Universidade de Brasília
Instituto de Física

Conformal Field Theories in Symplectic Manifolds

Gustavo Xavier Antunes Pretronilo

Universidade de Brasília
Instituto de Física

Conformal Field Theories in Symplectic Manifolds

Gustavo Xavier Antunes Petronilo
Advisors: Prof. Ademir Eugênio de Santana
Co-advisors: Prof. Sérgio Costa Ulhoa

Thesis presented to the Institute of Physics of the University of Brasília, to obtain the title of Doctor in Physics

License Permission

This thesis includes the following papers that have been reproduced with permission

- The Landau problem and nonclassicality, G. X. A. Petronilo, S. C. Ulhoa, K. V. S. Araújo, R. A. S. Paiva, R. G. G. Amorim, and A. E. Santana, International Journal of Modern Physics A, Vol. 35, No. 25, Copyright ©. 2020 World Scientific;

- Size-effect at finite temperature in a quark–antiquark effective model in phase space, G. X. A. Petronilo, R. G. G. Amorim, S. C. Ulhoa, A. F. Santos, A. E. Santana, and Faqir C. Khanna, International Journal of Modern Physics A, Vol. 36, No. 17, Copyright ©. 2021 World Scientific;

- Representations of Extended Carroll Group, G. X. A. Petronilo, S. C. Ulhoa, and A. E. Santana, Advances in Applied Clifford Algebras, Vol. 31, No. 41, Copyright © 2021, The Author(s), under exclusive licence to Springer Nature Switzerland AG.

- The Landau Problem and non-Classicality in Symplectic Carroll Symmetry, G. X. A. Petronilo, S.C. Ulhoa and A. E. Santana, Vol. 2667, Journal of Physics: Conference Series, XII International Symposium on Quantum Theory and Symmetries (QTS12), Copyright ©. 2023 Creative Commons Attribution 3.0 licence, IOPScience.

PUBLICATION LIST

Journal Articles

- Petronilo, G., Araújo, M., & Cruz, C. (2024). Simulating thermal qubits through thermofield dynamics: An undergraduate approach using quantum computing. *Revista Brasileira de Ensino de Física*, *45*, e20230287.
- Abu-Shady, M., Luz, R., Petronilo, G., Santana, A., & Amorim, R. (2023). On the fractional quark-antiquark confinement via symplectic quantum mechanics. *arXiv preprint arXiv:2310.18810*.
- Luz, R., Abu-Shady, M., Petronilo, G., Santana, A., & Amorim, R. (2023). Fractional effective quark-antiquark interaction in symplectic quantum mechanics. *Advances in High Energy Physics*, *2023*.
- Petronilo, G. (2023). Uma fascinante jornada pela simetria de carroll: O outro limite relativístico. *Jornal Mato-grossense de Física*, *3*(1), 85.
- Petronilo, G., Ulhoa, S., & Santana, A. (2023). The landau problem and non-classicality in symplectic carroll symmetry. *Journal of Physics: Conference Series*, *2667*(1), 012054.
- Dessano, H., Petronilo, G., Amorim, R., & Santana, A. (2022). Phase space schrödinger equation and quadratic friction. *Journal of Mathematical Physics*, *63*(2), 022101.
- Luz, R., Petronilo, G., De Santana, A., Costa, C., Amorim, R., & Paiva, R. (2022). Quark-antiquark effective potential in symplectic quantum mechanics. *Advances in High Energy Physics*, *2022*.
- Moreira, R., Petronilo, G., Santana, A., Amorim, R., & Belich, H. (2022). Cpt-even electrodynamics in a multidimensional torus: Casimir effect at finite temperature. *The European Physical Journal C*, *82*(11), 999.
- Petronilo, G., Ulhoa, S., & Santana, A. (2022). Compactified conformal field theories in symplectic manifolds in proceedings of the xxiv dae-brns high energy physics symposium, jatni, india. *277*.
- Petronilo, G., Amorim, R., Ulhoa, S., Santos, A., Santana, A., & Khanna, F. C. (2021). Size-effect at finite temperature in a quark–antiquark effective model in phase space. *International Journal of Modern Physics A*, *36*(17), 2150121.
- Petronilo, G., Ulhoa, S., & Santana, A. (2021). Representations of extended carroll group. *Advances in Applied Clifford Algebras*, *31*(3), 1–10.
- Petronilo, G., Ulhoa, S., Araújo, K., Paiva, R., Amorim, R., & Santana, A. (2020). The landau problem and nonclassicality. *International Journal of Modern Physics A*, *35*(25), 2050148.

Agradecimentos

Agradeço primeiramente a Deus, por ter me dado forças para finalizar mais uma etapa importante da minha vida.

Agradeço também ao meu orientador, o Dr. Prof. Ademir Eugênio Santana e coorientador Dr. Prof. Sérgio Ulhoa Costa, por terem me guiado com este tema, me ensinado os conceitos e fenômenos físicos, também por terem sido grandes amigos me ajudando em minhas dificuldades tanto acadêmicas como pessoais. Por ter se mostrado um exemplo de pessoa a ser seguido. Obrigado por tudo.

Agradeço aos meus colegas e professores que me ajudaram durante o processo de escrita deste trabalho. Cito aqui alguns que foram os que acompanharam de perto esta minha luta: Rendisley Aristoteles, Kayo Vaz, Obrigado!

Dedico essa conquista à minha família: aos meus pais, João Petronilo Filho e Gildete Xavier Antunes Petronilo, e aos meus irmãos Gabriela, Gadiel e Gabriel, que sempre estiveram comigo e me incentivaram a continuar. Agradeço a minha esposa Ysla Sabrina pelo carinho, apoio, amor e paciência. E por fim a CAPES por manter a bolsa durante o tempo de doutorado.

"A entropia há de vencer, mas hei de continuar lutando."

-Ademir Eugênio de Santana

Abstract

This work investigates the notion of a conformal group and derives a representation for symplectic quantum mechanics in the Galilean manifold, G, in a consistent manner using the Wigner function method. We study two non-Lorentzian conformal symmetries: the Conformal Carrollian group and the Schödinger group. A symplectic Hilbert space is built and unitary operators representing translations and rotations are studied, whose generators fulfill the Lie algebra in G. The Schrödinger (Klein-Gordon-like) equation for the wave functions in phase space is derived from this representation, where the variables have the contents of position and linear momentum. By means of the Moyal product, wave functions are linked to the Wigner function, so symbolizing a quasi-amplitude of probability. We establish the explicitly covariant form of the Levy-Leblond (Dirac-like) equation in phase-space. In conclusion, we demonstrate how the five-dimensional phase-space formalism and the standard formalism are equivalent. We next provide a solution that restores the standard (non-covariant) form of the Pauli-Schrödinger problem in phase-space. We investigate the non-relativistic part of the Stefan-Boltzmann law and the Casimir effect for the spin 0 and spin 1/2 particles with thermofield dynamics, also within the framework of Galilean covariance.

Key words: Conformal field theory, Galilean Covariance, Carrollian Covariance, Symplectic quantum mechanics, Wigner Function.

Resumo

Teoria de Campos Conformes em Variedades Simpléticas

Este trabalho investiga a noção de grupo conforme e deriva uma representação para a mecânica quântica simplética na variedade G de maneira consistente usando o método da função de Wigner. Estudamos duas simetrias conformes não Lorentzianas: o grupo Carrolliano Conforme e o grupo Schödinger. Um espaço de Hilbert simplético é construído e são estudados operadores unitários representando translações e rotações, cujos geradores cumprem a álgebra de Lie em G. A equação de Schrödinger (tipo Klein-Gordon) para as funções de onda no espaço de fase é derivada desta representação, onde as variáveis têm o conteúdo de posição e momento linear. Por meio do produto de Moyal, as funções de onda estão vinculadas à função de Wigner, simbolizando assim uma quase amplitude de probabilidade. Estabelecemos a forma explicitamente covariante da equação de Levy-Leblond (semelhante a Dirac) no espaço de fase. Concluindo, demonstramos como o formalismo do espaço de fase pentadimensional e o formalismo padrão são equivalentes. A seguir, fornecemos uma solução que restaura a forma padrão (não covariante) do problema de Pauli-Schrödinger no espaço de fases. Investigamos a parte não relativística da lei de Stefan-Boltzmann e o efeito Casimir para as partículas de spin 0 e spin 1/2 com dinâmica de campo térmico, também no âmbito da covariância de Galileu.

Palavras-chave: Teoria de campos conformes, Covariância Galileana, Covariância Carrolliana, Mecânica Quântica Simplética, Função de Wigner.

Contents

List of Figures

Chapter 1

Introduction

The idea of conformal symmetry in physics and mathematics is expressed as an extension of the Poincaré group. This extension includes special conformal transformations and dilations. In 1908, Bateman and Cunningham addressed the idea of a conformal group of spacetime [1–3]. They argued that kinematic groups are necessarily conformal, as they preserve the quadratic form of spacetime and are akin to orthogonal transformations, albeit with respect to an isotropic quadratic form. The freedoms of an electromagnetic field are not limited to kinematic movements but rather should only be locally proportional to a transformation that preserves the quadratic form. Bateman in 1910 studied the Jacobian matrix of a transformation that preserves the light cone and showed that it had the conformal property (proportional to a form preservation) [4]. Bateman and Cunningham showed that this conformal group is ”the largest group of transformations leaving the Maxwell equations structurally invariant” [5]. The conformal group of spacetime was denoted as C(1,3) [6].

Dirac, in 1936 [7], showed the invariance of conformal symmetry for massless relativistic spin 1/2 particles. In 1998, Maldacena [8] introduced the Anti-de Sitter/Conformal Field Theory correspondence (AdS/CFT). Discovered in the context of string theory, where it is common to treat field theories on hypersurfaces embedded in spaces of arbitrary dimensions, this correspondence conjectures connections between quantum field theory and gravity. In its original form, it related a Conformal Field Theory in 4-dimensional spacetime to the geometry of anti-de Sitter space in 5 dimensions. As it was studied further, the correspondence was extended to consider different situations, such as strong coupling in a quark-gluon plasma in quantum chromodynamics [9], thermodynamics of black holes [10], and in condensed matter physics, where systems are experimentally accessible, such as unitary fermions [11, 12]. Non-relativistic conformal symmetry in theories with Galilean covariance is the maximum symmetry that leaves the equations of motion invariant when written on the light cone of a de Sitter space (4,1) [13].

In 1988, Takahashi et al. [14] began a study of Galilean covariance, developing a non-relativistic field theory. With this formalism, the Schrödinger equation for spin-zero particles assumes a form similar to the Klein-Gordon equation but is written on the light cone of a de Sitter space (4,1) [15, 16], amenable to explicitly covariant tensorial formalism. With the advent of Galilean covariance, which is a particular case of conformal theory, it was possible to deduce the non-relativistic version of Dirac's theory, known in its usual form as the Pauli-Schrödinger equation. Subsequently, a version of this theory in phase space was constructed associated with Wigner's representation of quantum theories.

Wigner's quasi-probability distribution was introduced by Wigner in 1932 [17] to study quantum corrections to classical statistical mechanics. The goal was to relate the wave function appearing in the Schrödinger equation to a probability distribution in phase space. In 1927, Weyl presented a mapping of the density matrix into functions of real phase space and also of operators [18], in a context related to representation theory (Weyl quantization in physics). In fact, the Wigner function is the Wigner-Weyl transformation of the density matrix; hence the realization of this operator in phase space. In 1949, Moyal [19], independently, deduced the Wigner function as the generating functional of quantum momentum. This forms a basis for an elegant encoding of all expected values and thus of quantum mechanics in the phase space formulation (phase space representation).

In order to deduce a phase space representation for covariantly Galilean spin-1/2 particles, a symplectic representation for the Galilei group was used, which is associated with the Wigner approach [20–23]. These results were applied, in particular, to the Landau problem in phase space [24], where the statistics of Landau wave functions are analyzed through the Wigner function. Given the importance of conformal theories in gravity and field theory, it would be interesting to extend the previous analysis to (general) conformal theories. This has not been fullly explored in the literature, in particular for equilibrium real time finite temperature models as Thermofield dynamics.

Thermofield dynamics (TFD) is a real time, operator-based formalism of quantum field theory at finite temperature, proposed by Takahashi and Umezawa [37]. It incorporates thermal effects in the theory by enlarging the Hilbert (or Fock) space. The thermal average is measured by the expected value of arbitrary operator in a thermal vacuum, $\langle 0(\beta)|A|0(\beta)\rangle$. To create this thermal state, two elements are required: doubling of the Hilbert space, which comprises of the original and dual Hilbert spaces, and Bogoliubov transformations.

Dual conjugation principles govern doubling. Temperature effects are introduced by the Bogoliubov transformation, which is a rotation between the variables of the original and dual Hilbert spaces.

The present work aims to extend the previous results (deduced in [24]) to general conformal symmetries. The objective of this work is to explore the symplectic structure in conformal symmetries, considering the systems: Landau Model and Cornell Potential of quark-antiquark interaction.

1. the Landau model: describes electronic systems confined to a plane with a perpendicular external magnetic field

2. non-relativistic sector of quantum chromodynamics describing the quark-antiquark interaction (Cornell potential).

These systems have been considered in the literature due to their experimental importance. However, the analysis of the quantum nature of the states, such as non-classicality and quantum chaos, of these systems has only been partially explored. These characteristics can be studied using the Wigner function, with its symplectic structure. This work is organized as follows: In Chapter 2, a brief review of the Wigner formalism and the properties of the star product is presented. In Chapter 3, a review of Thermofield dynamics is presented. Conformal symmetries is analyzed, and the the Galilean and Carrollian covariant formalism is provided in chapter 4. Chapters 5, 6, 7 and 8 discusses the new results of this work. The Conclusion is given in chapter 9.

Chapter 2

Wigner Function

The Wigner quasi-probability distribution was first introduced by Eugene Wigner in 1932 [17]. Its primary objective was to establish a connection between the wave function appearing in the Schrödinger equation and a probability distribution in phase space. This distribution serves as a generating function for all spatial autocorrelation functions of a given quantum-mechanical wavefunction $\psi(x)$. As a result, it provides a mapping between the quantum density matrix and real phase space functions, along with Hermitian operators introduced by Hermann Weyl in 1927 [18]. Weyl's work has its roots in representation theory within the realm of mathematics, corresponding to the concept of Weyl quantization in physics. This essentially involves the transformation of the density matrix into phase space, known as the Wigner-Weyl transformation.

In 1949, José Enrique Moyal [19] recognized it as a functional generator of quantum moments and as the basis for an elegant formulation encompassing all expected values, thus providing a comprehensive representation of quantum mechanics within phase space.

The Wigner distribution finds applications across various fields such as statistical mechanics, quantum chemistry, quantum optics, classical optics, and signal analysis, extending to disciplines including electrical engineering and seismology [**?**, 26, 27]. This chapter offers a concise overview of the representation of quantum mechanics in phase space, along with the Wigner function. It emphasizes their properties, time evolution, the equivalence of Wigner's formalism for the product of two operators, and delves into the characteristics of the Moyal product. This chapter is based on the works [29–35].

2.1 The Density Matrix

In quantum mechanics we can do a statistical approach representing the macroscopic states through the density operator

$$\rho(t) = \sum_i \omega_i \left|\psi_i(t)\rangle\langle\psi_i(t)\right| ,$$

where $\{\psi_i\}$ are the microscopic states of the statistical ensemble and $\omega_i = \frac{N_i}{N}$, is the statistical weight for the quantum state $|\psi_i\rangle$. The density matrix is said to contain all the physically relevant information we can possibly obtain about the ensemble in question. For pure states we will have

$$\rho(t) = \left|\psi(t)\rangle\langle\psi(t)\right| . \tag{2.1}$$

The expected value of an operator A in the formulation of the usual statistical quantum mechanics is given by

$$\langle A\rangle = \langle\psi|A|\psi\rangle = Tr(\rho A) = Tr(A\rho). \tag{2.2}$$

The density matrix ρ has the following properties

- hermeticity: $\rho = \rho^\dagger$;
- trace: $\mathrm{Tr}\rho = 1$.

The equation that governs temporal evolution matrix densitye ρ it's called the Liouville-von Neumann equation, given by

$$i\hbar\frac{\partial\rho}{\partial t} = \left[H(t), \rho(t)\right] . \tag{2.3}$$

where H represents the total system energy.
It is possible to enter a quantum mechanics formulation in phase space from ρ. This formulation is known as the Wigner function method.

2.2 The Wigner's Function

The density operator ρ can receive many matrix representations, being the position representation, $\langle q|\rho|q'\rangle$, and momentum representation, $\langle p|\rho|p'\rangle$, the most common ones.

The Wigner representation is, in a way, between both. For one single particle in a dimension, it is defined as

$$f_w(q,p) = \Omega(\rho) = (2\pi\hbar)^{-1} \int dz \, \exp\left(\frac{ipz}{\hbar}\right) \left\langle q - \frac{z}{2} \middle| \rho \middle| q + \frac{z}{2} \right\rangle, \tag{2.4}$$

or yet

$$f_w(q,p) = \Omega(\rho) = (2\pi\hbar)^{-1} \int dk \, \exp\left(\frac{-iqk}{\hbar}\right) \left\langle p - \frac{k}{2} \middle| \rho \middle| p + \frac{k}{2} \right\rangle, \tag{2.5}$$

matching the mapping $\Omega : \rho \to f_w(q,p)$. Considering a quantum system described by a pure state so that $\rho = |\psi\rangle\langle\psi|$, Wigner's function can then be written as

$$f_w(q,p) = (2\pi\hbar)^{-1} \int dz \, \exp\left(\frac{ipz}{\hbar}\right) \psi^\dagger \left(q + \frac{z}{2}\right) \psi \left(q - \frac{z}{2}\right). \tag{2.6}$$

Wigner's function does not represent a probability distribution, since if f_ψ e f_ϕ are two Wigner functions associated respectively with the states $|\psi\rangle$ e $|\phi\rangle$, so

$$|\langle\psi|\phi\rangle|^2 = (2\pi\hbar) \int f_\psi(q,p;t) f_\phi(q,p;t) dqdp, \tag{2.7}$$

the left side of this equation is positive or null (in this case if the kets are orthogonal), in the latter case we have the integral, $f_\psi(q,p;t)f_\phi(q,p;t)$, is null, however $f_\psi(q,p;t)$ e $f_\phi(q,p;t)$ they are not necessarily null, forcing them to conclude that they can assume negative values. For this reason the Wigner function is called the quasi-probability distribution, since when integrated it can be interpreted as a probability distribution of physical variables, as we shall see below.

First, Wigner's function is limited.

Demonstration:

Take for example a pure state, given by Eq. (2.4)

$$f_w(q,p) = \Omega(\rho) = (2\pi\hbar)^{-1} \int dz \, \exp\left(\frac{ipz}{\hbar}\right) \left\langle q - \frac{z}{2} \middle| \rho \middle| q + \frac{z}{2} \right\rangle.$$

If we define the normalized wave functions

$$\varphi_1(z) = \frac{1}{\sqrt{2}} e^{\frac{ipz}{\hbar}} \psi^\dagger(q + \frac{z}{2}) \qquad \text{e} \qquad \varphi_2(z) = \frac{1}{\sqrt{2}} \psi(q - \frac{z}{2}),$$

we see that the wigner function can be interpreted as the scalar product

$$f_w(q,p) = \frac{1}{\pi\hbar}\int dz\, \varphi_1^\dagger(z)\varphi_2(z) = \frac{1}{\pi\hbar}\left\langle \varphi_1 | \varphi_2 \right\rangle,$$

and therefore,

$$|f_w(q,p)| = \frac{1}{\pi\hbar}|\langle \varphi_1|\varphi_2\rangle|.$$

Using Cauchy-Schwarz Inequality

$$\left|\langle \phi_1|\varphi_2\rangle\right|^2 \leq \left\langle \varphi_1|\varphi_1\right\rangle \left\langle \varphi_2|\varphi_2\right\rangle,$$

so, as φ_1 e φ_2 are normalized

$$\left|\langle \varphi_1|\varphi_2\rangle\right|^2 \leq 1.$$

Thus,

$$|f_w(q,p)| \leq \frac{1}{\pi\hbar}. \tag{2.8}$$

Inequality $|f_w(q,p)| \leq \frac{1}{\pi\hbar}$implies that the Wigner function is nonzero in a region whose phase space area is less than or equal to $h/2$ [28].Thus, Wigner's function for a pure state intrinsically carries information about the uncertainty scope ,q e p cannot be infinitely located at a single point in phase space.

It follows directly from (2.4) and (2.5) that

$$|\psi(q)|^2 = \int f_w dp \quad = \quad \langle q|\rho|q\rangle, \tag{2.9}$$

$$|\psi(p)|^2 = \int f_w dq \quad = \quad \langle p|\rho|p\rangle. \tag{2.10}$$

Demonstration:

to demonstrate the Eq.(2.9) just enter a Eq.(2.4) in $\int f_w dp$ which leads us to

$$\int dp f_w = (2\pi\hbar)^{-1}\int dpdz \left\langle q - \frac{z}{2}\middle|\rho\middle|q+\frac{z}{2}\right\rangle \exp\left(\frac{ipz}{\hbar}\right), \tag{2.11}$$

if you first integrate p, we have to

$$\int dz \left\langle q-\frac{z}{2}\middle|\rho\middle|q+\frac{z}{2}\right\rangle \left(\int dp\,(2\pi\hbar)^{-1}\exp\left(\frac{ipz}{\hbar}\right)\right), \tag{2.12}$$

where the term in parentheses is the Dirac delta, $\delta(z)$. We have

$$\int dz \left\langle q-\frac{z}{2}\middle|\rho\middle|q+\frac{z}{2}\right\rangle \delta(z) = \langle q|\rho|q\rangle = |\psi(q)|^2, \tag{2.13}$$

analogously replacing Eq.(2.10) in Eq.(2.4)

$$\int dq f_w = (2\pi\hbar)^{-1}\int dk dq \exp\left(\frac{-iqk}{\hbar}\right)\left\langle p-\frac{k}{2}\middle|\rho\middle|p+\frac{k}{2}\right\rangle, \tag{2.14}$$

if you first integrate q, we have to

$$\int dk \left\langle p-\frac{k}{2}\middle|\rho\middle|p+\frac{k}{2}\right\rangle \left(\int dq\,(2\pi\hbar)^{-1}\exp\left(\frac{-iqk}{\hbar}\right)\right), \tag{2.15}$$

where the term in parentheses is the Dirac delta, $\delta(k)$. With that we have

$$\int dz \left\langle p-\frac{k}{2}\middle|\rho\middle|p+\frac{k}{2}\right\rangle \delta(k) = \langle p|\rho|p\rangle = |\psi(p)|^2. \tag{2.16}$$

We will now show the normalization of the Wigner function, that is

$$\int f_w(q,p)dqdp = \mathrm{Tr}\rho = 1. \tag{2.17}$$

Demonstration:

Replacing Eq. (2.4) in (2.17), we obtain

$$\int f_w(q,p)dqdp = (2\pi\hbar)^{-1}\int dzdpdq \exp\left(\frac{ipz}{\hbar}\right)\left\langle q-\frac{z}{2}\middle|\rho\middle|q+\frac{z}{2}\right\rangle. \tag{2.18}$$

If we calculate first in p, we have

$$\int f_w(q,p)dqdp = \int dzdq \left\langle q-\frac{z}{2}\middle|\rho\middle|q+\frac{z}{2}\right\rangle \left((2\pi\hbar)^{-}1\int dp\, e^{i\frac{p}{\hbar}z}\right). \tag{2.19}$$

The term in parentheses is the Dirac delta. With that we have

$$\int f_w(q,p)dqdp = \int dzdq \left\langle q - \frac{z}{2}\middle|\rho\middle|q + \frac{z}{2}\right\rangle \delta(z), \tag{2.20}$$

$$= \int dq \left\langle q|\rho|q\right\rangle = \mathrm{Tr}\rho = 1, \tag{2.21}$$

as we wanted to demonstrate.

Now if phase space integration is performed on a two-state Wigner two-function product, characterized by ρ_1 and ρ_2, we will find a property that concerns the product trait of two arrays of density.

$$\int dqdpf_{w1}(q,p)f_{w2}(q,p) = \frac{1}{2\pi\hbar}\mathrm{Tr}(\rho_1\rho_2). \tag{2.22}$$

Demonstration:

Using Eq. (2.4), follows that

$$\int dqdpf_{w1}(q,p)f_{w2}(q,p) = \left(\frac{1}{2\pi\hbar}\right)^2 \int dqdpdz_1dz_2\, e^{\frac{ip}{\hbar}(z_1+z_2)} \times \left\langle q - \frac{z_1}{2}\middle|\rho_1\middle|q + \frac{z_1}{2}\right\rangle \left\langle q - \frac{z_2}{2}\middle|\rho_2\middle|q + \frac{z_2}{2}\right\rangle,$$

integrating in $\boldsymbol{p}$ give us a Dirac's delta $\delta(z_1+z_2)$, so that after integrating in z_2 we have

$$\int dqdpf_{w1}(q,p)f_{w2}(q,p) = \left(\frac{1}{2\pi\hbar}\right) \int dqdz_1 \left\langle q - \frac{z_1}{2}\middle|\rho_1\middle|q + \frac{z_1}{2}\right\rangle \left\langle q - \frac{z_1}{2}\middle|\rho_2\middle|q + \frac{z_1}{2}\right\rangle.$$

Making the change of variables

$$q' = q - \frac{z_1}{2}, \qquad q'' = q + \frac{z_1}{2},$$

we get

$$\int dqdpf_{w1}(q,p)f_{w2}(q,p) = \left(\frac{1}{2\pi\hbar}\right) \int dq'dq'' \left\langle q'|\rho_1|q''\right\rangle \left\langle q''|\rho_2|q'\right\rangle. \tag{2.23}$$

Using the completeness ratio, we have

$$\begin{aligned}\int dqdp f_{w1}(q,p) f_{w2}(q,p) &= \left(\frac{1}{2\pi\hbar}\right)\int dq' \langle q'|\rho_1\rho_2|q'\rangle\,, \\ &= \left(\frac{1}{2\pi\hbar}\right)\mathrm{Tr}(\rho_1\rho_2), \end{aligned} \tag{2.24}$$

q.e.d.

Now we wonder if it is possible to find for any quantum operator $A(Q,P)$, where Q and P are the position and momentum operators, a corresponding function, $A_w(q,p)$, in Wigner's representation. The answer is positive. Similarly to what was done in defining the Wigner function, we define the $A_w(q,p)$ functions associated with the $A(Q,P)$ operator given by,

$$A_w(q,p) = \int dz\ \exp\left(\frac{ipz}{\hbar}\right)\left\langle q - \frac{z}{2}\right| A(Q,P) \left| q + \frac{z}{2}\right\rangle, \tag{2.25}$$

or

$$A_w(q,p) = \int dz\ \exp\left(\frac{-iqk}{\hbar}\right)\left\langle p - \frac{k}{2}\right| A(Q,P) \left| p + \frac{k}{2}\right\rangle. \tag{2.26}$$

We will call these functions equivalent Wigner functions from the operators $A(Q,P)$. So we can say that the Wigner function is the equivalent Wigner function for the operator ρ

$$f_w = (2\pi\hbar)^{-1}\rho_w. \tag{2.27}$$

With the definition of Wigner equivalents to any quantum operators in the Wigner representation, we have the expected value of an observable state $|\psi\rangle$ is represented as

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \int dpdq A_w(q,p) f_w(q,p) = Tr(A\rho). \tag{2.28}$$

Demonstration:

Replacing Eq. (2.4) and (2.26) in (2.28), we have

$$\langle A\rangle = \int dpdq\, A_w(q,p) f_w(q,p) = Tr(\rho A) = \left(\frac{1}{2\pi\hbar}\right)\int dqdpdz'dz''\exp\left(\frac{ipz'}{\hbar}\right)$$
$$\times\left\langle q-\frac{z'}{2}\middle|A(Q,P)\middle|q+\frac{z'}{2}\right\rangle\left\langle q-\frac{z''}{2}\middle|A(Q,P)\middle|q+\frac{z''}{2}\right\rangle, \quad (2.29)$$

integrating into p results in a Dirac delta $\delta(z'+z'')$. With that, integrating into z''

$$\langle A\rangle = \int dqdz'\left\langle q-\frac{z'}{2}\middle|A(Q,P)\middle|q+\frac{z'}{2}\right\rangle\left\langle q-\frac{z'}{2}\middle|A(Q,P)\middle|q+\frac{z'}{2}\right\rangle. \quad (2.30)$$

Introducing changing variables,

$$q' = q-\frac{z_1}{2}, \qquad q'' = q+\frac{z_1}{2},$$

thus,

$$\langle A\rangle = \int dq'dq''\,\langle q'|A(Q,P)|q''\rangle\,\langle q''|\rho|q'\rangle = \mathrm{Tr}(\rho A), \quad (2.31)$$

q.e.d.

The problem now is to show the univocal correspondence between a quantum operator $A(Q,P)$ and the reciprocal representation of Wigner $A_w(q,p)$. This can be done via the Weyl quantization rule which is defined as follows. Given a function in phase space, $\alpha(\tau,\sigma)$, then there is a quantum operator in Hilbert space, $A(Q,P)$, associated with $\alpha(\tau,\sigma)$, such that

$$A(Q,P) = \frac{1}{2\pi\hbar}\int d\tau d\sigma\, e^{\frac{i(\sigma Q+\tau P)}{\hbar}}\alpha(\tau,\sigma), \quad (2.32)$$

where τ is associated with the position coordinate and σ with the momentum coordinate in phase space. If we write $A(Q,P)$ in terms of $A_w(q,p)$ we get the following result

$$\alpha(\tau,\sigma) = \int dqdp\, e^{\frac{i(\sigma Q+\tau P)}{\hbar}} A_w(q,p). \quad (2.33)$$

To verify this equivalence, it must be shown that the operator defined by $W(Q,P) = e^{\frac{i(\sigma Q+\tau P)}{\hbar}}$, satisfies a kind of orthogonality and completeness in the space of type operators $A(Q,P)$. Using Glauber's formula, given by $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ we rewrite then $W(P,Q)$

as

$$W(Q,P) = e^{\frac{i\sigma Q}{\hbar}} e^{\frac{i\tau P}{\hbar}} e^{\frac{i\sigma\tau}{2\hbar}},$$

where we use the fact that $[Q,P] = i\hbar$. We can then calculate the value of the expression

$$\langle q'|e^{\pm\frac{i}{\hbar}(\sigma Q+\tau P)}|q\rangle = \langle q'|e^{\pm\frac{i\sigma Q}{\hbar}} e^{\pm\frac{i\tau P}{\hbar}} e^{\pm\frac{i\sigma\tau}{2\hbar}}|q\rangle \,.$$

It is known that $Q\,|q\rangle = q\,|q\rangle$, so $e^{\pm\frac{i}{\hbar}\sigma Q}\,|q\rangle = e^{\pm\frac{i}{\hbar}\sigma q}\,|q\rangle$ and using the translation operator property, $e^{\frac{i\tau P}{\hbar}}\,|q\rangle = |q-\tau\rangle$. So, we get

$$\langle q'|e^{\pm\frac{i}{\hbar}(\sigma Q+\tau P)}|q\rangle = e^{\pm\sigma(\frac{i}{\hbar}q'\pm\frac{\tau}{2})}\delta(q'-q\pm\tau),$$

which implies

$$\mathrm{Tr}e^{-\frac{i}{2}(\sigma Q-\tau P)} = (2\pi\hbar)\delta(\sigma)\delta(\tau),$$

because, by definition $\mathrm{Tr}A = \int dqdp\,\langle q'|A|q\rangle = (2\pi\hbar)^{-1}\int dqdpA_w(q,p)$. Thus,

$$\begin{aligned}\mathrm{Tr}e^{-\frac{i}{\hbar}(\sigma Q-\tau P)} &= (2\pi\hbar)^{-1}\int dqdp\int dz\ \exp(ipz)\left\langle q-\frac{z}{2}\middle|e^{-\frac{i}{\hbar}(\sigma Q-\tau P)}\middle|q+\frac{z}{2}\right\rangle, \qquad (2.34)\\ &= (2\pi\hbar)^{-1}\int dqdp\int dz\ \exp(ipz)\exp(i\sigma(q-z-\tau))\delta(z+\tau).\end{aligned}$$

Using the Dirac delta and integrating in z, we have

$$\mathrm{Tr}e^{-\frac{i}{\hbar}(\sigma Q-\tau P)} = (2\pi\hbar)^{-1}\int dqdp\; e^{\frac{ip\tau}{\hbar}} e^{\frac{iq\sigma}{\hbar}}. \qquad (2.35)$$

Thus, we identified two deltas in the integral form, that is,

$$\mathrm{Tr}e^{-i\hbar(\sigma Q+\tau P)} = (2\pi\hbar)\delta(\sigma)\delta(\tau).$$

This leads us to orthogonality relations

$$\mathrm{Tr}e^{-\frac{i}{\hbar}(\sigma' Q-\tau' P)}e^{-\frac{i}{\hbar}(\sigma Q-\tau P)} = (2\pi\hbar)^3\delta(\sigma'-\sigma)\delta(\tau'-\tau). \qquad (2.36)$$

To prove equivalence between equations (2.26) e (2.27) and the equations (2.32) and (2.33), we assume that the expansion

$$A(Q,P) = \int d\sigma d\tau \ \alpha(\sigma,\tau) e^{\frac{i}{\hbar}(\sigma Q' + \tau P')}, \tag{2.37}$$

exists. So, using the orthogonality relationship shown earlier, we easily notice that

$$\alpha(\sigma,\tau) = \frac{1}{2\pi\hbar} \mathrm{Tr} \left\{ A(Q,P) e^{-\frac{i}{\hbar}(\sigma Q' + \tau P')} \right\}.$$

To prove the existence of the equation (2.37), we replace the equation

$$\alpha(\sigma,\tau) = \int dq dp e^{-\frac{i}{\hbar}(\sigma Q + \tau P)} A_w(q,p), \tag{2.38}$$

in the equation itself (2.37). Calculating the matrix elements in the position representation, we arrive at

$$\begin{aligned} \langle q|A(Q,P)|q'\rangle &= (2\pi\hbar)^{-1} \int d\sigma d\tau dq'' dq''' \, \langle q''|A(Q,P)|q'''\rangle \\ &\times \ \langle q'''|e^{\frac{i}{\hbar}(\sigma Q' + \tau P')}|q''\rangle \ \langle q|e^{\frac{i}{\hbar}(\sigma Q + \tau P)}|q'\rangle \,. \end{aligned}$$

And yet with the use of Eq. (2.36), we get the following identity

$$\langle q|A(Q,P)|q'\rangle = \langle q|A(Q,P)|q'\rangle \,. \tag{2.39}$$

What proves the existence of expansion (2.37). This also proves that it is possible to use the equations (2.37) and (2.38) to work in both directions: given $A(Q,P)$, we can determine $A_w(q,p)$ unequivocally and vice versa. versa.

2.3 Operator Equivalence in Wigner Representation

The purpose of this session is to demonstrate some equivalence properties between the operators written in the usual representation and their respective equivalents in the Wigner representation, which can be deduced from results already obtained.

If $A = A(P)$, that is independent of Q, so $A_w = A(p)$. That is, they will have the same form, with the exception that the P operators will be replaced by the p variables..
Demonstration:

An operator $A(P)$ can be expanded in series of P, as

$$A(P) = A(0) + PA'(0) + ... \tag{2.40}$$

Now, using Eq. (2.25), and replacing $A(Q,P)$ for the expansion (2.40), we have

$$A_w(q,p) = \int dk \, \exp\left(-\frac{iqk}{\hbar}\right) \left\langle p - \frac{k}{2}\middle| A(0) + PA'(0) + \frac{P^2}{2!}A''(0) + ... \middle| p + \frac{k}{2}\right\rangle . \tag{2.41}$$

Knowing that $P\,|p\rangle = p\,|p\rangle$ we have,

$$\begin{aligned} A_w(q,p) &= A(0)\int dk \, \exp\left(-\frac{iqk}{\hbar}\right)\left\langle p - \frac{k}{2}\middle| p + \frac{k}{2}\right\rangle \\ &+ A'(0)\int dk \, \exp\left(-\frac{iqk}{\hbar}\right)\left(p + \frac{k}{2}\right)\left\langle p - \frac{k}{2}\middle| p + \frac{k}{2}\right\rangle \\ &+ A(0)''\int dk \, \exp\left(-\frac{iqk}{\hbar}\right)\frac{(p+\frac{k}{2})^2}{2!}\left\langle p - \frac{k}{2}\middle| p + \frac{k}{2}\right\rangle + ... \end{aligned}$$

Noting also that $\langle p - \frac{k}{2}|p + \frac{k}{2}\rangle = \delta k$ and using the delta property to calculate the integral in k, one gets

$$A_w(p) = A(0) + pA'(0) + \frac{p^2}{2!}A''(0) + ... = A(p). \tag{2.42}$$

q.e.d.
Analogously, using Eq. (2.26), we arrive at, if $A = A(Q)$, so $A_w(q,p) = A(q)$.

If $A(Q,P) = 1c$, where c is a constant, that is $A(q,p)$ is a multiple of the identity operator, thus $A_w = c$.
Demonstration:

This property is demonstrated immediately. Just take the equation (2.25) and in place of $A(Q,P)$ put a constant c. Since a constant does not act on kets, we have

$$\begin{aligned} A_w(q,p) &= \int dz \, \exp\left(\frac{ipz}{\hbar}\right)\left\langle q - \frac{z}{2}\middle| c \middle| q + \frac{z}{2}\right\rangle , \\ A_w(q,p) &= c\int dz \, \exp\left(\frac{ipz}{\hbar}\right)\left\langle q - \frac{z}{2}\middle| q + \frac{z}{2}\right\rangle . \end{aligned} \tag{2.43}$$

Using the fact $\left\langle p - \frac{k}{2} \middle| p + \frac{k}{2} \right\rangle = \delta k$ and integrating in z,

$$A_w(q,p) = c \tag{2.44}$$

q.e.d.

$\mathrm{Tr}A = (2\pi\hbar)^{-1} \int dqdp\ A_w(q,p)$.

Demonstration:

Using $(2\pi\hbar)^{-1} \int dqdp\ A_w(q,p)$ and substituing in Eq. (2.25),

$$(2\pi\hbar)^{-1} \int dqdp\ A_w(q,p) = \int dqdp \int dz\ \exp\left(\frac{ipz}{\hbar}\right) \left\langle q - \frac{z}{2} \middle| A(Q,P) \middle| q + \frac{z}{2} \right\rangle .$$

Integrating in p, the Dirac delta function in its integral form is identified,

$$(2\pi\hbar)^{-3} \int dqdp\ A_w(q,p) = \int dqdp \left\langle q - \frac{z}{2} \middle| A(Q,P) \middle| q + \frac{z}{2} \right\rangle \int dz\ \exp\left(\frac{ipz}{\hbar}\right) .$$

Using the delta to integrate in z, we have

$$(2\pi\hbar)^{-1} \int dqdp\ A_w(q,p) = \int dqdz \left\langle q - \frac{z}{2} \middle| A(Q,P) \middle| q + \frac{z}{2} \right\rangle \delta(z). \tag{2.45}$$

We get

$$(2\pi\hbar)^{-1} \int dqdp\ A_w(q,p) = \int dq \left\langle q \middle| A(Q,P) \middle| q \right\rangle = \mathrm{Tr}A, \tag{2.46}$$

q.e.d.

From the above demonstration you can see that $\int dp\ A_w(q,p) = (2\pi\hbar)^{-1} \langle q|A|q\rangle$ e $\int dq\ A_w(q,p) = (2\pi\hbar)^{-1} \langle p|A|p\rangle$, we simply replace the equation (2.25) on the first property and the equation (2.26) in the second, using the same process as in the demonstration above.

Finally, we have $\langle q|A(Q,P)|q'\rangle = (2\pi\hbar)^{-2} \int d\sigma\ e^{i\sigma \frac{q+q'}{2\hbar}} \alpha(\sigma, q-q')$, where $\alpha(\sigma,\tau)$ is the Fourier transform of $A_w(q,p)$.

Demonstration:

Using the expression $A(Q,P) = \frac{1}{2\pi\hbar} \int d\sigma d\tau$, we have

$$\langle q|A(Q,P)|q'\rangle = \int d\sigma d\tau\ \alpha(\sigma,\tau) \langle q|e^{i\frac{\sigma Q + \tau P}{\hbar}}|q'\rangle . \tag{2.47}$$

And using the equation (2.36), it follow that

$$\langle q|A(Q,P)|q'\rangle = (2\pi\hbar)^{-2N}\int d\sigma\; e^{i\sigma\frac{q+q'}{2\hbar}}\alpha(\sigma, q-q'), \tag{2.48}$$

q.e.d.

Now that we know how operator equivalence occurs in the Wigner representation, our goal is to find out how the operator product equivalence is represented in the Wigner representation, as this is fundamental for the development of dynamics.

2.4 Weyl-Moyal Product

The product of two quantum operators AB in Wigner's representation is written in the form

$$(AB)_w = \int dz\; e^{i\frac{pz}{\hbar}}\left\langle q-\frac{z}{2}\middle|AB\middle|q+\frac{z}{2}\right\rangle, \tag{2.49}$$

introducing the closing relationship $\int dq\; |q\rangle\langle q| = 1$, we have

$$(AB)_w = \int dzdq\; e^{i\frac{pz}{\hbar}}\left\langle q-\frac{z}{2}\middle|A\middle|q\right\rangle\left\langle q\middle|B\middle|q+\frac{z}{2}\right\rangle. \tag{2.50}$$

Using a eq. (2.48)

$$\begin{aligned}(AB)_w = \quad & (2\pi\hbar)^{-2}\quad \int dzdq'\; e^{i\frac{pz}{\hbar}}\int d\sigma d\sigma' e^{i\frac{\sigma}{2\hbar}(q+q'(q+q'-\frac{z}{2}))}\alpha(\sigma, q'-q+\frac{z}{2})\\ \times \quad & \quad e^{i\frac{\sigma}{2\hbar}(q+q'(q+q'-\frac{z}{2}))}\beta(\sigma', q-q'+\frac{z}{2}).\end{aligned}$$

Making variable changes; $\tau = q'-q+\frac{z}{2}$ and $\tau' = q-q'+\frac{z}{2}$, we arrive to

$$(AB)_w = (2\pi\hbar)^{-2}\int d\sigma d\sigma' d\tau d\tau'\; e^{i\frac{\sigma t+\tau q}{\hbar}}\alpha(\sigma,\tau)e^{i\frac{\sigma'\tau+\sigma\tau'}{2\hbar}}\beta(\sigma',\tau')e^{i\frac{\sigma' q+\tau' p}{\hbar}}. \tag{2.51}$$

The factor $e^{i\frac{\sigma'\tau+\sigma\tau'}{2\hbar}}$ can be replaced in an equivalent way by $e^{i\hbar\frac{\mathbf{\Lambda}}{2}}$, where $\mathbf{\Lambda}$ is the bidiferential operator

$$\mathbf{\Lambda} = \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}.$$

The arrows indicate the direction in which the operators are to be applied. Therefore, using, $A_w = \int dq\, dp\, e^{i\frac{\sigma q+\tau p}{\hbar}} \alpha(\sigma, \tau)$ e $B_w = \int dqdpe^{i\frac{\sigma' q+\tau' p}{\hbar}} \beta(\sigma', \tau')$, the product of operators in the Wigner representation is written as

$$(AB)_w = A_w(q,p)e^{\frac{i\hbar\mathbf{\Lambda}}{2}} B_w(q,p),$$

or

$$(AB)_w = B_w(q,p)e^{-\frac{i\hbar\mathbf{\Lambda}}{2}} A_w(q,p).$$

Thus, the operation called star product is defined as

$$(AB)_w = A_w(q,p)e^{\frac{i\hbar\mathbf{\Lambda}}{2}} B_w(q,p) = A_w(q,p) \star B_w(q,p).$$

Note that the star product is not commutative, and relates the formalism proposed by Wigner to the quantization formalism proposed by Weyl.

2.5 Temporal Evolution

We can determine the temporal evolution of the Wigner function or any operator in the Wigner representation from the Liuoville Von-Neumann equation given by

$$i\hbar\partial_t\rho = H\rho - \rho H, \tag{2.52}$$

where ρ is the density matrix and H is the Hamiltonian. Using Wigner's application, Ω in this equation, we have

$$i\hbar\Omega(\partial_t\rho) = \Omega(H\rho) - \Omega(\rho H). \tag{2.53}$$

Como

$$i\hbar\frac{\partial f_w}{\partial t} = \{H_w, f_w\}_M, \tag{2.54}$$

where $\{H_w, f_w\}_M = H_w \star f_w - f_w \star H_w$ is Moyal's parenthesis. Moyal's parenthesis can also be written as follows,

$$\{a,b\}_M = a \star b - b \star a = 2ia(q,p)\,\text{sen}\left[\frac{\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] b(q,p), \tag{2.55}$$

where, we use the fact, $e^{i\hbar\mathbf{\Lambda}/2} - e^{-i\hbar\mathbf{\Lambda}/2} = 2i\,\text{sen}\left(\hbar\frac{\mathbf{\Lambda}}{2}\right)$.

Expanding in a series of powers the sine of the last expression that defines Moyal's parentheses, we obtain,

$$\text{sen}\left(\hbar\frac{\mathbf{\Lambda}}{2}\right) = \frac{\hbar\mathbf{\Lambda}}{2} - \frac{1}{3!}\left(\hbar\frac{\mathbf{\Lambda}}{2}\right)^3 + \frac{1}{5!}\left(\hbar\frac{\mathbf{\Lambda}}{2}\right)^5 + ... \tag{2.56}$$

At the limit where $\hbar \to 0$, we obtain as a result that the Wigner function obeys the classical Liouville equation, with H_w in place of the Hamiltonian function, that is

$$\frac{\partial f_w}{\partial t} = \frac{\partial H_w}{\partial q}\frac{\partial f_w}{\partial p} - \frac{\partial H_w}{\partial p}\frac{\partial f_w}{\partial q} = \{H_w, f_w\}, \tag{2.57}$$

and

$$\frac{\partial H_w}{\partial q} = -\dot{p} \qquad \text{e} \qquad \frac{\partial H_w}{\partial p} = \dot{q}. \tag{2.58}$$

So, Wigner's formalism recovers the canonical equations of classical mechanics, when we take the classical limit, which shows that this formalism is compatible with the principle of correspondence, strengthening the importance of Wigner's description in quantum mechanics in the study of the classical limit and in the development of semi-classical methods. The study presented on the Wigner method, so far, was based on Schrödinger's description of quantum mechanics, that is, considering that only states (and not operators) evolve over time. However, it is possible to develop an analogous treatment in terms of operators expressed in Heisenberg's description (where operators evolve over time, and states are static), without further problems.

2.6 Star Product Properties

The star product or Weyl product between two functions $f(q,p)$ and $g(q,p)$ is defined by

$$f(q,p) \star g(q,p) = f(q,p) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] g(q,p). \tag{2.59}$$

Below we will present some properties of the star product.

Let $c \in \mathbb{C}$. So

$$c \star f(q,p) = f(q,p) \star c = cf(q,p). \tag{2.60}$$

Demonstration:

Expanding the star product in a series of powers, we have

$$c \star f(q,p) = c\left\{1 + \frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right) + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)^2 + ...\right\} f(q,p).$$

Operators that act on the left side will cancel each other out, as c is a constant. The same happens if c is on the right hand side, leaving only the identity operator 1.

The star product is non-commutative, i.e.

$$f(q,p) \star g(q,p) \neq g(q,p) \star f(q,p). \tag{2.61}$$

That is, $f(q,p)e^{\frac{i\hbar\mathbf{\Lambda}}{2}}g(q,p) \neq g(q,p)e^{\frac{i\hbar\mathbf{\Lambda}}{2}}f(q,p)$. Because actually,

$$f(q,p)e^{\frac{i\hbar\mathbf{\Lambda}}{2}}g(q,p) = g(q,p)e^{-\frac{i\hbar\mathbf{\Lambda}}{2}}f(q,p).$$

Demonstration:

Case 1:

$$q \star p = \left(q + \frac{i\hbar}{2}\partial_p\right) p = qp + \frac{i\hbar}{2}.$$

Case 2:

$$p \star q = \left(p - \frac{i\hbar}{2}\partial_q\right) q = pq - \frac{i\hbar}{2},$$

q.e.d.

The star product carried out between two functions in the phase space promotes one of them as an operator category,

$$\begin{aligned} f(q,p) \star g(q,p) &= f\left(q + \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p}, p - \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q}\right) g(q,p) \\ &= f(q,p) g\left(q + \frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p}, p - \frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q}\right). \end{aligned}$$

Demonstration:

Letting $a \equiv \frac{\overrightarrow{\partial}}{\partial p}$ e $b \equiv \frac{\overrightarrow{\partial}}{\partial q}$, we obtain

$$f(q,p) \star g(q,p) = f(q,p) e^{\frac{i\hbar}{2}\left(a\frac{\overleftarrow{\partial}}{\partial q} - b\frac{\overleftarrow{\partial}}{\partial p}\right)} g(q,p).$$

Considering that $e^{a\partial_x} f(x) = f(x+a)$, we have

$$f(q,p) \star g(g,p) = f\left(q + \frac{i\hbar}{2}a, p - \frac{i\hbar}{2}b\right) g(p,q).$$

Thus

$$f(q,p) \star g(g,p) = f\left(q + \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p}, p - \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q}\right) g(p,q).$$

So we define the star operator,

$$\hat{f} = f(q,p) \star .$$

The complex conjugation changes the order of the star product,

$$(f \star g)^\dagger = g^\dagger \star f^\dagger. \tag{2.62}$$

Demonstration:

The eq. (2.59) can be rewritten as

$$f(q,p)\star g(q,p) = \exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial q}\frac{\partial}{\partial p'} - \frac{\partial}{\partial p}\frac{\partial}{\partial q'}\right)\right] f(q,p)g(q',p')\Big|_{q',p'=q,p} \tag{2.63}$$

Expanding the exponential in a series of powers, we have

$$\exp\left[\frac{i\hbar}{2}(\partial_q\partial_{p'} - \partial_p\partial_{q'})\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n(\partial_q\partial_{p'} - \partial_p\partial_{q'})^n.$$

And yet, writing and writing the expression $(\partial_q\partial_{p'} - \partial_p\partial_{q'})^n$ using Newton's binomial,

$$(\partial_q\partial_{p'} - \partial_p\partial_{q'})^n = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q\partial_{p'}\right]^{n-m}\left[\partial_p\partial_{q'}\right]^m. \tag{2.64}$$

Therefore, the star product can be written as

$$f(q,p)\star g(q,p) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m f(q,p)\right]\left[\partial_q^m\partial_p^{n-m}g(q,p)\right]. \tag{2.65}$$

Taking the complex conjugate of the above equation, we have

$$\begin{aligned}(f(q,p)\star g(q,p))^\dagger &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\left\{(-1)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m f^\dagger(q,p)\right]\right.\\ &\times\left.\left[\partial_q^m\partial_p^{n-m}g^\dagger(q,p)\right]\right\},\end{aligned} \tag{2.66}$$

where the term $(-1)^n$ arises from the complex conjugation of the term $(i\hbar/2)^n$. This term can be associated with the binomial, that is

$$(-1)^n(\partial_q\partial_{p'} - \partial_p\partial_{q'}) = (-\partial_q\partial_{p'} + \partial_p\partial_{q'}) = \sum_{m=0}^{\infty}(-1)^m\binom{n}{m}\left[\partial_p\partial_{q'}\right]^{n-m}\left[\partial_q\partial_{p'}\right]^m.$$

Applying these operators in two functions in the phase space, we have

$$(\partial_q\partial_{p'} - \partial_p\partial_{q'})f(q,p)g(q',p') = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m f(q,p)\right]\left[\partial_q^m\partial_p^{m-n}g(q,p)\right].$$

and

$$(-1)^n(\partial_q\partial_{p'}-\partial_p\partial_{q'})f(q,p)g(q',p')=\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m g(q,p)\right]\left[\partial_q^m\partial_p^{m-n}f(q,p)\right].$$

Comparing these last two equations, we obtain

$$\begin{aligned}(-1)^n \quad & \textstyle\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m f(q,p)\right]\left[\partial_q^m\partial_p^{m-n}g(q,p)\right]\\ = \quad & \textstyle\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m g(q,p)\right]\left[\partial_q^m\partial_p^{m-n}f(q,p)\right].\end{aligned} \tag{2.67}$$

Replacing the eq. (2.67) in (2.66)

$$\begin{aligned}\Big(f(q,p)\star g(q,p)\Big)^\dagger &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\left\{(-1)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}\left[\partial_q^{n-m}\partial_p^m g^\dagger(q,p)\right]\right.\\ &\times\left.\left[\partial_q^m\partial_p^{n-m}f^\dagger(q,p)\right]\right\},\\ &= g^\dagger(q,p)\star f^\dagger(g,p).\end{aligned}$$

As we wanted to demonstrate.

The star product is associative.
Considering f, g and h as functions in the phase space, we have

$$\Big(f(q.p)\star g(q,p)\Big)\star h(q,p)=f(q,p)\star\Big(g(q,p)\star h(q,p)\Big). \tag{2.68}$$

Demonstration:

We have that

$$\Big(f(q.p)\star g(q,p)\Big)\star h(q,p)=\left\{f\left(q+\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p},p-\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q}\right)g(q,p)\right\}h\left(q-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p},p+\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q}\right),$$

on the other hand,

$$f(q,p)\star\Big(g(q,p)\star h(q,p)\Big)=f\left(q+\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p},p-\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q}\right)\left\{g(q,p)h\left(q-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p},p+\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q}\right)\right\}.$$

The differential operators included here are associative. In this case, we can conclude that the star product is also associative.

Next chapter we will introduce the Termofield dynamics. Thermo field Dynamics is a formalism in Quantum Field Theory that incorporates thermal effects by enlarging the Hilbert (or Fock) space, allowing calculation of thermal averages using vacuum expectation values of local operators.

Chapter 3

Thermofield Dynamics

This chapter delves into thermofield dynamics (TFD), a particular operator-based formalism used in finite temperature quantum field theory. TFD incorporates thermal effects into quantum field theories, enabling the calculation of thermal averages using familiar quantum field theory techniques. This is achieved by enlarging the theory's Hilbert space, allowing thermal averages to be computed through local operator's thermal vacuum expectation values. For constructing this chapter we used the references [36–42].

3.1 Thermal Equilibrium and Ensemble Averages

In quantum statistical mechanics, the concepts of thermal equilibrium and ensemble averages hold immense importance. To understand the behavior of a system in thermal equilibrium, a key quantity known as the **ensemble average of an operator** plays a vital role.

For a system in thermal equilibrium, the ensemble average of an operator A is defined as

$$\langle A \rangle = \frac{1}{Z(\beta)} \mathrm{Tr}(e^{-\beta H} A) \tag{3.1}$$

where

- H represents the system's **Hamiltonian**, which encodes its total energy.
- $Z(\beta)$ is the **partition function**, a crucial quantity that encapsulates the system's statistical properties at a specific temperature.
- β denotes the **inverse temperature**, a parameter inversely proportional to the system's temperature.

- Tr refers to the **trace operation**, which essentially sums the diagonal elements of a matrix.

This equation provides a powerful tool for analyzing the average value of an observable quantity (represented by the operator A) in a system at thermal equilibrium. The partition function $Z(\beta)$ acts as a normalizing factor, ensuring that the average value accurately reflects the system's statistical state.

3.2 The Thermal State

In standard quantum theories, the average value of an operator is defined as

$$\langle A \rangle = \langle n|A|n\rangle \tag{3.2}$$

where n represents the state of the system.

We can define a state $|0(\beta)\rangle$, such that

$$\begin{aligned}\langle A \rangle &= \langle 0(\beta)|A|0(\beta)\rangle \\ &= \frac{1}{Z(\beta)}\sum_n e^{-\beta E_n}\langle n|A|n\rangle\end{aligned} \tag{3.3}$$

One way to make this definition possible is to introduce a duplication of Hilbert space, resulting in a tensor product of spaces. In this product space, a base vector is given by

$$|n, \widetilde{m}\rangle = |n\rangle \otimes |\widetilde{m}\rangle. \tag{3.4}$$

With that, we write

$$|0(\beta)\rangle = \sum_n f_n(\beta)|n, \widetilde{n}\rangle, \tag{3.5}$$

such that

$$\begin{aligned}\langle A \rangle &= \sum_{n,m} f_n^*(\beta) f_m(\beta)\langle n, \widetilde{n}|A|m, \widetilde{m}\rangle \\ &= \sum_n |f_n(\beta)|^2\langle n|A|n\rangle.\end{aligned} \tag{3.6}$$

Where we have assumed that the operator A acts only on the vectors of the non-tilde space, that is

$$\begin{aligned}\langle n,\widetilde{n}|A|m,\widetilde{m}\rangle &= \langle n|\otimes\langle\widetilde{n}|A|m\rangle\otimes|\widetilde{m}\rangle\\ &= \langle n|A|m\rangle\langle\widetilde{n}|\widetilde{m}\rangle\\ &= A_{nm}\delta_{nm}.\end{aligned}\tag{3.7}$$

In search of producing the thermal average, we now have

$$|f_n(\beta)|^2 = \frac{1}{Z(\beta)}e^{-\beta E_n}.\tag{3.8}$$

Which in turn allows us to write the solution

$$f_n(\beta) = \frac{1}{\sqrt{Z(\beta)}}e^{-\beta E_n/2}.\tag{3.9}$$

So the thermal state can be written as

$$|0(\beta)\rangle = \sum_n \frac{1}{\sqrt{Z(\beta)}}e^{-\beta E_n/2}|n,\widetilde{n}\rangle.\tag{3.10}$$

The duplication of Hilbert space is a key feature of TFD and is present in all thermal field theories. It allows for a convenient representation of thermal effects through the use of Lie algebras.

The thermal state can be transformed from its non-thermal counterpart, $|0,\widetilde{0}\rangle$, using a Bogoliubov transformation

$$|0(\beta)\rangle = U(\beta)|0,\widetilde{0}\rangle.\tag{3.11}$$

This transformation introduces thermal effects and is defined through a Bogoliubov operator $U(\beta)$.

3.3 Generators and symmetries

To establish a formalism based on $|0(\beta)\rangle$, we begin with the premise that the set of kinematical variables, $\mathcal{V}$, is a vector space of mappings in a Hilbert space designated by $\mathcal{H}_T$. The set $\mathcal{V}$ is made up of two subspaces and is expressed as $\mathcal{V} = \mathcal{V}_{obs}\otimes\mathcal{V}_{gen}$, where $\mathcal{V}_{obs}$ represents the set of kinematical observables and $\mathcal{V}_{gen}$ represents the set of kinematical generators of symmetries. In both quantum and classical theory, $\mathcal{V}_{obs}$ and $\mathcal{V}_{gen}$ are frequently identical with each other and with $\mathcal{V}$. Let us go over this subject in further detail. Often, each symmetry generator has a matching observable, and both are characterized by the same algebraic element. Here we consider the identical one-to-one correspondence between generators and observables, but this time we look at the scenario when $\mathcal{V}_{obs}$ and $\mathcal{V}_{gen}$ disagree [37]. In other words, $\mathcal{V}_{obs}$ and $\mathcal{V}_{gen}$ correspond to distinct mappings in $\mathcal{H}_T$. To underscore these points, we designate an arbitrary element of $\mathcal{V}_{obs}$ by A and the equivalent element in $\mathcal{V}_{gen}$ by $\widehat{A}$.

3.4 Thermal Algebras

Within the enlarged Hilbert space $\mathcal{H}_T$, we can define Lie products for operators as follows

$$[\widehat{A}_i, \widehat{A}_j] = iC_{ij}^k \widehat{A}_k; \tag{3.12}$$

$$[\widehat{A}_i, A_j] = iC_{ij}^k A_k; \tag{3.13}$$

$$[A_i, A_j] = iC_{ij}^k A_k, \tag{3.14}$$

where $A \in \mathcal{V}_{obs}$ and $\widehat{A} \in \mathcal{V}_{gen}$. From the above equations, we may find the same algebra written now in terms of the operators A and $\widetilde{A}$. Taking the genarators' comutator

$$[\widehat{A}_i, \widehat{A}_j] = \widehat{A}_i\widehat{A}_j - \widehat{A}_j\widehat{A}_i = iC_ijk\widehat{A}_k,$$

and substituting $\widehat{A} = A - \widetilde{A}$, we have

$$\begin{aligned} (A-\widetilde{A})_i(A-\widetilde{A})_j - (A-\widetilde{A})_j(A-\widetilde{A})_i &= iC_{ij}^k(A-\widetilde{A})_k \\ A_iA_j - A_i\widetilde{A}_j - \widetilde{A}_iA_j + \widetilde{A}_i\widetilde{A}_j - A_jA_i + A_j\widetilde{A}_i + \widetilde{A}_jA_i - \widetilde{A}_j\widetilde{A}_i &= iC_{ij}^kA_k - iC_ijk\widetilde{A}_k. \end{aligned}$$

Rearranging the terms, we write

$$[A_i, A_j] + [\widetilde{A}_i, \widetilde{A}_j] - [A_i, \widetilde{A}_j] - [\widetilde{A}_i, A_j] = iC_{ij}^kA_k - iC_{ij}^k\widetilde{A}_k$$

as we have $[A_i, \widetilde{A}_j] = -[\widetilde{A}_j, A_i]$, thus

$$[A_i, A_j] + [\widetilde{A}_i, \widetilde{A}_j] - 2[A_i, \widetilde{A}_j] \;=\; iC_{ij}^k A_k - iC_{ij}^k \widetilde{A}_k.$$

From this we can identify that

$$[A_i, A_j] \;=\; iC_{ij}^k A_k \tag{3.15}$$
$$[\widetilde{A}_i, \widetilde{A}_j] \;=\; -iC_{ij}^k \widetilde{A}_k \tag{3.16}$$
$$[A_i, \widetilde{A}_j] \;=\; 0, \tag{3.17}$$

where A_i and $\widetilde{A}_i$ represent operators acting on the original and conjugate spaces, respectively, and C_{ij}^k are structure constants. This properties are called *tilde conjugation rules*, and can be onbtained from the algebra (3.15-3.17). Thus, as it is a conjugation, $(A)\widetilde{\ } = \widetilde{A}$, so, from (3.15)

$$(iC_{ij}^k A_k)\widetilde{\ } \;=\; -iC_{ij}^k \widetilde{A}_k,$$

where we have used $(cA)\widetilde{\ } = c^* \widetilde{A}$, therefore

$$([A_i, A_j])\widetilde{\ } \;=\; [\widetilde{A}_i, \widetilde{A}_j]$$
$$(A_i A_j - A_j A_i)\widetilde{\ } \;=\; \widetilde{A}_i \widetilde{A}_j - \widetilde{A}_j \widetilde{A}_i.$$

For the last equality to be true we need that

$$(A_i A_j - A_j A_i)\widetilde{\ } = (A_i A_j)\widetilde{\ } - (A_j A_i)\widetilde{\ }, \quad \text{and} \quad (A_i A_j)\widetilde{\ } = \widetilde{A}_i \widetilde{A}_j. \tag{3.18}$$

These Lie products, along with Tilde conjugation rules, define the thermal algebra on the enlarged Hilbert space.

3.5 Thermal Propagator

The propagator at the thermal vacuum $|0(\alpha)\rangle = \mathcal{B}(\alpha)|0,\widetilde{0}\rangle$ is

$$\begin{aligned} G_0^{(ab)}(q-q',p-p';\alpha) &= i\langle 0(\alpha)|\tau[\psi^a(q,p)\psi^b(q',p')]|0(\alpha)\rangle, \\ &= i\int \frac{d^4p}{(2\pi)^4} e^{-ip(q-q')} G_0^{(ab)}(p;\alpha), \end{aligned} \tag{3.19}$$

where

$$G_0^{(ab)}(p;\alpha) = \mathcal{B}^{-1}(\alpha) G_0^{(ab)}(k)\mathcal{B}(\alpha), \tag{3.20}$$

with

$$G_0^{(ab)}(p) = \begin{pmatrix} G_0(p) & 0 \\ 0 & G_0^*(p) \end{pmatrix}. \tag{3.21}$$

Then $G_0(k)$ is given as

$$G_0(p) = \frac{1}{p^\mu p_\mu - m^2 + i\epsilon}. \tag{3.22}$$

Then the non-tilde variable is

$$G_0^{(11)}(p;\alpha) = G_0(p) + \xi v^2(p;\alpha)[G_0^*(k) - G_0(p)], \tag{3.23}$$

where $v^2(p;\alpha)$ is the generalized Bogoliubov transformation given as

$$v^2(p;\alpha) = \sum_{s=1}^{d}\sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1},\ldots,l_{\sigma_s}=1}^{\infty} (-\eta)^{s+\sum_{r=1}^{s} l_{\sigma_r}} \exp\left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} p^{\sigma_j}\right], \tag{3.24}$$

with d being the number of compactified dimensions, $\eta = 1(-1)$ for fermions (bosons) and $\{\sigma_s\}$ denotes the set of all combinations with s elements.

Next chapter we will give a brief introduction to conformal symmetries and in special conformal Galilean symmetry via Galilean Covariance.

Chapter 4

Conformal Symmetry

The studies of conformal symmetry in physics have gathered a lot of interest in the last fifteen years because they are relevant to at least three different areas of modern theoretical physics They serve as toy models for truly interacting quantum field theories, represent two-dimensional critical phenomena, and are important to string theory. Also, numerous aspects of contemporary mathematics, including as number theory, finite groups, low-dimensional topology, the theory of vertex operator algebras and Borcherds algebras, and finite groups, have been influenced by conformal field theories [44].

The first part of this chapter is devoted to a comprehensive review of conformal symmetries in physics, guided Fulton's approach [45]. It also draws upon several other significant works [13, 44, 46–48]. The second part focuses on a thorough review of Galilean covariance, based on the references [14, 16, 49–53]. The Conformal Carrollian Covariance is introduced in the third part and is based on the following works [54–56].

4.1 Conformal Transformations

Conformal transformations are a fascinating area of study in physics, particularly in the field of quantum field theory. In these transformations, angles are preserved, but not necessarily distances [57], which can lead to some intriguing results.

There are various forms of conformal transformations, each with its own set of characteristics and implications. Each form of transformation can have different physical interpretation on the system being studied. The diverse implications of these distinct transformations become particularly important when considering observers and observables.

Therefore, it's important to clarify the nature of the conformal transformation being used in any given context. Understanding the specific form of the transformation and its implications can help ensure that the physical interpretations drawn from the system are accurate and meaningful.

4.1.1 Conformal Point Transformation

Following Fulton's method [45], it is necessary to differentiate between spatial points and the coordinate system that defines their positions. To differentiate between points in space and their corresponding coordinate systems, we will use the notation x, $\bar{x}$, $\bar{\bar{x}}$, and so on to represent distinct points. Meanwhile, we will employ indices to represent the coordinate systems, indicating covariant and contravariant components of various quantities. For instance, x^{μ}, $x^{\nu\prime}$, and so forth represent the components of point x measured in coordinate systems S, S', and so on. Occasionally, we may use x and x' to refer to the point x measured in the respective coordinate systems S and S'.

Let's consider the point transformation, often referred to as an"active transformation,"

$$\bar{x} = f^{\mu}(x), \tag{4.1}$$

which determines the point component the $\bar{x}$ in the S coordinate system. By definition, this is a one-to-one and analytical transformation in a given domain D.

In the forthcoming discussion, our focus will primarily revolve around points in Riemann space and the metric tensor $g_{\mu\nu}(x)$. It is essential to note that this discussion encompasses a broader context; however, for the purposes of this work, our attention will be directed toward these specific aspects. Notably, the metric tensor is characterized by the fundamental condition that, in a local geodesic coordinate system, it assumes the form of Minkowski metric $\eta_{\mu\nu}$.

The expression that represents the line element of a curve with time-like characteristics can be expressed as follows,

$$d\tau^2(x) = -g_{\mu\nu}(x)dx^{\mu}dx^{\nu}, \tag{4.2}$$

where Einstein summation notation is used. The point transformation (4.1) give us

$$d\bar{x}^{\mu} = \partial_a \bar{x}^{\mu} dx^a, \tag{4.3}$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$. The mapping between two infinitesimally close points x and $x + dx$ can be used to express the connection that establishes the distinction between two components $\bar{x}$ and $\bar{x} + d\bar{x}$, according to Eq. (4.1).

A key feature of a conformal point transformation is its ability to link the line element $d\tau(\bar{x})$ at a given point $\bar{x}$ and the line element dx at point x through the mediation of a scalar function $\sigma(x)$,

$$d\tau(\bar{x}) = \sqrt{\sigma(x)}d\tau(x). \tag{4.4}$$

This implies

$$g_{\mu\nu}(\bar{x})d\bar{x}^{\mu}d\bar{x}^{\nu} = \sigma(x)g_{\mu\nu}(x)dx^{\mu}dx^{\nu} \tag{4.5}$$

with the restriction

$$\sigma(x) > 0.$$

Therefore, this relation can be expressed as

$$g_{\mu\nu}(\bar{x})\partial_{\alpha}\bar{x}^{\mu}\partial_{\beta}\bar{x}^{\nu} = \sigma(x)g_{\alpha}\beta(x), \tag{4.6}$$

It is evident that Eq. (4.4) is equivalent to eq. (4.6), and hence this equation can be used to define conformal point transformations.

It is important to highlight that all the quantities presented in the aforementioned equations relate to the same coordinate system. If we associate a coordinate system S with an observer, we can interpret it as a mapping from a domain D of points in the observer's space to a domain $\bar{D}$ of points.

Furthermore, it is crucial to distinguish the structure on the left-hand side of Equation (4.6) from a coordinate transformation $S \rightarrow S'$, where the same point P is represented by components x^{μ} and $x^{\mu'}$ in two different coordinate systems.

$$g_{\mu'\nu'}(x')\partial_{\alpha}x^{\mu'}\partial_{\beta}x^{\nu'} = g_{\alpha}\beta(x).$$

4.1.2 Conformal Coordinate Transformations

A coordinate transformation is defined by a one-to-one correspondence that establishes a bijective mapping between points in distinct coordinate systems. This mapping guarantees that each point in one coordinate system uniquely corresponds to a single point in another coordinate system, leaving no room for ambiguity or overlap.

The coordinate transformation can be mathematically represented as

$$x^{\mu'} = h^{\mu'}(x) \tag{4.7}$$

This equation captures the interrelation between the components of points x within a domain D, as observed by different observers or coordinate systems. It's important to note that we assume the transformation possesses the quality of analyticity, ensuring its mathematical rigor. Points that are infinitesimally close to each other transform according to

$$dx^{\mu'} = \partial_\alpha x^{\mu'} dx^\alpha, \tag{4.8}$$

while the metric tensor transforms as

$$g_{\mu'\nu'}(x') = \partial_{\mu'} x^\alpha \, \partial_{\nu'} x^\beta g_{\alpha\beta}(x). \tag{4.9}$$

These relationships imply that

$$d\tau^2(x') = d\tau^2(x), \tag{4.10}$$

in contradistiction to (4.4), unless $\sigma = 1$.

Equation (4.8) is a general feature of transformations for any tensor field $T^{\mu\nu\dots}_{\alpha\beta\dots}$, which transforms according to

$$T^{\mu'\nu'\dots}_{\alpha'\beta'\dots}(x') = \partial_\mu x^{\mu'} \partial_\nu x^{\nu'} \dots \partial_{\alpha'} x^\alpha \partial_{\beta'} x^\beta \dots T^{\mu\nu\dots}_{\alpha\beta\dots}(x). \tag{4.11}$$

To establish the definition of a conformal coordinate transformation correspodent of the conformal point tranformation (4.6),, we need to first establish the general relationship between coordinate transformations and point transformations.

A point transformation can be associated with each coordinate transformation by demanding the relationship

$$\bar{x}^{\mu'} \doteq x^\mu, \qquad \text{or} \qquad \bar{x}' \doteq x. \tag{4.12}$$

This indicates that a point $\bar{x}$ is linked with x in such a way that the components of $\bar{x}$ with respect to S' are the same as the components of x with respect to S for a particular relation of components of the point x in two coordinate systems. This equality only holds in the selected coordinate system, as indicated by the equal dot notation in (4.8). This establishes a clear association between the labels μ and μ' [45].

By Eq. (4.8) the relationship between the coordinate transformation (4.7) and the point transformation (4.1) is as follows

$$\bar{x}^{\mu'} = h^{\mu'}(\bar{x}) = h^{\mu'}(f(x)) \doteq x^{\mu}$$

This shows that the function h^{μ} in (4.7) is the inverse transformation to (4.1). If (4.1) implies

$$x^{\mu} = F^{\mu}(\bar{x}), \tag{4.13}$$

then (4.7) becomes

$$x^{\mu'} = F^{\mu'} = h^{\mu'}(x). \tag{4.14}$$

These relations can also be expressed as

$$\bar{x}^{\mu} = f^{\mu}(x) \doteq f^{\mu}(\bar{x}')$$

which, by using (4.12), leads to

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \doteq \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\alpha'}} \tag{4.15}$$

This establishes the connection between point transformations and coordinate transformations.

The conformal coordinate transformation can be inferred from the corresponding point translation (4.6). It takes the form

$$x^{\mu} = f^{\mu}(x') \tag{4.16}$$

where f^{μ} is the same function as (4.1), implying (4.6).

Importantly, we observe that (4.7) and (4.10) are mutually consistent equations. By replacing (4.15) into (4.6), we obtain

$$g_{\mu\nu}(\bar{x})\frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\alpha'}}\frac{\partial \bar{x}^{\nu}}{\partial \bar{x}^{\beta'}} = \sigma(x) g_{\alpha\beta}, \tag{4.17}$$

using (4.5) for $\bar{x}$, this leads to

$$g_{\alpha'\beta'}(\bar{x}') \doteq \sigma(x) g_{\alpha\beta}(x). \tag{4.18}$$

Hence, with (4.6),

$$d\tau^2(\bar{x}') \doteq \sigma(x) g_{\alpha\beta}(x) d\tau^2(x). \tag{4.19}$$

This confirms the coherence of (4.7) and (4.10).

4.1.3 Conformal Transformations of Tensor Fields

Let's consider a vector field $A_\mu(x)$. The covariant components of this field undergo a coordinate transformation as given by (4.11)

$$A_{\alpha'}(\bar{x}') = \bar{\partial}_\alpha \bar{x}^\mu A_\mu(\bar{x}), \tag{4.20}$$

where $\bar{\partial}_\alpha = \frac{\partial}{\partial \bar{x}^\alpha}$. By employing Eq. (4.15), this can be simplified to

$$A_{\alpha'}(\bar{x}') \doteq \partial_\alpha \bar{x}^\mu A_\mu(\bar{x}). \tag{4.21}$$

We can now introduce a new field $\bar{A}_\alpha(x)$ such that

$$A_{\alpha'}(\bar{x}') \doteq \bar{A}_\alpha(x). \tag{4.22}$$

However, it is now deduced from (4.6) and (4.22) that the equivalence between $\bar{A}^\alpha(x)$ and $A^\alpha(x)$, as well as $\bar{A}_\alpha(x)$ and $A_\alpha(x)$, is precluded. Instead, only one of the following can be established

$$\begin{cases} \bar{A}^\alpha(x) = A^\alpha(x) \\ \bar{A}_\alpha(x) = \sigma A_\alpha(x) \end{cases} \tag{4.23}$$

or

$$\begin{cases} \bar{A}^\alpha(x) = \frac{1}{\sigma} A^\alpha(x) \\ \bar{A}_\alpha(x) = A_\alpha(x). \end{cases} \tag{4.24}$$

The validity of identities (4.23), and similarly (4.24), can be verified by substituting the Eq. (4.23) or Eq. (4.24) into (4.6).

The outcomes (4.23) and (4.24) can be concisely expressed If the components of a field $A(x)$ transform as a covariant vector under a conformal point transformation, then the contravariant components transform analogously to an affine contravariant vector with a factor of σ^{-1},

$$A^\mu(\bar{x}) = \frac{1}{\sigma} \partial_\alpha \bar{x}^\mu A_\alpha(x). \tag{4.25}$$

Conversely, if a contravariant vector is subject to a conformal point transformation, then the corresponding covariant components transform as

$$A_\mu(\bar{x}) = \sigma\bar{\partial}_\mu x^\alpha A^\alpha(x). \tag{4.26}$$

As a result, the magnitude of a vector $A(x)$ transforms under a conformal point transformation (4.1) and (4.6) as

$$A_\mu(\bar{x})A^\mu(\bar{x}) = \frac{1}{\sigma(x)}A^\nu(x)A_\nu(x), \tag{4.27}$$

while the magnitude of a contravariant vector transforms as

$$A_\mu(\bar{x})A^\mu(\bar{x}) = \sigma(x)A^\nu(x)A_\nu(x). \tag{4.28}$$

These considerations can, of course, be extended to tensors of arbitrary rank.

4.1.4 Conformal Transformations of the Metric Tensor

Let's consider the conformal point transformations as defined by (4.6). When we utilize the symbol

$$g^c_{\mu\nu}(x) = \sigma(x)g_{\mu\nu}(x), \tag{4.29}$$

thus, Eq. (4.6) becomes

$$g^c_{\mu\nu}(x) = \partial_\mu\bar{x}^\alpha\partial_\nu\bar{x}^\beta g_{\alpha\beta}. \tag{4.30}$$

At first glance, this might seem reminiscent of a coordinate transformation. However, it's important to note that here, x and $\bar{x}$ refer to two distinct points within the same coordinate system, rather than belonging to different coordinate systems of the same point.

The conformal coordinate transformation characterized by Eq. (4.7), by means of Eq. (4.29) can be expressed as

$$g^c_{\mu\nu} \doteq g_{\mu'\nu'}(\bar{x}') \tag{4.31}$$

for the case of Eq. (4.12)

$$x \doteq \bar{x}'. \tag{4.32}$$

This insight points us towards a definition of conformal transformations that does not explicitly reference either point or coordinate transformations. Nevertheless, it remains consistent with both concepts

Given a metric manifold described by the metric tensor $g_{\mu\nu}$, we define

$$g^c_{\mu\nu}(x) = \sigma(x)g_{\mu\nu}(x),$$

$$g_c^{\mu\nu}(x) = \frac{1}{\sigma(x)}g^{\mu\nu}(x), \tag{4.33}$$

where σ is an arbitrary positive differentiable function of x. We refer to Eq. (4.33) as the conformal transformation of the metric tensor. These transformations form a group denoted as C_g.

The collection of all manifolds differing from each other solely due to elements of C_g constitutes a conformal space. It's important to recognize that the magnitudes of lengths $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in such a space lack absolute meaning. This is because comparing lengths at two different points involves the arbitrary function σ. However, the ratio of two infinitesimal lengths is well-defined when both lengths refer to the same point. Furthermore, angles maintain their well-defined nature at each point

$$\cos\alpha = \frac{g_{\mu\nu}dx^\mu \delta x^\nu}{(g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}(g_{\rho\sigma}\delta x^\rho \delta x^\sigma)^{1/2}}, \tag{4.34}$$

and this remains invariant under C_g, justifying the term "conformal."

As we delve deeper into the subject, we come to realize that the conformal point and coordinate transformations are combinations of the conformal transformation of the metric (4.29). We'll refer to the corresponding group transformation as C, which includes C_g and the entire coordinate transformation group as subgroups. We term this expanded group C as the extended conformal group.

From these insights, we draw the conclusion that when dealing with equations invariant under coordinate transformations, it suffices to assess transformations under C_g to ensure covariance under C.

In the ensuing discussion, our primary focus will be on C_g and coordinate transformations.

4.2 The Special Conformal Algebra

We narrow our attention to the subset C_o of special or restricted conformal transformations. This subset includes transformations in C that map flat space into flat space. This means that the functions $\sigma(x)$ are no longer entirely arbitrary; they are now constrained by

$$R^{\mu}_{\nu\sigma\rho} = 0 \quad \longrightarrow \quad R^{c\mu}_{\nu\sigma\rho} = 0,$$

where $R^{\mu}_{\nu\sigma\rho}$ is the Riemann curvature tensor. Haantjes [46] classified the corresponding transformations from Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$ to a flat space with metric $g^{c}_{\mu\nu} = \sigma\eta_{\mu\nu}$. These transformations form the special conformal group C_o. Moving forward, we'll restrict our focus to this significant case C_o, while for additional information about the C_g group, refer to [45].

The C_o group consists of a 15-parameter Lie group commonly referred to as conformal transformations. It aligns with the perspective of Bateman and Cunningham [2,3].
This group encompasses space-time translations

$$x'^{\mu} = x^{\mu} + b^{\mu}, \quad \text{(4 parameters)}$$

proper homogeneous Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \quad \text{(6 parameters)}$$

dilatation (or scale) transformations

$$x'^{\mu} = s x^{\mu}, \quad \text{(1 parameters)}$$

and acceleration transformations

$$x'' = (1 + 2a^{\alpha}x_{\alpha} + x^2 a^2)^{-1}(x + a^{\mu}x^2) \quad \text{(4 parameters)}$$

In the latter equation, $x^2 = x^{\alpha}x_{\alpha}$ and $a^2 = a^{\alpha}a_{\alpha}$ refer to Minkowski space. The 15 parameters encompass b_{μ}, $\Lambda^{\mu}{}_{\nu}$, s, and a^{μ}. Through an extension to Riemann spaces with non-definite metrics, Haantjes [46] established that every element of C_0 can be expressed using motions and inversions alone.

4.2.1 Invariance of Derivatives

In our exploration of the group C, particularly its subgroup C_0, it proves valuable to define quantities that lay bare their transformation properties under C_0. Therefore, the subsequent concepts will be useful.

A tensor in a *Weyl manifold* with weight n and k indices signifies a Riemannian manifold tensor with k indices, transforming as dictated by Eq. (4.33)

$$T_c = \sigma^n T. \tag{4.35}$$

Illustrative instances encompass the metric tensor $g_{\mu\nu}$, a tensor in a Weyl manifold of weight $+1$; dx^μ, a Weyl vector with weight zero; and $d\tau$, a Weyl scalar with weight $1/2$.

For the formulation of a conformally invariant covariant derivative, the presence of an affine connection in a Riemann space, derived from the metric tensor $g_{\mu\nu}$, is pivotal. The covariant derivative of a vector V^μ is then established as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + V^\alpha \begin{Bmatrix} \mu \\ \alpha\,\nu \end{Bmatrix}. \tag{4.36}$$

This left-hand side assumes the form of a Riemannian manifold tensor due to the Christoffel symbol being a solution to the transformation equation of linear connections

$$L'^\mu_{\alpha\beta} = \partial_\lambda x'^\mu (\partial'_\alpha x^\rho \partial'_\beta x^\sigma L^\lambda_{\rho\sigma} + \partial'_\alpha \partial'_\beta x^\lambda). \tag{4.37}$$

The fulfillment of Eq. (4.37) by the Christoffel symbol under the coordinate transformation (4.7) is evident. It's noteworthy that any other solution of (4.37), utilized in (4.36), will similarly yield a ∇_ν such that $\nabla_\nu V^\mu$ takes the form of a Riemannian manifold tensor.

The transformation under C_g can be expressed as

$$\nabla_\nu V^\mu = \partial_{\nu c} V^\mu + V^\alpha \begin{Bmatrix} \mu \\ \alpha\,\nu \end{Bmatrix} = \nabla_\nu V^\mu = \partial_\nu + \frac{1}{2}\Big(V^\mu s_\nu + \delta^\mu_\nu V^\alpha s_\alpha - s^\mu V_\nu\Big), \tag{4.38}$$

assuming that V^μ serves as a Weyl vector with weight zero.

Introducing the symmetric connection

$$\Gamma^\mu_{\alpha\beta} = \begin{Bmatrix} \mu \\ \alpha\,\nu \end{Bmatrix} - \frac{1}{2}(\delta^\mu_\alpha \kappa_\beta + \delta^\mu_\beta \kappa_\alpha - \kappa^\mu g_{\alpha\beta}), \tag{4.39}$$

this yields

$$\overset{c}{\Gamma}{}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} \tag{4.40}$$

given that k^μ transforms as

$$k^c_\mu = k_\mu + s_\mu \tag{4.41}$$

under C_g. Consequently, the expression

$$\nabla_\nu V^\mu \equiv \partial_\nu V^\mu + V^\alpha \Gamma^\mu_{\alpha\nu} \tag{4.42}$$

will be a tensor in a Weyl manifold with weight zero when $\Gamma^\mu_{\alpha\beta}$ fulfills the conditions stipulated by (4.37).

4.3 Conformal Invariance in Quantum theories

Let start with the massive Klein Gordon equation in flat space

$$P^\mu P_\mu \psi = m^2 \psi \tag{4.43}$$

or

$$\eta^{\mu\nu} P_\mu P_\nu \psi = m^2 \psi, \tag{4.44}$$

where m is a number and represents the mass of a particle. By Eq. (4.33) and (4.44) transform as

$$\frac{1}{\sigma} \eta^{\mu\nu} P_\mu P_\nu \psi = m^2 \psi, \tag{4.45}$$

Where we considered P_μ as a Weyl vector of weight zero, W_0. Clearly this equation is not form-invariant. A simple fix is to set $m = 0$, now Eq.(4.44) transforms as

$$\begin{aligned} \frac{1}{\sigma} \eta^{\mu\nu} P_\mu P_\nu \psi &= 0, \\ \eta^{\mu\nu} P_\mu P_\nu \psi &= 0. \end{aligned} \tag{4.46}$$

The Eq. (4.46) is form-invariant under C_g.

Haantjes [58] proposed a character $W_{-1/2}$ for m that makes the Eq. (4.46) invariant even with $m \neq 0$, but for this case m is not invariant under C_g, but takes on a continuum. Only $m = 0$ remains invariant under C_g.

4.4 Galilean Covariance

Galilean Covariance refers to a method of treating Galilean and Newtonian physics using the same tools used in Lorentzian physics. The theory is written in a Light cone of a five dimensional space with Minkowski metric.

4.4.1 Light cones coordinates

A five-dimensional Mikowski metric is given by

$$\eta_{AB} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.47}$$

Making the following changes

$$x^4 = \frac{x^0 + \sqrt{\epsilon}\, x^4}{\sqrt{2}} \quad \text{and} \quad x^5 = \frac{x^0 - \sqrt{\epsilon}\, x^4}{\sqrt{2}}. \tag{4.48}$$

Therefore, we have

$$ds^2 = dx^i dx_i - 2dx^4 dx^5. \tag{4.49}$$

The coordinates x^4 and x^5 are null coordinates and together with the spatial coordinates x^i they are known as the light-cone coordinates.

We can express Eq. (4.49) as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{4.50}$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{4.51}$$

with $\mu, \nu = 1, 2, 3, 4, 5$. So, this metric is know as the Minkowski metric in light-cone coordinates.

The dispersion relation in the Minkowski light-cone coordinates are

$$p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu = (p^i)^2 - 2p^4 p^5 = k^2, \tag{4.52}$$

this dispersion relation resembles the nonrelativistic dispersion relation, and as we will see can be interpreted as such in the appropriated embedding.

4.4.2 Embedding

1. The first embedding considered is defined by

$$\mathcal{E}_1\mathbf{A} \rightarrow A = \left(\mathbf{A}, A_4, \frac{\mathbf{A}^2}{2A_4}\right);\ \mathbf{A} \in \mathcal{E}_1,\ A \in \mathcal{G}. \tag{4.53}$$

   Thus, the square of the norm A is

$$(A|A) = \mathbf{A}^2 - 2A^4A^5 = 0. \tag{4.54}$$

   Embedding $\mathcal{E}_1$ establishes a correspondence of E_3 with null-norm vectors of $\mathcal{G}$. One example is the vectors associated with Galilean invariance in $\mathcal{G}$.

2. The second possible embedding is

$$\mathcal{E}_2\mathbf{A} \rightarrow A = (\mathbf{A}, A_4, 0). \tag{4.55}$$

   Hence the square of the norm of A is not null in $(A|A) = \mathbf{A}^2$. An example is x=($\mathbf{x}$,vt,0).

3. A third embedding possibility is

$$\mathcal{E}_3\mathbf{A} \rightarrow A = (\mathbf{A}, \frac{A_4}{\sqrt{2}}, \frac{A_4}{\sqrt{2}}). \tag{4.56}$$

   Thus, we have $(A|A) = \mathbf{A}^2 - (A_4)^2$. This embedding thus leads to a Minkowski space $\mathcal{M}_{3.1}$ in $\mathcal{G}$.

4.4.3 Galilei-Lie Algebra

Consider a vector $q^\mu \in G$ that obeys the set of linear transformations of the type

$$\bar{q}^\mu = G^\mu{}_\nu q^\nu + a^\mu. \tag{4.57}$$

Using the unitary representation under the function space in a point in $\mathcal{G}$, the generators are defined by

$$M_{\mu\nu} = (q_\mu p_\nu - q_\nu p_\mu), \tag{4.58}$$
$$P_\mu = p_\mu. \tag{4.59}$$

where $M_{\mu\nu}$ are the generators of homogeneous transformations and P_μ of non-homogeneous transformations. We then get the following Lie algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}), \tag{4.60}$$

$$[P_\mu, M_{\rho\sigma}] = -i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho), \tag{4.61}$$

$$[P_\mu, P_\sigma] = 0. \tag{4.62}$$

A particular case of interest of these transformation, given by

$$q^{i\prime} = R^i_j q^j + v^i q^4 + a^i \tag{4.63}$$

$$q^{4\prime} = q^4 + a^4 \tag{4.64}$$

$$q^{5\prime} = q^5 - (R^i_j q^j)v_i + \frac{1}{2}\mathbf{v}^2 q^4. \tag{4.65}$$

In the matrix form, the homogeneous transformations are written as

$$G^\mu{}_\nu = \begin{pmatrix} R^1{}_1 & R^1{}_2 & R^1{}_3 & v^i & 0 \\ R^2{}_1 & R^2{}_2 & R^2{}_3 & v^2 & 0 \\ R^3{}_1 & R^3{}_2 & R^3{}_3 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v_i R^i{}_j & v_i R^i{}_2 & v_i R^i{}_3 & \frac{\mathbf{v}^2}{2} & 1 \end{pmatrix}. \tag{4.66}$$

4.4.4 The Schrödinger Group

Consider the massless dispersion relation on the five dimensional Minkowski space (4.47)

$$p^A p_A = 0. \tag{4.67}$$

This equation is conformally invariant, using the light-cone coordinates defined by the metric (4.51), the dispersion relation becomes

$$p_i p_i - p_5 p_4 - p_4 p_5 = 0. \tag{4.68}$$

This is the non-relativistic dispersion relation.

Since the original dispersion relation has conformal symmetry, this means that the symmetry group of the Eq. (4.68) is a subgroup of the conformal group.

The conformal algebra is

$$\begin{aligned}
\left[\widetilde{M}_{\mu\nu}, \widetilde{M}_{\rho\sigma}\right] &= -i(\eta_{\nu\rho}\widetilde{M}_{\mu\sigma} - \eta_{\mu\rho}\widetilde{M}_{\nu\sigma} + \eta_{\mu\sigma}\widetilde{M}_{\nu\rho} - \eta_{\mu\rho}\widetilde{M}_{\nu\sigma}),\\
\left[\widetilde{P}_{\mu}, \widetilde{M}_{\rho\sigma}\right] &= -i(\eta_{\mu\rho}\widetilde{B}_{\sigma} - \eta_{\mu\sigma}\widetilde{P}_{\rho}),\\
\left[\widetilde{B}_{\mu}, \widetilde{M}_{\rho\sigma}\right] &= i(\eta_{\mu\rho}\widetilde{B}_{\sigma} - \eta_{\mu\sigma}\widetilde{B}_{\rho}),\\
\left[\widetilde{P}_{\mu}, \widetilde{D}\right] &= i\widetilde{P}_{\mu},\\
[\widetilde{D}, \widetilde{B}_{\mu}] &= i\widetilde{B}_{\mu}\\
\left[\widetilde{P}_{\mu}, \widetilde{B}_{\nu}\right] &= -2i(\eta_{\mu\nu}\widetilde{D} - \widetilde{M}_{\mu\nu}),
\end{aligned} \tag{4.69}$$

where $\widetilde{M}_{\mu\nu}$ are the generators of homogeneous transformations and $\widetilde{P}_{\mu}$ of non-homogeneous transformations, $\widetilde{D}$ generates scaling transformations (also known as dilatation) and $\widetilde{B}_{\mu}$ generates the special conformal transformations.

Doing the following identification

$$\begin{aligned}
J_i &= \frac{1}{2}\epsilon_{ijk}\widetilde{M}_{jk}, & K_i &= \widetilde{M}_{5i},\\
& & B &= \frac{\widetilde{B}_5}{2},\\
P_{\mu} &= \widetilde{P}_{\mu}, & D &= \widetilde{D} + \widetilde{M}_{54}.
\end{aligned} \tag{4.70}$$

Hence, the non-vanishing commutation relations can be rewritten as

$$\begin{aligned}
& & [J_i, K_j] &= i\epsilon_{ijk}K_k,\\
[J_i, J_j] &= i\epsilon_{ijk}J_k, & [K_i, C_j] &= i\delta_{ij}D + i\epsilon_{ijk}J_k,\\
[J_i, C_j] &= i\epsilon_{ijk}C_k, & [C_i, D] &= iC_i,\\
[D, K_i] &= iK_i, & [J_i, P_j] &= i\epsilon_{ijk}P_k,\\
[P_4, D] &= iP_4, & [P_i, C_j] &= i\delta_{ij}P_4,\\
[P_i, K_j] &= i\delta_{ij}P_5, & [P_5, C_i] &= iP_i.\\
[P_4, K_i] &= iP_i, & [P_i, D] &= iP_i,\\
[P_5, D] &= -iP_5, & [P_i, B] &= -iK_i,
\end{aligned} \tag{4.71}$$

A special subalgebra is given by

$$
\begin{array}{ll}
[J_i, J_j] = i\epsilon_{ijk} J_k, & [J_i, K_j] = i\epsilon_{ijk} K_k, \\
[D, K_i] = iK_i, & [J_i, P_j] = i\epsilon_{ijk} P_k, \\
[P_4, D] = 2iP_4, & [P_4, B] = iD, \\
[P_i, K_j] = i\delta_{ij} P_5, & [B, D] = -2iB, \\
[P_4, K_i] = iP_i, & [P_i, D] = iP_i, \\
 & [P_i, B] = -iK_i,
\end{array}
\tag{4.72}
$$

all others commutations relation being zero. This algebra can be interpretes as the Schrödinger-Lie algebra if we make the following associations, P_i are the generators of spacial translation, thus can be interpreted as linear momenta operators, $P^5 = -P_4$ is the time translation generator, terefore is intepreted as the Hamiltonian operator, J_i are related to rotations and can be associated with angular momenta, K_i are the generators associated with Galilean boosts, B and D are the generators of special conformal transformations and dilatations respectively. In fact, we can observe that eqs. (4.63) and (4.64) are the Galilei transformations with $x^4 = t$. The Eq. (4.65) is the compatibility condition which represents the embedding

$$
\mathcal{I}\mathbf{A} \to A = \left(\mathbf{A}, A_4, \frac{\mathbf{A}^2}{2A_4}\right); \quad \mathbf{A} \in \mathcal{E}_3, A \in \mathcal{G}.
$$

The commutation of K_i and P_i is naturally non-zero in his context, so P_5 can be interpreted in association with mass.

The invariants of this algebra in this context are

$$
I_1 = P_\mu P^\mu \tag{4.73}
$$

$$
I_2 = P_5 \tag{4.74}
$$

$$
I_3 = W_{5\mu} W_5^\mu, \tag{4.75}
$$

where $W_{\nu\mu}$ is the Pauli-Lubanski matrix in five dimensions.

The Schrödinger Equation

Using the Casimir invariants I_1, (4.73) e I_2, (4.74) and applying in Ψ, we have

$$\begin{aligned} p^\mu p_\mu \Psi &= 0, && (4.76) \\ p_5 \Psi &= -m\Psi. && (4.77) \end{aligned}$$

Using the correspondence, $p_\mu = -i\partial_\mu$, and applying Ψ, we have

$$\begin{cases} \partial_\mu \partial^\mu \Psi = 0 \\ \partial_5 \Psi = -im\Psi \end{cases}, \tag{4.78}$$

where m is a constant and using $\Psi(x_\mu) = exp(-imx^5)(\phi(\mathbf{x}, t)$, we obtain

$$-\frac{1}{2m}\nabla^2\phi(\mathbf{x}, t) = i\partial_t\phi(\mathbf{x}, t), \tag{4.79}$$

which is the time dependent Schrödinger equation of a free particle of mass m.

The 5-current is

$$j^\mu(x) = -\frac{i}{2m}\Big(\psi^*(x)\partial^\mu\psi(x) - \partial^\mu(\psi^*(x))\psi(x)\Big), \tag{4.80}$$

and is conserved because the 5-divergence is null, i.e.

$$\partial_\mu j^\mu = 0. \tag{4.81}$$

So the 5-current is equivalent to the usual 4-current,

$$\mathbf{j}(q) = -\frac{i}{2m}\Big[\Psi^*(q)\nabla\big(\Psi(q)\big) - \nabla(\Psi^*(q)\Psi(q))\Big],$$

$$j^4 = \rho(q) = -\frac{i}{2m}\Big[-\Psi^*(q)\partial_5(\Psi(q)) + \partial_5\Psi^*(q)\Psi(q))\Big] = |\Psi|^2,$$

where $\mathbf{j}(x)$ is the probability current and $\rho(q)$ is the probability density.

4.4.5 The Pauli-Schrödinger Equation

In this context, we present a construction of the spin wave equation 1/2, defining a new pentavetor γ^μ such that,

$$(\partial_\mu \partial^\mu) = (\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu) \tag{4.82}$$

so, for Eq. (4.82) to be valid γ^μ must obey Clifford's algebra, that is,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{4.83}$$

where $g^{\mu\nu}$ is our 5-dimensional metric.

Taking one of the parts and acting on the wave function $\psi(x)$

$$(\gamma^\mu \partial_\mu)\psi(x) = 0. \tag{4.84}$$

For convenience, we will use the following representations of γ^μ

$$\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{4.85}$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}; \tag{4.86}$$

where σ^i are the Pauli's matrices and $\sqrt{2}$ is the 2x2 identity matrix multiplied by $\sqrt{2}$.

Adding a potential V, we have

$$\begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & -(E-V)\sqrt{2} \\ \sqrt{2}m & -\boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix} \begin{pmatrix} \psi^L \\ \psi^S \end{pmatrix} = 0, \tag{4.87}$$

which leads us to

$$\boldsymbol{\sigma} \cdot \mathbf{p}\psi^L - (E - V)\sqrt{2}\psi^S = 0, \tag{4.88}$$

$$\boldsymbol{\sigma} \cdot \mathbf{p}\psi^S - \sqrt{2}m\psi^L = 0, \tag{4.89}$$

these are known as Levy-Leblond equation. In Eq. (4.89) we have

$$\psi^L = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{m\sqrt{2}}\psi^S. \tag{4.90}$$

Substituting in Eq. (4.88) and using the fact $(\boldsymbol{\sigma} \cdot \mathbf{p})(\boldsymbol{\sigma} \cdot \mathbf{p}) = p^2$, we have

$$E\psi^L = \frac{p^2}{2m}\psi^L + V\psi^L. \tag{4.91}$$

Similarly,

$$E\psi^S = \frac{p^2}{2m}\psi^S + V\psi^S. \tag{4.92}$$

What is the Schrödinger equation for ψ^S and ψ^L respectively.

4.5 Conformal Carrollian Covariance

Another subalgebra that can be achieved from Eq. (4.71) is given by

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, & [J_i, C_j] &= i\epsilon_{ijk}C_k, \\ [D, C_i] &= iC_i, & [J_i, P_j] &= i\epsilon_{ijk}P_k, \\ [P_5, D] &= 2iP_5, & [P_5, B] &= iD, \\ [P_5, C_i] &= iP_i, & [B, D] &= -2iB, \\ [P_i, C_j] &= i\delta_{ij}P_4, & [P_i, D] &= iP_i, \\ & & [P_i, B] &= -iC_i, \end{aligned} \tag{4.93}$$

This is a conformal Carrollian algebra isomorphic to the Schrödinger algebra (4.72).

Chapter 5

The Landau Problem and non-Classicality

In this chapter, we delve into the exploration of the extended Galilei group denoted as $\mathcal{G}$ and its representations within the context of field theories on a symplectic manifold. We establish a connection between these representations and the Wigner function. Specifically, we focus on the representation written on the light-cone coordinates of a de Sitter spacetime in five dimensions, constructing a Hilbert space equipped with a symplectic structure as the representation space for the Lie algebra of $\mathcal{G}$.

We derive the spin-zero Schrödinger equation, which describes wave functions in phase space. Additionally, we obtain the Pauli-Schrödinger equation in phase space, accounting for gauge symmetry and addressing spin $\frac{1}{2}$ particles. To demonstrate the practical application of this theory, we examine the behavior of an electron in the presence of an external field, leading to the recovery of the non-relativistic Landau Levels.

Furthermore, we embark on a study of the parameter of negativity associated with the system's non-classicality. By analyzing this parameter, we gain insight into the extent to which the system deviates from classical behavior. Overall, this chapter presents a comprehensive investigation of the extended Galilei group, its representations, and their relevance to field theories on a symplectic manifold, offering valuable insights into the quantum dynamics of physical systems.

The presentation of this chapter is organized as follows. In Section 5.1, we construct a symplectic structure within the Galilean manifold. Utilizing the commutation relations, we establish the Schrödinger equation in the phase space of a five-dimensional light-cone. By proposing a solution, we restore the Schrödinger equation in phase space to its non-covariant form in (3+1) dimensions.

In Section 5.2, we delve into the analysis of gauge symmetry for non-relativistic spin $\frac{1}{2}$ particles in phase space. We investigate the behavior of Galilean spin $\frac{1}{2}$ particles in the presence of an external field, proposing and discussing potential solutions. Furthermore, we calculate the negativity parameter and explore its physical implications.

Moving on to Section 5.3, we explicitly construct the covariant Pauli-Schrödinger equation. We focus on the representation of spin $\frac{1}{2}$ particles within the symplectic framework.

The results of this section are taken from work by Petronilo *et al.*(2020) [24].

5.1 The Galilei Group and Quantum Mechanics in Phase Space

To establish a connection between the Hilbert space $\mathcal{H}$ and the phase space Γ, we consider a set of complex-valued square-integrable functions, denoted as $\phi(q,p)$, defined on Γ. These functions satisfy the condition

$$\int dpdq; \phi^*(q,p)\phi(q,p) < \infty. \tag{5.1}$$

We can express $\phi(q,p)$ as $\langle q,p|\phi\rangle$ using the following relationship

$$\int dpdq; |q,p\rangle\langle q,p| = 1, \tag{5.2}$$

where $\langle\phi|$ represents the dual vector of $|\phi\rangle$. This space is called symplectic Hilbert space and is denoted as $H(\Gamma)$. Now, let us examine the Galilei group within the representation space $H(\Gamma)$. To accomplish this, we introduce unit transformations denoted as $U : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ that preserves the inner product $\langle\psi_1|\psi_2\rangle$.

Using the operator Λ, we define a mapping $e^{i\frac{\Lambda}{2}} = \star : \Gamma \times \Gamma \to \Gamma$, which can be expressed as

$$f \star g = f(q,p) \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}\right)\right] g(q,p),$$

where we have set $\hbar = 1$.

In order to establish a representation of the Galilei algebra within $\mathcal{H}$, we introduce the following operators

$$\widehat{P}^\mu = p^\mu\star = p^\mu - \frac{i}{2}\partial_{q_\mu}, \tag{5.3a}$$

$$\widehat{Q}^\mu = q^\mu\star = q^\mu + \frac{i}{2}\partial_{p_\mu}, \tag{5.3b}$$

and

$$\widehat{M}_{\nu\sigma} = M_{\nu\sigma}\star = \widehat{Q}_\nu \widehat{P}_\sigma - \widehat{Q}_\sigma \widehat{P}_\nu, \tag{5.3c}$$

whew $\mu = 1, 2, 3, 4, 5$. Here, $\widehat{M}_{\nu\sigma}$ represents the generators of homogeneous transformations, while $\widehat{P}_\mu$ corresponds to the generators of non-homogeneous transformations. Through straightforward calculations, we derive the following commutation relations from this set of unitary operators:

$$\begin{aligned}
\left[\widehat{M}_{\mu\nu}, \widehat{M}_{\rho\sigma}\right] &= -i(g_{\nu\rho}\widehat{M}_{\mu\sigma} - g_{\mu\rho}\widehat{M}_{\nu\sigma} + g_{\mu\sigma}\widehat{M}_{\nu\rho} - g_{\mu\sigma}\widehat{M}_{\nu\rho}),\\
\left[\widehat{P}_{\mu}, \widehat{M}_{\rho\sigma}\right] &= -i(g_{\mu\rho}\widehat{P}_\sigma - g_{\mu\sigma}\widehat{P}_\rho),\\
\left[\widehat{P}_{\mu}, \widehat{P}_{\sigma}\right] &= 0.
\end{aligned}$$

These relations give rise to a closed algebra, which is the Lie algebra of the extended Galilei group (Bargmann group). Within this context, we define $\widehat{J}_i = \frac{1}{2}\epsilon_{ijk}\widehat{M}_{jk}$ as the generators of rotations and $\widehat{K}_i = \widehat{M}_{5i}$ as the generators of pure Galilei transformations. Moreover, P_μ represents spatial and temporal translations. Notably, in this setting, the commutation between K_i and P_i is naturally non-zero, with P_5 being associated with mass.

The invariants of this algebra can be expressed as follows

$$I_1 = \widehat{P}_\mu \widehat{P}^\mu, \tag{5.4}$$

$$I_2 = \widehat{P}_5. \tag{5.5}$$

By utilizing the Casimir invariants I_1 and I_2 and applying them to Ψ, we obtain the following equations

$$\begin{aligned}
&\widehat{P}_\mu \widehat{P}^\mu \Psi = k^2\Psi,\\
&\widehat{P}_5 \Psi = -m\Psi.
\end{aligned}$$

From these equations, we derive the expression

$$\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\nabla^2 - k^2\right)\Psi = \left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\Psi,$$

where a solution to this equation is given by

$$\Psi = e^{-2i[(p_5+m)q_5+(p_4+E)t]}\Phi(q,p). \tag{5.6}$$

Consequently, we arrive at the following form

$$\frac{1}{2m}\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\boldsymbol{\nabla}^2\right)\Phi = \left(E + \frac{k^2}{2m}\right)\Phi,$$

which is the Schrödinger equation in phase space for a free particle with mass m [20], including an additional kinetic energy term of $\frac{k^2}{2m}$. We can always choose this term to be the zero energy reference.

This equation, along with its complex conjugate, can be obtained from the Lagrangian density in phase space (where $\partial^\mu = \partial/\partial q_\mu$)

$$\mathcal{L} = \frac{1}{4}\partial^\mu\Psi(q,p)\partial_\mu\Psi^*(q,p) + \frac{i}{2}p^\mu[\Psi(q,p)\partial^\mu\Psi^*(q,p) - \Psi^*(q,p)\partial^\mu\Psi(q,p)] + \left[p^\mu p_\mu - k^2\right]\Psi.$$

The connection between this representation and the Wigner formalism is established through the expression

$$f_w(q,p) = \Psi(q,p)\star\Psi^\dagger(q,p),$$

where $f_w(q,p)$ denotes the Wigner function.

The Wigner function, $f_w(q,p)$, satisfies the 5-dimensional Galilean covariant Liouville-von Neumann equation in phase space, which can be expressed as follows

$$p_\mu\partial_{q_\mu}f_w(q,p) = 0. \tag{5.7}$$

Moreover, in the Galilean covariant formalism, the Pauli-Schrödinger equation exhibits a structure reminiscent of the Dirac equation,

$$\left(\gamma^\mu\widehat{P}_\mu - k\right)\Psi(p,q) = 0 \tag{5.8}$$

or

$$\gamma^\mu\left(p_\mu - \frac{i}{2}\partial_\mu\right)\Psi(p,q) = k\Psi(p,q). \tag{5.9}$$

Equation (5.8) can be derived from the Lagrangian density governing the dynamics of the spin 1/2 particles in phase space, can be expressed as

$$\mathcal{L} = -\frac{i}{4}\left((\partial_\mu \bar{\Psi})\gamma^\mu \Psi - \bar{\Psi}(\gamma^\mu \partial_\mu \Psi)\right) - \bar{\Psi}(k - \gamma^\mu p_\mu)\Psi,$$

Here, $\bar{\Psi} = \zeta \Psi^\dagger$ where ζ is given by

$$\zeta = -\frac{i}{\sqrt{2}}\{\gamma^4 + \gamma^5\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the context of the Pauli-Schrödinger equation, the association with the Wigner function can be expressed as follows

$$f_w = \Psi \star \bar{\Psi},$$

where each component of the equation satisfies Eq. (5.7).

5.2 U(1) Gauge Theory in Phase Space

The Lagrangian density governing the dynamics of Galilean covariant spin 1/2 particles in phase space is given by

$$\mathcal{L} = \overline{\Psi}\gamma^\mu \widetilde{(p_\mu \star)}\Psi - k\overline{\Psi}\Psi, \tag{5.10}$$

here $A\widetilde{(p_\mu \star)}B = \frac{1}{2}[A(p_\mu \star B) - (p_\mu \star A)B]$.

The aim of this section is to examine the invariance of Eq. (5.10) under local gauge transformations

$$\Psi = e^{-i\Omega} \star \Psi, \quad \overline{\Psi} = \overline{\Psi} \star e^{i\Omega}, \tag{5.11}$$

where $\Omega \equiv \Omega(q, p)$. In the context of infinitesimal transformations, we can express the variations as $\delta\Psi = -i\Omega \star \Psi$ and $\delta\overline{\Psi} = i\overline{\Psi} \star \Omega$, such that

$$\delta(p_\mu \star \Psi) = -ip_\mu \star \Omega \star \Psi, \tag{5.12}$$

and

$$\delta(p_\mu \star \overline{\Psi}) = ip_\mu \star \overline{\Psi} \star \Omega. \tag{5.13}$$

It is important to acknowledge that the transformations of $\delta(p_\mu \star \Psi)$ and $\delta(p_\mu \star \overline{\Psi})$ do not exhibit covariance. To address this issue, we introduce the operator

$$D_\mu \star = p_\mu \star -iA_\mu \star, \tag{5.14}$$

which allows us to modify the Lagrangian density as follows

$$\mathcal{L} = \overline{\Psi}\gamma^{\mu}\widetilde{(D_{\mu}\star)}\Psi - k\overline{\Psi}\Psi. \quad (5.15)$$

By utilizing the identity $p(f \star g) = f \star (pg) - \frac{i}{2}(\partial_{\mu} f) \star g$, we can express the infinitesimal variation of $D_{\mu} \star \Psi$ as

$$\delta(D_{\mu} \star \Psi) = -i\Omega \star (p_{\mu} \star \Psi) - \partial_{\mu}\Omega \star \Psi - A_{\mu} \star (\Omega \star \Psi) - i(\delta(A_{\mu}) \star \Psi. \quad (5.16)$$

Considering the transformation of A_{μ} as

$$A'_{\mu} \rightarrow A_{\mu} + i\{A_{\mu}, \Omega\}_M + i\partial_{\mu}\Omega, \quad (5.17)$$

where $\{a, b\}_M = a \star b - b \star a$ denotes the Moyal Brackets, we find

$$\delta(D_{\mu} \star \Psi) = -i\Omega \star (D_{\mu} \star \Psi). \quad (5.18)$$

Likewise,

$$\delta(D_{\mu} \star \overline{\Psi}) = -i(D_{\mu} \star \overline{\Psi}) \star \Omega. \quad (5.19)$$

Therefore, the Lagrangian density presented in Eq. (5.15) is invariant under the transformation described in Eq.(5.11). Consequently, the rule for minimal coupling is to substitute $p_{\mu}\star$ with $D_{\mu}\star = p_{\mu} \star -iA_{\mu}\star$. In the subsequent section, we apply this formalism to analyze the Pauli-Schrödinger equation with electromagnetic interactions.

5.3 Electromagnetic Interactions in Pauli-Schrödinger fields

The equation describing the dynamics of spin 1/2 particle in the Galilean covariant phase space with electromagnetic interaction can be written as

$$\left[\gamma^{\mu}\left(\widehat{P}_{\mu} - e\widehat{A}_{\mu}\right) - k\right]\Psi = 0. \quad (5.20)$$

By introducing the following definition

$$\Psi = \left[\gamma^{\nu}(\widehat{P}_{\nu} - e\widehat{A}_{\nu}) + k\right]\psi, \quad (5.21)$$

where $\widehat{P}_\nu = (p_\nu - \frac{i}{2}\partial_\nu)$, we can express the Eq. (5.20) as

$$\Big[\gamma^\mu\gamma^\nu\big(\widehat{P}_\mu - e\widehat{A}_\mu\big)\big(\widehat{P}_\nu - e\widehat{A}_\nu\big) - k^2\Big]\psi = 0. \tag{5.22}$$

Where, $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$, and

$$\sigma^{\mu\nu} = \frac{1}{2}\Big(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\Big) = \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

Using these results, Eq. (5.22) can be rewritten as

$$\left(\widehat{P}^\mu\widehat{P}_\mu - e\left(\widehat{P}^\mu\widehat{A}_\mu + \widehat{A}^\mu\widehat{P}_\mu\right) - e\sigma^{\mu\nu}\left[\widehat{P}_\nu, \widehat{A}_\mu\right] + e^2\widehat{A}^\mu\widehat{A}_\mu\right)\psi = k^2\psi.$$

Letting $\widehat{A}^i = \frac{1}{2}e^{ijk}B_j\widehat{Q}_k$, where $\widehat{Q}_\mu = (q_\mu + \frac{i}{2}\partial_{p^\mu})$ and $A^4 = A^5 = 0$, with $\mathbf{B} = (0, 0, B)$ chosen as the magnetic field. Restricting the particle's motion to the plane (q_1, q_2), i.e., $\widehat{P}_3 = 0$, we obtain the following equation

$$\begin{aligned}
&- 2\left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\psi \\
&+ \left(p_1^2 + p_2^2 - \frac{1}{4}\Big(\frac{\partial^2}{\partial {q_1}^2} + \frac{\partial^2}{\partial {q_2}^2}\Big) - eB\left[\frac{i}{2}\Big(p_2\partial_{p_1} - p_1\partial_{p_2}\Big)\right.\right. \\
&+ \left.\frac{1}{4}\Big(\frac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \frac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\right] - i\Big(p_2\partial_{q_2} + p_1\partial_{q_2}\Big) - eB\left[(q_1p_2 - q_2p_1) - \frac{i}{2}\Big(q_1\partial_{q_2} - q_2\partial_{q_1}\Big)\right] \\
&+ \left.\frac{e^2B^2}{4}\left[\Big(q_1 + \frac{i}{2}\partial_{p_1}\Big)^2 + \Big(q_2 + \frac{i}{2}\partial_{p_2}\Big)^2\right] - ie\sigma^{12}B\right)\psi = k^2\psi,
\end{aligned} \tag{5.23}$$

where $\sigma^{12} = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$.

Letting

$$\psi = \begin{pmatrix} \Phi(q^\mu, p^\mu) \\ \Theta(q^\mu, p^\mu) \end{pmatrix},$$

We have obtained a pair of uncoupled equations, one for $\Phi(q^\mu, p^\mu)$ and the other for $\Theta(q^\mu, p^\mu)$, given by

$$-2\left(p_4 - \tfrac{i}{2}\partial_t\right)\left(p_5 - \tfrac{i}{2}\partial_5\right)\Phi(q^\mu, p^\mu)$$
$$+\left(p_1^2 + p_2^2 - \tfrac{1}{4}\left(\tfrac{\partial^2}{\partial q_1{}^2} + \tfrac{\partial^2}{\partial q_2{}^2}\right) - eB\left[\tfrac{i}{2}\left(p_2\partial_{p_1} - p_1\partial_{p_2}\right)\right.\right.$$
$$\left.+\tfrac{1}{4}\left(\tfrac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \tfrac{\partial^2}{\partial_{q_1}\partial_{p_2}}\right)\right] - i\left(p_2\partial_{q_2} + p_1\partial_{q_2}\right)$$
$$-eB\left[(q_1p_2 - q_2p_1) - \tfrac{i}{2}\left(q_1\partial_{q_2} - q_2\partial_{q_1}\right)\right] + \tfrac{e^2B^2}{4}\left[\left(q_1 + \tfrac{i}{2}\partial_{p_1}\right)^2 + \left(q_2 + \tfrac{i}{2}\partial_{p_2}\right)^2\right]$$
$$\left.+e\sigma^3 B\right)\Phi(q^\mu, p^\mu) = k^2\Phi(q^\mu, p^\mu),$$

and the equation for Θ follows a similar structure.
Taking $\Phi(q^\mu, p^\mu) = \varphi(q^i, p^i)\phi(q^4, q^5, p^4, p^5)$. Which result in

$$\left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\phi = mE\phi + k^2\phi, \tag{5.24a}$$

and

$$\begin{aligned}
&\left(p_1^2 + p_2^2 - \frac{1}{4}\left(\frac{\partial^2}{\partial q_1{}^2} + \frac{\partial^2}{\partial q_2{}^2}\right) - eB\left[\frac{i}{2}\left(p_2\partial_{p_1} - p_1\partial_{p_2}\right)\right.\right. \\
+ \;& \left.\frac{1}{4}\left(\frac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \frac{\partial^2}{\partial_{q_1}\partial_{p_2}}\right)\right] - i\left(p_2\partial_{q_2} + p_1\partial_{q_2}\right) \\
- \;& eB\left[(q_1p_2 - q_2p_1) - \frac{i}{2}\left(q_1\partial_{q_2} - q_2\partial_{q_1}\right)\right] \\
+ \;& \left.\frac{e^2B^2}{4}\left[\left(q_1 + \frac{i}{2}\partial_{p_1}\right)^2 + \left(q_2 + \frac{i}{2}\partial_{p_2}\right)^2\right] + e\sigma^3 B\right)\varphi \\
= \;& 2mE\varphi + k^2\varphi
\end{aligned} \tag{5.24b}$$

A solution of Eq. (5.24a) is of the form

$$\phi = C_1 e^{-2i[(p_5+m)q_5+(p_4+E)t]},$$

C_1 is a normalization constant. In order to address the solution of Eq. (5.24b), a transformation of variables will be introduced, denoted by

$$w(q_1, q_2, p_1, p_2) \quad = \quad p_1^2 + p_2^2 + eB(q_2 p_1 - q_1 p_2) + \frac{e^2 B^2}{4}(q_1^2 + q_2^2).$$

Following an extensive calculation, it becomes evident that the imaginary part of this equation is identically zero, yielding us

$$w\varphi - e^2 B^2 \frac{\partial \varphi(w)}{\partial \omega} - e^2 B^2 w \frac{\partial^2 \varphi(w)}{\partial w^2} = (2mE - esB + k^2)\varphi(w), \tag{5.25}$$

with $s\varphi = \sigma^3 \varphi$, with $s = \pm 1$. Consider $\omega = w/(eB)$, $\alpha = (2mE - seB + k^2)/eB$ and letting $f(w) \equiv e^w \phi(\omega)$, we have

$$\omega f''(\omega) + (1 - 2\omega) f'(\omega) - a f(\omega) = 0, \tag{5.26}$$

where $f'(x) = \frac{\partial f}{\partial \omega}$ and $a = (1 - \alpha)$. The Eq. (5.26) corresponds to the Kummer equation, a type of confluent hypergeometric equation. The physical solutions are expressed as follows,

$$f_n(\omega) = A_n U\left(\frac{1}{2} - \frac{\alpha}{2}, 1, 2\omega\right),$$

where $U(a, b, x)$ are the Kummer's function and A_n are constants. Nonetheless, it becomes evident that when $a = -n$ with $n = 0, 1, 2, \ldots$, the series $U(a, b, x)$ transforms into a polynomial series in x of degree no more than n. Therefore, expressing it as

$$\alpha - 1 = 2n,$$

We have the following eigenvalue relation.

$$E = \omega_c\left(n + \frac{1}{2} + \frac{s}{2}\right) - \frac{k^2}{2m},$$

with $\omega_c = \frac{eB}{m}$ and the corresponding eigenfunctions

$$f_n(w) = A_n U\left(-n, 1, \frac{2w}{eB}\right), \tag{5.27}$$

where A_n are normalization constants. As a result, the quasi-amplitudes is given by

$$\Phi_n = C_1 e^{-2i[(p_5+m)q_5+(p_4+E)t]}\left(A_n U\left(-n, 1, \frac{2w}{eB}\right)\exp\left(-\frac{w}{eB}\right)\right). \qquad (5.28)$$

The analog is valid for Θ. These calculations provide the correct value of the eigenstates of energy for fermions in an external magnetic field, the well-known problem of Landau. Therefore, formalism is consistent with the standard calculation. To find the corresponding Wigner function just do $\psi_n \star \bar{\psi}_n$.

To determine the expression for f_w, we will apply a similar procedure to that used for the harmonic oscillator. Note that in this case, we have $w = 2mh$, where

$$h = \frac{1}{2m}\left(p_1^2 + p_2^2 + eB(q_2p_1 - q_1p_2) + \frac{e^2B^2}{4}(q_1^2 + q_2^2)\right).$$

This step allows us to establish the appropriate relation for w in terms of the given parameters.

Therefore,

$$\psi_0 = C_0 e^{-2h/\omega_c}$$

Thus,

$$f_w^0 = C_0 e^{-2h/\omega_c} \star \psi_0 = C_0 e^{-2\hat{h}/\omega_c}\psi_0 = C_0 e^{-2E_0/\omega_c}\psi_0$$

Hence, the Wigner function describing the ground state of a spin-$\frac{1}{2}$ particle with spin-up and spin-down configurations can be expressed as follows

$$f_w^{0+} = (C_{0^+})^2 \frac{1}{e^2} e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB},$$

and

$$f_w^{0-} = (C_{0^-})^2 e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB}.$$

As for the general case,

$$f_w^{n^\pm} = (A_n^\pm)\Big(\frac{1}{n!\pi}\Big)e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))}U\left(-n,1,\frac{2(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))}{eB}\right) \quad (5.29)$$

The characteristics of the Wigner function in this system are depicted in Figures 1 through 4. From these plots, it becomes evident that the Wigner function corresponding to the ground level ($n = 0$), as depicted in Figure 1, lacks a negative component. Conversely, for higher energy levels, illustrated in Figures 2 through 4, the Wigner function exhibits a negative region. The graphical representations underscore a notable trend: the magnitude of the negative portion in the Wigner function becomes more prominent with increasing energy levels of the system under consideration.

Non-classicality Indicator

The Wigner function satisfies the normalization condition $\int_{-\infty}^{\infty} f_w(q,p), dq, dp = 1$. Hence

$$\begin{aligned}\eta(\psi) &= \int_{-\infty}^{\infty}\Big(|f_w(q,p)| - f_w(q,p)\Big)dqdp \\ &= \int_{-\infty}^{\infty}|f_w(q,p)|dqdp - 1,\end{aligned}$$

serves as an indicator of negativity for the vector $|\psi\rangle$ [59].

In the following, we conducted numerical calculations of this indicator for the Landau levels. The results of these calculations are shown in Table (5.1).

Table 5.1: *Parameter of negativity of the Landau Levels for* $n = 1, 2, 3$.

n	$\eta(\psi)$
0	0
1	0.42612
2	0.72899
3	0.97667

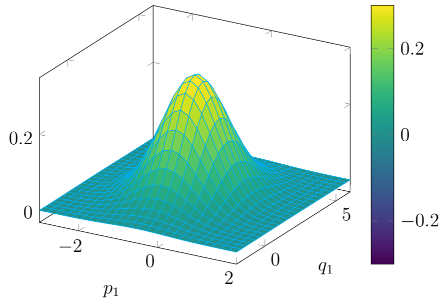

Figure 5.1: *Wigner Function (cut in q_1,p_1),Ground State.*

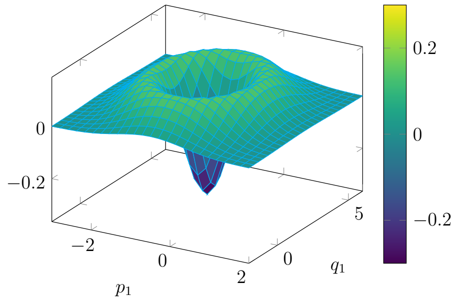

Figure 5.2: *Wigner Function (cut in q_1,p_1), First Excited State.*

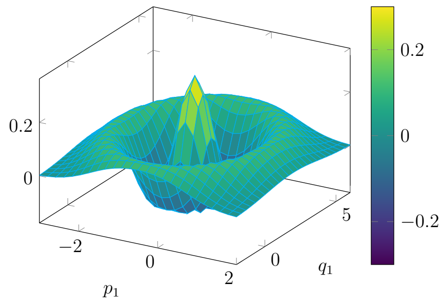

Figure 5.3: *Wigner Function (cut in q_1,p_1), Second Excited State.*

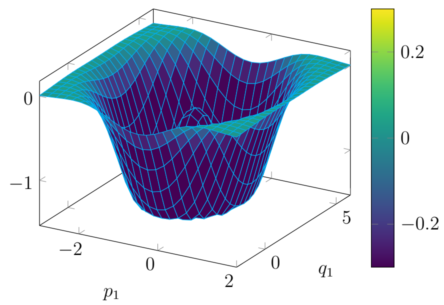

Figure 5.4: *Wigner Function (cut in q_1,p_1), Third Excited State.*

The data presented in Table (5.1) demonstrate a consistent pattern where the negativity indicator aligns with the system's energy level n, corroborating the trends depicted in FIG. 5.1 through FIG. 5.4. This observation suggests a potential connection between the growth of the indicator parameter and an increase in the quantum entanglement of the analyzed system. Such insights are of significance in fields like quantum computing, as evident in [60].

Chapter 6

Symplectic Galilean fields at finite temperature

In this chapter, the Galilean covariance in phase space is considered. The non-relativistic symplectic version of Klein-Gordon and Dirac equations are constructed in this framework. The propagator associated with both fields are determined. Then the non-relativistic Stefan-Boltzmann law and Casimir effect at finite temperature are calculated. The Thermofield Dynamics formalism is introduced, thus using thermal effects.

The main goal of this chapter is to explore the Galilean covariance in phase space and to investigate the size effect of non-relativistic quantum systems compactified in a torus. For quantum mechanics, the TFD formalism is used to deal with the spatial torus with temperature effects in the context of Wigner functions.

The formulation of classical mechanics in phase space is well-known. It leads to distinct physical meaning with a Hamiltonian function. In the case of quantum mechanics, this function is of paramount importance for the dynamical processes but the structure of the phase space is due to the non-commutativity between position and momentum. The formalism of quantum mechanics in phase space was introduced by Wigner. This allows the mapping of classical functions into quantum operators in phase space through the Moyal product. Wigner was interested in solving problems opposite to the one proposed by Weyl who established a quantization process based on classical functions. The formulation of the phase space has a well-defined classical limit. In contrast to the wave function, the Wigner function does not represent a true probability density. The formulation of a field in phase space has an interest in generalizing representations. It is important to note that observables are preserved in phase space. Another step in quantum phase space is given when the formalism could accommodate consistently the notion of gauge field with the Wigner function. This was accomplished with the notion of quasi-amplitude of probabilities, i.e. a wave function associated with the Wigner function. In order to consider the phase space structure of a quantum system in a torus, this formalism in phase space is considered with TFD.

This chapter is organized as follows. In Section 6.1, the spin 1/2 symplectic representation is introduced. The Dirac-like Lagrangian is obtained. In Section 6.2, a brief introduction to the TFD formalism is presented. In Section 6.3, some applications at finite temperature are calculated for the non-relativistic Stefan-Boltzmann law and the Casimir effect.

The results of this section are taken from work by Petronilo *et al.*(2021A) [61].

6.1 Spin 1/2 Symplectic Representation

This section provides a brief overview of the spin 1/2 symplectic representation. The Lagrangian density for a spin-1/2 field is given by:

$$\mathcal{L} = -\frac{i}{4}\Big((\partial_\mu\bar{\psi}(q,p))\gamma^\mu\psi(q,p) - \bar{\psi}(q,p)(\gamma^\mu\partial_\mu\psi(q,p))\Big) + \bar{\psi}(q,p)\gamma^\mu p_\mu\psi(q,p), \tag{6.1}$$

where $\bar{\psi} = \zeta\psi^\dagger$, with $\zeta = -\frac{i}{\sqrt{2}}\{\gamma^4 + \gamma^5\}$.

The Galilean covariant Pauli-Schrödinger equation is given by:

$$\gamma^\mu\left(p_\mu - \frac{i}{2}\partial_\mu\right)\psi(p,q) = 0. \tag{6.2}$$

The connection to the Wigner function is given by

$$f_w(q,p) = \psi(q,p) \star \bar{\psi}(q,p).$$

The energy-momentum tensor for the Dirac-like field in phase-space is obtained by using the Noether theorem as:

$$T_D^{\mu\nu}(q,p) = -\frac{i}{4}\left(-\bar{\psi}(q,p)\gamma^\mu\frac{\partial\psi(q,p)}{\partial q_\nu} + \frac{\partial\bar{\psi}(q,p)}{\partial q_\nu}\gamma^\mu\psi(q,p)\right) - \eta^{\mu\nu}\mathcal{L}(q,p). \tag{6.3}$$

The Green function, $G_D(q-q', p-p')$, in phase space is obtained from the equation

$$\gamma^\mu\left(\frac{i}{2}\partial_\mu - p_\mu\right)G_D(q-q',p-p') = \delta(q-q')\delta(p-p'). \tag{6.4}$$

For $q - q'$ it leads to

$$\gamma^\mu \left(\frac{1}{2}k_\mu - p_\mu\right)\overline{G}(k, p - p') = \delta(p - p'), \tag{6.5}$$

with $\overline{G}(k, p-p') = \frac{1}{(2\pi)^4}\int d^4q e^{ik^\mu(q_\mu - q'_\mu)} G_D(q - q', p - p')$. Then the field propagator is

$$\overline{G}(k, p - p') = \frac{\delta(p - p')}{\gamma^\mu \left[\frac{1}{2}k_\mu - \left(p_\mu - p'_\mu\right)\right]}. \tag{6.6}$$

This leads to

$$\begin{aligned} G_D(q - q', p - p') &= \int \frac{d^5k}{(2\pi)^5} e^{-ik^\mu(q_\mu - q'_\mu)} \overline{G}(k, p - p') \\ &= \int \frac{d^5k}{(2\pi)^5} e^{-ik^\mu(q_\mu - q'_\mu)} \frac{\delta(p - p')}{\gamma^\mu \left[\frac{1}{2}k_\mu - \left(p_\mu - p'_\mu\right)\right]}, \end{aligned} \tag{6.7}$$

which is the Green function for use at finite temperature.

6.2 Thermofield Dynamics

The Thermofield Dynamics (TFD) formalism is introduced here for study at finite temperatures [38–42]. The thermal average of an observable is considered as the vacuum expectation value in an extended Fock space, i.e., $\mathcal{S}_T = \mathcal{S} \otimes \tilde{\mathcal{S}}$, where $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are the original and tilde space respectively. This defines the Bogoliubov transformation. The relation between the tilde $\widetilde{\mathcal{X}}_i$ and non-tilde $\mathcal{X}_i$ operators is defined as

$$(\mathcal{X}_i\mathcal{X}_j)^\sim = \widetilde{\mathcal{X}}_i\widetilde{\mathcal{X}}_j, \quad (c\mathcal{X}_i + \mathcal{X}_j)^\sim = c^*\widetilde{\mathcal{X}}_i + \widetilde{\mathcal{X}}_j, \qquad (\mathcal{X}_i^\dagger)^\sim = \widetilde{\mathcal{X}}_i^\dagger, \qquad (\widetilde{\mathcal{X}}_i)^\sim = -\xi\mathcal{X}_i\,, \tag{6.8}$$

with $\xi = -1$ for bosons and $\xi = +1$ for fermions. The Bogoliubov transformation describes the finite temperature effect among variables. This leads to

$$\begin{pmatrix} \mathcal{X}(k, \alpha) \\ \widetilde{\mathcal{X}}^\dagger(k, \alpha) \end{pmatrix} = \mathcal{B}(\alpha) \begin{pmatrix} \mathcal{X}(k) \\ \widetilde{\mathcal{X}}^\dagger(k) \end{pmatrix}, \tag{6.9}$$

and the Bogoliubov transformation, $\mathcal{B}(\alpha)$, is

$$\mathcal{B}(\alpha) = \begin{pmatrix} u(\alpha) & -v(\alpha) \\ \xi v(\alpha) & u(\alpha) \end{pmatrix}, \tag{6.10}$$

where $u^2(\alpha) + \xi v^2(\alpha) = 1$. The α parameter is defined as the compactification parameter given by $\alpha = (\alpha_0, \alpha_1, \cdots \alpha_{D-1})$, where D is the dimension of spacetime. The temperature effect is described by the choice $\alpha_0 \equiv \beta$ and $\alpha_1, \cdots \alpha_{D-1} = 0$, where $\beta \propto \frac{1}{T}$ with T being the temperature.

The propagator for the scalar field is

$$G_0^{(ab)}(q-q', p-p'; \alpha) = i\langle 0, \widetilde{0}|\tau[\psi^a(q,p;\alpha)\psi^b(q',p';\alpha)]|0,\widetilde{0}\rangle, \tag{6.11}$$

where,

$$\psi(q,p;\alpha) \quad = \quad \mathcal{B}(\alpha)\psi(q,p)\mathcal{B}^{-1}(\alpha). \tag{6.12}$$

Here $a, b = 1, 2$ and τ is the time ordering operator. Then the propagator at the thermal vacuum $|0(\alpha)\rangle = \mathcal{B}(\alpha)|0,\widetilde{0}\rangle$ is

$$\begin{aligned} G_0^{(ab)}(q-q', p-p'; \alpha) &= i\langle 0(\alpha)|\tau[\psi^a(q,p)\psi^b(q',p')]|0(\alpha)\rangle, \\ &= i\int \frac{d^5k}{(2\pi)^5} e^{-ik(q-q')(p-p')} G_0^{(ab)}(k;\alpha), \end{aligned} \tag{6.13}$$

where

$$G_0^{(ab)}(k;\alpha) = \mathcal{B}^{-1}(\alpha)G_0^{(ab)}(k)\mathcal{B}(\alpha), \tag{6.14}$$

with

$$G_0^{(ab)}(k) = \begin{pmatrix} G_0(k) & 0 \\ 0 & G_0^*(k) \end{pmatrix}. \tag{6.15}$$

Then $G_0(k)$ is given as

$$G_0(k) = \frac{\delta^5(p-p')}{\frac{1}{4}k^2 - p^\mu k_\mu + p^\mu p_\mu + i\epsilon}, \tag{6.16}$$

where delta function is the Dirac delta [43]. Then the non-tilde variable is

$$G_0^{(11)}(k;\alpha) = G_0(k) + \xi v^2(k;\alpha)[G_0^*(k) - G_0(k)], \tag{6.17}$$

where $v^2(k;\alpha)$ is the generalized Bogoliubov transformation given as

$$v^2(k;\alpha) = \sum_{s=1}^{d} \sum_{\{\sigma_s\}} 2^{s-1} \sum_{l_{\sigma_1},\ldots,l_{\sigma_s}=1}^{\infty} (-\eta)^{s+\sum_{r=1}^{s} l_{\sigma_r}} \exp\left[-\sum_{j=1}^{s} \alpha_{\sigma_j} l_{\sigma_j} k^{\sigma_j}\right], \tag{6.18}$$

with d being the number of compactified dimensions, $\eta = 1(-1)$ for fermions (bosons) and $\{\sigma_s\}$ denotes the set of all combinations with s elements.

6.3 Non-relativistic Stefan-Boltzmann law and Casimir effect in phase space

In this section the Stefan-Boltzmann law and Casimir effect at finite temperature are obtained for Schrödinger (Klein-Gordon-like) and Pauli-Schrödinger (Dirac-like) equations in phase space.

6.3.1 Schrödinger equation

The Galilean Lagrangian density for covariant scalar field in phase space is given as

$$\begin{aligned} \mathcal{L}_\psi(q,p) &= \frac{1}{4}\partial^\mu\psi(q,p)\partial_\mu\psi^\dagger(q,p) + \frac{i}{2}p^\mu[\psi(q,p)\partial_\mu\psi^\dagger(q,p) \\ &- \psi^\dagger(q,p)\partial_\mu\psi(q,p)] + [p^\mu p_\mu]\,\psi(q,p)\psi^\dagger(q,p). \end{aligned} \tag{6.19}$$

The energy-momentum tensor, is given as

$$T^{\mu\nu}(q,p) = \frac{\partial\mathcal{L}_\psi(q,p)}{\partial(\partial_\mu\psi(q,p))}\partial^\nu\psi(q,p) - \eta^{\mu\nu}\mathcal{L}_\psi(q,p). \tag{6.20}$$

This leads to

$$T^{\mu\nu}(q,p) = \frac{1}{4}\partial^\mu\psi(q,p)\partial^\nu\psi^\dagger(q,p) + \frac{i}{2}p^\mu[\psi(q,p)\partial^\nu\psi^\dagger(q,p) - \psi^\dagger(q,p)\partial^\nu\psi(q,p)] \tag{6.21}$$

$$\begin{aligned} &- \eta^{\mu\nu}\left[\frac{1}{4}\partial^\lambda\psi(q,p)\partial_\lambda\psi^\dagger(q,p) + \frac{i}{2}p^\lambda[\psi(q,p)\partial_\lambda\psi^\dagger(q,p) - \psi^\dagger(q,p)\partial_\lambda\psi(q,p)]\right. \\ &+ \left.[p^\lambda p_\lambda]\,\psi(q,p)\psi^\dagger(q,p)\right]. \end{aligned} \tag{6.22}$$

It is necessary to re-write the energy-momentum tensor, at different space-time points as

$$
\begin{aligned}
T^{\mu\nu}(q,p) &= \lim_{(q'^{\mu},p'^{\mu})\to(q^{\mu},p^{\mu})} \tau\Bigg\{\frac{1}{4}\left(\frac{\partial\psi'^{\dagger}(q,p)}{\partial q'_{\mu}}\frac{\partial\psi(q,p)}{\partial q_{\nu}}+\frac{\partial\psi'^{\dagger}(q,p)}{\partial q'_{\nu}}\frac{\partial\psi(q,p)}{\partial q_{\mu}}\right)\\
&+ \frac{i}{2}p^{\mu}\left[\psi(q,p)\frac{\partial\psi'^{\dagger}(q,p)}{\partial q'_{\nu}}-\psi'^{\dagger}(q,p)\frac{\partial\psi(q,p)}{\partial q_{\nu}}\right]-\eta^{\mu\nu}\Bigg[\frac{1}{4}\partial^{\lambda}\psi(q,p)\partial_{\lambda}\psi^{\dagger}(q,p)\\
&+ \frac{i}{2}p^{\lambda}\left[\psi(q,p)\partial_{\lambda}\psi'^{\dagger}(q,p)-\psi^{\dagger}(q,p)\partial_{\lambda}\psi(q,p)\right]+p^{\lambda}p_{\lambda}\psi(q,p)\psi'^{\dagger}(q,p)\Bigg]\Bigg\}\\
&= \lim_{(q'^{\mu},p'^{\mu})\to(q^{\mu},p^{\mu})} \Gamma_1^{\mu\nu}\tau\{\psi(q,p)\psi^{\dagger}(q,p)\}
\end{aligned}
\tag{6.23}
$$

where τ is the time ordering operator and

$$
\begin{aligned}
\Gamma_1^{\mu\nu} &= \frac{1}{4}\left(\frac{\partial}{\partial q'_{\mu}}\frac{\partial}{\partial q_{\nu}}+\frac{\partial}{\partial q'_{\nu}}\frac{\partial}{\partial q_{\mu}}\right)+\frac{i}{2}p^{\mu}\left[\frac{\partial}{\partial q'_{\nu}}-\frac{\partial}{\partial q_{\nu}}\right]\\
&- \eta^{\mu\nu}\left[\frac{1}{4}\partial^{\lambda'}\partial_{\lambda}+\frac{i}{2}p^{\lambda}\left[\frac{\partial}{\partial q'^{\lambda}}-\frac{\partial}{\partial q^{\lambda}}\right]+\left[p^{\lambda}p_{\lambda}\right]\right].
\end{aligned}
\tag{6.24}
$$

The vacuum expectation value of the energy-momentum tensor is given as

$$
\langle T^{\mu\nu}(q,p)\rangle = \lim_{(q'^{\mu},p'^{\mu})\to(q^{\mu},p^{\mu})}\left\{\Gamma_1^{\mu\nu}G_0(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu})\right\}. \tag{6.25}
$$

The Green function propagator in phase space is

$$
\begin{aligned}
G_0(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu}) &= \left\langle 0\left|\tau\left[\psi(q,p)\psi'^{\dagger}(q',p')\right]\right|0\right\rangle,\\
&= \int\frac{d^5k}{(2\pi)^5}\frac{\delta^5(p-p')e^{ik_{\mu}(q^{\mu}-q'^{\mu})}}{\frac{1}{4}k^2-k_{\mu}(p^{\mu}-p'^{\mu})+(p^{\mu}-p'^{\mu})(p_{\mu}-p'_{\mu})}\\
&= \frac{\delta^3(\mathbf{p}-\mathbf{p}')\delta(p_4-p'_4)}{(2\pi)^3}\theta(t-t')e^{-2im(s-s')}\int d^3k\, e^{i\left[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-4(K^0+p_4)(t-t')\right]},
\end{aligned}
$$

with $K^0=\frac{1}{2m}(\frac{\mathbf{k}}{2}-\mathbf{p})^2$. After integration the propagator is

$$
\begin{aligned}
G_0(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu}) &= \delta^3(\mathbf{p}-\mathbf{p}')\delta(p_4-p'_4)\left(\frac{m}{2\pi}\right)^{3/2}\frac{\sqrt{i}}{(t-t')^{3/2}}\theta(t-t')\exp\left\{\frac{(p_4-p'_4)(t-t')}{m}\right\}\\
&\times \exp\left\{-im(s-s')+\frac{im(\mathbf{x}-\mathbf{x}')^2+2i(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{x}-\mathbf{x}')(t-t')}{2(t-t')}\right\}
\end{aligned}
\tag{6.26}
$$

The finite energy-momentum tensor with the α-parameter is

$$\mathbb{T}^{\mu\nu(ab)}(q,p;\alpha) = \lim_{(q'^{\mu},p'^{\mu})\to(q^{\mu},p^{\mu})} \left\{ \Gamma_1^{\mu\nu}\overline{G}_0^{ab}(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu};\alpha)\right\}, \tag{6.27}$$

where $\mathbb{T}^{\mu\nu(ab)}(q,p;\alpha) = \langle \mathcal{T}^{\mu\nu(ab)}(q,p;\alpha)\rangle - \langle \mathcal{T}^{\mu\nu(ab)}(q,p)\rangle$ and

$$\overline{G}_0^{(ab)}(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu};\alpha) = G_0^{(ab)}(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu};\alpha) - G_0^{(ab)}(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu}). \tag{6.28}$$

Some applications for the Klein-Gordon-like equation with different α-parameters are investigated. It is important to note that an average of components of the energy-momentum tensor diverges in the limit $q'^{\mu}\to q^{\mu}$ at zero temperature.

Non-relativistic Stefan-Boltzmann Law

To calculate the Stefan-Boltzmann law at finite temperature, β, the α-parameter has a value of $\alpha=(0,0,0,\beta,0)$. Then the generalized Bogoliubov transformation is

$$v^2(\beta) = \sum_{j=1}^{\infty} e^{-\beta k j} \tag{6.29}$$

and the Green function becomes

$$\overline{G}_0^{(ab)}(q^{\mu}-q'^{\mu},p^{\mu}-p'^{\mu};\beta) = 2\sum_{j=1}^{\infty} G_0(q^{\mu}-q'^{\mu}-i\beta j n_4,p^{\mu}-p'^{\mu}), \tag{6.30}$$

where $n_4=(0,0,0,1,0)$. Thus the component of energy-momentum tensor for $\mu=\nu=4$ is

$$\mathbb{T}^{44(11)}(\beta) \;=\; 2\lim_{(q'^{\mu},p'^{\mu})\to(q^{\mu},p^{\mu})}\sum_{j=1}^{\infty}\Gamma_1^{44}G_0^{(11)}\left(q^{\mu}-q'^{\mu}-i\beta j n_4,p^{\mu}-p'^{\mu}\right), \tag{6.31}$$

where

$$\Gamma_1^{44} = \frac{1}{2}\partial_5'\partial_5 + \frac{i}{2}p_5(\partial_5'-\partial_5). \tag{6.32}$$

Which leads to the form

$$\mathbb{T}^{44(11)}(\beta) \;=\; \delta^3(p-p')\delta(p_4-p_4')\,\frac{m^{7/2}\zeta(\frac{3}{2})}{(8\pi\beta)^{3/2}}, \tag{6.33}$$

where $p_5 = m$ and $\zeta(x)$ is the Riemann Zeta function. This is the non-relativistic Stefan-Boltzmann law in phase space.

Non-relativistic Casimir effect at finite temperature

Taking $\alpha = (0, 0, i2d, \beta, 0)$ the Bogoliubov transformation is

$$v^2(\beta, d) = \sum_{j=1}^{\infty} e^{-\beta kj} + \sum_{l=1}^{\infty} e^{-i2dkl} + 2\sum_{j,l=1}^{\infty} e^{-\beta kj - i2dkl}. \quad (6.34)$$

The first two terms are associated with the Stefan-Boltzmann law and the Casimir effect at zero temperature. The Green function of the third term is

$$\overline{G}_0(q^\mu - q'^\mu, p^\mu - p'^\mu; \beta, d) = 4\sum_{j,l=1}^{\infty} G_0\left(q^\mu - q'^\mu - i\beta j n_4 - 2dln_3, p^\mu - p'^\mu\right) \quad (6.35)$$

with $n_3 = (0, 0, 1, 0, 0)$. Then the energy-momentum tensor for $\mu = \nu = 3$ is

$$\mathbb{T}^{33(11)}(\beta, d) = 4 \lim_{(q'^\mu, p'^\mu)\to(q^\mu, p^\mu)} \sum_{j,l=1}^{\infty} \Gamma_1^{33} G_0^{(11)}\left(q^\mu - q'^\mu - i\beta j n_4 - 2dln_3, p^\mu - p'^\mu\right) \quad (6.36)$$

where

$$\begin{aligned}\Gamma_1^{33} &= \frac{1}{4}\left(\partial_3'\partial_3 - \partial_1'\partial_1 - \partial_2'\partial_2 + 2\partial_4'\partial_5\right) \quad (6.37)\\ &- \frac{i}{2}\left[p^1(\partial_1' - \partial_1) + p^2(\partial_2' - \partial_2) + p^4(\partial_4' - \partial_4) + p^5(\partial_5' - \partial_5)\right] - p^\lambda p_\lambda.\end{aligned}$$

Then the non-relativistic Casimir effect in phase space at finite temperature is given as

$$\begin{aligned}\left\langle 0(\beta)\left|T^{33(11)}\right|0(\beta)\right\rangle &= \delta^3(p - p')\delta(p_4 - p_4')\sum_{j,l=1}^{\infty}\left\{\frac{m^{3/2}}{64\sqrt{2\pi^3}(\beta j)^{7/2}} e^{\frac{-dl(dlm + 4i\beta jp_3)}{2\beta j}}\right.\\ &\times \left[m\left(5d^2l^2m - 11\beta j\right) + 4(2md^2l^2p_4 + i\beta jdlmp_3 - 4\beta^2 j^2 mp_5\right.\\ &+ \left.\left. 5(\beta jp_1)^2 + 5(\beta jp_2)^2 - (\beta jp_3)^2 - 6\beta jp_4)\right] - p^\lambda p_\lambda\right\}. \quad (6.38)\end{aligned}$$

6.3.2 Pauli-Schrödinger equation

The Pauli-Schrödinger equation in phase space is described by the Dirac-like Lagrangian density in the Galilean manifold as

$$\mathcal{L} = -\frac{i}{4}\left(\partial_\mu\bar{\psi}\gamma^\mu\psi - \bar{\psi}\gamma^\mu\partial_\mu\psi\right) - \bar{\psi}\gamma^\mu p_\mu\psi \tag{6.39}$$

The Galilean Dirac-like propagator is

$$\begin{aligned} G_D(q-q',p-p') &= \int \frac{d^5k}{(2\pi)^5} e^{-ik^\mu(q_\mu - q'_\mu)}\overline{G}(k,p-p') \\ &= \int \frac{d^5k}{(2\pi)^5} e^{-ik^\mu(q_\mu - q'_\mu)} \frac{\delta(p-p')}{\gamma^\mu\left[\frac{1}{2}k_\mu - \left(p_\mu - p'_\mu\right)\right]}, \end{aligned} \tag{6.40}$$

with some algebra we obtain

$$G_D(q-q',p-p') = 2e^{-i(p^\mu - p'^\mu)(q_\mu - q'_\mu)}\delta(p-p')\int \frac{d^5k}{(2\pi)^5}\frac{e^{-ik^\mu(q_\mu - q'_\mu)}}{\gamma^\mu k_\mu}. \tag{6.41}$$

This leads to the Green function for the Galilean covariant Pauli-Schrödinger equation in phase space,

$$G_D(q-q',p-p') = 2e^{-i(p^\mu - p'^\mu)(q_\mu - q'_\mu)} i\gamma^\mu\partial_\mu G_0(q^\mu - q'^\mu, p^\mu - p'^\mu), \tag{6.42}$$

where

$$\begin{aligned} G_0(q^\mu - q'^\mu, p^\mu - p'^\mu) &= \delta^3(\mathbf{p}-\mathbf{p}')\delta(p_4 - p'_4)\left(\frac{m}{2\pi}\right)^{3/2}\frac{\sqrt{i}}{(t-t')^{3/2}}\theta(t-t')\exp\left\{\frac{(p_4 - p'_4)(t-t')}{m}\right\} \\ &\times \exp\left\{-im(s-s') + \frac{im(\mathbf{x}-\mathbf{x}')^2 + 2i(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{x}-\mathbf{x}')(t-t')}{2(t-t')}\right\} \end{aligned} \tag{6.43}$$

is the scalar field propagator in phase space.

The vacuum average of the energy-momentum tensor associated with the Dirac-like Lagrangian is given by

$$\left\langle 0(\beta)\left|T^{\mu\nu(ab)}\right|0(\beta)\right\rangle = \lim_{(q'^\mu,p'^\mu)\to(q^\mu,p^\mu)} \Gamma_2^{\mu\nu} 2e^{-i(p^\mu - p'^\mu)(q_\mu - q'_\mu)} i\gamma^\alpha\partial_\alpha G_0(q^\mu - q'^\mu, p^\mu - p'^\mu;\alpha),$$

where

$$\Gamma_2^{\mu\nu} = -\frac{i}{4}\left[-\gamma^\mu \frac{\partial}{\partial q_\nu} + \gamma^\mu \frac{\partial}{\partial q'_\nu} - \eta^{\mu\nu}\left(\gamma^\lambda \frac{\partial}{\partial q^\lambda} - \gamma^\lambda \frac{\partial}{\partial q^\lambda} - \gamma^\lambda p_\lambda \right) \right].$$

Then the physical energy-momentum tensor is defined as

$$\mathcal{T}^{\mu\nu(ab)}(\alpha) \equiv \left\langle 0(\beta) \left| T^{\mu\nu(ab)} \right| 0(\beta) \right\rangle = \lim_{(q'^\mu, p'^\mu)\to(q^\mu,p^\mu)} \Gamma_2^{\mu\nu} \bar{G}_D(q^\mu - q'^\mu, p^\mu - p'^\mu; \alpha), \quad (6.44)$$

with $\bar{G}_D(q^\mu - q'^\mu, p^\mu - p'^\mu; \alpha) = G_D(q^\mu - q'^\mu, p^\mu - p'^\mu; \alpha) - G_D(q^\mu - q'^\mu, p^\mu - p'^\mu)$. Here a renormalization procedure has been used. Now results at finite temperature are considered.

Non-relativistic Stefan-Boltzmann law

For the study of the Stefan-Boltzmann law in phase space the parameter α is chosen as $\alpha = (0, 0, 0, \beta, 0)$. The Bogoliubov transformation is

$$v^2(k, \beta) = \sum_{j=1}^{\infty} (-1)^{j+1} e^{-\beta k j}. \quad (6.45)$$

The Green function for the Dirac-like field in phase space is

$$\begin{aligned} \overline{G}_D^{(ab)}(q - q', p - p'; \beta) &= \sum_{j=1}^{\infty} (-1)^{j+1} \big[G_D^*(q - q' + i\beta j n_4, p - p') \\ &- G_D(q - q' - i\beta j n_4, p - p') \big]. \end{aligned} \quad (6.46)$$

By taking $\mu = \nu = 4$, the energy-momentum tensor is

$$\begin{aligned} \mathcal{T}^{44(11)}(\beta) &= \lim_{(q',p')\to(q,p)} \sum_{j=1}^{\infty} (-1)^{j+1} \Gamma_2^{44} \big[G_D^*(q - q' + i\beta j n_4, p - p') \\ &- G_D(q - q' - i\beta j n_4, p - p') \big], \end{aligned} \quad (6.47)$$

where

$$\Gamma_2^{44} = -\frac{i\gamma^4}{4} (\partial_5 - \partial'_5). \quad (6.48)$$

The energy-momentum tensor is

$$\mathcal{T}^{44(11)}(\beta) = \frac{\delta^3(\mathbf{p}-\mathbf{p}')\delta(p_4-p_4')\gamma^4}{128}\frac{m}{\beta}\left(\frac{m}{2\pi\beta}\right)^{3/2}\left[4(\sqrt{2}-2)\beta\,\zeta\left(\frac{3}{2}\right)(-\gamma^5 m+\gamma^1 p_1 + \gamma^2 p_2+\gamma^3 p_3)+3(\sqrt{2}-4)\gamma^4\,\zeta\left(\frac{5}{3}\right)\right], \quad (6.49)$$

where we used the Clifford algebra $\{\gamma^m u, \gamma^n u\} = 2\eta^{\mu\nu}$. The non-relativistic Stefan-Boltzmann law associated with the Pauli-Schrödinger equation is in phase space.

Casimir effect at finite temperature

For $\alpha = (0, 0, i2d, \beta, 0)$, the generalized Bogoliubov transformation becomes

$$v^2(\beta, d) = \sum_{j=1}^{\infty}(-1)^{j+1}e^{-\beta k^0 j} + \sum_{l=1}^{\infty}(-1)^{l+1}e^{-i2dk^3 l} + 2\sum_{j,l=1}^{\infty}(-1)^{j+l}e^{-\beta k^0 j - i2dk^3 l}. \quad (6.50)$$

The first two terms of this expression corresponds to the Stefan-Boltzmann term and the Casimir effect at $T = 0$, respectively. The third term is associated with the Casimir effect analyzed for T not equal to 0. It leads to

$$\mathcal{T}^{33(11)}(\beta, d) = \lim_{q'\to q}\sum_{j,l=1}^{\infty}(-1)^{j+l}\Gamma_2^{33}[G_D^*(q-q'+i\beta j n_4+2dln_3, p-p') - G_D(q-q'-i\beta j n_4-2dln_3, p-p')], \quad (6.51)$$

with

$$\Gamma_2^{33} = -\frac{i}{4}\left[-\gamma^3\left(\partial_3-\partial_3'\right)-\gamma^\lambda\left(\partial_\lambda-\partial_\lambda'\right)+\not{p}\right], \quad (6.52)$$

where $\not{p} = \gamma^\mu p_\mu$. Then the non-relativistic Casimir effect at finite temperature is

$$\mathcal{T}^{33(11)}(\beta, d) = \delta^3(\mathbf{p}-\mathbf{p}')\delta(p_4-p_4')\sum_{j,l=1}^{\infty}\Big\{(-1)^{j+l}\tfrac{m^{3/2}}{128\sqrt{2}\,\pi^{3/2}(\beta j)^{11/2}} \times\left[-e^{-\frac{m(dl)^2}{2\beta j}}-\cosh(2idlp_3)\right]\Theta_1(\beta,d)-i(\beta j)^2\sinh(2idlp_3)\Theta_2(\beta,d)\Big\}, \quad (6.53)$$

where

$$\Theta_1(\beta,d) = (\beta j)^2\Big[(dl)^2(-2m^2\gamma^3+4m\not{p}\gamma^4)+15mi\gamma^3\gamma^4 dl+15(\gamma^4)^2-2m(\beta j(4\not{p}\beta j - 3idlm\gamma^3)+2((dl)^2m-3(\beta j))\gamma^4)\gamma^5\Big]+(\beta j)^3\Big[m(\gamma^1)^2+m(\gamma^2)^2-8im\not{p}dl\gamma^3 + 2m(\gamma^3)^2-12\not{p}\gamma^4\Big]+(\beta j)^4\Big[4(p_1\gamma^1)^2-2i\not{p}p_2\gamma^2+4(p_2\gamma^2)^2-2i\not{p}p_3\gamma^3+12p_2p_3\gamma^2\gamma^3 + 8(p_3\gamma^3)^2-2ip_1\gamma^1(\not{p}+4ip_2\gamma^2+6ip_3\gamma^3)+4m^2(\gamma^5)^2)\Big]-10m\beta j(\gamma^4 dl)^2-3im\gamma^3\gamma^4(dl)^3 + (m\gamma^4)^2(dl)^4 \tag{6.54}$$

and

$$\Theta_2(\beta,d) = (\beta j)\Big[-imdl\not{p}\gamma^3+8mp_3\gamma^3 dl-3\not{p}\gamma^4-18ip_3\gamma^3\gamma^4\Big]+2i(p_1\gamma^1+p_2\gamma^2)\Big(-3im\gamma^3 dl\beta j + 2(m(dl)^2-3\beta j)\gamma^4+4(\beta j)^2(\not{p}-m\gamma^5)\Big)+(\beta j)^2\Big[4\not{p}^2+16i\not{p}p_3\gamma^3-2m(\not{p}+6ip_3\gamma^3)\Big] + (dl)^2\left[m\not{p}\gamma^4+6imp_3\gamma^3\gamma^4\right]. \tag{6.55}$$

The results in Eq. (6.53) imply an effect of finite temperature and space compactification.

6.4 Results

By applying the Galilean Covariance formalism along with the thermodynamic formalism of Thermofields Dynamics (TFD), we were able to calculate the thermal Casimir effect for both scalar and spinorial fields. For bosons and fermions, we will plot the pressure vs temperature ($T^{33(11)}(\beta,d)_B \times T$) graphs as follows:

Bosons

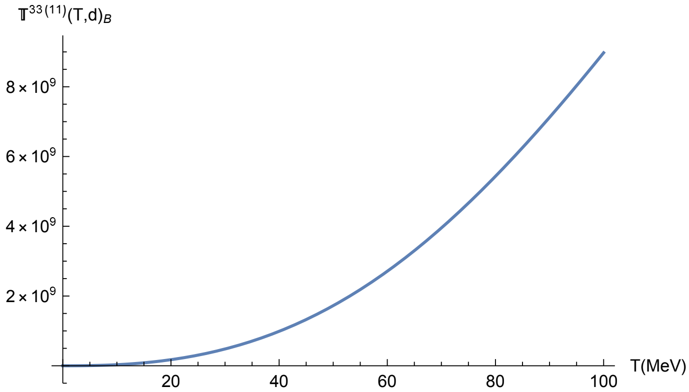

Figure 6.1: *Pressure, $T^{33(11)}(\beta,d)_B$, versus temperature for $d = 1fm = 0.005MeV^{-1}m = 350MeV$.*

Fermions

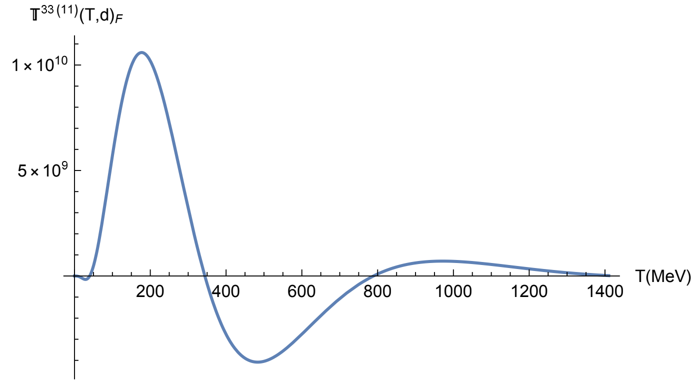

Figure 6.2: *Pressure* $(T^{33(11)}(\beta, d)_F)$ *versus temperature for* $d = 1fm = 0.005MeV^{-1}m = 350MeV$

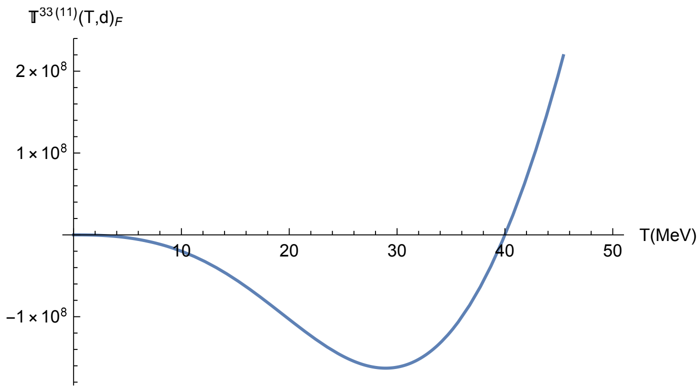

Figure 6.3: *This only shows a small part of Fig.*(6.2)*. This shows the first time for which temperature* $T^{33(11)}(\beta, d)F \to 0$ *is* $T \sim 40MeV$

The Galilean symmetry is assured along the calculations by writing the theory in the light-cone of a $(4 + 1)$-de Sitter space. The energy-momentum tensor associated with each field, Bosons and Fermions, is calculated. Considering the case of bosons, the size-effect term remains and goes to zero at zero-temperature, as it would be expected. However, the size-effect for high temperature for a fixed compactification lenght remains growing. This is an unexpected result and has motivated an analysis of a more realist model, by considering the spin of a quark. In this case, by increasing the temperature, from $T = 0$, the size-effect contribution to the energy momentum tensor is negative, diminishing the Stefan-Boltzmann effect. There is a minimum, and at $T \approx 40$ Mev the size-effect term of the energy momentum tensor is zero. In this case, the toroidal size-effect, d, as the hadron diameter, $d \approx 1fm$, and the quark mass as 350 MeV has been considered. In other words, the size-effect under this condition is no longer important, and the energy-momentum tensor is described only by the Stefan-Boltzmann law, for the (free) fermions. It is important to observe that such a temperature is in order of the estimated temperature for the chiral symmetry breaking, that gives rise to quark-gluon plasma. The connection of these two different results are not simple, but it demands more investigations. Finally, at high temperature, the size-effect term goes to zero, differently from the case of bosons. These results indicate that the spin is a crucial element for the quark-anti-quark effective model.

Chapter 7

Representations of Extended Carroll Group

Carroll's group is presented as a group of transformations in a 5-dimensional space ($\mathcal{C}$) obtained by embedding the Euclidean space into a (4,1) de Sitter space. Three of the five dimensions of $\mathcal{C}$ are related to $\mathcal{R}^3$, and the other two to mass and time. A covariant formulation of Caroll's group, analogous as introduced by Takahashi to Galilei's group, is deduced. Unit representations are studied.

This chapter is organized as follows, in section 7.1 the Carroll space embedded in the (4,1) de Sitter space is constructed. It is also shown that this space has an associated group that is an extended Carroll group. In section 7.2 the quantum representations of the extended Carroll group, namely the scalar and spinorial representations of Carroll fields are presented. In section 7.3 the Carrollian electric and magnetic limits using two different embedding of de Sitter space are shown.

The results of this section are taken from work by Petronilo *et al.*(2021B) [62].

7.1 The Carroll Group

The five-dimensional manifold with the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.1}$$

is a $(4+1)$ de Sitter space under the transformation $g_{\mu\nu} = U_{\mu}{}^{\alpha}\eta_{\alpha\beta}U^{\beta}{}_{\nu}$, where $(\eta_{\alpha\beta}) = (1,1,1,-1,1)$. This is easily seen by choosing the representation of U^{μ}_{ν} as

$$U^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{7.2}$$

The associated group of this manifold has a Lie algebra defined by the following commutation rule,

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\sigma}M_{\nu\rho}), \\ &[P_{\mu}, M_{\rho\sigma}] = -i(g_{\mu\rho}P_{\sigma} - g_{\mu\sigma}P_{\rho}), \\ &[P_{\mu}, P_{\sigma}] = 0. \end{aligned} \tag{7.3}$$

We can rewrite the generators, in a decomposition of (3+1+1) dimensions, as

$$\begin{aligned} J_i &= \frac{1}{2}\epsilon_{ijk}M_{jk}, \\ K_i &= M_{5i}, \\ C_i &= M_{4i}, \\ D &= M_{54}. \end{aligned} \tag{7.4}$$

Thus, the commutation relation becomes,

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k, & &[J_i, K_j] = i\epsilon_{ijk}K_k, \\ &[J_i, C_j] = i\epsilon_{ijk}C_k, & &[K_i, C_j] = i\delta_{ij}D + i\epsilon_{ijk}J_k, \\ &[D, K_i] = iK_i, & &[C_i, D] = iC_i, \\ &[P_4, D] = iP_4, & &[J_i, P_j] = i\epsilon_{ijk}P_k, \\ &[P_i, K_j] = i\delta_{ij}P_5, & &[P_i, C_j] = i\delta_{ij}P_4, \\ &[P_4, K_i] = iP_i, & &[P_5, C_i] = iP_i. \\ &[D, P_5] = iP_5, & & \end{aligned} \tag{7.5}$$

It is known that the Lie algebra of the extended Galilei group in $\mathcal{R}^3 \times \mathcal{R}$ is a sub-agebra of this algebra, with J_i, as generators of rotations, K_i of the pure Galilei boosts, and P_i spacial translations and P_4 is the temporal translations, associated with the energy, being P_5, in this context, a Casimir invariant associate with the mass, $P_5 = -mI$, where I is the identity matrix [14, 15, 51]. Another sub-algebra follows by setting the only non-zero commutation relations as

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, C_j] &= i\epsilon_{ijk}C_k, \\ [J_i, P_j] &= i\epsilon_{ijk}P_k, \end{aligned} \qquad \begin{aligned} [P_i, C_j] &= i\delta_{ij}P_4, \\ [P_5, C_i] &= iP_i. \end{aligned} \tag{7.6}$$

This is the algebra of Carroll group $\mathcal{C}$ with the addition of $[P_5, C_i] = iP_i$, that comes naturally of the structure of the five-dimensional manifold. In this context P_5 is not a Casimir invariant as is in the case of Galilei group $\mathcal{G}$. Indeed the Casimir invariants of this algebra are

$$I_1 = P^\mu P_\mu, \tag{7.7a}$$

$$I_2 = P_4, \tag{7.7b}$$

$$I_3 = W_{4\mu}W^4_\mu, \tag{7.7c}$$

where $W^{\mu\nu}$ is the 5-dimensional Pauli-Lubanski matrix. As P_4 is associated with the energy, E, in the Carroll symmetry, E is a Casimir invariant.

The coordinates transformations associated with this algebra are

$$q^{i\prime} = R^i_j q^j - v^i q^5 + a^i, \tag{7.8a}$$

$$q^{4\prime} = q^4 - (R^i_j q^j)v_i + \frac{1}{2}\mathbf{v}^2 q^5 + a^4, \tag{7.8b}$$

$$q^{5\prime} = q^5 + a^5, \tag{7.8c}$$

and

$$p^{i\prime} = R^i{}_j p^j - v^i p^5, \tag{7.8d}$$

$$p^{4\prime} = p^4 - \frac{v_i}{c'}(R^i_j p^j) + \frac{1}{2}\mathbf{v}^2 p^5, \tag{7.8e}$$

$$p^{5\prime} = p^5, \tag{7.8f}$$

Choosing $q^\mu = (\mathbf{q}, t, s)$ and $p^\mu = (\mathbf{p}, m\alpha, E)$, where $s \equiv \frac{\mathbf{q}^2}{2t}$ and α is related to the tranformations of p^4. These are the Carroll transformations in five dimensions. It is worthy noting that even though p_5 can not be interpreted as the invariant mass, it, nevertheless, carries mass information.

7.2 Representation of Quantum Mechanics

In this section we will construct the representations of quantum mechanics of the extended Carroll group.

Scalar Representation

For the scalar representation we take the invariants I_1 (7.7a) and I_2 (7.7b) and apply to a function ψ, and using the correspondence relation $p^\mu = -i\partial^\mu$ we have

$$\begin{cases} \partial_\mu \partial^\mu \Psi = k^2 \Psi \\ \partial_4 \Psi = -iE\Psi \end{cases}, \tag{7.9}$$

where k and E are constants. This is a non-relativistic Klein-Gordon-like equation with Carrollian symmetry.

Using $\Psi(x^\mu) = \exp\Big((-i(Et + m\,\alpha\,s))\psi(\mathbf{x})\Big)$, we have

$$-\nabla^2 \psi(\mathbf{x}) = 2m\alpha E \psi(\mathbf{x}). \tag{7.10}$$

In this context the 5-current is

$$j^\mu(x) = -\frac{i}{2m\alpha}\Big(\psi^*(x)\partial^\mu \psi(x) - \partial^\mu(\psi^*(x))\psi(x)\Big), \tag{7.11}$$

and is conserved because the 5-divergence is null, i.e.

$$\partial_\mu j^\mu = 0. \tag{7.12}$$

So the 5-current is equivalent to the usual 4-current,

$$\mathbf{j}(q) = -\frac{i}{2m\alpha}\Big[\Psi^*(q)\nabla\big(\Psi(q)\big) - \nabla\big(\Psi^*(q)\big)\Psi(q)\Big], \tag{7.13}$$

$$j^4 = \rho(q) = -\frac{i}{2m\alpha}\Big[-\Psi^*(q)\partial_s\big(\Psi(q)\big) + \partial_s\big(\Psi^*(q)\big)\Psi(q)\Big] = |\Psi|^2, \tag{7.14}$$

where $\mathbf{j}(x)$ is the probability current and $\rho(q)$ is the probability density.

Spinor Representation

In this context, we present a construction of the spin 1/2 wave equation, defining a new quadrivector γ^μ such that,

$$(\partial_\mu \partial^\mu - k^2) = (\gamma^\mu \partial_\mu + k)(\gamma^\nu \partial_\nu - k), \tag{7.15}$$

for (7.15) to be valid γ^μ must obey the Clifford algebra, that is

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{7.16}$$

where $g^{\mu\nu}$ is our penta-dimensional metric. Taking the plus-sign bracket and operating in the $\Psi(x)$ wave function, we get

$$(\gamma^\mu \partial_\mu + k)\Psi(x) = 0. \tag{7.17}$$

For convenience, we will use the following representations of γ^μ

$$\gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}.$$

where σ^i are Pauli's arrays and $\sqrt{2}$ is the 2x2 identity matrix multiplied by $\sqrt{2}$. We can write the Ψ object, as

$$\Psi = \begin{pmatrix} \varphi(\mathbf{x}, x^4, x^5) \\ \chi(\mathbf{x}, x^4, x^5) \end{pmatrix},$$

where φ and χ are 2-spinors dependent on $x^\mu; \mu = 1, ..., 5$. Therefore, in the representation where $k = 0$, Eq. (7.17) is reduced to

$$\boldsymbol{\sigma} \cdot \nabla\varphi + \sqrt{2}\partial_s \chi = 0, \tag{7.18a}$$

$$\sqrt{2}\partial_t \varphi + \boldsymbol{\sigma} \cdot \nabla\chi = 0. \tag{7.18b}$$

Eqs. (7.18) are the Carroll-Lévy-Leblond equations. The 5-current is

$$j^{\mu}(x) = \frac{1}{\sqrt{2}i}\left[\overline{\psi}(x)\gamma^{\mu}\psi(x)\right], \tag{7.19}$$

where $\overline{\psi} = \psi^{\dagger}\zeta$, with

$$\zeta = -\frac{i}{\sqrt{2}}(\gamma^4 + \gamma^5) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and j^{μ} is conserved, the 5-divergence is null

$$\partial_{\mu}J^{\mu} = 0.$$

In terms of φ e χ

$$j^i = \frac{1}{\sqrt{2}}\left[\chi^{\dagger}\boldsymbol{\sigma}\varphi + \varphi^{\dagger}\boldsymbol{\sigma}\chi\right], \tag{7.20}$$

$$j^4 = \varphi^{\dagger}\varphi, \qquad j^5 = \chi^{\dagger}\chi, \tag{7.21}$$

using Eq.(7.9), (7.18a) e (7.18b) we have

$$j^i = -\frac{i}{2m\alpha}\partial_i\left[\varphi^{\dagger}(x)\partial^i\varphi(x) - \partial^i(\varphi^{\dagger}(x))\varphi(x)\right] + \frac{1}{2m\alpha}\partial^j\left[\varphi^{\dagger}\sigma^k\varphi\right]\epsilon_{ijk},$$

and

$$\partial_5 j^5 = \partial_s(\chi^{\dagger}\chi) = 0.$$

The first term in j^i represents the probability current, given by Eq. (7.12), and the second is associated with the spin current, which results in the correct intrinsic magnetic moment value of the particle.

7.3 The Electric and Magnetic Limits

In this section we show the electric and magnetic Carrollian limits of Maxwell equations [54], using specific immersions.

In terms of the Faraday tensor, F_{AB}, the Maxwell equations are

$$\partial^A F_{AB} = j_B, \tag{7.22}$$

$$\partial_M F_{AB} + \partial_A F_{BM} + \partial_B F_{MA} = 0. \tag{7.23}$$

To obtain the differential equation in terms of the Electric and Magnetic fields we use the explicit form of the five-dimensional Faraday tensor

$$F_{AB} = \begin{pmatrix} 0 & B_2 & -B_2 & c_1 & d_1 \\ -B_3 & 0 & B_1 & c_2 & d_2 \\ B_2 & -B_1 & 0 & -c_3 & d_2 \\ -c_1 & -c_2 & -c_3 & 0 & R \\ -d_1 & -d_2 & -d_3 & -R & 0 \end{pmatrix}. \tag{7.24}$$

which applied to equations Eq. (7.22) and (7.23) results in

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0, && \nabla \cdot \mathbf{c} = j_4 + \partial_4 R, \\ &\nabla \times \mathbf{c} + \partial_4 \mathbf{B} = 0, && \nabla \cdot \mathbf{d} = j_5 + \partial_5 R, \\ &\nabla \times \mathbf{d} + \partial_5 \mathbf{B} = 0, && \nabla \times \mathbf{B} - \partial_4 \mathbf{d} - \partial_5 \mathbf{c} = \mathbf{j}. \end{aligned} \tag{7.25}$$

The fields are given by

$$\begin{aligned} \mathbf{c} &= \nabla A_4 - \partial_4 \mathbf{A}, \\ \mathbf{d} &= \nabla A_5 - \partial_5 \mathbf{A}, \end{aligned} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{7.26}$$

where $\mathbf{A}$ is the vector potential. Letting $\mathbf{c} = 0$, $R = 0$ and $\mathbf{d} = \mathbf{E}$, is the electric field.

We can obtain the Carrollian magnetic limit if we choose the following immersions

$$x^A = (\mathbf{x}, t, 0), \qquad A^A = (\mathbf{A}, 0, -\phi).$$

Thus, under Carrollian boost, we have

$$\bar{\mathbf{x}} = \mathbf{x}, \qquad \bar{t} = t - \mathbf{v} \cdot \mathbf{x}, \qquad \bar{x}^5 = 0,$$

So, $\partial_4 = \partial_t$ and, as A^A is a massless particle it is independent of α, then $\partial_5 \mathbf{A} = 0$. The resulting Maxwell equations are

$$\begin{aligned} &\nabla \cdot \mathbf{E} = \rho, && \nabla \cdot \mathbf{B} = 0, \\ &\nabla \times \mathbf{E} = 0, && \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j}. \end{aligned} \tag{7.27}$$

$$\mathbf{E} = -\nabla\phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The choice of gauge $\partial_\mu A^\mu = 0$ reduces to $\nabla \cdot \mathbf{A} = -\partial_t \phi$, the Lorenz gauge. The movement of electric charges is capable of creating a magnetic field, but a time-varying magnetic field would not create an electric field.

In the case of the Carrollian electric limit we made choose the following immersions

$$x^A = (\mathbf{x}, 0, t), \qquad A^A = (\mathbf{A}, 0, -\phi).$$

Thus, under Carrollian boost, we have

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{v}t, \qquad \bar{x}^4 = 0, \qquad \bar{t} = t,$$

and the obtained Maxwell equations are

$$\begin{aligned} &\nabla \cdot \mathbf{E} = \rho, && \nabla \cdot \mathbf{B} = 0, \\ &\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, && \nabla \times \mathbf{B} = \mathbf{j}. \end{aligned} \tag{7.28}$$

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The choice of gauge $\partial_\mu A^\mu = 0$ reduces to $\nabla \cdot \mathbf{A} = 0$, the Coulomb gauge. The temporal variation of the magnetic field creates an electric field but not the other way around. This results shown here are in accordance with the literature [54]. We note that the Carrollian electric limit has Galilean symmetry in its coordinates, as the Galilean magnetic limit has Carrollian symmetry.

The transformation of the fields are

$$\left.\begin{aligned} &\bar{\mathbf{E}} = \mathbf{E}. \\ &\bar{\mathbf{B}} = \mathbf{B} - \mathbf{v} \times \mathbf{E}. \end{aligned}\right\} \rightarrow \text{electric limit}, \qquad \left.\begin{aligned} &\bar{\mathbf{E}} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \\ &\bar{\mathbf{B}} = \mathbf{B}. \end{aligned}\right\} \rightarrow \text{magnetic limit}.$$

7.4 The interpretation of α

Choosing a reference frame where $\alpha = 1$, from Eq. (7.10) we have

$$p^2 - 2mE = 0, \tag{7.29}$$

this has the same form of Galilean mass shell condition. The difference over these two symmetries are that in the case of Carroll as the energy is invariant α varies with the relative velocity. In this way, Carroll particles can indeed move. Thus even though the momentum, p, transforms like (7.8d), the energy, $\frac{p^2}{2m}$, is invariant. If, in a inertial frame, the momentum is zero, we have the special case [54, 56]

$$E^2 - m^2 = 0,$$

where we have reintroduced the rest energy. Setting the speed of light $c = 1$, thus instead of the limit $c \to 0$, we will have $v \gg 1$, thus a Carrollian particle in this context will describe a tachyon. Here, the α parameter can be interpreted as a drag and the tachyon will acquire Carrollian symmetry in this limit. A experiment in the context of dual gravity/fluid can be proposed to study this drag, in this context a soliton should acquire Carrolian-like symmetry when $v \gg c'$, where c' is the sound speed in the fluid.

Chapter 8

The Landau Problem in Symplectic Carroll Symmetry

Carroll's group is shown as a group of transformations in a five-dimensional space ($\mathcal{C}$) obtained from the embedding of the Euclidean space into a (4,1)-de Sitter space. Three of the five dimensions of $\mathcal{C}$ are related to $\mathcal{R}^3$, and the other two to mass and time. A covariant formulation of Caroll's group is established in phase space. The Landau problem was studied. Finally, the negative parameter of the Wigner function is calculated.

The order in which this chapter will be presented is as follows. In Sec. 8.1 the construction of the Carrollian covariance is presented. Sec. 8.2, a symplectic structure is constructed in the Carrollian covariance formalism, and using the commutation relations the scalar equation in the light cone of five dimensions in phase space is constructed. In sec. 8.3 we analyze the gauge symmetry for spin 1/2 particle in phase space with Carrollian symmetry. In Sec. 8.4 we study the Carrollian spin 1/2 particle with an external field and solutions are proposed and discussed, after we calculated the negativity parameter and discuss the physical meaning.

8.1 The Carrollian Covariance

The Carroll transformations are given by

$$\begin{aligned}\overline{x}^i &= R^i_j x^j + a^i \\ \bar{t} &= t - (R^i_j x^j) v_i + b\,,\end{aligned}$$

where R represents three-dimensional Euclidean rotations, v represents the relative velocity defining Carrollian boosts, a represents spatial translations, and b represents time translations.

The following commutation rules describe the algebra associated with the formalism defined in the light cone of a five-dimensional de Sitter spacetime:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} + g_{\mu\sigma}M_{\nu\rho} - g_{\mu\sigma}M_{\nu\rho}),$$

$$[P_\mu, M_{\rho\sigma}] = -i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho), \tag{8.1}$$

$$[P_\mu, P_\sigma] = 0.$$

It is known that the Lie algebra of the extended Carroll group in $\mathcal{R}^3 \times \mathcal{R}$ is a subalgebra of this algebra, with J_i, as generators of rotations, C_i of the pure Carroll boosts, and P_μ spacial and temporal translations, being P_4, in this context, a Casimir invariant associate with the energy, $P_4 = -EI$, where I is the identity matrix [62].

The Casimir invariants of this algebra are

$$I_1 = p^\mu p_\mu, \tag{8.2a}$$
$$I_2 = p_4, \tag{8.2b}$$
$$I_3 = W_{4\mu}W^4_\mu, \tag{8.2c}$$

where $W^{\mu\nu}$ is the 5-dimensional Pauli-Lubanski matrix.

It is worth noting that even though p_5 can not be interpreted as the invariant mass, it, nevertheless, carries mass information [62].

8.2 representation of Quantum Mechanics in Phase Space

To associate the Hilbert space, H, with the phase space Γ, the set of square-integrable functions of complex value, $\phi(q,p)$, is considered in Γ, such that

$$\int dpdq\ \phi^*(q,p)\phi(q,p) < \infty. \tag{8.3}$$

Then we can write $\phi(q,p) = \langle q,p|\phi\rangle$, with the aid of

$$\int dpdq|q,p\rangle\langle q,p| = 1, \tag{8.4}$$

where $\langle\phi|$ is the dual vector of $|\phi\rangle$. This is the symplectic Hilbert space denoted by $H(\Gamma)$.

Now, let's consider the Carroll group with $H(\Gamma)$ as a representation space. To do so consider the unitary transformations, $U:\mathcal{H}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$, such that $\langle\psi_1|\psi_2\rangle$ is invariant. Using the operator Λ, we define a mapping $e^{i\frac{\Lambda}{2}} = \star:\Gamma \times \Gamma \rightarrow \Gamma$ called star product as

$$f \star g = f(q,p)\ \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}\right)\right] g(q,p),$$

where $\hbar = 1$. The following operators are defined to construct a representation of Galilei algebra in $\mathcal{H}$:

$$\widehat{P}^\mu = p^\mu \star = p^\mu - \frac{i}{2}\partial_{q_\mu}, \tag{8.5a}$$

$$\widehat{Q}^\mu = q^\mu \star = q^\mu + \frac{i}{2}\partial_{p_\mu}. \tag{8.5b}$$

and

$$\widehat{M}_{\nu\sigma} = M_{\nu\sigma}\star = \widehat{Q}_\nu \widehat{P}_\sigma - \widehat{Q}_\sigma \widehat{P}_\nu. \tag{8.5c}$$

Where $\widehat{M}_{\nu\sigma}$ are the generators of homogeneous transformations and $\widehat{P}_\mu$ of the non-homogeneous. After some simple calculations, we get the following set of commutations relations from this set of unitary operators

$$\begin{aligned}
\left[\widehat{M}_{\mu\nu}, \widehat{M}_{\rho\sigma}\right] &= -i(g_{\nu\rho}\widehat{M}_{\mu\sigma} - g_{\mu\rho}\widehat{M}_{\nu\sigma} + g_{\mu\sigma}\widehat{M}_{\nu\rho} - g_{\mu\sigma}\widehat{M}_{\nu\rho}),\\
\left[\widehat{P}_\mu, \widehat{M}_{\rho\sigma}\right] &= -i(g_{\mu\rho}\widehat{P}_\sigma - g_{\mu\sigma}\widehat{P}_\rho),\\
\left[\widehat{P}_\mu, \widehat{P}_\sigma\right] &= 0.
\end{aligned}$$

These relations form a closed algebra with the Carroll-Lie algebra as its subalgebra. Considering $\widehat{J}_i = \frac{1}{2}\epsilon_{ijk}\widehat{M}_{jk}$ the generators of rotations and $\widehat{C}_i = \widehat{M}_{4i}$ of the pure Carroll transformations, P_μ the spatial and temporal translations. The commutation of C_i and P_i is naturally non-zero in this context, being P_4 related to energy.

The invariants of this algebra are

$$I_1 = \widehat{P}_\mu \widehat{P}^\mu \quad (8.6a)$$
$$I_2 = \widehat{P}_4 \quad (8.6b)$$
$$I_3 = \widehat{W}_{4\mu} \widehat{W}^4_\mu, \quad (8.6c)$$

this is the phase space representation of the Casimir invariants of the group.

8.2.1 Scalar representation

Using the Casimir invariants I_1 and I_2 and applying to Ψ, we have

$$\widehat{P}_\mu \widehat{P}^\mu \Psi = k^2 \Psi,$$
$$\widehat{P}_4 \Psi = -E\Psi,$$

or explicitly

$$\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\nabla^2 - k^2\right)\Psi = \left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_5\right)\Psi,$$

a solution for this equation is

$$\Psi = e^{-2i[(p_5+m\alpha)q_5+(p_4+E)t]}\Phi(q,p). \quad (8.7)$$

Thus, we have

$$\frac{1}{2m\alpha}\left(p^2 - i\mathbf{p}\cdot\boldsymbol{\nabla} - \frac{1}{4}\boldsymbol{\nabla}^2\right)\Phi = \left(E + \frac{k^2}{2m\alpha}\right)\Phi,$$

with α a coefficient that depends on the reference frame [62], this is the Carrollian spin 0 equation in phase space, with the kinetic energy term of $\frac{k^2}{2m\alpha}$ added, that we may always use as the energy zero point.

This equation, and its complex conjugate, can be obtained by the Lagrangian density in phase space (we use $\partial^\mu = \partial/\partial q_\mu$)

$$\mathcal{L} = \frac{1}{4}\partial^\mu\Psi(q,p)\partial_\mu\Psi^*(q,p) + \frac{i}{2}p^\mu[\Psi(q,p)\partial^\mu\Psi^*(q,p) - \Psi^*(q,p)\partial^\mu\Psi(q,p)] + \left[p^\mu p_\mu - k^2\right]\Psi.$$

The association between this representation and Wigner formalism is established by

$$f_w(q,p) = \Psi(q,p) \star \Psi^\dagger(q,p)$$

where $f_w(q,p)$ is the Wigner function.

Which satisfies the five-dimensional Carrollian covariant Liouville-von Neumann equation in phase space given by

$$p_\mu \partial_{q_\mu} f_w(q,p) = 0. \tag{8.8}$$

8.2.2 spinorial representation

The Lévy-Leblond equation in Carrollian covariant formalism has the form of the Dirac equation,

$$\left(\gamma^\mu \widehat{P}_\mu - k\right) \Psi(p,q) = 0 \tag{8.9}$$

or

$$\gamma^\mu \left(p_\mu - \frac{i}{2}\partial_\mu\right) \Psi(p,q) = k\Psi(p,q). \tag{8.10}$$

Eq. (8.9) can be derived from the Lagrangian density for spin 1/2 particles in phase space, which is given by

$$\mathcal{L} = -\frac{i}{4}\Big((\partial_\mu \bar{\Psi})\gamma^\mu \Psi - \bar{\Psi}(\gamma^\mu \partial_\mu \Psi)\Big) - \bar{\Psi}(k - \gamma^\mu p_\mu)\Psi.$$

where $\bar{\Psi} = \zeta \Psi^\dagger$, with

$$\zeta = -\frac{i}{\sqrt{2}}\{\gamma^4 + \gamma^5\} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the case of the Lévy-Leblond equation, the association with the Wigner function is given by

$$f_w = \Psi \star \overline{\Psi},$$

with each component satisfying Eq. (8.8).

8.3 Gauge Theory for Carrollian Spin 1/2 particles in Phase Space

The Carrollian spin 1/2 Lagrangian in phase space can be expressed as

$$\mathcal{L} = \overline{\Psi}\gamma^\mu \widetilde{(p_\mu \star)}\Psi - k\overline{\Psi}\Psi, \tag{8.11}$$

where $A\widetilde{(p_\mu\star)}B = \frac{1}{2}[A(p_\mu \star B) - (p_\mu \star A)B]$. The purpose of this section is to examine the invariance of Eq.(8.11) under local gauge symmetry given by

$$\Psi = e^{-i\Omega} \star \Psi \qquad \overline{\Psi} = \overline{\Psi} \star e^{i\Omega}, \tag{8.12}$$

where $\Omega \equiv \Omega(q,p)$. For infinitesimal transformation, we have $\delta\Psi = -i\Omega\star\Psi$ and $\delta\overline{\Psi} = i\overline{\Psi}\star\Omega$, such that,

$$\delta(p_\mu \star \Psi) = -ip_\mu \star \Omega \star \Psi, \tag{8.13}$$

and

$$\delta(\overline{\Psi} \star p_\mu) = i\overline{\Psi} \star \Omega \star p_\mu. \tag{8.14}$$

It should be noted that $\delta(p_\mu \star \Psi)$ and $\delta(\overline{\Psi} \star p_\mu)$ do not transform covariantly. Thus, we define the operator to address this aspect as

$$D_\mu\star = p_\mu \star -iA_\mu\star, \tag{8.15}$$

resulting in the modified Lagrangian density

$$\mathcal{L} = \overline{\Psi}\gamma^\mu\widetilde{(D_\mu\star)}\Psi - k\overline{\Psi}\Psi. \tag{8.16}$$

Using the identity $p(f \star g) = f \star (pg) - \frac{i}{2}(\partial_\mu f) \star g$, the infinitesimal variation of $D_\mu \star \Psi$ is given by

$$\delta(D_\mu \star \Psi) = -i\Omega \star (p_\mu \star \Psi) - \partial_\mu\Omega \star \Psi - A_\mu \star (\Omega \star \Psi) - i(\delta(A_\mu) \star \Psi. \tag{8.17}$$

Considering that A_μ transforms by

$$A'_\mu \to A_\mu + i\{A_\mu, \Omega\}_M + i\partial_\mu\Omega, \tag{8.18}$$

where $\{a,b\}_M = a \star b - b \star a$ is the Moyal Brackets, we obtain

$$\delta(D_\mu \star \Psi) = -i\Omega \star (D_\mu \star \Psi). \tag{8.19}$$

Similarly, we have

$$\delta(\overline{\Psi} \star D_\mu) = -i(\overline{\Psi} \star D_\mu) \star \Omega. \tag{8.20}$$

In this sense, the Lagrangian density given in Eq.(8.16) is invariant under transformation in Eq.(8.12). Then, we have that the rule for minimal coupling is to replace $p_\mu\star$ by $D_\mu\star = p_\mu\star - iA_\mu\star$. The Pauli-Schrödinger equation with electromagnetic interactions is analyzed using the approach developed above in the following section.

8.4 Solution of the LL Equation with Electromagnetic Interactions

The equation describing a spin 1/2 particle in the Carrollian covariant phase space is given by

$$\left[\gamma^\mu\left(\widehat{P}_\mu - e\widehat{A}_\mu\right) - k\right]\Psi,$$

where A^μ is the 4-potential of the Carrollian electromagnetism [54].

Making the following definition

$$\Psi = \Big[\gamma^\nu\big(\widehat{P}_\nu - e\widehat{A}_\nu\big) + k\Big]\psi, \tag{8.21}$$

where $\widehat{P}_\nu = (p_\nu - \frac{i}{2}\partial_\nu)$. Thus, we have

$$\Big[\gamma^\mu\gamma^\nu\big(\widehat{P}_\mu - e\widehat{A}_\mu\big)\big(\widehat{P}_\nu - e\widehat{A}_\nu\big) - k^2\Big]\psi = 0. \tag{8.22}$$

Considering $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$, where

$$\sigma^{\mu\nu} = \frac{1}{2}\Big(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\Big) = \frac{1}{2}[\gamma^\mu, \gamma^\nu].$$

Using these results Eq. (8.22) becomes

$$\left(\widehat{P}^\mu\widehat{P}_\mu - e\left(\widehat{P}^\mu\widehat{A}_\mu + \widehat{A}^\mu\widehat{P}_\mu\right) - e\sigma^{\mu\nu}\left[\widehat{P}_\nu, \widehat{A}_\mu\right] + e^2\widehat{A}^\mu\widehat{A}_\mu\right)\psi = k^2\psi.$$

Letting $\widehat{A}^i = \frac{1}{2}e^{ijk}B_j\widehat{Q}_k$, with $\widehat{Q}_\mu = (q_\mu + \frac{i}{2}\partial_{p^\mu})$ and $A^4 = A^5 = 0$. Also, choosing the magnetic field as $\mathbf{B} = (0, 0, B)$. Keeping a particle's motion contained in a plane (q_1, q_2), i.e. $\widehat{P}_3 = 0$, the equation is as follows:

$$\begin{aligned}
- & \ 2\left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_s\right)\psi \\
+ & \ \left(p_1^2 + p_2^2 - \frac{1}{4}\Big(\frac{\partial^2}{\partial q_1{}^2} + \frac{\partial^2}{\partial q_2{}^2}\Big) - eB\left[\frac{i}{2}\Big(p_2\partial_{p_1} - p_1\partial_{p_2}\Big)\right.\right. \\
+ & \ \left.\frac{1}{4}\Big(\frac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \frac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\right] - i\Big(p_2\partial_{q_2} + p_1\partial_{q_2}\Big) - eB\left[(q_1p_2 - q_2p_1) - \frac{i}{2}\Big(q_1\partial_{q_2} - q_2\partial_{q_1}\Big)\right] \\
+ & \ \left.\frac{e^2B^2}{4}\left[\Big(q_1 + \frac{i}{2}\partial_{p_1}\Big)^2 + \Big(q_2 + \frac{i}{2}\partial_{p_2}\Big)^2\right] - ie\sigma^{12}B\right)\psi = k^2\psi,
\end{aligned} \tag{8.23}$$

where $\sigma^{12} = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$.

Letting

$$\psi = \begin{pmatrix} \Phi(q^\mu, p^\mu) \\ \Theta(q^\mu, p^\mu) \end{pmatrix},$$

we have two decoupled equations, one for $\Phi(q^\mu, p^\mu)$ and the other for $\Theta(q^\mu, p^\mu)$,

$$\begin{aligned}
-2\left(p_4 - \tfrac{i}{2}\partial_t\right)\left(p_5 - \tfrac{i}{2}\partial_s\right)\Phi(q^\mu, p^\mu) + \left(p_1^2 + p_2^2 - \tfrac{1}{4}\Big(\tfrac{\partial^2}{\partial q_1{}^2} + \tfrac{\partial^2}{\partial q_2{}^2}\Big) - eB\left[\tfrac{i}{2}\Big(p_2\partial_{p_1} - p_1\partial_{p_2}\Big)\right.\right. \\
\left.+\tfrac{1}{4}\Big(\tfrac{\partial^2}{\partial_{q_2}\partial_{p_1}} - \tfrac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\right] - i\Big(p_2\partial_{q_2} + p_1\partial_{q_2}\Big) - eB\left[(q_1p_2 - q_2p_1) - \tfrac{i}{2}\Big(q_1\partial_{q_2} - q_2\partial_{q_1}\Big)\right] \\
\left.+\tfrac{e^2B^2}{4}\left[\Big(q_1 + \tfrac{i}{2}\partial_{p_1}\Big)^2 + \Big(q_2 + \tfrac{i}{2}\partial_{p_2}\Big)^2\right] + e\sigma^3B\right)\Phi(q^\mu, p^\mu) = k^2\Phi(q^\mu, p^\mu),
\end{aligned}$$

and equation for Θ is analogous.

Taking $\Phi(q^\mu, p^\mu) = \varphi(q^i, p^i)\phi(q^4, q^5, p^4, p^5)$. This gives us the following equations

$$\left(p_4 - \frac{i}{2}\partial_t\right)\left(p_5 - \frac{i}{2}\partial_s\right)\phi = \alpha m E\phi + k^2\phi, \tag{8.24a}$$

and

$$\begin{aligned}
&\Bigg(p_1^2+p_2^2-\frac{1}{4}\Big(\frac{\partial^2}{\partial {q_1}^2}+\frac{\partial^2}{\partial {q_2}^2}\Big)-eB\Bigg[\frac{i}{2}\Big(p_2\partial_{p_1}-p_1\partial_{p_2}\Big)\\
&+\ \frac{1}{4}\Big(\frac{\partial^2}{\partial_{q_2}\partial_{p_1}}-\frac{\partial^2}{\partial_{q_1}\partial_{p_2}}\Big)\Bigg]-i\Big(p_2\partial_{q_2}+p_1\partial_{q_2}\Big)\\
&-\ eB\Bigg[(q_1p_2-q_2p_1)-\frac{i}{2}\Big(q_1\partial_{q_2}-q_2\partial_{q_1}\Big)\Bigg]\\
&+\ \frac{e^2B^2}{4}\Bigg[\Big(q_1+\frac{i}{2}\partial_{p_1}\Big)^2+\Big(q_2+\frac{i}{2}\partial_{p_2}\Big)^2\Bigg]+e\sigma^3B\Bigg)\varphi\\
&=\ 2\alpha mE\varphi+k^2\varphi.
\end{aligned} \tag{8.24b}$$

The solution of eq. (8.24a) is

$$\phi=C_1e^{-2i[(p_5+\alpha m)q_5+(p_4+E)t]},$$

where C_1 is a normalization constant.

To solve Eq. (8.24b) we will make a change of variables, defined by

$$\begin{aligned}
w(q_1,q_2,p_1,p_2)\ &=\ p_1^2+p_2^2+eB(q_2p_1-q_1p_2)\\
&+\ \frac{e^2B^2}{4}(q_1^2+q_2^2).
\end{aligned}$$

After a long calculation, it is shown that the imaginary part of this equation is identically null, which gives us

$$w\varphi(w)-e^2B^2\frac{\partial\varphi(w)}{\partial\omega}-e^2B^2w\frac{\partial^2\varphi(w)}{\partial w^2}=(2\alpha mE-esB+k^2)\varphi(w), \tag{8.25}$$

where $s\varphi=\sigma^3\varphi$, with $s=\pm1$. Letting $\omega=w/(eB)$, $\xi=(2m\alpha E-seB+k^2)/eB$ and defining $f(\omega)\equiv e^{\omega}\varphi(\omega)$, we have

$$\omega f''(\omega)+(1-2\omega)f'(\omega)-2af(\omega)=0, \tag{8.26}$$

with $f'(\omega)=\frac{\partial f}{\partial\omega}$ and $a=\frac{1}{2}(1-\xi)$. The equation (8.26) is a confluent hypergeometric equation, more specifically the Kummer equation, and the physical solutions are given by

$$f_n(\omega) = A_n U\left(\frac{1}{2} - \frac{\xi}{2}, 1, 2\omega\right),$$

where $U(a, b, x)$ are the Kummer's function and A_n are constants. However, it is realized that if $a = -n$ with $n = 0, 1, 2, ...$, the series $U(a, b, x)$ becomes a polynomial series in x not exceeding n. Thus, writing,

$$\xi - 1 = 2n,$$

we have the following relation of eigenvalue

$$E = \omega_c\left(n + \frac{1}{2} + \frac{s}{2}\right) - \frac{k^2}{2\alpha m},$$

with $\omega_c = \frac{eB}{\alpha m}$ and corresponding the following auto-functions

$$f_n(w) = A_n U\left(-n, 1, \frac{2w}{eB}\right), \tag{8.27}$$

such that A_n are normalization constants. Therefore, the quasi-amplitudes become,

$$\Phi_n = C_1 e^{-2i[(p_5+\alpha m)q_5+(p_4+E)t]}\left(A_n U\left(-n, 1, \frac{2w}{eB}\right)\exp\left(-\frac{w}{eB}\right)\right). \tag{8.28}$$

The analog is valid for Θ. To find the corresponding Wigner function just do $\psi_n \star \bar{\psi}_n$.

To find f_w, we will do the same procedure for the harmonic oscillator. Just realize that $w = 2\alpha m h$, with

$$h = \frac{1}{2\alpha m}\left(p_1^2 + p_2^2 + eB(q_2 p_1 - q_1 p_2) + \frac{e^2B^2}{4}(q_1^2 + q_2^2)\right).$$

Thus,

$$\psi_0 = C_0 e^{-2h/\omega_c}.$$

Therefore

$$f_w^0 = C_0 e^{-2h/\omega_c} \star \psi_0 = C_0 e^{-2\hat{h}/\omega_c} \psi_0 = C_0 e^{-2E_0/\omega_c} \psi_0.$$

Thus the ground state Wigner function for the spin particle $1/2$ and $-1/2$ are given respectively

$$f_w^{0+} = (C_{0^+})^2 \frac{1}{e^2} e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB},$$

and

$$f_w^{0-} = (C_{0^-})^2 e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2)+\frac{e^2B^2}{4}(q_1^2+q_2^2))/eB}.$$

For the general case, we have

$$f_w^{n^\pm} = (A_n^\pm)\left(\frac{1}{n!\pi}\right) e^{-(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))} U\left(-n, 1, \frac{2(p_1^2+p_2^2+eB(q_2p_1-q_1p_2))}{eB}\right). \quad (8.29)$$

It is worthwhile comparing with the case of Galilean covariance [24].

Chapter 9

Conclusions and Future Perspectives

In this work presents a new method for representing the phase space for particles satisfying the Galileo and Carroll conformal symmetries in a covariant symplectic representation. Based on the Wigner approach, this methodology allows analyzing the nature of the non-classicality of the states of compactified systems, such as the Landau model in the plane and fluctuations in vacuum at finite temperature and the confinement of the quark-antiquark system. In the latter case, the TFD formalism proved to be adequate in the study of vacuum fluctuations, allowing the analysis of confinement and the effect of temperature on confinement.

Our investigation delved into the spin 1/2 particle equation, particularly the Pauli equation within the framework of Galilean covariance. Leveraging this covariance, we constructed a phase space formalism, commencing with a comprehensive exposition on the Galilean manifold. This formalism allowed us to revisit the construction of Galilean covariance and the representations of quantum mechanics within this context.

By formulating the theory within the light-cone of a $(4+1)$-dimensional de Sitter space, we ensured the preservation of Galilean symmetry throughout our calculations. We computed the energy-momentum tensor associated with both Bosons and Fermions in each field. Notably, for Bosons, the size-effect term remains constant and diminishes to zero as temperature approaches zero. However, for a certain compactification length, the size-effect continues to increase at higher temperatures, prompting a reevaluation necessitating the consideration of a quark's spin. Remarkably, in this scenario, raising the temperature from $T = 0$ leads to a reduction in the Stefan-Boltzmann effect due to the negative contribution of the size-effect to the energy momentum tensor. The minimum of the energy-momentum tensor occurs when the size-effect term approaches zero at $T \approx 40$ MeV, a temperature comparable to the estimated threshold for quark-gluon plasma formation resulting from chiral symmetry violation. Further research is imperative to elucidate the intricate relationship between these disparate findings. Unlike Bosons, the size-effect term for Fermions diminishes at high temperatures, suggesting the critical role of spin in the quark-anti-quark effective model.

In the realm of Carrollian physics, we have developed a five-dimensional approach to Carrollian symmetry akin to that of Galilean Symmetry. The symmetry group formed by the subalgebra of the manifold G is isomorphic to the Lie-algebra of the Schrödinger group, albeit with the substitution of $t \to s$ and $p^4 \to p^5$. This subalgebra yields the Conformal Carroll algebra upon dimensional reduction, thus termed the extended Carroll algebra. We proceeded to investigate spin 0 and spin $1/2$ within the framework of Carrollian covariance, analyzing the Landau problem and comparing it with the case of Galilean covariance.

In conclusion, this work presents a novel method for a phase space representation for covariantly Conformal Galilean and Carroll particles employing a symplectic representation for the Galilei and Carroll group. Grounded in the Wigner approach, this methodology offers insights into the existence of mass and spin in nonrelativistic quantum mechanics. Through applications such as the Landau problem and the examination of the Size effect of quark-antiquark mesons via Thermofield Dynamics in symplectic field theory, we have demonstrated the versatility and efficacy of this approach.

Future Perspectives:

1. Continue studying the quark-antiquark system with the modified Cornell potential
2. Study the Cornell potential in Carroll symmetry
3. Incorporation of self-interaction in the studied systems
4. Deepen the study of chaoticity considering the self-interaction in the non-linear equation with the Cornell potential, via the negativity parameter.

By pursuing these future perspectives, we aim to advance our understanding of size effects in quantum systems, uncover new physics phenomena, and pave the way for transformative applications in various fields of science and technology.

Bibliography

[1] Bateman, H. "The conformal transformations of a space of four dimensions and their applications to geometrical optics". *Proc. London Math. Soc.*, 2 (1909), 70. 1

[2] Bateman, H. "The transformation of the electrodynamical equations". *Proc. London Math. Soc.*, 2 (1910), 223. 1, 39

[3] Cunningham, E. "The principle of relativity in electrodynamics and an extension thereof". *Proc. London Math. Soc.*, 2 (1910), 77. 1, 39

[4] Warwick, A. *Masters of theory: Cambridge and the rise of mathematical physics*. University of Chicago Press, Chicago (2003). 1

[5] Gilmore, R. *Lie groups, Lie algebras, and some of their applications*. Dover Publication, New York (2012). 1

[6] Kosyakov, B. *Introduction to the classical theory of particles and fields*. Springer Science & Business Media, New York (2007). 1

[7] Dirac, P. A. M. "Wave equations in conformal space". *Ann. Math.*, (1936), 429. 1

[8] Maldacena, J. "The large-n limit of superconformal field theories and supergravity". *Int. J. Theor. Phys.*, 38 (1999), 1113. 1

[9] Kovtun, P. K., Son, D. T., Starinets, A. O. "Viscosity in strongly interacting quantum field theories from black hole physics". *Phys. Rev. Lett.*, 94 (2005), 111601. 1

[10] Yamada, D. "Thermodynamics of black holes in the schrödinger space". *Class. Quantum Gravity*, 26 (2009), 075006. 1

[11] Son, D. T. "Toward an ads/cold atoms correspondence: a geometric realization of the schroedinger symmetry". *Phys. Rev. D*, 78 (2008), 046003. 1

[12] Nishida, Y. Son, D. T. "Nonrelativistic conformal field theories". *Phys. Rev. D*, 76 (2007), 086004. 1

[13] Melo, G. G. *Simetria conforme e covariância galileana*. Dissertation Thesis (2015). URL http://icts.unb.br/jspui/bitstream/10482/19882/1/2015_GustavoGarciaMelo.pdf 2, 31

[14] Takahashi, Y. "Towards the many-body theory with the galilei invariance as a guide. 1". *Fortschr. Phys.*, 36 (1988), 63. 2, 31, 81

[15] Omote, M., Kamefuchi, S., Takahashi, Y., Ohnuki, Y. "Galilean covariance and the schrödinger equation". *Fortschr. Phys.*, 37 (1989), 933. 2, 81

[16] Santana, A. E., Khanna, F. C., Takahashi, Y. "Galilei covariance and (4,1)-de Sitter space". *Prog. Theor. Phys.*, 99 (1998), 327. 2, 31

[17] Wigner, E. "Über das überschreiten von potentialschwellen bei chemischen reaktionen". *ZPC*, 19 (1932), 203. 2, 5

[18] Weyl, H. "Quantenmechanik und gruppentheorie". *Z. Phys.*, 46 (1927), 1. 2, 5

[19] Moyal, J. E. "Quantum mechanics as a statistical theory". "Math. Proc. Cambridge Philos. Soc.", vol. 45, 99–124. Cambridge University Press (1949). 2, 5

[20] Oliveira, M. D., Fernandes, M. C. B., Khanna, F. C., Santana, A. E., Vianna, J. D. M. "Symplectic quantum mechanics". *Ann Phys*, 312 (2004), 492. 2, 54

[21] Amorim, R. G. G., Khanna, F. C., Malbouisson, A. P. C., Malbouisson, J. M. C., Santana, A. E. "Quantum field theory in phase space". *Int. J. Mod. Phys. A*, (2019), 1950037. 2

[22] Dessano, H., Paiva, R. A. S., Amorim, R. G. G., Ulhoa, S. C., Santana, A. E. "Wigner function and non-classicality for oscillator systems". *Braz. J. Phys.*, (2019), 1. 2

[23] Paiva, R. A. S. Amorim, R. G. G. "Quantum physics in phase space: an analysis of simple pendulum". *ATCP*, 1 (2018), 1. 2

[24] Petronilo, G. X. A., Ulhoa, S. C., Araújo, K. V. S., Paiva, R. A. S., Amorim, R. G. G. "The landau problem and nonclassicality". *Int. J. Mod. Phys. A*, 35 (2020), 2050148. 2, 3, 52, 99

[25] Khanna, F. C. *Thermal quantum field theory: algebraic aspects and applications*. World Scientific (2009). 2, 25, 28

[26] Lee, H.-W. "Theory and application of the quantum phase-space distribution functions". *Phys. Rep.*, 259 (1995), 147. 5

[27] Leibfried, D., Meekhof, D., King, B., Monroe, C., Itano, W. M., "Experimental determination of the motional quantum state of a trapped atom". *PRL*, 77 (1996), 4281. 5

[28] Schleich, W. P. *Quantum optics in phase space*. John Wiley & Sons (2011). 8

[29] Amorim, R. G. G. "Formulação de teorias de campos via estruturas simpléticas and o produto de weyl" (2006). Dissertation Thesis (2006)
URL http://icts.unb.br/jspui/bitstream/10482/3492/1/2006_Ronni%20Geraldo%20Gomes%20de%20Amorim.pdf 5

[30] Amorim, R. G. G. *Geometria não-comutativa e teoria de campos simplética*. Dissertation Thesis (2009).
URL http://icts.unb.br/jspui/bitstream/10482/4561/1/2009_RonniGeraldoGomesdeAmorim.pdf 5

[31] Dessano, H. *Função de Wigner, quasi-amplitudes de probabilidades e sistemas dissipativos*. Dissertation Thesis (2014).
URL http://icts.unb.br/jspui/bitstream/10482/17897/1/2014_HaraDessanoFarias.pdf 5

[32] Cruz Filho, J. S. *Teoria quântica no espaço de fase: modelo de Hénon-Heiles e simetrias de calibre*. Dissertation Thesis (2016).
URL http://www.realp.unb.br/jspui/handle/10482/21972 5

[33] Ballentine, L. E. *Quantum mechanics: a modern development*. World Scientific Publishing Company (1998). 5

[34] Marchiolli, M. A. "Quantum mechanics in phase space: I. the weyl-wigner formalism". *Rev. Bras. Ens. Fis*, 24 (2002), 421. 5

[35] Wezeman, R. *Weyl quantization and Wigner distributions on phase space*. Faculty of Science and Engineering, Groningen (2014). Dissertation Thesis (2014).
URL http://fse.studenttheses.ub.rug.nl/11920/ 5

[36] Harsha, G., Henderson, T. M., Scuseria, G. E. "Thermofield theory for finite-temperature quantum chemistry". *J. Chem. Phys.*, 150 (2019). 25

[37] Khanna, F. C. *Thermal quantum field theory: algebraic aspects and applications*. World Scientific (2009). 2, 25, 28

[38] Matsumoto, H., Nakano, Y., Umezawa, H., Mancini, F., Marinaro, M. "Thermo field dynamics in interaction representation". *Prog. Theor. Phys.*, 70 (1983), 599. 25, 67

[39] Santana, A. E. Khanna, F. C. "Lie groups and thermal field theory". *Phys. Lett. A*, 203 (1995), 68. 25, 67

[40] Santana, A. E., Khanna, F. C., Chu, H., Chang, Y. C. "Thermal lie groups, classical mechanics, and thermofield dynamics". *Ann. Phys.*, 249 (1996), 481. 25, 67

[41] Takahashi, Y. Umezawa, H. "Coll. phenomena, 2: 55, 1975". *Reprinted in Int. J. Mod. Phys*, 10 (1755), 1996. 25, 67

[42] Umezawa, H. *Advanced field theory: Micro, macro, and thermal physics*. AIP (1995). 25, 67

[43] Abreu, L. M., de Montigny, M., Khanna, F. C., Santana, A. E. "Galilei-covariant path-integral quantization of non-relativistic complex scalar fields". *Ann. Phys.*, 308 (2003), 244. 69

[44] Gaberdiel, M. R. "An introduction to conformal field theory". *Rep. Prog. Phys.*, 63 (2000), 607. 31

[45] Fulton, T., Rohrlich, F., Witten, L. "Conformal invariance in physics". *RMP*, 34 (1962), 442. 31, 32, 34, 39

[46] Haantjes, J. *Conformal representation of an N-dimensional euclidean space with a non-definite fundamental form on itself* (1937). 31, 39

[47] Goldberger, W. D. "Ads/cft duality for non-relativistic field theory". *J. high Energy phys.*, 2009 (2009), 069. 31

[48] Blumenhagen, R. Plauschinn, E. *Introduction to conformal field theory: with applications to string theory*, vol. 779. Springer Science & Business Media (2009). 31

[49] Takahashi, Y. "Towards the many-body theory with the galilei invariance as a gluide part ii". *Fortschr. Phys.*, 36 (1988), 83. 31

[50] Pinski, G. "Galilean tensor calculus". *J. Math. Phys.*, 9 (1968), 1927. 31

[51] Santana, A. E. "Sobre covariância galileana and o campo de schrödinger". *Rev. Bras. Ens. Fis*, 19 (1997), 227. 31, 81

[52] Duval, C. Burdet, G. "K unzle, hp, perrin, m., bargmann structures and newton-cartan theory". *Phys. Rev. D*, 31 (1985), 1841. 31

[53] Duval, C., Horváthy, P., Palla, L. "Spinors in non-relativistic chern–simons electrodynamics". *Ann. Phys.*, 249 (1996), 265. 31

[54] Duval, C., Gibbons, G., Horvathy, P., Zhang, P. "Carroll versus newton and galilei: two dual non-einsteinian concepts of time". *Class. Quantum Gravity*, 31 (2014), 085016. 31, 84, 86, 87, 95

[55] Duval, C., Gibbons, G., Horvathy, P. "Conformal carroll groups". *J. Phys. A: Math. and Theor*, 47 (2014), 335204. 31

[56] Duval, C., Gibbons, G., Horvathy, P., Zhang, P. "Carroll symmetry of plane gravitational waves". *Class. Quantum Gravity*, 34 (2017), 175003. 31, 87

[57] Shaikh, A. A., Lone, M. S., Ghosh, P. R. "Conformal image of an osculating curve on a smooth immersed surface". *J. Geom. Phys*, 151 (2020), 103625. 31

[58] Schouten, J. A. Haantjes, J. "On the theory of the geometric object". *Proc. London Math. Soc.*, 2 (1937), 356. 41

[59] Kenfack, A. Życzkowski, K. "Negativity of the wigner function as an indicator of non-classicality". *Journal of Optics B: Quantum and Semiclassical Optics*, 6 (2004), 396. 61

[60] Veitch, V., Wiebe, N., Ferrie, C., Emerson, J. "Efficient simulation scheme for a class of quantum optics experiments with non-negative wigner representation". *New J. Phys.*, 15 (2013), 013037. 63

[61] Petronilo, G. X. A., Amorim, R. G. G., Ulhoa, S. C., Santos, A. F., Santana, A. E. "Size-effect at finite temperature in a quark–antiquark effective model in phase space". *Int. J. Mod. Phys. A*, (2021), 2150121. 66

[62] Petronilo, G. X. A., Ulhoa, S. C., Santana, A. E. "Representations of extended carroll group". *AACA*, 31 (2021), 1. 79, 90, 92