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DYNAMICS OF THERMOELASTIC BRESSE SYSTEMS: FOURIER LAW, OBSERVABILITY AND DELAYS

by

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In the labyrinth of numbers, where beauty and solitude intertwine,
we find that every solved equation
is an echo of questions
that will never cease to exist.

Irving R. Barreto

Dedicated to my family,

the only precious gift

in my life.

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Resumo

O sistema de Bresse é um modelo matemático para vigas circulares destacando as propriedades de força de cisalhamento, momento fletor e forças axiais. De acordo com trabalhos recentes de Jorge Silva e Ma (2023), estudaremos um sistema termoelástico do tipo Fourier onde a temperatura atua independentemente nas três propriedades acima mencionadas. Nossos resultados são os seguintes: a) Primeiramente, estudaremos a estabilidade exponencial de sistema termoelástico com a condição de fronteira de Dirichlet, sem adicionar hipóteses extras sobre os coeficientes do sistema. Mas por causa das dificuldades geradas pelos termos de fronteira, provaremos uma nova desigualdade de observabilidade. Isso nos permitirá aplicar uma caracterização de semigrupos exponencialmente estáveis de Gearhart e Prüss. b) Na presença de forças não lineares, provaremos a existência de um atrator global de dimensão fractal finita. c) Em seguida, perturbamos o sistema com um termo de retardo (delay) atuando no momento fletor. Provaremos que para um retardo suficientemente pequeno, a dissipação térmica ainda pode estabilizar o sistema exponencialmente. Notamos que na presença de um retardo, nosso sistema deixa de ser uniformemente dissipativo. Para contornar esse obstáculo apresentaremos algumas ideias novas. d) Finalmente, na presença de forças não lineares, comentaremos alguns trabalhos futuros sobre a dinâmica de longo prazo de tais sistemas com retardos.

Palavras-Chaves: *Equações diferenciais parciais, viga circular, sistema de Bresse, estabilidade exponencial, atrator global, análise de resolvante*.

Título em Português: *A Dinâmica de Sistemas de Bresse Termoelásticos: Lei de Fourier, Observabilidade e Retardos*

Abstract

The Bresse system is a mathematical model for circular beams that features shear force, bending moment and axial displacements. Following recent works of Jorge Silva and Ma (2023), we consider a thermoelastic Bresse beam where thermal effects satisfy Fourier Law and acts independently on above three features. Our main results are the following: a) First, we study the thermoelastic system with Dirichlet boundary condition. We prove that the thermal dissipation can drives the system exponentially to zero without adding special assumptions on the system's coefficients. To this end, because the difficulties coming from the boundary condition, we shall provide a suitable observability inequality. Then we apply a characterization of exponentially stability for linear semigroups by Gearhart and Prüss. b) Next, by adding a nonlinear foundation, we prove the existence of a global attractor. The main difficult is to show that the system is quasi-stable in the sense of Chueshov and Lasiecka. c) Then we perturb the thermoelastic Bresse system with a delay term acting on the bending moment. We prove that for a sufficiently small delay the thermal dissipation can still stabilize exponentially the system. Such kind of result was early proved for wave equations with frictional damping or with viscoelastic dissipation. Our result needs new arguments since thermal dissipation is essentially different from delay effect. In addition, our delay system is not uniformly dissipative. To deal with the delay term we use a method by Nicaise and Pignotti. d) Finally, in the framework of nonlinear foundation, we comment future ideas about the long time dynamics for the delay system.

Keywords: *Partial differential equations, circular beam, Bresse system, exponential stability, global attractor, resolvent analysis*.

Contents

Introduction

The use of arched structures is known since ancient times. Remarkable presence of such structure can be found in roman aqueducts and bridges. In many situations such structures can modeled through beam equations. Our objective is to study a class of arched beam proposed by Jacques Antonine Charles Bresse (1859) in the framework of thermoelasticity. As noticed in Jorge Silva and Ma [22], the effect of heat flux can act differently on the shear and bending components, and the producing new thermoelastic models.

Our work is to analyze the well-posedness, asymptotic stability and long-time dynamics of a thermoelastic Bresse beam featuring Fourier Law, as described in [22].

Literature. Proposed by Bresse [12], the arched beam has three main quantities to be taken into account, namely, vertical displacement (φ), rotation angle (ψ) and horizontal displacement (w). See Figure 1. The equations that describe its vibrations are given by,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) = 0, \\ \rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) = 0, \end{cases} \tag{1}$$

with $x \in (0, L)$ and $t \geq 0$. Here, $\rho_1, \rho_2, \rho_3, k, b, k_0$ are positive parameters of the model, accounting for density, modulus of elasticity, Poisson ratio and other physical quantities. We shall review some works close to ours.

- Modeling aspects of Bresse system can be found in Bresse [12], Lagnese, Leugering and Schmidt [24], Almeida Junior, Muñoz Rivera and Santos [3] and Jorge Silva and Ma [22].
- A celebrated paper that motivated many further analyses on the subject is given by Liu and Rao [25]. They presented a first analysis of the Bresse system by using Semigroup Theory. In a thermoelestic setting, they also studied the asymptotic behavior of the model. To this regard, they proved stability of the solution semigroup through resolvent analysis of Gearhart [20] and Prüss [31]. See also [26, Theorem 1.3.2].
- About thermoelastic Bresse systems, some early results where established by, for instance, Fatori and Muñoz Rivera [18], Bittencourt Moraes and Jorge Silva [11], and Dell'oro [16]. Almeida Junior, Muñoz Rivera and Santos [3], presented novel ideas where dissipative thermal effect appears only on the shear component, not on bending, including some modeling aspects. Our work is based in part on that paper. More recently, we find new results focusing longtime dynamics in Freitas et al. [19].
- Many papers deal with partially damped Bresse systems. Then exponential stability are achieved with some sort of equal wave condition. We mention, for instance, [1, 19, 8, 7, 16] and references therein.
- Some Bresse systems with viscoelastic properties can be found in [6, 29, 33] and references therein.
- Our work also consider a special class of delay system, which for wave equations, is precisely $\Box u + \mu u_t(t-\tau) = 0$. Note that delay term appears at the velocity level, as proposed by Nicaise and Pignotti [28]. In this direction, the main property is that a frictional damping can control the delay term in order to get exponential stability. In the same direction, a memory term can control the delay $\mu u_t(t-\tau)$ if $\mu > 0$ is small enough. See e.g. [2].

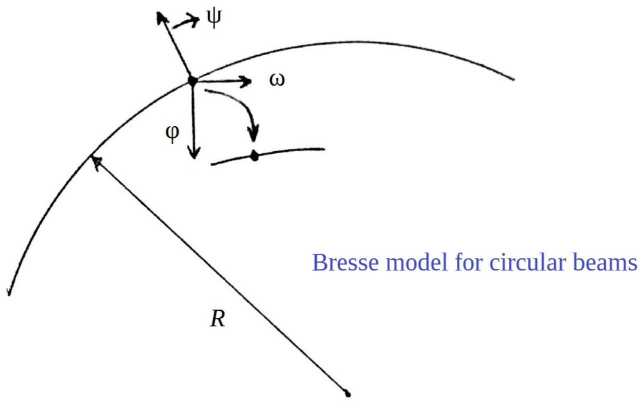

Figure 1: The Bresse arched beam.

On the modeling aspects. In what follows, let us denote:

$$Q = \text{shear force}, \quad M = \text{bending moment}, \quad N = \text{axial tension}.$$

Then the original linear Bresse model, without thermal effects, can be deduced from the constitutive equations

$$\begin{cases} Q = k(\varphi_x + \psi + \ell w), \\ M = b\psi_x, \\ N = k_0(w_x - \ell\varphi). \end{cases} \tag{2}$$

and the governing equations

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + \ell N, \\ \rho_2\psi_{tt} = M_x - Q, \\ \rho_1 w_{tt} = N_x - \ell Q, \end{cases} \tag{3}$$

Combining (2) and (3) we obtain the classical Bresse system (1).

Now, one introduces (distinct) heat effects $\theta^1, \theta^2, \theta^3$ on Q, M, N, respectively. Then we get new constitutive laws,

$$\begin{cases} Q = k(\varphi_x + \psi + \ell w) - m_1\theta^1, \\ M = b\psi_x - m_2\theta^2, \\ N = k_0(w_x - \ell\varphi) - m_3\theta^3. \end{cases} \tag{4}$$

For each heat component θ^j one has a proper Fourier heat equation. Then inserting new Q, M, N (4) into the governing equations (3), we finally obtain,

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 = 0, \\ \rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta 2_x - \ell m_1\theta^1 = 0, \\ \sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \\ \sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \\ \sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0, \end{cases} \tag{5}$$

defined with $x \in (0, L)$ and $t \geq 0$.

In the following we shall establish several results on the dynamics of this thermoelastic system.

Our results to the system with Dirichlet boundary condition. In the Chapter 2 we study the system (5) with Dirichlet boundary condition for all the equations (sometimes called full-Dirichlet). In this case, for a solution $(\varphi, \psi, w, \theta^1, \theta^2, \theta^3)$, the energy of the system (5) is defined by

$$\begin{aligned} E(t) = {}& \frac{k}{2}\|\varphi_x + \psi + \ell w\|^2 + \frac{b}{2}\|\psi_x\|^2 + \frac{k_0}{2}\|w_x - \ell\varphi\|^2 \\ & + \frac{\rho_1}{2}\|\varphi_t\|^2 + \frac{\rho_2}{2}\|\psi_t\|^2 + \frac{\rho_1}{2}\|w_t\|^2 \\ & + \frac{\sigma_1}{2}\|\theta^1\|^2 + \frac{\sigma_2}{2}\|\theta^2\|^2 + \frac{\sigma_3}{2}\|\theta^3\|^2. \end{aligned}$$

It follows that the energy space is $\mathcal{H} = H_0^1(0,L)^3 \times L^2(0,L)^6$. We can deduce that

$$E'(t) = -\Big(\gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2\Big),$$

which shows that thermal effect are the only dissipative components of the system.

Our contributions to this system are the following:

- We prove the Hadamard well-posedness to the system (5). This step is standard. We write the system as an abstract Cauchy Problem

$$\frac{dy}{dt} = Ay, \quad y(0) = y_0,$$

  defined in $\mathcal{H}$, and then apply Semigroup Theory. We also consider a nonlinear foundation to the system by adding nonlinear forces $f_j(\varphi, \psi, w)$, $j = 1, 2, 3$, on the first three equations, respectively. To keep the variational structure of the problem, we assume the forces are of gradient type, that is, there exists a potential $F : \mathbb{R}^3 \to \mathbb{R}$, such that $\nabla F = (f_1, f_2, f_3)$. Under some appropriate assumptions, we also prove the well-posedness of the semilinear problem. See Theorem 2.2.

- To the linear problem, we show that thermal dissipation drives the system exponentially to zero, without any special condition on the model parameters, including equal wave speeds. This is done by using the resolvent analysis by Gearhart and Prüss. The challenging point is that, with Dirichlet boundary condition, some boundary quantities are difficult to be estimated. To bypass this, we provide an observability inequality motivated by results in Bittencourt Moraes and Jorge Silva (see also [11], [4]). Our results are Theorem 2.3 and Lemma 2.5.

- To the semilinear problem, we establish the existence of a finite-dimensional global attractor. We show that the system is quasi-stable in the sense of Chueshov and Lasiecka

[15] by analyzing directly the variation of parameters formulae. This allow us taking advantage of the previous result where it was proved that the linear system corresponds to an exponentially stable contraction semigroup. It seems this approach is new in our context. See Theorem 2.7.

Our results to the system with delay at velocity level. In the Chapter 3 we consider the Bresse system (5) with a delay term at velocity level. As mentioned above, wave equations with frictional damping and delay at velocity level were studied by Nicaise and Pignotti [28]. To simplify a little the presentation, let us suppose that delay effect is effective only on the rotation angle ψ. Since it appears at velocity level, we shall modify the governing equation for ψ, that is,

$$\rho_2\psi_{tt} = M_x - Q + \mu\psi_t(x, t-\tau). \tag{6}$$

Then system (5) becomes

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + \mu\psi_t(x, t-\tau) = 0, \\ \rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 = 0, \\ \sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \\ \sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \\ \sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0. \end{cases} \tag{7}$$

To this delay system, we shall assume that equations for φ, ψ, w satisfy Dirichlet, Neumann, Neumann, boundary conditions, respectively. On the other hand, the equations for $\theta^1, \theta^2, \theta^3$ satisfy Neumann, Neumann, Dirichlet, boundary conditions, respectively. This assumption allow us to compare our results with some previous one in the literature and also makes observability inequality not necessary.

Our contributions are the following.

- The delay system needs a prescribed function g accounting $\psi_t(x,t)$ when $-\tau < t < 0$. Then using a change of variables proposed in Nicaise and Pignotti [28], namely,

$$z(x,\rho,t) = \psi_t(x, t-\rho\tau), \quad (x,\rho,t) \in (0,L) \times (0,1) \times \mathbb{R}^+,$$

  we can write system (7) into an equivalent one, defined only for $t \geq 0$, with variables $(\varphi, \psi, w, \varphi_t, \psi_t, w_t, \theta^1, \theta^2, \theta^3, z)$. As we will see in Chapter 3, this new system is no longer (uniformly) dissipative, that is, $\langle Az, z\rangle \leq 0$ does not hold. As a consequence, we cannot apply, for instance, the Lumer-Phillips Theorem to study its well posedness. Then, to prove the existence of result we show that associated linear operator is maximal monotone. See Theorem 3.1.

- By using energy method, we show that $\|\theta^2_x\|$ can control $\xi\|\psi_t\|$ for $\xi > 0$ sufficiently small. Also, $\xi\|\psi_t\|$ can control $\mu\|\psi_t(t-\tau)\|$ if $\mu > 0$ is sufficiently small. From this, we obtain an equivalent perturbed energy functional that decays exponentially to zero. This implies the exponential stability of the delay system. See Theorem 3.2.

- Finally, our last result establishes the existence of global solutions to the associated semilinear problem. We also discuss the steps to prove existence of global attractors. Because the associated linear operator is not uniformly dissipative, the system energy is not a strict Lyapunov functional. Then, to show the system is dissipative, we need construct an absorbing set. So, under certain assumptions, the existence of a finite dimensional global attractor can be obtained from the quasi-stability property. Upper semicontinuity of attractors for parameter $\mu \to 0$ can be also addressed.

Chapter 1

Preliminaries

This section is dedicated to introducing some known results that will be very helpful to understand appropriately the concepts, ideas and conclusions that will be presented throughout this work. This preliminaries involve concepts and classical definitions of L^p-spaces (see for more detail in [13, 32]). For Semigroup Theory and characterization of semigroups with additional properties, we use [30, Chapter 1] to obtain a better reference. Finally, we have a powerful tool which that we will use to show the existence and uniqueness for both of problems presented in this work (See [30, Chapter 4]).

1.1 Sobolev spaces

Given a open bounded set $\Omega \in \mathbb{R}^n$, we represent by $L^p(\Omega)$, with $1 \leq p < \infty$, the space of measurable functions $u : \Omega \to R$ such that $|u|^p$ is Lebesgue integrable over Ω, and $L^\infty(\Omega)$ the space of measurable functions such that $|u(x)|$ is bounded by a constant c almost everywhere in Ω. Such spaces is endowed with the norm:

$$\|u\|_p = \|u\|_{L^p(\Omega)} := \Big(\int_\Omega |u(x)|^p \Big)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

$$\|u\|_\infty = \|u\|_{L^\infty(\Omega)} := \sup_{x\in\Omega} \big\{|u(x)|\big\}$$

and they are Banach spaces. In particular, the space $L^2(\Omega)$, endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx$$

is a Hilbert space.

Given a bounded set $\Omega \in \mathbb{R}^n$, and taking a positive natural number m, we define the space:

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega)|\ D^\alpha u \in L^p(\Omega),\ \forall|\alpha| \leq m\}$$

where D^α represents the derivative in the sense of distributions. This space is endowed with the norm:

$$\|u\|_{W^{m,p}} := \|u\|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p dx\Big)^{\frac{1}{p}},\ \text{for } 1 \leq p < \infty$$

and it is a Banach space, and its called a Sobolev space. Similarly, if we take $p = 2$, the space $W^{m,2}(\Omega)$ is as Hilbert space, represent by $H^m(\Omega)$, and the inner product is given by:

$$(u, v)_{H^m(\Omega)} = \sum_{|a|\leq m} (D^\alpha u, D^\alpha v)_{L^2},\ \forall u, v \in H^m(\Omega).$$

This space is a Sobolev space of order m. When $m = 0$, we identify the space $H^0(\Omega)$ with $L^2(\Omega)$.

Definition 1.1. Given a bounded set Ω, the space $W_0^{m,p}(\Omega)$ is defined by the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, and the norm is defined by:

$$\|u\|_{W_0^{m,p}} := \Big(\sum_{|\alpha=m|} \int_\Omega |D^\alpha u(x)|^p dx\Big)^{\frac{1}{p}}$$

and it is equivalent to the norm $W_0^{m,p}(\Omega)$.

With this, we can define:

$$H_0^1(\Omega) = \{u \in L^2(\Omega) |\ D^\alpha u \in L^2(\Omega), \forall |\alpha| \leq 1\}$$

which norm is equivalent to the norm of space $H^1(\Omega)$, and its defined by

$$\|u\|_{H_0^1} = \|\nabla u\|_{L^2}.$$

1.1.1 Auxiliary results

Definition 1.2. Let X and Y two Hilbert spaces, with X being a subspace of Y. We say X is continuously embedded in Y, if there exist a positive constant M such that:

$$\|u\|_Y \leq M\|u\|_X, \quad \forall u \in X.$$

We denote this embed by $X \hookrightarrow Y$.

With this, we have that:

1. The distribution space $D(\Omega)$ is dense in $L^p(\Omega)$ and $D(\Omega) \hookrightarrow L^p(\Omega)$, for all $1 \leq p < +\infty$;

2. If Ω is bounded, and $1 \leq p < q < \infty$, then $L^q(\Omega) \hookrightarrow L^p(\Omega)$.

Proposition 1.1. *(Cauchy-Schwarz inequality)*

Let H an Hilbert space. Then, given $u, v_0 \in H$, we have

$$|(u, v)_H| \leq \|u\|_H \cdot \|v\|_H.$$

Proposition 1.2. *(Young's Inequality)*

Consider $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and a, b positive numbers. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

In particular, for any $\varepsilon > 0$, we have the following inequality

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q$$

where $C(\varepsilon)$ is a constant dependent on ε.

Proposition 1.3. *(Poincare's Inequality) Suppose Ω is an bounded open set in $\mathbb{R}^n$. Then, for every $1 \leq p < \infty$, there exists a constant, c_p, such that*

$$\|u\|_{W_0^{1,p}} \leq c_p \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega).$$

In particular, if we consider $\Omega = (0, L) \in \mathbb{R}$, we have the Poincare-Wirtinger inequality (See [27]):

$$\|u\|_{L^2} \leq \frac{L}{\pi} \|u_x\|_{L^2}, \quad \textit{and} \quad \|u\|_{H_0^1} = \|u_x\|_{L^2}$$

for any $u \in H_0^1(0, L)$.

Lemma 1.1. *(Grönwall type Lemma)*
Consider $m \in L^1(a, b)$ such that $m \geq 0$ almost everywhere in (a, b) e let $c \geq 0$. Take $f : [a, b] \to R$ satisfying:

$$f(t) \leq c + \int_a^t m(s) f(s) ds, \quad \forall t \in [a, b].$$

Then:

$$f(t) \leq c e^{\int_0^t m(s) ds}, \quad \forall t \in [a, b].$$

1.2 Semigroup theory

Consider X a Banach space. A one parameter family T, with $0 \leq t < \infty$, of bounded linear operators from X onto X is called a *semigroup of bounded linear operator on X* if:

1. $T(0) = I$, where I is the identity operator on X.

2. $T(t+s) = T(t)T(s)$, for every $t, s \geq 0$, (*the semigroup property*).

A semigroup of bounded linear operators, $T(t)$, is called *uniformly continuous* if satisfies:

$$\lim_{t \to 0^+} \|T(t) - I\| = 0.$$

The linear operator A defined by:

$$D(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt}\bigg|_{t=0} \qquad \text{for } x \in D(A),$$

is the *infinitesimal generator* of the semigroup $T(t)$, where $D(A)$ means for the domain of A.

Definition 1.3. A semigroup $T(t), 0 \leq t < \infty$ of bounded linear operators on X is called a *semigroup of class* C_0, or simply a C_0 semigroup if

$$\lim_{t \to 0^+} T(t)x = x, \qquad \text{for every } x \in X.$$

Theorem 1.1. *Let $T(t)$ be a C_0 semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that:*

$$\|T(t)\| \leq M e^{\omega t} \quad \textit{for } 1 \leq t < \infty.$$

Proof: See [30].

Corollary 1.1. *If $T(t)$ is a C_0 semigroup the for every $x \in X$, $t \to T(t)x$ is a continuous function from $\mathbb{R}_0^+$ into X.*

Proof: See [30].

1.3 The Hille-Yoshida and Lumer Phillips theorems

Let $T(t)$ be a C_0 semigroup. From the Theorem 1.1 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq; e^{\omega t}$, for $t \geq 0$. Of $\omega = 0$, $T(t)$ its called *uniformly bounded*. Moreover, if $M = 1$, the semigroup $T(t)$ is called a C_0 *semigroup of contractions*. In this section we give the characterization of the infinitesimal generators of C_0 semigroups of contractions.

The conditions on the operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0 semigroup of contractions, are given. Remembering that if A is a linear, not necessary bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ such that $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator on X. The family, $R(\lambda : A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operator is called the *resolvent of* A. With this, we state the first important Theorem of this section.

Theorem 1.2. *(Hille-Yoshida Theorem)*
The linear operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \geq 0$ if and only if

1. *A is closed and $\overline{D(A)} = X$.*

2. *The resolvent set $\rho(A)$, of A contains $\mathbb{R}^+$ and for every $\lambda > 0$,*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}$$

This theorem is very important since it helps to guarantee the existence of an operator A such that it is the infinitesimal generator of a C_0 semigroup $T(t)$. However, in many real models, is not easy to find the sufficient conditions to prove such a result. For this reason, we resort to a auxiliary Theorem that is more applicable to the general problems.
Let be X a Banach space, and let X^* be its dual. We denote the value $x^* \in X^*$ at $x \in X$ by

$\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. For each $x \in X$ we define the duality set $F(X) \subseteq X^*$ by:

$$F(X) = \left\{ x^* : x^* \in X^* \text{ and } \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{1.1}$$

Then, by the Hahn-Banach Theorem it follows that $F(x) \neq 0$ for every $x \in X$.

Definition 1.4. A linear operator A is called $dissipative$ if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that $\text{Re}\langle Ax, x^* \rangle \leq 0$.

With this, we enunciate a second important Theorem:

Theorem 1.3. *(Lumer-Phillips)*
Let A be a linear operator (non necessary bounded) with dense domain $D(A)$ in X.

1. *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

2. *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then the range $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, Re$\langle Ax, x^* \rangle \leq 0$..*

Finally, we state an auxiliary and very useful result as consequence of the Lumer-Phillips theorem.

Theorem 1.4. *Let A be an linear, not necessary bounded, dissipative operator with domain $D(A)$ dense in X. If $0 \in \rho(A)$, then A is a infinitesimal generator of a C_0 semigroup of contractions.*

Proof: See [1, 30].

1.4 The abstract Cauchy problem

Let X be a Banach space and let A be a linear operator (not necessary bounded) from $D(A) \subset X$ into X. Given $x \in X$ the homogeneous abstract Cauchy problem for A with initial data x consist of searching a solution $y(t)$ to the initial value problem:

$$\frac{dy(t)}{dt} = Ay(t), \qquad t > 0 \tag{1.2}$$
$$y(0) = x.$$

where by a solution we mean an valued function $y(t)$ in X such that $y(t)$ is continuous for $t \geq 0$, continuously differentiable and $y(t) \in D(A)$, for $t > 0$ and (1.2) is satisfied. Remember that since $y(t) \in D(A)$ for $t > 0$ and y is continuous at $t = 0$, (1.2) cannot have solution for $x \notin D(A)$.

From the results of the previous sections, it is clear that if A is the infinitesimal generator of a C_0 semigroup $T(t)$, the Homogeneous abstract Cauchy problem for A has a solution, namely $y(t) = T(t)x$, for every $x \in D(A)$. Its not difficult to show that for $x \in D(A)$, $y(t) \to T(t)x$ is the only solution of (1.2). The following theorem show us that, for every $x \in D(A)$, we have an unique solution.

Theorem 1.5. *Let A be a densely defined linear operator with a nonempty resolvent set $\rho(A)$. The initial value problem (1.2) has a unique solution $y(t)$, which is continuously differentiable on $[0, \infty)$, for every initial value $x \in D(A)$* ***if and only if*** *A is the infinitesimal generator of a C_0 semigroup $T(t)$.*

Proof: See [30].

The next theorem describes a situation in which the initial value problem (1.2) has a unique solution for every $x \in X$.

Theorem 1.6. *If A is a infinitesimal generator of a differentiable semigroup then for every $x \in X$ the initial value problem (1.2) has a unique solution.*

Proof: See [30].

Now, consider a function $f : [0, T] \to X$. The non homogeneous Cauchy problem is defined by:

$$\frac{dy(t)}{dt} = Ay(t) + f(t), \qquad t > 0 \tag{1.3}$$
$$y(0) = x.$$

We will assume throughout this section that A is th infinitesimal generator of a C_0 semigroup $T(t)$ so that the corresponding homogeneous equation (that is, $f = 0$), has a unique solution for every initial value $x \in D(A)$.

Definition 1.5. A function $y : [0, T) \to X$ is a (classical) solution of (1.3) on $[0, T]$ if y is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $y(t) \in D(A)$ for $0 < t < T$ and (1.3) is satisfied on $[0, T)$.

Let $T(t)$ be the C_0 semigroup generated by A (from the homogeneous problem), and let y be a solution of (1.3). Then the X-valued function $g(s) = T(t-s)y(s)$ is differentiable for $0 < s < t$ and

$$\begin{aligned}\frac{dg}{ds} &= -AT(t-s)y(s) + T(t-s)y'(s) \\ &= -AT(t-s)y(s) + T(t-s)Ay(s) + T(t-s)f(s) \\ &= T(t-s)f(s).\end{aligned}$$

If $f \in L^1(0, T : X)$ then, $T(t-s)f(s)$ is integrable and integrating the last inequality from 0 to t yields

$$y(t) = T(t)x + \int_0^t T(t-s)f(s)dx. \tag{1.4}$$

Corollary 1.2. *If $f \in L^1(0, T : X)$ then for every $x \in X$, the initial value problem (1.3) has at most one solution. If it has a solution, this solution is given by (1.4).*

Proof: See [30].

We conclude this section with a few observations concerning still another notion of solution of the initial value problem (1.3), namely *strong solution*.

Definition 1.6. A function $y : [0, T) \to X$ which is differentiable almost everywhere on $[0, T]$ such that $y' \in L^1(0, T : X)$ is called a *strong solution* of the initial value problem (1.3) if $y(0) = x$ and $y'(t) = A(u) + f(t)$, a.e on $[0, T]$.

As an important result that we will use in the next chapters, we have:

Theorem 1.7. *Let X be a reflexive Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If f is Lipschitz continuous on $[0, T]$ then for every $x \in D(A)$ the initial value problem (1.3) has a unique strong solution y on $[0, T]$, given by:*

$$y(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

Proof: See [30].

Chapter 2

Bresse-Fourier system with full thermal coupling

2.1 Well-posedness

In this section, we are concerned to study the well-posedness of solutions for a semilinear Bresse-Fourier system with thermal coupling in all variables, and we prove the existence and uniqueness of their solution, considering the nonlinear model as follows:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 + f_1(\varphi, \psi, w) = 0, \tag{2.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + f_2(\varphi, \psi, w) = 0, \tag{2.2}$$

$$\rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 + f_3(\varphi, \psi, w) = 0, \tag{2.3}$$

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \tag{2.4}$$

$$\sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \tag{2.5}$$

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0, \tag{2.6}$$

defined in $(0, L) \times \mathbb{R}^+$, where $f_i(\varphi, \psi, w)$, for $i = 1, 2, 3$, are nonlinear external forces. The system is subject to the totally-Dirichlet boundary conditions:

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) = 0, \quad t \geq 0, \tag{2.7}$$

$$\theta^1(0,t) = \theta^1(L,t) = \theta^2(0,t) = \theta^2(L,t) = \theta^3(0,t) = \theta^3(L,t) = 0, \quad t \geq 0, \tag{2.8}$$

and, for $i = 1, 2, 3$, initial data:

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x); \;\; \psi(x,0) = \psi_0(x); \;\; w(x,0) = w_0(x); \;\; \theta^i(x,0) = \theta^i_0(x); \\ &\varphi_t(x,0) = \varphi_1(x); \; \psi_t(x,0) = \psi_1(x); \; w_t(x,0) = w_1(x). \end{aligned} \tag{2.9}$$

Here, $\rho_1, \rho_2, \sigma_1, \sigma_2, \sigma_3, k, k_0, m_1, m_2, m_3$ and b are all positive structural constants coming from the physical model, $\gamma_1, \gamma_2, \gamma_1 > 0$ represents the damping coefficients, $L > 0$ is the length of the beam, $\ell > 0$ means the beam curvature, and the unknown variables φ, ψ, w stands for the vertical displacement, rotation angle, and longitudinal displacement, respectively. In addition, $\theta^1, \theta^2, \theta^3$ are the difference (in comparison with the environment) of temperatures of the thermal coupling. Since our problem has thermal damping terms in all of the equations (2.1)-(2.3) we shall not assume the equal wave speed assumption. (See [27]).

For the linear model, the equation is described by:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = 0, \tag{2.10}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 = 0, \tag{2.11}$$

$$\rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 = 0, \tag{2.12}$$

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \tag{2.13}$$

$$\sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \tag{2.14}$$

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0, \tag{2.15}$$

2.1.1 Phase space $\mathcal{H}$

Let us start with the development of the phase space $\mathcal{H}$ for the system (2.10)-(2.15) . First of all, notice that the system has the first usual three equations for a Bresse system, in addition with three heat equations on the next ones. So, equations $(2.10)-(2.12)$ are multiplied by φ_t, ψ_t and w_t, respectively, while $(2.13)-(2.15)$ are multiplied by $\theta^1, \theta^2, \theta^3$, respectively. Then, integrating on $(0, L)$, and using the Dirichlet boundary conditions we have:

$$\rho_1 \int_0^L \varphi_{tt}\varphi_t dx + k \int_0^L (\varphi_x + \psi + \ell w)(\varphi_x)_t dx + k_0 \int_0^L (w_x - \ell\varphi)(-\ell\varphi_t) dx + \underbrace{m_1 \int_0^L (\theta^1_x)(\varphi_t) dx + m_2 \int_0^L (\theta^2)(\ell\varphi)_t \, dx}_{A_1} = 0$$

$$\rho_2 \int_0^L \psi_{tt}\psi_t dx + b \int_0^L \psi_x(\psi_x)_t dx + k \int_0^L (\varphi_x + \psi + \ell w)(\psi_t) dx + \underbrace{m_3 \int_0^L (\theta^3_x)(\psi_t) dx - m_1 \int_0^L (\theta^1)(\psi_t) \, dx}_{A_2} = 0$$

$$\rho_1 \int_0^L w_{tt} w_t dx + k_0 \int_0^L (w_x - \ell\varphi)(w_x)_t dx + k \int_0^L (\varphi_x + \psi + \ell w)(\ell w)_t dx + \underbrace{m_2 \int_0^L (\theta^2_x)(w_t) dx - m_1 \int_0^L (\theta^1)(\ell w)_t \, dx}_{A_3} = 0$$

$$\sigma_1 \int_0^L \theta^1_t \cdot \theta^1 dx + \gamma_1 \int_0^L |\theta^1_x|^2 dx + \underbrace{m_1 \int_0^L (\varphi_x + \psi + \ell w)_t (\theta^1) \, dx}_{A_4} = 0$$

$$\sigma_2 \int_0^L \theta^2_t \cdot \theta^2 dx + \gamma_2 \int_0^L |\theta^2_x|^2 dx + \underbrace{m_2 \int_0^L (w_x - \ell\varphi)_t (\theta^2) \, dx}_{A_5} = 0$$

$$\sigma_3 \int_0^L \theta^3_t \cdot \theta^3 dx + \gamma_3 \int_0^L |\theta^3_x|^2 dx + \underbrace{m_3 \int_0^L (\psi_x)_t (\theta^3) \, dx}_{A_6} = 0$$

Using integration by parts and using the boundary conditions (2.8) on temperature variables, terms A_1 to A_6 are canceled from each other. Remarking that

$$\frac{d}{dt}\|\theta^i\|^2 = 2\int_0^L \theta^i_t \cdot \theta^i dx$$

Thus:

$$\sigma_i \int_0^L \theta^i_t \cdot \theta^i dx = \frac{d}{dt}\left(\frac{\sigma_i}{2}\|\theta^i\|^2\right)$$

If we don't consider the temperature terms, (2.10)-(2.12) are usually known and we can obtain the same terms as long as classical Bresse systems. Then, we obtain :

$$\frac{d}{dt}\frac{1}{2}\left(k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2\right) = 0$$

The next three equations give us:

$$\frac{d}{dt}\frac{1}{2}\left(\sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2\right) = -\Big(\gamma_1\|\theta^1_x\|^2 + \gamma_2\|\theta^2_x\|^2 + \gamma_3\|\theta^3_x\|^2\Big)$$

Adding this last two equations, we get:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\Big(&k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\ &+ \sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2\Big) \\ &\quad = -\gamma_1\|\theta^1_x\|^2 - \gamma_2\|\theta^2_x\|^2 - \gamma_3\|\theta^3_x\|^2 \le 0 \end{aligned} \tag{2.16}$$

Where the terms $\|\cdot\|^2$ means for the L^2-classical norm over $(0, L)$.

Remark: *Damping means dissipation coefficients that make energy decays. So, we can say, without loss of generality, that $\gamma_1, \gamma_2, \gamma_3$ are **damping coefficients**.*

Thus, for a vector $y = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \theta^1, \theta^2, \theta^3)$, the phase space $\mathcal{H}$ associated to the previous identity (2.16) is:

$$\mathcal{H} = (H_0^1(0,L))^3 \times (L^2(0,L))^6 \tag{2.17}$$

endowed with the norm $\|y\|_{\mathcal{H}}^2$ defined as:

$$\begin{aligned}\|y\|_{\mathcal{H}}^2 = k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\ +\sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2.\end{aligned} \tag{2.18}$$

Remembering that the usual norm for $\mathcal{H}$ is:

$$\|y\|_u^2 = \|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 + \|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2 + \|\theta^1\|^2 + \|\theta^2\|^2 + \|\theta^3\|^2. \tag{2.19}$$

Therefore, we can define the **linear energy** of the system as follows:

$$\begin{aligned}E(t) = \frac{1}{2}\Big(k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\ +\sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2\Big).\end{aligned} \tag{2.20}$$

Due to the Dirichlet conditions in all of their terms ([27]), the norms above are equivalent for any $\ell > 0$, with equivalence constants depending on such parameter, that is, there exists constants η_1 and η_2 such that:

$$\eta_1\|y\|_{\mathcal{H}}^2 \leq \|y\|_u^2 \leq \eta_1\|y\|_{\mathcal{H}}^2 \tag{2.21}$$

.

The norm $\|\cdot\|_{\mathcal{H}}$ comes from the inner product induced by the system in the space $\mathcal{H}$:

$$
\begin{aligned}
\langle y_1, y_2\rangle =& k\langle \varphi_x + \psi + \ell w, \tilde{\varphi}_x + \tilde{\psi} + \ell \tilde{w}\rangle + k_0\langle w_x - \ell\varphi, \tilde{w}_x - \ell\tilde{\varphi}\rangle + b\langle \psi_x, \tilde{\psi}_x\rangle \\
&+ \rho_1\langle \varphi_t, \tilde{\varphi}_t\rangle + \rho_2\langle \psi_t, \tilde{\psi}_t\rangle + \rho_1\langle w_t, \tilde{w}_t\rangle \\
&+ \sigma_1\langle \theta^1, \tilde{\theta^1}\rangle + \sigma_2\langle \theta^2, \tilde{\theta^2}\rangle + \sigma_3\langle \theta^3, \tilde{\theta^3}\rangle.
\end{aligned}
$$

for $y_1 = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \theta^1, \theta^2, \theta^3)$ and $y_2 = (\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{\varphi}_t, \tilde{\psi}_t, \tilde{w}_t, \tilde{\theta^1}, \tilde{\theta^2}, \tilde{\theta^3})$.

Thus, we can rewrite problem (2.10)-(2.15), with initial-boundary condition (2.7)-(2.9) as the abstract Cauchy problem:

$$
\begin{aligned}
\frac{d}{dt}y(t) &= Ay(t),\ t > 0, \\
y(0) &= y_0,
\end{aligned}
\tag{2.22}
$$

where

$$
y(t) = (\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t)) \in \mathcal{H},
$$

with

$$
\varphi' = \varphi_t,\ \psi' = \psi_t,\ w' = w_t.
$$

The initial data is determined by:

$$
y(0) = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \theta_0^1, \theta_0^2, \theta_0^3) = y_0.
$$

The unbounded operator $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is given by:

$$A\begin{bmatrix}\varphi\\ \psi\\ w\\ \varphi'\\ \psi'\\ w'\\ \theta^1\\ \theta^2\\ \theta^3\end{bmatrix}=\begin{bmatrix}\varphi'\\ \psi'\\ w'\\ \frac{k}{\rho_1}(\varphi_x+\psi+\ell w)_x+\frac{k_0\ell}{\rho_1}(w_x-\ell\varphi)-\frac{m_1}{\rho_1}\theta^1_x-\frac{m_2\ell}{\rho_1}\theta^2\\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi+\ell w)-\frac{m_3}{\rho_2}\theta^3_x+\frac{m_1}{\rho_2}\theta^1\\ \frac{k_0}{\rho_1}(w_x-\ell\varphi)_x-\frac{k\ell}{\rho_1}(\varphi_x+\psi+\ell w)-\frac{m_2}{\rho_1}\theta^2_x+\frac{\ell m_1}{\rho_1}\theta^1\\ \frac{\gamma_1}{\sigma_1}\theta^1_{xx}-\frac{m_1}{\sigma_1}(\varphi'_x+\psi'+\ell w')\\ \frac{\gamma_2}{\sigma_2}\theta^2_{xx}-\frac{m_2}{\sigma_2}(w'_x-\ell\varphi')\\ \frac{\gamma_3}{\sigma_3}\theta^3_{xx}-\frac{m_3}{\sigma_3}(\psi'_x)\end{bmatrix}$$

The boundary conditions in the problem allow us to define the domain of A as follows:

$$D(A)=\left(H^2(0,L)\cap H^1_0(0,L)\right)^3\times(H^1_0(0,L))^3\times\left(H^2(0,L)\cap H^1_0(0,L)\right)^3.$$

Results about linear Bresse systems showed that $D(A)$ is dense in $\mathcal{H}$ (cf. [6, 27, 7]). Since abstract Cauchy problem (2.22) has not force terms, we can easily see that:

$$Re\langle Ay,y\rangle=-\left(\gamma_1\|\theta^1_x\|^2+\gamma_2\|\theta^2_x\|^2+\gamma_3\|\theta^3_x\|^2\right)\le 0 \tag{2.23}$$

for all $y\in D(A)$, which proofs the dissipativity of A . It remains to prove that $0\in\rho(A)$. To this end, let's take $J(t)=(j_1,j_2,...,j_9)\in\mathcal{H}$. We want to prove that there exists a unique $y\in D(A)$ such that:

$$Ay=J.$$

The previous identity is equivalent to the following:

$$\begin{aligned}
\varphi' &= j_1, \\
\psi' &= j_2, \\
w' &= j_3, \\
k(\varphi_x + \psi + \ell w)_x + k_0\ell(w_x - \ell\varphi) - m_1\theta^1_x - \ell m_2\theta^2 &= \rho_1 j_4, \\
b\psi_{xx} - k(\varphi_x + \psi + \ell w) - m_3\theta^3_x + m_1\theta^1 &= \rho_2 j_5, \\
k_0(w_x - \ell\varphi)_x - k\ell(\varphi_x + \psi + \ell w) - m_2\theta^2_x + \ell m_1\theta^1 &= \rho_1 j_6, \\
\gamma_1\theta^1_{xx} - m_1(\varphi'_x + \psi' + \ell w') &= \sigma_1 j_7, \\
\gamma_2\theta^2_{xx} - m_2(w'_x - \ell\varphi') &= \sigma_2 j_8, \\
\gamma_3\theta^3_{xx} - m_3(\psi'_x) &= \sigma_3 j_9,
\end{aligned}$$

With the aim of obtaining what is desired, we deduce from the first three equations that:

$$\varphi' = j_1;\ \psi' = j_2;\ w' = j_3 \in H^1_0(0, L)$$

Without loss of generality, we can replace the right side of the remaining equations for functions $-h_i$, with $i = 1, ..., 6$, obtaining:

$$\begin{aligned}
k(\varphi_x + \psi + \ell w)_x + k_0\ell(w_x - \ell\varphi) - m_1\theta^1_x - \ell m_2\theta^2 &= \rho_1 j_4 = -h_1 \\
b\psi_{xx} - k(\varphi_x + \psi + \ell w) - m_3\theta^3_x + m_1\theta^1 &= \rho_2 j_5 = -h_2 \\
k_0(w_x - \ell\varphi)_x - k\ell(\varphi_x + \psi + \ell w) - m_2\theta^2_x + \ell m_1\theta^1 &= \rho_1 j_6 = -h_3 \\
\gamma_1\theta^1_{xx} &= m_1(j_{1,x} + j_2 + \ell j_3) + \sigma_1 j_7 = -h_4 \\
\gamma_2\theta^2_{xx} &= m_2(j_{3,x} - \ell j_1) + \sigma_2 j_8 = -h_5 \\
\gamma_3\theta^3_{xx} &= m_3(j_{2,x}) + \sigma_3 j_9 = -h_6
\end{aligned}$$

Since h_4, h_5 and h_6 are functions belongs to $L^2(0, L)$, and by the boundary conditions for temperature, we deduce that:

$$\theta_1, \theta_2, \theta_3 \in H^2(0, L) \cap H_0^1(0, L).$$

This reduces our previous expression to:

$$\begin{aligned} k(\varphi_x + \psi + \ell w)_x + k_0\ell(w_x - \ell\varphi) &= +m_1\theta_x^1 + \ell m_2\theta^2 - h_1 =: -\overline{h_1} \\ b\psi_{xx} - k(\varphi_x + \psi + \ell w) &= m_3\theta_x^3 - m_1\theta^1 - h_2 =: -\overline{h_2} \\ k_0(w_x - \ell\varphi)_x - k\ell(\varphi_x + \psi + \ell w) &= m_2\theta_x^2 - \ell m_1\theta^1 - h_3 =: -\overline{h_3} \end{aligned}$$

From this point, we will proceed to construct all necessary components for applying the Lax-Milgram theorem. Multiplying the previous equations by $\tilde{\varphi}, \tilde{\psi}, \tilde{w} \in H_0^1(0, L)$, respectively, give us:

$$-k\int_0^L (\varphi_x + \psi + \ell w)(\tilde{\varphi_x} + \tilde{\psi} + \ell\tilde{w})dx - k_0\int_0^L (w_x - \ell\varphi)(\tilde{w_x} - \ell\tilde{\varphi})dx - b\int_0^L \psi_x \cdot \tilde{\psi}_x dx$$
$$= -\int_0^L \Big(\overline{h_1}\tilde{\varphi} + \overline{h_2}\tilde{\psi} + \overline{h_3}\tilde{w})dx$$

Then

$$k\int_0^L (\varphi_x + \psi + \ell w)(\tilde{\varphi_x} + \tilde{\psi} + \ell\tilde{w})dx + k_0\int_0^L (w_x - \ell\varphi)(\tilde{w_x} - \ell\tilde{\varphi})dx + b\int_0^L \psi_x \cdot \tilde{\psi}_x dx$$
$$= \int_0^L \Big(\overline{h_1}\tilde{\varphi} + \overline{h_2}\tilde{\psi} + \overline{h_3}\tilde{w})dx \tag{2.24}$$

Considering the Hilbert space $V = (H_0^1(0, L))^3$, let us define a bilinear form a:

$$a : V \times V \longrightarrow \mathbb{R}$$

given by

$$a((\varphi,\psi,w),(\tilde{\varphi},\tilde{\psi},\tilde{w})) = k\int_0^L(\varphi_x+\psi+\ell w)(\tilde{\varphi}_x+\tilde{\psi}+\ell\tilde{w})dx + k_0\int_0^L(w_x-\ell\varphi)(\tilde{w}_x-\ell\tilde{\varphi})dx + b\int_0^L \psi_x\cdot\tilde{\psi}_x dx.$$

and a functional $\mathbf{F}: V \longrightarrow \mathbb{R}$ as:

$$F(\varphi,\psi,w) = \int_0^L \Big(\overline{h_1}\tilde{\varphi}+\overline{h_2}\tilde{\psi}+\overline{h_3}\tilde{w})dx.$$

Here, we can use the induced norm in V defined by

$$\|y\|_V := k\|\varphi_x+\psi+\ell w\|_{L^2} + b\|\psi_x\|_{L^2} + k_0\|w_x-\ell\varphi\|_{L^2},$$

and the usual norm, for every $y=(\varphi,\psi,w)$, by

$$\|y\| = \|\varphi_x\| + \|\psi_x\| + \|w_x\|,$$

Taking (2.24), we associate the left side one with the bilinear form $a(\ ,\ )$, and the right side with the functional $\mathbf{F}$ to obtain

$$a(y_1,y_2) = \mathbf{F}(y_2).$$

where $y_1=(\varphi,\psi,w),\ y_2=(\tilde{\varphi},\tilde{\psi},\tilde{w})\in V.$

By definition of both bilinear form a and functional $\mathbf{F}$, in addition with the equivalence of the norms mentioned above, is not difficult to prove the conditions for Lax-Milgram theorem, that is:

· a is coercive.

· a is continuous.

To show the continuity of $\mathbf{F}$, we can choose an arbitrary $y=(\varphi,\psi,w)$ in V, then, by Holder's inequality:

$$|\mathbf{F}(y)| \leq \|\overline{h_1}\|_{L^2} \cdot \|\varphi\|_{L^2} + \|\overline{h_2}\|_{L^2} \cdot \|\psi\|_{L^2} + \|\overline{h_3}\|_{L^2} \cdot \|w\|_{L^2}$$

Since the functions $\overline{h_1}, \overline{h_2}, \overline{h_3}$ are well known, there are limited by a universal constant C . By the Poincare inequality, and the property of equivalence of norms, give us the following result

$$|\mathbf{F}(x)| \leq C\|y\|_V.$$

In consequence, by Lax-Milgram Theorem, there exist an unique $y_1 = (\varphi, \psi, w)$ in V such that

$$a(y_1, y_2) = \mathbf{F}(y_2)$$

for all $y_2 \in V$.

In addition, due to elliptic regularity, we have that y in $\mathcal{H}$ also satisfies $y \in D(A)$. Then, $0 \in \rho(A)$, where $\rho(A)$ means the resolvent of the unbounded operator A.

With this result, the dissipativity of A, and the density of $D(A)$ into $\mathcal{H}$, we can conclude that A is an infinitesimal generator of a C_0-semigroup of contractions $S(t)$, for $t \geq 0$. Therefore, the existence theorem is given in terms of the equivalent Cauchy problem (2.22), as follows:

Theorem 2.1. *Under the above notations, let us assume that $\ell > 0$ and $\rho_1, \rho_2, k, k_0, b, \gamma_1, \gamma_2, \gamma_3$, $\sigma_1, \sigma_2, \sigma_3$ positive constants. Then for any initial data $y_0 \in \mathcal{H}$ and $T > 0$, problem (2.32) has a unique mild solution*

$$y \in C([0, T], \mathcal{H});\ y(0) = y_0,$$

which depends continuously on the initial data. In particular, if $y_0 \in D(A)$, then the solution is strong. Moreover, if $y \in D(A)$, and $y(t)$ is a local solution of (2.22) *in* $(0, T_{max})$*, then*

$$T_{max} = +\infty.$$

2.1.2 The nonlinear case

As we mentioned in the beginning of this section, we are interested about the existence, uniqueness and continuous dependence of solutions for the system (2.1)-(2.6). To this end, we will give the assumptions on the external force terms f_1, f_2 and f_3.
Lets start assume there exists a $C^2(0, L)$ function $F : \mathbb{R}^3 \to \mathbb{R}$ such that:

$$\nabla F = (f_1, f_2, f_3), \tag{2.25}$$

satisfying the following conditions: There exists a constant $m_F > 0$ such that

$$F(u, v, w) \geq -m_F, \;\; \forall u, v, w \in \mathbb{R}, \tag{2.26}$$

and there exists $p \geq 1$ and a constant $C_f > 0$ such that, for $i = 1, 2, 3$,

$$|\nabla f_i(u, v, w)| \leq C_f(1 + |u|^{p-1} + |v|^{p-1} + |w|^{p-1}), \;\; \forall u, v, w \in \mathbb{R}. \tag{2.27}$$

In particular this implies that there exists a constant $C_F > 0$ such that

$$|F_i(u, v, w)| \leq C_F(1 + |u|^{p+1} + |v|^{p+1} + |w|^{p+1}), \;\; \forall u, v, w \in \mathbb{R}. \tag{2.28}$$

Furthermore, we assume that, for all $u, v, w \in \mathbb{R}$,

$$\nabla F(u, v, w) \cdot (u, v, w) - F(u, v, w) \geq -m_F. \tag{2.29}$$

This information allow us to define the non-linear energy of the system (2.1)-(2.6) as

$$\mathcal{E}(t) := E(t) + \int_0^L F(\varphi, \psi, w) dx, \tag{2.30}$$

where $E(t)$ is the linear energy mentioned in the linear case section. Then, multiplying (2.1)-(2.6) by $\varphi_t, \psi_t, w_t, \theta^1, \theta^2, \theta^3$ respectively, we obtain by integration over $[0, L]$ the following identity:

$$\frac{d}{dt}\mathcal{E}(t) = -\int_0^L \big(\gamma_1|\theta^1_x(x,t)|^2 + \gamma_1|\theta^2_x(x,t)|^2 + \gamma_1|\theta^3_x(x,t)|^2\big)dx, \ t \geq 0. \tag{2.31}$$

The existence of global mild and strong solutions to the Bresse-Fourier system will be established through nonlinear semigroup theory [30, Theorem 4.1.6]. We shall write the system (2.1)-(2.9) as an abstract Cauchy Problem

$$\begin{aligned} \frac{d}{dt}y(t) &= Ay(t) + \mathcal{F}(y(t)), \ t > 0 \\ y(0) &= 0, \end{aligned} \tag{2.32}$$

where

$$y(t) = \big(\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t)\big) \in \mathcal{H},$$

with

$$\varphi' = \varphi_t, \ \psi' = \psi_t, \ w' = w_t,$$

and initial data given by

$$y(0) = (\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \theta^1_0, \theta^2_0, \theta^3_0) = y_0.$$

We see that $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ has the same form as the linear case, with domain

$$D(A) = \left(H^2(0,L) \cap H_0^1(0,L)\right)^3 \times (H_0^1(0,L))^3 \times \left(H^2(0,L) \cap H_0^1(0,L)\right)^3$$

and the forcing terms are described by a nonlinear function $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ defined by:

$$\mathcal{F}\begin{bmatrix} \varphi \\ \psi \\ w \\ \varphi' \\ \psi' \\ w' \\ \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f_1(\varphi,\psi,w)/\rho_1 \\ -f_2(\varphi,\psi,w)/\rho_2 \\ -f_3(\varphi,\psi,w)/\rho_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, our existence theorem is given in terms of equivalent problem (2.32).

Theorem 2.2. *Assume that $\ell > 0$ and the hypotheses (2.25)-(2.29) holds. Then for any initial data $y_0 \in \mathcal{H}$ and $T > 0$, problem (2.32) has a unique weak solution*

$$y \in C([0,T];\mathcal{H}), \text{ with } y(0) = y_0,$$

given by

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{F}(y(s))ds, \ t \in [0,T], \tag{2.33}$$

and depends continuously on the initial data, where $S(t)$ represents the semigroup associated to the linear problem (2.22). In particular, if $y_0 \in D(A)$ then the solution is strong.

Proof: First, we can see from (2.23) that A is dissipative and from previous results, the problem (2.22) has a unique solution. Then, we will prove that system (2.32) is a locally Lipschitz perturbation of (2.22).

To show that operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is locally Lipschitz, let B a bounded set of $\mathcal{H}$ and two vectors y^1, y^2 in B. Since the external forces act only on φ, ψ and w, we denote $z^1 = (\varphi^1, \psi^1, w^1)$ and $z^2 = (\varphi^2, \psi^2, w^2)$. From (2.27) we see that, for $i = 1, 2, 3$,

$$
\begin{aligned}
|f_i(z^1) - f_i(z^2)| \leq |\nabla f_i(\lambda z^1 + (1-\lambda) z^2)|^2 \cdot |z^1 - z^2|^2 \\
\leq C_f^2 (1 + |\varphi^1|^{p-1} + |\psi^1|^{p-1} + |w^1|^{p-1} + |\varphi^2|^{p-1} + |\psi^2|^{p-1} + |w^2|^{p-1})^2 \\
\times (|\varphi^1 - \varphi^2|^2 + |\psi^1 - \psi^2|^2 + |w^1 - w^2|^2)
\end{aligned}
$$

Thus, we deduce that, for some constant $C_B > 0$, and Poincare's inequality:

$$
\int_0^L |f_i(z^1) - f_i(z^2)|^2 dx \leq C_B \|z^1 - z^2\|^2_{(H_0^1)^3} \leq C_B \|y^1 - y^2\|^2_{\mathcal{H}}
$$

Summing this estimate on i, we obtain:

$$
\|\mathcal{F}(y^1) - \mathcal{F}(y^2)\|^2_{\mathcal{H}} = \sum_{i=1}^{3} \int_0^L |f_i(z^1) - f_i(z^2)|^2 dx \leq 3C_B \|y^1 - y^2\|^2_{\mathcal{H}}
$$

which proves that $\mathcal{F}$ is locally Lipschitz on $\mathcal{H}$.

Then, from classical results in [27], we obtain a local solution that is defined on a interval $[0, T_{max})$ where, if $T_{max} < \infty$, then:

$$
\lim_{t \to T_{max}} \|y(t)\|_{\mathcal{H}} = +\infty. \tag{2.34}
$$

To see that solution is global, that is, $T_{max} = +\infty$, we start the proof supposing by contradiction that time maximal is finite, and let $y(t)$ a mild solution with initial data $y_0 \in D(A)$. Then it is, in fact, a strong solution and so we can use the following estimate:

$$\begin{aligned}\mathcal{E}(t) &= E(t) + \int_0^L F(\varphi, \psi, w)dx \\ &\geq \frac{1}{2}\|y\|_{\mathcal{H}}^2 - \int_0^L m_F\, dx \\ &= \frac{1}{2}\|y\|_{\mathcal{H}}^2 - Lm_F,\ t \geq 0. \end{aligned} \tag{2.35}$$

and then $\|y\|_{\mathcal{H}}^2 \leq \frac{2}{\beta_0}(\mathcal{E}(t) + Lm_F)$, and thus, $\|y\|_{\mathcal{H}}^2$ doesn't blow up. By density, this inequality holds for mild solutions. Then, we can easily see that (2.34) does not hold and therefore $T_{max} = +\infty$.

Finally, using the variation of parameter formula (2.33), we can verify that for any initial data $y_0^1, y_0^2 \in \mathcal{H}$, the corresponding solutions y^1 and y^2 satisfy:

$$\begin{aligned}\|y^1(t) - y^2(t)\|_{\mathcal{H}}^2 &\leq 2\|S(t)(y_0^1 - y_0^2)\|_{\mathcal{H}}^2 + 2\Big\|\int_0^t S(t-s)[\mathcal{F}(y^1(s)) - \mathcal{F}(y^2(s))]ds\Big\|_{\mathcal{H}}^2 \\ &\leq C\|y_0^1 - y_0^2\|_{\mathcal{H}}^2\end{aligned}$$

for any $0 < t < T$ and a bounded set B.■

The previous results shows that, the semilinear Bresse-Fourier system (2.1)-(2.9) is well posed. Then, the solution operator $T(t) : \mathcal{H} \to \mathcal{H}$ is a C^0-semigroup on $\mathcal{H}$. Thus, we denote by $(\mathcal{H}, T(t))$ the dynamical system generated by the problem (2.1)-(2.9), meanwhile $(\mathcal{H}, S(t))$ is the dynamical system generated by the linear problem (2.10)-(2.15), with initial-boundary conditions (2.7)-(2.9).

The next section uses the results from Theorem 2.1 to show that the energy of the linear system decays exponentially.

2.2 Exponential stability

In this section we prove the exponential stability for the semigroup solution of Bresse-Fourier system $(2.10) - (2.15)$, with initial-boundary conditions (2.7)-(2.9). More precisely, we have:

Theorem 2.3. *Under the hypotheses of Theorem 2.1, and assuming* $\gamma_1, \gamma_2, \gamma_3 > 0$, *there exists positive constants* $a, C > 0$ *such that:*

$$\|y\|_{\mathcal{H}} \leq C\|y_0\|_{\mathcal{H}} \cdot e^{-at},\ t > 0, \tag{2.36}$$

where $\|y\|_{\mathcal{H}}$ *represents the norm in the Hilbert space* $\mathcal{H}$.

The proof of Theorem 2.3 will be concluded as a consequence of some important lemmas and the Gearhart-Pruss characterization of exponential stability for C_0-semigroups on Hilbert Spaces (see [7, 11, 4]).

Theorem 2.4. *Let* $\{S(t)\}_{t\geq 0} = \{e^{At}\}$ *be the* C_0*-semigroup of contractions on a Hilbert space* $\mathcal{H}$ *associated to* $(2.10) - (2.15)$, *with conditions* (2.7)-(2.9). *Then:*

$$\{S(t)\} \text{ is exponentially stable } \Leftrightarrow i\mathbb{R} \subseteq \rho(A) \text{ and } \limsup_{|\lambda|\to+\infty} \|(i\lambda I - A)^{-1}\| < \infty,$$

where $\rho(A)$ *means the resolvent of the unbounded operator A.*

Proof: Our starting point is the resolvent equation:

$$i\lambda y - Ay = f,\ \lambda \in \mathbb{R}, \tag{2.37}$$

where $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9) \in \mathcal{H}$ is given, and the solution is defined by

$$y = (\varphi, \psi, w, \varphi', \psi', w', \theta^1, \theta^2, \theta^3)$$

in $D(A)$, that is, term by term:

$$\begin{bmatrix} i\lambda\varphi \\ i\lambda\psi \\ i\lambda w \\ i\lambda\varphi' \\ i\lambda\psi' \\ i\lambda w' \\ i\lambda\theta^1 \\ i\lambda\theta^2 \\ i\lambda\theta^3 \end{bmatrix} - \begin{bmatrix} \varphi' \\ \psi' \\ w' \\ \dfrac{k}{\rho_1}(\varphi_x+\psi+\ell w)_x + \dfrac{k_0\ell}{\rho_1}(w_x-\ell\varphi) - \dfrac{m_1}{\rho_1}\theta^1_x - \dfrac{\ell m_2}{\rho_1}\theta^2 \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x+\psi+\ell w) - \dfrac{m_3}{\rho_2}\theta^3_x + \dfrac{m_1}{\rho_2}\theta^1 \\ \dfrac{k_0}{\rho_1}(w_x-\ell\varphi)_x - \dfrac{k\ell}{\rho_1}(\varphi_x+\psi+\ell w) - \dfrac{m_2}{\rho_1}\theta^2_x + \dfrac{\ell m_1}{\rho_1}\theta^1 \\ \dfrac{\gamma_1}{\sigma_1}\theta^1_{xx} - \dfrac{m_1}{\sigma_1}(\varphi'_x+\psi'+\ell w') \\ \dfrac{\gamma_2}{\sigma_2}\theta^2_{xx} - \dfrac{m_2}{\sigma_2}(w'_x-\ell\varphi') \\ \dfrac{\gamma_3}{\sigma_3}\theta^3_{xx} - \dfrac{m_3}{\sigma_3}(\psi'_x) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix}.$$

Then

$$i\lambda\varphi - \varphi' = f_1, \tag{2.38}$$

$$i\lambda\psi - \psi' = f_2, \tag{2.39}$$

$$i\lambda w - w' = f_3, \tag{2.40}$$

$$i\lambda\rho_1\varphi' - k(\varphi_x+\psi+\ell w)_x - \ell k_0(w_x-\ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = \rho_1 f_4, \tag{2.41}$$

$$i\lambda\rho_2\psi' - b\psi_{xx} + k(\varphi_x+\psi+\ell w) + m_3\theta^3_x - m_1\theta^1 = \rho_2 f_5, \tag{2.42}$$

$$i\lambda\rho_1 w' - k_0(w_x-\ell\varphi)_x + k\ell(\varphi_x+\psi+\ell w) + m_2\theta^2_x - \ell m_1\theta^1 = \rho_1 f_6, \tag{2.43}$$

$$i\lambda\sigma_1\theta^1 - \gamma_1\theta^1_{xx} + m_1(\varphi'_x+\psi'+\ell w') = \sigma_1 f_7, \tag{2.44}$$

$$i\lambda\sigma_2\theta^2 - \gamma_2\theta^2_{xx} + m_2(w'_x-\ell\varphi') = \sigma_2 f_8, \tag{2.45}$$

$$i\lambda\sigma_3\theta^3 - \gamma_3\theta^3_{xx} + m_3(\psi'_x) = \sigma_3 f_9. \tag{2.46}$$

In order to prove $i\mathbb{R} \subseteq \rho(A)$, we observe that, from the definition of $D(A)$, it is closed and compactly embedded in $\mathcal{H}$. Then, the spectrum

$$\sigma(A) = \mathbb{C}\backslash\rho(A)$$

has only eigenvalues.

Let suppose that A possesses an imaginary eigenvalue $\lambda = i\beta \in \sigma(A)$, with $\beta \neq 0$, and with their corresponding eigenvector

$$y = (\varphi, \psi, w, \varphi', \psi', w', \theta^1, \theta^2, \theta^3) \neq 0$$

From (2.37), taking inner product by $y \in D(A)$, with $f = 0$, and taking the real part, we have that:

$$\gamma_1\|\theta^1_x\|^2 + \gamma_2\|\theta^2_x\|^2 + \gamma_3\|\theta^3_x\|^2 = -Re\langle Ay, y\rangle = 0. \tag{2.47}$$

Using the Poincare's inequality, we easily conclude that $\theta^1 = \theta^2 = \theta^3 = 0$. Returning to equations (2.46) and (2.38), we have respectively that

$$\psi' = \psi = 0.$$

After that, we see from the remaining equations that $w' = w = 0$, and finally $\varphi' = \varphi' = 0$, which implies that $y = 0$ (This contradicts the fact that $y \neq 0$ is an eigenvector).

Hence, there are no purely imaginary eigenvalues in the spectrum $\sigma(A) = \mathbb{C}\backslash\rho(A)$, that is,

$$i\mathbb{R} \subseteq \rho(A).$$

The next goal for complete the proof of Theorem 2.3 is to prove the following estimate:

$$\|y\|_{\mathcal{H}} \leq C\|f\|_{\mathcal{H}} \tag{2.48}$$

for some constant $C > 0$.

Lemma 2.1. *Under the hypotheses of Theorem 2.3, there exist a constant $C > 0$ such that*

$$\sigma_i \|\theta_x^i\|^2 \leq C\|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} \tag{2.49}$$

for i=1,2,3. And consequently:

$$\sigma_i \|\theta^i\|^2 \leq C\|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} \tag{2.50}$$

Proof: From the resolvent equation, we have, for any $y \in D(A)$, that:

$$i\lambda\langle y, y\rangle - \langle Ay, y\rangle = \langle f, y\rangle.$$

Taking the real part

$$Re\big(i\lambda\langle y, y\rangle - \langle Ay, y\rangle\big) = Re\langle f, y\rangle,$$

and then

$$-Re\langle Ay, y\rangle \leq |\langle f, y\rangle|.$$

Equation (2.47) showed that:

$$-Re\langle Ay, y\rangle = \gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2.$$

Thus, by Cauchy-Schwarz inequality:

$$\gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2 \leq \|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}}.$$

Consequently, for each $i = 1, 2, 3$,

$$\gamma_i\|\theta_x^i\|^2 \leq \|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}}$$

Multiplying the inequality by σ_i and dividing by γ_i , we conclude

$$\sigma_i\|\theta_x^i\|^2 \leq \frac{\sigma_i}{\gamma_i}\|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} = C\|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}}.$$

For the therms $\sigma_i\|\theta^i\|^2$, the prove follows immediately by Poincare inequality (See [27]).

For the sake of brevity of computations, we need to state some estimates of norms that were previously shown. For this reason, we present the following argument that will be very useful in future calculations.

Proposition 2.1. *Under the assumptions of Theorem 2.3, given the functions $h \in L^2(0, L)$, $f \in \mathcal{H}$ and $y \in D(A)$, we have, for any $\varepsilon > 0$:*

1. *If $\|h\|_{L^2} \leq \|y\|_{\mathcal{H}}$, then:* $\|\theta_x^i\| \cdot \|h\| \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$

2. *If $\|h\|_{L^2} \leq \|f\|_{\mathcal{H}}$, then:* $\|\theta_x^i\| \cdot \|h\| \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$

The same argument applies for θ^i instead θ_x^i, by the Poincare inequality.

Proof: For the first item, observe that, from Young's inequality and Lemma 2.1, we obtain:

$$\begin{aligned}
\|\theta_x^i\| \cdot \|h\| &\leq c_\varepsilon\|\theta_x^i\|^2 + \frac{\varepsilon}{2}\|h\|^2 \\
&\leq \|y\|_{\mathcal{H}} \cdot c_\varepsilon\|f\|_{\mathcal{H}} + \frac{\varepsilon}{2}\|y\|_{\mathcal{H}}^2 \\
&\leq \frac{\varepsilon}{2}\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2 + \frac{\varepsilon}{2}\|y\|_{\mathcal{H}}^2 \\
&\leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.
\end{aligned}$$

Analogously, for the second one, we use the same arguments above to obtain:

$$\begin{aligned}\|\theta_x^i\| \cdot \|h\| &\leq \frac{1}{2}\|\theta_x^i\|^2 + \frac{1}{2}\|h\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2}\|\theta_x^i\|^2 + \frac{1}{2}\|f\|_{\mathcal{H}}^2 \\ &\leq \|y\|_{\mathcal{H}} \cdot C\|f\|_{\mathcal{H}} + \frac{1}{2}\|f\|_{\mathcal{H}}^2 \\ &\leq \varepsilon\|y\|_{\mathcal{H}}^2 + c_\varepsilon\|f\|_{\mathcal{H}}^2 + \frac{1}{2}\|f\|_{\mathcal{H}}^2 \\ &\leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2\end{aligned}$$

In order to show the same result for θ^i instead θ_x^i, we notice that:

$$\|\theta^i\| \leq \frac{\pi}{L}\|\theta_x^i\|$$

Thus:

$$\|\theta^i\| \cdot \|h\| \leq \frac{\pi}{L}\|\theta_x^i\| \cdot \|h\|$$

and we just take $\varepsilon \to \varepsilon \cdot \dfrac{\pi}{L}$, and h is satisfying hypotheses $1.$ or $2.$ ∎

Remark: The choose of ε can be different in each Young's inequality, but in principle, we abbreviate this constants in such a way that we can make the proofs of lemmas easier to read. Additionally, C, c_ε and $C_\varepsilon > 0$ represent several the constants that can be grouped in order to make easier the redaction.

2.2.1 Observability

Lemma 2.1 show us:

$$\|\theta^1\|^2 + \|\theta^2\|^2 + \|\theta^3\|^2 \le \varepsilon\|y\|^2_{\mathcal{H}} + C_\varepsilon\|f\|^2_{\mathcal{H}} \tag{2.51}$$

for any $\varepsilon > 0$ and for a $|\lambda| > 1$ large enough. Then, to prove (2.48), it remains to proof:

$$\|\varphi_x + \psi + \ell w\|^2 + \|\psi_x\|^2 + \|w_x - \ell\varphi\|^2 + \|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2 \le \varepsilon\|y\|^2_{\mathcal{H}} + C_\varepsilon\|f\|^2_{\mathcal{H}}.$$

Unlike another estimates (See [6]), this one presents some restrictions when performing the calculations. More specifically, totally Dirichlet type boundary conditions in all of terms prevent vanishing estimates on the boundary. Therefore, the methods used above cannot help to solve an appropriate estimate for all the terms mentioned above. For this reason, we will introduce an important observability criteria (see [11]).

First Step: We define an auxiliary cut-off function (for more details, see [4]). Indeed, let us consider $l_0 \in (0, L)$ and $\delta > 0$, arbitrary numbers such that $(l_0 - \delta, l_0 + \delta) \subset (0, L)$, and consider a function $s \in C^2(0, L)$ satisfying:

$$\begin{gathered} \mathit{supp}\ s \subset (l_0 - \delta, l_0 + \delta), \qquad 0 \le s(x) \le 1,\ x \in (0, L)\ \text{, and} \\ s(x) = 1, \ \text{ for } \ x \in [l_0 - \frac{\delta}{2}, l_0 + \frac{\delta}{2}]. \end{gathered} \tag{2.52}$$

Thus, with this new construction, we are able to state the following lemmas.

Lemma 2.2. *Assume the condition* (2.52) *and the hypotheses of Theorem 2.3 hold. Then, for every $\varepsilon > 0$, there exist a constant $C_\varepsilon > 0$, independent of λ, such that:*

$$\int_{l_0 - \frac{\delta}{2}}^{l_0 + \frac{\delta}{2}} k|\varphi_x + \psi + \ell w|^2 dx \le \varepsilon\|y\|^2_{\mathcal{H}} + C_\varepsilon\|f\|^2_{\mathcal{H}}\ , \tag{2.53}$$

Proof: From equations (2.38)-(2.40), we deduce that

$$(\varphi_x' + \psi' + \ell w') = i\lambda(\varphi_x + \psi + \ell w) + (f_{1,x} + f_2 + \ell f_3).$$

Using this equation into (2.44), we have:

$$i\lambda\sigma_1\theta^1 - \gamma_1\theta^1_{xx} + i\lambda m_1(\varphi_x + \psi + \ell w) - m_1(f_{1,x} + f_2 + \ell f_3) = \sigma_1 f_1.$$

$\Rightarrow$

$$\frac{\sigma_1}{m_1}\theta^1 - \frac{\gamma_1}{i\lambda m_1}\theta^1_{xx} + (\varphi_x + \psi + \ell w) - \frac{1}{i\lambda}(f_{1,x} + f_2 + \ell f_3) = \frac{\sigma_1}{i\lambda m_1} f_7.$$

Multiplying this equation by $k \cdot s(\overline{\varphi_x + \psi + \ell w})$, and integrating on $(0, L)$, we obtain

$$\begin{aligned} k\int_0^L s\left|\varphi_x + \psi + \ell w\right|^2 dx = &\frac{\sigma_1 k}{i\lambda m_1}\int_0^L s f_7(\overline{\varphi_x + \psi + \ell w})dx - \frac{\sigma_1 k}{m_1}\int_0^L s\theta^1(\overline{\varphi_x + \psi + \ell w})dx \\ &+ \frac{k}{i\lambda}\int_0^L s(f_{1,x} + f_2 + \ell f_3)(\overline{\varphi_x + \psi + \ell w})dx \\ &+ \frac{\gamma_1 k}{i\lambda m_1}\int_0^L s\theta^1_{xx}(\overline{\varphi_x + \psi + \ell w})dx. \end{aligned}$$

Integrating by parts the last term of the above identity, and using the fact that $s = 0$ in $\{0, L\}$, we obtain

$$\begin{aligned} k\int_0^L s\left|\varphi_x + \psi + \ell w\right|^2 dx = &\frac{\sigma_1 k}{i\lambda m_1}\int_0^L s f_7(\overline{\varphi_x + \psi + \ell w})dx - \frac{\sigma_1 k}{m_1}\int_0^L s\theta^1(\overline{\varphi_x + \psi + \ell w})dx \\ &+ \frac{k}{i\lambda}\int_0^L s(f_{1,x} + f_2 + \ell f_3)(\overline{\varphi_x + \psi + \ell w})dx \\ &- \frac{\gamma_1 k}{i\lambda m_1}\int_0^L \theta^1_x(s(\overline{\varphi_x + \psi + \ell w}))_x dx. \end{aligned}$$

Taking module and denoting for C for several constants, we have that

$$
\begin{aligned}
k\int_0^L s\big|\varphi_x+\psi+\ell w\big|^2dx \le{}& \frac{C}{|\lambda|}\int_0^L \big|sf_7(\overline{\varphi_x+\psi+\ell w})\big|dx + C\int_0^L \big|s\theta^1(\overline{\varphi_x+\psi+\ell w})\big|dx\\
&+\frac{C}{|\lambda|}\int_0^L \big|s(f_{1,x}+f_2+\ell f_3)(\overline{\varphi_x+\psi+\ell w})\big|dx\\
&+\frac{C}{|\lambda|}\int_0^L \big|\theta_x^1 s_x(\overline{\varphi_x+\psi+\ell w})\big|dx+\frac{C}{|\lambda|}\int_0^L \big|s\theta_x^1(\overline{\varphi_x+\psi+\ell w})_x\big|dx.
\end{aligned}
$$

Using Hölder, Young and Poincare's inequalities, the fact that $|\lambda|>1$, and $s\in C^2$, we have

$$
\begin{aligned}
k\int_0^L s\big|\varphi_x+\psi+\ell w\big|^2dx \le{}& C\int_0^L \big|f_7(\overline{\varphi_x+\psi+\ell w})\big|dx + C\int_0^L \big|\theta^1(\overline{\varphi_x+\psi+\ell w})\big|dx\\
&+C\int_0^L \big|(f_{1,x}+f_2+\ell f_3)(\overline{\varphi_x+\psi+\ell w})\big|dx\\
&+C\int_0^L \big|\theta_x^1(\overline{\varphi_x+\psi+\ell w})\big|dx+\frac{C}{|\lambda|}\int_0^L \big|\theta_x^1(\overline{\varphi_x+\psi+\ell w})_x\big|dx\\
\le{}& C\|f\|_{\mathcal{H}}\|y\|_{\mathcal{H}}+C\|\theta^1\|\|y\|_{\mathcal{H}}+\|f\|_{\mathcal{H}}\|y\|_{\mathcal{H}}+C\|\theta_x^1\|\|y\|_{\mathcal{H}}\\
&+\frac{C}{|\lambda|}\int_0^L \big|\theta_x^1(\overline{\varphi_x+\psi+\ell w})_x\big|dx\\
\le{}& \varepsilon\|y\|_{\mathcal{H}}^2+C_\varepsilon\|f\|_{\mathcal{H}}^2+\frac{C}{|\lambda|}\int_0^L \big|\theta_x^1(\overline{\varphi_x+\psi+\ell w})_x\big|dx.
\end{aligned}
$$

Seeing for the last inequality, we noticed an integral term. Analyzing it:

$$
\begin{aligned}
\frac{C}{|\lambda|}\int_0^L \big|\theta_x^1(\overline{\varphi_x+\psi+\ell w})_x\big|dx ={}& \frac{C}{|\lambda|}\int_0^L \big|\theta_x^1[i\lambda\rho_1\varphi'-k_0\ell(w_x-\ell\varphi)+m_1\theta_x^1+\ell m_2\theta^2-\rho_1 f_4]\big|dx\\
\le{}& \frac{C}{|\lambda|}\int_0^L |i\lambda\rho_1\theta_x^1\varphi'|dx+\frac{C}{|\lambda|}\int_0^L |k_0\ell\theta_x^1(w_x-\ell\varphi)|dx\\
&+\frac{C}{|\lambda|}\int_0^L |\theta_x^1|^2dx+\frac{C}{|\lambda|}\int_0^L |\ell m_2\theta_x^1\theta^2|dx\\
&+\frac{C}{|\lambda|}\int_0^L |\rho_1 f_4\theta_x^1|dx\\
\le{}& C\int_0^L |\theta_x^1\varphi'|dx+C\int_0^L |\theta_x^1(w_x-\ell\varphi)|dx+C\int_0^L |\theta_x^1|^2dx\\
&+C\int_0^L |\theta_x^1\theta^2|dx+C\int_0^L |\theta_x^1 f_4|dx.
\end{aligned}
$$

Here, we use Lemma 2.1 and Proposition 2.1, to reduce the last inequality into:

$$\frac{C}{|\lambda|}\int_0^L \left|\theta_x^1 \overline{(\varphi_x+\psi+\ell w)_x}\right|dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

With this, the initial expression is estimated, for any $|\lambda|>1$, by:

$$k\int_0^L s\left|\varphi_x+\psi+\ell w\right|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

By definition of s, and hypotheses (2.52), we conclude that:

$$k\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \left|\varphi_x+\psi+\ell w\right|^2 dx \le k\int_{l_0-\delta}^{l_0+\delta} s\left|\varphi_x+\psi+\ell w\right|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Thus:

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} k\left|\varphi_x+\psi+\ell w\right|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Lemma 2.3. *Under the conditions* (2.52) *and Theorem 2.3, given* $\varepsilon>0$*, there exist a constant* $C_\varepsilon>0$ *such that*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} b|\psi_x|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2\,, \tag{2.54}$$

for $|\lambda|>1$ *large enough.*

Proof: Deriving (2.39) in x and inserting this identity into equation (2.46), we have

$$\begin{aligned}
&i\lambda\sigma_3\theta^3-\gamma_3\theta_{xx}^3+m_3(i\lambda\psi_x-f_{2,x})=\sigma_3 f_9.\\
\Rightarrow\quad & i\lambda\sigma_3\theta^3-\gamma_3\theta_{xx}^3+i\lambda m_3\psi_x-m_2 f_{2,x}=\sigma_3 f_9.\\
\Rightarrow\quad & \sigma_3\theta^3-\frac{\gamma_3}{i\lambda}\theta_{xx}^3+m_3\psi_x-\frac{m_3}{i\lambda}f_{2,x}=\frac{\sigma_3}{i\lambda}f_9.
\end{aligned}$$

Multiplying by $b \cdot s\overline{\psi_x}$ and integrating over $(0, L)$, we obtain

$$\sigma_3 b \int_0^L s \cdot \theta^3 \overline{\psi_x} - \frac{\gamma_3 b}{i\lambda} \int_0^L s \cdot \theta^3_{xx} \overline{\psi_x} + m_3 b \int_0^L s|\psi_x|^2 - \frac{m_3 b}{i\lambda} \int_0^L s \cdot f_{2,x} \overline{\psi_x} = \frac{\sigma_3 b}{i\lambda} \int_0^L s \cdot f_9 \overline{\psi_x} dx.$$

Thus, integrating by parts

$$\begin{aligned} m_3 b \int_0^L s|\psi_x|^2 dx = \sigma_3 b \int_0^L (s\theta^3)_x \cdot \overline{\psi} dx + \frac{\gamma_3 b}{i\lambda} \int_0^L s\theta^3_{xx} \cdot \overline{\psi_x} dx \\ + \frac{m_3 b}{i\lambda} \int_0^L s \cdot f_{2,x} \overline{\psi_x} dx + \frac{\sigma_3 b}{i\lambda} \int_0^L s \cdot f_9 \overline{\psi_x} dx. \end{aligned}$$

Using again integration by parts on $s \cdot \theta^3_{xx} \overline{\psi_x}$, we obtain:

$$\begin{aligned} m_3 b \int_0^L s|\psi_x|^2 dx &= \sigma_3 b \int_0^L s_x \cdot \theta^3 \overline{\psi} + \sigma_3 b \int_0^L s \cdot \theta^3_x \overline{\psi} + \frac{\gamma_3 b}{i\lambda} \int_0^L s \cdot \theta^3_{xx} \overline{\psi_x} dx \\ &+ \frac{m_3 b}{i\lambda} \int_0^L s \cdot f_{2,x} \overline{\psi_x} + \frac{\sigma_3 b}{i\lambda} \int_0^L s \cdot f_9 \overline{\psi_x} dx \\ &= \sigma_3 b \int_0^L s_x \cdot \theta^3 \overline{\psi} + \sigma_3 b \int_0^L s \cdot \theta^3_x \overline{\psi} - \frac{\gamma_3 b}{i\lambda} \int_0^L \theta^3_x \cdot (s\overline{\psi_x})_x dx \\ &- \frac{m_3 b}{i\lambda} \int_0^L s \cdot f_{2,x} \overline{\psi_x} + \frac{\sigma_3 b}{i\lambda} \int_0^L s \cdot f_9 \overline{\psi_x} dx \\ &= \sigma_3 b \int_0^L s_x \cdot \theta^3 \overline{\psi} + \sigma_3 b \int_0^L s \cdot \theta^3_x \overline{\psi} - \frac{\gamma_3 b}{i\lambda} \int_0^L s_x \cdot \theta^3_x \cdot \overline{\psi_x} dx \\ &- \frac{\gamma_3 b}{i\lambda} \int_0^L s \cdot \theta^3_x \overline{\psi_{xx}} + \frac{m_3 b}{i\lambda} \int_0^L s \cdot f_{2,x} \overline{\psi_x} + \frac{\sigma_3 b}{i\lambda} \int_0^L s \cdot f_9 \overline{\psi_x} dx. \end{aligned}$$

Let's estimate all the terms of above identity. To this end, all several constants independent of λ, will be denoted by C, for simplicity of the computations. Indeed

$$\begin{aligned} \left|\sigma_3 b \int_0^L s_x \cdot \theta^3 \overline{\psi} dx\right| \leq C \left| \int_0^L \theta^3 \cdot \overline{\psi} \, dx \right| &\leq C\|\theta^3\| \cdot \|\psi\| \leq C\|\theta^3_x\| \cdot \|\psi_x\| \\ &\leq C\|\theta^3_x\| \cdot \|y\|_{\mathcal{H}} \leq \frac{1}{6}\varepsilon\|y\|^2_{\mathcal{H}} + C_\varepsilon\|f\|^2_{\mathcal{H}}, \end{aligned}$$

where were we used Hölder's inequality, Poincare's inequality, the fact that s_x is bounded in $(0, L)$ and Proposition 2.1. By the same arguments, we can estimate the other terms as follows:

$$\begin{aligned}\Big|\sigma_3 b\int_0^L s\cdot\theta_x^3\,\overline{\psi}\Big| &\le C\Big|\int_0^L \theta_x^3\cdot\overline{\psi}\Big| \le C\|\theta_x^3\|\cdot\|\psi\| \le C\|\theta_x^3\|\cdot\|\psi_x\|\\ &\le C\|\theta_x^3\|\cdot\|y\|_{\mathcal{H}} \le \frac{1}{6}\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.\end{aligned}$$

Since $|\lambda| > 1$, we estimate the third term to obtain:

$$\Big|\frac{\gamma_3 b}{i\lambda}\int_0^L s_x\cdot\theta_x^3\cdot\overline{\psi_x}\Big| \le \gamma_3 b\Big|\int_0^L \theta_x^3\cdot\overline{\psi_x}\Big| \le C\|\theta_x^3\|\cdot\|\psi_x\| \le \frac{1}{6}\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Similarly, the fifth term is limited by:

$$\begin{aligned}\Big|\frac{m_3 b}{i\lambda}\int_0^L s\cdot f_{2,x}\,\overline{\psi_x}\Big| &\le C\Big|\int_0^L f_{2,x}\cdot\overline{\psi_x}\Big| \le C\|f_{2,x}\|\cdot\|\psi_x\| \le C\|y\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\\ &\le \frac{1}{6}\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.\end{aligned}$$

Analagously, sixth term is bounded by:

$$\begin{aligned}\Big|\frac{\sigma_3 b}{i\lambda}\int_0^L s\cdot f_9\,\overline{\psi_x}\Big| &\le C\Big|\int_0^L f_9\cdot\overline{\psi_x}\Big| \le C\|f_9\|\cdot\|\psi_x\| \le C\|y\|_{\mathcal{H}}\|f\|_{\mathcal{H}}\\ &\le \frac{1}{6}\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.\end{aligned}$$

To estimate fourth term, we need to rewrite term $\overline{\psi_{xx}}$ using the equation (2.42), obtaining:

$$\begin{aligned}-\frac{\gamma_3 b}{i\lambda}\int_0^L s\cdot\theta_x^3\,\overline{\psi_{xx}} &= \frac{\gamma_3}{i\lambda}\int_0^L s\cdot\theta_x^3\,(-b\overline{\psi_{xx}})dx\\ &= \frac{\gamma_3}{i\lambda}\int_0^L s\theta_x^3\Big(\rho_2\overline{f_5} - k(\overline{\varphi_x+\psi+\ell w}) - m_3\theta_x^3 + m_1\overline{\theta^1} + i\lambda\rho_2\overline{\psi'}\Big)dx\\ &= \frac{\gamma_3\rho_2}{i\lambda}\int_0^L s\theta_x^3\,\overline{f_5} - \frac{\gamma_3 k}{i\lambda}\int_0^L s\theta_x^3 k(\overline{\varphi_x+\psi+\ell w}) - \frac{\gamma_3 m_3}{i\lambda}\int_0^L s|\theta_x^3|^2\\ &+ \frac{\gamma_3 m_1}{i\lambda}\int_0^L s\cdot\theta_x^3\,\overline{\theta^1}dx + \gamma_3\rho_2\int_0^L s\cdot\theta_x^3\,\overline{\psi'}dx.\end{aligned}$$

Taking module in all of those terms, and using the Proposition 2.1, we conclude that:

$$\Big|\frac{\gamma_3 b}{i\lambda}\int_0^L s\cdot\theta_x^3\,\overline{\psi_{xx}}\Big|dx \le \frac{1}{6}\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Combining all the estimates, and adapting ε to convenience, we conclude that:

$$b\int_0^L s\cdot|\psi_x|^2dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2. \tag{2.55}$$

By definition of the function s, we have:

$$b\int_{l_0-\delta}^{l_0+\delta} s|\psi_x|^2dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2. \tag{2.56}$$

Finally,

$$b\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\psi_x|^2dx \le b\int_{l_0-\delta}^{l_0+\delta} s|\psi_x|^2dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2,$$

for $|\lambda|$ large enough, which proves the lemma. ■

Lemma 2.4. *Under the conditions of previous Lemmas and Theorem 2.3, for any* $\varepsilon > 0$*, there exist a constant* $C_\varepsilon > 0$ *independent of* λ*, such that, for* $|\lambda| > 1$ *large enough:*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} k_0|w_x - \ell\varphi|^2 dx \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2, \tag{2.57}$$

Proof: Using equations (2.38) and (2.40), the following estimate is valid:

$$(w_x' - \ell\varphi') = i\lambda(w_x - \ell\varphi) - (f_{3,x} - \ell f_1).$$

Then, we use this identity into (2.45), to obtain

$$i\lambda\sigma_2\theta^2 - \gamma_2\theta_{xx}^2 + m_2\big(i\lambda(w_x - \ell\varphi) - (f_{3,x} - \ell f_1)\big) = \sigma_2 f_8.$$

Then

$$\sigma_2\theta^2 - \frac{\gamma_2}{i\lambda m_2}\theta_{xx}^2 + (w_x - \ell\varphi) - \frac{1}{i\lambda}(f_{3,x} - \ell f_1) = \frac{\sigma_2}{i\lambda m_2} f_8.$$

Multiplying by $sk_0(\overline{w_x - \ell\varphi})$ and integrating over $(0, L)$, we obtain:

$$\begin{aligned} k_0\int_0^L s|w_x - \ell\varphi|^2 dx = \frac{\sigma_2 k_0}{i\lambda}\int_0^L sf_8(\overline{w_x - l\varphi})dx - \sigma_2 k_0\int_0^L s\theta^2(\overline{w_x - l\varphi})dx \\ + \frac{\gamma_2 k_0}{i\lambda}\int_0^L s\theta_{xx}^2(\overline{w_x - l\varphi})dx + \frac{m_2 k_0}{i\lambda}\int_0^L (f_{3,x} - \ell f_1)(\overline{w_x - l\varphi})dx. \end{aligned}$$

Integrating by parts the integral with the term θ_{xx}^2, we obtain:

$$\begin{aligned} k_0\int_0^L s|w_x - \ell\varphi|^2 dx = \frac{\sigma_2 k_0}{i\lambda}\int_0^L sf_8(\overline{w_x - l\varphi})dx - \sigma_2 k_0\int_0^L s\theta^2(\overline{w_x - l\varphi})dx \\ - \frac{\gamma_2 k_0}{i\lambda}\int_0^L \theta_x^2 s_x(\overline{w_x - l\varphi})dx - \frac{\gamma_2 k_0}{i\lambda}\int_0^L s\theta_x^2(\overline{w_x - l\varphi})_x dx \\ + \frac{m_2 k_0}{i\lambda}\int_0^L (f_{3,x} - \ell f_1)(\overline{w_x - l\varphi})dx. \end{aligned}$$

Same as previous lemmas, the integral $\int_0^L s\theta_x^2(\overline{w_x-\ell\varphi})_x dx$ can be written as:

$$\int_0^L s\theta_x^2(\overline{w_x-l\varphi})_x dx = \int_0^L s\theta_x^2\big[-i\lambda\rho_1\overline{w'} - k\ell(\overline{\varphi_x+\psi+\ell w}) - m_2\overline{\theta_x^2} + \ell m_1\overline{\theta^1} + \rho_1\overline{f_6}\big]dx.$$

Including this identity on the previous one, we have:

$$\begin{aligned}
k_0\int_0^L s|w_x-\ell\varphi|^2dx = &\frac{\sigma_2 k_0}{i\lambda}\int_0^L sf_8(\overline{w_x-l\varphi})dx - \sigma_2 k_0\int_0^L s\theta^2(\overline{w_x-l\varphi})dx\\
&-\frac{\gamma_2 k_0}{i\lambda}\int_0^L \theta_x^2 s_x(\overline{w_x-l\varphi})dx + \frac{m_2k_0}{i\lambda}\int_0^L (f_{3,x}-\ell f_1)(\overline{w_x-l\varphi})dx\\
&+\frac{\gamma_2 k_0}{i\lambda}\int_0^L i\lambda\rho_1 s\theta_x^2\overline{w'}dx + \frac{\gamma_2 k_0}{i\lambda}\int_0^L sk\ell\theta_x^2(\overline{\varphi_x+\psi+\ell w})dx\\
&+\frac{\gamma_2 k_0}{i\lambda}\int_0^L m_2 s|\theta_x^2|^2dx - \frac{\gamma_2 k_0}{i\lambda}\int_0^L \ell m_1 s\theta_x^2\overline{\theta^1}dx - \frac{\gamma_2 k_0}{i\lambda}\int_0^L \rho_1 s\theta_x^2\overline{f_6}dx.
\end{aligned}$$

Combining all the constants except λ, and calling it by C, we can apply module on this identity to obtain:

$$\begin{aligned}
k_0\int_0^L s|w_x-\ell\varphi|^2dx \leq &\frac{C}{|\lambda|}\int_0^L |f_8(\overline{w_x-\ell\varphi})|dx + \frac{C}{|\lambda|}\int_0^L |\theta^2(\overline{w_x-\ell\varphi})|dx\\
&+\frac{C}{|\lambda|}\int_0^L |\theta_x^2(\overline{w_x-\ell\varphi})|dx + +\frac{C}{|\lambda|}\int_0^L |(f_{3,x}-\ell f_1)(\overline{w_x-l\varphi})|dx\\
&+C\int_0^L |\theta_x^2\overline{w'}|dx + \frac{C}{|\lambda|}\int_0^L |\theta_x^2(\overline{\varphi_x+\psi+\ell w})|dx + \frac{C}{|\lambda|}\int_0^L |\theta_x^2|^2dx\\
&+\frac{C}{|\lambda|}\int_0^L |\theta_x^2\overline{\theta^1}|dx + \frac{C}{|\lambda|}\int_0^L |\theta_x^2\overline{f_6}|dx
\end{aligned}$$

Since $|\lambda|>1$, then $\dfrac{C}{|\lambda|}\leq C$. Thus, we use the known inequalities, Lemma 2.1 and Proposition 2.1, to reduce the last inequality into:

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} k_0|w_x-\ell\varphi|^2dx \leq k_0\int_0^L s|w_x-\ell\varphi|^2dx \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

which complete the proof of the lemma. ■

Before establish estimates for the kinetic terms, we are going to state some preliminary results that will be very helpful to us in reducing terms for the next lemmas.

Proposition 2.2. *Consider $y(t)$ solution for the system (2.22) such that $y \in D(A)$, with $s(x)$ being the function defined in (2.52). Then, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that:*

$$\|s(\varphi_x + \psi + \ell w)\| \cdot \|h\| \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2,$$
$$\|s(w_x - \ell\varphi)\| \cdot \|h\| \leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

for any $h \in L^2(0, L)$ with $\|h\|_{L^2} \leq \|y\|_{\mathcal{H}}$ or $\|h\|_{L^2} \leq \|f\|_{\mathcal{H}}$.

Proof: We use Lemma 2.2 to calculated the first inequality as follows:

$$\begin{aligned}
\|s(\varphi_x + \psi + \ell w)\| \cdot \|h\| &= \int_0^L s^2|(\varphi_x + \psi + \ell w)|^2 dx \cdot \|h\| \\
&\leq \int_0^L s|(\varphi_x + \psi + \ell w)|^2 dx \cdot \|h\| \\
&\leq \sqrt{\delta^2\|y\|_{\mathcal{H}}^2 + C_\delta\|f\|_{\mathcal{H}}^2} \cdot \|h\| \\
&\leq \Big(\delta\|y\|_{\mathcal{H}} + C_\delta\|f\|_{\mathcal{H}}\Big) \cdot \|h\|_{L^2}.
\end{aligned}$$

If $\|h\| \leq \|y\|_{\mathcal{H}}$, we deduce that:

$$\begin{aligned}
\|s(\varphi_x + \psi + \ell w)\| \cdot \|h\| &\leq \delta\|y\|_{\mathcal{H}}^2 + C_\delta\|f\|_{\mathcal{H}}\|y\|_{\mathcal{H}} \\
&\leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2,
\end{aligned}$$

where $\varepsilon = 2\delta$. If $\|h\| \leq \|f\|_{\mathcal{H}}$, we take $\varepsilon = \delta^3$, in order to obtain:

$$\|s(\varphi_x + \psi + \ell w)\| \cdot \|h\| \leq \Big(\delta\|y\|_{\mathcal{H}}\Big)\|f\|_{\mathcal{H}} + C_\delta\|f\|_{\mathcal{H}}^2 \leq \varepsilon\|y\|_{\mathcal{H}}^2 + \|f\|_{\mathcal{H}}^2. \blacksquare$$

Remark: Same arguments are valid for $\|s(w_x - \ell\varphi)\|$ (Using Lemma 2.2 instead).

Lemma 2.5. *Under the conditions of previous Lemmas and Theorem 2.3, for any $\varepsilon > 0$, there exist a constant $C_\varepsilon > 0$ independent of λ, such that*

$$\rho_1 \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\varphi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2, \tag{2.58}$$

for $|\lambda| > 1$, large enough.

Proof: From (2.38), we deduce easily that:

$$i\lambda\varphi - \varphi' = f_1 \Longrightarrow \overline{\varphi} = \frac{i}{\lambda}(\overline{\varphi' + f_1}).$$

Then, we multiply (2.41) by $s\overline{\varphi}$ and integrate over $(0, L)$, and use the identity above to obtain:

$$\begin{aligned} i\lambda\rho_1 \int_0^L s\varphi' \cdot \frac{i}{\lambda}(\overline{\varphi' + f_1})dx - k\int_0^L s(\varphi_x + \psi + \ell w)_x \overline{\varphi} dx - k_0\ell \int_0^L s(w_x - \ell\varphi)\overline{\varphi} dx \\ + m_1 \int_0^L s\theta_x^1 \overline{\varphi} dx + \ell m_2 \int_0^L s\theta^2 \overline{\varphi} dx = \rho_1 \int_0^L s f_4 \overline{\varphi} dx. \end{aligned}$$

Thus

$$\begin{aligned} \rho_1 \int_0^L s|\varphi'|^2 dx =& -\rho_1 \int_0^L s f_4 \overline{\varphi} dx - \rho_1 \int_0^L s\varphi' \overline{f_1} dx - k\int_0^L s(\varphi_x + \psi + \ell w)_x \overline{\varphi} dx \\ & - k_0\ell \int_0^L s(w_x - \ell\varphi)\overline{\varphi} dx - \ell m_2 \int_0^L s\theta^2 \overline{\varphi} dx + m_1 \int_0^L s\theta_x^1 \overline{\varphi} dx. \end{aligned}$$

Integrating by parts in the integral with the term $(\varphi_x + \psi + \ell w)_x$, we obtain:

$$\begin{aligned} \rho_1 \int_0^L s|\varphi'|^2 dx =& -\rho_1 \int_0^L s f_4 \overline{\varphi} dx - \rho_1 \int_0^L s\varphi' \overline{f_1} dx + k\int_0^L s_x(\varphi_x + \psi + \ell w) \overline{\varphi} dx \\ & - k_0\ell \int_0^L s(w_x - \ell\varphi)\overline{\varphi} dx - \ell m_2 \int_0^L s\theta^2 \overline{\varphi} dx + m_1 \int_0^L s\theta_x^1 \overline{\varphi} dx \\ & + k\int_0^L s(\varphi_x + \psi + \ell w)\overline{\varphi_x} dx. \end{aligned}$$

Taking module in the last expression, and knowing that $|s_x|$ is bounded, we observe that:

$$
\begin{aligned}
\rho_1 \int_0^L s|\varphi'|^2 dx \leq \rho_1 \int_0^L |f_4 \overline{\varphi}| dx + \rho_1 \int_0^L |\varphi' \overline{f_1}| dx + k \int_0^L |(\varphi_x + \psi + \ell w)\overline{\varphi}| dx \\
+ k_0 \ell \int_0^L |s(w_x - \ell\varphi)\overline{\varphi}| dx - \ell m_2 \int_0^L |\theta^2 \overline{\varphi}| dx + m_1 \int_0^L |\theta_x^1 \overline{\varphi}| dx \\
+ k \int_0^L |s(\varphi_x + \psi + \ell w)\overline{\varphi_x}| dx. \qquad (2.59)
\end{aligned}
$$

Here, we use the known computations (Including Propositions 2.1 and 2.2) to estimate almost all of integrals, with except of:

$$
k \int_0^L |(\varphi_x + \psi + \ell w)\overline{\varphi}| dx.
$$

We will estimate this term. Using the representation of $\overline{\varphi} = \dfrac{i}{\lambda}\big(\overline{\varphi' + f_1}\big)$, and taking $|\lambda| > 1$

$$
\begin{aligned}
k \int_0^L |(\varphi_x + \psi + \ell w)\overline{\varphi}| dx &= k \int_0^L |(\varphi_x + \psi + \ell w)\frac{i}{\lambda}\big(\overline{\varphi' + f_1}\big)| dx \\
&\leq \frac{C}{|\lambda|} \int_0^L |(\varphi_x + \psi + \ell w)||\overline{\varphi'}| dx + \frac{C}{|\lambda|} \int_0^L |(\varphi_x + \psi + \ell w)||\overline{f_1'}| dx \\
&\leq \frac{C}{|\lambda|} \int_0^L |(\varphi_x + \psi + \ell w)||\overline{\varphi'}| dx + C \int_0^L |(\varphi_x + \psi + \ell w)||\overline{f_1'}| dx \\
&\leq \frac{C}{|\lambda|} \|y\|_{\mathcal{H}}^2 + C\|y\|_{\mathcal{H}} \|f\|_{\mathcal{H}}
\end{aligned}
$$

Taking $|\lambda|$ large enough, and use ε-Young type inequality, we conclude that:

$$
k \int_0^L |(\varphi_x + \psi + \ell w)\overline{\varphi}| dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2.
$$

Inserting this estimate into (2.59), we conclude that:

$$
\int_{l_0 - \frac{\delta}{2}}^{l_0 + \frac{\delta}{2}} \rho_1 |\varphi'|^2 dx \leq \rho_1 \int_0^L s|\varphi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2.
$$

Lemma 2.6. *Considering all the assumptions of previous Lemmas, given $\varepsilon > 0$, there exist a constant $C_\varepsilon > 0$, such that:*

$$\rho_2 \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\psi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2, \tag{2.60}$$

for $|\lambda| > 1$ large enough.

Proof: Under the above conditions of the cut-off function $s(x)$ we start the proof considering the equation (2.42)

$$i\lambda\rho_2\psi' - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta_x^3 - m_1\theta^1 = \rho_2 f_5,$$

and a variation of equation (2.39), as follows

$$\overline{\psi} = \frac{i}{\lambda}\big(\overline{f_2 + \psi'}\big). \tag{2.61}$$

Multiplying (2.42) by $s \cdot \overline{\psi}$ and integrating over $(0, L)$:

$$\begin{aligned} i\lambda\rho_2 \int_0^L s \cdot \psi' \, \overline{\psi} - b \int_0^L s \cdot \psi_{xx} \, \overline{\psi} + k \int_0^L s \cdot (\varphi_x + \psi + \ell w) \, \overline{\psi} \\ m_3 \int_0^L s \cdot \theta_x^3 \, \overline{\psi} - m_1 \int_0^L s \cdot \theta^1 \, \overline{\psi} \\ = \rho_2 \int_0^L s \cdot f_5 \, \overline{\psi} dx. \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} i\lambda\rho_2 \int_0^L s \cdot \psi' \, \overline{\psi} + b \int_0^L s \cdot \psi_x \, \overline{\psi_x} + \int_0^L s_x \cdot \psi_x \, \overline{\psi} + k \int_0^L s \cdot (\varphi_x + \psi + \ell w) \, \overline{\psi} \\ m_3 \int_0^L s \cdot \theta_x^3 \, \overline{\psi} - m_1 \int_0^L s \cdot \theta^1 \, \overline{\psi} = \rho_2 \int_0^L s \cdot f_5 \, \overline{\psi} dx. \end{aligned}$$

Replacing (2.61) into the first term of the above equation, we obtain

$$
\begin{aligned}
&-\rho_2 \int_0^L s \cdot \psi' \, \overline{\psi'} - \rho_2 \int_0^L s \cdot \psi' \, \overline{f_2} + \; b \int_0^L s \cdot \psi_x \, \overline{\psi_x} + b \int_0^L s_x \cdot \psi_x \, \overline{\psi} \\
&\qquad + k \int_0^L s \cdot (\varphi_x + \psi + \ell w) \, \overline{\psi} m_3 \int_0^L s \cdot \theta_x^3 \, \overline{\psi} - m_1 \int_0^L s \cdot \theta^1 \, \overline{\psi} dx = \rho_2 \int_0^L s \cdot f_5 \, \overline{\psi} dx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\rho_2 \int_0^L s \cdot |\psi'|^2 dx = &- \rho_2 \int_0^L s \cdot f_5 \overline{\psi} - \rho_2 \int_0^L s \cdot \psi' f_2 + b \int_0^L s \cdot |\psi_x|^2 + \overbrace{b \int_0^L s_x \cdot \psi_x \, \overline{\psi}}^{A_7} \\
&+ k \int_0^L s(\varphi_x + \psi + \ell w) \, \overline{\psi} + m_3 \int_0^L s \cdot \theta_x^3 \, \overline{\psi} - m_1 \int_0^L s \cdot \theta^1 \, \overline{\psi} dx. \qquad (2.62)
\end{aligned}
$$

Taking module, all terms with except of A_7 can be bounded easily by $\varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2$, using the previous methods, with the same arguments.

Let's construct an appropriate estimate for:

$$
b \int_0^L s_x \cdot \psi_x \, \overline{\psi} dx.
$$

To this end, using (2.61):

$$
\begin{aligned}
\Big| b \int_0^L s_x \cdot \psi_x \, \overline{\psi} \, dx \Big| &\leq C \Big| \int_0^L \frac{i}{\lambda} \psi_x \cdot (\overline{\psi' + f_2}) dx \Big| \\
&\leq \frac{C}{|\lambda|} \Big| \int_0^L \psi_x \cdot \overline{\psi'} dx \Big| + \frac{C}{|\lambda|} \Big| \int_0^L \psi_x \cdot \overline{f_2} dx \Big| \\
&\leq \frac{C}{|\lambda|} \|\psi_x\| \cdot \|\psi'\| + \frac{C}{|\lambda|} \|\psi_x\| \cdot \|f_2\| \\
&\leq \frac{C}{|\lambda|} \|y\|_{\mathcal{H}}^2 + C \|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}}.
\end{aligned}
$$

Combining all estimates, (2.62) is bounded by

$$
\rho_2 \int_0^L s \cdot |\psi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|} \|y\|_{\mathcal{H}}^2. \qquad (2.63)
$$

Taking $|\lambda|$ large enough, and as the same as Lemma 2.3, by definition of s, we have

$$\rho_2 \int_{l_0-\delta}^{l_0+\delta} s \cdot |\psi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2. \tag{2.64}$$

Therefore:

$$\rho_2 \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |\psi'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2,$$

ending the proof of the Lemma. ∎

The last Lemma is given on the same way as the another kinetic terms.

Lemma 2.7. *Assume that condition* (2.52) *and the hypotheses of Theorem 2.3 are hold. Given any $\varepsilon > 0$, there exists a constant C_ε, that not depends on λ, such that:*

$$\rho_1 \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |w'|^2 dx \leq \varepsilon \|y\|_{\mathcal{H}}^2 + C_\varepsilon \|f\|_{\mathcal{H}}^2. \tag{2.65}$$

Proof: We are going to proceed in the same way as the estimates for the kinetic terms. Using (2.40) and inserting into equation (2.44), we have

$$\begin{aligned}\rho_1 \int_0^L s|w'|^2 dx = & - \rho_1 \int_0^L s w' \overline{f_3} dx + k_0 \int_0^L s_x (w_x - \ell\varphi)\overline{w} dx + k_0 \int_0^L s(w_x - \ell\varphi)\overline{w_x} dx \\ & + k\ell \int_0^L s(\varphi_x + \psi + \ell w)\overline{w} dx + m_2 \int_0^L s\theta_x^2 \overline{w} dx \\ & - \ell m_1 \int_0^L s\theta^1 \overline{w} dx - \rho_1 \int_0^L s f_6 \overline{w} dx.\end{aligned}$$

Taking module on the last expression, the kinetic term w' can be estimated by

$$\begin{aligned}\rho_1 \int_0^L s|w'|^2 dx \leq & \rho_1 \int_0^L |s w' \overline{f_3}| dx + k_0 \int_0^L |s(w_x - \ell\varphi)\overline{w_x}| dx + k\ell \int_0^L |s(\varphi_x + \psi + \ell w)\overline{w}| dx \\ & + m_2 \int_0^L s\theta_x^2 \overline{w} dx - \ell m_1 \int_0^L s\theta^1 \overline{w} dx - \rho_1 \int_0^L s f_6 \overline{w} dx \\ & + k_0 \int_0^L |s_x (w_x - \ell\varphi)\overline{w}| dx.\end{aligned}$$

Almost all of the terms above can be estimated using classical computations, using the conclusion of Propositions 2.1 and 2.2. Our focus is then, estimate the following expression:

$$k_0 \int_0^L |s_x(w_x - \ell\varphi)\overline{w}| dx.$$

Since s_x is bounded on $(0, L)$, we deduce that

$$k_0 \int_0^L |s_x(w_x - \ell\varphi)\overline{w}| dx \le C \int_0^L |(w_x - \ell\varphi)\overline{w}| dx \le \frac{C}{|\lambda|}\|y\|_{\mathcal{H}}^2 + \frac{C}{|\lambda|}\|y\|\|f\|.$$

Then, using ε-Young type inequality and for $|\lambda| > 1$ large enough:

$$k_0 \int_0^L |s_x(w_x - \ell\varphi)\overline{w}| dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Then, we have

$$\rho_1 \int_0^L s|w'|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

Thus, by definition of $s(x)$, we conclude

$$\rho_1 \int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} |w'|^2 dx \le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.$$

which concludes the proof of Lemma and consequently, all the estimates for the norm terms. ∎

Corollary 2.1. *Under the hypotheses of Lemmas above, given an $\varepsilon > 0$, there exist a (universal) constant C_ε, that is not depend of λ, such that:*

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} k|\varphi_x + \psi + \ell w|^2 + k_0|w_x - \ell\varphi|^2 + b|\psi_x|^2 + \rho_1|\varphi'|^2 + \rho_2|\psi'|^2 + \rho_1|w'|^2 dx$$
$$\le \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2. \tag{2.66}$$

Proof: It follow as consequence of results of Lemmas (2.2)-(2.7).

Considering the classical Bresse system and using the resolvent equation (2.37), we can easily deduce that:

$$
\begin{aligned}
i\lambda\varphi - \varphi' = g_1 := f_1 \quad &\in \ H_0^1(0,L),\\
i\lambda\psi - \psi' = g_2 := f_2 \quad &\in \ H_0^1(0,L),\\
i\lambda w - w' = g_3 := f_3 \quad &\in \ H_0^1(0,L),\\
i\lambda\varphi'\rho_1 - k(\varphi_x + \psi + \ell w)_x - k_0(w_x - \ell\varphi) = g_4 := \rho_1 f_4 \quad &\in \ L^2(0,L),\\
i\lambda\psi'\rho_2 - b\psi_{xx} + k(\varphi_x + \psi + \ell w) = g_5 := \rho_2 f_5 \quad &\in \ L^2(0,L),\\
i\lambda w'\rho_1 - k_0(w_x - \ell\varphi)_x + \ell k(\varphi_x + \psi + \ell w) = g_6 := \rho_1 f_6 \quad &\in \ L^2(0,L).
\end{aligned}
$$

This new Bresse system is denoted as $\overline{(P)}$. Denoting by V and G the vector-valued functions $V = (\varphi, \psi, w, \varphi', \psi', w')^T$ and $G = (g_1, g_2, g_3, g_4, g_5, g_6)^T$, respectively. In addition, given any $a_1, a_2 \in [0, L]$ with $a_1 < a_2$ and $(a_1, a_2) \in [0, L]$. Finally, the notation $\| \ \|_{a_1,a_2}$ stands for:

$$
\|V\|^2_{a_1,a_2} := \int_{a_1}^{a_2} (|\varphi_x + \psi + \ell w|^2 + |w_x - \ell\varphi|^2 + |\psi_x|^2 + |\varphi'|^2 + |\psi'|^2 + |w'|^2)dx. \tag{2.67}
$$

Theorem 2.5. ***(Observability Result)*** *Under the above notations, let* $V = (\varphi, \psi, w, \varphi', \psi', w')^T$ *be a strong solution of* $(\overline{P})$*, for a vector* G *given, and suppose that* $0 \le a_1 < a_2 \le L$ *. Then, there exist constants* $C_0, C_1 > 0$ *such that, for* $i = 1, 2$*, and* $|\lambda| > 1$ *large enough, one has:*

$$
\begin{aligned}
&|\varphi_x(a_j) + \psi(a_j) + \ell w(a_j)|^2 + |w_x(a_j) - \ell\varphi(a_j)|^2 + |\psi_x(a_j)|^2\\
&\qquad + |\varphi'(a_j)|^2 + |\psi'(a_j)|^2 + |w'(a_j)|^2 \le C_0\|V\|^2_{a_1,a_2} + C_0\|G\|^2_{0,L} \,.
\end{aligned} \tag{2.68}
$$

and

$$
\begin{aligned}
\|V\|^2_{a_1,a_2} \le C_1\Big[&|\varphi_x(a_j) + \psi(a_j) + \ell w(a_j)|^2 + |w_x(a_j) - \ell\varphi(a_j)|^2 + |\psi_x(a_j)|^2\\
&+ |\varphi'(a_j)|^2 + |\psi'(a_j)|^2 + |w'(a_j)|^2\Big] + C_1\|G\|^2_{0,L} \,.
\end{aligned} \tag{2.69}
$$

As a consequence to our main result, we have the following corollary, which gives us the estimate required for the stability of the system (P)

Corollary 2.2. *Consider* V *a vector as the same as Theorem* 2.5*. If for some sub interval* $(a_1, a_2) \subset (0, L)$*, one has*

$$\|V\|^2_{a_1,a_2} \leq \Lambda, \tag{2.70}$$

with Λ *depending on* V, G *and* λ*, then, there exist a constant* $C > 0$ *such that*

$$\|V\|^2_{0,L} \leq C\Lambda + C\|G\|^2_{0,L} \tag{2.71}$$

Proof of Theorem 2.5:

Guided by the results in [11], the proof will be done in three steps.

Step 1: A key identity: Let us start by fixing three functions $q_1, q_2, q_3 \in C^1[a_1, a_2]$. Taking the term $q_1 k(\overline{\varphi_x + \psi + \ell w})$ and multiplying with the fourth equation of $(\overline{P})$ and integrating on (a_1, a_2), we get:

$$\begin{aligned}
&\int_{a_1}^{a_2} q_1 k g_4(\overline{\varphi_x + \psi + \ell w})dx \\
&\quad = \int_{a_1}^{a_2} q_1 \left(i\lambda\rho_1\varphi' - k(\varphi_x + \psi + \ell w)_x - k_0(w_x - \ell\varphi) \cdot k(\overline{\varphi_x + \psi + \ell w})\right)dx \\
&\quad = -\overbrace{\int_{a_1}^{a_2} q_1\ \rho_1 k \cdot \varphi'(\overline{i\lambda\varphi_x + \psi + \ell w})dx}^{J_1} \\
&\qquad + \overbrace{\int_{a_1}^{a_2} q_1\ (k(\varphi_x + \psi + \ell w))_x \cdot (\overline{k(\varphi_x + \psi + \ell w)})dx}^{J_2} \\
&\qquad + \overbrace{\int_{a_1}^{a_2} q_1\ k_0\ell(w_x - \ell\varphi) \cdot (\overline{\varphi_x + \psi + \ell w})dx}^{J_3}
\end{aligned} \tag{2.72}$$

Here, we are going to estimate all terms above. From the system $(\overline{P})$, we see that:

$$
\begin{aligned}
J_1 =& -\int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot \Big(\overline{(g_{1,x}+g_2+\ell g_3)+(\varphi'_x+\psi'+\ell w)}\Big)dx\\
=& -\int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot \overline{\varphi'_x} - \int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot (\overline{\psi'+\ell w'}) - \int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot (\overline{g_{1,x}+g_2+\ell g_3})dx\\
=& -\frac{1}{2}q_1\rho_1 k|\varphi'|^2\Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2}(q_1\ \rho_1 k)_x|\varphi'|^2 dx\\
& -\int_{a_1}^{a_2} q_1\ \rho_1 k|\varphi'|^2 - \int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot (\overline{g_{1,x}+g_2+\ell g_3})dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{Re } J_1 =& -\frac{1}{2}q_1\rho_1 k|\varphi'|^2\Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2}(q_1\rho_1 k)_x|\varphi'|^2 dx\\
& - \text{Re } \int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot (\overline{\psi'+\ell w'})dx - \text{Re } \int_{a_1}^{a_2} q_1\rho_1 k\ \varphi' \cdot (\overline{g_{1,x}+g_2+\ell g_3})dx
\end{aligned}
$$

and, analogously:

$$
\text{Re } J_2 = -\frac{1}{2}q_1 k^2|\varphi_x+\psi+\ell w|^2\Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2} q_{1,x}\cdot k^2|\varphi_x+\psi+\ell w|^2 dx
$$

Thus, taking the real part in (2.72), we obtain

$$
\begin{aligned}
&\frac{1}{2}\Big[-\big(q_1\rho_1 k|\varphi'|^2 + q_1 k^2|\varphi_x+\psi+\ell w|^2\big)\Big|_{a_1}^{a_2} + \int_{a_1}^{a_2}(q_1\rho_1 k)_x|\varphi'|^2 + q_{1,x}k^2|\varphi_x+\psi+\ell w|^2\Big]\\
&= \text{Re } \int_{a_1}^{a_2} q_1 k g_4(\overline{\varphi_x+\psi+\ell w})dx + \text{Re } \int_{a_1}^{a_2} q_1\rho_1 k\varphi(\overline{g_{1,x}+g_2+\ell g_3})dx\\
&+ \text{ Re } \int_{a_1}^{a_2} q_1\rho_1 k\varphi' \cdot (\overline{\psi'+\ell w'})dx + \text{Re } \int_{a_1}^{a_2} q_1 k_0 \ell k(w_x-\ell\varphi)\cdot(\overline{\varphi_x+\psi+\ell w})dx \qquad (2.73)
\end{aligned}
$$

Secondly, multiplying the third equation of $(\overline{P})$, by $q_2 b\overline{\psi_x}$ and integrating over (a_1, a_2):

$$\int_{a_1}^{a_2} q_2 b g_5 \overline{\psi_x} dx = \overbrace{-\int_{a_1}^{a_2} q_2 b \rho_2 \psi' (\overline{i\lambda\psi_x}) dx}^{J_3} \overbrace{-\int_{a_1}^{a_2} q_2 (b\psi_x)_x (\overline{b\psi_x}) dx}^{J_4}$$
$$\overbrace{+\int_{a_1}^{a_2} q_2 bk(\varphi_x + \psi + \ell w)\overline{\psi_x} dx}^{J_5}. \tag{2.74}$$

Thus, using the equations of $(\overline{P})$ and integrating by parts, J_3 and J_4 yields:

$$\begin{aligned}\text{Re } J_3 &= \text{Re } (-\int_{a_1}^{a_2} q_2 b \rho_2 (\psi' \overline{\psi'_x} + \psi' \overline{g_{2,x}}) dx) \\ &= -\frac{1}{2} q_2 \rho_2 b \cdot |\psi'|^2 \Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2} (q_2 \rho_2 b)_x \cdot |\psi'|^2 dx - \text{Re } \int_{a_1}^{a_2} q_2 \rho_2 b \ \psi' \cdot \overline{g_{2,x}} dx\end{aligned}$$

and

$$\text{Re } J_4 = -\frac{1}{2} q_2 b^2 |\psi_x|^2 \Big|_{a_1}^{a_2} + \frac{1}{2}\int_{a_1}^{a_2} q_{2,x} \cdot b^2 |\psi_x|^2 dx$$

In addition, integration by parts J_5 and using the equations of $(\overline{P})$, one has:

$$\begin{aligned}\text{Re } J_5 &= \text{Re } \Big(\int_{a_1}^{a_2} q_2 bk(\varphi_x + \psi + \ell w) \cdot \overline{\psi_x} dx\Big) \\ &= \text{Re } \Big(q_2 bk \cdot (\varphi_x + \psi + \ell w)\overline{\psi}\Big|_{a_1}^{a_2} - \int_{a_1}^{a_2} (q_2 bk \cdot (\varphi_x + \psi + \ell w))_x \cdot \overline{\psi} dx\Big)\end{aligned}$$

Analyzing the second part of the last identity, we noticed that, by the derivation of the product:

$$\begin{aligned}-\text{Re } \Big(\int_{a_1}^{a_2} &(q_2 bk \cdot (\varphi_x + \psi + \ell w))_x \cdot \overline{\psi} dx\Big) \\ &= -\text{ Re}\Big[\int_{a_1}^{a_2} q_{2,x} bk(\varphi_x + \psi + \ell w)(\frac{1}{\lambda})(\overline{\psi' + g_2}) dx\ \Big] \\ &\quad + \text{Re}\Big[\int_{a_1}^{a_2} q_2 b(g_4 + k_0 \ell (w_x - \ell\varphi) - i\lambda\rho_1\varphi')\overline{\psi}\ dx\Big]\end{aligned}$$

Remembering that, if $\mathbf{z} = a + ib$ is a complex number, then:

$$\text{Re } i\mathbf{z} = -\text{Im } \mathbf{z}$$

With this and some calculations, we obtain that:

$$\begin{aligned}
-\text{Re}\Big(\int_{a_1}^{a_2}(q_2bk\cdot(\varphi_x+\psi+\ell w))_x\cdot\overline{\psi}dx\Big) &= \frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_x\cdot k(\varphi_x+\psi+\ell w)\overline{\psi'}dx \\
&+\frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_x\cdot k(\varphi_x+\psi+\ell w)\overline{g_2}dx \\
&+\text{Re}\int_{a_1}^{a_2}q_2bg_4\cdot\overline{\psi'}dx \\
&-\text{Im}\int_{a_1}^{a_2}q_2bk_0\ell(w_x-\ell\varphi)\overline{g_2}dx \\
&-\text{Im}\int_{a_1}^{a_2}q_2bk_0(w_x-\ell\varphi)\overline{\psi'}dx \\
&+\text{Re}\int_{a_1}^{a_2}q_2b\rho_1\varphi'\overline{g_2}dx \\
&+\text{Re}\int_{a_1}^{a_2}q_2b\rho_1\varphi'\cdot\overline{\psi'}dx
\end{aligned}$$

Returning to (2.74), taking its real part and replacing these last three identities, we conclude that:

$$\begin{aligned}
&-\frac{1}{2}\Big(q_2b\rho_2|\psi'|^2+q_2b^2|\psi_x|^2\Big)\Big|_{a_1}^{a_2}+\frac{1}{2}\Big((q_2b\rho_2)_x|\psi'|^2+q_{2,x}b^2|\psi_x|^2\Big)\Big|_{a_1}^{a_2} \\
&\quad=\text{Re}\int_{a_1}^{a_2}q_2bg_5\overline{\psi_x}dx+\text{Re}\int_{a_1}^{a_2}q_2\rho_2b\psi'\cdot g_{2,x}dx-\text{Re}\Big(q_2bk(\varphi_x+\psi+\ell w)\overline{\psi}\Big)\Big|_{a_1}^{a_2} \\
&\quad-\frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_x\cdot k(\varphi_x+\psi+\ell w)\overline{g_2}dx-\frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_x\cdot k(\varphi_x+\psi+\ell w)\overline{\psi'}dx \\
&\quad-\text{Re}\int_{a_1}^{a_2}q_2bg_4\cdot\overline{\psi}dx+\frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}q_2bk_0\ell(w_x-\ell\varphi)\overline{g_2}dx-\text{Re}\int_{a_1}^{a_2}q_2b\rho_1\varphi'\cdot\overline{g_2} \\
&\quad+\frac{1}{\lambda}\text{Im}\int_{a_1}^{a_2}q_2kb_0\ell(w_x-\ell\varphi)\overline{\psi'}dx-\text{Re}\int_{a_1}^{a_2}q_2b\rho_1\psi'\cdot\overline{\psi'}dx \qquad (2.75)
\end{aligned}$$

Third, taking the multiplier $q_3k(\overline{w_x-\ell\varphi})$ in the last equation of $(\overline{P})$, and integrate over (a_1, a_2), we get:

$$\int_{a_1}^{a_2} q_3 k_0 g_6 (\overline{w_x - \ell\varphi}) dx = \overbrace{-\int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot (\overline{i\lambda (w_x - \ell\varphi)}) dx}^{J_6}$$
$$\overbrace{-\int_{a_1}^{a_2} q_3 (k_0 (w_x - \ell\varphi))_x (\overline{k_0 (w_x - \ell\varphi)}) dx}^{J_7}$$
$$+\int_{a_1}^{a_2} q_3 k_0 \ell k (w_x - \ell\varphi)(\overline{w_x - \ell\varphi}) dx \tag{2.76}$$

Here, we will make use of some previous results to obtain

$$J_6 = -\int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot \overline{w'} dx + \int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot \ell \overline{\varphi'} dx + \int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot (\overline{g_{3,x} - \ell g_1}) dx$$

Using integration by parts, J_6 is equal to:

$$J_6 = -\frac{1}{2} q_3 \rho_1 k_0 |w'| \Big|_{a_1}^{a_2} + \frac{1}{2} \int_{a_1}^{a_2} (q_3 \rho_1 k_0)_x |w'|^2 dx$$
$$+\int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot \ell \overline{\varphi'} dx + \int_{a_1}^{a_2} q_3 \rho_1 k_0 w' \cdot (\overline{g_{3,x} - \ell g_1}) dx.$$

and

$$\text{Re } J_7 = -\frac{1}{2} q_3 k_0^2 |w_x - \ell\varphi|^2 \Big|_{a_1}^{a_2} + \frac{1}{2} \int_{a_1}^{a_2} q_{3,x} k_0^2 |w_x - \ell\varphi|^2 dx$$

Then, taking the real part of (2.76), we obtain:

$$-\frac{1}{2} (q_3 \rho_1 k_0 |w'|^2 + q_3 k_0^2 |w_x - \ell\varphi|^2) \Big|_{a_1}^{a_2} + \frac{1}{2} \int_{a_1}^{a_2} \Big((q_3 \rho_1 k_0)_x |w'|^2 + q_{3,x} k_0^2 |w_x - \ell\varphi|^2 \Big) dx$$
$$= \text{Re} \int_{a_1}^{a_2} \Big((q_3 \rho_1 k_0)_x |w'|^2 + q_{3,x} k_0^2 |w_x - \ell\varphi|^2 \Big) dx - \text{Re} \int_{a_1}^{a_2} q_3 \rho_1 k_0 \ell w' \cdot \overline{\varphi'} dx$$
$$+ \text{Re} \int_{a_1}^{a_2} q_3 \rho_1 k_0 \ell (\overline{g_{3,x} - \ell g_1}) dx - \text{Re} \int_{a_1}^{a_2} q_3 k_0 \ell k (\varphi_x + \psi + \ell w)(\overline{w_x - \ell\varphi}) dx \tag{2.77}$$

Finally, combining the identities (2.73),(2.75) and (2.77), we arrive at:

$$\begin{aligned}
&\int_{a_1}^{a_2} \Big(q_{1,x}k^2|\varphi_x+\psi+\ell w|^2+(q_1\rho_1 k)_x|\varphi'|^2+q_{2,x}b^2|\psi_x|^2+(q_2\rho_2 b)_x|\psi'|^2 \\
&\qquad + q_{3,x}k_0^2|w_x-\ell\varphi|^2+(q_3\rho_1 k_0)_x|w'|^2\Big)dx \\
&=\big(q_1k^2|\varphi_x+\psi+\ell w|^2+q_1\rho_1 k|\varphi'|^2+q_2b^2|\psi_x|^2+q_2\rho_2 b|\psi'|^2\big)\Big|_{a_1}^{a_2} \\
&\qquad + \big(q_3k_0^2|w_x-\ell\varphi|^2+q_3\rho_1 k_0|w'|^2\big)\Big|_{a_1}^{a_2}+P(a_1,a_2)+J_{10}+J_{11}+J_{12}+J_{13}. \qquad (2.78)
\end{aligned}$$

for any $q_1, q_2, q_3 \in C^1[a_1, a_2]$, which denote:

$$\begin{aligned}
P(a_1,a_2) &= -2\text{Re}\left(q_2bk(\varphi_x+\psi+\ell w)\overline{\psi}\Big|_{a_1}^{a_2}\right) \\
J_{10} &= 2\text{Re}\int_{a_1}^{a_2}(q_1\rho_1 k-q_2\rho_1 b)\varphi'\cdot\overline{\psi'}dx+2\text{Re}\int_{a_1}^{a_2}\ell(q_1\rho_1 k-q_3\rho_1 k_0)w'\cdot\overline{\varphi'}dx \\
J_{11} &= -\frac{2}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_xk(\varphi_x+\psi+\ell w)\overline{\psi'}dx+\frac{2}{\lambda}\text{Im}\int_{a_1}^{a_2}q_2bk_0\ell(w_x-\ell\varphi)\overline{\psi'}dx \\
J_{12} &= -\frac{2}{\lambda}\text{Im}\int_{a_1}^{a_2}(q_2b)_xk(\varphi_x+\psi+\ell w)\overline{g_2}dx+\frac{2}{\lambda}\text{Im}\int_{a_1}^{a_2}q_2bk_0\ell(w_x-\ell\varphi)\overline{g_2}dx \\
&\quad +2\text{Re}\int_{a_1}^{a_2}q_3\rho_1 k_0w'(\overline{g_{3,x}-\ell g_1})dx+2\text{Re}\int_{a_1}^{a_2}q_1\rho_1 k\varphi'(\overline{g_{1,x}+g_2+\ell g_3})dx \\
&\quad +2\text{Re}\int_{a_1}^{a_2}q_3k_0g_6(\overline{w_x-\ell\varphi})dx+2\text{Re}\int_{a_1}^{a_2}q_1kg_4(\overline{\varphi_x+\psi+\ell w})dx \\
&\quad +2\text{Re}\int_{a_1}^{a_2}q_2\cdot b(g_4\overline{\psi_x}+\rho_2\psi'\overline{g_{2,x}}-\rho_1\varphi'\cdot\overline{g_2}-g_4\overline{\psi})dx \\
J_{13} &= 2\text{Re}\int_{a_1}^{a_2}k_0\ell k(q_1-q_3)(\varphi_x+\psi+\ell w)(\overline{w_x-\ell\varphi})dx.
\end{aligned}$$

Step 2: Conclusion of (2.70) - (2.71) for $j = 2$

Since (2.78) holds for any $q_1, q_2, q_3 \in C^1[a_1, a_2]$, let us choose them so that:

$$(q_1 k)(x) = (q_2 b)(x) = (q_3 k_0)(x) = \int_{a_1}^{x} e^{n\pi} d\tau,$$

for $x \in [a_1, a_2]$ and $n \in \mathbb{N}$ to be determined later.

Its easy to deduce that $J_{10} = 0$. Let us the estimate the remaining terms in (2.78).Indeed, since the constants $\rho_1, \rho_2, k, k_0, b$ are positive, from the Holder's inequality, there exist a constant $C_n > 0$ such that:

$$\begin{aligned}
|J_{11}| &\leq \frac{2}{\lambda}\Big|\int_{a_1}^{a_2}(q_2 b)_x k(\varphi_x + \psi + \ell w)\overline{\psi'}dx\Big| + \frac{2}{\lambda}\Big|\int_{a_1}^{a_2} q_2 b k_0(w_x - \ell\varphi)\overline{\psi'}dx\Big| \\
&\leq \frac{C_n}{\lambda}\Big|\int_{a_1}^{a_2}(\varphi_x + \psi + \ell w)\overline{\psi'}dx\Big| + \frac{C_n}{\lambda}\Big|\int_{a_1}^{a_2}(w_x - \ell\varphi)\overline{\psi'}dx\Big| \\
&\leq \frac{C_n}{\lambda}\int_{a_1}^{a_2}|\varphi_x + \psi + \ell w|^2 + |w_x - \ell\varphi|^2 + |\psi'|^2 dx \\
&\leq \frac{C_n}{\lambda}\|V\|^2_{a_1,a_2}.
\end{aligned}$$

For J_{12}, since $|\lambda| > 1$, the functions q_i are bounded in $[a_1, a_2]$, we can use the known techniques for estimates to obtain, in each integral, equivalent terms that are part of $\|V\|_{a_1,a_2}$ and $\|G\|_{0,L}$. Thus:

$$|J_{12}| \leq C_n\|V\|_{a_1,a_2} \cdot \|G\|_{0,L}\ . \tag{2.79}$$

Using Holder and Young inequalities and the embedding $H^1(a_1, a_2) \hookrightarrow L^\infty(a_1, a_2)$, one sees that:

$$\begin{aligned}
|P(a_1,a_2)| &= \Big| -2\text{Re}\,\Big(q_2bk(\varphi_x+\psi+\ell w)\overline{\psi}\Big|_{a_1}^{a_2}\Big)\Big| \\
&\le \frac{2}{|\lambda|}\Big| q_2b(\varphi_x+\psi+\ell w)(\overline{\psi'+g_2})\Big|_{a_1}^{a_2}\Big| \\
&\le \frac{C}{|\lambda|}\Big|(q_2b)(\varphi_x+\psi+\ell w)\psi'\Big|_{a_1}^{a_2}\Big| + \frac{C}{|\lambda|}\Big|(q_2b)(\varphi_x+\psi+\ell w)\overline{g_2}\Big|_{a_1}^{a_2}\Big| \\
&\le \frac{C}{|\lambda|}\Big|\int_{a_1}^{a_2} e^{n\tau}d\tau\cdot(\varphi_x+\psi+\ell w)(\overline{\psi'})(a_2) - \int_{a_1}^{a_1} e^{n\tau}d\tau\cdot(\varphi_x+\psi+\ell w)(\overline{\psi'})(a_1)\Big| \\
&+ \frac{C}{|\lambda|}\Big|\int_{a_1}^{a_2} e^{n\tau}d\tau\cdot(\varphi_x+\psi+\ell w)(\overline{g_2})(a_2) - \int_{a_1}^{a_1} e^{n\tau}d\tau\cdot(\varphi_x+\psi+\ell w)(\overline{g_2})(a_1)\Big| \\
&\le \frac{C_n}{|\lambda|}\Big|(\varphi_x+\psi+\ell w)(\overline{\psi'})(a_2)\Big| + \frac{C_n}{|\lambda|}\Big|(\varphi_x+\psi+\ell w)(\overline{g_2})(a_2)\Big|dx
\end{aligned}$$

Since $|\lambda| > 1$, we deduce that $\frac{1}{|\lambda|^2} < \frac{1}{|\lambda|}$, and then:

$$\begin{aligned}
|P(a_1,a_2)| &\le \frac{C_n}{|\lambda|}\Big|(\varphi_x+\psi+\ell w)(a_2)\Big|^2 + \frac{C_n}{|\lambda|}\Big|\psi'(a_2)\Big|^2 + \frac{C_n}{|\lambda|^2}\Big|(\varphi_x+\psi+\ell w)(a_2)\Big|^2 + C_n\Big|g_2(a_2)\Big|^2 \\
&\le \frac{C_n}{|\lambda|}\Big|(\varphi_x+\psi+\ell w)(a_2)\Big|^2 + \frac{C_n}{|\lambda|}\Big|\psi'(a_2)\Big|^2 + C_n\|g_2\|_\infty^2 \\
&\le \frac{C_n}{|\lambda|}\Big|(\varphi_x+\psi+\ell w)(a_2)\Big|^2 + \frac{C_n}{|\lambda|}\Big|\psi'(a_2)\Big|^2 + C_n\|G\|_{0,L}^2. \qquad (2.80)
\end{aligned}$$

Now, observing that:

$$(q_1-q_3)(x) = \frac{1}{k}\int_{a_1}^{x} e^{n\tau}d\tau - \frac{1}{k_0}\int_{a_1}^{x} e^{n\tau}d\tau = \frac{k-k_0}{k\cdot k_0}\int_{a_1}^{x} e^{n\tau}d\tau = \frac{k-k_0}{k\cdot k_0}\Big(\frac{e^{nx}-e^{na_1}}{n}\Big)$$

we infer:

$$(\ell k_0\cdot k)(q_1-q_3)(x) = \ell(k_0-k)\Big(\frac{e^{nx}-e^{na_1}}{n}\Big).$$

Returning to J_{13} and using the Young's inequality, we have that:

$$\begin{aligned}
|J_{13}| &= 2\Big|\int_{a_1}^{a_2} \ell k_0 \cdot k(q_1-q_3)(x)(\varphi_x+\psi+\ell w)(\overline{w_x-\ell\varphi})dx\Big| \\
&\leq \frac{M}{n}\Big|\int_{a_1}^{a_2}(e^{nx}-e^{na_1})(\varphi_x+\psi+\ell w)(\overline{w_x-\ell\varphi})dx\Big| \\
&\leq \frac{M}{n}\int_{a_1}^{a_2} e^{nx}\big(|\varphi_x+\psi+\ell w|^2+|w_x-\ell\varphi|^2\big)dx. \qquad (2.81)
\end{aligned}$$

Replacing (2.79)-(2.81) into (2.78), and using the fact that $(q_1k)(a_1)=(q_2b)(a_1)=(q_3k_0)(a_1)$=0, this estimate becomes:

$$\begin{aligned}
&\alpha_0\int_{a_1}^{a_2} e^{nx}\Big(|\varphi_x+\psi+\ell w|^2+|\varphi'|^2+|\psi_x|^2+|\psi'|^2+|w_x-\ell\varphi|^2+|w'|^2\Big)dx \\
&\leq \int_{a_1}^{a_2}\Big(q_{1,x}k^2|\varphi_x+\psi+\ell w|^2+(q_1\rho_1k)_x|\varphi'|^2+q_{2,x}b^2|\psi_x|^2+(q_2\rho_2b)_x|\psi'|^2 \\
&\quad + q_{3,x}k_0^2|w_x-\ell\varphi|^2+(q_3\rho_1k_0)_x|w'|^2\Big)dx \\
&\leq (q_1k)(a_2)k|\varphi_x+\psi+\ell w|^2(a_2)+(q_1k)(a_2)\rho_1|\varphi'|^2(a_2)+(q_2b)(a_2)b|\psi_x|^2 \\
&\quad + (q_2b)(a_2)\rho_2|\psi'|^2(a_2)+(q_3k_0)(a_2)k_0|w_x-\ell\varphi|^2(a_2)+(q_3k_0)(a_2)\rho_1|w'|^2(a_2) \\
&\quad + \frac{C_n}{|\lambda|}|(\varphi_x+\psi+\ell w)(a_2)|^2+\frac{C_n}{|\lambda|}|\psi'(a_2)|^2+C_n\|G\|_{0,L}^2+\frac{C_n}{|\lambda|}\|V\|_{a_1,a_2}^2 \\
&\quad + C_n\|V\|_{a_1,a_2}\cdot\|G\|_{0,L}+\frac{M}{n}\int_{a_1}^{a_2} e^{nx}\Big(|\varphi_x+\psi+\ell w|^2+|w_x-\ell\varphi|^2\Big)dx
\end{aligned}$$

where $\alpha_0 = min\{b,k,k_0,\rho_1,\rho_2\}$. Taking $n_0\in\mathbb{N}$ large enough such that:

$$\alpha_0-\frac{M}{n_0}>0$$

and denoting, for $j=1,2$:

$$I(a_j)=|\varphi_x(a_j)+\psi(a_j)+\ell w(a_j)|^2+|w_x(a_j)-\ell\varphi(a_j)|^2+|\psi_x(a_j)|^2+|\varphi'(a_j)|^2+|\psi'(a_j)|^2+|w'(a_j)|^2 \qquad (2.82)$$

We conclude that there exist a constants $C, \alpha_0 > 0$ so that:

$$Ce^{n_0 a_1}\|V\|_{a_1,a_2}^2 \leq C \cdot I(a_2) + \frac{C}{|\lambda|}\|V\|_{a_1,a_2}^2 + C\|V\|_{a_1,a_2} \cdot \|G\|_{0,L} + C\|G\|_{0,L}^2$$

Considering $|\lambda| > 1$ large enough and using Young's inequality with $\epsilon > 0$, there exist a constant $C_1 > 0$ such that:

$$\|V\|_{a_1,a_2}^2 \leq C_1 \cdot I(a_2) + C_1\|G\|_{0,L}^2.$$

which concludes (2.69) for $j = 2$. To conclude (2.68) for $j = 2$, we recall again the identity (2.78) and, in view of estimates (2.79)-(2.81), along the assumption that constants $b, k, k_0, \rho_1, \rho_2$ are positive, we use the positivity of the terms in (2.78) to estimate $I(a_2)$ as:

$$\begin{aligned} I(a_2) = \int_{a_1}^{a_2} \Big(& q_{1,x}k^2|\varphi_x + \psi + \ell w|^2 + (q_1\rho_1 k)_x|\varphi'|^2 + q_{2,x}b^2|\psi_x|^2 + (q_2\rho_2 b)_x|\psi'|^2 \\ & + q_{3,x}k_0^2|w_x - \ell\varphi|^2 + (q_3\rho_1 k_0)_x|w'|^2\Big)dx - P(a_1,a_2) - J_{10} - J_{11} - J_{12} - J_{13} \\ \leq & C\|V\|_{a_1,a_2}^2 + P(a_1,a_2) + J_{10} + J_{11} + J_{12} + J_{13} \\ \leq & C\|V\|_{a_1,a_2}^2 + \frac{C}{|\lambda|}|(\varphi_x + \psi + \ell w)(a_2)|^2 + \frac{C}{|\lambda|}|\psi'(a_2)|^2 + C\|G\|_{0,L}^2 + \frac{C}{|\lambda|}\|V\|_{a_1,a_2}^2 \\ & + C\|V\|_{a_1,a_2} \cdot \|G\|_{0,L} + Ce^{n_0 a_2}\|V\|_{a_1,a_2}^2 \end{aligned}$$

with $n_0 \in \mathbb{N}$ taking previously. Taking $|\lambda| > 1$ large enough and using Young's inequality, there exist a constant $C_0 > 0$ such that:

$$I(a_2) \leq C_0\|V\|_{a_1,a_2}^2 + C_0\|G\|_{0,L}^2.$$

This concludes the proof of (2.68) for $j = 2$. ■

Step 3. Conclusion of (2.68) and (2.69) for $j = 1$:

The proof is similar to the case $j = 2$ with some minimal changes. In fact, we starting choosing the functions q_1, q_2, q_3, specifically as:

$$(q_1k)(x) = (q_2b)(x) = (q_3k_0)(x) = -\int_x^{a_2} e^{-n\tau} d\tau$$

for $x \in [a_1, a_2]$ and $n \in \mathbb{N}$. Starting at (2.78) we see that $J_{10} = 0$ and the estimates (2.79) and (2.81) follow analogously (for more details, see [11]).

The next step proofs the extension result of the last theorem (Corollary 2.2).

Proof of Corollary 2.2:

From the last theorem, we proved that:

$$I(a_j) \le C_0 \|V\|^2_{a_1,a_2} + C_0\|G\|^2_{0,L}$$

for $j = 1, 2$. If we considering $(a_1, a_2) = (0, b_2)$ for some $0 < b_2 \le a_2$ we have:

$$I(b_j) \le C_0\Lambda + C_0\|G\|^2_{0,L}. \tag{2.83}$$

Taking $j = 2$, and using the previous estimate, we obtain:

$$\begin{aligned}&\int_0^{b_2} \Big(|\varphi_x + \psi + \ell w|^2 + |w_x - \ell\varphi|^2 + |\psi_x|^2 + |\varphi'|^2 + |\psi'|^2 + |w'|^2\Big)dx \\ &\quad C_1 \cdot I(b_2) + C_1\|G\|^2_{0,L} \le C_0 \cdot C_1\Lambda + C_1 \cdot C_0\|G\|^2_{0,L} + C_1\|G\|^2_{0,L} \\ &\quad \le C_2\Lambda + C_2\|G\|^2_{0,L}\end{aligned} \tag{2.84}$$

with $C_2 = C_0 \cdot C_1 + C_1$.

Analogously, using (2.69) with $a_1 = b_2, a_2 = L$ and (2.83) with $j = 2$. we also obtain:

$$\int_{b_2}^{L} \Big(|\varphi_x + \psi + \ell w|^2 + |w_x - \ell\varphi|^2 + |\psi_x|^2 + |\varphi'|^2 + |\psi'|^2 + |w'|^2\Big)dx \le C_2\Lambda + C_2\|G\|^2_{0,L} \tag{2.85}$$

Finally, combining (2.84) and (2.85), there exists a constant $C > 0$, independent of λ, such that:

$$\|V\|_{0,L}^2 \leq C\Lambda + C\|G\|_{0,L}^2 \blacksquare$$

Completion the proof of Theorem 2.4:

In order to prove the exponential stability and conclude Theorem 2.4, we will use last results in this section to show the desired result:

Consider $\varepsilon > 0$ be given. From corollary 2.1, we deduce that:

$$\int_{l_0-\frac{\delta}{2}}^{l_0+\frac{\delta}{2}} \Big(|\varphi_x + \psi + \ell w|^2 + |w_x - \ell\varphi|^2 + |\psi_x|^2 + |\varphi'|^2 + |\psi'|^2 + |w'|^2\Big) dx$$
$$\leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2 := \Lambda,$$

For some constant $C_\varepsilon > 0$.

In view of (2.38) - (2.43) the vector function $V := (\varphi, \psi, w, \varphi', \psi', w')^T$ is a solution of the system $(\overline{P})$ with $G = (g_1, g_2, g_3, g_4, g_5, g_6)^T$ given above, and the condition of Corollary 2.2 is verified with

$$a_1 = l_0 - \frac{\delta}{2} \quad \text{and} \quad a_2 = l_0 + \frac{\delta}{2}.$$

Thus, since G has terms that depends of f, we can estimate its norm easily by $\|G\|_{0,L}^2 \leq \|f\|_{\mathcal{H}}^2$, and then, by Corollary 2.2, we obtain

$$\|V\|_{0,L}^2 \leq C(\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2) + C\|G\|_{0,L}^2 \leq C\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2$$

Using the result of Lemma 2.1 and Young's inequality, we conclude

$$\begin{aligned}
\|y\|_{\mathcal{H}}^2 &\leq \|V\|_{0,L}^2 + \sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2 \\
&\leq C\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2 + \sigma_1\|\theta^1\|^2 + \sigma_1\|\theta^2\|^2 + \sigma_1\|\theta^3\|^2 \\
&\leq C\varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2 + C\|y\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} \\
&\leq \varepsilon\|y\|_{\mathcal{H}}^2 + C_\varepsilon\|f\|_{\mathcal{H}}^2.
\end{aligned}$$

for $|\lambda| > 1$ large enough.

Finally, with the aim to prove (2.48), take $\varepsilon > 0$ small enough in the last inequality to reach:

$$\|y\|_{\mathcal{H}}^2 \leq C\|f\|_{\mathcal{H}}^2,$$

and regarding the resolvent equation (2.37), we conclude:

$$\|(i\lambda - A)^{-1}f\|_{\mathcal{H}} \leq C\|f\|_{\mathcal{H}}, \text{ as } |\lambda| \to \infty.$$

Therefore, we complete the proof of Theorem 2.4 and show the exponential stability of the Bresse-Fourier system with full thermal coupling. ■

2.3 Gradient systems

From now on we will study the long time dynamics of the dynamical system associated with problem (2.1)-(2.9). We say that a dynamical system $(\mathcal{H}, T_\ell(t))$ is ***gradient*** if it admits a Lyapunov function, that is, a functional $\mathcal{G} : \mathcal{H} \to R$ such that $\mathcal{G}(T(t)y)$ is non-increasing with respect to $t \geq 0$, for any $y \in \mathcal{H}$; and whenever y satisfies

$$\mathcal{G}(T(t)y) = \mathcal{G}(y), \text{ for } t > 0$$

one has that y is a stationary point (that is $T(t)y = y$ for all $t \geq 0$). In this way, we obtain the next result:

Lemma 2.8. *Suppose that conditions (2.25)-(2.29) holds. Then the dynamical system* $(\mathcal{H}, T(t))$ *is gradient and, the associated Lyapunov functional* $\mathcal{G}$ *satisfies:*

$$\mathcal{G}(y) \to \infty \quad \textit{if and only if} \quad \|y\|_{\mathcal{H}} \to \infty \tag{2.86}$$

Proof: Let us define the strict Lyapunov function as nonlinear energy of the system (2.30). Given a $y_0 \in \mathcal{H}$, the trajectory of this vector is given by:

$$\mathcal{G}y_0(t) = (\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t)).$$

and thus,

$$\begin{aligned}\mathcal{G}(T(t)y_0) =& \frac{1}{2}\|\big(\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t)\big)\|^2_{\mathcal{H}} \\ &+ \int_0^L F(\varphi(t), \psi(t), w(t))dx.\end{aligned} \tag{2.87}$$

Remembering that semilinear energy is defined as:

$$\mathcal{E}(t) = E(t) + \int_0^L F(\varphi(t), \psi(t), w(t))dx.$$

Then, from (2.31), the Lyapunov functional satisfies the following estimate:

$$\frac{d}{dt}\mathcal{G}(T(t)y_0) = -\gamma_1\|\theta_x^1(t)\|^2 - \gamma_2\|\theta_x^2(t)\|^2 - \gamma_3\|\theta_x^3(t)\|^2 \leq 0, \tag{2.88}$$

Therefore, the functional $\mathcal{G}$ is non increasing, for $t \geq 0$.

Now suppose that for some $y_0 \in \mathcal{H}$, the functional is stationary, that is:

$$\mathcal{G}(T(t)y_0) = \mathcal{G}(y_0), \;\; t > 0. \tag{2.89}$$

Then

$$-\gamma_1\|\theta_x^1(t)\|^2 - \gamma_2\|\theta_x^2(t)\|^2 - \gamma_3\|\theta_x^3(t)\|^2 = \frac{d}{dt}\mathcal{G}(T(t)y_0) = \frac{d}{dt}\mathcal{G}(y_0) = 0$$

$\Rightarrow$

$$\gamma_1\|\theta_x^1(t)\|^2 + \gamma_2\|\theta_x^2(t)\|^2 + \gamma_3\|\theta_x^3(t)\|^2 = 0, \forall\, t > 0.$$

From which we can deduce that

$$\|\theta_x^1(t)\|_{L^2} = \|\theta_x^2(t)\|_{L^2} = \|\theta_x^3(t)\|_{L^2} = 0 \tag{2.90}$$

for any $t > 0$. Now, we gonna use the Poincare's inequality to obtain

$$\|\theta^1(t)\|_{L^2} = \|\theta^2(t)\|_{L^2} = \|\theta^3(t)\|_{L^2} = 0, \tag{2.91}$$

for $t > 0$. This implies that

$$\theta^i(x,t) = \theta^i_x(x,t) = 0. \tag{2.92}$$

for a.e $x \in (0, L)$, and for any $t > 0$. We conclude then, by continuity of t, that:

$$\frac{d}{dt}\theta^i(x,t) = \theta^i_t(x,t) = 0 \tag{2.93}$$

for $t \geq 0$, and for a.e $x \in (0, L)$.

This information is consistent with the fact that the energy remains stationary. In other words, there is no dissipation in the temperatures. We use this important result to show that kinetic terms φ_t, ψ_t, w_t are equal to 0. Observing the third heat equation of the Bresse-Fourier system, we note that:

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0 \quad \Rightarrow (\psi_t)_x = 0$$

for $t \geq 0$ and $x \in (0, L)$. Here, was used the Schwarz theorem in order to change the derivation order. From this, we have:

$$\frac{\pi}{L}\|\psi_t\| \leq \|(\psi_t)_x\| = 0,$$

which is valid for all $t > 0$. Thus:

$$\|\psi_t(t)\|_{L^2} = 0, \textit{ for } t \geq 0.$$

We conclude then that ψ not depends on t, that is:

$$\psi(x,t) = \psi(x) \text{ in } (0, L). \tag{2.94}$$

From the fourth equation of the nonlinear system, we can use (2.94) and the fact that temperature θ^1 is stationary, to obtain:

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0 \Rightarrow (\varphi_x + \ell w)_t = 0$$

in $(0, L) \times (0, \infty)$. Thus:

$$\varphi_x(x,t) + \ell w(x,t) = g_1(x). \tag{2.95}$$

Following the same argument in the fifth equation, we have:

$$w_x(x,t) - \ell\varphi(x,t) = g_2(x). \tag{2.96}$$

Now, differentiating (2.96) in x, multiplying (2.95) by ℓ and add both equations, we obtain:

$$w_{xx}(x,t) + \ell^2 w(x,t) = g_{2,x}(x) + \ell g_1(x) = g_3(x)$$

which represents an non-homogeneous second order ordinary differential equation that does not depend on t. Then, the last expression can be rewritten as:

$$w_{xx}(x) + \ell^2 w(x) = g_3(x). \tag{2.97}$$

Then, the solution of this ordinary differential equations, give us an function w that does not depend on t, from where

$$w(x,t) = w(x).$$

for $x \in (0, L)$ and $t > 0$.

Using this information into (2.96), we deduce that

$$\varphi(x,t) = \varphi(x)$$

for $x \in (0, L)$ and $t > 0$. Combining these last two results with (2.94) and (2.92), we conclude from assumption (2.89) that:

$$(\varphi, \psi, w, \varphi_t, \psi_t, w_t, \theta^1, \theta^2, \theta^3) = (\varphi(x), \psi(x), w(x), 0, 0, 0, 0, 0, 0).$$

i.e, the trajectory is stationary.

Remark: *We say the temperatures are equal to 0 does not mean that literally is zero, but the temperature coincides with the surrounding environment which the system is developed.*

To show the property (2.86), we use the properties of F and f to show that:

$$\begin{aligned}
\mathcal{E}(t) &= E(t) + \int_0^L F(\varphi, \psi, w)dx \\
&\leq \|y\|_{\mathcal{H}}^2 + C_F \int_0^L (1 + |\varphi|^{p+1} + |\psi|^{p+1} + |w|^{p+1})dx \\
&\leq \|y\|_{\mathcal{H}}^2 + C\Big(1 + \|\varphi\|_{p+1}^{p+1} + \|\psi\|_{p+1}^{p+1} + \|w\|_{p+1}^{p+1}\Big)
\end{aligned}$$

Using Sobolev embedding and equivalence of norms, we conclude that:

$$\mathcal{E}(t) \leq \|y\|_{\mathcal{H}}^2 + C(1 + \|y\|_{\mathcal{H}}^{p+1}) \tag{2.98}$$

Thus, if $G(y)$ goes to $+\infty$, then $\|y\|_{\mathcal{H}} \longrightarrow +\infty$.

Analogously, we see that:

$$\mathcal{E}(t) \geq \frac{1}{2}\|y\|_{\mathcal{H}}^2 - \int_0^L m_F dx = \frac{1}{2}\|y\|_{\mathcal{H}}^2 - Lm_F$$

Therefore:

$$\mathcal{E}(t) + Lm_f \geq \frac{1}{2}\|y\|_{\mathcal{H}}^2 \tag{2.99}$$

Here, if $\|y\|_{\mathcal{H}}$ goes to $+\infty$, the functional $G(y)$ tends to $+\infty$. Finally, (2.86) is showed and the proof of the lemma is concluded. ∎

2.4 Global attractors

2.4.1 Quasi-stability

In order to prove the existence of a global attractor for the Bresse-Fourier system, we must have a criteria of asymptotic compactness. The method of quasi-stability of Chueshov and Lasiecka has become a useful tool when demonstrating asymptotic compactness and other proprieties of dynamical system generated by wave type PDEs.

In simply words, the theory is applicable when the difference of two solutions (trajectories), in any bounded set that is forward invariant, is limited by a stable term and a compact term.
Lets start taking a bounded forward invariant set $B \subset \mathcal{H}$. Given $y^i \in B$ the corresponding solution trajectory is denoted by

$$T(t)y^i = \big(\varphi^i(t), \psi^i(t), w^i(t), \varphi^i_t(t), \psi^i_t(t), w^i_t(t), \theta^{1,i}(t), \theta^{2,i}(t), \theta^{3,i}(t)\big), \tag{2.100}$$

for $t \geq 0,\ i = 1, 2$. The difference of this solutions is written by

$$\varphi = \varphi^1 - \varphi^2,\ \ \psi = \psi^1 - \psi^2,\ \ w = w^1 - w^2, \theta^k = \theta^{k,1} - \theta^{k,2},\ k = 1, 2, 3.$$

By the results in [27], the quasi-stability property reduces to show the following estimate:

$$\begin{aligned} &\|T(t)y^1 - T(t)y^2\|^2_{\mathcal{H}} \\ &\qquad \leq C_B e^{-at}\|y^1 - y^2\|^2_{\mathcal{H}} + K_B \sup_{0\leq s\leq t}\Big(\|\varphi(s)\|^2_{p+2} + \|\psi(s)\|^2_{p+2} + \|w(s)\|^2_{p+2}\Big) \end{aligned} \tag{2.101}$$

for $t > 0$, where C_B, K_B are positive constants depending on B.

Theorem 2.6. *Suppose the conditions (2.25)-(2.29) hold. Then, the dynamical system* $(\mathcal{H}, T(t))$ *associated with* (2.1)-(2.9) *is quasi-stable on every bounded forward invariant set* $B \subset \mathcal{H}$*, that is, satisfies the condition (2.101).*

The proof of this theorem will use the result of exponential stability proved in the previous sections. Taking the difference of the solutions mentioned above, and by the corresponding notations, taking $(\varphi,\psi,w)=(\varphi^1,\psi^1,w^1)-(\varphi^2,\psi^2,w^2)$, we use the following notation:

$$F_i(\varphi,\psi,w)=f_i(\varphi^1,\psi^1,w^1)-f_i(\varphi^2,\psi^2,w^2),\ \ i=1,2,3.$$

Then the difference $(\varphi,\psi,w,\varphi_t,\psi_t,w_t,\theta^1,\theta^2,\theta^3)$ is the solution of:

$$\begin{aligned}
\rho_1\varphi_{tt}-k(\varphi_x+\psi+\ell w)_x-\ell k_0(w_x-\ell\varphi)+m_1\theta^1_x+\ell m_2\theta^2&=-F_1(\varphi,\psi,w),\\
\rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi+\ell w)+m_3\theta^3_x-m_1\theta^1&=-F_2(\varphi,\psi,w),\\
\rho_1 w_{tt}-k_0(w_x-\ell\varphi)_x+k\ell(\varphi_x+\psi+\ell w)+m_2\theta^2_x-\ell m_1\theta^1&=-F_3(\varphi,\psi,w),\\
\sigma_1\theta^1_t-\gamma_1\theta^1_{xx}+m_1(\varphi_x+\psi+\ell w)_t=0,&\\
\sigma_2\theta^2_t-\gamma_2\theta^2_{xx}+m_2(w_x-\ell\varphi)_t=0,&\\
\sigma_3\theta^3_t-\gamma_3\theta^3_{xx}+m_3(\psi_x)_t=0,&
\end{aligned}$$

with boundary conditions and initial condition:

$$(\varphi(0),\psi(0),w(0),\varphi_t(0),\psi_t(0),w_t(0),\theta^1(0),\theta^2(0),\theta^3(0))=y^1-y^2.$$

Since the homogeneous Bresse-Fourier system is exponentially stable, we have from the parameter variation formula that:

$$\begin{aligned}
\|T(t)y^1-T(t)y^2\|^2_{\mathcal{H}}&\leq 2\|S(t)y^1-S(t)y^2\|^2_{\mathcal{H}}+2\Big\|\int_0^t S(t-s)\Big[\mathcal{F}(y^1(s))-\mathcal{F}(y^2(s))\Big]ds\Big\|^2_{\mathcal{H}}\\
&\leq C_B e^{-at}\|y^1-y^2\|^2_{\mathcal{H}}+C_B\int_0^t e^{-a(t-s)}\|\mathcal{F}(y^1(s))-\mathcal{F}(y^2(s))\|^2_{\mathcal{H}}ds. \quad (2.102)
\end{aligned}$$

Where it was used the fact that the semigroup solution for the linear problem $S(t)$ is exponentially stable. By definition of $\mathcal{F}$, and hypotheses on f_i, we have that:

$$\begin{aligned}\|\mathcal{F}(y^1)-\mathcal{F}(y^2)\|_{\mathcal{H}}^2 &\leq C\int_0^L |F_1(\varphi,\psi,w)|^2+|F_2(\varphi,\psi,w)|^2+|F_3(\varphi,\psi,w)|^2dx\\ &\leq C\int_0^L |\nabla f_1|^2|(\varphi,\psi,w)|^2dx + C\int_0^L |\nabla f_2|^2|(\varphi,\psi,w)|^2dx\\ &\quad + C\int_0^L |\nabla f_3|^2|(\varphi,\psi,w)|^2dx\end{aligned}$$

where $\nabla f_i = (1+|\varphi^1|^{p-1}+|\psi^1|^{p-1}+|w^1|^{p-1}+|\varphi^2|^{p-1}+|\psi^2|^{p-1}+|w^2|^{p-1})$.

Applying Holder's inequality for p, and since both of trajectories are bounded on B, we can estimate all of terms ∇f_i by a constant that depends on B. Thus:

$$\|\mathcal{F}(y^1)-\mathcal{F}(y^2)\|_{\mathcal{H}}^2 \leq k_B\big[\|\varphi\|_{2p}+\|\psi\|_{2p}+\|w\|_{2p}\big]$$

Then, back into (2.102), we see that:

$$\begin{aligned}\|T(t)y^1-T(t)y^2\|_{\mathcal{H}}^2 \leq\ & C_Be^{-at}\|y^1-y^2\|_{\mathcal{H}}^2\\ &+C_B\cdot k_B\int_0^t e^{-a(t-s)}\big[\|\varphi(s)\|_{2p}+\|\psi(s)\|_{2p}+\|w(s)\|_{2p}\big]ds\\ \leq\ & C_Be^{-at}\|y^1-y^2\|_{\mathcal{H}}^2\\ &+C_Bk_B\sup_{0\leq s\leq t}\big[\|\varphi(s)\|_{2p}+\|\psi(s)\|_{2p}+\|w(s)\|_{2p}\big]\int_0^t e^{-a(t-s)}ds\\ \leq\ & C_Be^{-at}\|y^1-y^2\|_{\mathcal{H}}^2\\ &+\frac{C_Bk_B}{a}\sup_{0\leq s\leq t}\big[\|\varphi(s)\|_{2p}+\|\psi(s)\|_{2p}+\|w(s)\|_{2p}\big]ds\\ =\ & C_Be^{-at}\|y^1-y^2\|_{\mathcal{H}}^2+K_B\sup_{0\leq s\leq t}\big[\|\varphi(s)\|_{2p}+\|\psi(s)\|_{2p}+\|w(s)\|_{2p}\big]\end{aligned}$$

which proves estimate (2.101) and finishes the proof. ■

In the next, we will show an important estimate for stationary solutions of problem (2.1)-(2.9) in order to conclude the proof for existence of a global attractor.

Lemma 2.9. *The set $\mathcal{N}_\ell$ of stationary solutions of problem* (2.1)-(2.9) *is bounded.*

Proof: We start taking $y \in \mathcal{N}_\ell$. Then, it has the form $y = (\varphi, \psi, w, 0, 0, 0, \theta^1, \theta^2, \theta^3)$ with the property , $\theta^1_t = \theta^2_t = \theta^3_t = 0$. With this, for $y \in (0, L)$, it satisfies:

$$\begin{aligned}
-k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 + f_1(\varphi, \psi, w) &= 0,\\
-b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + f_2(\varphi, \psi, w) &= 0,\\
-k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 + f_3(\varphi, \psi, w) &= 0,\\
-\gamma_1\theta^1_{xx} &= 0,\\
-\gamma_2\theta^2_{xx} &= 0,\\
-\gamma_3\theta^3_{xx} &= 0.
\end{aligned} \tag{2.103}$$

Here, we deduce from the last three equations that $\theta^i(x) = a_i x + b_i$, for $i = 1, 2, 3$. Adding the boundary conditions on the temperature, we conclude that:

$$\theta^1(x) = \theta^2(x) = \theta^3(x) = 0, \textit{ for } x \in (0, L).$$

This information allow us to reduce the last system into:

$$-k(\varphi_x + \psi + \ell w)_x - k_0\ell(w_x - \ell\varphi) = -f_1(\varphi, \psi, w), \tag{2.104}$$

$$-b\psi_{xx} + k(\varphi_x + \psi + \ell w) = -f_2(\varphi, \psi, w), \tag{2.105}$$

$$-k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) = -f_3(\varphi, \psi, w). \tag{2.106}$$

Multiplying the equations (2.104),(2.105), (2.106) by φ, ψ, w respectively, and integrating over $[0, L]$, we obtain

$$\int_0^L \big(b\psi_x^2 + k(\varphi_x + \psi + \ell w) + k_0(w_x - \ell\varphi)\big)dx$$
$$= -\int_0^L \big(f_1(\varphi,\psi,w)\varphi + f_2(\varphi,\psi,w)\psi + f_3(\varphi,\psi,w)w\big)dx$$

Then, using the conditions about f and F, and the fact that $\|\theta^i\|_{L^2} = 0$, for $i = 1, 2, 3$, the last equation turns into:

$$\begin{aligned}\|y\|_{\mathcal{H}}^2 &= k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 \\ &\leq -\int_0^L \nabla F(\varphi,\psi,w)\cdot(\varphi,\psi,w)dx \\ &\leq -\int_0^L F(\varphi,\psi,w)dx + Lm_f \\ &\leq\ 2Lm_F.\end{aligned}$$

Therefore, $\mathcal{N}_\ell$ is bounded in $\mathcal{H}$. ■

Finally, with the results of Lemma 2.8, Theorem 2.6 and Lemma 2.9, we are in position to state our main theorem of this chapter:

Theorem 2.7. *Under the hypotheses* (2.25)-(2.29)*, for each* $\ell > 0$*, the dynamical system* $(\mathcal{H}, T(t))$ *generated bye the nonlinear problem (2.32) has a finite dimension global attractor* $\mathbf{A}_\ell$*. In addition, it is characterized by*

$$\mathbf{A}_\ell = \mathbb{M}_+(\mathcal{N}_\ell),$$

where $\mathbb{M}_+(\mathcal{N}_\ell)$ *is the unstable manifold emanating from* $\mathcal{N}_\ell$*, the set of stationary points of* $T(t)$*.*

Proof: Throughout this chapter we have explored the property of quasi-stable systems for a Bresse-Fourier system with full thermal coupling. Therefore, by [27, Theorem 4.1], it is asymptotically smooth. Furthermore, we have also proved that the system is gradient, satisfies

(2.86) and the set of stationary points is bounded. Thus, a classical result states that $(\mathcal{H}, T(t))$ has a global attractor $\mathbf{A}_\ell$ which coincides with the unstable manifold emanating from $\mathcal{N}_\ell$ (see references [27]).■

Remark: We note that the global attractor obtained from Theorem 2.7 has further standard properties, for instance, $\mathbf{A}_\ell \subset D(A)$ and it is upper semi-continuous with respect to ℓ. Indeed they are consequences of the quasi-stability estimate (Theorem 2.6). See Chueshov and Lasiecka [15, Chapter 7].

Chapter 3

A thermoelastic Bresse system with delay

3.1 Introduction

The presence of a time delay arises very frequently in many physical, economic, chemical, biological and thermal phenomena. It is well known that delay terms in the system may generate exponential instability [5], so various methods have been investigated in order to resolve and control this term [28]. One of the most known techniques is to insert a frictional dissipation within the delay term in the internal feedback, or equations with past memory terms. In both of cases, "small" delays can be controlled by the dissipation and the stability is guaranteed [5].

In this chapter, we deal with a similar model to (2.10)-(2.15), but adding a delay term in the shear angle (2.12), and considering different boundary conditions, and we will show that, under certain conditions for the parameter ℓ, we can still obtain a result of exponential stability, regardless of the wave speed condition [27]. In other words, we will prove that, using the multiplier method, the dissipation induced by Fourier's laws are sufficiently strong as to stabilize the thermal Bresse system, in the presence of a "small" delay.

3.2 Well-posedness

In this section we will obtain the existence and uniqueness result for the following system with delay term in the internal feedback:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - \ell k_0(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = 0, \tag{3.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + \mu\psi_t(x, t-\tau) = 0, \tag{3.2}$$

$$\rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 = 0, \tag{3.3}$$

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \tag{3.4}$$

$$\sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \tag{3.5}$$

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0, \tag{3.6}$$

where all equations are defined in $(0, L) \times \mathbb{R}^+$, and φ, ψ, w are subject to the Dirichlet-Neumann-Neumann boundary conditions:

$$\varphi(0, t) = \varphi(L, t) = \psi_x(0, t) = \psi_x(L, t) = w_x(0, t) = w_x(L, t) = 0, \quad t \geq 0. \tag{3.7}$$

The thermal terms have the Neumann-Dirichlet-Dirichlet boundary conditions:

$$\theta^1_x(0, t) = \theta^1_x(L, t) = \theta^2(0, t) = \theta^2(L, t) = \theta^3(0, t) = \theta^3(L, t) = 0, \quad t \geq 0, \tag{3.8}$$

with initial data, for $i = 1, 2, 3$:

$$\begin{aligned} &\varphi(x, 0) = \varphi_0(x);\ \ \psi(x, 0) = \psi_0(x);\ \ w(x, 0) = w_0(x);\ \ \theta^i(x, 0) = \theta^i_0(x); \\ &\varphi_t(x, 0) = \varphi_1(x);\ \psi_t(x, 0) = \psi_1(x);\ w_t(x, 0) = w_1(x). \end{aligned} \tag{3.9}$$

The delay's condition is given as follows:

$$\psi_t(x, t-\tau) = g_0(x, t-\tau) \text{ in } (0, L) \times (0, \tau), \tag{3.10}$$

where g_0 represents a history function. The relevance of this new variable lies in the fact that the function ψ_t cannot represent the effect of the values of t in the past (more specifically, values of t with a delay of τ), and we need to describe the events of the past through a function, since the effects of the changes are not felt immediately.

3.2.1 Construction of the phase space $\mathcal{H}$

Let start with a introduction of a new dependent variable for the delay:

$$\boldsymbol{z(x, \rho, t)} := \psi_t(x, t-\rho\tau), \text{ for } (x, \rho, t) \in (0, L) \times (0, 1) \times \mathbb{R}^+.$$

A simple derivation tell us that z satisfies:

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \text{ in } (0, L) \times (0, 1) \times \mathbb{R}^+,$$

with the following proprieties:

$$z(x, 0, t) = \psi_t(x, t) =: \psi_t, \ (x, t) \in (0, L) \times \mathbb{R}^+, \tag{3.11}$$

$$z(x, 1, t) = \psi_t(x, t-\tau), \quad (x, t) \in (0, L) \times \mathbb{R}^+, \tag{3.12}$$

$$z(x, \rho, 0) = g_0(x, -\rho\tau), \quad (x, \rho) \in (0, L) \times (0, 1). \tag{3.13}$$

With this new hypotheses, we have the following new system:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - \ell k_0(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 = 0, \tag{3.14}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + \mu z(x,1,t) = 0, \tag{3.15}$$

$$\rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 = 0, \tag{3.16}$$

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0, \tag{3.17}$$

$$\sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0, \tag{3.18}$$

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0, \tag{3.19}$$

$$\tau z_t + z_\rho = 0, \tag{3.20}$$

where the conditions (3.7)-(3.9) are preserved. The condition (3.13), that refers to the delay, is also added to the system.

In the same way as in the previous chapter, we multiply each equation of the system by φ_t, ψ_t, w_t, $\theta^1, \theta^2, \theta^3$, and $\mu z(x,\rho,\cdot)$, respectively. The last equation (3.20) and the condition (3.11) give us

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\mu\tau\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx\Big) &= -\mu\int_0^L\int_0^1 z_\rho(x,\rho,t)z(x,\rho,t)d\rho dx\\
&= -\mu\Big(\frac{1}{2}\int_0^L |z(x,1,t)|^2dx - \frac{1}{2}\int_0^L |z(x,0,t)|^2dx\Big)\\
&= \frac{\mu}{2}\int_0^L |\psi_t(x,t)|^2dx - \frac{\mu}{2}\int_0^L |z(1)|^2dx.
\end{aligned}$$

From the second equation (3.15), since we have the presence of the delay, the identity results into:

$$\begin{aligned}
\rho_2\int_0^L \psi_{tt}\psi_t dx + b\int_0^L \psi_x(\psi_x)_t + k\int_0^L (\varphi_x + \psi + \ell w)(\psi_t)dx\\
+ m_3\int_0^L \theta^3_x\cdot\psi_t dx - m_1\int_0^L \theta^1\cdot\psi_t dx + \boldsymbol{\mu\int_0^L z(1)\cdot\psi_t dx} = 0
\end{aligned}$$

Thus, after similar calculations to the previous chapter, and using the news results, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\Big(k\|\varphi_x+\psi+\ell w\|^2+k_0\|w_x-\ell\varphi\|^2+b\|\psi_x\|^2+\rho_1\|\varphi_t\|^2+\rho_2\|\psi_t\|^2+\rho_1\|w_t\|^2\\
&\quad+\sigma_1\|\theta^1\|^2+\sigma_2\|\theta^2\|^2+\sigma_3\|\theta^3\|^2+\mu\tau\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx\Big)\\
&\quad=-\Big(\gamma_1\|\theta^1_x\|^2+\gamma_2\|\theta^2_x\|^2+\gamma_3\|\theta^3_x\|^2\Big)-\mu\int_0^L z(1)\cdot\psi_t dx\\
&\qquad+\frac{\mu}{2}\int_0^L|\psi_t(x,t)|^2dx-\frac{\mu}{2}\int_0^L|z(1)|^2dx. \qquad (3.21)
\end{aligned}
$$

Using Hölder and Young inequalities, we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\Big(k\|\varphi_x+\psi+\ell w\|^2+k_0\|w_x-\ell\varphi\|^2+b\|\psi_x\|^2+\rho_1\|\varphi_t\|^2+\rho_2\|\psi_t\|^2+\rho_1\|w_t\|^2\\
&\quad+\sigma_1\|\theta^1\|^2+\sigma_2\|\theta^2\|^2+\sigma_3\|\theta^3\|^2+\mu\tau\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx\Big)\\
&\quad\le-\Big(\gamma_1\|\theta^1_x\|^2+\gamma_2\|\theta^2_x\|^2+\gamma_3\|\theta^3_x\|^2\Big)+\frac{\mu}{2}\int_0^L|z(1)|^2dx+\frac{\mu}{2}\int_0^L|\psi_t|^2dx\\
&\qquad+\frac{\mu}{2}\int_0^L|\psi_t(x,t)|^2dx-\frac{\mu}{2}\int_0^L|z(1)|^2dx\\
&\quad=-\Big(\gamma_1\|\theta^1_x\|^2+\gamma_2\|\theta^2_x\|^2+\gamma_3\|\theta^3_x\|^2\Big)+\mu\|\psi_t\|^2.
\end{aligned}
$$

Defining the linear energy as:

$$
\begin{aligned}
E(t)=&\frac{1}{2}\Big(k\|\varphi_x+\psi+\ell w\|^2+k_0\|w_x-\ell\varphi\|^2+b\|\psi_x\|^2+\rho_1\|\varphi_t\|^2+\rho_2\|\psi_t\|^2+\rho_1\|w_t\|^2\\
&+\sigma_1\|\theta^1\|^2+\sigma_2\|\theta^2\|^2+\sigma_3\|\theta^3\|^2+\mu\tau\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx\Big), \qquad (3.22)
\end{aligned}
$$

we arrive at:

$$
\frac{d}{dt}E(t)\le-\Big(\gamma_1\|\theta^1_x\|^2+\gamma_2\|\theta^2_x\|^2+\gamma_3\|\theta^3_x\|^2\Big)+\mu\|\psi_t\|^2. \qquad (3.23)
$$

Same as Chapter 2, we need to choose an appropriate space to energy make sense. First, we need to highlight the presence of a new variable within the energy, that is, $z(x, \rho, t)$. Since this variable tells that for each $\rho \in [0, 1]$, the function $z(x, \cdot, t)$ is in $L^2(0, L)$, the appropriate space for this variable is defined by:

$$L^2\big([0, 1]; L^2_*(0, L)\big)$$

On this form, and due to the boundary conditions given before, we define the phase space $\mathcal{H}$:

$$\mathcal{H} := H^1_0(0, L) \times (H^1_*(0, L))^2 \times L^2(0, L) \times (L^2_*(0, L))^2 \times L^2_*(0, L) \times (L^2(0, L))^2 \times L^2\big([0, 1]; L^2(0, L)\big)$$

where $L^2_*(0, L)$ and $H^1_*(0, L)$ are Banach spaces defined as:

$$\begin{aligned} L^2_*(0, L) &:= \{u \in L^2(0, L), \int_0^L u(x)dx = 0\}. \\ H^1_*(0, L) &:= H^1(0, L) \cap L^2_*(0, L). \end{aligned}$$

Thus, for a vector $y(t) = \big(\varphi(t), \psi(t), w(t), \varphi_t(t), \psi_t(t), w_t(t), \theta^1(t), \theta^2(t), \theta^3(t), z(\cdot, t)\big)$ in $\mathcal{H}$, the norm induced by this space is defined as:

$$\begin{aligned} \|y\|^2_{\mathcal{H}} := k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\ + \sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2 + \mu\tau \int_0^L \int_0^1 z^2(x, \rho, t)d\rho dx. \end{aligned} \tag{3.24}$$

Additionally, we define de usual norm by:

$$\begin{aligned} \|y\|^2 := \|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 + \|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2 + \|\theta^1\|^2 + \|\theta^2\|^2 + \|\theta^3\|^2 \\ + \int_0^L \int_0^1 z^2(x, \rho, t)d\rho dx. \end{aligned} \tag{3.25}$$

Lemma 3.1. *For each $\ell > 0$ with $\ell \neq \frac{n\pi}{L}$, norms (3.24) and (3.25) are equivalents, which constants η_1 and η_2 dependents of ℓ.*

From (3.24), we can easily deduce that:

$$E(t) = \frac{1}{2}\|y\|_{\mathcal{H}}^2.$$

An important observation about the estimate (3.23) is that we cannot conclude that energy is decreasing, because the presence of delay.
Thus, similar to [5], we introduce the vector function:

$$y(t) = \big(\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t), z(\cdot, t)\big)$$

where $\varphi' = \varphi_t$, $\psi' = \psi_t$ and $w' = w_t$.
In order to prove the existence of a solution for the problem (3.14)-(3.20), with boundary-initial conditions (3.7)-(3.9), and condition (3.13), we use the semigroup theory via Lumer-Phillips theorem, and construct an unbounded operator with the monotonicity property (see [5]). Unlike the Chapter 2, the presence of the delay does not allow to define an unbounded operator A such that it be monotone. Then, motivated by the argument explained by Nicasse and Pignotti ([28]), we add and subtract a frictional damping type term $\mu\psi'$ in the equation (3.15). Then, the system (3.14)-(3.20), with conditions (3.7)-(3.9), and n (3.13) can be rewritten as:

$$\frac{d}{dt}y(t) + [A+B]y(t) = 0,\ t > 0, \tag{3.26}$$

$$y(0) = y_0.$$

with $y_0 = \big(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \theta_0^1, \theta_0^2, \theta_0^3, g_0\big)$, and the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defined by:

$$A\begin{bmatrix}\varphi\\ \psi\\ w\\ \varphi'\\ \psi'\\ w'\\ \theta^1\\ \theta^2\\ \theta^3\\ z\end{bmatrix}=\begin{bmatrix}-\varphi'\\ -\psi'\\ -w'\\ -\Big(\frac{k}{\rho_1}(\varphi_x+\psi+\ell w)_x+\frac{k_0\ell}{\rho_1}(w_x-\ell\varphi)-\frac{m_1}{\rho_1}\theta^1_x-\frac{m_2\ell}{\rho_1}\theta^2\Big)\\ -\Big(\frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi+\ell w)-\frac{m_3}{\rho_2}\theta^3_x+\frac{m_1}{\rho_2}\theta^1\Big)+\frac{\mu}{\rho_2}\psi'+\frac{\mu}{\rho_2}z(1)\\ -\Big(\frac{k_0}{\rho_1}(w_x-\ell\varphi)_x-\frac{k\ell}{\rho_1}(\varphi_x+\psi+\ell w)-\frac{m_2}{\rho_1}\theta^2_x+\frac{\ell m_1}{\rho_1}\theta^1\Big)\\ -\Big(\frac{\gamma_1}{\sigma_1}\theta^1_{xx}-\frac{m_1}{\sigma_1}(\varphi'_x+\psi'+\ell w')\Big)\\ -\Big(\frac{\gamma_2}{\sigma_2}\theta^2_{xx}-\frac{m_2}{\sigma_2}(w'_x-\ell\varphi')\Big)\\ -\Big(\frac{\gamma_3}{\sigma_3}\theta^3_{xx}-\frac{m_3}{\sigma_3}(\psi'_x)\Big)\\ \frac{1}{\tau}z_\rho\end{bmatrix}$$

and the operator $B : D(B) = \mathcal{H} \to \mathcal{H}$ is defined by:

$$By(t)=\frac{\mu}{\rho_2}\begin{bmatrix}0\\0\\0\\0\\-\psi'\\0\\0\\0\\0\\0\end{bmatrix}.$$

The domain of A is defined by

$$\begin{aligned}D(A)=\Big\{&y\in\mathcal{H}|\ \varphi\in H^2(0,L)\cap H^1_0(0,L);\ \psi,w\in H^2(0,L)\cap H^1_*(0,L),\\ &\varphi',\theta^2,\theta^3\in H^1_0(0,L);\ \psi',w',\theta^1\in H^1_*(0,L)\\ &z,z_\rho\in L^2((0,1);L^2(0,L)),\ z(x,0)=\psi'(x)\Big\}.\end{aligned}$$

From [5], $D(A)$ is dense in $\mathcal{H}$. Therefore, we have the following existence and uniqueness result:

Theorem 3.1. *Let $y_0 \in \mathcal{H}$. Then there exists a local solution $y \in C([0, T_{max}]; \mathcal{H})$ for the Cauchy problem (3.26). Moreover, if $y_0 \in D(A)$, then $y \in C^1([0, T_{max}]; D(A))$, with $T_{max} < \infty$.*

Proof: We use the semigroup theory using the characterization of maximal monotone operators [27]. So, we prove that A is a maximal monotone operator and that B is a Lipschitz continuous operator. In what follows, we show that A is monotone. In that way, we use the information about the unbounded operator from the last chapter to see that:

$$\langle Ay, y\rangle = \gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2 + \mu\int_0^L |\psi'|^2 dx + \mu\int_0^L z(1)\cdot\psi' dx + \mu\int_0^L\int_0^1 zz_\rho d\rho dx.$$

We use the Young's inequality to show that the fifth term in the last identity gives

$$-\mu\int_0^L z(1)\cdot\psi' dx \le \frac{\mu}{2}\int_0^L z^2(1)dx + \frac{\mu}{2}\int_0^L (\psi')^2 dx,$$

which implies the following inequality

$$\mu\int_0^L z(1)\cdot\psi' dx \ge -\frac{\mu}{2}\int_0^L z^2(1)dx - \frac{\mu}{2}\int_0^L (\psi')^2 dx.$$

For the last term, we use integration by parts and the fact that $z(x, 0, t) = \psi'(x, t)$ to deduce that:

$$\int_0^L\int_0^1 zz_\rho d\rho dx = \frac{1}{2}\int_0^L z^2(1)dx - \frac{1}{2}\int_0^L (\psi')^2 dx$$

Consequently, $\langle Ay, y\rangle$ is estimated by:

$$\langle Ay, y\rangle \ge \gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2 \ge 0. \tag{3.27}$$

Therefore, A is monotone. Next, we prove that operator $I + A$ is surjective, where $I : \mathcal{H} \to \mathcal{H}$ means the identity operator. In other words, given $G = (g_1, ..., g_{10}) \in \mathcal{H}$ we prove that there exists $y \in D(A)$ satisfying:

$$(I + A)y = G, \tag{3.28}$$

That is:

$$-\varphi' + \varphi = g_1 \tag{3.29}$$

$$-\psi' + \psi = g_2 \tag{3.30}$$

$$-w' + w = g_3 \tag{3.31}$$

$$-\Big(\frac{k}{\rho_1}(\varphi_x + \psi + \ell w)_x + \frac{k_0 \ell}{\rho_1}(w_x - \ell\varphi) - \frac{m_1}{\rho_1}\theta^1_x - \frac{m_2 \ell}{\rho_1}\theta^2\Big) + \varphi' = g_4 \tag{3.32}$$

$$-\Big(\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi + \ell w) - \frac{m_3}{\rho_2}\theta^3_x + \frac{m_1}{\rho_2}\theta^1\Big) + \frac{\mu}{\rho_2}\psi' + \frac{\mu}{\rho_2}z(1) + \psi' = g_5 \tag{3.33}$$

$$-\Big(\frac{k_0}{\rho_1}(w_x - \ell\varphi)_x - \frac{\ell k}{\rho_1}(\varphi_x + \psi + \ell w) - \frac{m_2}{\rho_1}\theta^2_x + \frac{\ell m_1}{\rho_1}\theta^1\Big) + w' = g_6 \tag{3.34}$$

$$-\Big(\frac{\gamma_1}{\sigma_1}\theta^1_{xx} - \frac{m_1}{\sigma_1}(\varphi'_x + \psi' + \ell w')\Big) + \theta^1 = g_7 \tag{3.35}$$

$$-\Big(\frac{\gamma_2}{\sigma_2}\theta^2_{xx} - \frac{m_2}{\sigma_2}(w'_x - \ell\varphi')\Big) + \theta^2 = g_8 \tag{3.36}$$

$$-\Big(\frac{\gamma_3}{\sigma_3}\theta^3_{xx} - \frac{m_3}{\sigma_3}(\psi'_x)\Big) + \theta^3 = g_9 \tag{3.37}$$

$$\frac{1}{\tau}z_\rho + z = g_{10}. \tag{3.38}$$

Following a similar procedure as [5], we start the proof assuming $\varphi, \psi, w, \theta^1, \theta^2$ and θ^3 are given with the appropriate regularity. Thus, from equations (3.29),(3.30) and (3.31), we obtain:

$$\varphi' = \varphi - g_1 \in H^1_0(0, L), \tag{3.39}$$

$$\psi' = \psi - g_2 \in H^1_*(0, L), \tag{3.40}$$

$$w' = w - g_3 \in H^1_*(0, L). \tag{3.41}$$

The tenth equation (3.38) together with the (3.40) and the fact that $z(x,0)=\psi'$, give us :

$$z(x,\rho)=\psi(x)e^{-\tau\rho}-e^{-\tau\rho}g_2(x)+\tau e^{-\tau\rho}\int_0^{\rho}e^{\tau s}g_{10}(x,s)dx\in L^2_*(0,L). \tag{3.42}$$

Then, we use the above results to reduce our system to:

$$\begin{aligned}
-\Big(\frac{k}{\rho_1}(\varphi_x+\psi+\ell w)_x+\frac{k_0\ell}{\rho_1}(w_x-\ell\varphi)-\frac{m_1}{\rho_1}\theta^1_x-\frac{\ell m_2}{\rho_1}\theta^2\Big)+\varphi&=g_4+g_1,\\
-\Big(\frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi+\ell w)-\frac{m_3}{\rho_2}\theta^3_x+\frac{m_1}{\rho_2}\theta^1\Big)+\frac{\mu}{\rho_2}\psi-\frac{\mu}{\rho_2}g_2+\frac{\mu}{\rho_2}z(1)+\psi&=g_5+g_2,\\
-\Big(\frac{k_0}{\rho_1}(w_x-\ell\varphi)_x-\frac{\ell k}{\rho_1}(\varphi_x+\psi+\ell w)-\frac{m_2}{\rho_1}\theta^2_x+\frac{\ell m_1}{\rho_1}\theta^1\Big)+w&=g_6+g_3,\\
-\Big(\frac{\gamma_1}{\sigma_1}\theta^1_{xx}-\frac{m_1}{\sigma_1}(\varphi_x+\psi+\ell w)\Big)-\frac{m_1}{\sigma_1}\big(g_{1,x}+g_2+\ell g_3\big)+\theta^1&=g_7,\\
-\Big(\frac{\gamma_2}{\sigma_2}\theta^2_{xx}-\frac{m_2}{\sigma_2}(w_x-\ell\varphi)\Big)-\frac{m_2}{\sigma_2}\big(g_{3,x}-\ell g_1\big)+\theta^2&=g_8,\\
-\Big(\frac{\gamma_3}{\sigma_3}\theta^3_{xx}-\frac{m_3}{\sigma_3}(\psi_x)\Big)-\frac{m_3}{\sigma_3}g_{2,x}+\theta^3&=g_9.
\end{aligned}$$

Thus, $\varphi,\psi,w,\theta^1,\theta^2$ and θ^3 satisfying the following system:

$$\begin{aligned}
-k(\varphi_x+\psi+\ell w)_x-k_0\ell(w_x-\ell\varphi)+m_1\theta^1_x+\ell m_2\theta^2+\rho_1\varphi&=h_1\in L^2(0,L),\\
-b\psi_{xx}+k(\varphi_x+\psi+\ell w)+m_3\theta^3_x-m_1\theta^1+\tilde{\mu}\psi&=h_2\in L^2_*(0,L),\\
-k_0(w_x-\ell\varphi)_x+k\ell(\varphi_x+\psi+\ell w)+m_2\theta^2_x-\ell m_1\theta^1+\rho_1 w&=h_3\in L^2_*(0,L),\\
-\gamma_1\theta^1_{xx}+m_1(\varphi_x+\psi+\ell w)+\sigma_1\theta^1&=h_4\in L^2_*(0,L),\\
-\gamma_2\theta^2_{xx}+m_2(w_x-\ell\varphi)+\sigma_2\theta^2&=h_5\in L^2(0,L),\\
-\gamma_3\theta^3_{xx}+m_3(\psi_x)+\sigma_3\theta^3&=h_6\in L^2(0,L),
\end{aligned}\tag{3.43}$$

where

$$
\begin{aligned}
\tilde{\mu} &= \mu + \rho_2 + \mu e^{-\tau} \\
h_1 &= \rho_1(g_4 + g_1) \\
h_2 &= \tilde{\mu} g_2 + \rho_2 g_5 - \mu\tau e^{-\tau}\int_0^1 e^{-\tau s} g_{10} ds \\
h_3 &= \rho_1(g_6 + g_3) \\
h_4 &= \sigma_1 g_7 + m_1(g_{1,x} + g_2 + \ell g_3) \\
h_5 &= \sigma_2 g_8 + m_2(g_{3,x} - \ell g_1) \\
h_6 &= \sigma_2 g_9 + m_3(g_{2,x}).
\end{aligned}
$$

The variational formulation corresponding to the above system takes the form:

$$
\mathcal{B}(v, \tilde{v}) = \mathbf{F}(\tilde{v}). \tag{3.44}
$$

where $v = (\varphi, \psi, w, \theta^1, \theta^2, \theta^3)$, and $\tilde{v} = (\tilde{\varphi}, \tilde{\psi}, \tilde{w}, \tilde{\theta^1}, \tilde{\theta^2}, \tilde{\theta^3})$, $\mathcal{B}$ is the bilinear form defined by:

$$
\mathcal{B} : \Big[H_0^1(0, L) \times H_*^1(0, L) \times H_*^1(0, L) \times L_*^2(0, L) \times L^2(0, L) \times L^2(0, L) \Big] \longrightarrow \mathbb{R}
$$

$$
\begin{aligned}
\mathcal{B} :=& k \int_0^L (\varphi_x + \psi + w)(\tilde{\varphi}_x + \tilde{\psi} + \tilde{w}) dx + k_0 \int_0^L (w_x - \ell\varphi)(\tilde{w}_x - \ell\tilde{\varphi}) dx + b \int_0^L \psi_x \tilde{\psi}_x dx \\
&+ \rho_1 \int_0^L \varphi\tilde{\varphi} dx + \tilde{\mu} \int_0^L \psi\tilde{\psi} dx + \rho_1 \int_0^L w\tilde{w} dx + \sigma_1 \int_0^L \theta^1 \tilde{\theta}^1 dx + \sigma_2 \int_0^L \theta^2 \tilde{\theta}^2 dx \\
&+ \sigma_3 \int_0^L \theta^3 \tilde{\theta}^3 dx + \gamma_1 \int_0^L \theta_x^1 \tilde{\theta}_x^1 dx + \gamma_2 \int_0^L \theta_x^2 \tilde{\theta}_x^2 dx + \gamma_3 \int_0^L \theta_x^3 \tilde{\theta}_x^3 dx \\
&- m_1 \int_0^L \theta^1 (\tilde{\varphi}_x + \tilde{\psi} + \tilde{w}) dx - m_2 \int_0^L \theta^2 (\tilde{w}_x - \ell\tilde{\varphi}) dx - m_3 \int_0^L \theta^3 \tilde{\psi}_x dx \\
&+ m_1 \int_0^L (\varphi_x + \psi + w)\tilde{\theta}^1 dx + m_2 \int_0^L (w_x - \ell\varphi)\tilde{\theta}^2 dx + m_3 \int_0^L \psi_x \tilde{\theta}^3 dx.
\end{aligned}
$$

and $\mathbf{F}$: $\Big[H_0^1(0,L) \times H_*^1(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times L^2(0,L) \times L^2(0,L)\Big] \longrightarrow \mathbb{R}$ is a functional satisfying:

$$\mathbf{F}(\tilde{v}) = \int_0^L h_1\tilde{\varphi}dx + \int_0^L h_2\tilde{\psi}dx + \int_0^L h_3\tilde{w}dx + \int_0^L h_4\tilde{\theta}^1dx + \int_0^L h_5\tilde{\theta}^2dx + \int_0^L h_6\tilde{\theta}^3dx.$$

The norm for $V := H_0^1(0,L) \times H_*^1(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times L^2(0,L) \times L^2(0,L)$, is given by:

$$\|v\|_V^2 = \|\varphi_x + \psi + w\|^2 + \|\psi_x\|^2 + \|w_x - \ell\varphi\|^2 + \|\theta_x^1\|^2 + \|\theta_x^2\|^2 + \|\theta_x^3\|^2.$$

Thus, if $\ell \neq \frac{n\pi}{L}$, and using Holder,Young and Poincare's inequalities, one can see easily that $\mathcal{B}$ and $\mathbf{F}$ are bounded. Also, we see that:

$$\begin{aligned}
B(v,v) =& k\int_0^L |\varphi_x + \psi + w|^2dx + k_0\int_0^L |w_x - \ell\varphi|^2dx + b\int_0^L |\psi_x|^2dx \\
&+ \rho_1\int_0^L |\varphi|^2dx + \tilde{\mu}\int_0^L |\psi|^2dx + \rho_1\int_0^L |w|^2dx \\
&+ \sigma_1\int_0^L |\theta^1|^2dx + \sigma_2\int_0^L |\theta^2|^2dx + \sigma_3\int_0^L |\theta^3|^2dx \\
&+ \gamma_1\int_0^L |\theta_x^1|^2dx + \gamma_2\int_0^L |\theta_x^2|^2dx + \gamma_3\int_0^L |\theta_x^3|^2dx \geq C\|v\|_V^2.
\end{aligned}$$

Thus, $\mathcal{B}$ is coercive. Consequently, by Lax-Milgram theorem, the system (3.43) has a unique solution $v \in V$. Moreover, by elliptic regularity (see [7]), we can show additionally that $y \in D(A)$. Therefore, A is maximal. With this, we conclude that A is an maximal monotone operator. On the other hand, by definition of the operator B, it is Lipschitz continuous. Finally, by the results in [27, 30], the operator A is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. Hence, the existence of a local solution is guaranteed ([5]). ■

In order to proof the existence for a global solution, we need to show that the behavior of the local solution $y(t)$, for t large, is controlled. More specifically, we will proof the exponential stability of the solution, so that the solution is limited and thus, $T_{max} = +\infty$.

3.3 Global solution and exponential stability

In this section we discuss the asymptotic behavior of the solution for system (3.14)-(3.20). Remembering that he energy functional associated to this system was defined by:

$$E(t) = \frac{1}{2}\Big(k\|\varphi_x + \psi + \ell w\|^2 + k_0\|w_x - \ell\varphi\|^2 + b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2$$
$$+\sigma_1\|\theta^1\|^2 + \sigma_2\|\theta^2\|^2 + \sigma_3\|\theta^3\|^2 + \mu\tau\int_0^L\int_0^1 z^2(x,\rho,t)d\rho dx\Big).$$

An important remark about the energy is the fact that, by previous computations, is not decreasing at all, that is:

$$E'(t) \leq -\gamma_1\|\theta_x^1\|^2 - \gamma_2\|\theta_x^2\|^2 - \gamma_3\|\theta_x^3\|^2 + \mu\|\psi_t\|^2 \nleq 0.$$

However, we can state a stability result through the following theorem:

Theorem 3.2. *Let $y(t)$ a solution for the system (3.14)-(3.20), with conditions* (3.7)-(3.9) *and* (3.13)*, and assume that $\ell \neq \frac{n\pi}{L}$. For $\ell > 0$ small enough, the solution is global and we have that:*

$$E(t) \leq KE(0)e^{-\alpha t}, \qquad \forall t \geq 0, \tag{3.45}$$

where α, K are two positive constants.

To establish the proof of Theorem 3.2, we will make use of the multiplier method, so we need several lemmas.

Lemma 3.2. *Let $y(t)$ be the solution of (3.26). Then the energy functional, defined by (3.22), satisfies*

$$\frac{d}{dt}E(t) \leq -\Big(\gamma_1\|\theta_x^1\|^2 + \gamma_2\|\theta_x^2\|^2 + \gamma_3\|\theta_x^3\|^2\Big) + \mu\|\psi_t\|^2, \tag{3.46}$$

for all $t \in [0, T_{max})$.

Proof: Immediately from (3.23).

Remark: We see from the last inequality that energy is non decreasing in general. Thus, the system (3.26) is not necessarily dissipative.

Lemma 3.3. *Consider $y(t)$ be the solution of (3.26). Then the functional*

$$F_1(t) = \int_0^L (\rho_1 \varphi\varphi_t + \rho_2 \psi\psi_t + \rho_1 w w_t)\, dx$$

satisfies, for all $t \in [0, T_{max})$, the following estimate:

$$\begin{aligned} F_1'(t) \leq &- \frac{k}{2}\|\varphi_x + \psi + \ell w\|^2 - \frac{k_0}{2}\|w_x - \ell\varphi\|^2 - \frac{b}{2}\|\psi_x\|^2 \\ &+ \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 + c_1\|\theta_x^1\|^2 + c_1\|\theta_x^2\|^2 + c_1\|\theta_x^3\|^2 + c_1\mu^2\|z(1)\|^2. \end{aligned} \tag{3.47}$$

Proof: Taking the derivative in t for $F_1(t)$, and using (3.14), (3.15) and (3.16), we see that:

$$\begin{aligned} F_1'(t) = &\rho_1 \int_0^L |\varphi_t|^2 dx + \rho_2 \int_0^L |\psi_t|^2 dx + \rho_1 \int_0^L |w_t|^2 dx \\ &+ k \int_0^L (\varphi_x + \psi + \ell w)_x \varphi dx + k_0 \int_0^L (w_x - \ell\varphi)\varphi dx - m_1 \int_0^L \theta_x^1 \varphi dx + \ell m_2 \int_0^L \theta^2 \varphi dx \\ &+ b \int_0^L \psi_{xx}\psi dx - k \int_0^L (\varphi_x + \psi + \ell w)\psi dx - m_2 \int_0^L \theta_x^3 \psi dx + m_1 \int_0^L \theta^1 \psi dx - \mu \int_0^L z(1)\psi dx \\ &+ k_0 \int_0^L (w_x - \ell\varphi)_x w dx - k\ell \int_0^L (\varphi_x + \psi + \ell w) w dx - m_2 \int_0^L \theta_x^2 w dx + \ell m_2 \int_0^L \theta^1 w dx \\ = &- k\|\varphi_x + \psi + \ell w\|^2 - k_0\|w_x - \ell\varphi\|^2 - b\|\psi_x\|^2 + \rho_1 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\ &- m_1 \int_0^L \theta_x^1 \varphi dx + m_1 \int_0^L \theta^1 \psi dx + m_1 \int_0^L \theta^1 (\ell w) dx \\ &- m_2 \int_0^L \theta_x^2 w dx + \ell m_2 \int_0^L \theta^2 \varphi dx - m_3 \int_0^L \theta_x^3 \psi dx - \mu \int_0^L z(1)\psi dx. \end{aligned}$$

Here, we use integration by parts, boundary conditions (3.7) and (3.8), and several inequalities, to show that:

$$\begin{aligned}
F_1'(t) = & -k\|\varphi_x+\psi+\ell w\|^2 - k_0\|w_x-\ell\varphi\|^2 - b\|\psi_x\|^2 + \rho_1 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\
& + m_1\int_0^L \theta^1(\varphi_x+\psi+\ell w)\,dx + m_2\int_0^L \theta^2(w_x-\ell\varphi)dx + m_3\int_0^L \theta^3\psi_x dx - \mu\int_0^L z(1)\psi dx. \\
\leq & -k\|\varphi_x+\psi+\ell w\|^2 - k_0\|w_x-\ell\varphi\|^2 - b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 \\
& + \frac{k}{2}\|\varphi_x+\psi+\ell w\|^2 + \frac{k_0}{2}\|w_x-\ell\varphi\|^2 + \frac{b}{4}\|\psi_x\|^2 + c_1\mu^2\|z(1)\|^2 + \frac{b}{4}\|\psi_x\|^2 \\
& + c_1\|\theta_x^1\|^2 + c_1\|\theta_x^2\|^2 + c_1\|\theta_x^3\|^2.
\end{aligned}$$

Thus:

$$\boxed{\begin{aligned}
\boldsymbol{F_1'(t) \leq -\frac{k}{2}\|\varphi_x+\psi+\ell w\|^2 - \frac{k_0}{2}\|w_x-\ell\varphi\|^2 - \frac{b}{2}\|\psi_x\|^2} \\
+ \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2 + \rho_1\|w_t\|^2 + c_1\|\theta_x^1\|^2 + c_1\|\theta_x^2\|^2 + c_1\|\theta_x^3\|^2 + c_1\mu^2\|z(1)\|^2.
\end{aligned}}$$

where c_1 is a constant that have no relevance in the estimates. Terms in bold are important and their importance will be explained in detail later.

Lemma 3.4. *Consider $y(t)$ a solution from Cauchy problem (3.26), and define:*

$$F_2(t) = -\frac{\rho_1\sigma_2}{m_2}\int_0^L \theta^2\Big(\int_0^x w_t dy\Big)dx.$$

Thus, the functional satisfies, for all $\varepsilon_2, \bar{\varepsilon}_2 > 0$, and $t \in [0, T_{max})$, the following estimate:

$$\begin{aligned}
F_2'(t) \leq & -\frac{\rho_1}{2}\|w_t\|^2 + \rho_1 L\ell^2\|\varphi_t\|^2 + \varepsilon_2\|w_x-\ell\varphi\|^2 + \overline{\varepsilon_2}\|\varphi_x+\psi+\ell w\|^2 \\
& + c_2\Big(1+\frac{1}{\varepsilon_2}+\frac{1}{\bar{\varepsilon_2}}\Big)\|\theta_x^2\|^2 + c_2\|\theta_x^1\|^2.
\end{aligned} \tag{3.48}$$

Proof: Using equations (3.16) and (3.18), and integration by parts, we have that:

$$
\begin{aligned}
F_2'(t) &= \frac{\rho_1}{m_2}\int_0^L (-\sigma_2\theta_t^2)\Big(\int_0^x w_t dy\Big)dx + \frac{\sigma_2}{m_2}\int_0^L \theta^2\Big(\int_0^x (-\rho_1 w_{tt})dy\Big)dx \\
&= -\frac{\rho_1\gamma_2}{m_2}\int_0^L \theta_{xx}^2\int_0^x w_t dy dx + \frac{\rho_1 m_2}{m_2}\int_0^L (w_x-\ell\varphi)_t\Big(\int_0^x w_t dy\Big)dx \\
&\quad - \frac{\sigma_2 k_0}{m_2}\int_0^L \theta^2\Big(\int_0^x (w_x-\ell\varphi)_x dy\Big)dx + \frac{\sigma_2}{m_2}\int_0^L \theta^2\Big(\int_0^x k\ell(\varphi_x+\psi+\ell w)dy\Big)dx \\
&\quad + \frac{\sigma_2 m_2}{m_2}\int_0^L \theta^1\Big(\int_0^x \theta_x^2 dy\Big)dx - \frac{\sigma_2 m_1\ell}{m_2}\int_0^L \theta^1\Big(\int_0^x \theta^1 dy\Big)dx \\
&= \frac{\rho_1\gamma_2}{m_2}\int_0^L \theta_x^2 w_t dx + \boldsymbol{\rho_1\int_0^L w_{xt}\Big(\int_0^x w_t dy\Big)dx} - \rho_1\ell\int_0^L \varphi_t\Big(\int_0^x w_t dy\Big)dx \\
&\quad - \frac{\sigma_2 k_0}{m_2}\int_0^L \theta^2(w_x-\ell\varphi)dx + \frac{\sigma_2 k\ell}{m_2}\int_0^L \theta^2\Big(\int_0^x (\varphi_x+\psi+\ell w)dy\Big)dx \\
&\quad + \frac{\sigma_2 m_2}{m_2}\int_0^L \theta^1\Big(\int_0^x \theta_x^2 dy\Big)dx - \frac{\sigma_2 m_1\ell}{m_2}\int_0^L \theta^1\Big(\int_0^x \theta^1 dy\Big)dx
\end{aligned}
$$

Here, we use again several computations and Young's inequalities with ε_2 and $\bar{\varepsilon_2}$, and imply that:

$$
\begin{aligned}
F_2'(t) &\leq c_2\|\theta_x^2\|^2 + \frac{\rho_1}{4}\|w_t\|^2 \boldsymbol{- \rho_1\|w_t\|^2} + \frac{\rho_1}{4}\|w_t\|^2 + \rho_1 L\ell^2\|\varphi_t\|^2 + \varepsilon_2\|w_x-\ell\varphi\|^2 \\
&\quad + \frac{c_2}{\varepsilon_2}\|\theta_x^2\|^2 + \bar{\varepsilon_2}\|\varphi_x+\psi+\ell w\|^2 + \frac{c_2}{\bar{\varepsilon_2}} + c_2\|\theta_x^1\|^2 + c_2\|\theta_x^2\|^2
\end{aligned}
$$

Arranging the terms of the inequality, we have:

$$
\boxed{
\begin{aligned}
\boldsymbol{F_2'(t)} &\leq \boldsymbol{-\frac{\rho_1}{2}\|w_t\|^2} + \rho_1 L\ell^2\|\varphi_t\|^2 + \varepsilon_2\|w_x-\ell\varphi\|^2 + \overline{\varepsilon_2}\|\varphi_x+\psi+\ell w\|^2 \\
&\quad + c_2\Big(1+\frac{1}{\varepsilon_2}+\frac{1}{\overline{\varepsilon_2}}\Big)\|\theta_x^2\|^2 + c_2\|\theta_x^1\|^2
\end{aligned}
}
$$

where c_2 is a general constant.

Lemma 3.5. *Consider $y(t)$ be the solution of (3.26). Then the functional:*

$$
F_3(t) = -\frac{\sigma_3\rho_2}{m_3}\int_0^L \theta^3\Big(\int_0^x \psi_t dy\Big)dx
$$

satisfies, for all $\varepsilon_2, \bar{\varepsilon}_2 > 0$, and $t \in [0, T_{max})$, the following estimate:

$$\begin{aligned} F_3'(t) \leq & -\frac{\rho_2}{2}\|\psi_t\|^2 + \varepsilon_3\|\psi_x\|^2 + \overline{\varepsilon_3}\|\varphi_x + \psi + \ell w\|^2 + \frac{\mu^2}{2}\|z(1)\|^2 \\ & + c_3\|\theta_x^1\|^2 + c_3\Big(1 + \frac{1}{\varepsilon_3} + \frac{1}{\overline{\varepsilon_3}}\Big)\|\theta_x^3\|^2. \end{aligned} \tag{3.49}$$

where c_3 is a general constant that not depends on ℓ.

Proof: We start the proof with a derivation in t the functional $F_3(t)$, obtaining:

$$\begin{aligned} F_3'(t) = & -\frac{\rho_2}{m_3}\sigma_3\int_0^L \theta_t^3\Big(\int_0^x \psi_t dy\Big)dx - \frac{\sigma_3}{m_3}\rho_2\int_0^L \theta^3\Big(\int_0^x \psi_{tt} dy\Big)dx \\ = & \ \ \frac{\rho_2}{m_2}\int_0^L (-\gamma_3\theta_{xx}^3 + m_3\psi_{xt})\Big(\int_0^x \psi_t dy\Big)dx \\ & + \frac{\sigma_3}{m_3}\int_0^L \theta^3\Big(\int_0^x (-b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3 x - m_1\theta^1 + \mu z(1))dy\Big)dx \\ = & -\frac{\rho_2\gamma_3}{m_3}\int_0^L \theta_{xx}^3(\int_0^x \psi_t dy)dx + \boldsymbol{\rho_2\int_0^L \psi_{xt}\Big(\int_0^x \psi_t dy\Big)dx} - \frac{\sigma_3 b}{m_3}\int_0^L \theta^3\Big(\int_0^x \psi_{xx} dy\Big)dx \\ & + \frac{k\sigma_3}{m_3}\int_0^L \theta^3\Big(\int_0^x (\varphi_x + \psi + \ell w)dy\Big)dx + \frac{\sigma_3 m_3}{m_3}\int_0^L \theta^3\Big(\int_0^x \theta_x^3 dy\Big)dx \\ & - \frac{\sigma_3 m_1}{m_3}\int_0^L \theta^1\Big(\int_0^x \theta^3 dy\Big)dx + \frac{\sigma_3}{m_3}\mu\int_0^L \theta^3\Big(\int_0^x z(1)dy\Big)dx \\ = & +\frac{\rho_2\gamma_3}{m_3}\int_0^L \theta_x^3\psi_t dx - \boldsymbol{\rho_2\int_0^L \psi_t\psi_t dx} - \frac{\sigma_3 b}{m_3}\int_0^L \theta^3\psi_x dx \\ & + \frac{k\sigma_3}{m_3}\int_0^L \theta^3\Big(\int_0^x (\varphi_x + \psi + \ell w)dy\Big)dx + \frac{\sigma_3 m_3}{m_3}\int_0^L \theta^3\Big(\int_0^x \theta_x^3 dy\Big)dx \\ & - \frac{\sigma_3 m_1}{m_3}\int_0^L \theta^1\Big(\int_0^x \theta^3 dy\Big)dx + \frac{\sigma_3}{m_3}\mu\int_0^L \theta^3\Big(\int_0^x z(1)dy\Big)dx \end{aligned}$$

and again, using the known inequalities, we have that:

$$\begin{aligned} F_3'(t) \leq \ & c_2\|\theta_x^3\|^2 + \frac{\rho_2}{2}\|\psi_t\|^2 \boldsymbol{- \rho_2\|\psi_t\|^2} + \frac{c_3}{\varepsilon_3}\|\theta_x^3\|^2 + \varepsilon_3\|\psi_x\|^2 + \frac{c_3}{\overline{\varepsilon_3}}\|\theta_x^3\|^2 + \overline{\varepsilon_3}\|\varphi_x + \psi + \ell w\|^2 \\ & + c_3\|\theta_x^3\|^2 + c_3\|\theta_x^1\|^2 + \frac{\mu^2}{2}\|z(1)\|^2 \end{aligned}$$

Therefore:

$$\boldsymbol{F_3'(t) \leq -\frac{\rho_2}{2}\|\psi_t\|^2} + \varepsilon_3\|\psi_x\|^2 + \overline{\varepsilon_3}\|\varphi_x + \psi + \ell w\|^2 + \frac{\mu^2}{2}\|z(1)\|^2 + c_3\|\theta_x^1\|^2 + c_3\Big(1 + \frac{1}{\varepsilon_3} + \frac{1}{\overline{\varepsilon_3}}\Big)\|\theta_x^3\|^2$$

where $c_3 > 0$ is a general constant.

Lemma 3.6. *Consider, for a solution $y(t)$, the following functional:*

$$F_4(t) = -\frac{\sigma_1\rho_1}{m_1}\int_0^L \varphi_t\Big(\int_0^x \theta^1 dy\Big)dx$$

Then, this functional, satisfies the next estimate:

$$F_4'(t) \leq \frac{\rho_1}{2}\|\varphi_t\|^2 + \rho_1 L\ell^2\|w_t\|^2 + \varepsilon_4\|w_x - \ell\varphi\|^2 + \bar{\varepsilon_4}\|\varphi_x + \psi + \ell w\|^2 + c_4(1 + \frac{1}{\varepsilon_4} + \frac{1}{\bar{\varepsilon_4}})\|\theta_x^1\|^2 + c_4\|\theta_x^2\|^2, \tag{3.50}$$

for every ε_4, $\bar{\varepsilon_4} > 0$ and $t \in [0, T_{max})$.

Proof: Lets start the demonstration using the equations (3.14) and (3.17), and we use similar computations to previous lemmas:

$$\begin{aligned}F_4'(t) =& \frac{\sigma_1\rho_1}{m_1}\int_0^L \varphi_{tt}\Big(\int_0^x \theta^1 dy\Big)dx + \frac{\sigma_1\rho_1}{m_1}\int_0^L \varphi_t\Big(\int_0^x \theta_t^1 dy\Big)dx\\ =& \frac{\sigma_1}{m_1}\int_0^L \big(k(\varphi_x + \psi + \ell w)_x + k_0\ell(w_x - \ell\varphi) - m_1\theta_x^1 - \ell m_2\theta^2\big)\Big(\int_0^x \theta^1 dy\Big)dx\\ &+ \frac{\rho_1}{m_1}\int_0^L \varphi_t\Big(\int_0^x (\gamma_1\theta_{xx}^1 - m_1(\varphi_x + \psi + \ell w)_t)dy\Big)dx\end{aligned}$$

Using Holder, Poincare's and Young's inequalities with ε_4 and $\bar{\varepsilon_4}$, we infer that:

$$\begin{aligned}
F_4'(t) \leq & \varepsilon_4\|\varphi_x+\psi+\ell w\|^2+\bar{\varepsilon_4}\|w_x-\ell\varphi\|^2++c_4(1+\frac{1}{\varepsilon_4}+\frac{1}{\bar{\varepsilon_4}})\|\theta_x^1\|^2+c_4\|\theta_x^2\|^2 \\
& +\frac{\rho_1}{m_1}\int_0^L\varphi_t\Big(\int_0^x\big(\gamma_1\theta_{xx}^1-m_1(\varphi_x+\psi+\ell w)_t\big)dy\Big)dx \\
\leq & \varepsilon_4\|\varphi_x+\psi+\ell w\|^2+\bar{\varepsilon_4}\|w_x-\ell\varphi\|^2++c_4(1+\frac{1}{\varepsilon_4}+\frac{1}{\bar{\varepsilon_4}})\|\theta_x^1\|^2+c_4\|\theta_x^2\|^2 \\
& +\frac{\gamma_1\rho_1}{m_1}\int_0^L\varphi_t\theta_x^1dx-\rho_1\|\varphi_t\|^2-\rho_1\int_0^L\varphi_t\big(\int_0^x\psi_tdy\big)dx-\rho_1\ell\int_0^L\varphi_t\big(\int_0^x w_tdy\big)dx
\end{aligned}$$

Thus, we deduce that:

$$\begin{aligned}
F_4'(t) \leq & \varepsilon_4\|\varphi_x+\psi+\ell w\|^2+\bar{\varepsilon_4}\|w_x-\ell\varphi\|^2++c_4(1+\frac{1}{\varepsilon_4}+\frac{1}{\bar{\varepsilon_4}})\|\theta_x^1\|^2+c_4\|\theta_x^2\|^2 \\
& -\frac{\rho_1}{2}\|\varphi_t\|^2+c_4\|\psi_t\|^2+\rho_1L\ell^2\|w_t\|^2
\end{aligned}$$

and therefore, arranging the terms, we conclude that:

$$\begin{aligned}
\boldsymbol{F_4'(t)} \leq & -\boldsymbol{\frac{\rho_1}{2}\|\varphi_t\|^2}+\rho_1L\ell^2\|w_t\|^2+c_4\|\psi_t\|^2+\varepsilon_4\|\varphi_x+\psi+\ell w\|^2+\bar{\varepsilon_4}\|w_x-\ell\varphi\|^2 \\
& +c_4(1+\frac{1}{\varepsilon_4}+\frac{1}{\bar{\varepsilon_4}})\|\theta_x^1\|^2+c_4\|\theta_x^2\|^2.
\end{aligned}$$

where c_4 is a general constant.

Lemma 3.7. *Let $y(t)$, a solution for the Linear Cauchy problem (3.26). The following functional:*

$$F_5(t)=\tau\int_0^L\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx$$

defined for every $t\in[0,T_{max})$ satisfies the following estimate.

$$F_5'(t)\leq -m_0\Big(\|z(1)\|^2+\tau\|z\|_\rho^2\Big)+\|\psi_t\|^2, \tag{3.51}$$

for every $m_0 > 0$ *and* $t \geq 0$.

Proof: For this functional, we make use of equation (3.20) and the fact that $z(0) = \psi_t$, to obtain:

$$\begin{aligned}
F_5'(t) &= 2\tau \int_0^L \int_0^1 e^{-\tau\rho} z(x,\rho,t) z_t(x,\rho,t) d\rho dx \\
&= -2 \int_0^L \int_0^1 e^{-\tau\rho} z_\rho(x,\rho,t) z(x,\rho,t) d\rho dx \\
&= -\frac{d}{d\rho} \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx - \tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \\
&= -\Big[\int_0^L (z^2(1) e^{-\tau} - z(0)) dx \Big] - \tau \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \\
&\leq \int_0^L |\psi_t|^2 dx - c \int_0^L |z(1)|^2 dx - c\tau \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx \\
&\leq -m_0 \Big(\|z(1)\|^2 + \tau \|z\|_\rho^2 \Big) + \|\psi_x\|^2.
\end{aligned}$$

Thus:

$$\boxed{\boldsymbol{F_5'(t)} \leq \boldsymbol{-m_0 \Big(\|z(1)\|^2 + \tau \|z\|_\rho^2 \Big)} + \|\psi_t\|^2,}$$

where m_0 is a positive number. ■

Terms in bold from each Lemma presented are important since they represent a partial term of the negative linear energy. It is relevant since we want to construct a functional of Lyapunov that has the form:

$$\mathcal{L}(t) = NE(t) + G(t), \tag{3.52}$$

where N is a positive constant and $G(t)$ is a perturbation of the energy. With this, we search the following estimates:

$$\mathcal{L}'(t) \leq -CE(t)$$

where C is a positive general constant, and

$$c_1 E(t) \leq L(t) \leq c_2 E(t)$$

with c_1 and c_2 are positive numbers. Then, in order to show the last two inequalities, we state the following lemmas.

Lemma 3.8. *Suppose that $y(t)$ is a solution for (3.26). Additionally, suppose $\ell > 0$ small enough. Then, the functional of Lyapunov defined by:*

$$\mathcal{L}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t) + N_4 F_4(t) + \frac{\rho_2}{4} F_5. \tag{3.53}$$

for N and N_i are positive real number to be chosen appropriately later, satisfies:

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \quad \forall t \in [0, T_{max}) \tag{3.54}$$

for two positive constants c_1 and c_2.

Proof: Let N_1 equal to 1, and define:

$$\mathcal{G}(t) := F_1(t) + N_2 F_2(t) + N_3 F_3(t) + N_4 F_4(t) + \frac{\rho_1}{4} F_5.$$

Then:

$$\begin{aligned}|\mathcal{G}(t)| \leq &\rho_1 \int_0^L |\varphi_t \varphi| dx + \rho_2 \int_0^L |\psi_t \psi|^2 dx + \rho_1 \int_0^L |w_t w| dx + \frac{\rho_1 \sigma_2 N_2}{m_2} \int_0^L \left| \theta^2 \int_0^x w_t(y,t) dy \right| dx \\ &+ \frac{\rho_2 \sigma_3 N_3}{m_3} \int_0^L \left| \theta^3 \int_0^x \psi_t(y,t) dy \right| dx + \frac{\rho_1 \sigma_1 N_4}{m_1} \int_0^L \left| \varphi_t \int_0^x \theta^1(y,t) dy \right| dx \\ &+ \frac{\tau \rho_1}{4} \int_0^L \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx.\end{aligned}$$

By using Young, Hölder and Poincare's inequalities, and the equivalence of norms, we have that

$$\begin{aligned}|\mathcal{G}(t)| \leq &c \int_0^L \left(|\varphi_x|^2 + |\varphi_t|^2 + |\psi_x|^2 + |\psi_t|^2 + |w_x|^2 + |w_t|^2 + |\theta_x^1|^2 + |\theta_x^2|^2 + |\theta_x^3|^2 \right) dx \\ &+ \int_0^L \int_0^1 z^2(x,\rho,t) d\rho dx \\ \leq &cE(t).\end{aligned}$$

Consequently, by definition of $\mathcal{G}(t)$, we infer that:

$$|\mathcal{L}(t) - NE(t)| \leq cE(t),$$

which implies that:

$$(N - c)E(t) \leq \mathcal{L}(t) \leq (N + c)E(t)$$

By choosing of N large enough, estimate (3.54) is showed. ■

The next Lemma is important to show how the behavior of functional $\mathcal{L}(t)$ with respect to the energy.

Lemma 3.9. *Under the same Hypotheses as Lemma 3.8, we have the following estimate for* $\ell > 0$ *small enough:*

$$\mathcal{L}'(t) \leq -CE(t), \tag{3.55}$$

where C *is a general constant.*

Proof: We differentiate (3.53), and recall the previous lemmas to obtain:

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\frac{k}{2} - N_2\bar{\varepsilon_2} - N_3\bar{\varepsilon_3} - N_4\varepsilon_4\right]\|\varphi_x + \psi + \ell w\|^2 \\
& - \left[\frac{b}{2} - N_3\varepsilon_3\right]\|\psi_x\|^2 \\
& - \left[\frac{k_0}{2} - N_2\varepsilon_2 - N_4\bar{\varepsilon_4}\right]\|w_x - \ell\varphi\|^2 \\
& - \left[N_4\frac{\rho_1}{2} - \rho_1 - N_2\rho_1 L\ell^2\right]\|\varphi_t\|^2 \\
& - \left[N_3\frac{\rho_2}{2} - \rho_2 - \mu N - N_4 c_4 - \frac{\rho_2}{4}\right]\|\psi_t\|^2 \\
& - \left[N_2\frac{\rho_1}{2} - \rho_1 - N_4\rho_1 L\ell^2\right]\|w_t\|^2 \\
& - \left[N\gamma_1 - c_1 - N_2c_2 - N_3c_3 - c_4(1 + \frac{1}{\varepsilon_4} + \frac{1}{\bar{\varepsilon_4}})\right]\|\theta_x^1\|^2 \\
& - \left[N\gamma_2 - c_1 - c_2(1 + \frac{1}{\varepsilon_2} + \frac{1}{\bar{\varepsilon_2}}) - N_4c_4\right]\|\theta_x^2\|^2 \\
& - \left[N\gamma_3 - c_1 - c_3(1 + \frac{1}{\varepsilon_3} + \frac{1}{\bar{\varepsilon_3}})\right]\|\theta_x^3\|^2 \\
& - \left[\frac{\rho_2 m_0}{2} - c_1\mu^2 - N_3\frac{\mu^2}{2}\right]\|z(1)\|^2 \\
& - \left[\frac{\rho_2 m_0 \tau}{2}\right]\|z\|_\rho^2.
\end{aligned}$$

Here, we might choose carefully our constants. We set:

$$\varepsilon_2 = \frac{k_0}{8N_2},\ \bar{\varepsilon_2} = \frac{k}{8N_2},\ \varepsilon_3 = \frac{b}{4N_3},\ \bar{\varepsilon_3} = \frac{k}{8N_3},\ \varepsilon_4 = \frac{kb}{8N_4},\ \bar{\varepsilon_4} = \frac{k_0}{8N_4}.$$

With this, the last inequality reduces to

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\frac{k}{8}\|\varphi_x+\psi+\ell w\|^2-\frac{b}{4}\|\psi_x\|^2-\frac{k_0}{4}\|w_x-\ell\varphi\|^2-\frac{\rho m_0\tau}{2}\|z\|_\rho^2 \\
& -\Big[N_4\frac{\rho_1}{2}-\rho_1-N_2\rho_1 L\ell^2\Big]\|\varphi_t\|^2-\Big[N_3\frac{\rho_2}{2}-\rho_1-\mu N-N_4c_4-\frac{\rho_2}{4}\Big]\|\psi_t\|^2 \\
& -\Big[N_4\frac{\rho_1}{2}-\rho_1-N_2\rho_1 L\ell^2\Big]\|w_t\|^2-\Big[\frac{\rho_2 m_0}{2}-c_1\mu^2-N_3\frac{\mu^2}{2}\Big]\|z(1)\|^2 \\
& -\Big[N\gamma_1-c_1-N_2c_2-N_3c_3-c_4(1+\frac{8N_4}{k}+\frac{8N_4}{k_0})\Big]\|\theta_x^1\|^2 \\
& -\Big[N\gamma_2-c_1-c_2(1+\frac{8N_2}{k}+\frac{8N_2}{k_0})-N_4c_4\Big]\|\theta_x^2\|^2 \\
& -\Big[N\gamma_3-c_1-c_3(1+\frac{4N_3}{b}+\frac{8N_3}{k})\Big]\|\theta_x^3\|^2.
\end{aligned}$$

We take N_2 and N_4 large enough such that:

$$N_4\frac{\rho_1}{2}-\rho_1>0 \text{ and } =N_2\frac{\rho_1}{2}>0.$$

(For example, $N_2=N_4=0$). Then, we choose ℓ small enough to satisfies:

$$\beta_1=N_4\frac{\rho_1}{2}-\rho_1>0-N_2\rho_1L\ell^2 \quad \text{and} \quad \beta_2=N_2\frac{\rho_1}{2}-N_4\rho_1L\ell^2>0$$

After that, we can choose μ small enough such that numbers as μN and are still small. With this, we choose N_3 appropriately to obtain

$$\beta_3=N_3\frac{\rho_2}{2}-\rho_1-\mu N-N_4c_4-\frac{\rho_2}{4}>0.$$

Next, as μ is small, without loss of generality, we assume that μ^2N_3 continues smaller. So, we deduce that:

$$\beta_4=\frac{\rho_2 m_0}{2}-c_1\mu^2-N_3\frac{\mu^2}{2}>0.$$

Finally, for the dissipation terms, we can take N large enough such that $N\gamma_i$ continues larger and therefore, we conclude that:

$$\beta_5 = N\gamma_1 - c_1 - N_2c_2 - N_3c_3 - c_4(1 + \frac{8N_4}{k} + \frac{8N_4}{k_0}) > 0,$$
$$\beta_6 = N\gamma_2 - c_1 - c_2(1 + \frac{8N_2}{k} + \frac{8N_2}{k_0}) - N_4c_4 > 0,$$
$$\beta_7 = N\gamma_3 - c_1 - c_3(1 + \frac{4N_3}{b} + \frac{8N_3}{k}) > 0.$$

To conclude the estimate, we use Poincare's inequality for the thermal terms and take the minimum of all constants $\frac{k_0}{2}, \frac{k}{2}, \frac{b}{2}, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7$, to conclude that:

$$\mathcal{L}'(t) \leq - c\Big[\|\varphi_x + \psi + \ell w\|^2 + \|\psi_x\|^2 + \|w_x - \ell\varphi\|^2 + \|\varphi_t\|^2 + \|\psi_t\|^2 + \|w_t\|^2 + \|\theta^1\|^2 + \|\theta^2\|^2 + \|\theta^3\|^2 + \|z(1)\|^2 + \|z\|_\rho^2\Big].$$

Adapting the constants to the phase space norm and cancel the term $-\|z(1)\|^2 \leq 0$, we obtain

$$\mathcal{L}'(t) \leq -CE(t).$$

which shows the Lemma.

Conclusion the proof of Theorem 3.2:

Estimates (3.54) and (3.55), give us:

$$\mathcal{L}'(t) \leq -k_2\mathcal{L}(t) \tag{3.56}$$

where $k_2 = \dfrac{C}{c_2}$. By a Gronwall type Lemma, previous estimate implies that:

$$c_1E(t) \leq \mathcal{L}(t) \leq \mathcal{L}(0)e^{-k_2t} \leq c_2E(0)e^{-\alpha t}, \ \ \forall t \in [0, T_{max}).$$

As we see in Theorem 3.1, we obtain a local solution $y(t)$ for the abstract Cauchy problem (3.26)

; and the Theorem 3.2 tell us that the solution is controlled by a exponential function that decays to zero. That is, the solution does not explode when the time t is longer. Thus, by [30] we conclude that solution $y(t)$ is global and then $T_{max} = +\infty$, and proves (3.45), which ends the proof of stability exponential. ■

3.4 The nonlinear case

As we noticed in the Chapter 2, we can consider the presence of nonlinear forces in the system (7). More precisely, this functions act on the three main components of the system: φ, ψ and w. In this section, we will show the existence of a solution for a semilinear Bresse system, with the presence of the delay term acting on the rotation angle. Therefore, we consider three external forces $f_i(\varphi, \psi, w),\ i = 1, 2, 3$, satisfying conditions (2.25)-(2.29).
Then, the semilinear Bresse system with the presence of delay is given by:

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi + \ell w)_x - \ell k_0(w_x - \ell\varphi) + m_1\theta^1_x + \ell m_2\theta^2 + f_1(\varphi, \psi, w) = 0 \qquad (3.57)$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + \ell w) + m_3\theta^3_x - m_1\theta^1 + \mu z(x, 1, t) + f_2(\varphi, \psi, w) = 0 \qquad (3.58)$$

$$\rho_1 w_{tt} - k_0(w_x - \ell\varphi)_x + k\ell(\varphi_x + \psi + \ell w) + m_2\theta^2_x - \ell m_1\theta^1 + f_3(\varphi, \psi, w) = 0 \qquad (3.59)$$

$$\sigma_1\theta^1_t - \gamma_1\theta^1_{xx} + m_1(\varphi_x + \psi + \ell w)_t = 0 \qquad (3.60)$$

$$\sigma_2\theta^2_t - \gamma_2\theta^2_{xx} + m_2(w_x - \ell\varphi)_t = 0 \qquad (3.61)$$

$$\sigma_3\theta^3_t - \gamma_3\theta^3_{xx} + m_3(\psi_x)_t = 0 \qquad (3.62)$$

$$\tau z_t + z_\rho = 0 \qquad (3.63)$$

defined for $x \in (0, L)$ and $t \geq 0$. This system is subject under the Dirichlet-Neumann-Neumann hypotheses for the functions φ, ψ and w (condition (3.7)), and Neumann-Dirichlet-Dirichlet hypotheses for the temperature equations (condition (3.8)). We also assume the same initial data

(3.9):

$$y_0 = \big(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \theta_0^1, \theta_0^2, \theta_0^3, g_0\big).$$

Thus, the linear energy of the system is given by:

$$\mathcal{E}(t) := E(t) + \int_0^L F(\varphi, \psi, w)dx. \tag{3.64}$$

And, this functional satisfies the following estimate:

$$\mathcal{E}(t) \leq -\big(\gamma_1|\theta_x^1(x,t)|^2 + \gamma_1|\theta_x^2(x,t)|^2 + \gamma_1|\theta_x^3(x,t)|^2\big)dx + \mu\int_0^L |\psi_t(x,t)|^2 dx. \tag{3.65}$$

The previous chapter shows that, by certain conditions on the unbounded operator A and the functional $\mathcal{F}$, it is possible to proof the existence of local weak and strong solutions when using nonlinear semigroup theory (cf. [30, Theorem 4.1.6]). For this, we rewrite the system (3.57)-(3.63) as an Cauchy Problem:

$$\begin{cases} \dfrac{d}{dt}y(t) + Ay(t) = \overline{\mathcal{F}}y(t), \\ \qquad\qquad y(0) = y_0, \end{cases} \tag{3.66}$$

where

$$y(t) = \big(\varphi(t), \psi(t), w(t), \varphi'(t), \psi'(t), w'(t), \theta^1(t), \theta^2(t), \theta^3(t)\big) \in \mathcal{H},$$

with

$$\varphi' = \varphi_t,\ \psi' = \psi_t,\ w' = w_t.$$

The unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is the same as linear one obtained in this Chapter, with domain:

$$D(A) = \Big\{ y \in \mathcal{H} |\ \varphi \in H^2(0,L) \cap H_0^1(0,L);\ \psi, w \in H^2(0,L) \cap H_*^1(0,L), \\ \varphi', \theta^2, \theta^3 \in H_0^1(0,L);\ \psi', w', \theta^1 \in H_*^1(0,L) \\ z, z_\rho \in L^2((0,1); L^2(0,L)),\ z(x,0) = \psi'(x) \Big\}.$$

Now, the functional $\overline{\mathcal{F}}$ is composed by the terms of operator B and the external forces, as follows:

$$\overline{\mathcal{F}} y(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f_1(\varphi,\psi,w)/\rho_1 \\ -f_2(\varphi,\psi,w)/\rho_2 \quad -\mu\psi'/\rho_2 \\ -f_3(\varphi,\psi,w)/\rho_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The existence theorem is then given in terms of the problem (3.66).

Theorem 3.3. *(*Well-Posedness*). Consider $\ell \neq \frac{n\pi}{L}$, with $\ell > 0$. Assume that conditions for the linear delay system (3.14)-(3.20) and hypotheses about external forces (2.25)-(2.28) hold. Thus, for any initial data $y_0 \in \mathcal{H}$ and $T > 0$, the abstract problem (3.66) has a unique weak solution:*

$$y \in C([0,T], \mathcal{H}),\ with\ y(0) = y_0.$$

The solution is given by:

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}\overline{\mathcal{F}}(y(s))ds, \ t \in [0,T] \tag{3.67}$$

with depends continuously on the initial data. In particular, if $y_0 \in D(A)$ then the solution is strong.

Proof: Under the hypotheses on linear case, the operator A is maximal monotone in $\mathcal{H}$, and from standard theory (cf. [30]), the abstract Cauchy problem (3.26) has a unique solution. We will show that system (3.66) is a Lipschitz perturbation of (3.26). Then from [27], we obtain a local solution defined in an interval $[0, t_{max}]$, where if $t_{max} < \infty$, then

$$\lim_{t\to\infty} \|y(t)\|_{\mathcal{H}} = +\infty. \tag{3.68}$$

To show that operator $\overline{\mathcal{F}} :\to \mathcal{H}$ is locally Lipschitz, let $\mathcal{B}$ a bounded set on $\mathcal{H}$ and $y^1.y^2$ in $\mathcal{B}$. Then, we can see the functional $\overline{\mathcal{F}}$ as a sum of operator B and the functional $\mathcal{F}$ given in Chapter 2:

$$\overline{\mathcal{F}}y(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -f_1(\varphi,\psi,w)/\rho_1 \\ -f_2(\varphi,\psi,w)/\rho_2 \\ -f_3(\varphi,\psi,w)/\rho_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\mu\psi'/\rho_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The previous chapters show us that $\mathcal{F}$ is locally Lipschitz, and B represents a projection from $y(t)$ into his component ψ_t, when we can easily deduce that is a locally Lipschitz operator, for any bounded set $\mathcal{B} \in \mathcal{H}$. Thus, the sum continues to be locally Lipschitz in $\mathcal{H}$. ■

Now, in order to prove that the solution is global, that is, $t_{max} = \infty$, let $y(t)$ be a mild solution with initial data $y_0 \in D(A)$. Then, by [27], it is indeed a strong solution and so we can use energy estimate (2.35) to conclude that:

$$\|y\|_{\mathcal{H}}^2 \leq \frac{2}{\beta_0}(\mathcal{E}(t) + Lm_F)$$

Thus, $\|y\|_{\mathcal{H}}^2$ is bounded.

By density, this inequality holds for mild solutions. Then, we can see easily that (2.34) does not hold and therefore: $T_{max} = +\infty$.

Finally, using the variation of parameter formula (2.33), we can verify that for any initial data $y_0^1, y_0^2 \in \mathcal{H}$, the corresponding solutions y^1 and y^2 satisfy:

$$\begin{aligned} \|y^1(t) - y^2(t)\|_{\mathcal{H}}^2 &\leq 2\|e^{tA}(y_0^1 - y_0^2)\|_{\mathcal{H}}^2 + 2\Big\| \int_0^t e^{(t-s)A}[\mathcal{F}(y^1(s)) - \mathcal{F}(y^2(s))]ds \Big\|_{\mathcal{H}}^2 \\ &\leq C\|y_0^1 - y_0^2\|_{\mathcal{H}}^2 \end{aligned}$$

for any $0 < t < T$ and a bounded set B.■

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