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On the Maximum Principle and the Ricci Flow

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[1]O autor foi bolsista do CNPq e da CAPES durante a elaboração deste trabalho.

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On the Maximum Principle and the Ricci Flow

por

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To do Mathematics is to be, at once,
touched by fire and bound by reason,
Jordan Ellenberg

Abstract

Nesta dissertação, será apresentado um estudo sobre o princípio do máximo para escalares e fibrados vetoriais sobre variedades compactas e algumas aplicações sobre o fluxo de Ricci, tendo como objetivo final demonstrar importantes resultados obtidos em 1982 por Richard Hamilton. Iremos introduzir o fluxo de Ricci, calcular as equações de evolução de importantes objetos geométricos, demonstrar a existência e unicidade local do fluxo e procurar compreender os obstáculos para existência para todo tempo. Por fim, comentaremos o principal resultado do artigo de Richard Hamilton, que afirma que toda variedade Riemanniana de dimensão 3 compacta e sem bordo com curvatura de Ricci estritamente positiva admite uma métrica com curvatura seccional positiva constante e, portanto, é difeomorfa à esfera tridimensional (caso seja simplesmente conexa) ou ao quociente da esfera por algum grupo finito de isometrias agindo livremente na variedade. Os resultados apresentados apareceram em artigos publicados e esta dissertação é majoritariamente baseada nos artigos de 1982 e de 1984 de Richard Hamilton, nas notas sobre o fluxo de Ricci de Petter Topping e no livro de Bennet Chow e Dan Knopf sobre o fluxo de Ricci, assim como seus volumes subsequentes.

Abstract

In this dissertation, we will provide a study of the maximum principle both for scalars and for vector bundles on compact manifolds, as well as an introduction to the Ricci flow, with the goal of proving some important results due to Richard Hamilton, obtained in 1982 in his first paper on the Ricci flow. We shall introduce the Ricci flow, compute several evolution equations for some important geometric entities, prove short time existence and uniqueness of the Ricci flow and try to understand what are the obstacles for long time existence. Finally, we comment on Hamilton's main result from his seminal 1982 paper, that says that every three-dimensional closed Riemannian manifold with strictly positive Ricci curvature admits a metric with constant positive sectional curvature and, therefore, is diffeomorphic to the three dimensional sphere (if it's simply connected) or a quotient of the sphere by a finite group of isometries acting freely on it. All these results appeared in published papers and this dissertation is mainly based on Hamilton's 1982 and 1984 papers, Peter Topping's lecture notes on the Ricci flow and Bennet Chow's and Dan Knopf's book on the Ricci flow and its sequels.

Contents

Chapter 1

Introduction

The search for a canonical metric on a given manifold has always been a central question in differential geometry and this usually provides deep implications regarding the topology of the manifold. In dimension two, the Uniformization Theorem, for instance, shows that if a manifold is compact, then we always have a metric with constant curvature, which in turn gives us a complete topological classification of such manifolds. If the dimension of the manifold is 3 or higher, the search for an analogous result has been a big question in modern mathematics.

In this direction, W. Thurston introduced, in the late 70s, the Geometrization Conjecture, which basically says that every closed 3-manifold can be canonically decomposed into pieces such that each admits a unique geometric structure, i.e., a complete locally homogeneous Riemannian metric. The famous Poincaré Conjecture, which says that every simply connected closed 3-manifold is homeomorphic to the 3-sphere, is a corollary of the Geometrization Conjecture.

With the bold aim of proving the Poincaré Conjecture, Richard Hamilton introduced the Ricci flow in his seminal 1982's paper [14]. The Ricci flow is a geometric evolution equation in which one starts with a Riemannian manifold (M^n, g_0) and evolves its metric in the direction of the Ricci tensor by the equation

$$\frac{\partial}{\partial t} g = -2Ric,$$

which is a nonlinear reaction diffusion equation for metrics (often called heat-type equation), motivated by the harmonic heat flow introduced by Eells and Sampson in 1964 ([18]). After Hamilton's 1982 paper, a lot of innovations based on his work and subsequent papers have greatly impacted the field of geometric analysis. For example, the pinching estimates for 3-manifolds with positive Ricci curvature show that the eigenvalues of the curvature tensor become closer to each other as the curvature becomes large during the evolution of the flow. Another curvature estimate, due to Hamilton and Ivey, proves that the solutions that form singularities in dimension 3 must have nonnegative sectional curvature, which enables a detailed analysis of the formation of singularities in dimension three. Another application worth mentioning is the Li-Yau-Hamilton-type differential Harnack Inequality, which provides an *a priori* estimate for an expression which involves the curvature, its first and second spatial derivatives. For a good presentation of several results, see for example [3].

After a systematic development of the subject, a lot due to Hamilton himself, the Geometrization Conjecture was proved (and, therefore, the Poincaré Conjecture) by Grisha Perelman in his papers [24], [23] and [22], where he showed that the Ricci flow can be seen as a gradient flow for a certain functional. In his 1982's result, Hamilton showed that in the case of strictly positive Ricci curvature the Ricci flow develops a singularity simultaneously everywhere in the manifold as we approach a maximal time, which is finite.

One of the most important techniques when studying the Ricci flow is the maximum principle, which provides a lot of the estimates needed to prove the central results regarding the Ricci flow. The maximum principle was already a great tool to study second order elliptic and parabolic PDEs. However, in [14] and [12] Hamilton showed that one could still use the maximum principle on sections of vector bundles over compact manifolds.

With the Ricci flow, Hamilton introduced a very general method to study geometric evolution equations. Based on Hamilton's approach, not only mathematicians started working on Ricci flow, but also on other geometric flows, such as the curve shortening flow and mean curvature flow, with contributions by Huisken, Ecker, Grayson, Hamilton himself and others (for a brief introduction to these results, see Appendix C). In the following decades, the Ricci flow was widely used (and is still used) to study the topology, geometry and complex structure of manifolds.

In this dissertation, we aim to provide an introduction to the Ricci flow, discussing some of the most used techniques and a few classical results obtained with the use of this geometric flow. The work is organized as follows. In Chapter 2, we state important concepts from Riemannian Geometry and Lie algebras in order to provide the necessary background for the rest of the work.

In Chapter 3, we present evolution equations for several geometric quantities, such as the metric and its inverse, the Riemann, Ricci and scalar curvatures, the Levi-Civita connection and the volume form. Besides, we prove short-time existence for the Ricci flow, following DeTurck's work [9], which simplified Hamilton's original argument.

Chapter 4 is dedicated to the maximum principle, one of the central techniques on Hamilton's work on the Ricci flow. In his second paper regarding the Ricci flow, [12], Hamilton introduced the maximum principle on vector bundles, based on Weinberger's maximum principle for systems (see [29]). In this chapter, we present the maximum principle for scalars and for vector bundles, as well as key concepts from convex analysis that are necessary to understand the second one.

Finally, in Chapter 5 we deal with the Ricci flow on closed 3-manifolds with initially positive Ricci curvature. In his first paper on the Ricci flow, Hamilton proved that applying the Ricci flow to such a manifold, after a rescaling of the metric in order to keep the volume constant, one gets a limit metric which is smooth and has constant positive sectional curvature, which implies that the initial manifold is diffeomorphic to a quotient of $\mathbb{S}^3$ by finite groups of isometries acting freely on it. We prove some of the results that enabled Hamilton to prove his main theorem, but by using more recent techniques, developed after Hamilton's paper which are equivalent to his arguments. One essential result is to show that the only obstacle to continue the flow is the curvature becoming unbounded. After that, we obtain upper and lower estimates for the curvature, which enables us to prove that the sectional curvatures approach each other as we evolve the flow. In the end of the chapter, we make a brief comment on the normalized Ricci flow (rescaling the metric), which enabled Hamilton to complete the proof of his main result.

Chapter 2

Preliminary Results

In this chapter, we aim to give a brief introduction to some results that will be useful for us in the following chapters. We will assume some familiarity with the basic concepts from Riemannian Geometry, such as the definition of a manifold, a Riemannian metric and basic properties of the curvatures. The majority of the results in here can be found in [4],[5], [7], [16], [20], [21], [25] and [28].

Let $\{x^i\}$ be local coordinates in a neighborhood U of $p \in M$. In U, the vector fields $\{\frac{\partial}{\partial x^i}\}$ form a local basis of TM and $\{dx^i\}$ form a dual basis for T^*M. Then we may write the metric in local coordinates as $g = g_{ij} dx^i \otimes dx^j$, where we have used the Einstein summation convention (which will be used throughout the whole dissertation).

Definition 2.0.1. *We define the **(3,1)-Riemann curvature tensor**, denoted by Rm, as follows*

$$Rm(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components of the (3,1)-tensor Rm are given by

$$Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} := R^l_{ijk}\frac{\partial}{\partial x^l}.$$

We also define the (4,0)-Riemann curvature tensor by taking the inner product with an-

other vector field. Then its components are given by

$$R_{ijkl} = g_{lm} R_{ijk}^{m}.$$

Definition 2.0.2. *If $P \subset T_pM$ is a 2-plane, then the* ***sectional curvature*** *of P is defined by $K(P) \coloneqq g(Rm(e_1, e_2)e_2, e_1)$, where $\{e_1, e_2\}$ is an orthonormal basis of P. This definition is independent of the choice of such a basis.*

Definition 2.0.3. *The* ***Ricci tensor****, denoted by Ric, is defined as the trace of the Riemann curvature tensor $Ric(X, Y) \coloneqq trace(Z \mapsto Rm(Z, X)Y)$. Its components are given by*

$$R_{jk} = Ric\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \sum_{i=1}^{n} R_{ijk}^{i}.$$

The ***scalar curvature****, denoted by R, is the trace of the Ricci tensor, i.e.,*

$$R = g^{ij} R_{ij},$$

where $g^{ij} = (g^{-1})_{ij}$ is the inverse of the metric.

It is also important to define the **covariant derivative of a tensor**. If α is an (r,s)-tensor, we define its covariant derivative by:

$$\nabla_X \alpha(Y_1, \ldots, Y_r) \coloneqq \nabla_X \left(\alpha\left(Y_1, \ldots, Y_r\right)\right) - \sum_{i=1}^{r} \alpha\left(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r\right).$$

Hence, we may consider the covariant derivative as

$$\nabla : C^\infty(\otimes^{r,s} M) \longrightarrow C^\infty(\otimes^{r+1,s} M),$$

where $\nabla \alpha(X, Z_1, \ldots, Z_r) \coloneqq \nabla_X \alpha(Z_1, \ldots, Z_r)$.

Our first results, which will be important for many calculations later on, are the Bianchi identities. If R_{ijkl} are the components of the Riemann Curvature tensor on a given coordinate system, R_{ij} are the components of the Ricci tensor on this same coordinate system and R is the scalar curvature, then the **first and the second Bianchi identities** are:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0 \tag{2.1}$$

and

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0. \tag{2.2}$$

The **twice contracted second Bianchi identity** is

$$2g^{ij}\nabla_i R_{jk} = \nabla_k R. \tag{2.3}$$

Definition 2.0.4. *Let (M^n, g) be a connected Riemannian manifold. We say that (M^n, g) is an **Einstein manifold** if*

$$Ric = fg,$$

where $f : M \longrightarrow \mathbb{R}$ is a function.

Using the Bianchi identities, one can prove the following result.

Theorem 2.0.1. *Let (M^n, g) be an Einstein manifold. Then, if $n \geq 3$, we get that f is a constant. In particular, if $n = 3$, then M^3 has constant sectional curvature.*

We would also like to introduce the **Lie derivative**, which, in a certain sense, measures the lack of invariance of a tensor with respect to a family of diffeomorphisms generated by a vector field.

Definition 2.0.5. *Let α be a tensor and X a complete vector field generating a global 1-parameter group of diffeomorphisms φ_t. The **Lie derivative** of α with respect to X is defined by*

$$\mathcal{L}_X\alpha := \lim_{t\to 0}\frac{1}{t}(\alpha - (\varphi_t)_*\alpha), \tag{2.4}$$

where $(\varphi_t)_* = (\varphi_t^{-1})^* : T_p^*M \longrightarrow T^*_{\varphi_t(p)}M.$

Now, since we defined the Riemann curvature as the commutation of covariant derivatives acting on vector fields, we may also express the commutation of covariant derivatives acting on tensors in terms of the curvature. This is given by the **Ricci identities:**

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\, \alpha_{k_1 \dots k_r} = -\sum_{l=1}^{r} R_{ijk_l}^{m} \alpha_{k_1 \dots k_{l-1} m k_{l+1} \dots k_r}, \tag{2.5}$$

where α is a (s,r)-tensor. In particular, if α is a 2-tensor, then

$$\nabla_i \nabla_j \alpha_{kl} - \nabla_j \nabla_i \alpha_{kl} = -R_{ijk}^{p} \alpha_{pl} - R_{ijl}^{p} \alpha_{kp}. \tag{2.6}$$

Throughout this dissertation, it will be more convenient to do our computations in local coordinates rather than in an orthonormal moving frame. This is mostly due to the fact that the Ricci flow evolves the metric, so we can choose a fixed coordinate system. If we were doing our computations on a moving frame, we would have to take into account the evolution of the moving frame if we want it to remain orthonormal. Besides, since the majority of our equations are tensorial, we can always choose geodesic coordinates centered on a given point to do our calculations.

Now we define the so called **Kulkarni-Nomizu product**, which will be used to give a particularly useful decomposition of the Riemann curvature tensor.

Definition 2.0.6. *Let $S^2 M = T^*M \otimes_S T^*M$ be the bundle of symmetric 2-tensor. Then we define the* ***Kulkarni-Nomizu product*** $\odot$ *acting on $S^2M \times S^2M$ by*

$$(\alpha \odot \beta)_{ijkl} \coloneqq \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{ik}\beta_{jl} - \alpha_{jl}\beta_{ik}.$$

This gives us the following decomposition for the Riemann tensor:

$$Rm = \frac{R}{2n(n-1)}(g \odot g) + \frac{1}{n-2}\left(\overset{\circ}{Ric} \odot g\right) + W,$$

where $\overset{\circ}{Ric} = Ric - \frac{R}{n} g$ is the **trace-free part of the Ricci tensor** and W is the **Weyl tensor**, which is defined implicitly by the expression above.

Proposition 2.0.1. *If n=3, the Weyl tensor is identically zero.*

Proof. For a simple proof of this result, see [14]. □

Then, by the proposition above, we have the following expression, which relates the Riemann curvature tensor and the Ricci tensor on dimension 3:

$$R_{ijkl} = R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} - \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl}). \tag{2.7}$$

This last expression will be extremely important to prove that the Ricci flow preserves certain quantities, such as positive Ricci curvature.

In order to prove short-time existence for the Ricci flow, we will have to calculate the principal symbol of $Ric(g(t))$. To do so, we define the **linearization of a nonlinear differential operator**.

Definition 2.0.7. *Let* $F : C^\infty(E) \longrightarrow C^\infty(\tilde{E})$ *be a nonlinear differential operator* $F(p, \partial^k u)$, *where* $E, \tilde{E}$ *are vector bundles over* M, $p \in M$ *and* $u \in C^\infty(E)$. *Then the linearization of* F *at* u *is the linear operator*

$$Pv = \frac{d}{dt}F(p, \partial^k U(t))\bigg|_{t=0}, \tag{2.8}$$

where $U(t) \in C^\infty(E)$ *for all* t, $U(0) = u$ *and* $U'(0) = v$.

Now we state a result that shall be used in this dissertation, the famous **Bonnet-Myers theorem**, in which one assumes bounds on the Ricci tensor and gets information on the topology of the manifold.

Theorem 2.0.2. *Let* (M^n, g) *be a complete Riemannian manifold. If there is a constant* $k > 0$ *such that* $Ric \geq (n-1)k > 0$, *then* M^n *is compact and* $diam(M, g) \leq \frac{\pi}{\sqrt{k}}$.

In the rest of this chapter, we introduce the concept of a **Lie algebra** and provide a few details on the identification of tensor spaces and Lie algebras. This identification will be useful to us in Chapter 5 when trying to get a better grasp of the evolution equation of the Riemann curvature tensor under the Ricci flow.

Definition 2.0.8. *A* ***Lie algebra*** $\mathcal{G}$ *over a field* K *is a vector space over* K *together with a bi-linear map, called the Lie bracket,* $[\cdot,\cdot] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ *satisfying:*

1. $[v, v] = 0$ *for all* $v \in \mathcal{G}$,

2. $[u,[v,w]]+[w,[u,v]]+[v,[w,u]]=0$ *for all* $u,v,w\in\mathcal{G}$. *(Jacobi identity)*

Now suppose $\mathcal{G}$ has an inner product $\langle,\rangle$ and $\{\varphi^\alpha\}$ is an orthonormal basis for $\mathcal{G}$. We defined the **structure constants** $c_\gamma^{\alpha\beta}$ for the Lie bracket with respect to $\{\varphi^\alpha\}$ by

$$[\varphi^\alpha,\varphi^\beta]=c_\gamma^{\alpha\beta}\varphi^\gamma.$$

Since our basis is orthonormal, we have

$$c_\gamma^{\alpha\beta}=\langle[\varphi^\alpha,\varphi^\beta],\varphi^\gamma\rangle.$$

Using properties 1 and 2 of the Lie bracket, it is easy to check that the structure constants are fully anti-symmetric.

A special case of a Lie algebra arises when we have a vector space V and consider $E=End(V)$, the algebra of operator endomorphisms of V. Then E can be made into a Lie algebra over $\mathbb{R}$ by defining the bracket

$$[X,Y]:=X\cdot Y-Y\cdot X.$$

In this case, the Lie algebra is called the **general Lie algebra** $\mathfrak{gl}(V)$. If $V=\mathbb{R}^n$, then we have, for instance, the general linear Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of all $n\times n$ real matrices. Furthermore, the **special linear Lie algebra** $\mathfrak{sl}(n,\mathbb{R})$ is the set of real matrices of trace 0 and is a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$. More important to us will be the **special orthogonal Lie algebra** $\mathfrak{so}(n,\mathbb{R})=\{X\in\mathfrak{sl}(n,\mathbb{R});X^T=-X\}$, i.e., the set of skew-symmetric matrices.

Now we consider a real n-dimensional vector space V with an inner product $\langle,\rangle$. Let $\{e_\alpha\}_{\alpha=1}^n$ be an orthonormal basis for V. Consider the tensor space $V\otimes V$, which is the space of the linear applications defined by

$$x\otimes y:z\mapsto\langle y,z\rangle x, \tag{2.9}$$

for any $x, y \in V$. We may endow the tensor space $V \otimes V$ with an inner product

$$\langle x \otimes y, u \otimes v \rangle = \langle x, u \rangle \langle y, v \rangle.$$

Then $\{e_\alpha \otimes e_\beta\}$ forms an orthonormal basis for $V \otimes V$. In fact,

$$\langle e_\alpha \otimes e_\beta, e_\gamma \otimes e_\eta \rangle = \langle e_\alpha, e_\gamma \rangle \langle e_\beta, e_\eta \rangle = \delta_\alpha^\gamma \delta_\beta^\eta,$$

which shows that this basis is actually orthonormal.

Let $E_{\alpha\beta}$ be the matrix with 1 in the $(\alpha, \beta) - th$ entry and 0 in the other entries. Then $E_{\alpha\beta} E_{\lambda\eta} = \delta_{\beta\lambda} E_{\alpha\eta}$. We consider the following identification $\mathfrak{gl}(n, \mathbb{R}) \cong V \otimes V$, where $E_{\alpha\beta} \sim e_\alpha \otimes e_\beta$. Since any matrix $A \in \mathfrak{gl}(n, \mathbb{R})$ can be written as

$$A = \sum_{\alpha\beta} a_{\alpha\beta} E_{\alpha\beta},$$

where $a_{\alpha\beta} \in \mathbb{R}$, under this identification the inner product on $\mathfrak{gl}(n, \mathbb{R})$ can be given by

$$\begin{aligned}
\langle A, B \rangle &= \langle a_{\alpha\beta} E_{\alpha\beta}, b_{\gamma\eta} E_{\gamma\eta} \rangle \\
&= \sum_{\alpha,\beta,\gamma,\eta} a_{\alpha\beta} b_{\gamma\eta} \langle E_{\alpha\beta}, E_{\gamma\eta} \rangle \\
&= \sum_{\alpha,\beta,\gamma,\eta} a_{\alpha\beta} b_{\gamma\eta} \langle e_\alpha \otimes e_\beta, e_\gamma \otimes e_\eta \rangle \\
&= \sum_{\alpha,\beta,\gamma,\eta} a_{\alpha\beta} b_{\gamma\eta} \delta_\alpha^\gamma \delta_\beta^\eta \\
&= \sum_{\alpha,\eta} a_{\alpha\eta} b_{\alpha\eta} = tr(A^T B).
\end{aligned}$$

for any $A, B \in \mathfrak{gl}(n, \mathbb{R})$.

Define the **second exterior power of V**, denote by $\Lambda^2 V = V \otimes V / \mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $x \otimes x$ for every $x \in V$. We define the linear transformation

$$x \wedge y : z \mapsto \langle y, z \rangle x - \langle x, z \rangle y, \tag{2.10}$$

for all $x, y \in V$ and observe that $x \wedge y = -y \wedge x$. We also define the inner product on $\Lambda^2 V$ by

$$\langle x \wedge y, u \wedge v\rangle = \langle x, u\rangle\langle y, v\rangle - \langle x, v\rangle\langle y, u\rangle. \tag{2.11}$$

With respect to this inner product, $\{e_\alpha \wedge e_\beta\}_{\alpha<\beta}$ forms an orthonormal basis for the vector space $\Lambda^2 V$. Using the linear transformation above, we may identify

$$\Lambda^2 V \cong \mathfrak{so}(n) \cong R^m,$$

where $m = \frac{n(n-1)}{2}$. This can also be seen by mapping $e_\alpha \wedge e_\beta$ to the skew-symmetric matrix $E_{\alpha\beta} - E_{\beta\alpha}$. With this identification, the inner product on $\mathfrak{so}(n)$ is given by

$$\langle A, B\rangle = \frac{1}{2} tr(A^T B). \tag{2.12}$$

In fact, if $A = \sum_{\alpha<\beta} a_{\alpha\beta}(E_{\alpha\beta} - E_{\beta\alpha})$ and $B = \sum_{\lambda<\eta} b_{\lambda\eta}(E_{\lambda\eta} - E_{\eta\lambda})$, with $a_{\alpha\beta}, b_{\lambda\eta} \in \mathbb{R}$, then

$$\langle A, B\rangle = \sum_{\alpha<\beta} a_{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} tr(A^T B) = -\frac{1}{2} tr(AB).$$

If we look at $A = \sum_{\alpha<\beta} a_{\alpha\beta}(E_{\alpha\beta} - E_{\beta\alpha}) \cong (a_1 2, \ldots, a_{(m-1)m})$, we may identify $\mathfrak{so}(n)$ with $\mathbb{R}^m$, $m = \frac{n(n-1)}{2}$, with the Euclidean product.

A particular case of the above formulation is when $n = 3$ and $V = \mathbb{R}^3$. Then $\Lambda^2 V \cong \mathfrak{so}(3) \cong \mathbb{R}^3$, where the Lie Bracket in $\mathbb{R}^3$ is given by the usual cross product.

Finally, we consider the dual space $\Lambda^2 V^*$. Let $\{\omega^\alpha\}$ be the dual basis to $\{e_\alpha\}$. Then we define $\omega^\alpha \wedge \omega^\beta$ by

$$\omega^\alpha \wedge \omega^\beta(e_\gamma, e_\eta) := \omega^\alpha(e_\gamma)\omega^\beta(e_\eta) - \omega^\alpha(e_\eta)\omega^\beta(e_\gamma) = \delta^\gamma_\alpha \delta^\eta_\beta - \delta^\eta_\alpha \delta^\gamma_\beta.$$

The inner product on this space is given by (2.11), but now it is applied to dual vectors, i.e.,

$$\langle \omega^\alpha \wedge \omega^\beta, \omega^\gamma \wedge \omega^\eta \rangle = \langle \omega^\alpha, \omega^\gamma \rangle \langle \omega^\beta, \omega^\eta \rangle - \langle \omega^\alpha, \omega^\eta \rangle \langle \omega^\beta, \omega^\gamma \rangle.$$

Then we define the Lie Bracket

$$[\omega^\alpha \wedge \omega^\beta, \omega^\gamma \wedge \omega^\eta] = \delta_{\alpha\eta}\omega^\beta \wedge \omega^\gamma + \delta_{\beta\gamma}\omega^\alpha \wedge \omega^\eta - \delta_{\alpha\gamma}\omega^\beta \wedge \omega^\eta - \delta_{\beta\eta}\omega^\alpha \wedge \omega^\gamma.$$

Any $\varphi \in \Lambda^2 V^*$ can be written as

$$\varphi = \frac{1}{2}\sum_{\alpha,\beta} \varphi_{\alpha\beta}\omega^\alpha \wedge \omega^\beta = \sum_{\alpha<\beta} \varphi_{\alpha\beta}\omega^\alpha \wedge \omega^\beta, \tag{2.13}$$

where $\varphi_{\alpha\beta} := \varphi(e_\alpha, e_\beta)$. Then the definition of the Lie bracket above enables us to define the components of the bracket, with respect to this basis, by

$$[\varphi, \psi]_{\alpha\beta} := \varphi_{\alpha\gamma}\psi_{\gamma\beta} - \psi_{\alpha\gamma}\varphi_{\gamma\beta}, \tag{2.14}$$

for any $\varphi, \psi \in \Lambda^2 V^*$. We identify $\Lambda^2 V^*$ to $\mathfrak{so}(n)$ by considering $\omega^\alpha \wedge \omega^\beta \mapsto E_{\alpha\beta} - E_{\beta\alpha}$. With this identification, the inner product in $\mathfrak{so}(n)$ is given by

$$\langle A, B \rangle = \frac{1}{2} tr(A^T B),$$

for all $A, B \in \mathfrak{so}(n)$, just like we did it above.

Now suppose $\{\varphi^\alpha\}$ is an orthonormal basis for $\Lambda^2 V^*$, with structure constants $c^{\alpha\beta}_\gamma$, and suppose $\{\sigma_\alpha\}$ is an orthonormal basis for $\Lambda^2 V$ dual to $\{\varphi^\alpha\}$. Then the corresponding structure constants for this dual basis are given by

$$[\sigma_\alpha, \sigma_\beta] = c^\gamma_{\alpha\beta}\sigma_\gamma.$$

Identifying $\Lambda^2 V$ with $\Lambda^2 V^*$, we get that $c^\gamma_{\alpha\beta} = c^{\alpha\beta}_\gamma$.

We now consider a Lie algebra $\mathcal{G}$ endowed with an inner product $\langle , \rangle$. Let $\{\varphi^\alpha\}$ be a basis of $\mathcal{G}$ and let $C_\gamma^{\alpha\beta}$ denote its structure constants. Also, let $\{\varphi_\alpha^*\}$ denote the basis algebraically dual to $\{\varphi^\alpha\}$, so that $\varphi_\alpha^*(\varphi^\beta) = \delta_{\alpha\beta}$. Let L be a symmetric bilinear form on $\mathcal{G}^*$. Then we may regard L as the element of $\mathcal{G} \otimes_S \mathcal{G}$ whose components are given by

$$L_{\alpha\beta} := L(\varphi_\alpha^*, \varphi_\beta^*).$$

Furthermore, we may define an operation $\#$ on $\mathcal{G} \otimes_S \mathcal{G}$, which is commutative, bi-linear and is given by

$$(L\#M)_{\alpha\beta} := C_\alpha^{\gamma\varepsilon} C_\beta^{\delta\xi} L_{\gamma\delta} M_{\varepsilon\xi}. \tag{2.15}$$

Then we define the **Lie algebra square** $L^\# \in \mathcal{G} \otimes_S \mathcal{G}$ of L by

$$(L^\#)_{\alpha\beta} := (L\#L)_{\alpha\beta} = C_\alpha^{\gamma\varepsilon} C_\beta^{\delta\xi} L_{\gamma\delta} L_{\varepsilon\xi}.$$

Now we prove a result that will soon be useful for us.

Lemma 2.0.3. *If $L \geq 0$, i.e., if for every $u \in \mathcal{G}$, $L(u,u) \geq 0$, then $L^\# \geq 0$.*

Proof. Let $\{\varphi^\alpha\}$ be a basis for $\mathcal{G}$ such that L is diagonal in this basis, so that $L_{\alpha\beta} = \delta_{\alpha\beta} L_{\alpha\alpha}$. Then for $v = v^\alpha \varphi_\alpha^*$ in $\mathcal{G}^*$, we get

$$L^\#(v,v) = (v^\alpha C_\alpha^{\gamma\delta})(v^\beta C_\beta^{\varepsilon\xi}) L_{\gamma\varepsilon} L_{\delta\xi} = (v^\alpha C_\alpha^{\gamma\delta})^2 L_{\gamma\gamma} L_{\delta\delta}.$$

Therefore, $L^\# \geq 0$. □

Remark. *$\langle , \rangle$ on $\mathcal{G}$ defines a metric isomorphism $\mathcal{G} \longrightarrow \mathcal{G}^*$ by $v \mapsto \langle v, \ \rangle$. Hence we are able to consider $L : \mathcal{G} \longrightarrow \mathcal{G}$ as a self-adjoint endomorphism.*

In chapter 5, we will study the case when $V = TM^n$, where M^n is a Riemannian manifold. Then $\wedge^2 T_pM \cong \wedge^2\mathbb{R}^n \cong \mathfrak{so}(n) \cong R^m$ for each $p \in M^n$, where $m = \dfrac{n(n-1)}{2}$. In particular, we consider the case when $n = 3$ and M^3 is a closed manifold. We would like to get an expression for the $\#$ operator. Let $\{e_i\}$ be a globally defined orthonormal moving frame and $\{\omega^k\}$ be the dual frame to $\{e_i\}$, so that $\omega^k(e_i) = \delta_k^i$. Then the dimension of $\wedge^2 TM^3$ is $m = 3$ and we may write a basis $\{\theta^k\}$ of $\wedge^2 TM^3$ by

$$\theta^1 = \frac{1}{\sqrt{2}} e_2 \wedge e_3 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\theta^2 = \frac{1}{\sqrt{2}} e_3 \wedge e_1 \sim \begin{pmatrix} 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix},$$

$$\theta^3 = \frac{1}{\sqrt{2}} e_1 \wedge e_2 \sim \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the matrices above are just the normalized versions of $E_{23} - E_{32}$, $E_{13} - E_{31}$ and $E_{12} - E_{21}$. Note also that when $n = 3$, the Lie algebra bracket on $\mathfrak{so}(3)$ with inner product being just the Euclidean inner product corresponds to the cross product on $\mathbb{R}^3$. Then, using equation (2.15), we may compute the Lie algebra square:

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}^{\#} = \begin{pmatrix} df - e^2 & ce - bf & be - cd \\ ce - bf & af - c^2 & bc - ae \\ be - cd & bc - ae & ad - b^2 \end{pmatrix}$$

since $\langle [\theta^i, \theta^j], \theta^k \rangle = C_{ij}^k$ is fully alternating in i, j, k and the Lie structure constants are given by

$$C_{\alpha\beta}^{\gamma} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } (\alpha\beta\gamma) \text{ is a positive permutation of } (123) \\ \frac{-1}{\sqrt{2}} & \text{if } (\alpha\beta\gamma) \text{ is a negative permutation of } (123) \\ 0 & \text{for all the other cases.} \end{cases} \tag{2.16}$$

Therefore, the matrix $(B^{\#})$ is just the adjoint matrix of B :

$$B^{\#} = \det(B)(B^{-1})^T. \tag{2.17}$$

Chapter 3

Short Time Existence for the Ricci Flow

In this chapter, we will prove that a solution for the Ricci flow always exists on a short time interval, regardless of the initial metric $g(0)$. Given a smooth family of metrics $g(t)$, we can compute the variations of the Levi-Civita connection and its associated curvature tensors. First, we will consider a general variation $\frac{\partial}{\partial t} g_{ij} = v_{ij}$, where v is any symmetric 2-tensor. Note that if $v = -2Ric$, then we have the Ricci flow. After that, we shall see that the Ricci tensor can be seen as a nonlinear-partial differential operator on the space of positive definite symmetric (2,0)-tensors, i.e., Riemannian metrics on M.

Inspired by that, we will calculate the linearization of the Ricci Tensor and its principal symbol. This will allow us to prove that the Ricci flow is not strictly parabolic. Motivated by that, we will introduce the Ricci DeTurck flow, that modifies the Ricci Flow into a strictly parabolic equation, which has a solution on a short interval because of the theory of parabolic PDEs. Finally, we will show that using the solution to this modified flow, we get a unique solution to the original Ricci flow. The variation formulas below can be found in [1] and the results on the Ricci flow can be found in [14], [9], [5] and [28].

3.1 Variation Formulas

As mentioned above, we will consider the following differential equation for a smooth family of metrics $g(t)$ on M^n

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= v_{ij}, \\ g(0) &= g_0, \end{aligned} \tag{3.1}$$

where v is any symmetric 2-tensor on M.

First, we recall that, although a connection is not a tensor, the difference of two connections, say ∇^t and ∇^{t_0}, is a tensor. In fact, if $f \in C^\infty(M)$ and X, Y are vector fields over M, then

$$\begin{aligned} \left(\nabla^{t_0}_Y - \nabla^t_Y\right)(fX) &= \nabla^{t_0}_Y(fX) - \nabla^t_Y(fX) \\ &= Y(f)X + f\nabla^{t_0}_Y X - Y(f)X - f\nabla^t_Y X = f\left(\nabla^{t_0}_Y - \nabla^t_Y\right)(X). \end{aligned}$$

In particular, if we take the limit $t \to t_0$, it is not difficult to see that $\frac{\partial}{\partial t}\nabla$ is a tensor.

Lemma 3.1.1. *Let $g(t)$ be a family of metrics such that $g(t)$ solves* (3.1). *Then the inverse of the metric, $g^{-1}(t)$, evolves by*

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} v_{kl}.$$

Proof. Just remember that $\delta_{il} = g^{ik} g_{kl}$. This gives us

$$0 = \left(\frac{\partial}{\partial t} g^{ik}\right) g_{kl} + g^{ik} v_{kl},$$

consequently

$$\frac{\partial}{\partial t} g^{ij} = \left(\frac{\partial}{\partial t} g^{ik}\right) g_{kl} g^{jl} = -g^{ik} g_{jl} v_{kl}.$$

□

Lemma 3.1.2. *For a solution $g(t)$ of* (3.1)*, the variation of the Christoffel symbols is given by*

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right).$$

Proof. Recall that in local coordinates $\{x_i\}$, if $\partial_i = \frac{\partial}{\partial x_i}$, we have $\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$. If we take geodesics coordinates centered at $p \in M$, then

$$\Gamma_{ij}^k(p) = \partial_i g_{jk}(p) = \nabla_{\partial_i} \partial_j(p) = 0.$$

In particular, if A is a 2-tensor, then

$$(\nabla_i A)(\partial_j, \partial_k)(p) = \partial_i (A(\partial_j, \partial_k))(p) - A(\nabla_i \partial_j(p), \partial_k) - A(\partial_j, \nabla_i \partial_k(p)),$$

so $\nabla_i A_{jk} = \partial_i A_{jk}$ at p. Thus, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma_{ij}^k(p) &= \frac{1}{2} \left(\frac{\partial}{\partial t} g^{kl} \right) \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)(p) \\ &+ \frac{1}{2} g^{kl} \left(\partial_i \frac{\partial}{\partial t} g_{jl} + \partial_j \frac{\partial}{\partial t} g_{il} - \partial_l \frac{\partial}{\partial t} g_{ij} \right)(p) \\ &= \frac{1}{2} g^{kl} \left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij} \right)(p). \end{aligned}$$

Since both sides of the above expression are components of a tensor (as we stated in the beginning of the section), this equation is valid for any $p \in M$ in any coordinate system. □

Lemma 3.1.3. *If $g(t)$ is a solution to equation* (3.1)*, then the evolution of the Riemann curvature tensor Rm is given by*

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}^l =& \frac{1}{2} g^{lp} (\nabla_i \nabla_j v_{kp} + \nabla_i \nabla_k v_{jp} - \nabla_i \nabla_p v_{jk} \\ & - \nabla_j \nabla_i v_{kp} - \nabla_j \nabla_k v_{ip} + \nabla_j \nabla_p v_{ik}). \end{aligned}$$

Proof. In local coordinates $\{x_i\}$, we have the formula

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^p \Gamma_{ip}^l - \Gamma_{ik}^p \Gamma_{jp}^l,$$

then we can apply the same reasoning as in Lemma 3.1.2 to infer:

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijk}^l =& \partial_i \frac{\partial}{\partial t} \Gamma_{jk}^l - \partial_j \frac{\partial}{\partial t} \Gamma_{ik}^l + \frac{\partial}{\partial t} \left(\Gamma_{jk}^p \right) \Gamma_{ip}^l \\ & + \Gamma_{jk}^p \frac{\partial}{\partial t} \left(\Gamma_{ip}^l \right) - \frac{\partial}{\partial t} \left(\Gamma_{ik}^p \right) \Gamma_{jp}^l - \Gamma_{ik}^p \frac{\partial}{\partial t} \left(\Gamma_{jp}^l \right). \end{aligned}$$

Using geodesic coordinates centered at $p \in M$, the Christoffel symbols vanish at p and we get from Lemma 3.1.2

$$\begin{aligned}\frac{\partial}{\partial t} R_{ijk}^{l}(p) = &\nabla_i \left[\frac{1}{2} g^{lp} \left(\nabla_j v_{kp} + \nabla_k v_{jp} - \nabla_p v_{jk}\right)\right] \\ &- \nabla_j \left[\frac{1}{2} g^{lp} \left(\nabla_i v_{kp} + \nabla_k v_{ip} - \nabla_p v_{ik}\right)\right].\end{aligned}$$

Since $\nabla_i g^{lp}(p) = 0$, the result follows.

$\square$

Remark. *If we commute the derivatives in Lemma 3.1.3 and use the Ricci identities* (2.6)*, we can also write*

$$\frac{\partial}{\partial t} R_{ijk}^{l} = \frac{1}{2} g^{lp}(\nabla_i \nabla_k v_{jp} + \nabla_j \nabla_p v_{ik} - \nabla_i \nabla_p v_{jk} - \nabla_j \nabla_k v_{ip} - R_{ijk}^{q} v_{qp} - R_{ijp}^{q} v_{kq}).$$

Lemma 3.1.4. *The evolution of the Ricci tensor Ric is given by*

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{lm} \left(\nabla_l \nabla_j v_{km} + \nabla_l \nabla_k v_{jm} - \nabla_l \nabla_m v_{jk} - \nabla_j \nabla_k v_{lm}\right). \tag{3.2}$$

Proof. Using Lemma 3.1.3,

$$\begin{aligned}\frac{\partial}{\partial t} R_{ljk}^{l} = &\frac{1}{2} g^{lm} \left(\nabla_l \nabla_j v_{km} + \nabla_l \nabla_k v_{jm} - \nabla_l \nabla_m v_{jk}\right) \\ &+ \frac{1}{2} g^{lm} \left(-\nabla_j \nabla_k v_{lm} - \nabla_j \nabla_l v_{km} + \nabla_j \nabla_m v_{lk}\right).\end{aligned}$$

Note that since we have a summation over l and m and v is symmetric, the last two terms vanish. Hence, the result follows. $\square$

Remark. *Recall that the* ***divergence of a (2,0)-tensor*** v *is given by*

$$(\delta v)_k := n - (\mathrm{div} v)_k = -g^{ij} \nabla_i v_{jk},$$

and denote the ***Lichnerowicz Laplacian*** *of a (2,0)-tensor by*

$$(\Delta_L v)_{jk} := \Delta v_{jk} + 2 g^{qp} R_{qjk}^{r} v_{rp} - g^{qp} R_{jp} v_{qk} - g^{qp} R_{kp} v_{jq}. \tag{3.3}$$

If we denote the trace of v by

$$V := tr_g(v) = g^{pq} v_{pq}, \tag{3.4}$$

then it is possible to write the evolution of the Ricci tensor in the form

$$\frac{\partial}{\partial t} R_{jk} = -\frac{1}{2} \left[(\Delta_L v)_{jk} + \nabla_j \nabla_k V + \nabla_j (\delta v)_k + \nabla_k (\delta v)_j \right]. \tag{3.5}$$

In fact, it follows from (3.2) that

$$\frac{\partial}{\partial t} R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j v_{kp} + \nabla_q \nabla_k v_{jp} - \nabla_q \nabla_p v_{jk} - \nabla_j \nabla_k v_{qp} \right).$$

Then the third term is already the Laplacian $-\frac{1}{2}(\Delta v)_{jk}$. Also, for the first two terms, we would like to commute the derivatives $q \leftrightarrow j$ and $q \leftrightarrow k$. One can do this by using the Ricci identity, then one gets

$$\begin{aligned}
\frac{\partial}{\partial t} R_{jk} =& -\frac{1}{2} \Delta v_{jk} + \frac{1}{2} g^{pq} \left(\nabla_j \nabla_q v_{kp} + \nabla_k \nabla_q v_{jp} \right) - \frac{1}{2} \nabla_j \nabla_k V \\
& - \frac{1}{2} g^{pq} \left(R^r_{qjk} v_{rp} + R^r_{qjp} v_{kr} \right) - \frac{1}{2} g^{pq} \left(R^r_{qkj} v_{rp} + R^r_{qkp} v_{jr} \right) \\
=& -\frac{1}{2} \left[\Delta v_{jk} + g^{pq} R^r_{qjp} v_{kr} + g^{pq} R^r_{qkp} v_{jr} \right] - \frac{1}{2} \nabla_j \nabla_k V - \frac{1}{2} \nabla_j (\delta v)_k \\
& - \frac{1}{2} \nabla_k (\delta v)_j - \frac{1}{2} g^{pq} (R^r_{qjk} v_{rp} + R^r_{qkj} v_{rp}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{\partial}{\partial t} R_{jk} =& -\frac{1}{2} \left[\Delta v_{jk} + \nabla_j \nabla_k V + \nabla_j (\delta v)_k + \nabla_k (\delta v)_j \right] \\
& - \frac{1}{2} g^{pq} \left[R^r_{qjp} v_{kr} + R^r_{qkp} v_{jr} + (R^r_{qjk} + R^r_{qkj}) v_{rp} \right].
\end{aligned}$$

We observe that $g^{pq} R^r_{qkj} v_{rp} = g^{sr} g^{pq} R_{qkjs} v_{rp} = g^{sr} g^{pq} R_{sjkq} v_{rp}$. Now, since there is a summation over s, r, p and q, we interchange s and q as well as r and p. Then

$$g^{pq} R^r_{qkj} v_{rp} = g^{pq} g^{sr} R_{qjks} v_{pr}.$$

In addition, we observe that

$$g^{pq}R^r_{qjp}v_{kr} = g^{rs}g^{pq}R_{qjps}v_{rk} = -g^{rs}g^{pq}R_{qjsp}v_{rk} = -g^{rs}R_{js}v_{rk} = -g^{pq}R_{jp}v_{qk}.$$

In the same way, we get

$$g^{pq}R^r_{qjp}v_{rk} = -g^{pq}R_{kp}v_{jq}.$$

This shows that equation (3.5) holds.

Lemma 3.1.5. *For a solution $g(t)$ of (3.1), the evolution of the scalar curvature function R is given by*

$$\frac{\partial}{\partial t}R = -\Delta V + \nabla^p\nabla^q v_{pq} - \langle v, Ric\rangle,$$

where V is given by (3.4).

Proof. Since $R = g^{jk}R_{jk}$, we just have to use Lemmas 3.1.1 and 3.1.4:

$$\begin{aligned}\frac{\partial}{\partial t}R &= \left(\frac{\partial}{\partial t}g^{jk}\right)R_{jk} + g^{jk}\left(\frac{\partial}{\partial t}R_{jk}\right)\\ &= -g^{ij}g^{kl}v_{il}R_{jk} + \frac{1}{2}g^{jk}g^{lm}\left(\nabla_l\nabla_j v_{km} + \nabla_l\nabla_k v_{jm} - \nabla_l\nabla_m v_{jk} - \nabla_j\nabla_k v_{lm}\right).\end{aligned}$$

Remember that there is a summation over i, j, k, l and m. So we can interchange indexes and write

$$\begin{aligned}\frac{\partial}{\partial t}R =& -g^{ij}g^{kl}\left(v_{ik}R_{jl}\right) + \frac{1}{2}g^{jk}g^{li}\left(\nabla_l\nabla_j v_{ki} + \nabla_l\nabla_k v_{ji} - \nabla_l\nabla_i v_{jk} - \nabla_j\nabla_k v_{li}\right)\\ &= -g^{ij}g^{kl}\left(v_{ik}R_{jl}\right) + \frac{1}{2}g^{ij}g^{kl}\left(\nabla_l\nabla_j v_{ik} + \nabla_l\nabla_i v_{jk} - \nabla_l\nabla_k v_{ij} - \nabla_j\nabla_i v_{kl}\right)\\ &= -g^{ij}g^{kl}\left(v_{ik}R_{jl} + \nabla_i\nabla_j v_{kl} - \nabla_k\nabla_i v_{jl}\right),\end{aligned}$$

where we used that v and g are symmetric several times. Finally, this last equality gives us

$$\frac{\partial}{\partial t}R = -\langle v, Ric\rangle - \Delta V + \mathrm{div}(\mathrm{div} v).$$

□

Lemma 3.1.6. *The volume element $d\mu$ of $(M, g(t))$ evolves by*

$$\frac{\partial}{\partial t} d\mu = \frac{V}{2} d\mu,$$

where V is given by (3.4).

Proof. We consider oriented local coordinates. Then

$$d\mu = \sqrt{det(g)} dx_1 \wedge \cdots \wedge dx_n.$$

Hence, we get

$$\begin{aligned}
\frac{\partial}{\partial t} d\mu =& \frac{\partial}{\partial t}(\sqrt{det(g)}) dx_1 \wedge \cdots \wedge dx_n = \frac{1}{2\sqrt{det(g)}} \frac{\partial}{\partial t}(det(g)) dx_1 \wedge \cdots \wedge dx_n \\
=& \frac{1}{2} \frac{1}{\sqrt{det(g)}} tr\left(g^{-1} \frac{\partial}{\partial t} g\right) det(g) dx_1 \wedge \cdots \wedge dx_n \\
=& \frac{1}{2}\left(g^{ij} \frac{\partial}{\partial t} g_{ij}\right) d\mu = \frac{1}{2}\left(g^{ij} v_{ij}\right) d\mu \\
=& \frac{V}{2} d\mu,
\end{aligned}$$

where we have used (3.1) and (3.4). □

Corollary 3.1.6.1. *For a compact manifold M, the total scalar curvature $\int_M R d\mu$ evolves by*

$$\frac{\partial}{\partial t}\left(\int_M R d\mu\right) = \int_M \left(\frac{1}{2} RV - \langle Ric, v\rangle\right) d\mu.$$

Proof. This is a direct consequence of the two lemmas above and Stokes' Theorem. □

If we choose v to be

$$v = -2Ric \tag{3.6}$$

then $V = -2R$ and we say that $g(t)$ satisfies **Ricci flow**, i.e.,

$$\begin{aligned}
&\frac{\partial}{\partial t} g(t) = -2Ric(g(t)), \qquad (3.7)\\
&g(0) = g_0.
\end{aligned}$$

In this section, we will suppose that we have a solution for the Ricci flow on a short time interval $[0, \varepsilon)$, for some $\varepsilon > 0$. Later in this chapter, we shall prove this fact. The next proposition is a direct consequence of the results above, in the particular case where v is given by (3.6).

Proposition 3.1.1. *Suppose $g(t)$ is a solution of the Ricci flow. Then, we have*

1. *The Levi-Civita connection $\Gamma(t)$ of $g(t)$ evolves by*

$$\frac{\partial}{\partial t}\Gamma_{ij}^{k} = -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right), \tag{3.8}$$

2. *The (3,1)-Riemann curvature tensor $Rm(t)$ of $g(t)$ evolves by*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{l} = g^{lp}[&-\nabla_i\nabla_j R_{kp} - \nabla_i\nabla_k R_{jp} + \nabla_i\nabla_p R_{jk} + \nabla_j\nabla_i R_{kp} \\ &+ \nabla_j\nabla_k R_{ip} - \nabla_j\nabla_p R_{ik}],\end{aligned} \tag{3.9}$$

3. *The Ricci tensor $Ric(t)$ of $g(t)$ evolves by*

$$\frac{\partial}{\partial t}R_{jk} = \Delta R_{jk} + \nabla_j\nabla_k R - g^{pq}(\nabla_q\nabla_j R_{kp} + \nabla_q\nabla_k R_{jp}), \tag{3.10}$$

4. *The scalar curvature $R(t)$ of $g(t)$ evolves by*

$$\frac{\partial}{\partial t}R = 2\Delta R - 2g^{jk}g^{pq}\nabla_q\nabla_j R_{kp} + 2|Ric|^2, \tag{3.11}$$

5. *The volume form $d\mu(t)$ of $g(t)$ evolves by*

$$\frac{\partial}{\partial t}d\mu = -Rd\mu. \tag{3.12}$$

By applying some curvature identities, we will obtain more useful forms of the equations above. In fact, we shall see that in dimension $n \geq 2$, the Riemann curvature, the Ricci curvature and the scalar curvature all satisfy reaction-diffusion equations (often called heat-type equations).

Proposition 3.1.2. *The Scalar Curvature under the Ricci flow* (3.7) *evolves by*

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2. \tag{3.13}$$

Proof. Using the contracted second Bianchi identity (2.3), we get

$$\nabla_k R = 2g^{il}\nabla_l R_{ik}.$$

Applying this to (3.11), we get

$$\begin{aligned}\frac{\partial}{\partial t}R =& 2\Delta R - g^{pq}\nabla_q\nabla_p R + 2|Ric|^2 \\ =& \Delta R + 2|Ric|^2.\end{aligned}$$

□

Proposition 3.1.3. *Under the Ricci flow, the Ricci tensor evolves by*

$$\frac{\partial}{\partial t}R_{jk} = \Delta_L R_{jk} = \Delta R_{jk} + 2g^{pq}g^{rs}R_{pjkr}R_{qs} - 2g^{pq}R_{jp}R_{qk}, \tag{3.14}$$

where Δ_L *is defined by* (3.3).

Proof. We just have to use the contracted second Bianchi identity again in (3.5). Then

$$\begin{aligned}\frac{\partial}{\partial t}R_{jk} =& \Delta_L R_{jk} + \nabla_j\nabla_k R - g^{pq}(\nabla_j\nabla_p R_{qk} + \nabla_k\nabla_p R_{jq}) \\ =& \Delta_L R_{jk} + \nabla_j\nabla_k R - \nabla_j(\frac{1}{2}\nabla_k R) - \nabla_k(\frac{1}{2}\nabla_j R) = \Delta_L R_{jk},\end{aligned}$$

where we used that R is a scalar function in the last equality. □

Remark. *The presence of the Riemann tensor* (Rm) *in equation (3.14) is a big obstacle to showing that nonnegative Ricci curvature is preserved. Therefore, we would like to get a better understanding of the contribution of the Riemann tensor to the evolution of the Ricci tensor.*

Since the Weyl tensor $W = 0$ in dimension $n = 3$, we get the following

Proposition 3.1.4. *In dimension $n = 3$, the Ricci tensor of a solution to the Ricci flow evolves by*

$$\frac{\partial}{\partial t} R_{jk} = \Delta R_{jk} + 3RR_{jk} - 6g^{pq} R_{jp} R_{qk} + (2|Ric|^2 - R^2) g_{jk}. \tag{3.15}$$

Proof. We just have to write the Riemann tensor Rm as a combination of the Ricci tensor Ric. Then it follows from (2.7) that

$$\begin{aligned} 2g^{pq} g^{rs} R_{pjkr} R_{qs} &= 2g^{pq} g^{rs} [R_{pr} g_{jk} + R_{jk} g_{pr} - R_{pk} g_{jr} - R_{jr} g_{pk} - \frac{R}{2}(g_{pr} g_{jk} - g_{pk} g_{jr})] R_{qs} \\ &= 2g^{pq} g^{rs} R_{pr} R_{qs} g_{jk} + 2g^{pq} g^{rs} g_{pr} R_{qs} R_{jk} - 2g^{pq} g^{rs} g_{jr} R_{pk} R_{qs} \\ &- 2g^{pq} g^{rs} g_{pk} R_{jr} R_{qs} - g^{pq} g^{rs} g_{pr} g_{jk} R_{qs} R + g^{pq} g^{rs} g_{pk} g_{jr} R_{qs} R \\ &= 2|Ric|^2 g_{jk} + 2g^{qs} R_{qs} R_{jk} - 2g^{pq} R_{pk} R_{qj} - 2g^{rs} R_{jr} R_{ks} \\ &- g^{qs} R_{qs} g_{jk} R + R_{kj} R \\ &= (2|Ric|^2 - R^2) g_{jk} + 3R_{kj} R - 4g^{pq} R_{jp} R_{qk}, \end{aligned}$$

where we used the fact the the Ricci tensor is symmetric and since there is a summation over p, q, r and s on $-2g^{pq} R_{pk} R_{qj} - 2g^{rs} R_{jr} R_{ks}$, we can add those terms together. Substituting this in equation (3.14) yields the formula in (3.15).

□

We would also like to check if the Riemann curvature tensor satisfies a reaction-diffusion equation.

Proposition 3.1.5. *Under the Ricci flow, the (3,1)-Riemann curvature tensor evolves by*

$$\begin{aligned} \frac{\partial}{\partial t} R^l_{ijk} = \Delta R^l_{ijk} + g^{pq} \left(R^r_{ijp} R^l_{rqk} - 2R^r_{pik} R^l_{jqr} + 2R^l_{pir} R^r_{jqk} \right) - R^p_i R^l_{pjk} \\ - R^p_j R^l_{ipk} - R^p_k R^l_{ijp} + R^l_p R^p_{ijk}. \end{aligned} \tag{3.16}$$

Proof. Using the second Bianchi identity (2.2) and the fact that $R^l_{jqk} = g^{lm} R_{jqkm}$, we have

$$\Delta R^l_{ijk} = g^{pq} \nabla_p \nabla_q R^l_{ijk} = g^{pq} \nabla_p (-\nabla_i R^l_{jqk} - \nabla_j R^l_{qik}).$$

Now commuting the covariant derivatives, from (2.5) we get

$$\begin{aligned}\Delta R^l_{ijk} =& g^{pq}\big[-\nabla_i\nabla_p R^l_{jqk} + R^r_{pij}R^l_{rqk} + R^r_{piq}R^l_{jrk} + R^r_{pik}R^l_{jqr}\\ &- R^l_{pir}R^r_{jqk} - \nabla_j\nabla_p R^l_{qik} - R^r_{pji}R^l_{rqk} - R^r_{pjq}R^l_{irk}\\ &- R^r_{pjk}R^l_{iqr} + R^l_{pjr}R^r_{iqk}\big].\end{aligned}$$

Observe that in the terms $-R^l_{pir}R^r_{jqk}$ and $R^l_{pjr}R^r_{iqk}$, we have switched the indexes r and l, this is why this terms have different signs. Applying the second Bianchi identity again, we have

$$\begin{aligned}g^{pq}\nabla_p R^l_{jqk} =& g^{pq}g^{lm}(-\nabla_k R_{jqmp} - \nabla_m R_{jqpk}) = g^{pq}g^{lm}(\nabla_k R_{qjmp} - \nabla_m R_{qjkp})\\ =& \nabla_k R^l_j - \nabla^l R_{jk},\end{aligned}$$

where we used the fact that $R^l_{jqk} = g^{lm}R_{jqkm}$ and the fact that g is a parallel tensor. Doing the same computation for $\nabla_p R^l_{qik}$, we can rewrite

$$\begin{aligned}\Delta R^l_{ijk} =& -\nabla_i\nabla_k R^l_j + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R^l_i - \nabla_j\nabla^l R_{ik}\\ &+ g^{pq}\big[R^r_{pij}R^l_{rqk} + R^r_{piq}R^l_{jrk} + R^r_{pik}R^l_{jqr} - R^l_{pir}R^r_{jqk}\\ &- R^r_{pji}R^l_{rqk} - R^r_{pjq}R^l_{irk} - R^r_{pjk}R^l_{iqr} + R^l_{pjr}R^r_{iqk}\big].\end{aligned}$$

By the first Bianchi identity, we know that

$$R^r_{pij}R^l_{rqk} - R^r_{pji}R^l_{rqk} = -R^r_{ijp}R^l_{rqk}.$$

Then ΔR^l_{ijk} can be written as

$$\begin{aligned}\Delta R^l_{ijk} =& -\nabla_i\nabla_k R^l_j + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R^l_i - \nabla_j\nabla^l R_{ik}\\ & R^r_i R^l_{jrk} + R^r_j R^l_{irk} + g^{pq}\big[-R^r_{ijp}R^l_{rqk} + R^r_{pik}R^l_{jqr}\\ &- R^l_{pir}R^r_{jqk} - R^r_{pjk}R^l_{iqr} + R^l_{pjr}R^r_{iqk}\big].\end{aligned}$$

Regarding the equation above, observe that since we are contracting on g^{pq}, we have

$$g^{pq}R^r_{pik}R^l_{jqr} = g^{pq}R^l_{pjr}R^r_{iqk}$$

and

$$g^{pq}R^l_{pir}R^r_{jqk} = g^{pq}R^r_{pjk}R^l_{iqr}.$$

Finally, considering formula (3.9) and rewriting

$$g^{pq}(\nabla_j\nabla_i R_{kp} - \nabla_i\nabla_j R_{kp}) = g^{lp}(R^q_{ijk}R_{qp} + R^q_{ijp}R_{kp}),$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}R^l_{ijk} =& -\nabla_i\nabla_k R^l_j - \nabla_j\nabla^l R_{ik} + \nabla_i\nabla^l R_{jk} + \nabla_j\nabla_k R^l_i + g^{lp}(R^q_{ijk}R_{qp} + R^q_{ijp}R_{kq}) \\ =& \Delta R^l_{ijk} + g^{pq}(R^r_{ijp}R^l_{rqk} - 2R^r_{pik}R^l_{jqr} + 2R^l_{pir}R^r_{jqk}) - R^p_i R^l_{pjk} - R^p_j R^l_{ipk} \\ &- g^{lq}R_{kp}R^q_{ijp} + R^l_p R^p_{ijk},\end{aligned}$$

where we interchanged the indexes p and q on the last two terms since there is a contraction on g^{pq}.

□

As a direct corollary we get

Corollary 3.1.6.2. *Under the Ricci flow, the (4,0)-Riemann curvature tensor satisfies the following reaction-diffusion equation:*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + g^{pq}\left(R^r_{ijp}R_{rqkl} - 2R^r_{pik}R_{jqrl} + 2R_{pirl}R^r_{jqk}\right) \\ &- \left(R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^l_p R_{ijkp}\right).\end{aligned} \tag{3.17}$$

Proof. Just remember that $R_{ijkl} = g_{lm}R^m_{ijk}$. Hence,

$$\frac{\partial}{\partial t}R_{ijkl} = \frac{\partial}{\partial t}(g_{lm}R^m_{ijk}) = (-2R_{lm})R^m_{ijk} + g_{lm}(\frac{\partial}{\partial t}R^m_{ijk}),$$

then the result follows from Proposition 3.1.5.

□

We define the (4,0)-tensor B by

$$B_{ijkl} \coloneqq -g^{pr}g^{qs}R_{ipjq}R_{krls} = -R_{pij}^{q}R_{qlk}^{p}. \tag{3.18}$$

Note that B satisfies

$$B_{ijkl} = B_{jilk} = B_{klij}. \tag{3.19}$$

Proposition 3.1.6. *Under the Ricci flow, the (4,0)-Riemann curvature tensor evolves by*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ & - (R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_p^l R_{ijkp})\end{aligned} \tag{3.20}$$

Proof. First we see that

$$-2g^{pq}R_{pik}^{r}R_{jqrl} = -2g^{pq}g^{rs}R_{ipks}R_{jqlr} = 2B_{ikjl}$$

and

$$2g^{pq}R_{pirl}R_{jqk}^{r} = 2g^{pq}g^{rs}R_{iprl}R_{jqkr} = -2B_{iljk}.$$

Then using the first Bianchi identity in the second term of (3.16) , we get

$$\begin{aligned}g^{pq}R_{ijp}^{r}R_{rqkl} =& g^{pq}g^{rs}R_{ijpr}R_{sqkl} = g^{pq}g^{rs}R_{rpji}R_{sqkl} \\ &= g^{pq}g^{rs}(-R_{rjip} - R_{ripj})(-R_{sklq} - R_{slqk}) \\ &= -B_{jikl} + B_{jilk} + B_{ijkl} - B_{ijlk} = 2B_{ijkl} - 2B_{ijlk}.\end{aligned}$$

Substituting this into equation (3.17), the proposition follows.

□

Remark. *Although the tensor B does not satisfy the first Bianchi identity, if we define the tensor C by*

$$C_{ijkl} \coloneqq B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk},$$

then C satisfies the Bianchi identity. Similarly, although $B_{jikl} \neq -B_{ijkl}$, we get $C_{jikl} = -C_{ijkl}$.

3.2 The Linearization of the Ricci Curvature

In this section, we consider the Ricci tensor Ric_g as a nonlinear partial differential operator on the metric g, i.e.,

$$Ric : C^\infty(S_2^+T^*M^n) \longrightarrow C^\infty(S_2T^*M^n),$$

where $C^\infty(S_2^+T^*M^n)$ denotes the space of positive definite symmetric (2,0)-tensors (i.e., Riemannian metrics) and $C^\infty(S_2T^*M^n)$ is the space of symmetric (2,0)-tensors.

First we introduce the concept of parabolicity of linear differential operators on vector bundles. Let E be a smooth vector bundle over a closed manifold M^n. Let $v \in C^\infty(E)$ be a smooth section of E. Locally we may write $v = v^\alpha e_\alpha$ for some local frame $\{e_\alpha\}$. Then we consider

$$\frac{\partial v}{\partial t} = L(v),$$

where L is a linear second order differential operator $L : C^\infty(E) \longrightarrow C^\infty(E)$ that may be given locally, in terms of coordinates $\{x^i\}$ on M^n and the local frame $\{e_\alpha\}$, by

$$L(v) = \left(a^{ij}_{\alpha\beta} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} v^\beta + b^i_{\alpha\beta} \frac{\partial}{\partial x^i} v^\alpha + c_\alpha v^\beta \right) e_\alpha.$$

Definition 3.2.1. *Let $\Pi : T^*M \longrightarrow M$ be the bundle projection over M^n and $\Pi^*(E)$ a vector bundle over T^*M whose fibre at $(p,\xi) \in T^*M$ is E_p, i.e., $(\Pi^*(E))_{(p,\xi)} = E_p$. Thus, we define the* ***principal symbol*** *of L, denoted by $\sigma(L) : \Pi^*(E) \longrightarrow \Pi^*(E)$, at $(p,\xi) \in T^*M$ as*

$$\sigma(L)(p,\xi)v = \left(a^{ij}_{\alpha\beta} \xi_i \xi_j v^\beta \right) e_\alpha.$$

We say that $\dfrac{\partial v}{\partial t} = L(v)$ is ***strictly parabolic*** *if there exists $\lambda > 0$ such that, for any fibre* *metric,*

$$\langle \sigma(L)(p,\xi)v, v\rangle \geq \lambda|\xi|^2|v|^2$$

$\forall (p,\xi) \in T^*M$ *and* $v \in C^\infty(E)$.

As we mentioned above, the Ricci curvature is nonlinear. In order to calculate its principal symbol, first we need to compute its linearization, which is defined in the following way.

Definition 3.2.2. *Let* E *and* F *be vector bundles over* M^n*. For a nonlinear differential operator* $F(x, \partial^k u) : C^\infty(E) \longrightarrow C^\infty(F)$*, its* ***linearization*** *at* $u \in C^\infty(E)$ *is defined to be the linear operator*

$$P_u v = \frac{d}{ds} F\left(x, \partial^k(U(s))\right)\bigg|_{s=0},$$

where $U(s) \in C^\infty(E)$*,* $U(0) = u$ *and* $U'(0) = v$*.*

For the Ricci curvature, we shall denote its linearization by

$$DRic : C^\infty(S_2T^*M) \longrightarrow C^\infty(S_2T^*M),$$

given by

$$(DRic)_g v = \frac{d}{ds} Ric(g(s))\bigg|_{s=0}$$

where $g(s)$ is a family of metrics such that $g(0) = g$ and $g'(0) = v$, which is the directional derivative of Ric in the direction of the variation of the metric. We may denote $E = -2Ric$. Then we have

Proposition 3.2.1. *The linearization of the differential operator*

$$E = -2Ric : C^\infty(S_2^+T^*M^n) \longrightarrow C^\infty(S_2T^*M^n)$$

can be written as

$$(DE_g v)_{ij} = g^{kl}\left(\nabla_i\nabla_j v_{kl} - \nabla_k\nabla_i v_{jl} - \nabla_k\nabla_j v_{il} + \nabla_k\nabla_l v_{ij}\right). \tag{3.21}$$

Proof. From the definition of linearization of a nonlinear differential operator on vector bundles, we have

$$DE_g v = -2(DRic)_g v = -2\frac{d}{ds}Ric(g(s))\bigg|_{s=0}.$$

Since $g(s)$ is a family of metrics with $g(0) = g$ and

$$g'(0) = \frac{\partial}{\partial s}g(s)\bigg|_{s=0} = v,$$

one sees that we can use the evolution equation for the Ricci tensor and equation (3.2) yields

$$(DE_g v)_{ij} = -2\frac{d}{ds}Ric(g(s))\bigg|_{s=0} = -2\frac{1}{2}g^{kl}\left(-\nabla_i\nabla_j v_{kl} + \nabla_k\nabla_i v_{jl} + \nabla_k\nabla_j v_{il} - \nabla_k\nabla_l v_{ij}\right).$$

Hence, the result follows. □

Now we are ready to compute the principal symbol of DE.

Corollary 3.2.0.1. *The principal symbol of the linear differential operator DE in the direction $\xi = (\xi_1, ..., \xi_n)$ is*

$$\sigma(DE(g)\xi)v_{ij} = g^{kl}\left(\xi_i\xi_j v_{kl} + \xi_k\xi_l v_{ij} - \xi_i\xi_k v_{jl} - \xi_k\xi_j v_{il}\right). \tag{3.22}$$

In particular, the Ricci Flow is not strictly parabolic.

Proof. As previously defined, we just have to replace ∇_i by the variable ξ_i to get (3.22), since our equation is tensorial. Now considering equation (3.22), we may assume that $\|\xi\| = 1$ and, since σ is a tensor, we can choose coordinates at a point such that

$$g_{ij} = \delta_{ij},$$
$$\xi = (1, 0, ..., 0).$$

Then

$$\sigma\left(DE(g)\right)(\xi)v_{ij} = v_{ij} + \delta_{i1}\delta_{ji}\left(v_{11} + v_{22} + ... + v_{nn}\right) - \delta_{i1}v_{1j} - \delta_{j1}v_{1i}.$$

So we get

$$[\sigma(DE(g))(\xi)v_{ij}]_{11} = v_{22} + ... + v_{nn},$$
$$[\sigma(DE(g))(\xi)v_{ij}]_{1k} = 0, \text{ if } k \neq 1,$$
$$[\sigma(DE(g))(\xi)v_{ij}]_{lk} = v_{lk}, \text{ if } k \neq 1 \text{ and } l \neq 1.$$

In particular,

$$(v_{ij}) = \begin{pmatrix} * & * & \dots & * \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix}$$

are eigenvectors of σ with eigenvalue 0. This shows that the Ricci flow cannot be strictly parabolic. □

3.3 Short Time Existence

The lack of strict parabolicity of the Ricci flow is our motivation to consider the modified Ricci flow (Ricci-DeTurck flow). First we rewrite the linearization of the Ricci tensor as

$$\begin{aligned}(DE_g v)_{jk} &= -2[D(Ric(g))v]_{jk} = -2\frac{\partial}{\partial s}R_{jk} \\ &= \Delta v_{jk} - \nabla_j(g^{pq}\nabla_q v_{pk}) - \nabla_k(g^{pq}\nabla_q v_{pj}) + \nabla_j\nabla_k(g^{pq}v_{qp}) \\ &+ 2g^{pq}R^r_{qjk}v_{rp} - g^{pq}R_{jp}v_{kq} - g^{pq}R_{kp}v_{jq}.\end{aligned}$$

Now we define the 1-form $H = H(g,v)$ given by

$$H_k \coloneqq g^{pq}\nabla_p v_{qk} - \frac{1}{2}\nabla_k(g^{pq}v_{pq}). \tag{3.23}$$

We observe that $V = g^{pq}v_{pq}$ is a scalar function, so $\nabla_j\nabla_k V = \nabla_k\nabla_j V$. Then we may write

$$-2[D(Ric(g))v]_{jk} = \Delta v_{jk} - \nabla_j H_k - \nabla_k H_j + S_{jk}, \tag{3.24}$$

where $S = S(g, v)$ is the symmetric 2-tensor defined by

$$S_{jk} := 2g^{pq}R^{r}_{qjk}v_{rp} - g^{pq}R_{jp}v_{kq} - g^{pq}R_{kp}v_{jq}. \tag{3.25}$$

We check that S involves no derivatives of v. Also, note that we may rewrite H as

$$\begin{aligned} H_k &= \frac{1}{2}g^{pq}\left(\nabla_p v_{qk} + \nabla_q v_{pk} - \nabla_k v_{pq}\right) \\ &= g^{pq}g_{kr}\left[D\left(\Gamma_g\right)v\right]^{r}_{pq}, \end{aligned}$$

where $D\left(\Gamma_g\right) : C^{\infty}(S_2T^*M^n) \longrightarrow C^{\infty}(S_2T^*M^n \otimes TM^n)$ denotes the linearization of the Levi-Civita connection and is given by

$$\left[D\left(\Gamma_g\right)v\right]^{k}_{ij} = \frac{\partial}{\partial s}\Gamma^{k}_{ij}\bigg|_{s=0}, \tag{3.26}$$

when $\frac{\partial}{\partial s}g\big|_{s=0} = v$.

Now we fix a background metric $\tilde{g}$ (we could always consider $\tilde{g} = g_0$, our initial metric on the manifold) and consider its Levi-Civita connection $\tilde{\Gamma}^{k}_{ij}$. We define a vector field $W = W(g, \tilde{\Gamma})$ by

$$W^k := g^{pq}\left(\Gamma^{k}_{pq} - \tilde{\Gamma^{k}_{pq}}\right). \tag{3.27}$$

Using the fact that the difference of two connections is a tensor, we get that $W \in \mathcal{X}(M^n)$ is globally well-defined. If we define $P = P(\tilde{\Gamma}) := \mathcal{L}_W g$, we see that P is a second order partial differential operator on g since W contains order one derivatives of g.

Since

$$\left(\mathcal{L}_W g\right)_{ij} = \nabla_i W^k g_{kj} + \nabla_j W^k g_{ik} + W^k \nabla_k(g_{ij}),$$

we can consider a geodesic frame and write

$$\left(\mathcal{L}_W g\right)_{ij} = \nabla_i W^j + \nabla_j W^i. \tag{3.28}$$

Then

$$(\mathcal{L}_W g)_{ij} = \nabla_i \left(g^{pq} \left(\Gamma^j_{pq} - \tilde{\Gamma^j_{pq}} \right) \right) + \nabla_j \left(g^{pq} \left(\Gamma^i_{pq} - \tilde{\Gamma^i_{pq}} \right) \right).$$

This, together with equation (3.26), shows that the linearization of P in v is

$$[DP(v)]_{jk} = \nabla_k H_j + \nabla_j H_k + T_{jk}, \tag{3.29}$$

where T_{jk} appears when we take $\frac{\partial}{\partial t}$ of g^{pq} and is a linear first order expression in v, so it will not be important for our purpose. Also, note that $\tilde{\Gamma}$ is fixed, so it vanishes when we take $\frac{\partial}{\partial t}$ of this expression.

This leads us to consider the modified Ricci operator

$$Q \coloneqq -2Ric + P : C^\infty(S_2 T^* M^n) \longrightarrow C^\infty(S_2 T^* M^n). \tag{3.30}$$

Proposition 3.3.1. *Q is an elliptic operator.*

Proof. From (3.24),

$$-2\left[D\left(Ric(g)\right)v\right]_{jk} = \Delta v_{jk} - \nabla_j H_k - \nabla_k H_j + S_{jk},$$

where H_k and S_{jk} are given by (3.23) and (3.25). Then it follows from (3.29) that the linearization of Q is given by

$$DQ(v) = \Delta v + U,$$

where $U_{jk} = S_{jk} + T_{jk}$ is a first order operator on g. This implies that the principal symbol of $DQ(v)$ is given by

$$\sigma\left[DQ(\xi)v\right] = |\xi|^2 v, \tag{3.31}$$

which shows that Q is elliptic. □

This construction leads us to define the **Ricci-DeTurck** flow on M^n by

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W + \nabla_j W, \\ g(0) &= g_0, \end{aligned} \tag{3.32}$$

where W is the 1-form g-dual to the vector field defined in (3.27). In particular,

$$W_j = g_{jk} W^k = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right) \tag{3.33}$$

depends on $g(t)$, $\Gamma(t)$ and the fixed background connection $\tilde{\Gamma}$.

Proposition 3.3.2. *The Ricci-DeTurck flow, defined in equation* (3.32)*, is strictly parabolic. Moreover, there exists an* $\varepsilon = \varepsilon(g_0) > 0$ *such that* (3.32) *has a unique solution* $g(t)$ *for* $t \in [0, \varepsilon)$.

Proof. Equation (3.31) shows that the Ricci-DeTurck flow is strictly parabolic. Since M is compact, it is a standard result from PDE theory that for any smooth initial metric g_0, there exists and $\varepsilon > 0$, depending on g_0, such that a unique solution $g(t)$ to (3.32) will exist on $[0, \varepsilon)$. For a brief discussion of the existence an uniqueness of solutions to parabolic partial differential equations on vector bundles, see Appendix A. □

If $t \in [0, \varepsilon)$, it is clear that the family of vector fields $W(t)$ is well defined. Then we consider the following family of maps

$$\begin{aligned} \frac{\partial}{\partial t} \phi_t(p) &= -W(\phi_t(p), t), \\ \phi_0 &= id, \end{aligned} \tag{3.34}$$

$\forall p \in M^n$, $\forall t \in [0, \varepsilon)$, where $\phi_t : M \longrightarrow M$ for each t.
The next lemma says that (3.34) always has a unique solution in $[0, \varepsilon)$.

Lemma 3.3.1. *If* $\{X_t : 0 \leq t < T \leq \infty\}$ *is a continuous time-dependent family of vector fields on a compact manifold* M^n*, then there exists an one-parameter family of diffeomorphisms*

$$\{\phi_t : M^n \longrightarrow M^n : 0 \leq t < T \leq \infty\}$$

defined on the same interval such that

$$\frac{\partial}{\partial t}\phi_t(p) = X_t(\phi_t(p)),$$
$$\phi_0(p) = p,$$

$\forall p \in M^n$, $\forall t \in [0, T)$.

Proof. We assume there is $t_0 \in [0, T)$ such that the solution exists for all $t \in [0, t_0]$ and $p \in M^n$. Let $t_1 \in (t_0, T)$ be given. If we prove that ϕ_t exists for all $t \in [t_0, t_1]$, the lemma follows since t_1 is arbitrary. For $p_0 \in M^n$ given, we consider local coordinates (U, x) and (V, y) such that $p_0 \in x(U)$ and $\phi_{t_0}(p_0) \in y(V)$.

If $p = x(q) \in x(U)$ and $\phi_t(p) \in y(V)$, then $\frac{\partial}{\partial t}\phi_t(p) = X_t(\phi_t(p))$ is equivalent to

$$\frac{\partial}{\partial t}\left(y^{-1} \circ \phi_t \circ x(q)\right) = y_*^{-1}\left[\frac{\partial}{\partial t}\phi_t(x(q))\right],$$

for $q \in U$ such that $\phi_t \circ x(q) \in y(V)$, where y_*^{-1} represents the differential of y^{-1}. Setting $z_t = y^{-1} \circ \phi_t \circ x$ and $F_t = y_*^{-1} X_t \circ x$, we get $\frac{\partial}{\partial t} z_t = F_t(z_t)$. Hence, our system is locally equivalent to a nonlinear ODE in $\mathbb{R}^n$. So for all $p \in x(U)$ such that $\phi_{t_0}(p) \in y(V)$, a unique solution exists for $t \in [t_0, t_0 + \varepsilon]$ for some $\varepsilon > 0$.

Since X_t is uniformly bounded on $M^n \times [t_0, t_1]$, there exists an $\varepsilon_0 > 0$, independent of $p \in M^n$ and of $t \in [t_0, t_1]$, such that there exists a unique solution $\phi_t(p)$ for $t \in [t_0, t_0 + \varepsilon_0]$. We see that this still holds for the flow starting at $\phi_{t_0+\varepsilon_0}$, so a simple iteration proves the lemma.

□

As a corollary, we get a unique solution for (3.34). Now we are ready to prove the existence of solutions to the Ricci flow.

Proposition 3.3.3. *Let* (M^n, g_0) *be a closed Riemannian manifold and let Ric denote its*

Ricci curvature tensor. Then the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -2Ric(g(t)), \\ g(0) &= g_0, \end{aligned} \tag{3.35}$$

has a solution for a short time $[0, \varepsilon)$, *for some* $\varepsilon > 0$.

Proof. Let $g(t)$ be a solution to (3.32) and let ϕ_t be a unique solution for (3.34) on $M \times [0, \varepsilon)$. We consider the family of metrics given by

$$\overline{g}(t) \coloneqq \phi_t^* g(t), \qquad t \in [0, \varepsilon) \tag{3.36}$$

We have $\overline{g}(0) = id^* g(0) = g_0$. In addition,

$$\begin{aligned} \frac{\partial}{\partial t}\overline{g}(t) = \frac{\partial}{\partial t}\left(\phi_t^* g(t)\right) = \frac{\partial}{\partial s}\left(\phi_{t+s}^* g(t+s)\right)\Big|_{s=0} &= \lim_{s\to 0} \frac{\phi_{t+s}^* g(t+s) - \phi_t^* g(t)}{s} \\ &= \lim_{s\to 0} \frac{\phi_{t+s}^* g(t+s) - \phi_t^* g(t+s)}{s} + \lim_{s\to 0} \frac{\phi_t^* g(t+s) - \phi_t^* g(t)}{s} \\ &= \phi_t^* \left(\lim_{s\to 0} \frac{(\phi_t^*)^{-1} \circ \phi_{t+s}^* - Id}{s}\right)(g(t+s)) + \phi_t^* \left(\lim_{s\to 0} \frac{g(t+s) - g(t)}{s}\right) \\ &= \phi_t^* \left(\lim_{s\to 0} \frac{(\phi_t^*)^{-1} \circ \phi_{t+s}^* - Id}{s}\right)(g(t)) + \phi_t^* \left(\frac{\partial}{\partial t} g(t)\right) \\ &= -\phi_t^* \left(\mathcal{L}_{W(t)} g(t)\right) + \phi_t^* \left(-2Ric(g(t)) + \mathcal{L}_{W(t)} g(t)\right) \\ &= -2\phi_t^* Ric(g(t)) = -2Ric\left(\phi_t^* g(t)\right) = -2Ric(\overline{g}(t)), \end{aligned}$$

where we used (3.32), (3.28) and in the last equality we used the invariance of the Ricci tensor under the family of diffeomorphisms ϕ_t^*. This shows the existence of the solution $\overline{g}(t)$ for the Ricci flow (3.35) for $t \in [0, \varepsilon)$. □

3.4 Uniqueness of the Ricci Flow

In order to prove uniqueness for the Ricci flow, Hamilton [14] used the sophisticated Nash-Moser Theorem. This was used because the Ricci flow is weakly parabolic, due to its invariance under diffeomorphisms. In this dissertation, we will follow the argument introduced by Hamilton in [13]. To do so, we need to define the **harmonic map flow**, firstly introduced by Eells and Sampson in [18].

Let $f : M^n \longrightarrow N^m$ be a smooth map between two Riemannian manifolds (M^n, g) and (N^m, h). We already know that the derivative of f, df, is an element of the vector bundle $C^\infty(T^*M^n \otimes f^*TN^m)$, where f^*TN^m is the pullback bundle over M^n.

Now let $\{x^i\}$ and $\{y^\alpha\}$ be local coordinates on M^n and N^m, respectively. We shall also denote the Levi-Civita connections of g and h respectively by $\Gamma(g)^k_{ij}$ and $\Gamma(h)^\gamma_{\alpha\beta}$. Thus,

$$df = (df)^\alpha_j \left(dx^j \otimes \frac{\partial}{\partial y^\alpha} \right) = \frac{\partial f^\alpha}{\partial x^j} \left(dx^j \otimes \frac{\partial}{\partial y^\alpha} \right).$$

We may also induce the following connection

$$\nabla : C^\infty(T^*M^n \otimes f^*T^*N^m) \longrightarrow C^\infty(T^*M^n \otimes T^*M^n \otimes f^*T^*N^m)$$

by

$$(f^*\Gamma)^\gamma_{i\beta} = \frac{\partial f^\alpha}{\partial x^i} \left(\Gamma_h \circ f \right)^\gamma_{\alpha\beta}.$$

Thus we get

$$\nabla(df) = \sum_{i,j,\alpha} (\nabla df)^\alpha_{ij} dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha},$$

where

$$(\nabla df)^\alpha_{ij} = \nabla_i d_j f^\alpha = \frac{\partial}{\partial x^i} \left(\frac{\partial f^\alpha}{\partial x^j} \right) - (\Gamma_g)^k_{ij} \left(\frac{\partial f^\alpha}{\partial x^k} \right) + \frac{\partial f^\beta}{\partial x^i} (\Gamma_h \circ f)^\alpha_{\beta\gamma} \left(\frac{\partial f^\gamma}{\partial x^j} \right).$$

Hence we may define the **harmonic map Laplacian** with respect to the domain metric g and codomain metric h as the trace

$$\Delta_{g,h} f \coloneqq tr_g(\nabla(df)) \in C^\infty(f^*TN^m).$$

In components, we get $(\Delta_{g,h}f)^\gamma = g^{ij}(\nabla df)^\alpha_{ij}$, i.e

$$(\Delta_{g,h}f)^\gamma = g^{ij}\left[\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - (\Gamma_g)^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + \left((\Gamma_h)^\gamma_{\alpha\beta}\circ f\right)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right]. \tag{3.37}$$

Now we are ready to define the harmonic map flow.

Definition 3.4.1. *Given a diffeomorphism $f_0 : M^n \longrightarrow N^m$, the* ***harmonic map flow*** *is given by*

$$\frac{\partial}{\partial t} f = \Delta_{g,h} f \tag{3.38}$$

$$f(0) = f_0 \tag{3.39}$$

Eells and Sampson ([18]) showed that this is a parabolic equation, so we have a unique solution on a short time interval. The following lemma gives us a useful way to rewrite the Laplacian operator.

Lemma 3.4.1. *Let $f : (M^n, g) \longrightarrow (N^m, h)$ be a diffeomorphism of Riemannian manifolds. Then*

$$(\Delta_{g,h}f)^\gamma(p) = [(f^{-1})^* g]^{\alpha\beta}\left(-\Gamma(f^{-1})^*g)^\gamma_{\alpha\beta} + \Gamma(h)^\gamma_{\alpha\beta}\right)(f(p)).$$

Proof. For simplicity, let $\kappa = (f^{-1})^* g$. Then, considering local coordinates $\{x^i\}$ and $\{y^\alpha\}$ on M^n and N^m, respectively, we shall compute the Christoffel symbols of κ.

$$\begin{aligned}\Gamma(f^*\kappa)^k_{ij}\frac{\partial}{\partial x^k} &= \nabla_{\frac{\partial}{\partial x^i}}(f^*\kappa)\frac{\partial}{\partial x^j} = (f^{-1})_*\left(\nabla_{f^*(\frac{\partial}{\partial x^i})} f_*(\frac{\partial}{\partial x^j})\right) = (f^{-1})_*\left(\nabla_{\frac{\partial f^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}}(\kappa)\left(\frac{\partial f^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\right)\right)\\ &= (f^{-1})_*\left(\frac{\partial^2 f^\beta}{\partial x^i\partial x^j}\frac{\partial}{\partial y^\beta} + (\Gamma(\kappa))^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\frac{\partial}{\partial y^\gamma}\right)\\ &= \left(\frac{\partial^2 f^\beta}{\partial x^i\partial x^j}\frac{\partial (f^{-1})^k}{\partial y^\beta} + (\Gamma(\kappa))^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\frac{\partial (f^{-1})^k}{\partial y^\gamma}\right)\frac{\partial}{\partial x^k}.\end{aligned}$$

Now this gives us

$$(\Gamma(f^*\kappa))^k_{ij}\frac{\partial f^\gamma}{\partial x^k} = \frac{\partial^2 f^\gamma}{\partial x^i\partial x^j} + (\Gamma(\kappa))^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}.$$

Multiplying it by $(f^*\kappa)^{ij}$ and using that

$$\kappa^{\alpha\beta} = (f^*\kappa)^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}, \tag{3.40}$$

we get

$$-\kappa^{\alpha\beta} (\Gamma(\kappa))^\gamma_{\alpha\beta} = (f^*\kappa)^{ij} \left(-\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - (\Gamma(f^*\kappa))^k_{ij} \frac{\partial f^\gamma}{\partial x^k} \right). \tag{3.41}$$

Now, we recall that $\kappa = (f^{-1})^* g$. Therefore, substituting (3.41) and (3.40) into (3.37) gives us

$$(\Delta f)^\gamma = \left((f^{-1})^* g\right)^{\alpha\beta} \left[-\left(\Gamma((f^{-1})^* g)\right)^\gamma_{\alpha\beta} + (\Gamma(h))^\gamma_{\alpha\beta} \right] f,$$

so the lemma follows. □

We then consider the case where $M = N$. The idea is to combine the Ricci flow with the harmonic map flow. Let us recall that the diffeomorphisms ϕ_t, defined by (3.34), satisfy the following equation

$$\frac{\partial}{\partial t} \phi_t = -W \circ \phi_t = g^{pq} \left(-\Gamma^k_{pq} + \tilde{\Gamma}^k_{pq} \right).$$

Hence it follows from Lemma 3.4.1 and the fact that $\overline{g} = (\phi_t)^* g$ that

$$\frac{\partial}{\partial t} \phi_t = g^{pq} \left(-\Gamma^k_{pq} + \tilde{\Gamma}^k_{pq} \right) = \left((\phi_t^{-1})^* \overline{g} \right)^{pq} \left(-\Gamma^k_{pq} + \tilde{\Gamma}^k_{pq} \right) = \Delta_{\overline{g(t)}, \tilde{g}} \phi_t.$$

This implies that the DeTurck diffeomorphisms satisfy the harmonic map flow. We are finally ready to prove uniqueness of the Ricci flow.

Proposition 3.4.1. *Under the same conditions of Proposition 3.3.3, the Ricci flow has a unique solution for a short time interval.*

Proof. We have already proved existence of solutions to the Ricci flow. Now, suppose that we have two solutions for the Ricci flow, $\overline{g_1}(t)$ and $\overline{g_2}(t)$, such that $\overline{g_1}(0) = \overline{g_2}(0)$. Let

ϕ_t^i be the solution to the harmonic map flow with respect to $\overline{g_i}(t)$ and $\tilde{g}$, with $\phi_0^i = id$ and $i = 1, 2$. Then $g_1(t) = (\phi_t^1)_* \overline{g_1}(t)$ and $g_2(t) = (\phi_t^2)_* \overline{g_2}(t)$ are both solutions of the Ricci-DeTurck flow with

$$g_1(0) = (\phi_0^1)_* \overline{g_1}(0) = (id)_* \overline{g_1}(0) = (\phi_0^2)_* \overline{g_2}(0) = g_2(0).$$

Since we have a unique solution for the Ricci-DeTurck flow, we get that $g_1(t) = g_2(t)$ as long as both exist. However, this implies that ϕ_t^1 and ϕ_t^2 are both solutions to the same autonomous ODE

$$\frac{\partial}{\partial t}(\phi_t^i)(p) = -W\left(\phi_t^i(p), t\right),$$

$i = 1, 2$, generated by the same vector field

$$W^k = (g_i)^{pq}\left(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k\right).$$

Thus, $\phi_t^1 = \phi_t^2$ for all times t where both are well defined. In particular,

$$\overline{g_1}(t) = (\phi_t^1)^* g_1(t) = (\phi_t^2)^* g_2(t) = \overline{g_2}(t),$$

and we have uniqueness for the Ricci flow. □

Chapter 4

The Maximum Principle

The Maximum Principle has been one of the techniques most used in the field of Geometric Analysis in the past decades. In this chapter, we provide an overview of the Maximum Principle in the scalar case and on Vector Bundles. The results in this chapter can be found in [12], [6] and [8].

4.1 The Scalar Case

For future comparison, we will start with the first and second derivative test, from differential calculus. Let $u : (0, l) \times [0, T) \longrightarrow \mathbb{R}$ be a C^2 function satisfying the following inequality

$$\frac{\partial^2}{\partial x^2} u(x,t) - \frac{\partial}{\partial t} u(x,t) > 0, \tag{4.1}$$

on a region $E = (0, l) \times (0, T)$. The function u cannot attain a local maximum on any interior point $(x_0, t_0) \in E$ because otherwise we would have for $(x_0, t_0) \in E$:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} u(x_0, t_0) &\leq 0 \\ \frac{\partial}{\partial t} u(x_0, t_0) &= 0, \end{aligned}$$

a contradiction with (4.1). This basic idea will guide us through the following results.

We start with a closed Riemannian manifold M^n and a family of Riemannian metrics $g(t)$, $t \in [0,T)$, on M. Also, let $X(t) = X_t$ be a familly of smooth vector fields on M. We say that a C^2 function $u : M^n \times [0,T) \longrightarrow \mathbb{R}$ is a **supersolution** of

$$\frac{\partial}{\partial t} v = \Delta_{g_t} v + \langle X_t, \nabla v \rangle \tag{4.2}$$

in $(p,t) \in M^n \times [0,T)$ if

$$\frac{\partial}{\partial t} u(p,t) \geq \Delta_{g_t} u(p,t) + \langle X_t, \nabla u \rangle (p,t). \tag{4.3}$$

Analogous to that, we say that u is a **subsolution** if

$$\frac{\partial}{\partial t} u(p,t) \leq \Delta_{g_t} u(p,t) + \langle X_t, \nabla u \rangle (p,t), \tag{4.4}$$

where $\Delta_{g_t} = g^{ij} \nabla_i \nabla_j$ is the Laplacian on the metric $g(t)$.

Theorem 4.1.1. *Let g_t and X_t as above, $t \in [0,T)$. Let $u : M^n \times [0,T) \longrightarrow \mathbb{R}$ be a C^2 function, where M^n is a closed Riemannian manifold. Suppose that there exists $\alpha \in \mathbb{R}$ such that $u(p,0) \geq \alpha$, $\forall p \in M$ and that u is a supersolution of* (4.2) *at every point $(p,t) \in M \times [0,T)$ where $u(p,t) < \alpha$. Then $u(p,t) \geq \alpha$ for all $(p,t) \in M \times [0,T)$.*

Proof. We define the auxiliary function

$$H \coloneqq u - \alpha + \varepsilon t + \varepsilon,$$

for any $\varepsilon > 0$. Since u is a supersolution of 4.2 at every point where $u < \alpha$, we get

$$\frac{\partial}{\partial t} H = \frac{\partial}{\partial t} u + \varepsilon \geq \Delta u + \langle X_t, \nabla u \rangle + \varepsilon = \Delta H + \langle X_t, \nabla H \rangle + \varepsilon, \tag{4.5}$$

at those points. Moreover, $H(p,0) \geq \varepsilon$ since $u(p,0) \geq \alpha$.

We claim that $H(p,t) > 0$ for all $p \in M$ and all $t \in [0,T)$. In fact, suppose that there exists a point $(p_1,t_1) \in M \times [0,T)$ such that $H(p_1,t_1) \leq 0$. Let us consider the function

$$F : [0,T) \longrightarrow \mathbb{R}$$

given by $F(t) := \inf_{p\in M} H(p,t)$.

Since M is compact, this infimum is always attained. Now F is obviously continuous, $F(0) > 0$ and $F(t_1) \leq 0$. Therefore there must be a first time $t_0 \in (0,t_1]$ such that $F(t_0) = 0$. Let $p_0 \in M$ be such that $F(t_0) = H(p_0,t_0)$. Thus $H(p_0,t_0) = 0$ gives us

$$u(p_0,t_0) = \alpha - \varepsilon(1+t_0) < \alpha,$$

so u is a supersolution at (p_0,t_0). Since H is a C^2 function and (p_0,t_0) is a point and time where H attains its minimum among all $p \in M$ and all $t \in [0,t_0]$, that is,

$$H(p_0,t_0) = \min_{M\times[0,t_0]} H,$$

we have the following conditions on its derivatives

$$\begin{aligned}
&\frac{\partial}{\partial t}H(p_0,t_0) \leq 0,\\
&\nabla H(p_0,t_0) = 0,\\
&\Delta H(p_0,t_0) \geq 0.
\end{aligned}$$

Combining this with (4.5) implies

$$0 \geq \frac{\partial}{\partial t}H(p_0,t_0) \geq \Delta H(p_0,t_0) + \langle X, \nabla H\rangle(p_0,t_0) + \varepsilon \geq \varepsilon > 0,$$

which is a contradiction. Hence our claim holds. It follows from the definition of H that $u(p,t) + \varepsilon t + \varepsilon > \alpha$ for all $p \in M$ and all $t \in [0,T)$. Since $\varepsilon > 0$ is arbitrary, this proves the theorem. □

Remark. *This is the simplest case of the scalar maximum principle and it is, basically, the original heat equation since* ∇u *vanishes at local maximum and minimum.*

Now we consider the heat equation with a linear reaction term. Let $\beta : M^n \times [0,T) \longrightarrow$

$\mathbb{R}$ be a given function and consider the following equation

$$\frac{\partial}{\partial t}v = \Delta_{g_t}v + \langle X_t, \nabla v\rangle + \beta v.$$

We define **supersolution** and **subsolution** in the same way as before, i.e., v is a supersolution of the equation above if

$$\frac{\partial}{\partial t}v \geq \Delta_{g_t}v + \langle X_t, \nabla v\rangle + \beta v$$

and a subsolution if

$$\frac{\partial}{\partial t}v \leq \Delta_{g_t}v + \langle X_t, \nabla v\rangle + \beta v.$$

Proposition 4.1.1. *Let $u : M^n \times [0,T) \longrightarrow \mathbb{R}$ be a C^2 function, where M^n is a closed Riemannian manifold, satisfying*

$$\frac{\partial}{\partial t}u \geq \Delta_{g_t}u + \langle X_t, \nabla u\rangle + \beta u. \tag{4.6}$$

Suppose that for each $\tau \in [0,T)$, there exists a constant $0 < C(\tau) < \infty$ such that $\beta(p,t) \leq C(\tau)$ $\forall (p,t) \in M \times [0,\tau]$. If $u(p,0) \geq 0$ $\forall p \in M$, then $u(p,t) \geq 0$ $\forall (p,t) \in M \times [0,T)$.

Proof. We start by defining the auxiliary function $J(p,t) := e^{-C(\tau)t}u(p,t)$, for each $\tau \in (0,T)$, where $C(\tau)$ is defined as in the hypothesis. Thus,

$$\frac{\partial}{\partial t}J = -C(\tau)J + e^{-C(\tau)t}\frac{\partial}{\partial t}u \geq \Delta_{g_t}J + \langle X_t, \nabla J\rangle + (\beta - C(\tau))J.$$

Now, fix an arbitrary $\tau \in (0,T)$. Suppose there exists a point $(p_0,t_0) \in M \times [0,\tau)$ such that $J(p_0,t_0) < 0$. Therefore, on a neighborhood $U \in M \times (0,\tau)$ of (p_0,t_0) we have $J(p,t) < 0$. This yields

$$\frac{\partial}{\partial t}J(p,t) \geq \Delta_{g_t}J(p,t) + \langle X_t, \nabla J\rangle(p,t),$$

$\forall (p,t) \in U$ since $\beta - C(\tau) \leq 0$ in $M \times [0,\tau]$.
By hypothesis $u(p,0) \geq 0$. Hence, using Theorem 4.1.1, we get that $J \geq 0$ in $M \times [0,\tau]$, which says that $u \geq 0$ in $M \times [0,\tau]$. Since $\tau \in (0,T)$ is arbitrary, the result follows. □

Finally, we are ready for our main result. Now we would like to consider a *non-linear*

reaction term. This is the most common case when we are studying the evolution of geometric quantities under the Ricci flow or the mean curvature flow.
We consider the following semi-linear heat equation

$$\frac{\partial}{\partial t}v = \Delta_{g_t}v + \langle X_t, \nabla v\rangle + F(v), \tag{4.7}$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function. We define **supersolution** and **subsolution** in the same way as before: u is a supersolution of (4.7) if

$$\frac{\partial}{\partial t}u \geq \Delta_{g_t}u + \langle X_t, \nabla u\rangle + F(u)$$

and a subsolution if

$$\frac{\partial}{\partial t}u \leq \Delta_{g_t}u + \langle X_t, \nabla u\rangle + F(u).$$

Theorem 4.1.2. *Let $u : M^n \times [0,T) \longrightarrow \mathbb{R}$ be a C^2 function and a supersolution for (4.7) on a closed manifold. Suppose that there exists a constant C_1 such that $u(p,0) \geq C_1$ $\forall p \in M$ and let ϕ_1 be the solution of the initial value problem*

$$\frac{\partial}{\partial t}\phi_1 = F(\phi_1),$$
$$\phi_1(0) = C_1.$$

Then $u(p,t) \geq \phi_1(t)$ $\forall p \in M$ and $\forall t \in [0,T)$ where $\phi_1(t)$ exists. Analogously, if u is a subsolution of (4.7) and $u(p,0) \leq C_2$ $\forall p \in M$, where $C_2 \in \mathbb{R}$ is a constant. Let $\phi_2(t)$ be the solution of the initial value problem

$$\frac{\partial}{\partial t}\phi_2 = F(\phi_2),$$
$$\phi_2(0) = C_2.$$

Then $u(p,t) \leq \phi_2(t)$ $\forall p \in M$ and $\forall t \in [0,T)$ where $\phi_2(t)$ exists.

Proof. For the first part, we have

$$\frac{\partial}{\partial t}(u - \phi_1) \geq \Delta_{g_t}(u - \phi_1) + \langle X_t, \nabla_t(u - \phi_1)\rangle + F(u) - F(\phi_1),$$

where $u - \phi_1 \geq 0$ when $t = 0$. Let $\tau \in (0,T)$ be arbitrary. Since M is compact, given

$\tau \in (0,T)$, there exists $0 < C(\tau) < \infty$ such that $|u(p,t)| < C(\tau)$ and $|\phi_1| < C(\tau)$ $\forall p \in M$ and $\forall t \in [0,\tau]$. Now since F is locally Lipschitz, there exists $0 < L(\tau) < \infty$ such that $|F(u) - F(\phi_1)| < L(\tau)|u - \phi_1|$, $\forall u, \phi_1$ such that $|u(p,t)| < C(\tau)$ and $|\phi_1| < C(\tau)$ $\forall p \in M$ and $\forall t \in [0,\tau]$. Therefore,

$$\frac{\partial}{\partial t}(u - \phi_1) \geq \Delta_{g_t}(u - \phi_1) + \langle X_t, \nabla_t(u - \phi_1)\rangle - L(\tau)sgn(u - \phi_1)(u - \phi_1),$$

because

$$|F(u) - F(\phi_1)| < L(\tau)|u - \phi_1|$$

implies

$$|F(\phi_1) - F(u)| < L(\tau)sgn(u - \phi_1)(u - \phi_1),$$

and therefore

$$F(u) - F(\phi_1) \geq -|F(\phi_1) - F(u)| > -L(\tau)sgn(u - \phi_1)(u - \phi_1).$$

If $\beta := -L(\tau)sgn(u - \phi_1)$, then $\beta(p,t) \leq L(\tau)$ and $u(p,0) - \phi_1(0) \geq 0$. Therefore it follows from Proposition 4.1.1 that $u - \phi_1 \geq 0$ $\forall (p,t) \in M \times [0,T)$ where $\phi_1(t)$ exists.
Now, for the second part, we have

$$\frac{\partial}{\partial t}(\phi_2 - u) \geq \Delta_{g_t}(\phi_2 - u) + \langle X_t, \nabla_t(\phi_2 - u)\rangle + F(\phi_2) - F(u),$$

with $\phi_2 - u \geq 0$ at $t = 0$. Then, the result is totally analogous to the first part. □

4.2 The Maximum Principle for Vector Bundles

In 1986, Hamilton ([12]) introduced the maximum principle for systems, which says that given a heat-type equation for sections of a vector bundle over a manifold, if the solution is initially in a closed convex subset, invariant under parallel translation, and if the ODE associated to the PDE preserves the subset, then the solution of PDE remains in the subset for positive time. This result is given in details in Theorem 4.2.14. In 2004, Bennet Chow and Peng Lu ([6]) presented a more general maximum principle. In their paper, the subsets are time-dependent.

We shall introduce the techniques that are needed to prove Chow and Lu's result, formulate the vector bundle that we will need, present the result and prove it. This section is mainly based on the theory developed in [8].

4.2.1 Spatial Maximum Functions and Its Dini Derivatives

Definition 4.2.1. *Let $f(t)$ be a function defined on (a, b). The **upper Dini derivative** is the* lim sup *of forward difference quotients:*

$$\frac{d^+ f}{dt}(t) := \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h},$$

*the **lower Dini derivative** is the* lim inf *of the same quotients:*

$$\frac{d^- f}{dt}(t) := \liminf_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

*We may also define the **upper converse Dini derivative** by*

$$\frac{d_+ f}{dt}(t) := \limsup_{h \to 0^+} \frac{f(t) - f(t-h)}{h}$$

*and the **lower converse Dini derivative** by*

$$\frac{d_- f}{dt}(t) := \liminf_{h \to 0^+} \frac{f(t) - f(t-h)}{h}.$$

We say that $f(t)$ is **right upper semi-continuous** at τ if

$$\limsup_{t \to \tau^+} f(t) \leq f(\tau).$$

Similarly, $f(t)$ is **left lower semi-continuous** at τ if

$$\liminf_{t \to \tau^-} f(t) \geq f(\tau).$$

Finally, we say that $f(t)$ is right upper semi-continuous (resp. left lower semi-continuous) if it is right upper semi-continuous (resp. left lower semi-continuous) at t for all $t \in (a, b)$. For instance, if $\dfrac{d^+ f}{dt} < \infty$ then f is right upper semi-continuous.

Lemma 4.2.1. *Assume $f : [0, T) \longrightarrow \mathbb{R}$ is right upper semi-continuous and left lower semi-continuous, with $f(0) \leq c$. If $\dfrac{d^+ f}{dt} \leq 0$, $\forall t \in [0, T)$ where $f(t) > c$, then $f(t) \leq c$ for all $t \in [0, T)$.*

Proof. Given $\varepsilon > 0$, let $f_\varepsilon(t) \coloneqq f(t) - \varepsilon - \varepsilon t$. Let $t_0 \in (0, T]$ be defined by

$$t_0 \coloneqq \sup \left\{ \tau \in [0, T]; f_\varepsilon(t) \leq c, \forall t \in [0, \tau) \right\}.$$

Since $f_\varepsilon(0) = c - \varepsilon < c$ and f_ε is right upper semi-continuous, we get $t_0 > 0$. Suppose that $t_0 < T$ for some $\varepsilon > 0$. First we observe that $f_\varepsilon(t_0) = c$. In fact, since f_ε is right upper semi-continuous and left lower semi-continuous and $t_0 < T$, we have

$$c \leq \limsup_{t \to {t_0}^+} f_\varepsilon(t) \leq f_\varepsilon(t_0)$$

and

$$c \geq \liminf_{t \to {t_0}^-} f_\varepsilon(t) \geq f_\varepsilon(t_0),$$

which implies the equality.

Then we can consider a sequence $\{t_i\} \subset [0, T]$, $t_i \searrow t_0$ such that $f_\varepsilon(t_i) > c = f_\varepsilon(t_0)$. Thus,

$$0 \leq \frac{f_\varepsilon(t_i) - f_\varepsilon(t_0)}{t_i - t_0} = \frac{f(t_i) - f(t_0)}{t_i - t_0} - \varepsilon.$$

Therefore,

$$\frac{d^+}{dt} f(t_0) \geq \liminf_{i \to \infty} \frac{f(t_i) - f(t_0)}{t_i - t_0} \geq \varepsilon > 0.$$

Since $f(t_0) = c + \varepsilon + \varepsilon t_0 > c$, we get that $\dfrac{d^+}{dt} f(t_0) \leq 0$ by hypothesis, which is a contradiction. Hence $t_0 = T$ for all $\varepsilon > 0$. Therefore $f_\varepsilon(t) \leq c$ $\forall t \in [0, T)$. If we let $\varepsilon \to 0$, we prove the lemma. □

The following result is a direct consequence of the lemma above.

Corollary 4.2.1.1. *If $f : [0, T) \longrightarrow \mathbb{R}$ is right upper semi-continuous and left lower semi-continuous and $\dfrac{d^+ f}{dt} \leq 0$, $\forall t \in [0, T)$, then $f(t)$ is nonincreasing on $[0, T)$.*

The next result shows that under certain conditions, exponential growth preserves $f \leq 0$.

Corollary 4.2.1.2. *Let $f : [0,T) \longrightarrow \mathbb{R}$ be right upper and left lower semi-continuous with $f(0) \leq 0$. If there exists c, $0 < c < \infty$ such that $\frac{d^+ f}{dt} \leq cf(t)$ $\forall t \in [0,T)$ where $f(t) > 0$, then $f(t) \leq 0$ $\forall t \in [0,T)$.*

Proof. Consider $e^{-ct}f(t)$. Then we have

$$\begin{aligned}\frac{d^+}{dt}(e^{-ct}f(t)) =& e^{-ct}\frac{d^+}{dt}f(t) - ce^{-ct}f(t)\\ &= e^{-ct}\left[\frac{d^+}{dt}f(t) - cf(t)\right] \leq 0,\end{aligned}$$

$\forall t \in [0,T)$ where $f(t) > 0$. Then it follows from Lemma 4.2.1 that $e^{-ct}f(t) \leq 0$ $\forall t \in [0,T)$. Since $e^{-ct} > 0$, the result follows.

□

Now, let $\mathcal{S}$ be a topological space and let $g : \mathcal{S} \times [0,T) \longrightarrow \mathbb{R}$ be a function. We define the **spatial maximum function** $f : [0,T) \longrightarrow \mathbb{R}$ relative to g by

$$f(t) := \sup_{s \in \mathcal{S}} g(s,t).$$

The following lemma will be useful to help us characterize when an ODE preserves a closed convex set and when a PDE preserves a set that is closed and convex in each fiber.

Lemma 4.2.2. *If $\mathcal{S}$ is a sequentially compact topological space and if g and $\frac{\partial g}{\partial t}$ are continuous in s and t, then f is locally Lipschitz and*

$$\frac{d^+ f}{dt}(t) = \sup\left\{\frac{\partial g}{\partial t}(s,t); s \in \mathcal{S} \text{ satisfies } g(s,t) = f(t)\right\}.$$

Proof. First, since $\mathcal{S}$ is sequentially compact, we get that g is uniformly Lipschitz in t on $\mathcal{S} \times [0, T-\varepsilon]$ for every $\varepsilon > 0$, therefore f is locally Lipschitz. Now, let (t_i) be a sequence of times such that $t_{i+1} \leq t_i$ and $t_i \to t$ with

$$\lim_{i\to\infty} \frac{f(t_i) - f(t)}{t_i - t} = \frac{d^+ f}{dt}(t).$$

Since $\mathcal{S}$ is compact, for each $i \in \mathbb{N}$ we get $s_i \in \mathcal{S}$ such that

$$g(s_i, t_i) = f(t_i).$$

Also, we can find a subsequence such that $s_i \to s_\infty$ for some $s_\infty \in \mathcal{S}$. Then

$$g(s_\infty, t) = \lim_{i\to\infty} g(s_i, t_i) = \lim_{i\to\infty} f(t_i) = f(t).$$

Hence, we get, using the Mean Value Theorem,

$$\frac{d^+f}{dt}(t) = \lim_{i\to\infty} \frac{g(s_i, t_i) - g(s_\infty, t)}{t_i - t} \leq \lim_{i\to\infty} \frac{g(s_i, t_i) - g(s_i, t)}{t_i - t} = \lim_{i\to\infty} \frac{\partial g}{\partial t}(s_i, \tau_i),$$

for some $\tau_i \in [t, t_i]$, where the inequality follows from $g(s_\infty, t) = \sup_{s\in\mathcal{S}} g(s, t) \geq g(s_i, t)$. Now, since $s_i \to s_\infty$ and $\tau_i \to t$, we have

$$\lim_{i\to\infty} \frac{\partial g}{\partial t}(s_i, \tau_i) = \frac{\partial g}{\partial t}(s_\infty, t),$$

because $\frac{\partial g}{\partial t}$ is continuous on both variables by hypothesis. Then $\frac{d^+f}{dt}(t) \leq \frac{\partial g}{\partial t}(s_\infty, t)$ for some $s_\infty \in \mathcal{S}$ with $f(t) = g(s_\infty, t)$.
On the other hand, let $s_0 \in \mathcal{S}$ such that $g(s_0, t) = f(t)$ and

$$\frac{\partial g}{\partial t}(s_0, t) = \sup\left\{\frac{\partial g}{\partial t}(s, t); s \in \mathcal{S} \text{ satisfies } g(s, t) = f(t)\right\}.$$

Let (δ_i) be a sequence of positive real numbers such that $\delta_{i+1} \leq \delta_i$, $\delta_i \to 0$ and

$$\frac{d^+f}{dt}(t) = \lim_{i\to\infty} \frac{f(t+\delta_i) - f(t)}{\delta_i} \geq \lim_{i\to\infty} \frac{g(s_0, t+\delta_i) - g(s_0, t)}{\delta_i} = \frac{\partial g}{\partial t}(s_0, t).$$

Then $\frac{d^+f}{dt}(t) \geq \frac{\partial g}{\partial t}(s_0, t) \geq \frac{\partial g}{\partial t}(s_\infty, t)$ and the lemma follows. □

4.2.2 Convex Sets and Support Functions

In this section, we introduce important properties of convex sets, which are going to be useful for our maximum principle.

Definition 4.2.2. *A set $\Gamma \subset \mathbb{R}^k$ is a **cone** with vertex $u \in \mathbb{R}^k$ if, for every $\omega \in \Gamma$, we* *have* $u + t(\omega - u) \in \Gamma$, $\forall t \in [0, \infty)$.

In what follows, $J \subset \mathbb{R}^k$ is a closed convex set. We state the next lemma, which is a classical result and will help us in dealing with half-spaces containing a given convex set.

Lemma 4.2.3. *(i) For any* $\omega \notin J$*, there exists a* ***unique*** $v \in \partial J$ *such that* $d(\omega, v) = \inf_{x \in J} d(\omega, x) = d(\omega, J)$.

(ii) Let $J(t)$, $t \in [0, T]$, *be a continuous family of convex sets. Given* $\omega \notin J(t)$, *let* $v(\omega, t)$ *be the point defined by* ω *and* $J(t)$ *as in* (i). *Then* v *is a continuous function of* (ω, t).

Proof. See [2], Theorem 5.2 on page 132. □

Lemma 4.2.4. *Given* $\omega \notin J$ *and* $v \in \partial J$ *such that* $d(\omega, J) = d(v, \omega)$, *the half-space*

$$H_\omega := \left\{ x \in \mathbb{R}^k ; \langle x - v, \omega - v \rangle \le 0 \right\}$$

contains J *and* $\omega \notin H_\omega$. *Hence any convex set is equal to the intersection of a family of half-spaces.*

Proof. See [2], Theorem 5.2 on page 132. □

Definition 4.2.3. *Given* $v \in \partial J$, *the* ***tangent cone*** $C_v J$ *of* J *at* v *is the intersection of all closed half-spaces containing* J *and with* v *on the boundary of the half-space.*

Remark. *Observe that this definition only makes sense for convex sets.*

Lemma 4.2.5. *If* Γ *is a closed convex cone with vertex* u, *then* Γ *is an intersection of half-spaces with* u *contained on their boundaries.*

Proof. The result follows from the fact that $\forall w \notin \Gamma$, there exists a half-space H_ω containing Γ with $u \in \partial H_\omega$ and $w \notin H_\omega$. In fact,$\forall w \in \Gamma$ it follows from Lemmas 4.2.3 and 4.2.4 that there is a unique $v \in \partial\Gamma$ closest to w. Let

$$H := \left\{ x \in \mathbb{R}^k ; \langle x - v, w - v \rangle \le 0 \right\}.$$

Then H_ω is a closed half-space with $\Gamma \subset H$ and $w \notin H$. Also, it is clear that $v \in \partial H$. The ray $R = \{u + t(v - u); t \ge 0\}$ is in Γ. If $u \notin \partial H$, then for some t sufficiently close to 1, $u + t(v - u) \in \Gamma$ is closest to w than v, a contradiction. Hence $u \in \partial H$. □

Lemma 4.2.6. *The tangent cone $C_v J$ is the smallest closed convex cone with vertex v containing J.*

Proof. It follows directly from the lemma above and the definition of a tangent cone. □

Definition 4.2.4. *A linear function $l : \mathbb{R}^k \longrightarrow \mathbb{R}$ is a* ***support function*** *for J at $v \in \partial J$ if*

(i) $|l| := \sup\limits_{x, |x|=1} l(x) = 1$*;*

(ii) $l(\omega) \leq l(v)$, $\forall \omega \in J$.

Definition 4.2.5. *Given a support function l for J at $v \in \partial J$, define the* ***associated closed half-space*** $H_l \subset \mathbb{R}^k$ *by*

$$H_l := \left\{\omega \in \mathbb{R}^k ; l(\omega) \leq l(v)\right\}.$$

Remark. *H_l is the closed half-space whose boundary is equal to the hyperplane passing through v and perpendicular to N_l, where N_l is the outward unit normal to ∂J at v. Also, (ii) says that $J \subset H_l$.*

From now on, let H be any closed half-space of $\mathbb{R}^k$. For every $v \in \partial H$, the unit outward normal is the same and will be denoted by N_H.

Definition 4.2.6. *The linear function $l_H : \mathbb{R}^k \longrightarrow \mathbb{R}$ associated to a half-space $H \subset \mathbb{R}^k$ is defined by*

$$l_H(\omega) := \langle \omega, N_H \rangle, \tag{4.8}$$

for all $\omega \in \mathbb{R}^k$.

Lemma 4.2.7. *Let $v \in \partial J$ and let H be a closed half-space containing J, with $v \in \partial H$. Then the linear function l_H is a support function for J at v.*

Proof. We have that $l_H(\omega) \leq l_H(v)$ $\forall \omega \in H$, in particular $\forall \omega \in J$. Also, $|l_H| \leq 1$ and $l_H(N_H) = 1$, so $|l_H| = 1$. □

Given $v \in \partial J$, let

$$S_v J := \text{"set of support functions of } J \text{ at } v\text{"}$$

and

$$H_v J := \text{"set of closed half-spaces containing } J \text{ with } v \text{ on their boundaries".}$$

If we define $\phi : S_v J \longrightarrow H_v J$ by $\phi(l) := H_l$ and $\psi : H_v J \longrightarrow S_v J$ by $\psi(H) := l_H$, then it is clear that $\phi^{-1} = \psi$ and $\psi^{-1} = \phi$. The next lemma offers a criterion for a vector to be in the tangent cone.

Lemma 4.2.8. *For any $v \in \partial J$, we have $X \in C_v J$ if and only if $l(X) \leq l(v)$, $\forall l \in S_v J$.*

Proof. In fact, it is easy to check that $X \in C_v J$ is equivalent to any of the following

1. $X \in H$, $\forall H \in H_v J$;
2. $l_H(X) \leq l_H(v)$, $\forall H \in H_v J$;
3. $l(X) \leq l(v)$, $\forall l \in S_v J$.

□

Corollary 4.2.8.1. *For every $\omega \notin J$, there exists $v \in \partial J$ and $H \in H_v J$ such that $\omega \notin H$ and*

$$d(\omega, J) = d(\omega, H) = l_H(\omega - v) = \langle \omega - v, N_H \rangle.$$

Proof. It follows directly from Lemmas 4.2.3 and 4.2.4. □

Let $s : \mathbb{R}^k \longrightarrow [0, \infty)$ be the **distance function** to J :

$$s(\omega) = d(\omega, J).$$

Lemma 4.2.9. *If $\omega \notin J$, then*

$$s(\omega) = \sup \{ l(\omega - v); v \in \partial J \text{ and } l \in S_v J \}. \tag{4.9}$$

Hence, $s(\omega)$ is a convex function. Moreover, the supremmum is attained and $s(\omega)$ is a continuous function of ω.

Proof. Let $\sigma(\omega) := \sup\{l(\omega - v); v \in \partial J \text{ and } l \in S_v J\}$. From Corollary 4.2.8.1, $\exists v_0 \in \partial J$ and $H_0 \in H_v J$ such that $\omega \notin H_0$ and

$$s(\omega) = d(\omega, J) = d(\omega, H_0) = l_{H_0}(\omega - v_0) \leq \sigma(\omega).$$

On the other hand, for any $v \in \partial J$ and $H \in H_v J$, we have $J \subset H$, so

$$l_H(\omega - v) = d(\omega, H) \leq d(\omega, J) = s(\omega).$$

Hence $\sigma(\omega) \leq s(\omega)$ and the equality follows. Since l is convex for every $l \in S_v J$, $s(\omega)$ is also convex.

□

Lemma 4.2.10. *Let $J \subset \mathbb{R}^k$ be a closed convex set.*

(i) The function s is C^1 on $\mathbb{R}^k - J$. For $\omega \notin J$, let $v \in \partial J$ be the unique point such that $s(\omega) = d(\omega, v)$. Then the gradient $\nabla s(\omega)$ is equal to the unit vector pointing in the direction from v to ω.

(ii) For $\omega \notin J$, define the closed convex set $\tilde{J} := s^{-1}([0, s(\omega)])$. Then $\partial \tilde{J}$ is C^1 and the gradient $\nabla s(\omega)$ is equal to the unit outward normal to $\tilde{J}$ at ω.

(iii) Let $J(t)$, $t \in [0, T]$, be a continuous family of convex sets. Given $\omega \notin J(t)$, let $s(\omega, t) := d(\omega, J(t))$ be the distance function. Then the gradient $\nabla s(\omega, t)$ is a continuous function of (ω, t).

Proof. In order to prove (i), consider a Euclidean coordinate system $\{y_i\}$ on $\mathbb{R}^k$ such that the origin is in v and the positive y_k axis is in the direction from v to ω. Then

$$J \subset \left\{(y_1, \dots, y_k) \in \mathbb{R}^k; y_k \leq 0\right\} := H_-.$$

Moreover, $\omega = (0, \dots, 0, \overline{y_k})$, for some $\overline{y_k} > 0$. If $y \notin H_-$, then

$$d(y, H_-) = s(y) \leq d(y, 0),$$

i.e., if $y = (y_1, \dots, y_k)$,

$$y_k \leq s(y_1, \dots, y_k) \leq \sqrt{{y_1}^2 + \dots {y_k}^2}.$$

Therefore, $s(y_1, \dots, y_k)$ is differentiable at $(0, \dots, 0, \overline{y_k})$ and its gradient is given by

$$\nabla s(0, \dots, 0, \overline{y_k}) = (0, \dots, 0, 1).$$

This proves (i).

Item (ii) follows directly from (i); we just have to observe that $\tilde{J}$ is the $s(\omega)$-neighborhood of J.

Finally, using Lemma 4.2.3, $v(\omega, t)$ is continuous on (ω, t). Then $\omega - v(\omega, t)$ is continuous. Since

$$\nabla s(\omega, t) = \frac{\omega - v(\omega, t)}{|\omega - v(\omega, t)|},$$

the result (iii) follows. □

4.2.3 Vector Bundle Formulation

Let $\pi : E^r \longrightarrow M^n$ be a rank r real vector bundle over a closed manifold M. Consider a family of Riemannian metrics on M, $g(t)$, $t \in [0, T)$, a fixed bundle metric h on E and time dependent connections

$$\overline{\nabla}(t) = \overline{\nabla}^t : C^\infty(E) \longrightarrow C^\infty(E \otimes T^*M),$$

compatible with h, i,e,

$$X(h(u, v)) = h\left(\overline{\nabla}^t_X u, v\right) + h\left(\overline{\nabla}^t_X v, u\right),$$

for all vector fields $X \in TM^n$, sections $u, v \in C^\infty(E)$ and time $t \in [0, T)$. In coordinates, we may consider a local basis $\{v_i\}$ of E and, therefore, for any section $u \in C^\infty(E)$ we can write $u = \sum_{i=1}^{r} u^i v_i$, where u^i are smooth functions on M. Hence, we get the following expression

$$\overline{\nabla}^t_X \left(\sum_{i=1}^{r} u^i v_i \right) = \sum_{i=1}^{r} \left(X(u^i) v_i + u^i \overline{\nabla}^t_X v_i \right).$$

We may also define $\hat{\nabla}(t) = \hat{\nabla}^t : C^\infty(E \otimes T^*M) \longrightarrow C^\infty(E \otimes T^*M \otimes T^*M)$ by

$$\hat{\nabla}^t_X (\phi \otimes \xi) = \left(\overline{\nabla}^t_X \phi \right) \otimes \xi + \phi \otimes \nabla^t_X \xi,$$

for all $X \in TM$, $\xi \in T^*M$ and $\phi \in C^\infty(E)$, where ∇^t is the Levi-Civita connection on M^n with respect to $g(t)$.

Then we may define the **time-dependent bundle Laplacian** $\hat{\Delta}(t)$ acting on sections of E :

$$\hat{\Delta}_t \phi := tr_g \hat{\nabla} \left(\overline{\nabla} \phi \right). \tag{4.10}$$

The Laplacian at $(p, t) \in M \times [0, T)$ can be expressed as follows. Let $\gamma : [0, b] \longrightarrow M$ be a differentiable curve on M. We say that a section $v(s) \in E_{\gamma(s)}$ is **parallel** along γ if

$$\overline{\nabla}^t_{\gamma(s)'} v(s) = 0.$$

For every $\gamma : [0, b] \longrightarrow M$ and vector $v_0 \in E_{\gamma(0)}$, there exists a unique parallel section $v(s) \in E_{\gamma(s)}$ along $\gamma(s)$ with $v(0) = v_0$ (we call it a **parallel lift** on $\gamma(s)$). Given a vector $u_0 \in E_{p_0}$, $p_0 \in M$, we can extend u_0 to a section u of E over the normal neighborhood $B_\rho(p_0) \subset M$, of p_0, where ρ is the injectivity radius of $p_0 \in (M^n, g(t))$. In fact, for every geodesic γ starting at p_0, parallel translate u_0 along γ using the connection $\overline{\nabla}^t$. This gives a well-defined section u on $B_\rho(p_0)$.

Given any $X_0 \in T_{p_0}M$, since u is a parallel section we have that

$$\left(\overline{\nabla}^t_{X_0} u\right)(p_0) = 0$$

and the diagonal part of the second covariant derivative also vanishes:

$$\left(\overline{\nabla^t}^2_{X_0} u\right)(p_0) = 0.$$

In fact, let $\gamma : [0, \frac{\rho}{|X_0|}] \longrightarrow M$ be the constant speed geodesic with $\gamma(0) = p_0$ and $\gamma'(0) = X_0$. Then we get:

$$\left(\overline{\nabla}^2_{\gamma',\gamma'}(t)\right) u = \overline{\nabla}^t_{\gamma'}\left(\overline{\nabla}^t_{\gamma'} u\right) - \left(\overline{\nabla}^t_{\gamma'}\gamma'\right) u = 0,$$

since u is parallel along γ and γ is a geodesic.

Now consider a basis $\{v_i(p_0)\}_{i=1}^r$ of E_{p_0}. We extend this basis to a basis of local sections v_i, defined on a neighborhood U of p_0, by parallel transport along geodesics emanating from p_0. If $u \in C^\infty(E)$, then we may write, on U,

$$u := \sum_{i=1}^r u^i v_i,$$

where $u^i : U \longrightarrow \mathbb{R}$. Given $X_0 \in T_{p_0}M$, let γ denote the constant speed geodesic with $\gamma(0) = p_0$ and $\gamma'(0) = X_0$. Let $X = \gamma'$ along γ. Since

$$\overline{\nabla}_X X = 0 \text{ and } \overline{\nabla}_X v_i = 0 \text{ along } \gamma,$$

we have

$$\overline{\nabla}^2_{X,X}(t)u = \overline{\nabla}_X\left(\overline{\nabla}_X u\right) - \left(\overline{\nabla}_X X\right) u = X(X(u^i))v_i.$$

Choosing $\{e_j\}_{j=1}^n \subset T_{p_0}M$ an orthonormal frame of tangent vectors at p_0 and taking $X = e_j$, we have

$$\left(\hat{\Delta} u\right)(p) = \sum_{j=1}^n \left(\overline{\nabla}^2_{e_j,e_j} u\right)(p) = \sum_{i=1}^r \sum_{j=1}^n e_j\left(e_j(u^i)\right) v_i(p) = \sum_{i=1}^r \left(\Delta u^i\right)(p) v^i(p),$$

as we expected. Now that we understand the Laplacian $\hat{\Delta}$, we would like to consider heat-type equations for sections of E. Suppose that $u(t) \in C^\infty(E)$, $t \in [0,T)$, satisfies

$$\frac{\partial u}{\partial t} = \hat{\Delta^t} u + \overline{\nabla^t}_{X(t)} u + F(u,t), \tag{4.11}$$

where $X(t)$ is a time dependent vector field on M and $F : E \times [0,T) \longrightarrow E$ is a fiber-preserving map for each $t \in [0,T)$.
We may consider the system of ODE on E_p related to (4.11), for each $p \in M$:

$$\frac{du}{dt} = F_p(u,t), \tag{4.12}$$

where $F_p \coloneqq F\big|_{E_p \times [0,T)} : E_p \times [0,T) \longrightarrow E_p$.

Definition 4.2.7. *Let $K \subset E$ be a subset of E and denote $K_p = K \cap E_p$, for $p \in M$. We say that K is **invariant under parallel translation** if for every differentiable curve $\gamma : [0,b] \longrightarrow M$ and vector $v \in K_{\gamma(0)}$, the unique parallel section $v(s) \in E_{\gamma(s)}$, $s \in [0,b]$, along $\gamma(s)$ with $v(0) = v$ is contained in K.*

Remark. *The maximum principle for the scalar case (Theorem 4.1.2) basically says that solutions to the associated ODE give bounds for the solutions to the PDE. We wish to generalize this result to systems. The analogue, for systems, of the initial pointwise bounds $c_1 \leq u(p,0) \leq c_2$, is the requirement that the initial data lies in a closed subset $K \subset E$ which is invariant under parallel translation in E with respect to $\overline{\nabla}^t$, $\forall t \in [0,T)$, and is convex in each fiber, i.e., $K_p = K \cap E_p$ is a convex subset of E_p, $\forall p \in M$.*

If, for instance, $E = M \times \mathbb{R}$ is the trivial line bundle, invariance under parallel translation is the same as $[c_1, c_2]$ being independent of $p \in M$. Convexity in the fibers corresponds to $[c_1, c_2] \subset \mathbb{R}$ being convex.

Remark. *If $M = [0,1]$ and $T = \infty$, let $K = M \times \mathbb{R}^k$ be a higher rank trivial vector bundle. Suppose $u : [0,1] \times [0,\infty) \longrightarrow \mathbb{R}^k$ is a solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$ and the values at 0 and 1 are fixed, $u(0) = a$ and $u(1) = b$. Then the heat equation smooths out (or averages) the function u to the linear function $u_\infty(s) = (1-s)a + sb$ as $t \nearrow \infty$.*

Intuitively, this is why we need K to be convex in each fiber.

In applications to the Ricci flow, we are going to consider vector bundles of the form $E = V \otimes_S V$, where V is a vector bundle and $\otimes_S$ is the symmetric tensor product. Let $End_{SA}(V)$ be the bundle of self-adjoint endomorphisms of V. Thus, using the metric on V, we may identify V with V^* and $E \cong End_{SA}(V)$.

Now let $u_0 \in E_{p_0}$ for some $p_0 \in M$. For a given path $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p_0$, let $u(t)$ be the parallel lift of γ such that $u(0) = u_0$. Let $\omega_0 \in V_{p_0}$ be an eigensection of u_0 with eigenvalue $\lambda_0 \in \mathbb{R}$, i.e., $u_0(\omega_0) = \lambda_0\omega_0$, using our identification $E \cong End_{SA}(V)$. Let $\omega : [0,1] \longrightarrow V$ be the unique parallel lift of γ such that $\omega(0) = \omega_0$. We claim that ω is an eigensection of u with the same eigenvalue λ_0. Indeed, $\nabla_{\gamma'}(u(\omega) - \lambda_0\omega) = 0$ since u and ω are parallel lifts. Furthermore, $(u(\omega) - \lambda_0\omega)(0) = 0$, then $u(\omega) = \lambda_0\omega$ $\forall t \in [0,1]$. Now let $r := rank(V)$ and, given $u_0 \in E_{p_0}$, let

$$\lambda_1(u_0) \geq \cdots \geq \lambda_r(u_0)$$

denote the ordered eigenvalues of u_0. Let

$$\Gamma := \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r; \lambda_1 \geq \cdots \geq \lambda_r\}. \tag{4.13}$$

Under these conditions, the following is true.

Lemma 4.2.11. *Suppose that $G : \Gamma \longrightarrow \mathbb{R}$ is a function, where Γ is given by* (4.13)*. Given $c \in \mathbb{R}$, let*

$$K_c := \{u \in E; G(\lambda_1(u), \ldots, \lambda_r(u)) \leq c\}. \tag{4.14}$$

Then $K_c \subset E$ is invariant under parallel translation.

Proof. According to the construction above, let $\omega_1, \ldots, \omega_r \in V_p$ be unit eigensections of $u \in E_p$ corresponding to $\lambda_1 \geq \cdots \geq \lambda_r$, so that

$$u = \sum_{a=1}^{r} \lambda_a \omega_a \otimes \omega_a.$$

Given any path $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = p$, let $\overline{\omega_a} : [0,1] \longrightarrow V$ be the parallel lift of γ with $\overline{\omega_a}(0) = \omega_a$. Then $\overline{u} = \sum_{a=1}^{r} \lambda_a \overline{\omega_a} \otimes \overline{\omega_a}$ is a parallel lift of γ with $\overline{u}(0) = u$. Hence $\lambda_a(\overline{u}) = \lambda_a(u)$ for $a = 1, \ldots, r$ and the lemma follows. □

4.2.4 Understanding when ODEs preserve convex sets

Now we consider a solution $u(t)$ of a given ODE $\frac{d}{dt}u(t) = F(u,t)$. Suppose that the initial value $u(0) = u_0$ is inside a closed convex set $J \subset \mathbb{R}^k$. We would like to understand when solutions to the ODE remain in J. The basic idea, when J is independent of time, is that if the vector field F, which drives the ODE, points into J at the boundary ∂J, then J is **preserved under the ODE**, i.e., the solution to the ODE remains in J. This is the content of the next result. We say that the solution for the ODE $\frac{du}{dt} = F(u,t)$ preserves the set J if $u(t_0) \in J$ for some t_0 implies $u(t) \in J$ for all $t \geq t_0$ such that the solution exists.

Proposition 4.2.1. *Let $J \subset \mathbb{R}^k$ be a closed convex set and let $F : \mathbb{R}^k \times [0,T) \longrightarrow \mathbb{R}^k$ be a continuous function which is locally Lipschitz in $\mathbb{R}^k$. The ODE*

$$\frac{du}{dt} = F(u,t) \tag{4.15}$$

preserves J if and only if

$$v + F(v,t) \in C_v J,$$

$\forall v \in \partial J$ and $\forall t \in [t_0, T)$, which is equivalent to

$$l(F(v,t)) \leq 0,$$

$\forall v \in \partial J,\ t \in [t_0, T)$ and $\forall l \in S_v J$.

Proof. Assume the ODE preserves J, we want to prove that $l(F(v,t)) \leq 0$ $\forall v \in \partial J,\ t \in [t_0, T)$ and $\forall l \in S_v J$. Suppose $l_0(F(v_0, t_0)) > 0$ for some $v_0 \in \partial J$, $t_0 \in [0, T)$ and $l_0 \in S_{v_0} J$.

Let $u(t)$ be a solution of (4.15) with $u(t_0) = v_0$. Then

$$\frac{d}{dt} l_0(u(t))\bigg|_{t=t_0} = l_0(\frac{d}{dt} u(t_0)) = l_0(F(v_0, t_0)) > 0.$$

This implies that if $t > t_0$ is close enough to t_0, then $l_0(u(t)) > l_0(v_0)$. Hence, $u(t) \notin H_{l_0}$ and $u(t) \notin J$. Therefore, $u(t) \in J$ for all $t \in [t_0, T)$ implies

$$l(F(v, t)) \leq 0,$$

$\forall v \in \partial J,\ t \in [t_0, T)$ and $\forall l \in S_v J$.

In order to prove the converse, let $u_0 \in J$, $t_0 \in [0, T)$ and $t_1 \in (t_0, T)$. Consider a solution $u(t)$ to (4.15), with $t \in [t_0, t_1]$ and $u(t_0) = u_0$. Then, there exists $R > 0$ such that $u(t) \in B_R(0)$, $\forall t \in [t_0, t_1]$. Let $J_R \coloneqq J \cap B_R(0)$, a convex and compact set. We will prove that $u(t) \in J_R$ $\forall t \in [t_0, t_1]$, which clearly proves the converse. First, we introduce the following definition. The space of support functions for J is

$$S(J) \coloneqq \{(v, l); v \in \partial J, l \in S_v J\}.$$

Claim: If J is compact, then $S(J)$ is compact. In particular, $S_R = S(J_R)$ is compact.

In order to prove the claim, let $(v_i, l_i) \subset S(J)$ be any sequence. Since ∂J is compact, there exists a subsequence of (v_i), which will still be represented by (v_i), such that $v_i \to v_\infty \in \partial J$, for some v_∞. Since the unit sphere $S_1^{k-1}(0) \subset \mathbb{R}^k$ is compact, there exists a further subsequence such that the outward unit normal vectors N_i of l_i converge to some $N_\infty \in S_1^{k-1}(0)$. Then $l_i \to l_\infty$, where $l_\infty(\omega) \coloneqq \langle \omega, N_\infty \rangle$. Since l_∞ is the limit of a sequence of support functions, we get that $|l_\infty|$=1. Moreover, since $J \subset H_{l_i}$ $\forall i \in \mathbb{N}$ and $H_{l_\infty} = \lim H_i$, we have $J \subset H_{l_\infty}$. Hence, $(v_\infty, l_\infty) \in S(J)$ and S(J) is compact. This proves the claim.

Now, if $\omega \in B_R(0) - J$, since $u_0 \in J \cap B_R(0)$, we have

$$d(\omega, J) = d(\omega, J_R) = \sup\{l(\omega - v); v \in \partial J_R, l \in S_v J_R\}.$$

Define $g : S_R \times [t_0, t_1] \longrightarrow \mathbb{R}$ by $g(v, l, t) = l(u(t) - v)$. Let $s(t) = d(u(t), J)$. Then, whenever $s(t) > 0$, by our choice of R we have

$$s(t) = \sup_{(v,l)\in S_R} g(v, l, t) = \sup_{(v,l)\in S_R} l(u(t) - v),$$

because $u(t) \in B_R(0)$. We also have

$$\frac{d}{dt}\left(l(u(t) - v)\right) = l\left(\frac{du}{dt}\right) = l(F(u(t), t)).$$

Hence, from Lemma 4.2.2, whenever $s(t) > 0$ we get

$$\frac{d^+}{dt}s(t) = \sup\left\{l(F(u(t), t)); (v, l) \in S_R \text{ with } l(u(t) - v) = s(t)\right\}.$$

Then we consider $(v, l) \in S_R$ with respect to which the above supremum is taken. We have

$$l(u(t) - v) = d(u(t), v) = s(t),$$

with $v \in \partial J$ and $l \in S_v J$. By the assumption, we have $l(F(v, t)) \leq 0$. Since $|l| = 1$, this implies that for $(v, l) \in S$ as above,

$$\begin{aligned} l(F(u(t), t)) =& l(F(u(t), t) - F(v, t)) + l(F(v, t)) \\ &\leq |F(u(t), t) - F(v, t)| \leq cd(u(t), v) = cs(t), \end{aligned}$$

provided that $s(t) > 0$, where we have used that F is locally Lipschitz in the first entry. Then $\frac{d^+}{dt}s(t) \leq cs(t)$ whenever $s(t) > 0$ in $[t_0, t_1]$. Since $s(t_0) = 0$, it follows from Corollary 4.2.1.2 that $s(t) = 0$ $\forall t \in [t_0, t_1]$, i.e., $u(t) \in J_R$ $\forall t \in [t_0, t_1]$. Since $t_1 \in [t_0, T)$ is arbitrary, it follows that $u(t) \in J$ $\forall t \in [t_0, T)$. □

Remark. *The condition $v + F(v, t) \in C_v J$ shows that $F(v, t)$ should be thought of as based at v.*

Now, we let our set J to depend on time. Let $J(t) \subset \mathbb{R}^k$, $t \in [0, T)$, be a family of time-dependent closed sets of the Euclidean space. We define the **space-time track** of

$J(t)$ by

$$\mathcal{L} := \left\{(v,t) \in \mathbb{R}^k \times \mathbb{R}; v \in J(t), 0 \le t < T\right\}.$$

Definition 4.2.8. *Given $(v,t) \in \mathcal{L}$, we define the* ***set of forward looking directions*** *$D_{(v,t)}\mathcal{L}$ as the set of all $\omega \in \mathbb{R}^k$ such that for all decreasing sequences of real numbers $h_i \to 0$, there exists a subsequence (h_{i_j}) and a sequence of vectors $\omega_{i_j} \in \mathbb{R}^k$ converging to ω such that*

$$v + h_{i_j}\omega_{i_j} \in J(t + h_{i_j}),$$

i.e.,

$$(v,t) + h_{i_j}(\omega_{i_j}, 1) \in \mathcal{L}.$$

Example. *Given $a < b$, let $J(t) = \{v \in \mathbb{R}; at \le v \le bt\}$, $t \in [0,\infty)$. Then*

$$D_{(0,0)}\mathcal{L} = [a,b]$$

since $a(t+h_{i_j}) \le v + h_{i_j}\omega_{i_j} \le b(t+h_{i_j})$ for sufficiently big i_j, where $\omega_{i_j} \to 0$ and $h_{i_j} \to 0$.

Lemma 4.2.12. *If $J(t)$ is convex for each t, then the set $D_{(v,t)}\mathcal{L} \subset \mathbb{R}^k$ is convex.*

Proof. Let $\omega, x \in D_{(v,t)}\mathcal{L}$. Let $(h_i) \subset \mathbb{R}$ be an arbitrary sequence such that $h_i \searrow 0$. Then there exists a subsequence of (h_i), still represented by (h_i), and $\omega_i \to \omega$ such that

$$v + h_i\omega_i \in J(t + h_i).$$

Also, there exists a further subsequence (h_i) and $x_i \to x$ such that

$$v + h_i x_i \in J(t + h_i).$$

Observe that, for all $\varepsilon \in [0,1]$, we have

$$v + h_i[(1-\varepsilon)\omega_i + \varepsilon x_i] = (1-\varepsilon)(v + h_i\omega_i) + \varepsilon(v + h_i x_i) \in J(t + h_i)$$

since $J(t + h_i)$ is convex. Also, since $(1 - \varepsilon)\omega_i + \varepsilon x_i \to (1 - \varepsilon)\omega + \varepsilon x$, it follows that

$$(1 - \varepsilon)\omega + \varepsilon x \in D_{(v,t)}\mathcal{L},$$

$\forall \varepsilon \in [0, 1]$.

□

Lemma 4.2.13. *Let $u(t)$ be a solution to (4.15) with $u(t_0) \in J(t_0)$ for some $0 < t_0 < T$. Suppose that there exists a $t_2 \in [0, T)$, with $t_2 > t_0$, such that $u(t_2) \notin J(t_2)$. Also, suppose that $F(v, t) \in D_{(v,t)}\mathcal{L}$, for all $(v, t) \in \partial\mathcal{L}$ and $t < T$. Moreover, let $t_1 \in [t_0, t_2)$ such that $u(t_1) \in J(t_1)$ and $u(t) \notin J(t)$, $\forall t \in (t_1, t_2]$. Finally, let*

$$s(t) = d(u(t), J(t)). \tag{4.16}$$

Then, $s(t)$ is right continuous and left lower semi-continuous on $(t_1, t_2]$.

Proof. First, we will prove that $s(t)$ is lower semi-continuous (LSC). Suppose $\bar{t} \in (t_1, t_2)$ and let $(\overline{t_i})$ be a sequence of times with $\overline{t_i} \to \bar{t}$. For each $i \in \mathbb{N}$, there exists $v_i \in J(\overline{t_i})$ such that $d(u(\overline{t_i}), v_i) = s(\overline{t_i})$, since $J(\overline{t_i})$ is closed. Consider a subsequence $(\overline{t_{i_j}})$ of $(\overline{t_i})$ such that

$$\lim_{j\to\infty} s(\overline{t_{i_j}}) = \liminf_{i\to\infty} s(\overline{t_i}).$$

Since $\mathcal{L}$ is also closed, there exists a subsequence (v_{i_j}) that converges to an element of $J(\bar{t})$. Let $v_\infty = lim v_{i_j}$. Then,

$$\begin{aligned}\liminf_{i\to\infty} s(\overline{t_i}) = \lim_{j\to\infty} s(\overline{t_{i_j}}) &= \lim_{j\to\infty} d(u(\overline{t_{i_j}}), v_{i_j}) = d(u(\bar{t}), v_\infty) \\ &\geq s(\bar{t}).\end{aligned}$$

So $s(t)$ is lower semi-continuous.

In order to prove that $s(t)$ is right continuous, we only need to prove that it is right upper semi-continuous on $[t_1, t_2]$. Since t_1 is such that $(u(t_1), t_1) \in \mathcal{L}$ and $(u(t), t) \notin \mathcal{L}$ $\forall t \in (t_1, t_2]$, then $(u(t_1), t_1) \in \partial\mathcal{L}$. Let $v_{t_1} = u(t_1)$. For any $t \in (t_1, t_2)$ there exists a

$v_t \in J(t)$ such that $s(t) = d(u(t), v_t)$ and $(v_t, t) \in \partial \mathcal{L}$, because $u(t) \notin J(t)$.
By hypothesis, $F(v_t, t) \in D_{(v_t,t)}\mathcal{L}$. Given $t \in [t_1, t_2)$, let (h_i) be a sequence such that $h_i \searrow 0$ and

$$\lim_{i\to\infty} s(t + h_i) = \limsup_{\tau\to t^+} s(\tau).$$

By definition, we get a subsequence of (h_i) such that $v_t+h_i\omega_i \in J(t+h_i)$ and $\omega_i \to F(v_t, t)$, where the subsequence is still represented by (h_i). Then

$$s(t + h_i) \leq d(u(t + h_i), v_t + h_i\omega_i).$$

If we let $i \to \infty$, we get

$$\limsup_{\tau\to t^+} s(\tau) = \lim_{i\to\infty} s(t + h_i) \leq d(u(t), v_t) = s(t).$$

□

Proposition 4.2.2. *Let $J(t) \subset \mathbb{R}^k$, $0 \leq t < T$, be a family of nonempty closed convex sets such that $\mathcal{L}$ is closed in $\mathbb{R}^k \times [0, T)$. Consider the ODE (4.15), where $F : \mathbb{R}^k \times [0, T) \longrightarrow \mathbb{R}^k$ is continuous in (u, t) and locally Lipschitz in u. Then the following conditions are equivalent:*

(i) For any initial time $t_0 \in [0, T)$ and any solution of the ODE (4.15) *such that $u(t_0) \in J(t_0)$, the solution $u(t) \in J(t)$ for all $t \in [t_0, T)$.*

(ii) $F(v, t) \in D_{(v,t)}\mathcal{L}$, $\forall(v, t) \in \partial\mathcal{L}$ and $t < T$.

Proof. First we prove that (i) implies (ii). Consider any $(v_0, t_0) \in \partial\mathcal{L}$ and $u(t)$ a solution to (4.15) with $u(t_0) = v_0$. Then, (i) implies that $u(t_0 + t) \in J(t_0 + t)$, $\forall t \in [0, T - t_0)$. This implies that for every sequence $h_i \in \mathbb{R}$ with $h_i \to 0^+$, we get

$$F_i := \frac{u(t_0 + h_i) - u(t_0)}{h_i} \to \frac{du}{dt}(t_0) = F(v_0, t_0)$$

and

$$v_0 + h_iF_i = u(t_0 + h_i) \in J(t_0 + h_i).$$

This proves that $F(t_0, v_0) \in D_{(v_0,t_0)}\mathcal{L}$.

Now, we prove that (ii) implies (i) : . We will argue by contradiction. Suppose that $u(t)$ is a solution of (4.15) with $u(t_0) \in J(t_0)$. Assume there exists $t_1 < t_2$ such that $u(t_1) \in J(t_1)$ and $u(t) \notin J(t)$ for all $t \in (t_1, t_2]$. Then $s(t_1) = 0$ and $s(t) > 0$, $\forall t \in (t_1, t_2]$.

Claim: $\dfrac{d^+s}{dt}(t) \le cs(t)$ for $t \in (t_1, t_2)$. This will imply that $s(t) = 0$ for $t \in [t_1, t_2)$ by Corollary 4.2.1.2, which is a contradiction.

In order to prove our claim, we consider

$$S(t) = \{(v, l); v \in \partial J(t) \text{ and } l \in S_v J(t)\}.$$

Since

$$s(t) = \sup_{(v,l)\in S(t)} g(v, l, t),$$

where $g(v, l, t) = l(u(t) - v)$, there exist $v_i \in \partial J(t + h_i)$ and $l_i \in S_{v_i} J(t + h_i)$ such that

$$g(v_i, l_i, t + h_i) = s(t + h_i) = |u(t + h_i) - v_i|.$$

There also exist $v_\infty \in \partial J(t)$ and $l_\infty \in S_{v_\infty} J(t)$ such that

$$g(v_\infty, l_\infty, t) = |u(t) - v_\infty|.$$

Hence, we compute

$$\begin{aligned}
\frac{d^+s}{dt} &= \lim_{i\to\infty} \frac{g(v_i, l_i, t + h_i) - g(v_\infty, l_\infty, t)}{h_i} \\
&= \lim_{i\to\infty} \frac{l_i(u(t + h_i) - v_i) - l_\infty(u(t) - v_\infty)}{h_i} \\
&= \lim_{i\to\infty} \frac{l_i(u(t + h_i) - u(t)) + l_i(u(t) - v_i) - l_\infty(u(t) - v_\infty)}{h_i}.
\end{aligned} \tag{4.17}$$

From (ii), $F(v_\infty, t) \in D_{(v_\infty,t)}\mathcal{L}$, so there exists a subsequence (h_{i_j}) and a sequence of F_{i_j}

with $F_{i_j} \to F(v_\infty, t)$ such that

$$v_\infty + h_{i_j} F_{i_j} \in J(t + h_{i_j}),$$

so that $l_{i_j}(v_\infty + h_{i_j} F_{i_j} - v_{i_j}) \leq 0$. Let us denote (h_{i_j}) by (h_i). Then it follows from (4.17) that

$$\begin{aligned}
\frac{d^+ s}{dt} &\leq \limsup_{i\to\infty} l_i \left(\frac{u(t+h_i) - u(t)}{h_i} - F_i \right) + \limsup_{i\to\infty} \frac{l_i(v_\infty + h_i F_i - v_i)}{h_i} \\
&+ \limsup_{i\to\infty} \frac{1}{h_i}(l_i - l_\infty)(u(t) - v_\infty) \\
&\leq \lim_{i\to\infty} \left| \frac{u(t+h_i) - u(t)}{h_i} - F_i \right| = \big| F(u(t), t) - F(v_\infty, t) \big| \\
&\leq c|u(t) - v_\infty| = cs(t),
\end{aligned}$$

where we used the fact that $|l_i| = 1$ and

$$(l_i - l_\infty)(u(t) - v_\infty) = l_i(u(t) - v_\infty) - |u(t) - v_\infty| \leq 0.$$

This ends the proof. □

We are finally ready to prove our main theorem.

Theorem 4.2.14. *Under the vector bundle formulation described in Subsection 4.2.3, let $K(t) \subset E$ be a family of subsets which are invariant under parallel translation with respect to $\overline{\nabla}(t)$, for each $t \in [0, T)$. We require $K_p(t) := K(t) \cap E_p$ to be closed and convex and the space-time track*

$$\mathcal{T} := \{(v, t) \in E \times \mathbb{R}; v \in K(t) \text{ and } t \in [0, T)\}$$

also to be closed in $E \times [0, T)$. Assume that $F(u, t) : E \times [0, T) \longrightarrow E$ is continuous in (u, t) and locally Lipschitz in u. Suppose that for any $p \in M$ and initial time $t_0 \in [0, T)$, any solution $u(t)$ of

$$\frac{du}{dt} = F_p(u, t),$$

which starts in $K_p(t_0)$, will remain in $K_p(t)$ for $t \in [t_0, T)$. Then the solution $u(t)$, $t \in$

$[0, T)$, *of the PDE*

$$\frac{\partial u}{\partial t} = \hat{\Delta} u + \overline{\nabla}_{X(t)} u + F(u, t) \tag{4.18}$$

will remain in $K(t)$ *for all* $t \in [t_0, T)$, *provided* $u(t_0) \in K(t_0)$.

Proof. Given $u \in E_p = \pi^{-1}(p)$, let $d(u, K_p(t))$ denote the distance from u to $K_p(t)$ with respect to the metric h. Let $u(p, t)$ be a solution to (4.18) with $u(p, 0) \in K(0)$ $\forall p \in M$ and let

$$s(t) := \sup_{p \in M} d(u(p, t), K_p(t))$$

be the maximum distance from $u(t)$ to the set $K(t)$. We shall argue by contradiction. Suppose there exist $p_2 \in M$ and $t_2 \in (0, T)$ such that $u(p_2, t_2) \notin K(t_2)$. Since $\mathcal{T}$ is closed, there exists a time $t_1 \geq 0$ such that $u(p, t_1) \in K(t_1)$ $\forall p \in M$ and for any $t \in (t_1, t_2]$, we can find a $\overline{p} \in M$ such that $u(\overline{p}, t) \notin K(t)$. So we have $s(t_1) = 0$ and $s(t) > 0$ $\forall t \in (t_1, t_2]$. We then make two claims.

Claim 1: $s(t)$ is left lower semi-continuous and right continuous on $[t_1, t_2]$.

Claim 2: $s(t)$ grows at most exponentially:

$$\frac{d^+ s}{dt}(t) \leq C s(t)$$

for all $t \in (t_1, t_2]$ and for some constant $C < \infty$.

If our claims are true, the result will follow. In fact, since $s(t_1) = 0$, from Corollary 4.2.1.2 we get that $s(t) \equiv 0$, $\forall t \in (t_1, t_2]$, which is a contradiction.

Now, we prove our first claim. First, we prove that $s(t)$ is lower semi-continuous. Since $s(t_1) = 0$ and $s(t) > 0$ $\forall t \in (t_1, t_2]$, $s(t)$ is obviously lower semi-continuous at t_1. Now consider an arbitrary $\overline{t} \in (t_1, t_2]$. Thus, $s(\overline{t}) > 0$. Fix $\overline{p} \in M$ such that $s(\overline{t}) = d(u(\overline{p}, \overline{t}), K_{\overline{p}}(\overline{t}))$. Since $\mathcal{T}$ is closed, we get an $\varepsilon > 0$ such that $u(\overline{p}, t) \notin K_{\overline{p}}(t)$ for every $t \in (\overline{t} - \varepsilon, \overline{t} + \varepsilon)$.

Restricting ourselves to $E_{\overline{p}}$, we may apply Lemma 4.2.13 to conclude that $d\left(u(\overline{p}, .), K_{\overline{p}}(.)\right)$

is lower semi-continuous at $\bar{t}$. Thus, if $t_i \to 0$, we get

$$\begin{aligned}\liminf_{i\to\infty} s(\bar{t}+t_i) &\geq \liminf_{i\to\infty} d\left(u(\overline{p},\bar{t}+t_i), K_{\overline{p}}(\bar{t}+t_i)\right)\\ &\geq d(u(\overline{p},\bar{t}), K_{\overline{p}}(\bar{t})) = s(\bar{t}).\end{aligned}$$

This proves that $s(t)$ is lower semi-continuous at $\bar{t}$ and, therefore, on $[t_1, t_2]$.

Now we prove that $s(t)$ is upper right semi-continuous, which will be sufficient to prove its right continuity since we have already proved that $s(t)$ is lower semi-continuous and, in particular, lower right semi-continuous.

Again, let $\bar{t} \in [t_1, t_2)$ be an arbitrary time. Also, consider $t_i \to 0^+$. We shall prove that there is a subsequence of t_i, also denoted by t_i, such that

$$\lim_{i\to\infty} s(\bar{t}+t_i) \leq s(\bar{t}).$$

In fact, by considering a subsequence if necessary, we may assume the existence of $\lim_{i\to\infty} s(\bar{t}+t_i)$. Then, for each i, let $p_i \in M$ be such that

$$s(\bar{t}+t_i) = d\left(u(p_i,\bar{t}+t_i), K_{p_i}(\bar{t}+t_i)\right).$$

Since M^n is compact, we may assume that $p_i \to p_\infty \in M^n$, again by passing to a subsequence if necessary. Then, let $v_\infty \in K_{p_\infty}(\bar{t})$ such that

$$d\left(u(p_\infty,\bar{t}), K_{p_\infty}(\bar{t})\right) = d(u(p_\infty,\bar{t}), v_\infty),$$

whose existence is guaranteed by the fact that $K_{p_\infty}(\bar{t})$ is closed.

Now, since $s(\bar{t}+t_i) > 0$, $\mathcal{T}_p \coloneqq \mathcal{T} \cap (E_p \times [0,T])$ is invariant under parallel translation and $\mathcal{T}$ is closed, it follows that $(v_\infty, \bar{t}) \in \partial\mathcal{T}_{p_\infty}$. If follows from our hypothesis and from Proposition 4.2.2 that $D_{(v_\infty,\bar{t})}\mathcal{T}_{p_\infty} \neq \emptyset$. Then we get $(v_\infty + t_i\omega_i) \in K_{p_\infty}(\bar{t}+t_i)$ with $\omega_i \to \omega$

for some $\omega \in E_{p_\infty}$. This implies that

$$d\left(u(p_\infty, \bar{t} + t_i), v_\infty + t_i\omega_i\right) \geq d\left(u(p_\infty, \bar{t} + t_i), K_{p_\infty}(\bar{t} + t_i)\right). \tag{4.19}$$

Since $u(p, \bar{t})$ is continuous in p and $K_p(\bar{t} + t_i)$ is invariant under parallel translation with respect to $\nabla(\bar{t} + t_i)$ for all $p \in M$, we may consider

$$d\left(u(p_\infty, \bar{t} + t_i), K_{p_\infty}(\bar{t} + t_i)\right)$$

arbitrarily close to $d\left(u(p_i, \bar{t} + t_i), K_{p_i}(\bar{t} + t_i)\right)$ when i is large enough.

So the right hand-side of (4.19) goes to the limit of $s(\bar{t}+t_i)$. However, the left hand-side approaches

$$d(u(p_\infty, \bar{t}), v_\infty) = d\left(u(p_\infty, \bar{t}), K_{p_\infty}(\bar{t})\right) \leq s(\bar{t}).$$

Hence,

$$\lim_{i\to\infty} s(\bar{t} + t_i) \leq s(\bar{t})$$

and we have proved the upper right semi-continuity of $s(t)$. **This proves the first claim.**

In order to prove the second claim, let

$$S_v K_p(t) \subset (E_p)^*$$

be the set of support functions of $K_p(t)$ at $v \in \partial K_p(t)$ and let

$$S_p(t) = \{(v, l); v \in \partial K_p(t) \text{ and } l \in S_v K_p(t)\}$$

be the set of all support functions of $K_p(t)$. Let

$$R(t) := \bigcup_{p\in M} S_p(t),$$

which is a subset of $E \otimes E^*$. Define

$$g(v, l, p, t) := l(u(p, t) - v),$$

for $(v, l, p) \in R(t)$ and $t \in [t_1, t_2]$. Then, for $t \in (t_1, t_2]$ we may write $s(t)$ as

$$s(t) = \sup_{p \in M} \left\{ \sup_{(v,l) \in S_p(t)} l(u(p, t) - v) \right\} = \sup_{(v,l,p) \in R(t)} g(v, l, p, t).$$

Let $t \in (t_1, t_2)$ be fixed and consider (h_i) a sequence of real numbers such that $h_i \searrow 0$ and

$$\frac{d^+ s}{dt} = \lim_{i \to \infty} \frac{s(t + h_i) - s(t)}{h_i}.$$

Then by Corollary 4.2.8.1 we have sequences $(p_i) \subset M$, (v_i), with $v_i \in \partial K_{p_i}(t + h_i)$ and $l_i \in S_{v_i} K_{p_i}(t + h_i)$, such that

$$g(v_i, l_i, p_i, t + h_i) = s(t + h_i).$$

Since M is compact and $\mathcal{T}$ is closed, we get a subsequence of p_i, still represented by p_i, such that $p_i \to p_\infty \in M$, for some p_∞. Also, from the equality above and since $\lim s(t+h_i) \leq s(t)$ and $s(t)$ is right continuous, we get that $s(t+h_i)$ is uniformly bounded. This implies that the sequence v_i does not diverge, so we get subsequences such that $v_i \to v_\infty \in \partial K_{p_\infty}(t)$ and $l_i \to l_\infty \in S_{v_\infty} K_{p_\infty}(t) \subset (E_{p_\infty})^*$.

By the continuity of g and the right-continuity of $s(t)$, the first claim gives us

$$s(t) = g\left(v_\infty, l_\infty, p_\infty, t\right) = l_\infty \left(u(p_\infty, t) - v_\infty\right). \tag{4.20}$$

So

$$\begin{aligned} \frac{d^+ s}{dt} &= \lim_{i \to \infty} \frac{g(v_i, l_i, p_i, t + h_i) - g\left(v_\infty, l_\infty, p_\infty, t\right)}{h_i} \\ &= \lim_{i \to \infty} \frac{l_i\left(u(p_i, t + h_i) - v_i\right) - l_\infty\left(u(p_\infty, t) - v_\infty\right)}{h_i}. \end{aligned} \tag{4.21}$$

For each $i \in \mathbb{N}$, let $\overline{v_i} \in \partial K_{p_i}(t)$ and $\overline{l_i} \in S_{\overline{v_i}} K_{p_i}(t)$ be such that

$$d\left(u(p_i,t), K_{p_i}(t)\right) = \overline{l_i}\left(u(p_i,t) - \overline{v_i}\right) = |u(p_i,t) - \overline{v_i}|_h. \tag{4.22}$$

Now let $U_i(t)$ be the solution to the ODE in E_{p_i} restricted to $[t_0, t_0 + \varepsilon)$.

$$\frac{dU_i}{dt} = F(U_i, t), \tag{4.23}$$

$$U_i(t) = \overline{v_i}, \tag{4.24}$$

where $\varepsilon > 0$ is independent of i. Define $F_i \in E_{p_i}$ by

$$\overline{v_i} + h_i F_i = U_i(t + h_i). \tag{4.25}$$

Note that $U_i(t + h_i) \in K_{p_i}(t + h_i)$ by our hypothesis. Thus

$$l_i\left(\overline{v_i} + h_i F_i - v_i\right) \leq 0. \tag{4.26}$$

It follows from (4.23) and (4.25) that if we let $i \to \infty$, then

$$|F_i - F(\overline{v_i}, t)| = \left|\frac{U_i(t + h_i) - U_i(t)}{h_i} - \frac{dU_i}{dt}(t)\right| \to 0.$$

Passing to a subsequence if necessary, we get $\overline{v_i} \to \overline{v}_\infty \in K_{p_\infty}(t)$ and $\overline{l_i} \to \overline{l}_\infty$. We claim that $\overline{v}_\infty = v_\infty$.

From (4.22), letting $i \to \infty$ we have

$$\begin{aligned} |u(p_\infty,t) - \overline{v}_\infty|_h &= \overline{l}_\infty\left(u(p_\infty,t) - \overline{v}_\infty\right) = d\left(u(p_\infty,t), K_{p_\infty}(t)\right) \\ &= |u(p_\infty,t) - v_\infty|, \end{aligned}$$

by (4.20). Since $K_{p_\infty}(t)$ is convex, $v_\infty \in \partial K_{p_\infty}(t)$ is the **unique** point in $K_{p_\infty}(t)$ closest to $u(p_\infty, t)$, so $v_\infty = \overline{v}_\infty$. Using (4.21), we have

$$\begin{aligned}\frac{d^+s}{dt} &= \lim_{i\to\infty}\left[l_i\left(\frac{u(p_i,t+h_i)-u(p_i,t)}{h_i}-F_i\right)+\frac{l_i(\overline{v}_i+h_iF_i-v_i)}{h_i}\right]\\ &\quad+\lim_{i\to\infty}\frac{l_i(u(p_i,t)-\overline{v}_i)-l_\infty(u(p_\infty,t)-v_\infty)}{h_i}.\end{aligned}$$

However, since $|l_i| = 1$ for each $i \in \mathbb{N}$ we get

$$\begin{aligned}l_i(u(p_i,t)-\overline{v}_i) &\le |u(p_i,t)-\overline{v}_i|_h = d(u(p_i,t),K_{p_i}(t))\\ &\le s(t) = d\left(u(p_\infty,t),K_{p_\infty}(t)\right) = l_\infty\left(u(p_\infty,t)-v_\infty\right),\end{aligned}$$

where we used the fact that $s(t)$ is defined as a supremum of those distances. From (4.20), we also get that $(v_\infty, l_\infty, p_\infty) \in R(t)$ satisfies

$$l_\infty(u(p_\infty,t)-v_\infty) = s(t).$$

Consider the parallel translation of v_∞ and l_∞ along geodesics emanating from p_∞ with respect to $g(t)$. Then $(v_\infty(p), l_\infty(p), p) \in R(t)$, for p in a neighborhood of p_∞. Since l_∞ is linear, $\hat{\Delta}u(p_\infty,t) \in E_{p_\infty}$ and $l_\infty(u(t)-v_\infty)$ is a real-valued function in a neighborhood of p_∞ which achieves a local maximum at p_∞, we have

$$0 = \overline{\nabla}\left(l_\infty(u(p,t)-v_\infty)\right)(p_\infty) = l_\infty\left(\overline{\nabla}u(p_\infty,t)\right)$$

and

$$0 \ge \hat{\Delta}\left(l_\infty(u(t)-v_\infty)\right)(p_\infty) = l_\infty\left(\hat{\Delta}u(p_\infty,t)\right).$$

Hence, from (4.26), since $l_i(\overline{v}_i + h_iF_i - v_i) \le 0$ and $l_i(u(p_i,t)-\overline{v}_i) \le l_\infty(u(p_\infty,t)-v_\infty)$, we have

$$\begin{aligned}\frac{d^+s}{dt} &= \lim_{i\to\infty}\left[l_i\left(\frac{u(p_i,t+h_i)-u(p_i,t)}{h_i}-F_i\right)+\frac{l_i(\overline{v}_i+h_iF_i-v_i)}{h_i}\right]\\ &\quad+\lim_{i\to\infty}\frac{l_i(u(p_i,t)-\overline{v}_i)-l_\infty(u(p_\infty,t)-v_\infty)}{h_i}\\ &\le l_\infty\left(\frac{\partial u}{\partial t}(p_\infty,t)-F(\overline{v}_\infty,t)\right).\end{aligned}$$

Hence, since we are assuming that u is a solution of (4.18), we get

$$\begin{aligned}
\frac{d^+s}{dt}(t) &\leq l_\infty\left(\hat{\Delta}u(p_\infty,t)+\overline{\nabla}_{X(t)}u(p_\infty,t)+F\left(u(p_\infty,t),t\right)-F(\overline{v}_\infty,t)\right)\\
&= l_\infty\left(\hat{\Delta}u(p_\infty,t)\right)+l_\infty\left(\overline{\nabla}_{X(t)}u(p_\infty,t)\right)+l_\infty\left(F\left(u(p_\infty,t),t\right)-F(\overline{v}_\infty,t)\right)\\
&\leq |l_\infty||F\left(u(p_\infty,t),t\right)-F(\overline{v}_\infty,t)|\\
&\leq C|u(p_\infty,t)-\overline{v}_\infty|_h = c|u(p_\infty,t)-v_\infty|_h = cs(t),
\end{aligned}$$

where we used the fact that $l_\infty\left(\hat{\Delta}u(p_\infty,t)\right) \leq 0$ and $l_\infty\left(\overline{\nabla}_{X(t)}u(p_\infty,t)\right) = 0$ at p_∞, $|l_\infty| = 1$ and $F(u,t)$ is Lipschitz in u. This proves the second claim and, therefore, the theorem. □

Chapter 5

Three-Manifolds with Positive Ricci Curvature

Our goal in this chapter is to prove a result on long time existence of the Ricci flow. We will start by explaining a formulation which enables us to write the evolution of the Riemann curvature tensor in a particularly nice form. After that, we will use the discussion about Lie algebras on Chapter 2 to understand more about this evolution equation. This will make it possible to get local and, later on, global estimates for the curvature tensor. These estimates will show that the family of metrics $g(t)$, $t \in [0, T)$, which is a solution to the Ricci flow, converges uniformly to an Einstein metric. In the end of the chapter, we make a brief comment on what would be the next steps to prove Hamilton's main theorem from his first paper [14], using the important results obtained in this chapter.

5.1 Uhlenbeck's Trick

Now we would like to simplify the evolution equations of curvatures, in order to understand them better. In particular, we shall find a nice form for equation (3.20):

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ & - (R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_p^l R_{ijkp}).\end{aligned}$$

With this goal in mind, we will consider an orthonormal frame (with respect to the initial metric g_0) and evolve it in a way that it remains orthonormal. The idea of evolving a frame field to compensate for the evolution of $g(t)$ has a more abstract formulation and is due to Karen Uhlenbeck. The Uhlenbeck's trick gives us a particularly nice form for the evolution equation above. We now present the formulation. We start with the following motivation:

Let $(M^n, g(t))$ be a solution to the Ricci flow $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$ and let $\{e_a^0\}$, $a = 1, \dots, n$, be a local orthonormal frame field w.r.t g_0 defined on an open set $U \subset M^n$. Consider the following ODE system in T_pM, for each $p \in M$:

$$\begin{aligned} &\frac{d}{dt}e_a(p,t) = Ric(e_a(p,t)), \\ &e_a(p,0) = e_a^0(p), \end{aligned} \tag{5.1}$$

where the Ricci tensor in the metric $g(t)$ is regarded as a (1,1)-tensor, $Ric : TM^n \longrightarrow TM^n$. Since (5.1) is a system of n ODE's with an initial value, there exists a unique solution as long as the solution $g(t)$ of the Ricci flow exists.

Lemma 5.1.1. *Assume $g(t)$ is a solution of the Ricci flow and $e_a(p,t)$ is a solution of (5.1) for each $a = 1, ..., n$. If $\{e_a^0(p)\}$ is orthonormal for each $p \in M$, then $\{e_a(t)\}$ is orthonormal for each t.*

Proof. We just have to see that

$$\begin{aligned} \frac{\partial}{\partial t}(g(e_a, e_b)) =& (\frac{\partial}{\partial t}g)(e_a, e_b) + g(\frac{\partial}{\partial t}e_a, e_b) + g(e_a, \frac{\partial}{\partial t}e_b) \\ =& -2Ric(e_a, e_b) + g(Ric(e_a), e_b) + g(e_a, Ric(e_b)) \\ =& -2Ric(e_a, e_b) + Ric(e_a, e_b) + Ric(e_a, e_b) = 0. \end{aligned}$$

Then $g(e_a(t), e_b(t)) = g_0(e_a^0, e_b^0) = \delta_{ab}$, $\forall t$.

□

Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow $\dfrac{\partial}{\partial t}g = -2Ric(g(t))$ with $g(0) = g_0$. Let V be a bundle over M^n such that $(\iota_0)_p : V_p \longrightarrow T_pM^n$ is a vector space isomorphism for each $p \in M^n$, depending smoothly on $p \in M^n$. Define $h = (\iota_0)^*(g_0)$. Then $\iota_0 : (V, h) \longrightarrow (TM^n, g_0)$ is a bundle isometry.

Suppose we evolve the isometry ι_0 by

$$\begin{aligned} \frac{\partial}{\partial t}\iota(t) &= Ric_{g(t)} \circ \iota, \\ \iota(0) &= \iota_0. \end{aligned} \tag{5.2}$$

For each $p \in M^n$, (5.2) is a system of linear ordinary differential equations. Hence there exists a unique solution for $t \in [0, T)$. Furthermore, $i(t)$ remains a smooth bundle isomorphism for all $t \in [0, T)$. In fact, we get more.

Proposition 5.1.1. *For each $t \in [0, T)$ the solution $\iota(t) : (V, h) \longrightarrow (TM, g(t))$ of (5.2) is a bundle isometry.*

Proof. We will prove that $\iota(t)$ is an isometry by showing that h is the pullback of $g(t)$ via $\iota(t)$, i.e., $h = (\iota(t))^*(g(t))$, $\forall t \in [0, T)$. Since h is fixed and $\iota(0)$ is an isometry, it suffices to show that $(\iota(t))^*(g(t))$ remains constant in time. Let $p \in M^n$ and $X, Y \in V_p$ be arbitrary. Then

$$\begin{aligned} \frac{\partial}{\partial t}\left((\iota^* g)(X, Y)\right) =& \frac{\partial}{\partial t}\left[g(\iota(X), \iota(Y))\right] \\ =& -2Ric(\iota(X), \iota(Y)) + g(Ric(\iota(X)), \iota(Y)) + g(\iota(X), Ric(\iota(Y))) = 0, \end{aligned}$$

as we required. Therefore $\iota(t)$ remains an isometry.

□

Now we may define connections on V by the pull-back of the Levi-Civita connections $\nabla(t)$ on M^n :

$$D(t) := \iota(t)^* \nabla(t) : C^\infty(TM^n) \times C^\infty(V) \longrightarrow C^\infty(V),$$

where for each $X \in C^\infty(TM^n)$ and $\xi \in C^\infty(V)$, we have

$$D(t)(X, \xi) = (D(t))_X(\xi) = (\iota^* \nabla)_X(\xi) = \nabla_X(\iota(\xi)).$$

We can also define connections on tensor product bundles of TM^n, V and their dual bundles T^*M^n and V^*, using the usual product rule, as it was done on Section 4.2.3

of the previous chapter. These connections will be denoted by

$$\nabla^t_D : C^\infty(TM^n) \times C^\infty\left(V \otimes T^*M\right) \longrightarrow C^\infty\left(V \otimes T^*M \otimes T^*M\right),$$

and are given by

$$\nabla^t_D(X, \xi \otimes \phi) = (\nabla_X(\iota(\xi))) \otimes \phi + \xi \otimes \nabla_X \phi,$$

for all $X \in TM$, $\xi \in C^\infty(V)$ and $\phi \in T^*M$. Any other connection on tensor product bundles of the above mentioned bundles will also be denoted by ∇^t_D. Now we pull-back the Riemann curvature tensor to V. For an arbitrary $p \in M$ and $\xi, \eta, \alpha, \beta \in V_p$, we define $(\iota^* Rm)(t) \in C^\infty(\wedge^2 V \otimes_S \wedge^2 V)$ by

$$(\iota^* Rm)(\xi, \eta, \alpha, \beta) \coloneqq Rm\left(\iota(\xi), \iota(\eta), \iota(\alpha), \iota(\beta)\right). \tag{5.3}$$

Consider local coordinates $\{x^k\}$, $k = 1, \ldots, n$, on an open set $U \subset M^n$ and let $\{e_a\}$ be a basis of sections of V restricted to U. Then we define the components of ι^k_a of $(\iota)(t) : (V, h) \longrightarrow (TM^n, g(t))$ by

$$\iota(e_a) \coloneqq \sum_{k=1}^n \iota^k_a \frac{\partial}{\partial x^k}.$$

Accordingly, the components R_{abcd} of $\iota^* Rm$ are given by

$$R(t)_{abcd} \coloneqq (\iota^* Rm)(e_a, e_b, e_c, e_d)(t) = \sum_{i,j,k,l=1}^n \iota^i_a \iota^j_b \iota^k_c \iota^l_d R_{ijkl}. \tag{5.4}$$

Now we would like to understand how the evolution equation for $\iota^*(Rm)$ relates to the evolution equation of Rm. First, we define the Laplacian acting on tensor product bundles of $(TM^n, g(t))$ and (V, h).

$$\Delta^t_D \coloneqq tr_g(\nabla^t_D \circ D(t)) = \sum_{i,j=1}^n g^{ij} (\nabla^t_D)_i (D(t))_j,$$

where $(\nabla^t_D)_j(\xi) = \nabla^t_j(\iota(\xi))$. Then we get the following result.

Proposition 5.1.2. *Let $g(t)$ be a solution of the Ricci flow and $\iota(t)$ a solution of (5.2).*

Then $\nabla^t_D(\iota) = 0$ *and* $\iota^* Rm$*, defined as in (5.3), evolves by*

$$\frac{\partial}{\partial t} R_{abcd} = \Delta^t_D R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}), \tag{5.5}$$

where $B_{abcd} \coloneqq -h^{ei}h^{fl}R_{aebf}R_{cidl}$.

Proof. First, we recall from (5.2) that $\frac{\partial}{\partial t}\iota^k_a = R^k_l \iota^l_a$. Then

$$\begin{aligned}\frac{\partial}{\partial t} R_{abcd} = &\sum_{i,j,k,l=1}^{n} \frac{\partial}{\partial t}\left(\iota^i_a \iota^j_b \iota^k_c \iota^l_d R_{ijkl}\right)\\
&= (R^i_m \iota^m_a)\iota^j_b \iota^k_c \iota^l_d R_{ijkl} + \iota^i_a (R^j_m \iota^m_b)\iota^k_c \iota^l_d R_{ijkl}\\
&+ \iota^i_a \iota^j_b (R^k_m \iota^m_c)\iota^l_d R_{ijkl} + \iota^i_a \iota^j_b \iota^k_c (R^l_m \iota^m_d) R_{ijkl}\\
&+ \iota^i_a \iota^j_b \iota^k_c \iota^l_d [\Delta^t R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\\
&- (R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^l_p R_{ijkp})]\\
&= \iota^i_a \iota^j_b \iota^k_c \iota^l_d \left[\Delta^t R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\right].\end{aligned}$$

The last equality follows from the fact that $(R^i_m \iota^m_a)\iota^j_b \iota^k_c \iota^l_d R_{ijkl} = \iota^i_a \iota^j_b \iota^k_c \iota^l_d R^p_i R_{pjkl}$ and the respective equalities for the other terms. Now, we note that $\nabla^t_D(\iota) = 0$. In fact, consider $\iota(t)$ as an element of $C^\infty(V^* \otimes TM)$. Then, by definition,

$$\iota((\nabla^t_D)_X \xi) = \nabla^t_X(\iota(\xi)) = (\nabla^t_D)_X(\iota(\xi)) = ((\nabla^t_D)_X \iota)(\xi) + \iota((\nabla^t_D)_X \xi),$$

which implies that $(\nabla^t_D)_X \iota = 0$ for all $X \in V_p$, $\forall p \in M$. Then we get from (5.4) that

$$\Delta^t_D R_{abcd} = \iota^i_a \iota^j_b \iota^k_c \iota^l_d \Delta^t R_{ijkl}$$

and the proposition follows. □

5.2 The Structure of the Evolution Equation for the Curvature

In order to get a better grasp of the evolution equation for the Riemann tensor, we look more closely at the structure of the Riemann curvature operator. In order to so, it is better to look at Rm as the operator on 2-forms

$$Rm : \wedge^2 T^*M \longrightarrow \wedge^2 T^*M$$

defined for all $u \in \wedge^2 T^*M$ by

$$[Rm(u)]_{ij} \coloneqq -g^{kp}g^{lq}R_{ijkl}u_{pq}.$$

If we define the **inner product** on $\wedge^2 T^*M$ as

$$\langle u, v\rangle \coloneqq g^{ik}g^{jl}u_{ij}v_{kl}, \tag{5.6}$$

then Rm is self adjoint:

$$\begin{aligned}\langle Rm(u), v\rangle =& - g^{ik}g^{jl}g^{kp}g^{lq}R_{ijkl}u_{pq}v_{kl} = -g^{ik}g^{jl}g^{pk}g^{ql}R_{klij}v_{kl}u_{pq}\\ &= g^{pk}g^{ql}(-g^{ik}g^{jl}R_{klij}v_{kl})u_{pq} = g^{pk}g^{ql}[Rm(v)]_{kl}u_{pq}\\ &= \langle Rm(v), u\rangle.\end{aligned}$$

Now we consider the operator

$$Rm^2 = Rm \circ Rm : \wedge^2 T^*M \longrightarrow \wedge^2 T^*M$$

given in local coordinates by

$$(Rm^2(u))_{ij} = g^{km}g^{ln}g^{pq}g^{rs}R_{ijps}R_{rqkl}u_{mn}. \tag{5.7}$$

Considering equation (5.5), we would like to get more information about the last term

$$2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}).$$

In order to do so, we consider $\wedge^2 T_p^*M$ as a **Lie algebra**, for each $p \in M^n$. In what follows, we explain this formulation.

If we fix $p \in M^n$, we may introduce in $\wedge^2 T_p^*M^n$ a Lie algebra structure. In fact, given

$u, v \in \wedge^2 T_p^* M^n$, we define their Lie bracket by

$$[u, v]_{ij} := g^{kl}(u_{ik}v_{lj} - v_{ik}u_{lj}). \tag{5.8}$$

Let $\{e_i\}$ be a local orthonormal frame field on M and let $\{\omega^i\}$ be a dual basis for $\{e_i\}$. Then, as it was explained in Chapter 2, any 2-form u may be naturally identified with an anti-symmetric matrix (u_{ij}), whose entries are the components of u, i.e., $u = \sum_{i<j} u_{ij}\omega_i \wedge \omega_j$. For each $p \in M^n$, this gives a Lie algebra isomorphism between $\mathcal{G} = \left(\wedge^2 T_p^* M, [,]\right)$ and $\mathfrak{so}(n)$, exactly as we explained in Chapter 2. The inner product on $\mathcal{G}$ is defined as in (5.6). Hence formula (5.8) corresponds to

$$[u, v]_{ij} = (uv - vu)_{ij}. \tag{5.9}$$

If we take local coordinates $\{x^i\}$ around $p \in M$ then $\{dx^i \wedge dx^j; 1 \leq i < j \leq n\}$ is a basis for $\mathcal{G}$. Moreover, the structure constants $C_{(ij)}^{(pq)(rs)}$ (see Chapter 2) are defined by

$$[dx^p \wedge dx^q, dx^r \wedge dx^s] = \sum_{(ij)} C_{(ij)}^{(pq)(rs)} dx^i \wedge dx^j.$$

Since

$$dx^p \wedge dx^q = \frac{1}{2}(dx^p \otimes dx^q - dx^q \otimes dx^p) = \frac{1}{2}(\delta_k^p \delta_l^q - \delta_k^q \delta_l^p) dx^k \otimes dx^l,$$

we can use (5.8) to get an explicit formula for the structure constants:

$$\begin{aligned}
C_{(ij)}^{(pq),(rs)} &= \frac{1}{4} g^{kl}\big[(\delta_i^p\delta_k^q - \delta_i^q\delta_k^p)(\delta_l^r\delta_j^s - \delta_l^s\delta_j^r) - (\delta_i^r\delta_k^s - \delta_i^s\delta_k^r)(\delta_l^p\delta_j^q - \delta_l^q\delta_j^p)\big] \\
&= \frac{1}{4}\big[g^{qr}\delta_i^p\delta_j^s - g^{qs}\delta_i^p\delta_j^r - g^{pr}\delta_i^q\delta_j^s + g^{ps}\delta_i^q\delta_j^r - g^{sp}\delta_i^r\delta_j^q \\
&+ g^{sq}\delta_i^r\delta_j^p + g^{rp}\delta_i^s\delta_j^q - g^{rq}\delta_i^s\delta_j^p\big] \\
&= \frac{1}{4}\big[g^{qr}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p) + g^{qs}(\delta_i^r\delta_j^p - \delta_i^p\delta_j^r) \\
&+ g^{pr}(\delta_i^s\delta_j^q - \delta_i^q\delta_j^s) + g^{ps}(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q)\big].
\end{aligned}$$

Therefore, the definition of the Lie algebra square (see Chapter 2) gives us

$$(Rm^{\#})_{ijkl} = C_{(ij)}^{(pq),(rs)} C_{(kl)}^{(uv),(wy)} R_{pquv} R_{rswy}. \tag{5.10}$$

Now we are able to write the evolution of the Riemann curvature tensor in a way which enables us to apply the maximum principle.

Theorem 5.2.1. *Let $g(t)$ be a solution of the Ricci flow. Then the curvature $\iota^*(Rm)$ defined in equation (5.3) evolves by*

$$\frac{\partial}{\partial t}(\iota^* Rm) = \Delta_D(\iota^* Rm) + (\iota^* Rm)^2 + (\iota^* Rm)^{\#}. \tag{5.11}$$

Proof. First, we look at equation (5.7) and use the first Bianchi identity to see that

$$\begin{aligned}(Rm^2)_{ijkl} &= g^{pq}g^{rs}R_{ijps}R_{qrlk} = g^{pq}g^{rs}(-R_{ipsj} - R_{isjp})(-R_{qkrl} - R_{qlkr}) \\ &= g^{pq}g^{rs}(R_{pijs} + R_{jpis})(R_{lrkq} + R_{rklq}) \\ &= (R^r_{pij} - R^r_{pji})(R^p_{rkl} - R^p_{rlk}) \\ &= -B_{ijlk} + B_{ijkl} + B_{jilk} - B_{jikl} \\ &= 2(B_{ijkl} - B_{ijlk})\end{aligned}$$

since $B_{ijkl} = -R^r_{pij}R^p_{rlk}$.

On the other hand, since the structure constants are fully antisymmetric, i.e., antisymmetric in (ij), (pq) and (rs), we get that

$$\begin{aligned}R_{pquv}C_{(ij)}^{(pq),(rs)} = \; &\frac{R_{pquv}}{4}g^{qr}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p) + \frac{R_{pquv}}{4}g^{qs}(\delta_i^r\delta_j^p - \delta_i^p\delta_j^r) \\ &+ \frac{R_{pquv}}{4}g^{pr}(\delta_i^s\delta_j^q - \delta_i^q\delta_j^s) + \frac{R_{pquv}}{4}g^{ps}(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q).\end{aligned}$$

Then, since we are adding on p and q and $R_{pquv} = -R_{qpuv}$, we get

$$R_{pquv}C_{(ij)}^{(pq),(rs)} = \frac{1}{2}R_{pquv}\left[g^{qr}(\delta_i^p\delta_j^s - \delta_i^s\delta_j^p) + g^{ps}(\delta_i^q\delta_j^r - \delta_i^r\delta_j^q)\right]$$

and

$$R_{pquv}R_{rswy}C^{(pq),(rs)}_{(ij)} = R_{pquv}R_{rswy}g^{qr}(\delta^p_i\delta^s_j - \delta^s_i\delta^p_j).$$

From (5.10), $(Rm^{\#})_{ijkl} = C^{(pq),(rs)}_{(ij)}C^{(uv),(wy)}_{(kl)}R_{pquv}R_{rswy}$, we get

$$\begin{aligned}(Rm^{\#})_{ijkl} &= R_{pquv}R_{rswy}g^{qr}(\delta^p_i\delta^s_j - \delta^s_i\delta^p_j)g^{vw}(\delta^u_l\delta^y_k - \delta^x_l\delta^u_k)\\ &= R^r_{uvp}R^v_{sry}(\delta^p_i\delta^s_j - \delta^s_i\delta^p_j)(\delta^u_l\delta^y_k - \delta^y_l\delta^u_k)\\ &= R^r_{lvi}R^v_{jrk} - R^r_{kvi}R^v_{jrl} - R^r_{lvj}R^v_{irk} + R^r_{kvj}R^v_{irl}\\ &= -B_{likj} + B_{kilj} + B_{ljki} - B_{kjli}\\ &= 2(B_{ikjl} - B_{iljk})\end{aligned}$$

since $B_{ijkl} = B_{jilk} = B_{klij}$.

Then pulling back by the bundle isomorphism ι defined in (5.2), the theorem follows by using Proposition 5.1.2.

□

Our aim is to apply the maximum principle for systems in order to study equation (5.11) by considering the associated ODE

$$\frac{d}{dt}B = B^2 + B^{\#},$$

where B is a self adjoint linear transformation in $Sym^2(\wedge^2\mathbb{R}^n)$.

Let (M^n, g) be a Riemannian manifold and let $\{e_i\}$ be an orthonormal moving frame on an open subset $U \subset M^n$. Then $\{e_i\}$ defines a dual frame $\{\omega^i\}$ such that $\omega^i(e_k) = \delta^k_i$ and $\{\omega^i \wedge \omega^j\}_{i<j}$ gives a basis on $\wedge^2 TU$. Therefore, given $p \in U$, we get a Lie algebra isomoprhism

$$\varphi_p : \wedge^2 T_pM^n \longrightarrow \wedge^2\mathbb{R}^n,$$

which takes an ordered basis $(\omega^1 \wedge \omega^2, \ldots, \omega^{(n-1)} \wedge \omega^n)$ to $\beta = (\beta_1, \ldots, \beta_m)$, an ordered basis of $\wedge^2\mathbb{R}^n$, where $m = \frac{n(n-1)}{2}$. From now on, let us denote the ordered basis $\{\omega^i \wedge \omega^j\}_{i<j}$

of $\Lambda^2 T_p M^n$ by $\{\theta^k = \theta^k_{ij}\omega^i \wedge \omega^j, 1 \leq i < j \leq n\}_{k=1}^m$. Let $\mathcal{R}$ denote the **space of self adjoint linear transformations in $Sym^2(\wedge^2\mathbb{R}^n)$ that obey the 1^{st} Bianchi identity**.

Let B be a solution of

$$\frac{d}{dt}B = B^2 + B^{\#},$$
$$B(0) \in \mathcal{R},$$

which is the ODE that corresponds to (5.11). Using the basis β, we may represent B by a $n \times n$ matrix B_β defined by $B(\beta_j) = \beta_i (B_\beta)_{ij}$, where B_β is symmetric. Also, the structure constants for $\wedge^2 T_p M \cong \wedge^2\mathbb{R}^n \cong \mathfrak{so}(n)$, C_β, on the basis β are given by

$$[\beta_j, \beta_k] = \beta_i (C_\beta)_{ijk}.$$

Finally, if we define by Q to be the following transformation

$$B \mapsto Q := B^2 + B^{\#},$$

then Q is given by

$$(Q_\beta)_{ij} = (B_\beta)_{ik}(B_\beta)_{kj} + (C_\beta)_{ipq}(C_\beta)_{jrs}(B_\beta)_{pr}(B_\beta)_{qs}.$$

Now, we consider the case when $n = 3$ and M^3 is a closed manifold. Equation (2.17) from Chapter 2 gives us an expression for the # operator. Therefore, if the matrix B is diagonal, i.e.,

$$B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

then

$$B^{\#} = \begin{pmatrix} \beta\gamma & 0 & 0 \\ 0 & \alpha\gamma & 0 \\ 0 & 0 & \alpha\beta \end{pmatrix}.$$

Now we identify Rm with the following quadratic form B on $\wedge^2 TM^3$:

$$B(e_i \wedge e_j, e_l \wedge e_k) = \langle R(e_i, e_j)e_k, e_l \rangle.$$

Thus, considering the matrix (B_{pq}), related to B, given by

$$\langle R(e_i, e_j)e_k, e_l \rangle = B_{pq}\theta_{ij}^p \theta_{lk}^q \tag{5.12}$$

on each fiber $\wedge^2 T_p M^3$ of $\wedge^2 TM^3$, and evolving $\{e_i\}$ so that it remains orthonormal (Uhlenbeck's trick) we get that the PDE (5.11) corresponds to the ODE

$$\frac{d}{dt}B = B^2 + B^{\#}, \tag{5.13}$$

satisfied by B in each fiber.

If we choose $\{e_i\}$ so that $B(0)$ is diagonal at $p \in M^3$ with eigenvalues $\lambda(0) \geq \mu(0) \geq \nu(0)$, then, by uniqueness of solution to (5.13), the elements outside the diagonal remain 0. In particular, $B(t)$ remains diagonal. Therefore, equation (5.13) is given by

$$\frac{d}{dt}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} = \begin{pmatrix} \lambda^2 + \mu\nu & 0 & 0 \\ 0 & \mu^2 + \lambda\nu & 0 \\ 0 & 0 & \nu^2 + \lambda\mu \end{pmatrix}, \tag{5.14}$$

a system in $\mathbb{R}^3$. Hence

$$\begin{aligned} \frac{d}{dt}(\lambda - \mu) &= \lambda^2 + \mu\nu - \mu^2 - \lambda\nu = (\lambda - \mu)(\lambda + \mu - \nu) \\ \frac{d}{dt}(\mu - \nu) &= \mu^2 + \lambda\nu - \nu^2 - \lambda\mu = (\mu - \nu)(\mu + \nu - \lambda). \end{aligned} \tag{5.15}$$

Let $(0, T]$ be an interval where a solution of (5.14) exists. Then the inequality $\lambda(t) \geq \mu(t) \geq \nu(t)$ is preserved for all $t \in (0, T]$. In fact, we either have strict inequality for all times or the equality holds for all $t \in (0, T]$. In case of strict inequality, (5.15) gives $\lambda - \mu$ and $\mu - \nu$ in terms of exponentials.

We may define the greatest eigenvalue of Rm, λ, as

$$\lambda(B) = \max_{|v|=1} B(v, v).$$

Similarly, we define the smallest eigenvalue, ν, and $(\mu + \nu)$ by

$$\begin{aligned} \nu(B) &= min_{|v|=1} B(v, v) \\ (\mu + \nu)(B) &= \min_{|v_1|=|v_2|=1, \langle v_1, v_2 \rangle = 0} \left(B(v_1, v_1) + B(v_2, v_2) \right). \end{aligned}$$

Proposition 5.2.1. *With regard to the definitions above, $\lambda : \mathcal{R} \longrightarrow \mathbb{R}$ is a convex function and ν and $(\mu + \nu)$ are concave functions. Moreover, the eigenvalues λ, μ and ν are twice the sectional curvatures, i.e.*

$$\begin{aligned} \lambda &= 2R_{2323}, \\ \mu &= 2R_{1313}, \\ \nu &= 2R_{1212}. \end{aligned}$$

Proof. Let $B, N \in \mathcal{R}$ and let $\theta \in [0, 1]$ be arbitrary. Then

$$\theta B(v, v) + (1 - \theta) N(v, v) \leq \theta \lambda(B) + (1 - \theta) \lambda(N).$$

Considering the maximum of the left-hand side over all $|v| = 1$, we get that λ is a convex function. In the same way, we have for ν :

$$\theta B(v, v) + (1 - \theta) N(v, v) \geq \theta \nu(B) + (1 - \theta) \nu(N).$$

Now, considering the minimum of the left-hand side over all $|v| = 1$, we prove that ν is concave. The same reasoning applies to prove that $(\mu + \nu)$ is concave.

Finally, observe that $\lambda = B_{11}$. Then equation (5.12) implies

$$B_{11} \theta^1_{ij} \theta^1_{lk} = R_{ijkl}.$$

Since we already know the structure constants by (2.16), we get

$$\lambda \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = R_{2323},$$

which gives us $\lambda = 2R_{2323}$. The other eigenvalues follow by a totally analogous reasoning. □

Therefore, the Ricci tensor may be regarded as the matrix

$$Ric = \frac{1}{2} \begin{pmatrix} \mu + \nu & 0 & 0 \\ 0 & \lambda + \nu & 0 \\ 0 & 0 & \lambda + \mu \end{pmatrix}. \tag{5.16}$$

Recalling Definition (2.0.6), where we defined the trace-free part of the Ricci tensor, we write the trace-free parts of Rm and Ric for later reference:

$$\overset{\circ}{Rm} = \frac{1}{3} \begin{pmatrix} 2\lambda - \mu - \nu & 0 & 0 \\ 0 & 2\mu - \lambda - \nu & 0 \\ 0 & 0 & 2\nu - \lambda - \mu \end{pmatrix} = -2\overset{\circ}{Ric}. \tag{5.17}$$

5.3 Local Estimates

In this section, we will provide two estimate results for 3-manifolds with positive Ricci curvature. The first one shows that a comparison of the curvatures is preserved and the second one shows that it actually improves it. This second result shows that a solution to the Ricci flow on a closed 3-manifold with positive Ricci curvature is nearly Einstein at any point where the scalar curvature is large.

First, we will introduce a few results for the Ricci flow as a direct consequence of the maximum principle for systems introduced in Chapter 4. Again, let $\lambda(t) \geq \mu(t) \geq \nu(t)$ be the eigenvalues of Rm of a solution to the Ricci flow $(M^3, g(t))$ on a closed 3-manifold. Also, recall that these eigenvalues are twice the sectional curvature. Finally, throughout this section, let $E = Sym^2(\wedge^2 TM^3)$.

Proposition 5.3.1. *Let $(M^3, g(t))$, $t \in [0,T)$, be a solution to the Ricci flow on a closed 3-manifold, such that the scalar curvature at $t = 0$ $R_{g_0} \geq C_0$, for some constant C_0. Then $R_{g(t)} \geq C_0$ for all $t \in [0,T)$.*

Proof. Let

$$K^t = \left\{P \in E; \lambda^t(P) + \mu^t(P) + \nu^t(P) \geq C_0\right\}. \tag{5.18}$$

We see that K^t is closed, invariant under parallel translation by Lemma 4.2.11 and since $\lambda(t) + \mu(t) + \nu(t)$ is convex (the trace is actually linear), it follows that K^t is convex in each fiber. When we consider the associated ODE, we know that

$$\begin{aligned} \frac{d}{dt}\lambda &= \lambda^2 + \mu\nu, \\ \frac{d}{dt}\mu &= \mu^2 + \lambda\nu, \\ \frac{d}{dt}\nu &= \nu^2 + \lambda\mu. \end{aligned} \tag{5.19}$$

Therefore,

$$\frac{d}{dt}(\lambda + \mu + \nu) = \frac{1}{2}\left((\lambda+\mu)^2 + (\lambda+\nu)^2 + (\mu+\nu)^2\right) \geq 0$$

Then it follows that K^t_p is preserved by the ODE for each $p \in M$. By the maximum principle for systems (Theorem (4.2.14)), the Ricci flow preserves $R_{g(t)} \geq C_0$.

□

Proposition 5.3.2. *Let $(M^3, g(t))$, $t \in [0,T)$, be a solution to the Ricci flow on a closed 3-manifold such that $Ric(g_0) \geq 0$. Then $Ric(g(t)) \geq 0$ for all $t \in [0,T)$.*

Proof. First, we note that the smallest eigenvalue of *Ric* is $\mu + \nu$. So, if we prove that $\nu \geq 0$ is preserved, the result follows. Hence, let us define

$$K^t = \left\{P \in E; \nu^t(P) \geq 0\right\} = \left\{P \in E; -\nu^t(P) \leq 0\right\}$$

K^t is closed, invariant by parallel translation from Lemma 4.2.11 and since $-\nu^t$ is convex (because ν^t is concave), K^t_p is convex for each $p \in M$. Looking at the associated ODEs (5.19), we consider the following options:

(i) $\nu(0) = 0$ and $\mu(0) > 0$: then $\frac{d}{dt}\nu(0) = \lambda(0)\mu(0) > 0$, so $\nu(t) > \nu(0) = 0$;

(ii) $\nu(0) > 0$ and $\mu(0) > 0$: then $\frac{d}{dt}\nu(0) = \nu^2(0) + \lambda(0)\mu(0) > 0$, so $\nu(t) > \nu(0) > 0$;

(iii) $\lambda(0) = \mu(0) = \nu(0) = 0$: then $\lambda(t) = \mu(t) = \nu(t) = 0$ for all $t \in [0, T)$;

(iv) $\nu(0) = \mu(0) = 0$ and $\lambda(0) > 0$: then $\lambda(t) > 0$ for all $t \in [0, T)$.

Therefore, $Ric \geq 0$ is preserved.

□

Lemma 5.3.1. *Let $(M^3, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. If there exists a constant $0 < C < \infty$ such that $\lambda(0) \leq C(\mu(0) + \nu(0))$, then*

$$\lambda(t) \leq C(\mu(t) + \nu(t)). \tag{5.20}$$

Proof. Since $\lambda \geq \mu \geq \nu$, it is true that $\lambda \geq \frac{1}{2}(\mu + \nu) > 0$, by hypothesis. Considering the associated ODEs (5.19), we may look at $\log(\frac{\lambda}{\mu + \nu})$. Then we have

$$\begin{aligned}
\frac{d}{dt}\log(\frac{\lambda}{\mu+\nu}) &= \frac{\mu+\nu}{\lambda}\left(\frac{d\lambda}{dt}\frac{1}{\mu+\nu} - \frac{\lambda}{(\mu+\nu)^2}\frac{d}{dt}(\mu+\nu)\right) \\
&= \frac{1}{\lambda(\mu+\nu)}\left(\frac{d\lambda}{dt}(\mu+\nu) - \lambda\frac{d}{dt}(\mu+\nu)\right) \\
&= \frac{1}{\lambda(\mu+\nu)}\left[(\lambda^2+\mu\nu)(\mu+\nu) - \lambda(\nu^2+\lambda\mu+\mu^2+\lambda\nu)\right] \\
&= \frac{1}{\lambda(\mu+\nu)}\left[\lambda^2\nu+\lambda^2\mu+\mu\nu^2+\nu\mu^2-\lambda\nu^2-\lambda^2\mu-\lambda\mu^2-\lambda^2\nu\right] \\
&= \frac{\mu^2(\nu-\lambda)+\nu^2(\mu-\lambda)}{\lambda(\mu+\nu)} \leq 0,
\end{aligned}$$

since $\nu(t) \leq \lambda(t)$ and $\mu(t) \leq \lambda(t)$.

Let $\lambda(P) \geq \mu(P) \geq \nu(P)$ denote the eigenvalues of

$$P \in (\wedge^2 T^*M^3 \otimes_S \wedge^2 T^*M^3)_p.$$

Define $K^t \subset (\wedge^2 T^* M^3 \otimes_S \wedge^2 T^* M^3)_p$ by

$$K^t := \left\{P; \lambda^t(P) - C(\nu^t(P) + \mu^t(P)) \leq 0\right\}.$$

It follows from Lemma 4.2.11 that K^t is invariant under parallel translation. Now, since

$$\lambda^t(P) - C(\nu^t(P) + \mu^t(P)) = \max_{|U|=1} P(U,U) + C \max_{|V|=|W|=1, \langle V,W\rangle=0} (-P(V,V) - P(W,W))$$

is a convex function, the set K^t is convex in each fiber. Moreover, K^t is closed. Then it follows from $\frac{d}{dt}\log(\frac{\lambda^t}{\mu^t + \nu^t}) \leq 0$ that if $0 < C < \infty$ is sufficiently large so that $B(0) \in K^0\ \forall p \in M^3$, then $B(t)$ remains in K^t, where $B \in (\wedge^2 T^* M^3 \otimes_S \wedge^2 T^* M^3)_p$ is the quadratic form that corresponds to $Rm(g(t))$, given by (5.12). From the maximum principle (Theorem (4.2.14)), since by our hypothesis $\lambda(0) \leq C(\mu(0) + \nu(0))$, it follows that $\lambda(t) - C(\nu(t) + \mu(t)) \leq 0$ and the lemma follows. □

Corollary 5.3.1.1. *Let $(M^3, g(t))$, $t \in [0,T)$, be a solution of the Ricci flow on a closed 3-manifold M^3 such that $Ric(g_0) > 0$. Let*

$$R_{min}(t) := \inf_{p \in M^3} R(p,t).$$

Then there exists $\beta > 0$, depending only on g_0, such that at all points of M^3,

$$Ric(g(t)) \geq 2\beta^2 R(t) g(t) \geq 2\beta^2 R(t)_{min} g(t),$$

for all $t \in [0,T)$.

Proof. From the Lemma (5.3.1) and equation (5.16), we can find $C > 0$ depending on g_0 such that the following holds $\forall p \in M^3$:

$$Ric \geq \frac{\mu + \nu}{2} g \geq \frac{\lambda}{2C} g \geq \frac{\lambda + \mu + \nu}{6C} g \geq \frac{R_{min}}{6C} g.$$

Considering $2\beta^2 = \frac{1}{6C}$, the result follows. □

The following theorem proves that the metric is nearly Einstein at points where the scalar curvature is large enough. This shows that the upper and lower estimates get better as $R \to \infty$.

Theorem 5.3.2. *Let $(M^3, g(t))$ be a solution to the Ricci flow on a closed 3-manifold such that the initial metric g_0 has strictly positive Ricci curvature. Then there exist positive constants $\delta < 1$ and C depending only on g_0 such that*

$$\frac{\lambda - \nu}{\lambda + \mu + \nu} \leq \frac{C}{(\lambda + \mu + \nu)^{\delta}}. \tag{5.21}$$

Remark. *Before we prove our theorem, we observe that $\dfrac{\lambda - \nu}{\lambda + \mu + \nu}$ is invariant under homotheties of the metric $g(t)$, while $\dfrac{C}{(\lambda + \mu + \nu)^{\delta}}$ tends to 0 as $R = \lambda + \mu + \nu \to \infty$. Therefore, equation (5.21) shows that λ tends to ν, which is the smallest eigenvalue. Since $\lambda \geq \mu \geq \nu$, this actually shows that, at points where the scalar curvature R goes to ∞, the eigenvalues all approach each other. Further ahead, we will clarify what happens on the manifold as a whole.*

Proof. We may assume that $\lambda(B) > \nu(B)$ for $B(t) \in (\wedge^2 T^* M^3 \otimes_S \wedge^2 T^* M^3)_p$, where B is the quadratic form corresponding to $Rm(g(t))$, given by (5.12). Now we calculate, for the associated ODEs (5.19),

$$\frac{d}{dt} \log(\lambda - \nu) = \lambda - \mu + \nu$$

and

$$\begin{aligned} \frac{d}{dt} \log(\lambda + \mu + \nu) =& \frac{1}{\lambda + \mu + \nu} \left[\lambda^2 + \mu\nu + \nu^2 + \lambda\mu + \mu^2 + \lambda\nu\right] \\ =& \frac{1}{\lambda + \mu + \nu} \left[\mu(\nu + \mu) + \lambda(\mu - \nu) + (\lambda + \nu)^2\right]. \end{aligned}$$

Let $0 < \delta < 1$ be a constant that will be chosen later. Hence,

$$\frac{d}{dt}\log\left(\frac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right) = \frac{d}{dt}\left[\log(\lambda-\nu) - (1-\delta)\log(\lambda+\mu+\nu)\right]$$
$$= \lambda-\mu+\nu-(1-\delta)\frac{\mu(\nu+\mu)+\lambda(\mu-\nu)+(\lambda+\nu)^2}{\lambda+\mu+\nu}$$
$$= \delta(\lambda-\mu+\nu) - \frac{(1-\delta}{\lambda+\mu+\nu}\left[\mu(\nu+\mu)+\lambda(\mu-\nu)+(\lambda+\nu)^2+(\mu-\lambda-\nu)(\lambda+\mu+\nu)\right]$$
$$= \delta(\lambda-\mu+\nu) - \frac{(1-\delta}{\lambda+\mu+\nu}\left[\mu(\mu+\nu)+\lambda(\mu-\nu)+\mu^2\right]$$
$$\leq \delta(\lambda-\mu+\nu) - (1-\delta)\frac{\mu^2}{\lambda+\mu+\nu},$$

since $\nu \leq \mu$. Let $0 < C < \infty$ such that $\lambda(0) \leq C(\mu(0)+\nu(0))$, where C depends only on the initial metric g_0. By our hypothesis, $Ric(g(0)) > 0$. Hence, using Lemma 5.3.1, we get

$$\lambda+\nu-\mu \leq \lambda \leq 2C\mu.$$

Since

$$\frac{2\mu}{\mu+\lambda+\nu} \geq \frac{\mu+\nu}{\mu+\lambda+\nu} \geq \frac{\mu+\nu}{3\lambda},$$

we get

$$\frac{\mu}{\mu+\lambda+\nu} \geq \frac{\mu+\nu}{6\lambda}.$$

Therefore,

$$\frac{\mu}{\lambda+\mu+\nu} \geq \frac{\mu+\nu}{6\lambda} \geq \frac{1}{6C}.$$

Then we choose $\delta > 0$ small enough so that

$$\frac{\delta}{1-\delta} \leq \frac{1}{12C^2}$$

in order to get

$$\frac{d}{dt}\log\left(\frac{\lambda-\nu}{(\lambda+\mu+\nu)^{1-\delta}}\right) \leq 0.$$

Now we define the set

$$K^t := \left\{P; \lambda(P) - \nu(P) - C[\lambda(P) + \mu(P) + \nu(P)]^{1-\delta} \leq 0\right\}.$$

Following the same reasoning as before, we get that K^t is invariant under parallel translation, closed and convex in each fiber. Then, by the maximum principle (Theorem 4.2.14) , the result follows. In particular, if $Ric(g_0) > 0$, then there exist constants $0 < C_0 < \infty$ and $\delta > 0$ such that

$$\frac{\lambda(Rm) - \nu(Rm)}{R^{1-\delta}} \leq C_0.$$

□

In his 1982's paper [14], Hamilton proved the following result, which follows from the theorem above.

Corollary 5.3.2.1. *There exist constants $\overline{\delta} > 0$ and $C < \infty$ depending only on g_0 such that*

$$\frac{4|\overset{\circ}{Ric}|^2}{R^2} \leq CR^{-\overline{\delta}}, \tag{5.22}$$

where $\overset{\circ}{Ric}$ is the trace-free part of the Ricci tensor.

Proof. Observe that

$$Ric - \frac{1}{3}Rg = \frac{1}{6}\begin{pmatrix} \mu + \nu - 2\lambda & 0 & 0 \\ 0 & \lambda + \nu - 2\mu & 0 \\ 0 & 0 & \lambda + \mu - 2\nu \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\frac{\left|Ric - \frac{1}{3}Rg\right|^2}{R^2} &= \frac{1}{36(\lambda+\mu+\nu)^2}\left[(\mu+\nu-2\lambda)^2 + (\lambda+\nu-2\mu)^2 + (\lambda+\mu-2\nu)^2\right] \\
&= \frac{1}{36(\lambda+\mu+\nu)^2}\left[6(\lambda^2+\mu^2+\nu^2) - 6(\lambda\mu+\lambda\nu+\mu\nu)\right] \\
&= \frac{1}{36(\lambda+\mu+\nu)^2}\left[3\left((\lambda-\mu)^2+(\mu-\nu)^2+(\lambda-\nu)^2\right)\right] \\
&= \frac{(\lambda-\mu)^2+(\mu-\nu)^2+(\lambda-\nu)^2}{12(\lambda+\mu+\nu)^2} \\
&\le \frac{3(\lambda-\nu)^2}{12(\lambda+\mu+\nu)^2} \le \frac{1}{4}CR^{-\overline{\delta}},
\end{aligned}$$

where $\overline{\delta} = 2\delta$.

□

5.4 Estimating the Gradient of the Scalar Curvature

In the previous section, we obtained estimates that compare curvatures at the same point, which tells us that the sectional curvatures approach each other if the scalar curvature goes to ∞ somewhere in our manifold. Since these estimates are punctual, this is not enough to conclude that the sectional curvatures approach each other everywhere. In this section, we shall obtain an estimate on the gradient of the scalar curvature, which enables us to compare sectional curvatures at different points. As a motivation, according to Theorem 2.0.1, if g is an Einstein metric on a manifold M^n, then $Ric = fg$ for some function f on M^n. Besides, we have

$$\nabla_k R = \nabla_k(g^{ij}R_{ij}) = \nabla_k(g^{ij}fg_{ij}) = n\nabla_k f.$$

On the other hand, the contracted second Bianchi identity gives us the following

$$\nabla_k(R) = 2\nabla^j R_{jk} = 2\nabla^j(fg_{jk}) = 2\nabla_k f.$$

So if $n > 2$, then $(n-2)\nabla f = 0$ implies that $f = \frac{R}{n}$ is constant.

Now, if we consider a solution $(M^3, g(t))$ of the Ricci flow on a closed 3-manifold, equation (5.22) can be written as

$$\frac{|Ric - \frac{1}{3}Rg|^2}{R^2} \leq CR^{-\bar{\delta}}. \tag{5.23}$$

The right-hand side is small when the scalar curvature is large, so $g(t)$ is getting closer to an Einstein metric. Our calculation above shows that R should be close to being constant. Therefore, it is natural to expect that we will be able to get a bound on $|\nabla R|$. In fact, this is exactly what happens. In order to prove this result, first we have to obtain several evolution equations, and that is what we shall do in the next lemmas.

Lemma 5.4.1. *If $(M^n, g(t))$ is a solution of the Ricci flow, then the evolution of $|\nabla R|^2$ is given by*

$$\frac{\partial}{\partial t}|\nabla R|^2 = \Delta|\nabla R|^2 - 2|\nabla\nabla R|^2 + 4\langle\nabla R, \nabla|Ric|^2\rangle. \tag{5.24}$$

Proof. We recall the evolution equation for the scalar curvature (equation (3.13)):

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2.$$

Hence

$$\frac{\partial}{\partial t}|\nabla R|^2 = \frac{\partial}{\partial t}\left(g^{ij}\nabla_i R \nabla_j R\right) = 2Ric(\nabla R, \nabla R) + 2\langle\nabla R, \nabla\left(\Delta R + 2|Ric|^2\right)\rangle.$$

Now we recall the Bochner-Weitzenböck formula (see [20], Lemma 3.4 on page 27):

$$\Delta|\nabla R|^2 = 2|\nabla\nabla R|^2 + 2\langle\nabla R, \Delta\nabla R\rangle + 2Ric(\nabla R, \nabla R).$$

Comparing the terms, the lemma follows. □

Lemma 5.4.2. *Let $(M^n, g(t))$, with $t \in [0, T)$, be a solution of the Ricci flow such that $R(0) > 0$. Then the following holds for all $t \in [0, T)$:*

$$\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) = \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{|\nabla R|^2}{R}\right)\right|^2 - 2\frac{|\nabla R|^2}{R^2}|Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle. \tag{5.25}$$

Proof. We already know that $R > 0$ is preserved. Now, using the evolution equation for R and Lemma 5.4.1, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) =& \frac{1}{R}\left[\Delta|\nabla R|^2 - 2|\nabla\nabla R|^2 + 4\langle\nabla R, \nabla|Ric|^2\rangle\right] \\ &- \frac{|\nabla R|^2}{R^2}\left(\Delta R + 2|Ric|^2\right).\end{aligned}$$

Now we recall that for any smooth functions u and v, we have

$$\Delta\left(\frac{u}{v}\right) = \frac{\Delta u}{v} - \frac{u\Delta v}{v^2} - \frac{2}{v^2}\langle\nabla u, \nabla v\rangle + \frac{2u}{v^3}|\nabla v|^2.$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) =& \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2\left(\frac{|\nabla R|^4}{R^3} - \frac{\langle\nabla|\nabla R|^2, \nabla R\rangle}{R^2} + \frac{|\nabla\nabla R|^2}{R}\right) \\ &- \frac{2}{R^2}|\nabla R|^2|Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle \\ =& \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left(\frac{|\nabla R|^4}{R^4} - \frac{\langle\nabla|\nabla R|^2, \nabla R\rangle}{R^3} + \frac{|\nabla\nabla R|^2}{R^2}\right) \\ &- \frac{2}{R^2}|\nabla R|^2|Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle \\ =& \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - \frac{2}{R^2}|\nabla R|^2|Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle.\end{aligned}$$

□

Lemma 5.4.3. *If $(M^n, g(t))$ is a solution of the Ricci flow, then*

$$\frac{\partial}{\partial t}R^2 = \Delta R^2 - 2|\nabla R|^2 + 4R|Ric|^2 \tag{5.26}$$

and

$$\frac{\partial}{\partial t}|Ric|^2 = \Delta|Ric|^2 - 2|\nabla Ric|^2 + 4R^{il}R^{jk}R_{ijkl}. \tag{5.27}$$

Proof. The first equation follows from the evolution equation of R since

$$\begin{aligned}\frac{\partial}{\partial t}R^2 =& 2R\left(\Delta R + 2|Ric|^2\right) \\ =& \Delta R^2 - 2|\nabla R|^2 + 4R|Ric|^2.\end{aligned}$$

On the other hand, using the evolution equation for the Ricci tensor (3.14), we get

$$\begin{aligned}\frac{\partial}{\partial t}|Ric|^2 =&\frac{\partial}{\partial t}(g^{ij}g^{kl}R_{ik}R_{jl}) = 2R_{ij}g^{kl}R_{ik}R_{jl} + 2g^{ij}R_{kl}R_{ik}R_{jl}\\ &+ g^{ij}g^{kl}(\Delta_L Ric)_{ik}R_{jl} + g^{ij}g^{kl}R_{ik}(\Delta_L Ric)_{jl}\\ &= 4tr_g(Ric^3) + 2\langle Ric, \Delta_L Ric\rangle\\ &= \Delta|Ric|^2 - 2|\nabla Ric|^2 + 4R^{il}R^{jk}R_{ijkl},\end{aligned}$$

since the term $4tr_g(Ric^3)$ is cancelled by the last term of equation (3.14). □

Corollary 5.4.3.1. *If $(M^3, g(t))$ is a solution of the Ricci flow on a 3-manifold, then*

$$\begin{aligned}\frac{\partial}{\partial t}\left(|Ric|^2 - \frac{1}{3}R^2\right) =&\Delta\left(|Ric|^2 - \frac{1}{3}R^2\right) - 2\left(|\nabla Ric|^2 - \frac{1}{3}|\nabla R|^2\right)\\ &- 8tr_g(Ric^3) + \frac{26}{3}R|Ric|^2 - 2R^3.\end{aligned} \tag{5.28}$$

Proof. In dimension 3, we may use the fact that the Weyl tensor vanishes (see Chapter 2) to write the Riemann tensor in terms of the Ricci tensor. Then, equation (5.27) can be written as

$$\frac{\partial}{\partial t}|Ric|^2 = \Delta|Ric|^2 - 2|\nabla Ric|^2 - 2R^3 - 8tr_g(Ric^3) + 10R|Ric|^2.$$

The result follows from this. □

In order to prove our main result of this section, we would like to show that the term $\frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle$ on the evolution equation of $\frac{|\nabla R|^2}{R}$ (equation (5.25)) can be controlled. In fact, this term will be eliminated by computing the evolution of $\dfrac{|\nabla R|^2}{R} + |Ric|^2 - \dfrac{1}{3}R^2$. This is what we shall do in the next results.

Lemma 5.4.4. *In dimension $n = 3$, we have*

$$\left(1 - \frac{1}{37}\right)\left|\nabla Ric\right|^2 - \frac{1}{3}\left|\nabla R\right|^2 \geq 0. \tag{5.29}$$

Proof. First, we define a (3,0)-tensor X by

$$X_{ijk} \coloneqq \nabla_i R_{jk} - \frac{1}{3}g_{jk}\nabla_i R$$

and a (1,0)-tensor Y by

$$Y_k := g^{ij} X_{ijk}.$$

Observe that using the second contracted Bianchi identity, we get

$$Y_k = g^{ij} X_{ijk} = g^{ij} \nabla_i R_{jk} - \frac{1}{3} g^{ij} g_{jk} \nabla_i R = \frac{1}{2} \nabla_k R - \frac{1}{3} \delta_i^k \nabla_i R.$$

Then

$$Y_k = \frac{\nabla_k R}{6}$$

and

$$|Y|^2 = \frac{1}{36} |\nabla R|^2.$$

In any dimension n, we have the following estimate for a (2,0)-tensor Z :

$$|Z|^2 \geq \frac{1}{n} \left(tr_g Z \right)^2 .$$

Hence

$$\begin{aligned}
\frac{1}{3}|Y|^2 \leq |X|^2 &= g^{ip} g^{js} g^{kl} X_{ijk} X_{psl} \\
&= g^{ip} g^{js} g^{kl} \left(\nabla_i R_{jk} - \frac{1}{3} g_{jk} \nabla_i R \right) \left(\nabla_p R_{sl} - \frac{1}{3} g_{sl} \nabla_p R \right) \\
&= g^{ip} g^{js} g^{kl} \left(\nabla_i R_{jk} \nabla_p R_{sl} - \frac{1}{3} g_{jk} \nabla_i R \nabla_p R_{sl} - \frac{1}{3} g_{sl} \nabla_p R \nabla_i R_{jk} + \frac{1}{9} g_{jk} g_{sl} \nabla_i R \nabla_p R \right) \\
&= |\nabla Ric|^2 - \frac{2}{3} |\nabla R|^2 + \frac{3}{9} |\nabla R|^2 \\
&= \left|\nabla Ric\right|^2 - \frac{1}{3} \left|\nabla R\right|^2.
\end{aligned}$$

Then we get

$$\left|\nabla Ric\right|^2 \geq \frac{1}{3}(1 + \frac{1}{36}) \left|\nabla R\right|^2 = \frac{37}{108} \left|\nabla R\right|^2 \tag{5.30}$$

and the result follows.

□

Corollary 5.4.4.1. *If $(M^3, g(t))$ is a solution of the Ricci flow on a 3-manifold, then*

$$\begin{aligned} \frac{\partial}{\partial t}\left(|Ric|^2 - \frac{1}{3}|R|^2\right) \leq& \Delta\left(|Ric|^2 - \frac{1}{3}|R|^2\right) - \frac{2}{37}\left|\nabla Ric\right|^2 \\ & - 8tr_g(Ric^3) + \frac{26}{3}R|Ric|^2 - 2R^3. \end{aligned} \tag{5.31}$$

Proof. This is a direct result. We just have to use Corollary 5.4.3.1 and substitute the inequality (5.29) into equation (5.28). □

Now, let us consider equation (5.25) again. On a 3-manifold with positive Ricci curvature, we have $|Ric| \leq R$. Also, the following holds:

$$\left|\nabla|Ric|^2\right| \leq 2\left|\nabla Ric\right||Ric|.$$

Then, if we consider (5.30) and the term $\frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle$ on (5.25), by applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle \leq \frac{4}{R}\left|\nabla R\right|\left|\nabla|Ric|^2\right| &\leq 8\left|\nabla R\right|\left|\nabla Ric\right|\frac{|Ric|}{R} \\ &\leq 8\sqrt{\frac{108}{37}}\left|\nabla Ric\right|^2 \leq 8\sqrt{3}\left|\nabla Ric\right|^2. \end{aligned}$$

Now we consider

$$V := \frac{|\nabla R|^2}{R} + \frac{37}{2}(8\sqrt{3}+1)\left(|Ric|^2 - \frac{1}{3}R^2\right),$$

which will provide an upper bound for $\frac{|\nabla R|^2}{R}$.

Lemma 5.4.5. *If $(M^3, g(t))$ is a solution of the Ricci flow on a 3-manifold whose Ricci curvature is initially positive, then*

$$\begin{aligned} \frac{\partial}{\partial t}V \leq& \Delta V - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - 2\frac{|Ric|^2}{R^2}\left|\nabla R\right|^2 - \left|\nabla Ric\right|^2 \\ & + \frac{37}{2}(8\sqrt{3}+1)\left(\frac{26}{3}R|Ric|^2 - 8tr_g(Ric^3) - 2R^3\right). \end{aligned}$$

Proof. Using Lemma 5.4.4, we have that

$$\left|\nabla Ric\right|^2 - \frac{1}{3}\left|\nabla R\right|^2 \geq \frac{1}{37}\left|\nabla Ric\right|^2.$$

By Corollary 5.4.4.1, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(|Ric|^2 - \frac{1}{3}|R|^2\right) \leq& \Delta\left(|Ric|^2 - \frac{1}{3}|R|^2\right) - \frac{2}{37}\left|\nabla Ric\right|^2 \\ &- 8tr_g(Ric^3) + \frac{26}{3}R|Ric|^2 - 2R^3,\end{aligned}$$

and by Lemma 5.4.2 we get

$$\frac{\partial}{\partial t}\left(\frac{|\nabla R|^2}{R}\right) = \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{|\nabla R|^2}{R}\right)\right|^2 - 2\frac{|\nabla R|^2}{R^2}|Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle.$$

Hence, we have

$$\begin{aligned}\frac{\partial}{\partial t}V \leq& \Delta\left(\frac{|\nabla R|^2}{R}\right) - 2R\left|\nabla\left(\frac{\nabla R}{R}\right)\right|^2 - 2\frac{|\nabla R|^2}{R^2}|Ric|^2 + 4\frac{\langle\nabla R, \nabla|Ric|^2\rangle}{R} \\ &+ \frac{37}{2}(8\sqrt{3}+1)\left[\Delta(|Ric|^2 - \frac{1}{3}R^2) - \frac{2}{37}|\nabla Ric|^2 - 8tr_g(Ric^3) + \frac{26}{3}R|Ric|^2 - 2R^3\right] \\ &= \Delta V - \frac{2|Ric|^2}{R^2}|\nabla R|^2 - (8\sqrt{3}+1)|\nabla Ric|^2 + \frac{4}{R}\langle\nabla R, \nabla|Ric|^2\rangle \\ &\leq \nabla V - \frac{2|Ric|^2}{R^2}|\nabla R|^2 - (8\sqrt{3}+1)|\nabla Ric|^2 + 8\sqrt{3}|\nabla Ric|^2,\end{aligned}$$

so the lemma follows. □

We will show that the term

$$W := \frac{26}{3}R|Ric|^2 - 8tr_g(Ric^3) - 2R^3,$$

which appears in the evolution of V is small when the metric is close to an Einstein metric because of Corollary 5.4.3.1.

Lemma 5.4.6. *On a 3-manifold of positive Ricci curvature, one has*

$$W \leq \frac{50}{3}R\left(|Ric|^2 - \frac{1}{3}R^2\right).$$

Proof. First, let $X := -8\langle Ric - \frac{1}{3}Rg, Ric^2\rangle$. Then $X = \frac{8}{3}R|Ric|^2 - 8tr_g(Ric^3)$ and W can be written as

$$W = X + 6R(|Ric|^2 - \frac{1}{3}R^2).$$

Now let us define a (2,0)-tensor Y by $Y := Ric^2 - \frac{1}{9}R^2g$. We note that

$$Y_{ij} := R_i^k R_{kj} - \frac{1}{9}R^2 g_{ij} = g^{kl}\left(R_{ik} - \frac{1}{3}Rg_{ik}\right)\left(R_{jl} + \frac{1}{3}Rg_{jl}\right).$$

Then we may use Cauchy-Schwarz to estimate X. Since

$$\begin{aligned}\langle Ric - \frac{1}{3}Rg, Ric^2 - \frac{1}{9}R^2g\rangle &= tr_g(Ric^3) - \frac{1}{3}R|Ric|^2 - \frac{1}{9}R^2tr_g(Ric) + \frac{1}{27}R^33 \\ &= tr_g(Ric^3) - \frac{1}{3}R|Ric|^2 - \frac{1}{9}R^3 + \frac{1}{9}R^3 = -\frac{X}{8},\end{aligned}$$

we have

$$X \leq 8|Ric - \frac{1}{3}Rg|^2|Ric + \frac{1}{3}Rg|.$$

Now since $Ric > 0$, we get $X \leq \frac{32}{3}R\left(|Ric|^2 - \frac{1}{3}R^2\right)$. Finally,

$$\begin{aligned}W &= X + 6R\left(|Ric|^2 - \frac{1}{3}R^2\right) \\ &\leq \left(\frac{32}{3} + \frac{18}{3}\right)R\left(|Ric|^2 - \frac{1}{3}R^2\right) = \frac{50}{3}R\left(|Ric|^2 - \frac{1}{3}R^2\right).\end{aligned}$$

□

We now introduce a result that will be used later and will be needed for the main result of this section.

Lemma 5.4.7. *Let $(M^n, g(t))$, with $t \in [0,T)$, be a solution of the Ricci flow, where M^n is a closed manifold. If there are $t_0 \geq 0$ and $\rho > 0$ such that*

$$\inf_{p\in M^n} R(p, t_0) = \rho,$$

then $g(t)$ becomes singular in finite time.

Proof. The proof is a straightforward application of the Maximum Principle. In fact, from equation (3.13), we have

$$\frac{\partial}{\partial t}R = \Delta R + 2|Ric|^2 \geq \Delta R + \frac{2}{n}R^2.$$

Then we may consider the solution of the corresponding ODE

$$r(t) = \frac{2}{n}r^2,$$
$$r(t_0) = \rho > 0,$$

which is given by

$$r(t) = \frac{\rho n}{n - 2\rho(t - t_0)}.$$

Since $R(p, t_0) \geq \rho$, it follows from the maximum principle (Theorem (4.2.14)) that for all $p \in M^n$

$$R(p, t) \geq \inf_{p \in M} R(p, t) \geq r(t)$$

as long as the solution $r(t)$ exists. However, if we let $\bar{t} = t_0 + \dfrac{n}{2\rho}$, then

$$\lim_{t \to \bar{t}} r(t) = \infty,$$

then

$$\lim_{t \to \bar{t}} R(t) = \infty.$$

Hence, $g(t)$ becomes singular for some $t \leq \bar{t}$. □

Finally, we are ready to prove our theorem.

Theorem 5.4.8. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a closed 3-manifold with $g(0) = g_0$. If $Ric(g_0) > 0$, then there exist constants $\overline{\beta}, \overline{\delta} > 0$ depending only on g_0 such that for any $\beta \in [0, \overline{\beta}]$, there exists a constant $C > 0$ depending only on β and g_0 such that*

$$\frac{|\nabla R|^2}{R^3} \leq \beta R^{-\frac{\overline{\delta}}{2}} + CR^{-2}.$$

Remark. *Here, the left-hand side is scale invariant. On the other hand, the right-hand side is small when the scalar curvature is large.*

Proof. We may use Lemmas 5.4.5 and 5.4.6 to write

$$\begin{aligned}\frac{\partial}{\partial t}V &\leq \Delta V - |\nabla Ric|^2 + \frac{37}{2}(8\sqrt{3}+1)\frac{50}{3}R\left(|Ric|^2 - \frac{1}{3}R^2\right) \\ &= \Delta V - |\nabla Ric|^2 + \frac{7400\sqrt{3}+925}{3}R\left(|Ric|^2 - \frac{1}{3}R^2\right).\end{aligned}$$

Now, using equation (5.23), we get a slightly better estimate

$$\frac{\partial}{\partial t}V \leq \Delta V - |\nabla Ric|^2 + CR^{3-2\gamma},$$

where C and $\gamma \equiv \frac{\overline{\delta}}{2}$ depend only on g_0. It follows directly from (3.13) that

$$\frac{\partial}{\partial t}R^{2-\gamma} = \Delta(R^{2-\gamma}) - (2-\gamma)(1-\gamma)R^{-\gamma}\left|\nabla R\right|^2 + 2(2-\gamma)R^{1-\gamma}|Ric|^2.$$

Now let $\overline{\beta}$ be such that

$$0 < \overline{\beta} \leq \frac{(R_{min}(0))^{\gamma}}{3(2-\gamma)(1-\gamma)}$$

and recall that

$$\left|\nabla R\right|^2 \leq 3\left|\nabla Ric\right|^2.$$

Then, for any $\beta \in [0, \overline{\beta}]$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(V - \beta R^{2-\gamma}\right) \leq &\Delta\left(V - \beta R^{2-\gamma}\right) + \left[\beta(2-\gamma)(1-\gamma)R^{-\gamma}\left|\nabla R\right|^2 - |\nabla Ric|^2\right] \\ &+ CR^{3-2\gamma} - 2\beta(2-\gamma)R^{1-\gamma}|Ric|^2.\end{aligned}$$

Observe that the second term is non-positive because of the constants that we have chosen. For the rest of the terms, we see that

$$CR^{3-2\gamma} - 2\beta(2-\gamma)R^{1-\gamma}|Ric|^2 \leq CR^{3-2\gamma} - \overline{C}R^{3-\gamma},$$

where $\overline{C}$ is a constant. For R large enough this term is dominated by the second term, which is negative. Therefore we can get a uniform upper bound C_1 for it. Hence, we have

$$\frac{\partial}{\partial t}\left(V - \beta R^{2-\gamma}\right) \leq \Delta\left(V - \beta R^{2-\gamma}\right) + C_1$$

From the maximum principle for the scalar case (Theorem (4.1.2)) we get

$$V - \beta R^{2-\gamma} \leq C_1 t + C_2.$$

On the other hand, (3.13) implies that

$$\frac{\partial}{\partial t}R \geq \Delta R + \frac{2}{3}R^2.$$

From Lemma 5.4.7, this means that there is a time $T < \infty$ at which the solution becomes singular. Thus,

$$\frac{|\nabla R|^2}{R} \leq V \leq \beta R^{2-\gamma} + C_1 T + C_2.$$

Let $C := C_1 T + C_2$. This proves the theorem. □

5.5 Long-Time Existence and Finite Time Blow Up

We already know that on a compact manifold M^n with an initial metric g_0, there exists a unique solution $g(t)$ of the Ricci flow with $g(0) = g_0$ on a short time interval. Hence, there must be a maximal time interval $[0, T)$, with $0 < T \leq \infty$, on which the solution exists. In this section, we would like to understand what happens if $T < \infty$. In fact, we shall prove that if the maximum curvature remains bounded, then $T = \infty$.

First, we need to obtain some derivative estimates of the curvature, due to Bernstein, Bando and Shi (see, for example, [26] and [27]).

Lemma 5.5.1. *If $(M^n, g(t))$ is a solution of the Ricci flow, then we have the following evolution equation for the square of the norm of its curvature tensor*

$$\frac{\partial}{\partial t}|Rm|^2 = \Delta|Rm|^2 - 2\left|\nabla Rm\right|^2 + 4g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}),$$

where $B_{ijkl} = -R^q_{pij}R^p_{qlk}$. *In particular, we have*

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 - 2\,|\nabla Rm|^2 + C|Rm|^3,$$

where C *depends only on* n.

Proof. We know that the (4,0)-Riemann curvature tensor evolves by

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\\ &- (R^p_i R_{pjkl} + R^p_j R_{ipkl} + R^p_k R_{ijpl} + R^l_p R_{ijkp}).\end{aligned}$$

Then we can check that

$$\begin{aligned}\frac{\partial}{\partial t}|Rm|^2 =& \frac{\partial}{\partial t}\left(g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}R_{ijkl}\right) = 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\left[\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\right]\\ &- 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}(R^m_i R_{mjkl} + R^m_j R_{imkl} + R^m_k R_{ijml} + R^m_l R_{ijkm})\\ &+ 2g^{sj}g^{pk}g^{ql}(g^{tr}g^{im}R_{tm})R_{rspq}R_{ijkl} + 2g^{ri}g^{pk}g^{ql}(g^{ts}g^{jm}R_{tm})R_{rspq}R_{ijkl}\\ &+ 2g^{ri}g^{sj}g^{ql}(g^{tp}g^{km}R_{tm})R_{rspq}R_{ijkl} + 2g^{ri}g^{pk}g^{sj}(g^{qt}g^{lm}R_{tm})R_{rspq}R_{ijkl}.\end{aligned}$$

Since the last four terms cancel out with

$$-2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}(R^m_i R_{mjkl} + R^m_j R_{imkl} + R^m_k R_{ijml} + R^m_l R_{ijkm}),$$

we get

$$\frac{\partial}{\partial t}|Rm|^2 = \frac{\partial}{\partial t}\left(g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}R_{ijkl}\right) = 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\left[\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk})\right].$$

Since

$$\Delta|Rm|^2 = 2g^{ri}g^{sj}g^{pk}g^{ql}R_{rspq}\Delta R_{ijkl} + 2|\nabla Rm|^2,$$

the lemma follows. □

Corollary 5.5.1.1. *If* $(M^n, g(t))$, *with* $t \in [0, T)$, *is a solution of the Ricci flow on a* *compact manifold and*

$$Z(t) := \sup_{p \in M^n} |Rm(p,t)|_{g(p,t)}\,,$$

then there exists $C > 0$ depending only on the dimension n such that

$$Z(t) \leq 2Z(0)$$

*for all t within $0 \leq t < min\{T, \frac{C}{Z(0)}\}$. This result is known as the **Doubling-Time Estimate**.*

Proof. By Lemma 5.5.1, $Z(t)$ is Lipschitz on t and satisfies

$$\frac{dZ}{dt} \leq \frac{CZ^3}{2Z} = \frac{C}{2}Z^2,$$

where C depends only on the dimension n. To see this, just drop the Laplacian and the gradient terms on the lemma (we can do this because $Z(t)$ is defined as the supremum of Rm over the manifold, for each time t). Then we get

$$Z(t) \leq \frac{1}{\frac{1}{Z(0)} - \frac{C}{2}t}$$

as long as $t \in [0, T)$ satisfies $t < \frac{2}{CZ(0)}$. If we choose C as $\frac{1}{C}$, then we have the result. □

Corollary 5.5.1.2. *If (M^n, g_0) is a Riemannian manifold such that $|Rm(g_0)|_{g_0} \leq K$, then the unique solution of the Ricci flow with $g(0) = g_0$ exists at least for $t \in [0, \frac{C}{K}]$, where $C > 0$ is a constant depending only on n.*

Proof. In order to obtain this result, we just have to combine the Doubling Time Estimate and Theorem 5.5.4. □

Theorem 5.5.2. *Let $(M^n, g(t))$ be a solution of the Ricci flow, where M^n is a closed manifold. For any $\alpha > 0$ and $m \in \mathbb{N}$, suppose there exists a constant $K > 0$ such that*

$$|Rm(p,t)|_{g(p,t)} \leq K,$$

$\forall p \in M^n$ and $t \in [0, \frac{\alpha}{K}]$, then there exists a positive constant C_m depending only on m, n and $max\{\alpha, 1\}$ such that

$$|\nabla^m Rm(p,t)|_{g(p,t)} \leq \frac{C_m K}{t^{\frac{m}{2}}},$$

$\forall p \in M^n$ and $t \in (0, \frac{\alpha}{K}]$.

Proof. Although this result is essential to our goal in this dissertation, its proof is very long and technical. In order to provide a better reading experience, we provide the proof of this theorem in Appendix B. □

Remark. *We observe that these estimates do not hold when $t = 0$. This is actually expected since bounds on an arbitrary curvature tensor do not necessarily tell us anything about its derivatives. However, the Bando-Bernstein-Shi (BBS) estimates (Theorem* (5.5.2)*) show that after the flow starts, the derivatives of the curvature tensor instantly begin to be brought under control.*

Corollary 5.5.2.1. *Let $(M^n, g(t))$ be a solution of the Ricci flow, where M^n is compact. If there are $\beta > 0$ and $K > 0$ such that*

$$|Rm(p,t)| \leq K,$$

$\forall p \in M$ and $\forall t \in [0, T]$, where $T > \frac{\beta}{K}$, then there exists, for each $m \in \mathbb{N}$, a constant C_m depending only on m, n and $min\{\beta, 1\}$ such that

$$|\nabla^m Rm| \leq C_m K^{1+\frac{m}{2}},$$

$\forall p \in M$ and $\forall t \in [\frac{min\{\beta,1\}}{K}, T]$.

Proof. First, let $\beta_0 := min\{\beta, 1\}$. We fix $t_0 \in [\frac{\beta_0}{K}, T]$, set $T_0 = t_0 - \frac{\beta_0}{K}$ and $\bar{t} = t - T_0$. Now, let $\bar{g}(\bar{t})$ solve the Ricci flow equation with $\bar{g}(0) = g(T_0)$. By the uniqueness of solutions to the Ricci flow, given $\bar{t} \in [0, \frac{\beta}{K}]$, we have $\bar{g}(\bar{t}) = g(t)$. Hence, $|\overline{Rm}(p, \bar{t})|_{\bar{g}} \leq K$ by the hypothesis, for all $p \in M^n$ and $\bar{t} \in [0, \frac{\beta}{K}]$.

Using theorem 5.5.2, let $\alpha = \beta_0$. Then we get constants $\overline{C_m}$ depending only on m and n such that

$$|\overline{\nabla}^m \overline{Rm}(p, \bar{t})|_{\bar{g}} \leq \frac{\overline{C}_m K}{\bar{t}^{\frac{m}{2}}},$$

$\forall p \in M$ and $\bar{t} \in (0, \frac{\beta_0}{K}]$. When $\bar{t} \in [\frac{\beta_0}{2K}, \frac{\beta_0}{K}]$, we have

$$\bar{t}^{\frac{m}{2}} \geq \beta_0^{\frac{m}{2}} 2^{\frac{-m}{2}} K^{\frac{-m}{2}}.$$

Then, if $\bar{t} = \frac{\beta_0}{K}$, we have

$$|\nabla^m Rm(p, t_0)|_g \leq \left(\frac{2^{\frac{m}{2}} \overline{C}_m}{\beta_0^{\frac{m}{2}}} \right) K^{1+\frac{m}{2}},$$

$\forall p \in M^n$. Since t_0 is arbitrary in $[\frac{\beta_0}{K}, T]$, our proof is completed. □

Before we can state another gradient estimate, we need a few results. Here, we will work with a half-open interval $[0, T)$ since the application we are interested in is to help us understand what are the obstacles to long-time existence of the Ricci flow.

Lemma 5.5.3. *Let M^n be a closed manifold. For $0 \leq t < T \leq \infty$, let $g(t)$ be a one-parameter family of metrics on M^n depending smoothly on space and time. If there exists a constant $C < \infty$ such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(p,t) \right|_{g(t)} dt \leq C,$$

$\forall p \in M^n$, then

$$e^{-C} g(p, 0) \leq g(p, t) \leq e^C g(p, 0),$$

$\forall p \in M$ and $t \in [0, T)$. Moreover, as $t \nearrow T$, $g(t)$ converges uniformly to a continuous metric $g(T)$ such that

$$e^{-C} g(p, 0) \leq g(p, T) \leq e^C g(p, 0),$$

$\forall p \in M^n$.

Proof. Let $p \in M^n$ and $t_0 \in [0, T)$ be arbitrary. Also, consider an arbitrary vector $v \in T_pM^n$. Since $|A(u,u)| \leq |A|_g$ for any 2-tensor A and any unit vector u, we get

$$\begin{aligned}
\left|\log\left(\frac{g(p,t_0)(v,v)}{g(p,0)(v,v)}\right)\right| &= \left|\int_0^{t_0} \frac{\partial}{\partial t}\log(g(p,t)(v,v))dt\right| \\
&= \left|\int_0^{t_0} \frac{\frac{\partial}{\partial t}g(p,t)(v,v)}{g(p,t)(v,v)}dt\right| \\
&\leq \int_0^{t_0} \left|\frac{\frac{\partial}{\partial t}g(p,t)(v,v)}{|v|_g^2}\right| dt \\
&= \int_0^{t_0} \left|\frac{\partial}{\partial t}g(p,t)\left(\frac{v}{|v|}, \frac{v}{|v|}\right)\right| dt \leq \int_0^{t_0} \left|\frac{\partial}{\partial t}g(p,t)\right| dt \leq C.
\end{aligned}$$

Thus, the uniform bounds follow from considering the exponential of the inequality above. In particular, this shows that the metrics $g(t)$ are all uniformly equivalent. Therefore, we have

$$\int_0^T \left|\frac{\partial}{\partial t}g(p,t)\right|_{g(0)} dt \leq \overline{C},$$

for some $\overline{C} > 0$. Observe that now we are taking the norm with respect to the fixed metric $g(0)$. Let us define

$$g(p,T) := g(p,0) + \int_0^T \frac{\partial}{\partial t}g(p,t)dt.$$

This integral is well defined because our family of metrics is smooth and the bound above tells us that the integrand is absolutely integrable with respect to the norm induced by $g(0)$. Thus

$$|g(p,t) - g(p,T)|_{g(0)} \leq \int_0^T \left|\frac{\partial}{\partial t}g(p,t)\right|_{g(0)} dt \to 0$$

as $t \to T$ for each fixed $p \in M$. The convergence above is uniform due to the compactness of M. Therefore $g(T)$ is continuous. The last bound of the lemma follows directly just by taking the limit on $e^{-C}g(p,0) \leq g(p,t) \leq e^C g(p,0)$. This shows that $g(T)$ is positive definite and, therefore, $g(t)$ converges to a continuous Riemannian metric $g(T)$, uniformly equivalent to $g(0)$. $\square$

Corollary 5.5.3.1. *Let $(M^n, g(t))$ be a solution of the Ricci flow. If there exists $K > 0$ such that $|Ric| \leq K$ on $[0, T]$, then*

$$e^{-2KT} g(p, 0) \leq g(p, t) \leq e^{2kT} g(p, 0),$$

$\forall p \in M^n$ and $\forall t \in [0, T]$.

Proof. Just remember that $\frac{\partial}{\partial t} g = -2Ric$. Then, for an arbitrary $t_0 \in [0, T)$, we have

$$\int_0^{t_0} \left| \frac{\partial}{\partial t} g(p, t) \right| dt = \int_0^{t_0} |-2Ric(g(t))| \, dt \leq \int_0^{t_0} 2K dt \leq 2KT.$$

□

We have just shown that there is a limit metric $g(T)$, which is continuous. Now, in our result on the long-time existence of the Ricci flow, we will need to show that this limit metric is actually smooth. To do so, we need to make sure that the spatial derivatives of $g(t)$ are controlled when we are approaching the time T. This is the content of our next two results, that follow from Theorem 5.5.2.

Proposition 5.5.1. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a compact manifold with a fixed background metric $\overline{g}$ and a connection $\overline{\nabla}$. If there exists $K > 0$ such that*

$$|Rm(p, t)|_g \leq K$$

$\forall p \in M^n$ and $\forall t \in [0, T]$, then for every $m \in \mathbb{N}$, there exists a constant C_m that depends on $m, n, K, T, g_0 = g(0)$ and the pair $(\overline{g}, \overline{\nabla})$ such that

$$\left| \overline{\nabla}^m g(p, t) \right|_{\overline{g}} \leq C_m,$$

$\forall p \in M^n$ and $\forall t \in [0, T]$.

Proof. Similarly to Theorem (5.5.2), we provide a proof for this proposition in Appendix B. □

Corollary 5.5.3.2. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a compact manifold* *with a fixed background metric $\overline{g}$ and a connection $\overline{\nabla}$. If there exists $K > 0$ such that*

$$|Rm(p,t)|_g \leq K$$

$\forall p \in M^n$ and $\forall t \in [0,T]$, then for every $m \in \mathbb{N}$, there exists a constant C'_m that depends on $m, n, K, T, g_0 = g(0)$ and the pair $(\overline{g}, \overline{\nabla})$ such that

$$\left|\overline{\nabla}^m Ric(p,t)\right|_{\overline{g}} \leq C'_m,$$

$\forall p \in M^n$ and $\forall t \in [0,T]$.

Proof. This is a result established in the proof of Proposition 5.5.1. In order to check the proof in details, see Appendix B. □

Now we state and prove a theorem that shows that the only obstacle to long-time existence of the Ricci flow is the curvature becoming unbounded.

Theorem 5.5.4. *If g_0 is a smooth metric on a compact manifold M^n, the Ricci flow with $g(0) = g_0$ has a unique solution $g(t)$ on a maximal time interval $t \in [0,T)$, with $T \leq \infty$. Moreover, if $T < \infty$, then*

$$\lim_{t \nearrow T} \left(\sup_{p \in M^n} |Rm(p,t)| \right) = \infty. \tag{5.32}$$

Proof. Let us define

$$Z(t) \coloneqq \sup_{p \in M^n} |Rm(p,t)|.$$

We already know that there exists a unique solution $g(t)$ of the Ricci flow satisfying the initial condition $g(0) = g_0$ on a short time interval $[0, \varepsilon)$. First, we will prove the claim that the lim sup of $Z(t)$ goes to ∞.

Suppose that the solution exists on the maximal finite time interval $[0,T)$, with $T < \infty$. We claim that

$$\sup_{0 \leq t < T} Z(t) = \infty. \tag{5.33}$$

Suppose by contradiction that there is a constant $K > 0$ such that

$$\sup_{0 \le t < T} Z(t) \le K.$$

We will show that, given this condition, we are able to define the flow beyond T. Consider local coordinates $\{x^i\}$ on an open set $U \subset M^n$ around an arbitrary point $p \in M^n$. Let $\tau \in (0, T)$ be also arbitrary. Using Lemma 5.5.3, we get a continuous limit metric $g(T)$ that can be written as

$$g_{ij}(p, T) = g_{ij}(p, \tau) - 2 \int_{\tau}^{T} R_{ij}(p, t) dt.$$

Let α be any multi-index with $|\alpha| = m \in \mathbb{N}$. Then it follows from Proposition 5.5.1 and Corollary 5.5.3.2 that $\frac{\partial^m}{\partial x^\alpha} g_{ij}$ and $\frac{\partial^m}{\partial x^\alpha} R_{ij}$ are uniformly bounded on $U \times [0, T)$. Thus

$$\left(\frac{\partial^m}{\partial x^\alpha} g_{ij} \right)(p, T) = \left(\frac{\partial^m}{\partial x^\alpha} g_{ij} \right)(p, \tau) - 2 \int_{\tau}^{T} \left(\frac{\partial^m}{\partial x^\alpha} R_{ij} \right)(p, t) dt.$$

This shows that our limit metric is smooth. Moreover,

$$\left| \left(\frac{\partial^m}{\partial x^\alpha} g_{ij} \right)(p, T) - \left(\frac{\partial^m}{\partial x^\alpha} g_{ij} \right)(p, \tau) \right| \le C(T - \tau)$$

for some constant $C < \infty$. So $g(\tau) \to g(T)$ uniformly in any C^m norm as $\tau \nearrow T$.

Due to the smoothness of $g(T)$, we know that there is a unique solution of the Ricci flow, $\overline{g}(t)$, with $\overline{g}(0) = g(T)$ on a short time interval $[0, \overline{\varepsilon})$. Since $g(\tau) \to g(T)$ smoothly, we have that

$$\tilde{g}(t) := \begin{cases} g(t), & t \in [0, T) \\ \overline{g}(t - T), & t \in [T, T + \overline{\varepsilon}) \end{cases}$$

is a solution of the Ricci flow with $g(0) = g_0$. This is a contradiction with the fact that T is maximal. Hence, if $T < \infty$, then (5.33) holds.

Suppose (5.32) is false, i.e., suppose there exists $K_0 < \infty$ such that

$$\lim_{t \nearrow T} \left(\sup_{p \in M^n} |Rm(p,t)| \right) \leq K_0.$$

Then, there exists a sequence $t_i \nearrow T$ such that $Z(t_i) \leq K_0$. Using the doubling-time estimate of Corollary 5.5.1.1, we get a constant $C = C(n) > 0$ such that

$$Z(t) \leq 2Z(t_i) \leq 2K_0,$$

$\forall t \in \left[t_i, \min\{T, t_i + \frac{C}{K_0}\}\right)$. Since $t_i \nearrow T$, we get a large enough index i_0 such that $t_{i_0} + \frac{C}{K_0} \geq T$. Thus

$$\sup_{t_{i_0} \leq t < T} Z(t) \leq 2K_0,$$

which contradicts the claim (5.33), previously established on this proof. This completes the proof of the theorem.

□

Corollary 5.5.4.1. *Any solution $(M^3, g(t))$ of the Ricci flow on a compact manifold whose Ricci curvature is initially positive exists on a maximal time interval $0 \leq t < T < \infty$ and the following holds*

$$\lim_{t \nearrow T} \left(\sup_{p \in M^3} |Rm(p,t)| \right) = \infty.$$

Proof. In order to obtain this result, we just have to combine Theorem 5.5.4 and Lemma 5.4.7. □

Now we obtain new global estimates for the curvature.

Proposition 5.5.2. *Let $(M^3, g(t))$ be a solution of the Ricci flow on a compact manifold whose Ricci curvature is initially positive. Then the solution becomes singular at some $T < \infty$. Moreover, it obeys the following a priori estimates:*

1. *Let $R_{min}(t) = \inf_{p \in M^3} R(p,t)$ and $R_{max}(t) = \sup_{p \in M^3} R(p,t)$. There exist positive constants*

C and α depending only on g_0 such that

$$\frac{R_{min}}{R_{max}} \geq 1 - \frac{C}{R_{max}^{\alpha}} \tag{5.34}$$

for all times $0 \leq t < T$. In particular, $\frac{R_{min}}{R_{max}} \to 1$ as $t \nearrow T$.

2. *For $p \in M^3$ and $t \in [0,T)$, let $\lambda(p,t) \geq \mu(p,t) \geq \nu(p,t)$ denote the eigenvalues of the curvature operator at (p,t). Then for any $\varepsilon \in (0,1)$, there exists $T_\varepsilon \in [0,T)$ such that*

$$\min_{p \in M^3} \nu(p,t) \geq (1-\varepsilon)\left[\max_{p \in M^3} \lambda(p,t)\right] > 0$$

for all times $t \in [T_\varepsilon, T)$. In particular, the solution eventually attains positive sectional curvature everywhere.

Proof. Lemma 5.4.7 tells us that the solution becomes singular at some time $T < \infty$. Additionally, since our dimension is $n = 3$, we know Ric completely determines Rm. Therefore, we have $c|Ric| \leq |Rm| \leq C|Ric|$ for some positive constants c and C. So it follows from Theorem 5.5.4 that

$$\lim_{t \nearrow T} \left(\sup_{p \in M^3} |Ric(p,t)| \right) = \infty. \tag{5.35}$$

Theorem 5.4.8 provides positive constants A, B and α such that

$$|\nabla R|^2 \leq \frac{1}{2} A^2 R_{max}^{3-2\alpha} + B^2 R_{max}.$$

Using equation (5.35), the fact that $|Ric|^2 \leq R^2$ and $R > 0$, we are able to find a $\tau \in [0,T)$ such that

$$|\nabla R| \leq A R_{max}^{\frac{3}{2}-\alpha}$$

$\forall t \in (\tau, T)$. Now, for $t \in (\tau, T)$ fixed, there exists $\overline{p}(t) \in M^3$ such that $R_{max}(t) = R(\overline{p}, t)$, because M^3 is compact.

Consider the geodesic ball $B(\overline{p}, L)$, where $L < \infty$ is given by

$$L(t) := \frac{1}{\varepsilon\sqrt{R_{max}(t)}},$$

for a given ε.

If γ is a minimizing geodesic from $\overline{p}$ to $p \in B(\overline{p}, L)$, we have the following estimate

$$R_{max} - R(p) \leq \int_\gamma |\nabla R|\, ds \leq A R_{max}^{\frac{3}{2}-\alpha} L \leq \frac{A}{\varepsilon} R_{max}^{1-\alpha}.$$

This gives us a lower bound on $B(\overline{p}, L)$:

$$R \geq R_{max}\left(1 - \frac{A}{\varepsilon}\left(\frac{1}{R_{max}}\right)^{\alpha}\right). \tag{5.36}$$

Since $R \to \infty$ as $t \to T$, it follows that there exists $\overline{t} \in (\tau, T)$ depending on A, α and ε such that

$$R \geq (1-\varepsilon)R_{max} \tag{5.37}$$

on $B(\overline{p}, L)$, $\forall t \in [\overline{t}, T)$. We shall prove that for $\varepsilon > 0$ small enough, $B(\overline{p}, L)$ is actually all of M^3. Since from (5.36) we already have our estimate on $B(\overline{p}, L)$, this will prove the first item.

From Corollary 5.3.1.1, we get a constant $\beta > 0$ that depends only on g_0 such that

$$Ric \geq 2\beta^2 R g,$$

then equation (5.37) implies that

$$Ric \geq 2\beta^2(1-\varepsilon)R_{\max} g \tag{5.38}$$

holds for all points of $B(\overline{p}, L)$. We first observe that if γ is a geodesic emanating from $\overline{p}$ with length $l(\gamma) \leq L$, then the estimate above holds for all points in γ. Moreover, it is a consequence from the proof of Bonnet-Myers' theorem (see, for example, [4]) that in a complete manifold (M^n, g), if $Ric \geq (n-1)Hg$ along a geodesic of length at least $\frac{\pi}{\sqrt{H}}$, where $H > 0$ is a constant, then this geodesic has conjugate points. Finally, we choose $\varepsilon \in (0,1)$ so that

$$\frac{\pi}{\beta\sqrt{(1-\varepsilon)R_{max}}} < \frac{1}{\varepsilon\sqrt{R_{max}}} = L,$$

which is always possible.

Now we suppose there exists $p_0 \in M^3$ such that $p_0 \notin B(\overline{p}, L)$. Then let γ be the minimizing geodesic starting at $\overline{p}$ and connecting it to p_0, so that $d(\overline{p}, p_0) = l(\gamma)$; whose existence is guaranteed because M^3 is compact, hence complete. Since p_0 is not in $B(\overline{p}, L)$, we know that γ intersects the boundary of $B(\overline{p}, L)$. Let p_1 be that point of intersection. Then the length of γ from $\overline{p}$ to p_1 is $L > \dfrac{\pi}{\beta\sqrt{(1-\varepsilon)R_{max}}}$ and, from our observation above and estimate (5.38), it follows that γ has a conjugate point within $B(\overline{p}, L)$. However, this contradicts the fact that γ is a minimizing geodesic from $\overline{p}$ to p_0. Therefore, we must have $p_0 \in B(\overline{p}, L)$. Since $p_0 \in M^3$ is arbitrary, it follows that $M^3 = B(\overline{p}, L)$ and this proves the first item.

Now, for our second item, we use Theorem 5.3.2. Since we have positive constants C and $\delta < 1$, depending only on g_0, such that

$$\nu \geq \lambda - C(\lambda + \mu + \nu)^{1-\delta},$$

$\forall p \in M^3$, we get the pointwise inequality

$$\frac{\nu}{\lambda} \geq 1 - 3CR^{-\delta} \geq 1 - 3CR_{min}^{-\delta}. \tag{5.39}$$

Then, let $p, q \in M^3$ and $1 > \eta > 0$ be given. By (5.35), (5.37) and (5.39), there exists

$T_\eta \in [\bar{t}, T)$ such that

$$\begin{aligned} \nu(p,t) &\geq (1-\eta)\lambda(p,t) \geq \frac{1-\eta}{3} R(p,t) \\ &\geq \frac{(1-\eta)^2}{3} R(q,t) \geq \frac{(1-\eta)^2}{3}[\lambda + 2(1-\eta)\lambda](q,t) \\ &\geq (1-\eta)^3 \lambda(q,t), \end{aligned}$$

$\forall t \in [T_\eta, T)$. If we take the infimum over $p \in M^3$ and the supremum over $q \in M^3$, the claim follows. □

In particular, we may conclude that $g(t)$ approaches an Einstein metric uniformly as $t \nearrow T$:

Corollary 5.5.4.2. *If $(M^3, g(t))$, with $t \in [0,T)$, is a solution of the Ricci flow on a compact manifold with strictly positive Ricci curvature at $t = 0$, then*

$$\lim_{t \nearrow T} \left(\sup_{p \in M^3} \frac{|\overset{\circ}{Ric}|^2}{R^2} \right) = 0.$$

Proof. Just apply the estimate of Theorem 5.3.2 in the form of equation (5.22). Thus, there are positive constants C and $\bar{\delta}$ such that

$$\frac{|\overset{\circ}{Ric}|^2}{R^2} \leq C R^{-\bar{\delta}} \leq R_{min}^{-\bar{\delta}}.$$

Since $R_{max}(t) \to \infty$ and $\frac{R_{min}}{R_{max}} \to 1$ as $t \nearrow T$, the result follows from Proposition (5.5.1). □

After having established long time existence for the Ricci flow and having showed that $g(t)$ approaches an Einstein metric uniformly as t goes to the maximal time T, we would need to define the **normalized Ricci flow**, which is just a rescale of $g(t)$ in order to keep the volume of $(M^n, g(t))$ equal to 1. After that, one can show that this new flow exists for all time and asymptotically approaches an Einstein metric. Then one would need to prove some of the results already established in this dissertation for this normalized Ricci flow. Using these results, one can show that this convergence is exponential in every C^k norm. Therefore, every compact 3-manifold with initially strictly positive Ricci curvature admits a metric with constant positive sectional curvature. For more details, we refer the reader to [14], [12] and [5].

Appendix A

Existence Theory for Parabolic PDEs

In this appendix, we will present the existence results for PDEs used throughout the text. This is mainly based on [28]. Consider a vector bundle $\pi : E \longrightarrow M$, where M is a Riemannian manifold, with a fixed bundle metric h. We are interested in the following type of PDEs:

$$\begin{aligned} \frac{\partial}{\partial t} u &= L(u), \\ u(p, 0) &= u_0(p), \end{aligned} \tag{A.1}$$

where $u : M \times [0, T) \longrightarrow E$ is a section of E and $L : C^\infty(E) \longrightarrow C^\infty(E)$ is a differential operator.

First, we fix some notation.

1. $x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_i \in \mathbb{R}$,

2. $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \alpha_i \in \mathbb{N}$,

3. $|\alpha| = \alpha_1 + \dots + \alpha_n$,

4. $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

We use ∂^α to denote the derivative operator of order $|\alpha|$ such that if $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function, then

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

Now let L be a linear differential operator. Then, using the multi-index notation above, we may write

$$L(u) = \sum_{|\alpha| \leq k} L_\alpha \partial^\alpha u,$$

where k is the order of L and $L_\alpha \in Hom(E, E)$. For instance, if $M = \mathbb{R}^n$, $E = \mathbb{R}^n \times \mathbb{R}$, and $k = 2$, then

$$L(u) = \sum_{i,j} a_{ij} \partial^i \partial^j u + \sum_i b_i \partial^i u + cu,$$

where $a_{ij}, b_i, c : \mathbb{R}^n \longrightarrow \mathbb{R}$ are smooth functions.

Definition A.0.1. *We say that the second order operator L is* ***elliptic*** *if the coefficients a_{ij} are uniformly positive definite, which means that there exists some $\lambda > 0$ such that*

$$a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2,$$

$\forall \xi \in \mathbb{R}^n$ *(or, in the case of vector bundles, for all sections in E).*

Definition A.0.2. *We say that equation* (A.1) *is* ***parabolic*** *if L is elliptic.*

Remark. *In chapter 3, we defined strict parabolicity using the principal symbol of the operator. We observe that these definitions are equivalent and the word strict (sometimes we also use strong) is used to distinguish it from more general definitions.*

The Ricci flow mainly gives us **non-linear PDEs**. Hence, we need to know what it means for a non-linear PDE to be parabolic. To do so, we define the linearization of the non-linear operator L, which has already been done in Chapter 3. For this kind of equation, we say that $\frac{\partial}{\partial t} u = L(u)$ is parabolic if $\frac{\partial}{\partial t} u = [DL(v)]u$ is parabolic. Then we get the following result, which will be used in all existence proofs in this dissertation.

Theorem A.0.1. *If equation* (A.1) *is (strictly) parabolic at u_0 (L being linear or non-linear), then there exists a solution on a time interval $[0, \varepsilon)$, for some $\varepsilon > 0$, which is unique as long as it exists.*

This is a classical result on the theory of parabolic PDEs. See, for example, [19].

Appendix B

Proof of Some Results

In this Appendix, we prove two technical results that were not proved in the text. Theorem 5.5.2 shows that if we have bounds on the Riemann curvature tensor, then the Ricci flow provides us bounds on its derivatives as soon as we apply it to our manifold. Proposition 5.5.1, in the other hand, shows that if the Riemann curvature tensor is bounded with regards to the metric $g(t)$, then the higher order derivatives of $g(t)$ with respect to a fixed background metric $\overline{g}$ are also bounded.

Before we begin our proof, let us consider a simple problem, that will be useful on our next calculations. If $Q(t)$ is a 1-parameter family of (1,0)-tensor fields on a solution $(M^n, g(t))$ of the Ricci flow, then we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i Q_j &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_i}Q_j - \Gamma_{ij}^k Q_k\right)\\ &= \nabla_i(\frac{\partial}{\partial t}Q)_j + (\nabla_i R_j^k + \nabla_j R_i^k - \nabla^k R_{ij})Q_k.\end{aligned}$$

Hence, using the evolution equation of g^{-1}, we get

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla Q|^2 &= \frac{\partial}{\partial t}\left(g^{ik}g^{jl}\nabla_i Q_j \nabla_k Q_l\right)\\ &= 2\nabla^i Q^j \nabla_i(\frac{\partial}{\partial t}Q)_j + 2\nabla^i Q^j(\nabla_i R_j^k + \nabla_j R_i^k - \nabla^k R_{ij})Q_k\\ &+ 2R^{ik}\nabla_i Q^j \nabla_k Q_j + 2R^{jl}\nabla^i Q_j \nabla_i Q_l.\end{aligned}$$

The point here is that when we compute the time derivative of a quantity such as $|\nabla Q|^2$, we have to take into consideration how the metric and its Levi-Civita connection evolve.

Finally, we just introduce a notation convention for the theorem. Consider two tensors, A and B, on a Riemannian manifold. Then we denote by $A * B$ any quantity obtained from $A \otimes B$ by one or more of the following operations:

1. Summation over pairs of matching upper and lower indices;
2. Contraction on upper indices with respect to the metric;
3. Contraction on lower indices with respect to the inverse of the metric;
4. Multiplication by constants that depend only on n, $rank(A)$ and $rank(B)$.

Furthermore, $(A^*)^k$ will denote any k-fold product $A * \cdots * A$.

Theorem B.0.1. *Let $(M^n, g(t))$ be a solution of the Ricci flow, where M^n is a closed manifold. For any $\alpha > 0$ and $m \in \mathbb{N}$, suppose there exists a constant $K > 0$ such that*

$$|Rm(p,t)|_{g(p,t)} \leq K,$$

$\forall p \in M^n$ and $t \in [0, \frac{\alpha}{K}]$, then there exists a positive constant C_m depending only on m, n and $max\{\alpha, 1\}$ such that

$$|\nabla^m Rm(p,t)|_{g(p,t)} \leq \frac{C_m K}{t^{\frac{m}{2}}},$$

$\forall p \in M^n$ and $t \in (0, \frac{\alpha}{K}]$.

Proof. **We will apply complete induction on m. First, let m=1.** Then, we see that the evolution equation for $|\nabla Rm|^2$ is

$$\frac{\partial}{\partial t}|\nabla Rm|^2 = 2\langle \nabla \left(\frac{\partial}{\partial t} Rm\right), \nabla Rm\rangle + \nabla Ric * Rm * \nabla Rm + Ric * [(\nabla Rm)^*]^2,$$

using the notation introduced above and the evolution equations for $|\nabla Q|^2$ previously described. Now, we want to understand $\nabla(\frac{\partial}{\partial t} Rm)$. Before that, we must understand how the commutator $[\nabla, \Delta]$ acts on a tensor. Let A be any tensor. The technique for commuting derivatives gives us the following:

$$\begin{aligned}[\nabla_k, \Delta]A &= \nabla_k \Delta A - \Delta \nabla_k A = g^{ij}\left(\nabla_k \nabla_i \nabla_j A - \nabla_i \nabla_j \nabla_k A\right) \\ &= g^{ij}\left([\nabla_k, \nabla_i]\nabla_j A + \nabla_i \nabla_k \nabla_j A - \nabla_i \nabla_j \nabla_k A\right) = g^{ij}\left([\nabla_k, \nabla_i]\nabla_j A + \nabla_i([\nabla_k, \nabla_j]A)\right).\end{aligned}$$

Thus, using the formula for commuting covariant derivatives, we see that $[\nabla, \Delta]$ is of the form

$$[\nabla, \Delta]A = Rm * \nabla A + \nabla(Rm * A) = Rm * \nabla A + \nabla Rm * A.$$

Using the second Bianchi identity, we have

$$[\nabla, \Delta]A = Rm * \nabla A + \nabla Ric * A.$$

Then, using formula (3.20) and replacing the instances of Rc with Rm, one gets

$$\nabla\left(\frac{\partial}{\partial t}Rm\right) = \nabla\left(\Delta Rm + (Rm)^{*2}\right) = \Delta \nabla Rm + Rm * \nabla Rm.$$

Since $\Delta|A|^2 = \Delta(\langle A, A\rangle) = \langle \Delta A, A\rangle + 2\langle \nabla A, \nabla A\rangle$ for any tensor, we conclude that

$$\frac{\partial}{\partial t}|\nabla Rm|^2 = \Delta|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + Rm * (\nabla Rm)^{*2}. \tag{B.1}$$

To obtain a good estimate for $|\nabla Rm|^2$ from the equation above, we must control two possible difficulties. The first one is the term $Rm * (\nabla Rm)^{*2}$ and the other is the fact that, *a priori*, we have no control on $|\nabla Rm|^2$ at $t = 0$. With this goal in mind, we introduce a new quantity:

$$F \coloneqq t|\nabla Rm|^2 + \beta|Rm|^2,$$

where β is a constant that will be chosen later.

At $t = 0$, we get an upper bound $F \leq \beta K^2$. Also, when t is small, the term $Rm * (\nabla Rm)^{*2}$, obtained by differentiating $|\nabla Rm|^2$, can be compensated by a new term we get when we differentiate $|Rm|^2$, $-2\beta|\nabla Rm|^2$. Taking the derivative of F, we get

$$\begin{aligned}\frac{\partial}{\partial t}F &= |\nabla Rm|^2 + t\left(\frac{\partial}{\partial t}|\nabla Rm|^2\right) + \beta\frac{\partial}{\partial t}(|Rm|^2)\\ &= |\nabla Rm|^2 + t\left(\Delta|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + Rm * (\nabla Rm)^{*2}\right) + \beta\left(\Delta|Rm|^2 - 2|\nabla Rm|^2 + (Rm)^{*3}\right)\\ &= \Delta F + (1-2\beta)|\nabla Rm|^2 + tRm * (\nabla Rm)^{*2} - 2t|\nabla^2 Rm|^2 + \beta(Rm)^{*3}\\ &\leq \Delta F + (1 - 2\beta + c_1 t|Rm|)|\nabla Rm|^2 + c_2\beta|Rm|^3,\end{aligned}$$

where c_1 and c_2 depend only on the dimension n. Since $|Rm| \leq K$ for all $t \in [0, \frac{\alpha}{K}]$ by the hypothesis, we get

$$\frac{\partial}{\partial t}F \leq \Delta F + (1 + c_1\alpha - 2\beta)|\nabla Rm|^2 + c_2\beta K^3.$$

Let β be any constant such that $\beta \geq \frac{1+c_1\alpha}{2}$ (note that β depends only on n and α). Then,

$$\frac{\partial}{\partial t}F \leq \Delta F + c_2\beta K^3,$$

$\forall t \in [0, \frac{\alpha}{K}]$. Using the maximum principle, we have

$$\sup_{p\in M} F(p,t) \leq \beta K^2 + c_2\beta K^3 t \leq (1 + c_2\alpha)\beta K^2 \leq {C_1}^2 K^2,$$

$\forall t \in [0, \frac{\alpha}{K}]$, where C_1 is a constant again depending only on n and α. Thus,

$$|\nabla Rm| \leq \sqrt{\frac{F}{t}} \leq \frac{C_1 K}{t^{\frac{1}{2}}},$$

for $0 < t \leq \frac{\alpha}{K}$. **This proves the case m=1.**

Now we prove the inductive step. Suppose that $|\nabla^j Rm|$ is estimated for all $1 \leq j < m$. Let $1 \leq k \leq m$. First, we see that

$$\frac{\partial}{\partial t}|\nabla^k Rm|^2 = 2\langle\frac{\partial}{\partial t}\left(\nabla^k Rm\right), \nabla^k Rm\rangle + Ric * (\nabla^k Rm)^{*2}. \tag{B.2}$$

Now we need to calculate $\frac{\partial}{\partial t}(\nabla^k Rm)$. In fact, by formulas (3.20) and (3.8), we get

$$\begin{aligned}\frac{\partial}{\partial t}\left(\nabla^k Rm\right) &= \nabla^k\left(\frac{\partial}{\partial t}Rm\right) + \sum_{j=0}^{k-1}\nabla^j\left(\nabla Ric * \nabla^{k-1-j}Rm\right)\\ &= \nabla^k\left(\Delta Rm + (Rm)^{*2}\right) + \sum_{j=1}^{k}\nabla^j Rm * \nabla^{k-j}Rm\\ &= \nabla^k \Delta Rm + \sum_{j=0}^{k}\nabla^j Rm * \nabla^{k-j}Rm.\end{aligned}$$

Similar to what we did in the case m=1, we shall substitute $\nabla^k\Delta Rm$ by $\Delta\nabla^k Rm$ and add a compensation term. For any tensor A, we have

$$[\nabla^k, \Delta]A = \nabla^k\Delta A - \Delta\nabla^k A = \sum_{j=0}^{k}\nabla^j Rm * \nabla^{k-j}A.$$

Hence,

$$\frac{\partial}{\partial t}\left(\nabla^k Rm\right) = \Delta\nabla^k Rm + \sum_{j=0}^{k}\nabla^j Rm * \nabla^{k-j}Rm.$$

Finally, we get from (B.2) and $\Delta|A|^2 = \Delta(\langle A, A\rangle) = \langle\Delta A, A\rangle + 2\langle\nabla A, \nabla A\rangle$ that

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k Rm|^2 &= 2\langle\Delta\nabla^k Rm + \sum_{j=0}^{k}\nabla^j Rm * \nabla^{k-j}Rm, \nabla^k Rm\rangle + Ric * (\nabla^k Rm)^{*2}\\ &= \Delta|\nabla^k Rm|^2 - 2|\nabla^{k+1}Rm|^2 + \sum_{j=0}^{k}\nabla^j Rm * \nabla^{k-j}Rm * \nabla^k Rm.\end{aligned} \tag{B.3}$$

Now, if $k = m$, we get

$$\frac{\partial}{\partial t}|\nabla^m Rm|^2 \leq \Delta|\nabla^m Rm|^2 + \sum_{j=0}^{m} C_{mj}|\nabla^j Rm||\nabla^{m-j}Rm||\nabla^m Rm|,$$

where C_{mj} depends only on j, m and n, $\forall 0 \leq j \leq m$.

Using the inductive hypothesis, we can estimate $|\nabla^{m-j}Rm|$ for all $0 \leq j \leq m$ and

$|\nabla^j Rm|$ for all $0 \leq j < m$. Thus, we have

$$\frac{\partial}{\partial t}|\nabla^m Rm|^2 \leq \Delta|\nabla^m Rm|^2 + (C_{m0} + C_{mm})K|\nabla^m Rm|^2 + \left(\sum_{j=1}^{m-1} C_{mj} \frac{C_j}{t^{\frac{j}{2}}} \frac{C_{(m-j)}}{t^{\frac{(m-j)}{2}}}\right) K^2 |\nabla^m Rm|$$
$$\leq \Delta|\nabla^m Rm|^2 + K\left(C'_m|\nabla^m Rm|^2 + \frac{C''_m}{t^{\frac{m}{2}}} K|\nabla^k Rm|\right),$$

$\forall t \in (0, \frac{\alpha}{K}]$, where C'_m and C''_m depend only on m and n. Now, if we regard the term in parenthesis as an incomplete square, we get a new constant $\overline{C}_m$, also depending only on m and n, such that

$$\frac{\partial}{\partial t}|\nabla^m Rm|^2 \leq \Delta|\nabla^m Rm|^2 + \overline{C}_m K\left(|\nabla^m Rm|^2 + \frac{K^2}{t^m}\right). \tag{B.4}$$

Analogous to the case $m = 1$, we define a new quantity

$$G := t^m|\nabla^m Rm|^2 + \beta_m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} t^{m-k} |\nabla^{m-k} Rm|^2.$$

Defined like this, we observe that G satisfies

$$G \leq \beta_m (m-1)! K^2$$

when $t = 0$. Using equation (B.3), we see that there are constants $\overline{C}_k$, by the inductive hypothesis, such that for any $1 \leq k < m$, we have

$$\frac{\partial}{\partial t}|\nabla^k Rm|^2 \leq \Delta|\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 + \frac{\overline{C}_k K^3}{t^k}, \tag{B.5}$$

$\forall t \in (0, \frac{\alpha}{K}]$. We observe that we retained the term $-2|\nabla^{k+1} Rm|$ on (B.5), although the same term was dropped on (B.4). This term will be helpful further on the proof. Now, we compute the evolution equation for G.

$$
\begin{aligned}
\frac{\partial}{\partial t}G &= mt^{m-1}|\nabla^m Rm|^2 + t^m\left(\frac{\partial}{\partial t}|\nabla^m Rm|^2\right) + \beta_m\sum_{k=1}^{m}\frac{(m-1)!}{(m-k-1)!}t^{m-k-1}|\nabla^{m-k}Rm|^2 \\
&+ \beta_m\sum_{k=1}^{m}\frac{(m-1)!}{(m-k)!}t^{m-k}\left(\frac{\partial}{\partial t}|\nabla^{m-k}Rm|^2\right) \\
&\le mt^{m-1}|\nabla^m Rm|^2 + t^m\left[\Delta|\nabla^m Rm|^2 + \overline{C}_m K\left(|\nabla^m Rm|^2 + \frac{K^2}{t^m}\right)\right] \\
&+ \beta_m\sum_{k=1}^{m}\frac{(m-1)!}{(m-k-1)!}t^{m-k-1}|\nabla^{m-k}Rm|^2 \\
&+ \beta_m\sum_{k=1}^{m}\frac{(m-1)!}{(m-k)!}t^{m-k}\left[\Delta|\nabla^{m-k}Rm|^2 - 2|\nabla^{m-k+1}Rm|^2 + \frac{\overline{C}_{m-k}K^3}{t^{m-k}}\right].
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\frac{\partial}{\partial t}G \le& \Delta G + \overline{C}_m K t^m|\nabla^m Rm|^2 + mt^{m-1}|\nabla^m Rm|^2 + \overline{C}_m K^3 \\
&+ \beta_m\sum_{k=1}^{m}\frac{(m-1)!}{(m-k)!}\left[-2t^{m-k}|\nabla^{m-k+1}Rm|^2 + \overline{C}_{m-k}K^3 + (m-k)t^{m-k-1}|\nabla^{m-k}Rm|^2\right],
\end{aligned}
$$

which gives us the estimate

$$
\frac{\partial}{\partial t}G \le \Delta G + \left(\overline{C}_m K t + m - 2\beta_m\right)t^{m-1}|\nabla^m Rm|^2 + \left(\overline{C}_m + \beta_m\overline{C}_m^{'}\right)K^3,
$$

where

$$
\overline{C}_m^{'} := \sum_{k=1}^{m}\frac{(m-1)!}{(m-k)!}\overline{C}_{m-k}.
$$

In the estimate above, the terms $-2\frac{(m-1)!}{(m-k)!}t^{m-k}|\nabla^{m-k+1}Rm|^2$ compensate the terms

$$
\frac{(m-1)!}{(m-k+1)!}(m-k+1)t^{m-k}|\nabla^{m-k+1}Rm|^2
$$

since

$$\sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!}(m-k)t^{m-(k-1)}|\nabla^{m-k}Rm|^2 - \sum_{k=2}^{m} 2\frac{(m-1)!}{(m-k)!}t^{m-k}|\nabla^{m-(k-1)}Rm|^2 =$$
$$= \sum_{k=2}^{m}\left[\frac{(m-1)!}{(m-k+1)!}(m-k+1)t^{m-k}|\nabla^{m-(k-1)}Rm|^2 - 2\frac{(m-1)!}{(m-k)!}t^{m-k}|\nabla^{m-(k-1)}Rm|^2\right]$$
$$= -\sum_{k=2}^{m} \frac{(m-1)!}{(m-k)!}t^{m-k}|\nabla^{m-(k-1)}Rm|^2,$$

which is a nice term for our estimate. This is why we work with G, not $|\nabla^m Rm|^2$ directly.

Now, if we choose $\beta_m \geq \frac{(\overline{C}_m\alpha+m)}{2}$, we have

$$\frac{\partial}{\partial t}G \leq \Delta G + \left(\overline{C}_m + \beta\overline{C'}_m\right)K^3,$$

$\forall t \in [0, \frac{\alpha}{K}]$. Also, we observe that β_m depends only on n,m and α. Since $G \leq \beta_m(m-1)!K^3$ at $t=0$, the scalar case of the maximum principle gives us

$$\begin{aligned}\sup_{p\in M} G(p,t) &\leq \beta_m(m-1)!K^2 + \left(\overline{C}_m + \beta_m\overline{C'}_m\right)K^3 t \\ &\leq \beta_m(m-1)!K^2 + \left(\overline{C}_m + \beta_m\overline{C'}_m\right)K^3\frac{\alpha}{K} \\ &= \left[\beta_m(m-1)! + \left(\overline{C}_m + \beta_m\overline{C'}_m\right)\alpha\right]K^2,\end{aligned}$$

for $0 \leq t \leq \frac{\alpha}{K}$. Thus, if $C_m := \sqrt{\beta_m(m-1)! + \left(\overline{C}_m + \beta_m\overline{C'}_m\right)\alpha}$, we get

$$|\nabla^m Rm| \leq \sqrt{\frac{G}{t^m}} \leq \frac{C_m K}{t^{\frac{m}{2}}},$$

for $0 < t \leq \frac{\alpha}{K}$. This proves the inductive step and, therefore, the theorem.

□

Finally, we prove **Proposition 5.5.1**.

Proposition B.0.1. *Let $(M^n, g(t))$ be a solution of the Ricci flow on a compact manifold* *with a fixed background metric* $\overline{g}$ *and a connection* $\overline{\nabla}$*. If there exists* $K > 0$ *such that*

$$|Rm(p,t)|_g \leq K$$

$\forall p \in M^n$ *and* $\forall t \in [0,T]$*, then for every* $m \in \mathbb{N}$*, there exists a constant* C_m *that depends on* $m, n, K, T, g_0 = g(0)$ *and the pair* $(\overline{g}, \overline{\nabla})$ *such that*

$$\left|\overline{\nabla}^m g(p,t)\right|_{\overline{g}} \leq C_m,$$

$\forall p \in M^n$ *and* $\forall t \in [0,T]$.

Proof. First, we use the compactness of M^n to get a finite atlas in which we have uniform estimates on the derivatives of the local charts. Then, we fix one of these charts, $\varphi : U \subset M^n \longrightarrow \mathbb{R}^n$. Now, since the pair $(\overline{g}, \overline{\nabla})$ is fixed, we only need to prove that for each $m \in \mathbb{N}$, we can find a constant C_m depending on m, n, K, Γ and g_0 so that the following holds

$$|\partial^m g(p,t)| \leq C_m,$$

$\forall p \in U$ and $\forall t \in [0,T)$, where $|.| = |.|_\delta$ is taken with respect to the Euclidean metric δ in U. Also, we shall regard Γ as a tensor in U, being the difference of the Levi-Civita connection on g and the background flat metric in U. As expected, we will complete our proof by induction on m.

In what follows, C will be a generic constant that may change from line to line, but it will always depend only on m, n, K, Γ and g_0. Let $\beta = \beta(K,T)$ so that $0 < \beta < min\{KT, 1\}$.
Using Corollary 5.5.3.1, we get uniform pointwise estimates for $g(t)$ on $(0,T]$. Now, we shall estimate the first derivatives of the metric, i.e., the case $m = 1$.

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_i} g_{jk}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial t} g_{jk}\right) = -2\frac{\partial}{\partial x_i} R_{jk} = -2\left(\nabla_i R_{jk} + \Gamma^l_{ij} R_{lk} + \Gamma^l_{ik} R_{jl}\right).$$

Since $|Rm(p,t)|_g \leq K$ by hypothesis, we have

$$\left|\frac{\partial}{\partial t}\partial g\right| = 2|\partial Ric| \leq 2|\nabla Ric| + CK|\Gamma|. \tag{B.6}$$

Additionally, we know from equation (3.8) that

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right).$$

This gives us another estimate:

$$\left|\frac{\partial}{\partial t}\Gamma\right| \leq C\left|\nabla Ric\right|.$$

Now, Corollary 5.5.2.1 says that there exists a constant $B = B(m,n,K,\beta)$ such that $|\nabla Ric| \leq B$ holds on $(\frac{\beta}{K}, T)$. Furthermore, since $|\Gamma|$ is bounded on $[0, \frac{\beta}{K}]$ by some constant $A = A(K,\beta,g_0)$, we see that

$$|\Gamma(p,t)| \leq A + BC(T - \frac{\beta}{K}) \leq C \tag{B.7}$$

$\forall p \in M$ and $\forall t \in [0,T)$.
Then, since $|\nabla Ric|$ is bounded on $[0, \frac{\beta}{K}]$ by some $D = D(K,\beta,g_0)$, (B.6) gives us

$$|\partial g| \leq |\partial g_0| + CD\frac{\beta}{K} + (2B + C)(T - \frac{\beta}{K}) \leq C$$

by the maximum principle.

Now, we prove the inductive step. Let $\alpha = (a_1, \ldots, a_r)$ be any multi-index with $|\alpha| = m$. Then, since

$$\frac{\partial}{\partial t}\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}\right) = -2\left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}\right),$$

we just need a bound for $|\partial^\alpha Ric|$. We first consider the case $m = 2$. Then

$$\begin{aligned}\partial_i\partial_j R_{kl} =& \nabla_i\nabla_j R_{kl} + \left[\Gamma_{ij}^p\nabla_p R_{kl} + \Gamma_{ik}^p\nabla_j R_{pl} + \Gamma_{il}^p\nabla_j R_{kp} + \Gamma_{jk}^p\nabla_i R_{pl} + \Gamma_{jl}^p\nabla_i R_{kp}\right] \\ &+ \left[\Gamma_{ip}^q\Gamma_{jl}^p R_{kq} + \Gamma_{ip}^q\Gamma_{jk}^p R_{ql} + \Gamma_{il}^p\Gamma_{jk}^q R_{qp} + \Gamma_{ik}^p\Gamma_{jl}^q R_{pq}\right] + \left[\partial_i\Gamma_{jk}^p R_{pl} + \partial_i\Gamma_{jl}^p R_{kp}\right].\end{aligned}$$

Then, considering the general case, we see that

$$|\partial^m Ric| \leq \sum_{i=0}^{m} C_i \left|\Gamma^i\right| \left|\nabla^{m-i} Ric\right| + \sum_{i=1}^{m-1} C_i^{'} \left|\partial^i \Gamma\right| \left|\partial^{m-1-i} Ric\right|. \tag{B.8}$$

Now, by Corollary 5.5.2.1 and using our estimate on $|\Gamma|$, (B.7), we get

$$\sum_{i=0}^{m} \left|\Gamma^i\right| \left|\nabla^{m-i} Ric\right| \leq C \sup_{0 \leq t \leq \frac{\beta}{K}} \left[\sum_{i=0}^{m} \left|\nabla^{m-i} Ric\right|\right] + C \sum_{i=0}^{m} C_{m-i} K^{1+\frac{m-i}{2}} \leq C.$$

For the other term on B.8, we may apply the inductive step, i.e., suppose that $|\partial_t \partial^p g|$ (equivalently $|\partial^p Ric|$) has been estimated for all $0 \leq p < m-1$. Then (B.8) implies that we have bounds for $|\partial^i \Gamma|$, with $1 \leq i \leq m-2$. So we just need to estimate $|\partial^{m-1}\Gamma|$ and our proof is done. Using (3.8) again, we see that

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial^{m-1}}{\partial x^{p_1} \dots \partial x^{p_{m-1}}} \Gamma_{ij}^k\right) &= \frac{\partial^{m-1}}{\partial x^{p_1} \dots \partial x^{p_{m-1}}} \left(\frac{\partial}{\partial t} \Gamma_{ij}^k\right) \\ &= \frac{\partial^{m-1}}{\partial x^{p_1} \dots \partial x^{p_{m-1}}} \left[-g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right)\right]. \end{aligned}$$

Hence, using the inductive hypothesis and Corollary 5.5.3.1, we have

$$\begin{aligned} \left|\frac{\partial}{\partial t} \partial^{m-1} \Gamma\right| &\leq C \sum_{i=0}^{m-1} \left|\partial^{m-1-i}(g^{-1})\right| \left|\partial^i \nabla Ric\right| \\ &\leq C \sum_{i=0}^{m-1} \left|\partial^{m-1-i} g\right| \left|\partial^i \nabla Ric\right| \\ &\leq C \sum_{i=0}^{m-1} \left|\partial^i \nabla Ric\right|, \end{aligned} \tag{B.9}$$

since g being bounded implies that g^{-1} is bounded and the same is true for the derivatives of each one because

$$\frac{\partial}{\partial x_i} g^{jk} = -g^{jp} g^{kq} \frac{\partial}{\partial x_i} g_{pq}.$$

Similarly to what we did in (B.8), we get

$$\left|\partial^i \nabla Ric\right| \leq \sum_{j=0}^{i} \overline{C}_j \left|\Gamma^j\right| \left|\nabla^{i+1-j} Ric\right| + \sum_{j=1}^{i-1} \overline{C'}_j \left|\partial^j \Gamma\right| \left|\partial^{i-j} Ric\right|,$$

where $\overline{C}_j$ and $\overline{C'}_j$ depend only on p and n. Finally, we apply this to (B.9) to get

$$\left|\frac{\partial}{\partial t}\partial^{m-1}\Gamma\right| \leq C\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}\left|\Gamma^j\right|\left|\nabla^{i+1-j}Ric\right| + \sum_{j=1}^{i-1}\left|\partial^j\Gamma\right|\left|\partial^{i-j}Ric\right|\right).$$

Since all the terms on the right-hand side have already been bounded, we get a bound on $\left|\frac{\partial}{\partial t}\partial^{m-1}\Gamma\right|$ and therefore, $|\partial^{m-1}\Gamma| \leq C + CT$ and this completes the proof.

□

Appendix C

The Maximum Principle and Other Geometric Flows

In this Appendix, we briefly introduce the curve shortening flow (CSF) and the mean curvature flow (MCF), with the aim of showing how the maximum principle and the ideas developed by Hamilton can be used in other geometric flows. In fact, the CSF can be seen as the one dimensional case of the MCF. However, it is interesting to study both cases separately. In particular, we observe the similarity between the results for the Ricci flow presented in this work and the results for the curve shortening flow in this appendix. The work presented in here is mainly based on [17], [11], [10] and [15].

Definition C.0.1. *A one-parameter family of embedded curves* $\{\Gamma_t \subset \mathbb{R}^2\}_{t\in I}$ *moves by curve shortening flow if the normal velocity at each point is given by the curvature vector, i.e., if we consider embeddings* $\gamma = \gamma(.,t) : S^1 \times I \longrightarrow \mathbb{R}^2$ *with* $\Gamma_t = \gamma(S^1, t)$, *then*

$$\frac{\partial}{\partial t}\gamma(x,t) = \kappa(x,t)N(x,t), \tag{C.1}$$

where κ *is the curvature of the curve and* N *is its inward pointing unit normal vector.*

Remark. *Equation* (C.1) *can also be written as*

$$\frac{\partial}{\partial t}\gamma = \frac{\partial^2}{\partial s^2}\gamma,$$

where s is the arc length of γ. This is almost the heat equation, but the arc length depends on (x,t) in a nonlinear way, so the curve shortening flow is actually a nonlinear PDE.

Theorem C.0.1. *Let $\gamma_0 : S^1 \longrightarrow \mathbb{R}^2$ be an embedded curve. Then there exists a unique smooth solution $\gamma : S^1 \times [0,T) \longrightarrow \mathbb{R}^2$ of the curve shortening flow*

$$\partial_t \gamma = \partial_s^2 \gamma,$$
$$\gamma(.,0) = \gamma_0(.),$$

defined on a maximal time interval $[0,T)$. Besides, the maximal existence time is characterized by

$$\sup_{S^1 \times [0,T)} |\kappa(x,t)| = \infty.$$

Proof. For a comprehensive proof of this result, with the use of the Maximum Principle, see [15]. □

Proposition C.0.1. *Let the closed curve $\gamma(t)$ be a solution to the CSF and let $L(t)$ be the length of the solution at time t, given by*

$$L(t) = \int_{S^1} \sqrt{\langle \partial_x \gamma, \partial_x \gamma \rangle} dx.$$

Then

$$\frac{d}{dt} L(t) = -\int_{\Gamma_t} \kappa^2 ds,$$

that is, the curve shortening flow is the gradient flow of the length functional. Therefore, it shortens curves in the fastest way possible.

Proof. See [10], Lemma 3.1.2. □

Proposition C.0.2. *Let the closed curve $\gamma(t)$ be a solution to the CSF and let $A(t)$ be the area enclosed by the solution at time t, then*

$$\frac{d}{dt} A(t) = -2\pi.$$

In particular, $A(t) = A(0) - 2\pi$ *and* $T \leq \frac{A(0)}{2\pi}$.

Proof. See [10], Lemma 3.1.7. □

Proposition C.0.3. *If* $\{\Gamma_t \subset \mathbb{R}^2\}$ *evolves by the curve shortening flow, then its curvature evolves by*

$$\kappa_t = \kappa_{ss} + \kappa^3,$$

where s *denotes arc length.*

Proof. See [10], Lemma 3.1.6. □

Corollary C.0.1.1. *Convexity is preserved under curve shortening flow, i.e., if* $\kappa > 0$ *at* $t = 0$ *then* $\kappa > 0$ *for all* $t \in [0, T)$.

Proof. See [10]. This is a direct consequence of the proposition above and the maximum principle. In fact, if $\kappa_{min}(t) := \min_{\Gamma_t} \kappa$ is positive at $t = 0$, then it is nondecreasing in time and satisfies

$$\kappa_{min}(t) \geq \frac{\kappa_{min}(0)}{1 - 2t\kappa_{min}^2(0)}.$$

□

Theorem C.0.2. *There exist constants* $C_m(K, T) < \infty$ *such that if* $\{\Gamma_t \subset \mathbb{R}^2\}$ *is a solution of the curve shortening flow with*

$$\sup_{t \in [0,T)} |\kappa| \leq K,$$

then

$$\sup_{\Gamma_t} |\partial_s^m \kappa| \leq \frac{C_m}{t^{\frac{m}{2}}}.$$

Proof. For a proof using the maximum principle, see [15]. □

Theorem C.0.3. *If* $\Gamma \subset \mathbb{R}^2$ *is a closed embedded curve, then the curve shortening flow* $\{\Gamma_t\}_{t \in [0,T)}$ *with* $\Gamma_0 = \Gamma$ *exists until* $T = \frac{A(\Gamma)}{2\pi}$ *and converges for* $t \to T$ *to a round point,* *i.e., there exists a unique point $x_0 \in \mathbb{R}^2$ such that the rescaled flows*

$$\Gamma_t^\lambda := \lambda\Gamma_{(T+\lambda^{-2}t - x_0)}$$

converge for $\lambda \to \infty$ to the round shrinking circle $\{\partial B_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$.

Proof. See [11]. □

Now we state some results regarding the mean curvature flow. Consider a closed surface $M^n = M_0^n$, $n \geq 2$, which is uniformly convex and embedded in $\mathbb{R}^{n+1}$. Let $F_0 : U \subset \mathbb{R}^n \longrightarrow M_0$ be a local chart to M_0. Then we say that a family of hypersurfaces $M(t) \subset \mathbb{R}^n$ evolves by the mean curvature flow if there exist local charts satisfying

$$\frac{\partial}{\partial t}F(x,t) = \Delta_t F(X,t) = -H(x,t)\nu(x,t), \tag{C.2}$$

$$F(.,0) = F_0, \tag{C.3}$$

where Δ_t is the Laplace-Beltrami operator on the manifold M_t, given by $F(.,t)$, $H(.,t)$ is the mean curvature of M_t and $\nu(.,t)$ is the outer unit normal on M_t.

Theorem C.0.4. *The evolution equation C.2 has a solution M_t for a short time with any smooth closed initial surface $M = M_0$ at $t = 0$.*

Proof. See [17]. □

Lemma C.0.5. *The metric of M_t satisfies*

$$\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij},$$

where h_{ij} are the coefficients of the second fundamental form $A(t)$ of M_t.

Proof. See [17], Lemma 3.2. □

Lemma C.0.6. *The unit normal to M_t satisfies $\frac{\partial}{\partial t}\nu = \nabla H$.*

Proof. See [17], Lemma 3.3. □

Proposition C.0.4. *The mean curvature of M_t satisfies*

$$\frac{\partial}{\partial t}H = \Delta H + |A|^2 H.$$

Proof. See [17], Corollary 3.5. □

Remark. *Observe that H satisfies a heat-type equation and we have the following estimate*

$$\frac{\partial}{\partial t}H \geq \Delta H + \frac{1}{n}H^3. \tag{C.4}$$

Then, if $H(x,0) > 0\ \forall x \in M^n$, then we can consider $\varphi(t)$ solution of the associated ODE

$$\varphi'(t) = \frac{1}{n}\varphi^3(t), \tag{C.5}$$

$$\varphi(0) = H_{min}(0) > 0. \tag{C.6}$$

The maximum principle says that $H(x,t) \geq \varphi(t)$ as long as the solution exists. Therefore,

$$H(x,t) \geq \frac{H_{min}(0)}{\sqrt{1 - \frac{2}{n}H^2_{min}(0)t}} > 0.$$

Inspired by Hamilton's approach, Huisken studied the eigenvalues of the second fundamental form for the MCF, showed that we have bounds on the gradient of the mean curvature and for the higher derivatives of A, proved that the mean curvature becomes constant as $t \to T$, T being a maximum time interval for the existence of the mean curvature flow. Finally, with the help of the maximum principle, Huisken proved the following result.

Theorem C.0.7. *Let $n \geq 2$ and assume that M_0 is uniformly convex, i.e., the eigenvalues of its second fundamental form are strictly positive everywhere. Then the evolution equation* (C.2) *has a smooth solution on a finite time interval $[0,T)$ and M_t converges to a single point as $t \to T$. The normalized version of the flow has a solution for all positive time and M_t converges exponentially to a sphere of area $A(0)$ in any C^k-norm.*

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