\,) = Re(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ and $\beta(\,\cdot\,,\,\cdot\,) = Im(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ so that the restriction of the operator D on H is called the *polarizer* of the symplectic form $Im(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ with respect to the real scalar product $Re(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ and D_H is called the polarizer of H.

A remark is necessary here. From a real vector space H together with a scalar product α and a symplectic form β, it is possible to construct a complex Hilbert space $\mathcal{H}$ such that there exists a real linear map κ in such a way that: $\kappa(H) + i\kappa(H)$ is dense in $\mathcal{H}$, i.e., it serves as a basis for $\mathcal{H}$. Then, by identifying a polarizer for a real subspace H of $\mathcal{H}$, it is possible to turn the real part of the product into the imaginary part of the product. It is clear that two different subspaces are being identified: the subspace H and the subspace iH. These elements belong to $\kappa(H)$ and $i\kappa(H)$.

Define the E_H *orthogonal projection* onto H, such that

$$\beta(h, k) = \alpha(h, D_H k) = \alpha(h, (-E_h i)k) = -Re(h, ik), \quad h, k \in H. \tag{6.4}$$

Follows that

$$D_H = -E_H i|_H. \tag{6.5}$$

In the above is clear that the two different notions of subspaces H and iH has been used and that's why the orthogonal name for the projector E_H. Define $H' = (iH)^{\perp_{\mathbb{R}}}$ as the *symplectic complement* of H and $\perp$ as orthogonal with respect to the scalar product. Now several different spaces can be defined.

Let $H \subset \mathcal{H}$ a closed subspace. Then one have

$$\begin{aligned}
&H_0 = (H + iH)^{\perp}, \\
&H_\infty = H \cap iH, \\
&H_a = H \cap H', \quad \beta \text{ is degenerated}, \\
&H_a = H_a + iH_a, \\
&H_f = H_0 + H_a + H_\infty, \quad \beta \text{ is non-degenerated}, \\
&H_f = H_f \cap H.
\end{aligned} \tag{6.6}$$

Is clear that H_a is complex orthogonal to H_∞ and real orthogonal to iH_a. The a stands for Abelian and f stands for $factorial$.

As $D_H = -E_h i|_H$ then $D_H^2 = E_H i E_H i|_H = -E_H E_{iH}|_H$ and now, one have the orthogonal projection onto iH identified. If $h \in H$ then

$$(D_H^2 + 1) = h = 0 \iff E_H E_{iH} h = h \iff h \in H \cap iH. \tag{6.7}$$

In some sense it means that $h \in H_\infty$. Also notice that

$$Ker\beta = ran(D_H)^{\perp} = Ker(D_H^*) = Ker(D_H), \tag{6.8}$$

so clearly $Ker(\beta) = H \cap H' \equiv H_a$, and that's why for H_a the symplectic form is degenerated.

Lemma 6.1.1 *For*

$$Ker(D_H^2 + 1) = H \cap iH, \tag{6.9}$$

the subspace H is separating if and only if $Ker(D_H^2 + 1) = \{0\}$. Also H is factorial, i.e. $H \cap H' = \{0\}$, if and only if $Ker(D_H) = \{0\}$.

To finish the section, rest to define what means a subspace to be standard and how it relates to the above definitions.

Let $\mathcal{H}$ be a complex Hilbert space closed, real linear subspace. The subspace H

is cyclic if $H + iH$ is dense in $\mathcal{H}$ and H separating if $H \cap iH = \{0\}$. Follows that H is standard if it is both cyclic and separating. For H_s a standard subspace, follows that

$$H_s = H_a + H_f, \tag{6.10}$$

and

$$H_s \cap iH_s = \{0\}, H_s + iH_s \text{ is dense in } H_s. \tag{6.11}$$

6.2 Zoo of projectors

Here the point start is the desire to define the dynamic operator and to do that one invoke the Tomita operator. A great fact (see [10]) is that one can define the previous orthogonal projector in terms of the modular objects.

Given a standard subspace H_s of $\mathcal{H}$. Introduce the Tomita operator S_H associated to H_s that is closed, densely defined and an anti-linear involution on $\mathcal{H}$ such that

$$S_H : h + ik \mapsto h - ik, \quad h, k \in H_s. \tag{6.12}$$

From the polar decomposition $S_H = J_H \Delta^{1/2}$ define J_H and Δ_H the modular conjugation and the modular operator of H_s. From now on the subscript s for standard will be omitted. The modular operator Δ_H is a non-singular, positive self-adjoint operator and the modular conjugation J_H is an antiunitary involution. The following holds

$$\begin{aligned} &J_H \Delta_H J_H = \Delta_H^{-1}, \\ &\Delta_H^{is} = H, \quad J_H H = H', \quad s \in \mathbb{R}. \end{aligned} \tag{6.13}$$

Denote by

$$L_H = \log \Delta_H, \tag{6.14}$$

the modular Hamiltonian of H.

Now assume H to be standard and factorial. Define P_H as the cutting projection

$$P_H : h + h' \mapsto h, \quad h \in H, h' \in H'. \tag{6.15}$$

One have $P_H : D(P_H) \subset \mathcal{H} \to \mathcal{H}$, a closed, densely defined, real linear operator with domain $D(P_H) = H + H'$. Now follows the great result that the cutting

projector P_H can be written as a function of the modular objects. First introduce

$$a(\lambda) = (1-\lambda)^{-1}, \quad b(\lambda) = \lambda^{1/2} a(\lambda), \tag{6.16}$$

such that

$$P_H = (a(\Delta_H) + J_H b(\Delta_H))^{-}, \quad \Delta_f = \Delta_H|_{H_f}, \tag{6.17}$$

or

$$P_H = (1+S_H)(1-\Delta_H)^{-1}, \tag{6.18}$$

so that for

$$E_H = (1+S_H)(1+\Delta_H)^{-1}, \tag{6.19}$$

one have the cutting projector as a function of the modular Hamiltonian

$$P_H = E_H(1+\Delta_H)(1-\Delta_H)^{-1} = E_H \coth L_H/2. \tag{6.20}$$

In fact the cutting projector also can be written as an complex-linear operator

$$Q = P - iPi - 1 = \frac{1+\Delta}{1-\Delta} = \coth \frac{\log \Delta}{2}. \tag{6.21}$$

By the same line of arguments one can write

$$1 - E + iEi = \frac{\Delta - 1}{\Delta + 1} = \tanh \frac{\log \Delta}{2}. \tag{6.22}$$

It allows to determine properties about boundness of Δ and of $\log \Delta$. For instance one can see that H is $i \tanh \log \Delta/2$ is invariant since the hyperbolic tangent function is bounded and then $i \tanh \log \Delta/2|_H$ is bounded and skew-selfadjoint in H. By the same arguments follows that its inverse $-i \coth \log \Delta/2|_H$ is a skew-selfadjoint operator on H. The domain of Q self-adjoint closure contains at least

$$dom(P) \cap dom(iPi) = (H + H') \cap (iH + iH'). \tag{6.23}$$

Follows now finally the relation

$$\log \Delta = -2 \operatorname{arcoth} Q = -2 \operatorname{arcoth} (P - iPi - 1). \tag{6.24}$$

Of course there is a lot of open question at this moment. The main thing that is missing here is how one can localize the Klein-Gordon solution or even better

which scheme of localization one can use, because it is clear that one can choose to work on either $\mathbb{R}^3$ or use the unitary Weyl representation to describe fields as elements $W(f)$ where the support of f localizes.

6.3 The question on localization

First, consider the whole $\mathbb{R}^3$ after a slice time. Now impose some criteria such that it selects the desired test functions, i.e., for a given projector $Q_\sigma = [\mathcal{L}^{\sigma/2}]$ that localizes via a characteristic function χ_σ the test functions to be supported on $\mathcal{O} \subset \mathbb{R}^3$ the convergence criteria is

$$\begin{aligned} &f \in \mathcal{L}^{-1/2}(\mathcal{O}), \quad g \in \mathcal{L}^{1/2}(\mathcal{O}), \\ &||x||_\sigma = ||\omega^\sigma x|| < \infty. \end{aligned} \tag{6.25}$$

Also follows that $\mathcal{L}^{\sigma/2}(\mathcal{O}) < \mathcal{L}^{\sigma/2}(\mathcal{O})$ and $\sigma = \pm 1$. Follows from the properties of the Local Sobolev space (see section 5.4 for a discussion) that Q_σ is the same for any mass greater than zero, $m > 0$, but differ from Q_σ corresponding to mass zero, $m = 0$.

Now is clear the tuple of functions to be taken. Consider $(f, g) \in \mathcal{L}(\mathcal{O}) = \mathcal{L}_\phi(\mathcal{O}) \oplus \mathcal{L}_\pi(\mathcal{O})$ where one can identify $+1/2$ with ϕ and $-1/2$ with π. Now finally the standard subspace can be identified.

$$(K_\phi, K_\pi) = (\omega^{-1/2}\mathcal{L}^{-1/2}(\mathcal{O}), \omega^{1/2}\mathcal{L}^{1/2}(\mathcal{O})), \tag{6.26}$$

such that

$$\omega^{-1/2}\mathcal{L}^{-1/2}(\mathcal{O}) \supset \omega^{1/2}\mathcal{L}(\mathcal{O}) = \{0\}. \tag{6.27}$$

Again, it is worth recalling the importance of identifying the standard subspace from the theorem by Figliolini and Guido. It is related to the vacuum vector state being cyclic and separating with respect to the second quantization. The standard subspace is identified with respect to the local space, and it is of interest to *represent* these elements in the whole $\mathbb{R}^3$. This is because, in general, the operators (in our case, the Tomita operator and its adjoint) will act on $\mathcal{O}$ and $\mathcal{O}^c$, where c stands for the spatial complement, and these are dense in $\mathbb{R}^3$.

Now introduce P the projector in $\mathcal{H}$ with image $\mathcal{L}$, kernel $\mathcal{L}'$ and P is closed.

The domain of P is such that

$$dom(P) = \mathcal{L} + \mathcal{L}' = \mathcal{L}^{1/2}(\mathcal{O}) + \mathcal{L}^{1/2}(\mathcal{O}^c) \oplus \mathcal{L}^{-1/2}(\mathcal{O}) + \mathcal{L}^{-1/2}(\mathcal{O}^c). \tag{6.28}$$

In some sense it is playing the same role of the previous elements introduced, i.e., it is taken from the whole $\mathbb{R}^3$ the test functions that converges, so is clear that if one introduce the operators $\omega^{\sigma/2}$ that delocalize the test functions, one can identify these elements on $K = \mathcal{H} = L^2(\mathbb{R}^3)$. To do that one have

$$P = \chi_{1/2} \oplus \chi_{-1/2}, \tag{6.29}$$

$$U = \omega^{1/2} \oplus \omega^{-1/2}, \tag{6.30}$$

$$UPU^{-1} = \omega^{1/2}\chi_{1/2}\omega^{-1/2} \oplus \omega^{-1/2}\chi_{-1/2}\omega^{1/2}. \tag{6.31}$$

Before presenting the striking final result of this section, attributed to Bostelmann et al. [37], let us take a step back. If one considers the domain of dependence $\mathcal{D}_\mathcal{O}$ of $\mathcal{O} \subset \mathbb{R}^3$, it is known that the von Neumann algebra $\mathcal{R}(\mathcal{O})$ of the local region is the same as the von Neumann algebra of its domain of dependence, $\mathcal{R}(\mathcal{D}_\mathcal{O})$ (strictly speaking, this is Haag duality).

It follows that if one takes a local region from the entire spatial region after a slice time and describes the von Neumann algebra in such a precise manner, as previously done, one obtains the complete algebra of its domain of dependence for the Hermitian scalar field, ensuring that its Cauchy data evolves in a physical manner. Additionally, the Weyl unitary representation for the von Neumann algebra associated with the region encapsulates the entire CCR structure and the information regarding the causal structure.

Next, we invoke the Tomita-Takesaki theorem and realize that its action is such that the Tomita operator maps elements and returns the elements associated with the causally separated algebra, i.e., the action of the modular operator J. Furthermore, it respects the time parameter, specifically the modular Hamiltonian in the second quantization, and is continuous with respect to the mass.

The conclusion is that once one identifies, from the polar decomposition, the action of the modular operator, the dynamics of the local region is determined. As discussed previously, the entanglement entropy can then be evaluated, and, for instance, energy bounds can be analyzed.

Recall the polar decomposition $\Delta = e^{\delta}, J = e^{j}, S = J\,\Delta^{1/2}$ such that $\delta = s^* s$ one

have

$$\begin{aligned} dom(s) &= \mathcal{L}^{-1/2}(\mathcal{O}) \oplus \mathcal{L}^{1/2}(\mathcal{O}), \\ dom(s^*)&\mathcal{L}^{-1/2}(\mathcal{O}^c) \oplus \mathcal{L}^{1/2}(\mathcal{O}^c). \end{aligned} \tag{6.32}$$

The necessity to identify the local Standard subspace to the global Standard subspace is clear, now both s and s^* can act.

Now follows the strong final result to be discussed on this section and it is due Bostelmann, Cadamuro and Minz [37]. Consider the one-particle structure $(\mathcal{H}, \mathcal{L}, \omega)$ for an operator B, 6.21, on $\mathcal{H}$ such that

$$B = \omega^{1/2}\chi_{1/2}\omega^{-1/2} + \omega^{-1/2}\chi_{-1/2}\omega^{1/2} - 1, \tag{6.33}$$

and its domain is such that

$$dom(B) = (\omega^{1/2}(\mathcal{L}^{1/2} + \mathcal{L}^{\perp,1/2})) \cap (\omega^{-1/2}(\mathcal{L}^{-1/2} + \mathcal{L}^{\perp,-1/2})). \tag{6.34}$$

Follows that

$$\begin{aligned} \log\Delta &= i_\omega \begin{pmatrix} 0 & M_- \\ -M_+ & 0 \end{pmatrix}, \\ M_\pm &= 2\,\omega^\pm \operatorname{arcoth}(B)\,\omega^\pm. \end{aligned} \tag{6.35}$$

The importance of the equation (6.35) is outstanding. If one introduce a space of functions $\mathcal{L}$ and take a set of test functions, the Hilbert space can be discretized in such a way the problem in determine the modular Hamiltonian is now the problem in evaluating $M_\pm$, in special M_- since from that one the M_+ element is determined, and in fact it is a matrix problem.

Chapter 7

Calculating the Relative Entropy

The goal here is to reproduce the recent description concerning relative entropies. The results primarily stem from Ciolli et al. [10], Longo and Morsella [31], and Bostelmann et al. [36]. As we shall see, the entire concept of standard subspace and its factorization into different subspaces is under consideration. The Bekenstein bound, as discussed in section 3.2, is also tested. Furthermore, the last section addresses the notion of information theory and the realization of operators as entropy operators, as referenced in [38].

7.1 Entropy of a Vector

First, recall the definitions in section 6.1. Given a vector space $\mathcal{H}$ over $\mathbb{R}$, introduce two bilinear forms in $\mathcal{H}$, such that now one have a separable Hilbert space $(\mathcal{H}, \tau)$ and a symplectic space $(\mathcal{H}, \sigma)$. In that way, one has a symplectic Hilbert space $(\mathcal{H}, \tau, \sigma)$ such that $\tau(f,g) = Re < f,g >$ and $\sigma(f,g) = Im < f,g >$, where $< \cdot, \cdot >$ is the complex scalar product of $\mathcal{H}$ as the Hilbert space over $\mathbb{C}$. Introducing now a subspace $H \subset \mathcal{H}$ and decomposing it as in (6.6). Define the standard subspace as in (6.10) and (6.11).

Define $K_H = -\log \Delta_H$, extend it by 0 in H and as undefined in $H_\infty^\oplus \setminus \{0\}$. Denote the modular group by $U_H(x) = \exp\{-i^\oplus x K_H\}$ defined in $H_0^\oplus \oplus H_s^\oplus$.

Now we introduce the Fock-Bose space associated to the Hilbert space. To that Fock-Bose space introduce the Weyl unitaries representation $W(f), f \in \mathcal{H}$ such that

$$W(f)W(g) = e^{-i\sigma(f,g)} W(f+g), \tag{7.1}$$

$$W(f)^* = W(-f). \tag{7.2}$$

For $H \subset \mathcal{H}$, define $\mathcal{A}_\mathcal{H}^\oplus = CCR(\mathcal{H}^\oplus, \sigma^\oplus) \supset \mathcal{A}_\mathcal{H}$ and $A_H = CCR(H, \sigma) \subset \mathcal{A}_\mathcal{H}$, etc.

On $\mathcal{A}_\mathcal{H}$, the bilinear form τ induces the quasifree state ω (vanishing one-point

function)

$$\omega(W(f)) = e^{-\tau(f,f)/2}. \tag{7.3}$$

For each $g \in \mathcal{H}$, consider a coherent state (quasifree with nonvanishing one-point function)

$$\omega_g = \omega(W(g)^* \cdot W(g)), \tag{7.4}$$

also, $\omega_0 = \omega$.

For $(\mathcal{H}, \tau, \sigma)$ a symplectic Hilbert space, let $H \subset \mathcal{H}$ then one have the following factorization

$$K^{\oplus} \simeq H_0^{\oplus} \oplus H_a^{\oplus} \oplus H_f^{\oplus} \oplus H_\infty^{\oplus}, \tag{7.5}$$

$$H \simeq 0 \oplus H_a \oplus H_f \oplus H_\infty. \tag{7.6}$$

Now since pure quasifree states are faithful on the respective subalgebras, the algebra $\mathcal{A}_{\mathcal{H}}^{\oplus}$ is isomorphic to the spatial tensor product of C^*-algebras

$$\mathcal{A}_{\mathcal{H}}^{\oplus} \simeq \mathcal{A}_0^{\oplus} \otimes \mathcal{A}_a^{\oplus} \otimes \mathcal{A}_f^{\oplus} \otimes \mathcal{A}_\infty^{\oplus}, \tag{7.7}$$

under this isomorphism

$$\mathcal{A}_H \simeq \mathbb{C}\mathbb{1} \otimes \mathcal{A}_a \otimes \mathcal{A}_f \otimes \mathcal{A}_\infty, \tag{7.8}$$

and

$$\omega_g \simeq \omega_{P_0^{\oplus} g} \otimes \omega_{P_a^{\oplus} g} \otimes \omega_{P_f^{\oplus} g} \otimes \omega_{P_\infty^{\oplus} g}. \tag{7.9}$$

Due to the additivity of the relative entropy, we have

$$S_{\mathcal{A}_H}(\omega_g||\omega) = S_{\mathcal{A}_a}(\omega_{P_a^{\oplus} g}||\omega) + S_{\mathcal{A}_f}(\omega_{P_f^{\oplus} g}||\omega) + S_{\mathcal{A}_\infty}(\omega_{P_\infty^{\oplus} g}||\omega). \tag{7.10}$$

Now rest to calculate the three terms in equation (7.10).

First, let us calculate the relative entropy of the abelian part: $S_{\mathcal{A}_a}(\omega_g||\omega)$. Consider any $g \in H_a^{\oplus}$. The (real-linear) projector P_a is onto H_a. Recall

$$\begin{aligned} H_a &= H \cap H', \\ H_a^{\oplus} &= H_a + i^{\oplus} H_a. \end{aligned} \tag{7.11}$$

Under the considerations of the algebra, $\mathcal{A}_a$, the state ω_g coincides with $\omega_{\hat{g}}$, where $\hat{g} = (1 - P_a)g$. Here H_a is real-orthogonal for $i^{\oplus} H_a$. For the calcula-

tion of the entropy $S_{\mathcal{A}_a}(\omega_g||\omega)$, without loss of generality, one can assume that $g \in (1-P_a)H_a^{\oplus} = i^{\oplus}H_a$. The GNS representation π for $(\mathcal{A}_a, \omega)$ acts on $L^2(\mathbb{R}^n, d\mu)$, where $d\mu = (2\pi)^{-n/2}\exp\{-||x||^2/2\}\, d^n x$, with $\pi(W(f))$ being multiplication with $\exp\{i < f, \cdot >\}$, and $\pi(\mathcal{A}_a)'' = L^{\infty}(\mathbb{R}^n, d\mu)$. The states here are ω and ω_g such that $\Omega(x) = 1, \Omega_g(x) = \exp\{< i^{\oplus}g, x > -(||g||^{\oplus})^2\}$. The modular group turns out to act by multiplication with $\exp\{-2it < i^{\oplus}g, x > +2it(||g||^{\oplus})^2\}$. Now follows the relative entropy for the abelian part H_a of the subspace $H \subset \mathcal{H}$.

$$S_{\mathcal{A}_a}(\omega_g||\omega) = 2(||(1-P_a)g||^{\oplus})^2. \tag{7.12}$$

Second, let us calculate the relative entropy of the factorial part: $S_{\mathcal{A}_f}(\omega_g||\omega)$. Consider any $g \in H_f^{\oplus} \cap dom(K_H)$. Recall the usefulness of K_H from ($K_H = -\log\Delta$). Since $(H_f^{\oplus}, \tau^{\oplus}, \sigma^{\oplus})$ is pure, the GNS representation π of $(\mathcal{A}_f, \omega)$ acts on the Fock space over $H_f^{\oplus}$ and in that representation both ω and ω_g are vector states, ω corresponding to the Fock vacuum vector Ω and ω_g corresponding to the vector $\Omega_g = \pi(W(g))\Omega$. The vector Ω is cyclic and separating for $\pi(\mathcal{A}_f)''$, the associated Tomita-Takesaki modular group is $\Delta_{\Omega}^{it} = \Gamma(\Delta_H^{it})$, the second quantization of the unitary $\Delta_H^{it} \restriction H_f^{\oplus}$. Let $g \in H_f \cap dom(K_H)$ and $W(g) \in \mathcal{A}_f$. Follows now that

$$\begin{aligned} S_{\mathcal{A}_f}(\omega_g||\omega) &= i\frac{d}{dt} < \Omega_g, \Delta_{\Omega,\Omega_g}^{it}\Omega_g > \Big|_{t=0} = \\ &= i\frac{d}{dt} < \Omega, \pi(W(g))^*\Delta_{\Omega}^{it}\pi(W(g))\Omega > \Big|_{t=0} = \\ &= i\frac{d}{dt} < \Omega, \pi(W(g))^*\Delta_{\Omega}^{it}\pi(W(g))\Delta_{\Omega}^{-it}\Omega > \Big|_{t=0}. \end{aligned} \tag{7.13}$$

Using that

$$\begin{aligned} \pi(W(g))^*\Delta_{\Omega}^{it}\pi(W(g))\Delta_{\Omega}^{-it} &= \pi(W(g))^*\pi(W(\Delta_H^{it}g)) = \\ &= \pi(W(\Delta_H^{it}g - g))\, \exp\Big\{i\sigma^{\oplus}(g, \Delta_H^{it}g)\Big\}. \end{aligned} \tag{7.14}$$

The relative entropy with respect to the abelian part is given.

$$\begin{aligned} S_{\mathcal{A}_f}(\omega_g||\omega) &= i\frac{d}{dt}\, e^{-(||\Delta_H^{it}g - g||^{\oplus})^2}\, e^{i\sigma^{\oplus}(g, \Delta_H^{it}g)}\Big|_{t=0} = \\ &= \sigma^{\oplus}(g, i^{\oplus}K_H g). \end{aligned} \tag{7.15}$$

Holding for $g \in H_f \cap dom(K_H)$. Also holds for $g \in H_f' \cap dom(K_H)$, vanishing. For

the general case $g \in H_f^{\oplus} \cap dom(K_H)$, recall that the projector P_f is defined where $\Delta_f = \Delta_H \restriction H_f^{\oplus}$.

Now the last entropy calculation, $S_{\mathcal{A}_\infty}(\omega_g||\omega)$. Take $g \in H_\infty^{\oplus}$. As before, $(H_\infty, \tau^{\oplus}, \sigma^{\oplus})$ is pure. The GNS representation π of $(\mathcal{A}_\infty, \omega)$ is irreducible, ω and ω_g are given by the vector states Ω and $\Psi = \pi(W(g))\Omega$. The support projections of these states are hence projectors P_Ω and P_Ψ. The relative entropy with respect to the $\mathcal{A}_\infty$ part is given.

$$S_{\mathcal{A}_\infty}(\omega_g||\omega) = \begin{cases} 0, & \text{if } g = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{7.16}$$

7.2 Entropy of a Wave Packet

Now it is of interest in calculating the previous construction for the case of a wave, i.e., a Klein-Gordon solution. We shall start from $\mathbb{R}^{d+1}$ solution and move to the $\mathbb{R}^d$, the time-zero solution. In that case, due to the conserved current give rise to the symplectic form, we are on the factorial case.

Let $\Phi \in S'(\mathbb{R}^{d+1})$ be a solution of the Klein-Gordon equation. Due to the split of frequency, one have $h_m \cup -h_m$, where $h_m = \{p : p \cdot p + m^2 = 0,\ p_0 \geq 0\}$ is the positive Lorentz hyperboloid. A vector of the Hilbert space $\mathcal{H} = L^2(h_m, d\Omega_m)$ is defined to be $[\Phi] = \sqrt{2}\hat{\Phi}|_{h_m}$. Again, due to the split of frequency of the solution, $\Phi = \Phi^- + \Phi^+$, where $\Phi^{\pm} = \hat{\Phi}$ on $\pm h_m$ and $\Phi^{\pm} = 0$ on $\mp h_m$. The map of the solutions to vectors $\Phi \mapsto [\Phi]$ is one to one. The one-particle product between waves is given by

$$([\Phi], [\Psi]) = \int_{h_m} \overline{\hat{\Phi}(p)}\hat{\Psi}(p)\, d\Omega_m, \tag{7.17}$$

also follows that

$$\overline{([\Phi], [\Psi])} = -\int_{-h_m} \overline{\hat{\Phi}(p)}\hat{\Psi}(p)\, d\Omega_m. \tag{7.18}$$

Now the identification of the symplectic form is straightforward,

$$Im([\Phi], [\Psi]) = \frac{i}{2}(([\Phi], [\Psi]) - \overline{([\Phi], [\Psi])}) = \frac{1}{2}\int_{x^0} (\Psi\Phi' - \Phi\Psi')dx. \tag{7.19}$$

The Φ' is to be understood as the time derivative $\partial^0\Phi$.

Now consider a spacetime region $\mathcal{O} \in \mathbb{R}^{d+1}$. The closed real Hilbert space $H(\mathcal{O})$ is made up of smooth real functions on $\mathbb{R}^{d+1}$ compactly supported in $\mathcal{O}$.

The Cauchy data $\Phi_0 = \Phi|_{x^0=0}$ and $\Phi_0' = \Phi'|_{x^0=0}$ are defined by

$$\begin{aligned}\Phi_0(\mathbf{x}) &= \frac{1}{2} Re \int C(\omega(\mathbf{p}), \mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{x}}\, \frac{d\mathbf{p}}{\omega(\mathbf{p})},\\ \Phi_0'(\mathbf{x}) &= \frac{1}{2} Im \int C(\omega(\mathbf{p}), \mathbf{p})\, e^{i\mathbf{p}\cdot\mathbf{x}}\, d\mathbf{p}.\end{aligned} \tag{7.20}$$

Now is clear that given $\mathcal{O} \in \mathbb{R}^{d+1}$ and $H(\mathcal{O})$, one have

$$[\Phi] \in H + H', \tag{7.21}$$

such that $P_H[\Phi] = [\Phi^+] \in \mathcal{H}$ corresponding to $\Phi_0^+, \Phi_0'^+$, where

$$\Phi^+ \in H(\mathcal{O}), \quad \Phi^- \in H(\mathcal{O}'). \tag{7.22}$$

Consider now a real smooth wave Φ solution of $(\Box + m^2)\Phi = 0$ with Cauchy data $\Phi_0 = f, \Phi_0' = g$. Denotes Γ and Λ the waves with Cauchy data respectively $\Gamma_0 = f, \Gamma_0' = 0$ and $\Lambda_0 = 0, \Lambda_0' = g$. The entropy is given by

$$\begin{aligned} S_\Phi &= Im(\Phi, P_H A \Phi)\\ &= Im(\Gamma + \Lambda, P_h A(\Gamma + \Lambda))\\ &= Im(\Gamma, P_H A(\Gamma)) + Im(\Gamma, P_H A(\Lambda)) + Im(\Lambda, P_H A(\Gamma)) + Im(\Lambda, P_H A(\Lambda))\\ &= S_\Gamma + S_\Lambda + Im(\Gamma, P_H A(\Lambda)) + Im(\Lambda, P_H A(\Gamma)).\end{aligned} \tag{7.23}$$

Now just recall the role of A in the above, i.e., $A = -iK$, where

$$i = \begin{bmatrix} 0 & \omega_m^{-1} \\ -\omega_m & 0 \end{bmatrix}, \tag{7.24}$$

defines the complex structure ($i^2 = -1$) and

$$K : (f, g) \mapsto (K^{(1)} g, K^{(2)} f). \tag{7.25}$$

The conclusion is that one have two ways in evaluating the entropy: by directly introducing the modular Hamiltonian operator K or introducing the modular flow of $H(\mathcal{O})$.

7.3 Application: Rindler Wedge

For the first case, let us consider the Rindler Wedge case in $1+1$ dimension and the Bisognano Wichmann result for the modular operator.

Let W denote the wedge in Minkowski space,

$$W = \{x \in \mathbb{R}^{1+1} : x^1 > |x^0|\}, \tag{7.26}$$

and all others wedges are from Poincaré translations of the above. Now invoke the Bisognano and Wichmann result,

$$U(\Lambda_W(2\pi s)) = \Delta_{H(W)}^{-is}, \tag{7.27}$$

where U is the unitary representation of the Poincaré group on $\mathcal{H}$ and $H(W) \subset \mathcal{H}$ is the standard subspace associated with W. Introduce $f \in S'(\mathbb{R}^{1+1})$ so that,

$$\partial^W f = \frac{d}{ds} f\Lambda_W(s). \tag{7.28}$$

The one-parameter subgroup $\Lambda(s)$ of special transformation, i.e., boosts in x^1 direction, is given as

$$\Lambda(s) = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix} \tag{7.29}$$

that transform W into itself, see Ciolli et al. [10] and Haag [7]. If $U(\Lambda(s))$ and $U(x)$ are unitary operators implementing respectively the boost and the spacetime translations, then,

$$U(\Lambda(s))U(x)U(\Lambda(s))^{-1} = U(x(s)), \tag{7.30}$$

where,

$$x^0(s) = x^0 \cosh s + x^1 \sinh s, \tag{7.31}$$

$$x^1(s) = x^0 \sinh s + x^1 \cosh s. \tag{7.32}$$

Finally, one obtain

$$\begin{aligned} \partial_0^W &= \partial_s^W|_{s=0} = \frac{d}{ds} f\Lambda_W(s)|_{s=0} \\ &= x^1 \partial_{x^1} + x_0 \partial_{x^1}. \end{aligned} \tag{7.33}$$

Where ∂_{x^k} is the partial derivative with respect to x^k and now on it will be written as ∂_k for the same.

Now, following Ciolli et al. [10], back to the equation (7.23) and recalling the symplectic product (7.19), define $\Gamma_0 = f, \Gamma_0' = 0$ and $\Lambda_0 = 0, \Lambda_o' = g$ for the waves at $x^0 = 0$. For $Im(\Gamma, P_H A\Lambda)$ it is obtained,

$$Im(\Gamma, P_H A\Lambda) = 2\pi \frac{1}{2} \int_{x^0=0} (\Gamma' \partial_W \Lambda - (\partial_W \Lambda)' \Gamma)\, dx = 0. \tag{7.34}$$

Since $\Gamma_0' = 0$ and also one can verify that,

$$\begin{aligned} (\partial_W \Lambda)' &= (x^0 \partial 1 \Lambda + x^1 \partial_0 \Lambda)' \\ &= \partial_1 \Lambda + x^0 \partial_1 \Lambda' + x^1 (\partial_1^2 - m^2)\Lambda \\ &= 0 \quad \text{at } x^0 = 0. \end{aligned} \tag{7.35}$$

For $Im(\Lambda, P_H A\Gamma)$ one have the same

$$Im(\Lambda, P_H A\Gamma) = 2\pi \frac{1}{2} \int_{x^0=0} (\Lambda' \partial_W \Gamma - (\partial_W \Gamma)' \Lambda)\, dx = 0. \tag{7.36}$$

Since $\Lambda_0 = 0$ and also one can verify by the same argument as before that

$$\partial_W \Gamma = x^0 \partial_1 \Gamma + x^1 \partial_0 \Gamma = 0, \tag{7.37}$$

at $x^0 = 0$ and $\Gamma_0' = 0$.

Now the entropy is given by

$$S_\Phi = S_\Gamma + S_\Lambda. \tag{7.38}$$

For the first entropy term one have, (using that $\Gamma_0' = 0$),

$$\begin{aligned} S_\Gamma &= Im(\Gamma, P_H A\Gamma) \\ &= -2\pi \frac{1}{2} \int_{x^0=0} (\partial_1 \Gamma + x^1 (\partial_1^2 - m^2)\Gamma)\Gamma\, dx \\ &= 2\pi \frac{1}{2} \int_{x^0=0} x^1 ((\partial_1 f)^2 + m^2 f^2)\, dx. \end{aligned} \tag{7.39}$$

Now, for the second term one have, (using that $\Lambda_0' = 0$),

$$\begin{aligned} S_\Lambda &= Im(\Lambda, P_H A \Lambda) \\ &= 2\pi \frac{1}{2} \int_{x^0=0} \Lambda' \partial_W \Lambda \, dx \\ &= 2\pi \frac{1}{2} \int_{x^0=0} g x^1 g \, dx. \end{aligned} \tag{7.40}$$

The entropy of a wave can be written in a very interesting way if one recall the energy density of a wave from the energy-momentum tensor

$$T_{00}(x) = \frac{1}{2}((\Phi')^2 + |\partial_1 \Phi|^2 + m^2 \Phi^2), \tag{7.41}$$

so that

$$\begin{aligned} S_\Phi &= 2\pi \int_{x^0=0} x^1 T_{00}(x) dx \\ &= 2\pi \int_{x^0=0} x^1 \frac{1}{2}((\partial_0 \Phi)^2 + (\partial_1 \Phi)^2 + m^2 \Phi^2) \, dx. \end{aligned} \tag{7.42}$$

It is very clear that the entropy of a wave is characterized by its momentum entropy S_Λ and its field entropy S_Γ.

Also, now one can write the modular Hamiltonian operator nicely,

$$K : (f, g) \mapsto (x^1 g, x^1 (\partial_1^2 - m^2) f + \partial_1 f), \tag{7.43}$$

only by considering the Bisognano and Wichmann result.

7.4 Application: Massless Double Cone

Now consider the double cone case and massless Hermitian scalar free fields solutions of the wave equation. Denote $\mathcal{O}$ the double cone in Minkowski spacetime, $\mathbb{R}^{1+3}$, with base the open unit ball B centered at the origin in the time zero hyperplane $\mathbb{R}^3$. Here $H(\mathcal{O}) = H(B)$ is the standard subspace in the one-particle Hilbert space $\mathcal{H}$. The modular group $V(2\pi s)$ and the modular Hamiltonian K_0 associated to $H(B)$ is the one discussed in section 5.3, based on Longo and Morsella [31] and Hislop and Longo [29]. Since the modular Hamiltonian is known, one

only have to invoke it

$$\begin{aligned}(K_0\Phi)(x^0,r) &= \frac{1}{2}(1-((x^0)^2+r^2))\partial_0\Phi - x^0 r\partial_r\Phi - Dx_0\Phi,\\ (\partial_0 K_0\Phi)(x^0,r) &= -x^0\partial_0\Phi + \frac{1}{2}(1-((x^0)^2+r^2))\partial_0^2\Phi - r\partial_r\Phi - D\Phi.\end{aligned} \tag{7.44}$$

At time zero $x^0 = 0$ one have,

$$(K_0\Phi)(0,r) = \frac{1}{2}(1-r^2)\partial_0\Phi\Big|_{x^0=0}, \tag{7.45}$$

$$(\partial_0 K_0\Phi)(0,r) = \frac{1}{2}(1-r^2)\partial_r^2\Phi - r\partial_r\Phi - D\Phi\Big|_{x^0=0}. \tag{7.46}$$

Because of the projectors P_H, the action can be written as

$$K_0 : (f,g) \mapsto (\frac{1}{2}(1-r^2)g, \frac{1}{2}(1-r^2)\partial_r^2 f - r\partial_r f - Df). \tag{7.47}$$

As before, consider the wave packet to have Cauchy data as $\Phi = \Gamma + \Lambda$ such that $\Gamma_0 = f, \Gamma_0' = 0$ and $\Lambda_0 = 0, \Lambda_0' = g$. Let us notice first that some terms of the entropy (7.10) vanish. First, one have,

$$\begin{aligned}Im(\Gamma, P_h A\Lambda) &= \int_{x^0=0}(0 - (K_0\Lambda)'\Gamma)\,dx\\ &= -\int_{x^0=0}((\partial_0 K_0)\Lambda)\Gamma + (\Lambda' K_0\Gamma)\,dx\\ &= -\int_{x^0=0} g(\frac{1}{2}(1-r^2)\partial_0)\Gamma\,dx\\ &= 0.\end{aligned} \tag{7.48}$$

As well, one have,

$$\begin{aligned}Im(\Lambda, P_H A\Gamma) &= \int_{x^0=0} g(\frac{1}{2}(1-r^2)\partial_0)\Gamma\,dx\\ &= 0.\end{aligned} \tag{7.49}$$

Now let us calculate the non-zero terms of the entropy. For Γ one have

$$\begin{aligned} S_\Gamma &= Im(\Gamma, P_H A\Gamma) \\ &= \int_{x^0=0} (\Gamma' K_0 \Gamma - (k_0\Gamma)'\Gamma)\, dx \\ &= -\int_{x^0=0} f(\frac{1}{2}(1-r^2)\partial_r^2 - r\partial_r - D)f\, dx \\ &= -\int_{x^0=0} \frac{1}{2}(1-r^2) f\partial_r^2 f\, dx + \int_{x^0=0} rf\partial_r f\, dx + D\int_{x^0=0} f^2\, dx. \end{aligned} \tag{7.50}$$

For Λ one have

$$\begin{aligned} S_\Lambda &= Im(\Lambda, P_H A\Lambda) \\ &= \int_{x^0=0} (\Lambda' K_0 \Lambda)\, dx \\ &= \int_{x^0=0} f(\frac{1}{2}(1-r^2)g)\, dx. \end{aligned} \tag{7.51}$$

To write in a better way, [see 31, Appendix A], use the identity

$$\int \frac{1}{2}(1-r^2)|\partial_r f|^2\, dx = -\int \frac{1}{2}(1-r^2) f\partial_r^2 f\, dx + \int rf\partial_0 f\, dx, \tag{7.52}$$

to write the full entropy as follows

$$S_\Phi = \frac{1}{2}\int (1-r^2)(g^2 + |\partial_r f|^2)\, dx + D\int f^2\, dx, \tag{7.53}$$

or even better

$$S_\Phi = \frac{1}{2}\int (1-r^2)((\partial_0\Phi)^2 + |\partial_r\Phi|^2)\, dx + \frac{D}{2}\int f^2\, dx. \tag{7.54}$$

Recall the energy density (7.41) for massless fields, $m = 0$, and write finally the entropy of a wave as

$$S_\Phi = \pi\int_B (1-r^2) T_{00}\, dx + \pi D\int \Phi^2\, dx. \tag{7.55}$$

Again, one could define the S_Γ entropy as the field entropy and S_Λ entropy as the momentum entropy. But in fact, these concepts need more polish, and that is the motivation for the next section.

7.5 General comments

From the discussion of the last section, the entropy of a vector $\Phi \in \mathcal{H}$ with respect to the standard subspace $H \subset \mathcal{H}$ is defined by

$$\begin{aligned} S_\Phi &= Im(\Phi, P_H A\Phi) \\ &= Im(\Phi, P_H(i\log\Delta)\Phi) \\ &= (\Phi, iP_H i\log\Delta\Phi). \end{aligned} \tag{7.56}$$

Where is used that $A_H = -i\log\Delta$. Note that $iP_H i\log\Delta$ is a real linear, positive and self-adjoint operator whose expectation values give the entropy of states. [38]

Definition 7.5.1 *The entropy operator $\mathcal{E}_H$ is defined by*

$$\mathcal{E}_H = iP_H i\log\Delta, \tag{7.57}$$

so that the entropy is the expectation value of the entropy operator,

$$S_\Phi = (\Phi, \mathcal{E}\Phi), \quad \Phi \in \mathcal{H}. \tag{7.58}$$

$$\log\Delta_B = -2\pi i \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\partial_r^2 - r\partial_r - 2D & 0 \end{bmatrix} = -2\pi i \begin{bmatrix} 0 & M \\ L_D & 0 \end{bmatrix} \tag{7.59}$$

One can define $L_D = L - 2D$ and explicit write the Legendre operator as an entropy operator. Also, the advantage in doing that is that one can recall from information theory (or any usual Fourier course) that the following measures the energy of a wave/signal in a spacetime region B,

$$S_\Phi = 2\pi D \int |f|^2\, dx = 2\pi D(f, E_B f) = 2\pi D||f||_B^2. \tag{7.60}$$

That one we can call the Born type entropy. Now, for $f \in L^2(\mathbb{R}^d)$ we set

$$\pi(f, Mf)_B = \pi \int_B (1-r^2) f^2\, dx, \tag{7.61}$$

as the *Parabolic entropy of f in B*. This is equal to the entropy S_Φ of the flat wave and in fact is related to the *momentum entropy* S_g accordingly to the Cauchy data.

Similarly, we set

$$-\pi(f, Lf)_B = \pi \int_B (1 - r^2)|\partial_r f|^2 \, dx, \tag{7.62}$$

as the *Legendre entropy of f in B*. This is equal to the entropy S_Φ of the stationary wave and as well is related to the *field entropy* S_f accordingly to the Cauchy data.

A nice realization from Longo [38] is that one can introduce *prolate entropy* W as

$$-W + M = -L + 1. \tag{7.63}$$

For a given $f \in S(\mathbb{R}^d)$ one have

$$-\pi(f, Wf)_B + \pi(f, Mf)_B = -\pi(f, Lf)_B + \pi(f, f)_B. \tag{7.64}$$

The sum of the prolate entropy and the parabolic entropy is equal to the sum of the Legendre entropy and the Born entropy, all with respect to B [38]. Also, one can write the entropy density of a wave for the Double Cone case in a suitable way,

$$S_\Phi = \pi(-(f, Wf)_B + (f, Mf)_B + (g, Mg)_B + \frac{d-2}{2}||f||_B^2). \tag{7.65}$$

A notable realization from this is the presence of the parabolic distribution $(1 - r^2)$ in both the parabolic and the Legendre entropy expressions. Near the origin, the parabolic entropy closely resembles the Born-type entropy. Conversely, near the boundary of B, the prolate entropy approaches the Born-type entropy. The significance of this realization stems from the fact that the modular Hamiltonian, modular flow, and entropy of a massive free scalar Hermitian field are currently unknown. From the discussion above, one expects that a similar analysis can be conducted in the massive case.

From a Cauchy slice, one introduces the Cauchy data for a field. Building on the earlier discussion regarding the standard subspace in relation to the Hilbert space, one introduces the Weyl unitary representation. Now, we turn to the Tomita-Takesaki discussion. The anti-linear Tomita operator is an involutive and closed operator that acts in two ways: it evolves in time, and for each infinitesimal step, it maps localized vectors to a causally separated region. In fact, it can be decomposed in a modular (or polar) manner, yielding the modular operator and the modular conjugation, which respectively act in the two ways described earlier. If one understands from its geometric realization how the modular flow acts, one can describe how the time evolution of the states occurs. Additionally, one can describe how the entropy of the field is localized and evolves.

Since we have the entropy for a wave in the Rindler Wedge and for the Double Cone case, one can test the bounds discussed in Chapter 3. The first bound one can see directly is from the equation (7.65) and can be announced in a Corollary way as follows, [see 38].

Corollary 7.5.1 *The entropy $\Phi = f \oplus g$ in some region B and $f, g \in L^2(B)$, is lower bounded by*

$$S_\Phi \geq 2\pi D ||f||_B^2. \tag{7.66}$$

Also, the inequality is an equality if $f = \chi_B$ and $g = 0$ and for this case

$$S_{f\oplus g} = 2\pi Vol(B)\, D. \tag{7.67}$$

Consider the entropy of a wave packet in a region B of radius $R > 0$ and center $\bar{\mathbf{x}}$, [see 38],

$$S_\Phi(R) = \pi \int_{B_R} d\mathbf{x}\, \frac{R^2 - r^2}{R} < T_{00} >_\Phi + \frac{\pi D}{R} \int_{B_R} d\mathbf{x}\, \Phi^2. \tag{7.68}$$

Here is considered $r = |\mathbf{x} - \bar{\mathbf{x}}|$. If one now consider the large R case, $S_\Phi(R)/R$ gets proportional to the total local energy $E = \int_{B_R} d\mathbf{x} < T_{00} >_\Phi (t, \mathbf{x})$. That is

$$\frac{S_\Phi(R)}{R} \sim \pi E, \tag{7.69}$$

as $R \to \infty$. That is in agreement with the Bekenstein Bound, see Chapter 3,

$$S_\Phi(R) \leq \pi\, E\, R. \tag{7.70}$$

Now for the small R case, at fixed time t. The local entropy get proportional to

$$S_\Phi(R, \mathbf{x}) = \pi D \Phi^2(t, \mathbf{x}) V_d R^{d-1} + \ldots = \pi \frac{D}{d} A_{d-1}(R) \Phi^2(t, \mathbf{x}) + \ldots, \tag{7.71}$$

where is used $A_{d-1}(R) = 2\frac{\pi^{d/2}}{\Gamma(d/2)} R^{d-1}$, the area of the d-dimensional sphere ∂B_R and $V_d = A_{d-1}(1)/d$ is the volume of B. The final conclusion is that the entropy density of a wave packet Φ around a point gets proportional to the area of the sphere boundary B_R, as expected by holographic area theorems for the entropy. The discussion for the Black Hole case is considered in Chpater 3.

Chapter 8

Numerical Approach

As motivation for this chapter, we recall section 6.3. In fact, the last assertion was that starting from the expression (6.35), one could discretize the Hilbert space and obtain the spectrum of the components M_- and M_+ of the modular Hamiltonian via a numerical approach, since the analytical calculation is very challenging. This approach is utilized by Bostelmann et al. [37], and it will be discussed in detail in the following sections. Considering massive fields in the double cone case, the two main questions to be addressed are: Is the component M_- a multiplicative operator? Is the component M_- mass-dependent? In the last section, we expect to employ the discretization approach to uncover analytical properties, such as the behavior of the derivative with respect to the mass. If it is a differential operator, we will investigate whether the derivative is of first, second, or another order and if one can formulate an ansatz.

8.1 Discretization of the Hilbert space

Start from

$$\log \Delta = i_A \begin{pmatrix} 0 & M_- \\ -M_+ & 0 \end{pmatrix}, \tag{8.1}$$

where,

$$M_\pm = 2A^{\pm 1/4} \operatorname{arcoth} B \, A^{\pm 1/4}, \tag{8.2}$$

$$B = \overline{A^{1/4}\chi A^{-1/4}} + \overline{A^{-1/4}\chi A^{1/4}} - 1. \tag{8.3}$$

Hence, Δ is determined solely from χ and A, where A is the Helmholtz operator: $A = \partial^2 + m^2$. The goal now is to use numerical approximation for $A^{\pm 1/4}$ and χ to write it as finite dimensional matrices.

Define the discretized objects as

$$\begin{aligned} \mathcal{H}_r^{(n)} &= P^{(n)}\mathcal{H}_r, \quad H_r^{(n)} = P^{(n)}H_r, \\ A^{(n)} &= (P^{(n)}A^{\mp 1/4}P^{(n)})^{\pm 1/4}, \end{aligned} \tag{8.4}$$

for a suitable finite-dimensional orthogonal projectors $P^{(n)}$ in $\mathcal{H}_r$.

Choose an orthonormal basis $\{e^{(n)_j}\}$ of $\mathcal{H}_r^{(n)} = H_r^{(n)} + H_r^{\perp(n)}$. For sufficiently regular test functions h and h' to be introduced, it is expected that

$$\Sigma_{j,k} < h, e_j^{(n)} >< e_j^{(n)}, M_-^{(n)} e_k^{(n)} >< e_k^{(n)}, h' > \longrightarrow \int\int h(x) M_-(x,y) h'(y)\, d\mu(x) d\mu(y), \tag{8.5}$$

for n going to infinity. Also, one should care of the spectrum of the discretized operator $B^{(n)}$ that falls into $(-\infty, -1) \cup (1, \infty)$. Because of that, the algorithm should consider high floating points during its eigendecomposition. For the matrix inversion of $A^{-1/4}$ also one should care that for high floating points the matrix $A^{+1/4}$ is indeed its inverse.

The next step is to introduce the boxes. Introduce some interval $\{0, \ldots, n-1\}$ of size $n \in \mathbb{N}$ and points a_i such that $a_0 < \ldots < a_n$ and define $b_i = a_{i+1}$. Define a orthogonal basis functions $e_i^{(n)}$ to be supported in $[a_i, b_i] \subset \mathbb{R}$. For instance, one wants a box as basis of size $[a_i, b_i]$. It will be given by

$$e_i^{(n)}(x) = n_i \theta(x - a_i)\theta(b_i - x), \tag{8.6}$$

i.e., a region of points greater than a_i and less than b_i, such that

$$< e_i^{(n)}, e_i^{(n)} >= 1. \tag{8.7}$$

In fact, one wants initially a equally spaced grid, i.e., $a_i = -b + \frac{2i}{n}b$.

Example. Suppose one wants to discretize the interval $[-b, b]$ with $b = 1$, using $n = 10$ box functions $e_i^{(n,b)} = e_i^{(10,1)}$. Hence, it is obtained for $i = 0$: $[a_0, b_0] = [a_0, a_1] = [-1, -0.8]$ and $e_0^{(10)}(x) = n_0\theta(x+1)\theta(-0.8 - x)$. The grid spacing is $2b/n = 0.2$. Call i the resolution and n the discretization size. With that definition one have a nice way in determining the discretization of a function in a finite dimensional basis.

The last step in to determine the discretization of the Helmholtz operator, i.e., the kernel

$$A^{-1/4} = (p^2 + m^2)^{-1/4}. \tag{8.8}$$

In configuration space it is given by

$$A^{-1/4}(x,y) = (\pi^{1/2}\,\Gamma(1/4))^{-1}\left(\frac{2m}{|x-y|}\right)^{1/4} K_{-1/4}(m|x-y|). \tag{8.9}$$

One can write that kernel as $A^{-1/4}(x,y) = f(|x-y|)$ with f integrable on $[0,\infty)$. Over the boxes it is

$$\begin{aligned}(A^{-1/4})_{ij}^{(n,b)} &=< e_i^{(n,b)}, A^{-1/4\,(n,b)} e_j > \\ &= n_i n_j \int\int_{[a_i,b_i]\times[a_j,b_j]} f(|x-y|)\,dxdx.\end{aligned} \tag{8.10}$$

In few words, the integration is over the boxes determined by the indices i and j. As argued in [37], one can separate in a diagonal and non-diagonal part.

The discretization step is over now. One just have to introduce the standard subspace and a set of test functions to localize it inside the boxes. The algorithm will take care of the discretization, eigendecomposition of the operator B and the spectrum with respect to the position of the component M_- will be obtained.

8.2 The Wedge

Consider the $(1+1)$ dimensional Minkowski space. The Wedge region is given by

$$\mathcal{O} = \{(t,x) \in \mathbb{R}^2 \,|\, x > |t|\}. \tag{8.11}$$

For $\mathcal{H}_r = L^2_{\mathbb{R}}(\mathbb{R})$ one have the standard subspace given by

$$H_r = \{f \in \mathcal{H}_r \,|\, suppf \subset [0,\infty]\}. \tag{8.12}$$

For $M_-^{(n,b)}$, take $n = 256$ and $b = 4$. The resolution will be $0 \leq i \leq 40$. The aim is to evaluate $< h_i, M_- h_i >$. Introduce a set of Gaussian functions with a fixed width (standard deviation) σ and a position parameter μ_i,

$$h_i(x) = \frac{1}{(\pi\sigma^2)^{1/4}}\,\exp\left\{-\frac{(x-\mu_i)^2}{2\sigma^2}\right\}. \tag{8.13}$$

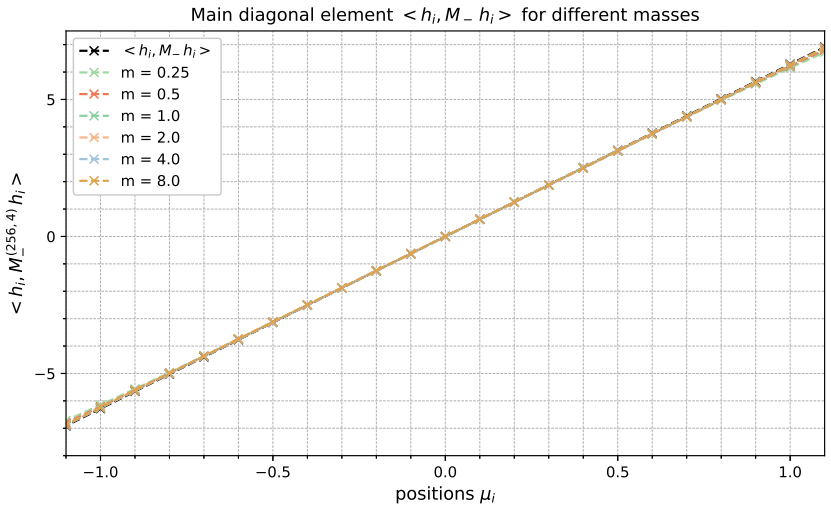

Figure 8.1: Right Wedge. The parameters values are $\sigma = 6/32$, $n = 256$ and 41 position points. Graph reproduced from the numerical algorithm of [37].

These are normalized with respect to the inner product of $\mathcal{H}_r$. To be effective, the width σ has to be larger than the grid spacing,

$$\sigma > \frac{2}{n} b, \tag{8.14}$$

i.e., $\sigma > \frac{1}{32}$. Choose $\sigma = 6/32$ and μ_i over $[-2, 2]$ equally spaced.

The result is expressed in Figure 8.1. In fact, is observed that,

$$< h_i, M_- h_j > \approx < h_i, M_- h_i > = 2\pi\mu_i. \tag{8.15}$$

Some observation we make are

- The graph is highly concentrated in the diagonal part,
- Massless,
- Proportional to 2π,
- Non-differential operator.

In that way one can conclude it is indeed the M_- component of the modular Hamiltonian operator corresponding to the modular flow as described by Bisognano and Wichmann. It is a multiplicative operator, mass independent.

8.3 The Double Cone

First, consider the $(1+1)$ dimensional scalar field. As usual, for $\mathcal{H}_r = L^2_{\mathbb{R}}(\mathbb{R})$ one have the following standard subspace

$$H_r = \{f \in \mathcal{H}_r \,|\, suppf \subset [-1,1]\}. \tag{8.16}$$

But now, since we are in a region of different size, keep the grid points a_i to be equally spaced only in $[-1,1]$. For $b = 4$, choose a spacing outside such that increases linearly towards the cut-off at $\pm b$ starting from the fixed value of the inner spacing. Hence, a quarter value of the basis functions is supported to the left of the interval, the half is supported inside the interval and the another quarter is supported to the right of the interval. Use $n = 256$. The non-equal grid spacing yields twice the resolution over the interval region when compared to the case of the right wedge. Use $\sigma = \frac{6}{64}$. The set of Gaussians functions still given by (8.13).

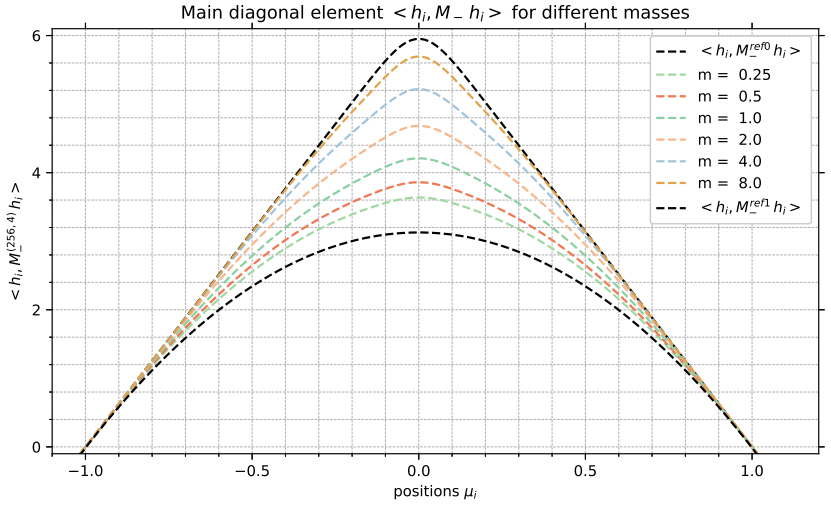

Figure 8.2: Double Cone $(1+1)$-dimensional. The parameters values are $\sigma = 6/64$, $n = 256$ and 2009 position points. Graph reproduced from the numerical algorithm of [37].

The result is expressed in Figure 8.2. Some facts concerning the graph are,

- It is highly concentrated in the diagonal part,
- Is mass dependent,
- For $m \to 0$: $< M_- > \approx \pi(1 - x^2 - \sigma^2/2)$.

For non-zero mass the spectrum is obtained and for mass near zero it is indeed the M_- component of the modular Hamiltonian operator corresponding to the modular flow as described by Hislop and Longo. The question on whether it is a multiplicative operator or a differential operator still open. If it were the case of a differential operator, would be expected the spectrum to be scattered outside the diagonal part, in some sense there would be derivatives of the Dirac delta.

Consider the $(1+3)$ dimensional scalar field. For $\mathcal{H}_r = L^2_{\mathbb{R}}(\mathbb{R}^3)$, one have the following standard subspace,

$$H_r = \{f \in \mathcal{H}_r \,|\, suppf \subset \mathcal{B}_1\}, \tag{8.17}$$

where $\mathcal{B}_1$ is the ball of radius 1 around the origin. Express the Helmholtz operator in spherical coordinates,

$$A = -\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{r^2} + m^2, \tag{8.18}$$

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta^2}\frac{\partial^2}{\partial\phi^2}. \tag{8.19}$$

Where L^2 is the square angular momentum that can be decomposed on its eigenbasis, i.e., the spherical harmonics Y_l^k for $l \in \mathbb{N}_0, k \in \mathbb{Z}, -l \leq k \leq l$ with eigenvalues $l(l+1)$. Our spaces are now identified,

$$\mathcal{H}_r = \bigoplus_{l,k} L^2_{\mathbb{R}}((0,\infty), r^2 dr), \quad H_r = \bigoplus_{l,k} \{f \,|\, suppf \subset (0,1]\}, \quad A = \bigoplus_{l,k} A_l. \tag{8.20}$$

In that way one can introduce the modified spherical Bessel operator

$$A_l = -\frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} - m^2r^2 - l(l+1)\right). \tag{8.21}$$

Now we have an additional parameter, the eigenvalue l, but the algorithm can work in a suitable way.

For the boxes, use the discretization range $[0, b]$ being the radial direction. The measure of the Hilbert space is now $r^2 dr$ and it will change the normalization of the box. The discretization grid is taken as $r \in [0, 1]$. Use $n = 256$, $b = 4$ and the spacing is such that increases linearly from $r = 1$ and $r = b$.

For the discretization of the Helmholtz operator, start from $A_l^{-1/4}$ such that,

$$(A_l A_l^{-1})(r,s) = \frac{1}{r^2}\delta(r-s). \tag{8.22}$$

The expression for $A^{-1}(r,s)$ is well known in terms of modified Bessel functions of the first and second kind, $I_{l+1/2}$ and $K_{l+1/2}$ respectively,

$$A_l^{-1}(r,s) = \sqrt{\frac{1}{rs}}(\Theta(r-s)K_{l+1/2}(mr)I_{l+1/2}(ms) + \Theta(s-r)I_{l+1/2}(mr)K_{l+1/2}(ms)). \tag{8.23}$$

The required discretization of $A_l^{-1/4}$ by computing the fractional power of the matrix $(A_l^{-1})_{ij}^{(n,b)}$ numerically. Following, for $A_l^{1/4}$ the procedure is the same as before, using high floating numbers and making sure it is in fact the inverse of $A_l^{-1/4}$ with sufficient numerical precision.

Introduce the set of (normalized) log-Gaussian test functions, with parameters σ and μ_i as before,

$$h_i(r) = \frac{1}{(\pi \log \alpha_i)^{1/4}}\sqrt{\frac{1}{r^3}}\exp\left\{-\frac{\log^2\left(\alpha_i \frac{r}{\mu_i}\right)}{4\log\alpha_i}\right\}, \quad \alpha_i = \sqrt{1+\frac{\sigma^2}{\mu_i^2}}. \tag{8.24}$$

For this one, set $\sigma = \frac{6}{128}$.

The results are expressed in Figure 8.3 for $l = 1$. Some facts concerning the graph are,

- Highly concentrated in diagonal part,
- Mass dependent,
- For small radii, the mass dependence is more pronounced,
- For large r, e.g. $r > 0.7$, it becomes mass-independent,
- As it is getting close to $r = 1$, it gets similar to the case of a left Wedge at $r = 1$,
- The maximum moves to the left as the mass increases,
- The dependence on l is very small.

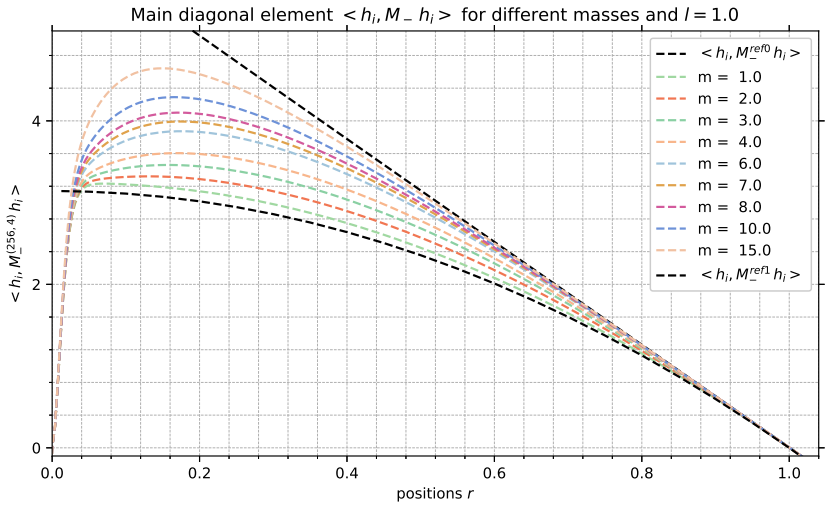

Figure 8.3: Double Cone $(1+3)$-dimensional. The parameters values are $\sigma = 6/128$, $n = 256$ and 325 position points. Graph reproduced from the numerical algorithm of [37].

Also, as the mass decreases to zero, one have the reference for the massless solution,

$$M(r,s) = \pi(1-r^2)\frac{1}{r^2}\delta(r-s). \tag{8.25}$$

The question on whether M_- is a differential operator or a multiplicative operator still open. By the same line of arguments one can say due the spectrum to be concentrated in diagonal part, it is expected to be a multiplicative operator.

8.4 In search of an Ansatz

Suppose for a moment that the Hislop and Longo result is unknown, meaning there is no geometrical interpretation for the modular flow in the double cone case. However, suppose the method discussed earlier still exists. Would it be possible to obtain the M_- component solely through the previous algorithm? There would be no assumptions regarding the mass dependence, whether it is a multiplicative or a differential operator, or if it diverges for some mass or position point. In what follows, the idea is to exploit which parameters indeed have explicit influence on the spectrum of the modular Hamiltonian operator.

Question: Is there any kind of derivative? To answer the question suppose a

test function as before,

$$h(y) = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left\{-\frac{(y-x)^2}{2\sigma^2}\right\}. \tag{8.26}$$

Then a dictionary is made:

1. $< h, ch >= c, \quad c \in \mathbb{R}$,
2. $< h, xh >= x, \quad$ for x position point,
3. $< h, y^2 h >= x^2 + \frac{\sigma^2}{2}$,
4. $< h, \partial_y h >= 0$,
5. $< h, \partial_y^2 h >= -\frac{1}{2\sigma^2}$,
6. $< h, y\partial_y h >= -\frac{1}{2}$,
7. $< h, y^2\partial_y h >= -x$,
8. $< h, y\partial_y^2 h >= -\frac{x}{2\sigma^2}$,
9. $< h, y^2\partial_y^2 h >= \frac{1}{4} - \frac{x^2}{2\sigma^2}$.

It is a fact that if there is an even number of derivatives, it will be exploited by the σ dependence of the result. There is no presence of any term above resulting in x after integration since the graph is symmetric.

For the supposition the Hislop and Longo result is unknown, one would observe that $< M_- >= -ax^2 - b\sigma^2 + c$ for a, b and c real constants. In fact, there is no such derivative. By varying σ and x one would obtain easily the correct component: $M_- = \pi(1 - x^2)$. By doing the same for the massive case, we obtain that it must have some kind of derivative. Unfortunately, no progress in that direction is possible at the moment, see Figure 8.4 and the dictionary defined above.

Another interesting idea is to look for analytical behavior of the component M_-. For the $(1+1)$-dimensional case, it can be found in Figure 8.5. We take the difference of the massive diagonal element$< M_-^m >$ by the massless reference $< M_-^0 >$ for different position points. We found that the tangent line of the graph is going to infinity as $m \to 0$. That indicates a possible divergence at $m = 0$ for the component M_- of the modular Hamiltonian. By considering equation (6.35), we see the only possible mass dependence of M_- is from $A^{\pm 1/2}$ and χ. Since the role

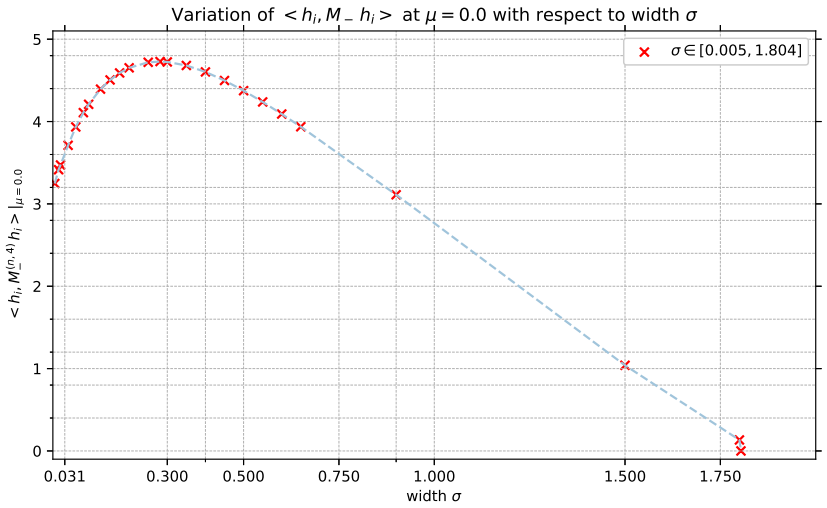

Figure 8.4: Double Cone $(1+1)$-dimensional. Parameter $b = 4$ fixed, position $\mu = 0.0$ fixed and parameter σ varying in $[0.005, 1.804]$. Graph reproduced from the numerical algorithm of [37].

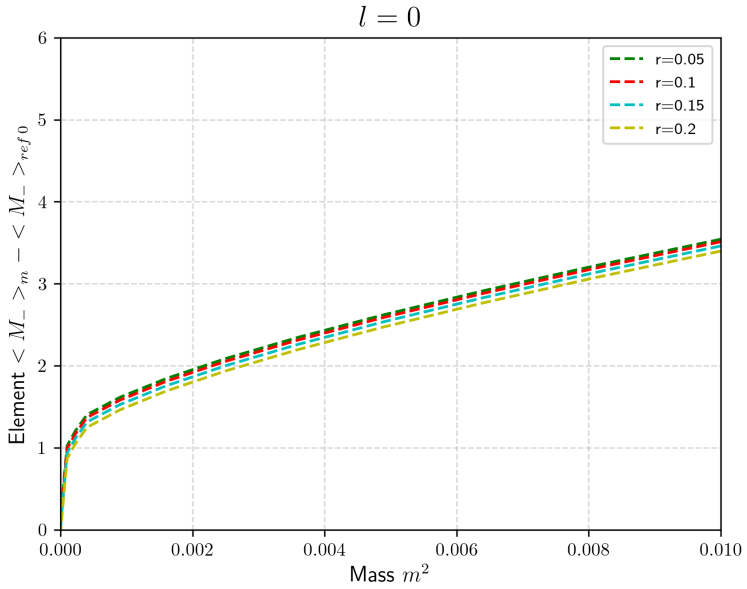

Figure 8.5: Double Cone $(1+3)$-dimensional. Graph of the difference $< M_-^m > - < M_-^0 >$ for different position points and parameter $l = 0$. The tangent line to the graph is going to infinity as mass m goes to zero value. Graph reproduced from the numerical algorithm of [37].

of χ projectors is only to localize, it is natural to suppose it to be massless. If is the case, the kernel $A^{\pm}$ must be divergent at $m = 0$. That's the motivation for the next chapter.

Chapter 9

Analytical properties of the Bogoliubov transformations

In the last chapter, we were led to state that the first derivative of the modular Hamiltonian operator is divergent at mass zero. Now, the goal is to address the origin of that divergence. Using the expression (6.35) for the M_- component, we shall study the analytical structure of the operator μ_m. The operator μ_m has significant applications in Quantum Field Theory. The same operator is employed in the analysis of the local equivalence of vacuum states of free Hermitian scalar fields of different masses [39]. Additionally, in a recent work [40], a similar construction was used, but within the context of modular theory and standard subspace.

9.1 Isomorphism between local Algebras

For a bounded region $\mathcal{O} \subseteq \mathbb{R}^d$, one can define an algebra associated to the second quantization, i.e., the CCR in Weyl form. Now, follows the prescription of the standard subspace. Let $\mu = (p^2 + m^2)^{1/2}$ be a real positive self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ such that for some dense $K_\mu \subseteq L^2(\mathbb{R})$, define $\mathcal{D}_\mu = K_\mu + iK_\mu$ one have

$$\mathcal{D}(\mu^{\pm 1/2}) \supseteq \mathcal{D}. \tag{9.1}$$

Now, introduce two operators μ_1^{-1} and μ_2^{-1}. Assume that for some dense $K \subseteq L^2(\mathbb{R}^d)$, one has $\mathcal{D} = K + iK$,

$$\mathcal{D}(\mu_1^{\pm 1/2}) \supseteq \mathcal{D} \cup \mu_2^{\mp 1/2}\mathcal{D}, \quad \mathcal{D}(\mu_2^{\pm 1/2}) \supseteq \mathcal{D} \cup \mu_1^{\mp 1/2}\mathcal{D} \tag{9.2}$$

In that way, one have the definition of the von Neumann algebra associated to a standard subspace associated to m_1 and m_2. See Definition 5.4.1, Theorem 5.4.1 and equation (5.90).

Introduce the Bogoliubov transformation $\beta = (\beta_+, \beta_-)$ on $L^2(\mathbb{R}^d)$ satisfying

$$\beta_+^*\beta_+ - \beta_-^*\beta_- = 1, \quad \beta_+^*\beta_- = \beta_-^*\beta_+. \tag{9.3}$$

Recall (5.70) and (5.71), the action of the Bogoliubov transformation is the following,

$$a_\beta^*(f) = a^*(\beta_+ f) + a(\beta_- f), \tag{9.4}$$
$$a_\beta(f) = a^*(\beta_- f) + a(\beta_+ f), \tag{9.5}$$
$$\tag{9.6}$$

for a given β and $f \in \mathcal{D}(\beta_\pm)$. For μ_1 and μ_2 satisfying (9.2), we set

$$\beta_\pm = \frac{1}{2}(\mu_2^{-1/2}\mu_1^{1/2} \pm \mu_2^{1/2}\mu_1^{-1/2}). \tag{9.7}$$

From that, a map between second quantization is defined: $(\phi_{\mu_1}(f), \pi_{\mu_1}(g)) \to (\phi_{\mu_2}(f), \pi_{\mu_2}(g))$. The goal of [39] was then to find the minimum criteria to say when those von Neumann algebras will be isomorphic or unitary equivalent. There the minimum criteria are proved, the Bogoliubov coefficients have to be Hilbert-Schmidt, essentially $L^2(\mathbb{R}^d)$.

In a very recent work [40], the same approach is considered but within the context of modular theory. Given two standard subspaces associated with μ_1 and μ_2, respectively, and a linear bijection between them, it is shown that the resolvents of the local modular operators on the one-particle space depend continuously on the field mass with respect to the Hilbert-Schmidt norm of the relevant local standard subspace. Furthermore, the resolvent of the modular operator at mass m_1 (non-zero) is a perturbation of the resolvent of the modular operator at mass $m_2 = 0$. This establishes the continuity mentioned earlier.

9.2 Motivation for some Analytical Expression

Starting from (6.35), one can recall that $\delta = \log \Delta$. Hence, if the first derivative of M_- diverges at some mass value, one should expect some relation with δ diverging as well. From the section 5.4, define

$$B_m^{-1} = \frac{\delta_m - 1}{\delta_m + 1} = 1 - \frac{2}{\delta_m + 1} = \begin{pmatrix} 0 & iQ_-\mu_m Q_+ \\ -iQ_+\mu_m^{-1}Q_- & 0 \end{pmatrix}. \tag{9.8}$$

Then, follows

$$B_m^{-1} - B_0^{-1} = \begin{pmatrix} 0 & iQ_-(\mu_m - \mu_0)Q_+ \\ -iQ_+(\mu_m^{-1} - \mu_0^{-1})Q_+ & 0 \end{pmatrix}. \tag{9.9}$$

Assuming $Q_\pm$ do not have mass dependence.

If μ_m is differentiable, i.e.,

$$\mu_m = \mu_0 + m^2 D, \tag{9.10}$$

for some D, then B_m^{-1} would be differentiable as well,

$$B_m^{-1} = B_0^{-1} + m^2\gamma, \tag{9.11}$$

for some γ such that

$$\gamma = \begin{pmatrix} 0 & iQ_-DQ_+ \\ -iQ_+DQ_+ & 0 \end{pmatrix}. \tag{9.12}$$

That is the aim in studying the analytical structure of μ_m and it will be discussed as follows.

Suppose one wants to differentiate the μ_m operator, i.e., calculate the expression

$$\frac{\mu_m - \mu_0}{m^2} = \mu_0(\mu_m\mu_0^{-1} - 1)m^{-2} \tag{9.13}$$

Worth to recall the Bogoliubov coefficient β_- from (9.7) and express it like the following,

$$\beta_- = \frac{1}{2}((\mu_m^{1/2}\mu_0^{-1/2} - 1) - (\mu_0^{1/2}\mu_m^{-1/2} - 1)). \tag{9.14}$$

Worth to recall that the behavior of μ_m do not change for any power of that, so change the analysis from μ_m to $\overset{\pm 1/2}{\mu}$ is advantageous because it is related to the Bogoliubov coefficients.

The conclusion at this point is that if one has an analytical expression for $\mu_m^{1/2}\mu_0^{-1/2} - 1$ contained in (9.14), then it is possible to analyze its analytical behavior. In particular, one can divide by m^2 and take the limit as m^2 approaches zero. Additionally, since our assertion that the derivative of M_- is divergent at mass zero relies on configuration space, it is essential to have an explicit expression for the above relations.

9.3 $(1+1)$-dimensional Case

We shall obtain the analytical expression for the terms of the Bogoliubov coefficient of the equation (9.14). To write it in the configuration space, one uses the method of Mellin transformation and Meijer G-functions as described in B.

The first term is $(\mu_m^{1/2}\mu_0^{-1/2}-1)$. Following Appendix B, it is found to be,

$$\begin{aligned}(\mu_m^{1/2}\mu_0^{-1/2}-1)(x,y) &= \int_{-\infty}^{\infty} dp \left[\left(\frac{p^2+m^2}{p^2}\right)^{1/4}-1\right] e^{ipx} \\ &= -\frac{\sqrt{\pi}}{\Gamma(-\frac{1}{4})} m G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\,\begin{matrix}\frac{3}{4}\\ \frac{1}{2},0,-\frac{1}{2}\end{matrix}\right).\end{aligned} \tag{9.15}$$

For the second term $(\mu_0^{1/2}\mu_m^{-1/2}-1)$ one just have to change the parameters of the result found in Appendix B, i.e., $\mu=\nu=\frac{1}{4}$. Hence, the expression found is the following,

$$\begin{aligned}(\mu_0^{1/2}\mu_m^{-1/2}-1)(x,y) &= \int_{-\infty}^{\infty} dp \left[\left(\frac{p^2}{p^2+m^2}\right)^{1/4}-1\right] e^{ipx} \\ &= -\frac{\sqrt{\pi}}{\Gamma(\frac{1}{4})} m G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\,\begin{matrix}\frac{1}{4}\\ \frac{1}{2},0,-\frac{1}{2}\end{matrix}\right).\end{aligned} \tag{9.16}$$

Follows now the final expression for the β_- Bogoliubov coefficient in $(1+1)$ dimension,

$$\begin{aligned}\beta^m_-(x,y) = \frac{\sqrt{\pi}\,m}{2}\Bigg(&\frac{1}{\Gamma(\frac{1}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\,\begin{matrix}\frac{1}{4}\\ \frac{1}{2},0,-\frac{1}{2}\end{matrix}\right) \\ &- \frac{1}{\Gamma(-\frac{1}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\,\begin{matrix}\frac{3}{4}\\ \frac{1}{2},0,-\frac{1}{2}\end{matrix}\right)\Bigg).\end{aligned} \tag{9.17}$$

9.4 $(1+3)$-dimensional Case

Now, let us do the same procedure but in $(1+3)$ dimension.

The first term is $(\mu_m^{1/2}\mu_0^{-1/2}-1)$. Following Appendix B, it is found to be,

$$\begin{aligned}(\mu_m^{1/2}\mu_0^{-1/2}-1) &= \int_{-\infty}^{\infty} d^3p \left[\left(\frac{m^2+p^2}{p^2}\right)^{1/4}-1\right] e^{ip\cdot x} \\ &= -\frac{4\pi}{x}\int_0^{\infty} dp \left[\left(\frac{m^2+p^2}{p^2}\right)^{1/4}-1\right] p\,\sin px \\ &= \frac{2\pi^{3/2}m^2}{x\Gamma(-\frac{1}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\, \begin{matrix}\frac{1}{4}\\ \frac{1}{2},0,-1\end{matrix}\right).\end{aligned} \tag{9.18}$$

For the second term, $(\mu_0^{1/2}\mu_m^{-1/2}-1)$, one just have to change the parameters of the result found in Appendix B, i.e., $\mu=\nu=\frac{1}{4}$. Hence, the expression found is the following,

$$\begin{aligned}(\mu_0^{1/2}\mu_m^{-1/2}-1) &= \int_{-\infty}^{\infty} d^3p \left[\left(\frac{p^2}{p^2+m^2}\right)^{1/4}-1\right] e^{ip\cdot x} \\ &= -\frac{4\pi}{x}\int_0^{\infty} dp \left[\left(\frac{p^2}{p^2+m^2}\right)^{1/4}-1\right] p\,\sin px \\ &= \frac{2\pi^{3/2}m^2}{x\Gamma(-\frac{3}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\, \begin{matrix}\frac{3}{4}\\ \frac{1}{2},0,-1\end{matrix}\right).\end{aligned} \tag{9.19}$$

Follows now the final expression for the β_- Bogoliubov coefficient in $(1+3)$ dimension,

$$\begin{aligned}\beta_-^m(x,y) = \frac{\pi^{3/2}}{x}\, m^2 \Bigg(&\frac{1}{\Gamma(-\frac{1}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\, \begin{matrix}\frac{1}{4}\\ \frac{1}{2},0,-1\end{matrix}\right) \\ &- \frac{1}{\Gamma(-\frac{3}{4})} G^{2,1}_{1,3}\left(\frac{m^2(|x-y|)^2}{4}\,\middle|\, \begin{matrix}\frac{3}{4}\\ \frac{1}{2},0,-1\end{matrix}\right)\Bigg).\end{aligned} \tag{9.20}$$

9.5 General comments

Motivated by the presence of a divergence in the derivative of the component M_- of the modular Hamiltonian operator, a schematic analysis of the kernel operator μ_m has been developed in previous sections. In fact, if one considers the derivative of this kernel operator, it can be associated with the Bogoliubov coefficient β_-. By utilizing Mellin transformations and Meijer G-functions, it is possible to express its formulation in configuration space to study its analytical behavior. The conclusion is that the kernel operator β_- is well-behaved for any

mass value. One is led to conclude that the divergence with respect to the mass is not due to the kernel operator μ_m; rather, it stems from the projectors $Q_\pm$, which were assumed to be mass-independent. These projectors remain mysterious objects in the construction. Since section 5.4, it has been possible to argue that their behavior exhibits a jump from the massless to the massive case. This jump is a potential origin of the divergence. Of course, this is a strong assertion, and a study concerning these projectors is necessary. Another point to consider is the difference $(\mu_m^{1/2} - \mu_0^{-1/2})$, which may affect the convergence of $f \in H_m^{1/2}(\mathcal{O})$. This will be addressed in the future [41].

Chapter 10

Outlook and Perspectives

We have examined many definitions, properties, and applications of modular theory. In this work, the term "algebraic" in Algebraic Quantum Field Theory (AQFT) was extensively studied. Using the definition of entropy of a wave as motivation, we reviewed the significant results of Bisognano and Wichmann for the Rindler wedge, as well as those of Hislop and Longo for the Double Cone, and discussed how to define the entropy of a wave for these cases. As a final step, several contributions to the field are presented.

After an extensive introduction to the algebra of operators, we have examined how the Cauchy data of a scalar field generates the standard subspace. As a result, various consequences arise. The von Neumann algebra associated with this standard subspace is standard with respect to the vacuum, and through the GNS construction, we obtain a representation of the Fock space in which the CCR holds. Since Quantum Field Theory involves infinite degrees of freedom, what we actually measure is some difference—specifically, vacuum subtraction. At this point, the concept of relative entropy becomes relevant. By introducing the Araki-Uhlmann formula, we can evaluate the relative entropy of two states in the context of infinite degrees of freedom. Using the Cauchy data of the field that generates the standard subspace, we can assess the relative entropy through its intrinsic dynamics, which follow the KMS condition. The presence of the cutting projectors plays a crucial role, as they cut the Cauchy data, providing a geometric interpretation for the modular Hamiltonian. For a massless field associated with a ball of a certain radius in space, its evolution is constrained by the causal structure. As described, the entropy of a wave in this case is found to be the sum of the so-called prolate entropy and the parabolic entropy. These are related to the concept that the entropy is partially attributable to the field and partially to the momentum field.

Unfortunately, there is a lack of geometric interpretation for the thermal time flow, as defined by the KMS condition, in the case of massive fields. We have made progress in addressing this problem by exploring different approaches. The

first idea is to extract analytical properties of the spectrum resulting from recent numerical analyses. We have found indicative evidence suggesting that one of the components of the modular Hamiltonian behaves as a differential operator. Additionally, for the same component, we have found evidence that the first derivative diverges at a mass value of zero. Despite the advancements in formal expressions for the modular Hamiltonian in the massive field case, we have not yet been able to formulate a conjecture for that component.

Assuming the cutting projector to be massless, we study the kernel associated with the dispersion relation for the field, as it could be the source of the previously mentioned divergence. Furthermore, this kernel is related to the non-diagonal part of the Bogoliubov transformations. Since the numerical approach focuses on the global aspects of the theory, understanding that divergence can be enhanced by expressing the relevant kernel in configuration space. To achieve this, we utilized the Mellin transform and the Meijer G-functions to determine its Fourier transform. We hope that introducing this integration tool into the field for evaluating formal expressions becomes a standard practice. Additionally, we note the difficulty of writing the Bogoliubov coefficients for different non-zero masses [41].

To summarize, future studies should focus on the projectors. If one were to hypothesize or anticipate the behavior of the massive modular Hamiltonian, it would likely take the form $m^2 \cdot \log m^2$. A numerical approach to the problem of entropy is also suggested, particularly in light of recent developments in the information problem [38], which could shed light on the questions surrounding prolate, Legendre, and parabolic distributions in the massive case. Furthermore, obtaining the spectrum of the other component of the modular Hamiltonian would enhance our understanding of the differential presence in the expressions. For the known cases, that component is indeed a differential operator. Describing how various terms influence the spectrum is an important objective for future research.

Appendix A

Types of the von Neumann Algebras

The goal of this Appendix is to develop the concept of Types of the von Neumann Algebras. We shall start from the very basic definition of a von Neumann algebra, following, there will be defined the concept of factors and finally the concept of types and its main properties. To finish the Appendix, we shall see how it impacts the entropy concept due to the notion of trace and traceless class. For more details of the construction, see Haag [7] and Sorce [42].

A.1 Factors and Types

First, recall some concepts. Introduce an algebra, make it to be involutive, and call it a $*$-algebra, giving some operation $*$. If it has a norm, it is a normed vector space. It will be a Banach algebra if its normed induced topology converges. That Banach algebra will be a C^*-algebra if its norm satisfies the triangle inequality and the concept of square of the norm is well-defined. The $*$-algebra is a von Neumann algebra if the double commutant relation holds. If $\mathcal{R}$ is a von Neumann algebra, then $\mathcal{R}'$ is a von Neumann algebra. Define the prime, $\prime$, to be the commutant.

Definition A.1.1 *The von Neumann algebra $\mathcal{R}$ is called a factor if its center is trivial, contains only multiples of the identity element,*

$$\mathcal{Z} = \mathcal{R} \cap \mathcal{R}' = \lambda \mathbb{I}, \quad \lambda \in \mathbb{R}. \tag{A.1}$$

Denote $\mathcal{B}(\mathcal{H})$ the space of all bounded operators defined in the Hilbert space $\mathcal{H}$. Also, $\mathcal{R}$ will be a factor if only $\mathcal{A}$ and $\mathcal{A}'$ together generate all of $\mathcal{B}(\mathcal{H})$, i.e., $\mathcal{R} \vee \mathcal{R}' = \mathcal{B}(\mathcal{H})$. The symbol $\vee$ means the smallest algebra. A self adjoint set $S \subset \mathcal{B}(\mathcal{H})$ is irreducible if and only if either its commutant is made up of multiples of the identity or its double commutant is equal to the space of all bounded operators defined in the Hilbert space. It follows from Schur's Lemma. For a von Neumann algebra $\mathcal{R}$, introduce P an orthogonal projector in $\mathcal{B}(\mathcal{H})$. Then

$P\mathcal{H}$ is an invariant subsapce for $\mathcal{R}$, i.e. $\mathcal{R}P\mathcal{H} \subseteq \mathcal{B}(\mathcal{H})$, if and only if $P \in \mathcal{R}'$. Take a bounded operator $T \in \mathcal{B}(\mathcal{H})$. It has a polar decomposition if it has an extension $T = V|T|$, where $|T| = \sqrt{T^*T}$, and V is a partial isometry. The isometry V means its domain is restricted to the orthogonal complement of its kernel, V^*V is an orthogonal projection. That decomposition is unique.

If one introduce an operator-valued distribution $\phi[f]$, where $f(x)$ is support in some spacetime region $\mathcal{O}$. It is expected that for some von Neumann algebra $\mathcal{R}(\mathcal{O})$, associated to operators-valued distributions with support in $\mathcal{O}$, it does have a factor. The center would consist of observables. For matter fields carrying charges, no such arguments can be made.

Define two projectors P_1 and P_2 such that $\mathcal{H}_1 = P_1\mathcal{H}$ and $\mathcal{H}_2 = P\mathcal{H}$ respectively. These projectors P_i are equivalent, $P_1 \sim P_2$, if $P_1 = V^*V$ and $P_2 = VV^*$. The projector V is a partial isometry between $\mathcal{H}_1$ and $\mathcal{H}_2$. Write $P_1 > P_2$ if the P_i are not equivalent but there exist a subspace of $\mathcal{H}_1$ whose projectors P_1 is equivalent to P_2. For $P_i \in \mathcal{R}$, one of the following holds: $P_1 > P_2, P_1 \sim P_2, P_2 > P_1$. If $\mathcal{R}$ is a factor, for $A \in \mathcal{R}$ and $B \in \mathcal{R}$, $AB = 0$ implies that either $A = 0$ or $B = 0$.

Define *dimension relative* to be a positive number, possibly infinity, associated to a projector, and it is relative to a von Neumann algebra. Two projectors $P_1, P_2 \in \mathcal{R}$ are said to have the same dimension relative to $\mathcal{R}$ if $P_1 \sim P_2$. Also follows that the dimension relative of P_1 to $\mathcal{R}$ is greater (less) than the dimension relative of P_2 to $\mathcal{R}$ if $P_1 > (<) P_2$. If P_1 is orthogonal to P_2, $P_1P_2 = 0$, then the dimension relative of $(P_1 + P_2)$ is the sum of the dimension relatives. The last property is that the dimension relative of 0 is 0.

A projector $P \in \mathcal{R}$ is called minimal if it is not zero but $\mathcal{R}$ contains no nonzero projector $P_1 < P$, strictly speaking it means P is equivalent in some of its subspace only for the space of the zero projector. A projection $P \in \mathcal{R}$ is finite-dimensional relative to $\mathcal{R}$ if for every proper subprojection $Q \in \mathcal{R}$, both have not the same dimensional relative value, i.e., Q is inequivalent to P. Also, P will be infinite-dimensional relative to $\mathcal{R}$ if there exist a projector $Q \in \mathcal{R}$ that both do not have the same dimensional relative, i.e. Q is inequivalent to P. Hence, follows that for $P, Q \in \mathcal{R}$, if both are equivalent, one of them is finite, then the other is also finite.

If $P \in \mathcal{R}$ is finite, follows that for any $Q \in \mathcal{R} < P$ is finite as well. The main conclusion is that a factor is finite if every projection is finite, i.e., every projection have the same dimensional relative number as its proper subprojection. A factor is infinite if it contains at least one infinite projection, i.e., there contains a projector that do not have the same dimension relative to other projection.

A factor $\mathcal{R}$ is:

Type I if it contains a nonzero minimal projector.

Type II if it contains nonzero finite projector, but no nonzero minimal projectors.

Type III if it contains no nonzero finite projectors.

By the definition, the Type I differs from an irreducible algebra only by the tensoring with some degeneracy space, i.e., $\mathcal{R} = \mathcal{B}(\mathcal{H}_{irr}) \otimes \mathcal{I}_{deg}$. Also follows for Type III that if $\mathcal{H}$ is separable, all the projectors (infinite) are equivalent.

The Type II can be divided into two types: Type II_1 if it is finite and Type II_∞ if it is of Type II and has no nonzero central finite projection.

Introduce now the trace for operators in $\mathcal{R}$. If $\mathcal{R}$ is a factor, the trace is uniquely determined up to some normalization. By the definition, in Type III all the nonzero elements have infinite trace. For Type I one can normalize such that minimal projectors have trace 1.

Appendix B

Meijer G-functions

B.1 Definitions and Mellin Transform

First, let us define the Meijer G-function, see Gradshteyn and Ryzhik [43].

Definition B.1.1

$$G_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1,a_p\\ b_1,b_q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\,\prod_{i=1}^{n}\Gamma(1-a_i+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\,\prod_{i=n+1}^{p}\Gamma(a_i-s)}\,z^s ds. \tag{B.1}$$

Where C is contour, and it is such that leaves all poles $\Gamma(b_j-s)$ for $j=\overline{1,m}$ to the right of all the poles of $\Gamma(1-a_i+s)$ for $i=\overline{1,n}$. Also, one can say that $a_k-b_j\neq 1,2,3,\ldots$ for the non intersection of poles definition. The coefficients m,n,p,q are all integers numbers.

The choice of the path integral is as follows.

1. C runs from $-i\infty$ to $i\infty$ such that all poles of $\Gamma(b_j-s)$ are on the right and all the poles of $\Gamma(1-a_k+s)$ are on the left. That integral converges for $|arg\,z|<\delta\,\pi$, where $\delta=m+n-\frac{1}{2}(p+q)$ and $\delta>0$. Also, the integral converges for $|arg\,z|>\delta\pi\geq 0$ if $(q-p)(\sigma+\frac{1}{2})>Re(\nu)+1$, for σ representing $Re(s)$, the integration variable. The ν is defined to be $\nu=\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j$. For $|arg\,z|=\delta\pi$ and $p=q$ the integral converge independent of σ whenever $Re(\nu)<-1$.

2. For either $q>p\geq 0$ or $q=p>0$ and $|z|<1$ the path is such that begins and ends at $-\infty$ encircling all the poles of $\Gamma(b_j-s)$ exactly once in the negative direction, but not encircling any pole of $\Gamma(1-a_k+s)$. It also converges for $|z|=1$ if $Re(\nu)<-1$. Where ν is as defined before.

3. For either $p>q\geq 0$ or $p=q>0$ and $|z|>1$ the path is such that begins and ends at $-\infty$ encircling all the poles of $\Gamma(1-a_k+s)$ exactly once in the positive direction, but not encircling the poles of $\Gamma(b_j-s)$. For $p=q$ the integral converges for $|z|=1$ when $Re(\nu)<-1$, for ν defined in item 1.

The motivation to study *Meijer G-function* is the *Mellin transformation* of a convolution.

Definition B.1.2 *The Mellin transformation of the function $f(x)$, denoted by $\hat{f}(s)$, is defined by the integral*

$$\hat{f}(s) = \int_0^\infty dx\, f(x)\, x^{s-1}. \tag{B.2}$$

The inversion of the Mellin transformation is given by,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \hat{f}(s)\, x^{-s}. \tag{B.3}$$

The Mellin transformation preserves the convolution on the multiplicative group $\mathbb{R}_+$, i.e.,

$$(f * g)(z) = \int_0^\infty dy\, f(y) g(\frac{y}{z}) y^{-1}. \tag{B.4}$$

In that way, one can say the Mellin transform of the Mellin convolution is the product of the Mellin transforms.

$$\hat{f(x)g}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+\infty} du\, \hat{f}(u) \hat{g}(s-u). \tag{B.5}$$

Now is easy to establish the Mellin transform of the Meijer G-function

$$\hat{G}^{m,n}_{p,q}\left[z \middle| \begin{matrix} a_1, a_p \\ b_1, b_q \end{matrix} \right] = \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{i=1}^n \Gamma(1 - a_i + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{i=n+1}^p \Gamma(a_i - s)}. \tag{B.6}$$

Now follows a series of propositions concerning the Mellin transforms and the usual functions. For some of them, see Erdélyi and et al. [44].

B.2 Applications

Before the calculations, worth to recall the poles of the Gamma functions. If one write a Gamma function as follows,

$$\Gamma(-s) = \int_0^\infty dt\, t^{-s-1} e^{-t}, \tag{B.7}$$

it does have poles because of t^{-s-1} of the integrand. If it is in the denominator of a fraction, then there are no poles. Now one can use the residue theorem to write

it, as

$$\operatorname*{Res}_{s=n\geq 0} \Gamma(-s) = \frac{(-1)^n}{n!}. \tag{B.8}$$

In that way one can change the integral for a summation of the residues of the Gamma functions contained inside the Meijer G-function.

Also, worth to recall the *Pochhammer symbol*,

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x-n+1)}. \tag{B.9}$$

In special, the property we are interesting is the following,

$$(1+t)^x = \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!}. \tag{B.10}$$

Proposition B.2.1

$$\cos(z) = \sqrt{\pi}\, G^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}-\\ 0,\frac{1}{2}\end{matrix}\right]. \tag{B.11}$$

□

Here we have $p > q$, the path of the integral is such that encircles all the poles of $\Gamma(b_j - s)$.

$$\begin{aligned} G^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}-\\ 0,\frac{1}{2}\end{matrix}\right] &= \frac{1}{2\pi i}\int_C ds\, \frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)}\left(\frac{z}{2}\right)^{2s} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{\sqrt{\pi}(2n)!} \\ &= \frac{\cos(z)}{\sqrt{\pi}}. \end{aligned} \tag{B.12}$$

Where is used that $\Gamma(n+\frac{1}{2}) = 2^{-2n}\sqrt{\pi}\frac{(2n)!}{n!}$. ■

Proposition B.2.2

$$\sin(z) = \sqrt{\pi}\, G^{1,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}-\\ \frac{1}{2},0\end{matrix}\right]. \tag{B.13}$$

□ Follows from the same line of arguments of the $\cos(z)$ case. ■

Proposition B.2.3

$$I_n(z^{1/2}) = 2^{-n} z^{n/2}\, G^{1,0}_{0,2}\left[-\frac{z}{4}\,\middle|\,\begin{matrix}-\\ 0,-n\end{matrix}\right]. \tag{B.14}$$

□ Here again: $q > p$.

$$\begin{aligned} G_{0,2}^{1,0}\left[-\frac{z}{4}\middle|\begin{matrix}-\\0,-n\end{matrix}\right] &= \frac{1}{2\pi i}\int_C ds\,\frac{\Gamma(-s)}{\Gamma(1+n+s)}\left(\frac{-z}{4}\right)^s \\ &= \sum_{k=0}^{\infty}\frac{z^k z^{k/2} z^{-k/2} 2^n 2^{-n}}{k!\Gamma(1+n+k)2^{2k}} \\ &= \left(\sum_{k=0}^{\infty}\frac{z^{k+n/2}2^{-2k-n}}{k!\Gamma(1+n+k)}\right) z^{-n/2}2^n \\ &= I_n(z^{1/2})z^{-n/2}2^n. \end{aligned} \tag{B.15}$$

■

Proposition B.2.4

$$J_{a-b}(2\,z^{1/2}) = z^{-\frac{1}{2}(a+b)}\,G_{0,2}^{1,0}\left[z\middle|\begin{matrix}-\\a,b\end{matrix}\right]. \tag{B.16}$$

□ Here again: $q > p$.

$$\begin{aligned} J_{a-b}(2\,z^{1/2}) &= \left(\sum_{k=0}^{\infty}(-1)^k\frac{z^{\frac{1}{2}(a-b)+k}}{k!\Gamma(1+a-b+k)}\right) \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k(z)^k z^{\frac{1}{2}(a-b)}z^{\frac{1}{2}a}z^{-\frac{1}{2}a}}{k!\Gamma(1+a-b+k)} \\ &= \frac{1}{2\pi i}\int_C ds\,\frac{\Gamma(a-s)}{\Gamma(1-b+s)}z^s z^{-\frac{1}{2}(a+b)} \\ &= G_{0,2}^{1,0}\left[z\middle|\begin{matrix}-\\a,b\end{matrix}\right] z^{-\frac{1}{2}(a+b)}. \end{aligned} \tag{B.17}$$

■

Proposition B.2.5

$$z^{\rho-1}(z+1)^{-\sigma} = \frac{1}{\Gamma(\sigma)}G_{1,1}^{1,1}\left[z\middle|\begin{matrix}\rho-\sigma\\\rho-1\end{matrix}\right]. \tag{B.18}$$

□

Here one have $p = q$. Hence, it is necessary to take account of the contours:

1. $|z| < 1$: take contour such that encircle all the poles of $\Gamma(b_j - s)$,

2. $|z| > 1$: take contour such that encircle all the poles of $\Gamma(1-a+s)$.

Also, one should guarantee that $Re(b) > Re(a) - 1$.

1)$|\mathbf{z}| < \mathbf{1}$.

$$\begin{aligned} G_{1,1}^{1,1}\left[z\left|\begin{matrix}a\\b\end{matrix}\right.\right] &= \frac{1}{2\pi i}\int_C ds\,\Gamma(b-s)\Gamma(1-a-s)z^s \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(1-a+b+n)z^{b+n} \\ &= \Gamma(b-a+1)\sum_{n=0}^{\infty}\frac{z^{b+n}}{n!}(a-b-1)_n \\ &= \Gamma(b-a+1)z^b(1+z)^{a-b-1}. \end{aligned} \tag{B.19}$$

Where $(a)_n$ is the Pochhammer symbol introduced in the beginning of the section.

1)$|\mathbf{z}| > \mathbf{1}$.

$$\begin{aligned} G_{1,1}^{1,1}\left[z\left|\begin{matrix}a\\b\end{matrix}\right.\right] &= \frac{1}{2\pi i}\int_C ds\,\Gamma(b-s)\Gamma(1-a-s)z^s \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma(1-a+b+n)z^{-1+a-n} \\ &= \Gamma(b-a-1)z^{a-1}\sum_{n=0}^{\infty}\frac{z^{-n}}{n!}(a-b-1)_n \\ &= \Gamma(b-a-1)z^b(1+z)^{a-b-1}. \end{aligned} \tag{B.20}$$

Holding whenever $Re(b) > Re(a-1)$ and $|arg\,\| < \pi$. One can take $a = \rho - \sigma$ and $b = \rho - 1$ since $\sigma > 0$.

■

Proposition B.2.6

$$\int_0^{\infty} dx\,(x^2+\beta^2)^{-\mu-1}\cos(ax) = \frac{\sqrt{\pi}}{2\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\,G_{1,3}^{2,1}\left[\frac{a^2\beta^2}{4}\,\middle|\,\begin{matrix}-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2}, 0, 1/2\end{matrix}\right]. \tag{B.21}$$

□

Here $q > p$.

$$\begin{aligned}\int_0^\infty dx\,(x^2+\beta^2)^{-\mu-1}\cos(ax) &= \sqrt{\pi}\int_0^\infty x^{2\nu}(x^2+\beta^2)^{-\mu-1}G_{0,2}^{1,0}\left[\frac{a^2x^2}{4}\,\middle|\,\begin{matrix}-\\0,\frac{1}{2}\end{matrix}\right]\\ &= \frac{\pi}{2}\int_0^\infty dx\,x^{\nu-\frac{1}{2}}(x+\beta^2)^{-\mu-1}G_{0,2}^{1,0}\left[\frac{a^2x}{4}\,\middle|\,\begin{matrix}-\\0,\frac{1}{2}\end{matrix}\right]\\ &= \frac{\sqrt{\pi}}{2\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}\\ &\times\int_0^\infty dx\,G_{1,1}^{1,1}\left[x\,\middle|\,\begin{matrix}\nu-\mu-\frac{1}{2}\\ \nu-\frac{1}{2}\end{matrix}\right]G_{0,2}^{1,0}\left[\frac{a^2\beta^2}{4}x\,\middle|\,\begin{matrix}-\\0,\frac{1}{2}\end{matrix}\right]\\ &= \frac{\sqrt{\pi}}{2\Gamma(\mu+1)}\beta^{2\nu-2\mu-1}G_{1,3}^{2,1}\left[\frac{a^2\beta^2}{4}\,\middle|\,\begin{matrix}\nu-\frac{1}{2}\\ \mu-\nu+\frac{1}{2},0,\frac{1}{2}\end{matrix}\right]\end{aligned} \tag{B.22}$$

Holding $a>0, Re(\beta)>0$ and $-\frac{1}{2}<Re(\nu)<Re(\mu+1)$. ■

Proposition B.2.7

$$z^{\nu-\frac{1}{2}}[(z+1)^{-\mu}-z^{-\mu}] = -\frac{1}{\Gamma(\mu)}G_{1,1}^{1,1}\left[z\,\middle|\,\begin{matrix}\nu-\mu+\frac{1}{2}\\ \nu-\frac{1}{2}\end{matrix}\right]. \tag{B.23}$$

□

Here $q=p$ and again one should take care of the contour as explained previously.

$|\mathbf{z}|>\mathbf{1}$

$$\begin{aligned}z^{\nu-\frac{1}{2}}[(z+1)^{-\mu}-z^{-\mu}] &= z^{\nu-\frac{1}{2}-\mu}[(1+z^{-1})^{-\mu}-1]\\ &= z^{\nu-\frac{1}{2}-\mu}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!z^n}n!\right)\\ &= z^{\nu-\frac{1}{2}-\mu}\left(\frac{1}{n!}\frac{(-\mu)_n}{z^n}\right)\\ &= -\frac{1}{\Gamma(\mu)}z^{\nu-\mu-\frac{1}{2}}\sum_{n=0}^{\infty}z^{-n-1}\frac{\Gamma(\mu+n+1)\Gamma(n+1)}{\Gamma(n+2)},\end{aligned} \tag{B.24}$$

where one can identify the sum as the sum of residues of the gamma function

corresponding to $s = -n$. Follows that

$$\begin{aligned} z^{\nu-\frac{1}{2}}[(z+1)^{-\mu} - z^{-\mu}] &= -\frac{1}{\Gamma(\mu)}\frac{1}{2\pi i}\int_C ds\, z^{\nu-\mu-\frac{3}{2}}\frac{\Gamma(\mu+1-s)\Gamma(1-s)\Gamma(+s)}{\Gamma(1-s+1)}z^s \\ &= -\frac{1}{\Gamma(\mu)}z^{\nu-\mu-\frac{3}{2}}G^{1,1}_{1,1}\left[z\,\middle|\,\begin{matrix}1\\ \mu+1\end{matrix}\right]. \end{aligned} \tag{B.25}$$

$|\mathbf{z}| < \mathbf{1}$

From the unicity of the integral one have,

$$\begin{aligned} G^{2,1}_{2,2}\left[z\,\middle|\,\begin{matrix}\nu-\mu-\frac{1}{2}, \nu-\mu+\frac{1}{2}\\ \nu-\frac{1}{2}, \nu-\mu-\frac{1}{2}\end{matrix}\right] &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{\nu-\frac{1}{2}+n}\frac{\Gamma(-\mu-n)\Gamma(\mu+n+1)}{\Gamma(-\mu+1-n)} \\ &+ \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^{\nu-\mu-\frac{1}{2}}z^n\frac{\Gamma(\mu-n)\Gamma(n+1)}{\Gamma(1-n)} \\ &= \Gamma(\mu)z^{\nu-\frac{1}{2}}(z^{-\mu} - (1+z)^{-\mu}). \end{aligned} \tag{B.26}$$

■

B.3 The Integral for the $(1+1)$-dimensional Case

As discussed in the main text, one have the integral to be solved,

$$\begin{aligned} I^{(1+1)} &= \int_0^{\infty} dp\,[p^{2\nu}(p^2+\beta^2)^{-\mu} - p^{2\nu-2\mu}]\,\cos(px) \\ &= \frac{1}{2}\beta^{2\nu-2\mu+1}\int_0^{\infty} dz\, z^{\nu-\frac{1}{2}}[(z+1)^{-\mu} - z^{-\mu}]\,\cos(a\sqrt{z}). \end{aligned} \tag{B.27}$$

For $a = x\beta$ and $\{\mu - \frac{1}{2} < \nu < \mu + \frac{1}{2}\} \cap \{\nu - \frac{1}{2}\}$. One just has to insert the respectively Meijer G-function of each term inside the integral.

$$\begin{aligned} I^{(1+1)} &= -\frac{\sqrt{\pi}}{2\Gamma(\mu)}\beta^{2\nu-2\mu+1}\int_0^{\infty} dz\, G^{2,1}_{2,2}\left[z\,\middle|\,\begin{matrix}\nu-\mu-\frac{1}{2}, \nu-\mu+\frac{1}{2}\\ \nu-\frac{1}{2}, \nu-\mu-\frac{1}{2}\end{matrix}\right] G^{1,0}_{0,2}\left[\frac{x^2\beta^2}{4}z\,\middle|\,\begin{matrix}-\\ 0, \frac{1}{2}\end{matrix}\right] \\ &= -\frac{\sqrt{\pi}}{2\Gamma(\mu)}\beta^{2\nu-2\mu+1}G^{2,2}_{2,4}\left[\frac{x^2\beta^2}{4}\,\middle|\,\begin{matrix}-\nu+\frac{1}{2}, \mu-\nu+\frac{1}{2}\\ \mu-\nu+\frac{1}{2}, 0, \frac{1}{2}, -\nu+\mu-\frac{1}{2}\end{matrix}\right] \end{aligned} \tag{B.28}$$

It is possible to simplify the Meijer G-function found since there equals a and b values.

$$I^{(1+1)} = -\frac{\sqrt{\pi}}{2\Gamma(\mu)}\beta^{2\nu-2\mu+1}G^{2,1}_{1,3}\left[\frac{x^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \frac{1}{2},0,-\nu+\mu-\frac{1}{2}\end{matrix}\right] \tag{B.29}$$

B.4 The Integral for the $(1+3)$-dimensional Case

For the $(1+3)$-dimensional case, the procedure is very similar to the $(1+1)$-dimensional case.

$$\begin{aligned}
I^{(1+3)} &= -\frac{4\pi}{x}\int_0^\infty dp\,[p^{2\nu}(p^2+\beta^2)^{-\mu} - p^{2\nu-2\mu}]\,\sin(px)\\
&= -\frac{4\pi}{x}\frac{\beta^{2\nu-2\mu+2}}{2}\int_0^\infty dz\,G^{2,1}_{2,2}\left[z\middle|\begin{matrix}\nu-\mu-\frac{1}{2},\nu-\mu+\frac{1}{2}\\ \nu-\frac{1}{2},\nu-\mu-\frac{1}{2}\end{matrix}\right]G^{1,0}_{0,2}\left[\frac{x^2\beta^2}{4}z\middle|\begin{matrix}-\\ \frac{1}{2},0\end{matrix}\right]\\
&= \frac{\pi^{3/2}}{\Gamma(\mu)}\frac{\beta^{2\nu-2\mu+2}}{x}G^{2,2}_{2,4}\left[\frac{x^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2},-\nu+\mu+\frac{1}{2}\\ -\nu+\mu+\frac{1}{2},\frac{1}{2},0,-\nu+\mu-\frac{1}{2}\end{matrix}\right]\\
&= \frac{\pi^{3/2}}{\Gamma(\mu)}\frac{\beta^{2\nu-2\mu+2}}{x}G^{2,1}_{1,3}\left[\frac{x^2\beta^2}{4}\middle|\begin{matrix}-\nu+\frac{1}{2}\\ \frac{1}{2},0,\nu-\mu-\frac{1}{2}\end{matrix}\right].
\end{aligned} \tag{B.30}$$

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